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LINEAR FEEDBACK GUIDANCE

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ABSTRACT

The problem of determining the coefficients for a linear feedback guidance system with polynomial gains is investigated. The solution to the linearized equations of motion is employed to determine the first partial derivatives of the optimum thrust angle deviations with respect to position and velocity coordinates. The partial derivatives are approximated by quadratic time functions, which in turn are the first approximation to a time-variable set of gains. This system is then analyzed, and two of the weighting functions are linearized with respect to perturbations in the polynomial coefficients. Changes in these coefficients are determined to minimize the weighted sum of squares of these expressions over several time points. The resulting changes were made, and a satisfactory reduction in magnitude of these weighting functions resulted. This, coupled with a linear feedback of thrust acceleration and a small time function derived to cancel the effect of initial conditions, produced the final guidance function which performed exceptionally well on a 100 n.m. orbital mission of an early SA-6 second stage vehicle. Of particular interest is a demonstration of the effectiveness of the numerical methods used which are generally applicable to a wide variety of guidance and control feedback problems.

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TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

DEFINITION OF SYMBOLS

Symbol	Definition
$A =$	$\begin{bmatrix} 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \\ \frac{\partial \ddot{x}_g}{\partial x} & \frac{\partial \ddot{x}_g}{\partial y} & 0 & 0 \\ \frac{\partial y_g}{\partial x} & \frac{\partial y_g}{\partial y} & 0 & 0 \end{bmatrix}$
$\bar{A} =$	$A + H_1B$
$\hat{A} =$	$\bar{A} + \Delta A$
$B =$	$[a(t) \quad b(t) \quad c(t) \quad d(t)]$
$\hat{B} =$	$[\hat{a}(t) \quad \hat{b}(t) \quad \hat{c}(t) \quad \hat{d}(t)]$
$F_i =$	$\begin{bmatrix} f_i \\ g_i \end{bmatrix}$
$\bar{F}_i =$	$\begin{bmatrix} \bar{f}_i \\ \bar{g}_i \end{bmatrix}$
$\hat{F}_i =$	$\begin{bmatrix} \hat{f}_i \\ \hat{g}_i \end{bmatrix}$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$F_o^* =$	$\begin{bmatrix} f_o^* \\ g_o^* \end{bmatrix}$
$H_o =$	$\begin{bmatrix} 0 \\ 0 \\ \sin \chi_s \\ \cos \chi_s \end{bmatrix}$
$H_{2i+1} =$	$(\pi/180)^{2i+1} \dfrac{(-1)^i}{(2i+1)!} f/m_s \begin{bmatrix} 0 \\ 0 \\ \cos \chi_s \\ -\sin \chi_s \end{bmatrix} \quad i = 0,1,2 \ldots$
$H_{2i} =$	$(\pi/180)^{2i} \dfrac{(-1)^i}{2i!} f/m_s \begin{bmatrix} 0 \\ 0 \\ \sin \chi \\ \cos \chi \end{bmatrix}$
K	a 1 x 4 matrix defined in equation (5.10)
$M(t) =$	$\int_t^{t_n} F_1(t)\, P(t)\, dt$

Symbol	Definition
$P(t) =$	$[p_1(t) \quad p_2(t)]$
T	a 2 x 4 matrix derived from approximating cutoff deviations by a linear transformation on the deviations at standard cutoff time, t_n, so that $$\Delta R = T \Delta X(t_n)$$
$T_1 =$	$\frac{1}{v_n} [0 \quad 0 \quad \dot{x}_n \quad \dot{y}_n]$
$U(t) =$	$TU(t_n, t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$
$U_o(t)$	a 1 x 4 matrix such that $\Delta t = U_o(t)\ \Delta X(t)$
$U_1(t)$	a 1 x 4 matrix such that $\Delta r = U_1(t)\ \Delta X(t)$
$U_2(t)$	a 1 x 4 matrix such that $\Delta\theta = U_2(t)\ \Delta X(t)$
$U(t, t_o)$	solution to the matrix differential equation $$\dot{U}(t, t_o) = A(t)\ U(t, t_o),\ U(t_o, t_o) = I$$
$\bar{U}(t, t_o)$	solution to the matrix differential equation $$\dot{\bar{U}}(t, t_o) = \bar{A}(t)\ U(t, t_o),\ \bar{U}(t_o, t_o) = I$$
$\dot{W}$	propellant flow rate

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$X =$	$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$
$a(t)$	quadratic approximation to the function $\partial X/\partial x$
a_i	coefficient of the i^{th} powered term of $a(t)$
$\hat{a}(t) =$	$a(t) + \Delta a(t)$
$\hat{a}_i =$	$a_i + \Delta a_i$
$b(t)$	quadratic approximation to the function $\partial X/\partial y$
b_i	coefficient of the i^{th} powered term of $b(t)$
$\hat{b}(t) =$	$b(t) + \Delta b(t)$
$\hat{b}_i =$	$b_i + \Delta b_i$
$c(t)$	quadratic approximation to the function $\partial X/\partial \dot{x}$
c_i	coefficient of the i^{th} powered term of $\dot{c}(t)$
$\hat{c}(t) =$	$c(t) + \Delta c(t)$
$\hat{c}_i =$	$c_i + \Delta c_i$
$d(t)$	quadratic approximation to the function $\partial X/\partial \dot{y}$
d_i	coefficient of the i^{th} powered term of $d(t)$
$\hat{d}(t) =$	$d(t) + \Delta d(t)$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\hat{d}_i =$	$d_i + \Delta d_i$
$e(t)$	quadratic function of time representing the time variable gain for f/m
e_i	coefficient of the i^{th} ordered term of $e(t)$
f	thrust force
f/m	thrust acceleration
$f(\Delta X, \Delta f/m, t)$	defined by equation (2.5)
f_i	the top element of the vector $TU(t_n,t)\ H_i(t)$
$\bar{f}_i$	the top element of the vector $T\bar{U}(t_n,t)\ H_i(t)$
$\hat{f}_i$	the top element of the vector $T[\bar{U}(t_n,t) + \Delta U(t_n,t)]\ H_i(t)$
f_o^*	linear approximation to $\hat{f}_o$
$g(\Delta X, \Delta f/m, t)$	defined by equation (2.6)
g_i	the bottom element of the vector $TU(t_n,t)\ H_i(t)$
$\bar{g}_i$	the bottom element of the vector $T\bar{U}(t_n,t)\ H_i(t)$
$\hat{g}_i$	the bottom element of the vector $T[\bar{U}(t_n,t) + \Delta U(t_n,t)]\ H_i(t)$
g_o^*	linear approximation to $\hat{g}_o$
$h(\Delta X, \Delta f/m, t)$	defined in equation (2.4)
$h_i =$	$- \frac{T_1}{T_1 \dot{X}_n} U(t_n,t)\ H_i(t)$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$k(t)$	polynomial of time defined by equation (5.13)
k_i	coefficient of the i^{th} degree term in $k(t)$
$\hat{k}(t) =$	$k(t) + \Delta k(t)$
$\hat{k}_i =$	$k_i + \Delta k_i$
$p_1 =$	$-\dfrac{f_1}{100(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)}$
$p_2 =$	$-\dfrac{g_1}{h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2}$
r	magnitude of the radius vector measured from the center of the earth
s	a subscript denoting that the function is evaluated on the standard trajectory
t	time
t_o	second stage ignition time
t_n	cutoff time on the standard trajectory
t_c	cutoff time on any trajectory
v	velocity
v_n	velocity at standard cutoff time
x, y	Cartesian coordinates with origin at the center of the earth $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ represent their first and second time derivatives
$\ddot{x}_g$	x-component of gravitational acceleration
$\ddot{y}_g$	y-component of gravitational acceleration

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\Delta A =$	$H_1 B$
$\Delta B =$	$[\Delta a(t) \quad \Delta b(t) \quad \Delta c(t) \quad \Delta d(t)]$
$\Delta F_o =$	$T \Delta U(t_n, t)\ H_o(t)$
$\Delta F_o^* =$	$T \Delta \bar{U}(tn, t)\ H_o(t)$
ΔK	defined in equation (5.10)
$\Delta \hat{K}$	matrix used to determine $\Delta \hat{k}_i$ such that $\begin{bmatrix} \Delta k_o \\ \Delta k_6 \end{bmatrix} = \Delta \hat{K}\ \Delta X(t_o)$
$\Delta R =$	$\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix}$
$\Delta U(t, t_1)$	solution to the matrix differential equation $\Delta \dot{U}(t, t_1) = \hat{A}(t)\ \Delta U(t, t_1),\ \Delta U(t_1, t_1) = 0$
ΔW	the weight of propellant required for a given example minus the propellant that would have been required for the same example to reach the nominal end condition requirements
$\Delta X(t) =$	$X(t) - X_s(t)$, t measured from second stage ignition on each trajectory
$\Delta a(t) =$	$\Delta a_o + \Delta a_1 \tau + \Delta a_2 \tau^2$
$\Delta b(t) =$	$\Delta b_o + \Delta b_1 \tau + \Delta b_2 \tau^2$
$\Delta c(t) =$	$\Delta c_o + \Delta c_1 \tau + \Delta c_2 \tau^2$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\Delta d(t) =$	$\Delta d_o + \Delta d_1\tau + \Delta d_2\tau^2$
$\Delta a_o,\ \Delta a_1,\ \Delta a_2$ $\Delta b_o,\ \Delta b_1,\ \Delta b_2$ $\Delta c_o,\ \Delta c_1,\ \Delta c_2$ $\Delta d_o,\ \Delta d_1,\ \Delta d_2$	changes in the polynomial gain coefficients chosen to minimize the magnitude of the elements of $\hat{F}_o$
$\Delta k(t) =$	$K(t)\ \Delta X(t_o)$
$\Delta\hat{k}(t)$	polynomial approximation to $\Delta k(t)$
Δk_i	coefficient of the i^{th} degree term of $\Delta\hat{k}(t)$
Δr	deviation of radius vector at cutoff
Δt	additional burning time required beyond that required for the standard trajectory
$\Delta x,\ \Delta y$	deviations in position coordinates compared at equal time from second stage ignition
$\Delta\dot{x},\ \Delta\dot{y}$	time derivatives of Δx and Δy
$\Delta\lambda_1,\ \Delta\lambda_2$	changes in the value of the Lagrange multipliers required to meet the end conditions with a non-standard trajectory
$\Delta\chi$	deviation of thrust angle from its standard value compared at equal times from second stage ignition
$\Lambda =$	$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
$\alpha =$	$\dfrac{100\Delta\lambda_1}{2}$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\beta =$	$\frac{\Delta\lambda_2}{2}$
δX	actual thrust angle minus the value predicted by the guidance function
θ	angle of the velocity vector measured from local vertical
λ_1, λ_2	Lagrange multipliers required to fulfill the constraint that $h_1 + \lambda_1 f_1 + \lambda_2 g_1 = 0$
$\bar{\lambda}_1$, $\bar{\lambda}_2$	values of λ_1 and λ_2 for the standard trajectory
$\tau =$	$\frac{t - t_o}{100}$

TECHNICAL MEMORANDUM X-53362

LINEAR FEEDBACK GUIDANCE

SUMMARY

The problem of determining the coefficients for a linear feedback guidance system with polynomial gains is investigated. The solution to the linearized equations of motion is employed to determine the first partial derivatives of the optimum thrust angle deviations with respect to position and velocity coordinates. The partial derivatives are approximated by quadratic time functions, which in turn are the first approximation to a time-variable set of gains. This system is then analyzed, and two of the weighting functions are linearized with respect to perturbations in the polynomial coefficients. Changes in these coefficients are determined to minimize the weighted sum of squares of these expressions over several time points. The resulting changes were made, and a satisfactory reduction in magnitude of these weighting functions resulted. This, coupled with a linear feedback of thrust acceleration and a small time function derived to cancel the effect of initial conditions, produced the final guidance function which performed exceptionally well on a 100 n.m. orbital mission of an early SA-6 second stage vehicle. Of particular interest is a demonstration of the effectiveness of the numerical methods used which are generally applicable to a wide variety of guidance and control feedback problems.

INTRODUCTION

The problem of determining a closed loop guidance function with linear, time-variable feedback is investigated by studying the behavior of the equations of motion in the neighborhood of a standard calculus of variations solution. The linearized Euler-Lagrange equations are used to determine a good approximation to the optimum thrust angle deviation, ΔX. The solution gives ΔX as a linear combination of known functions of time. The particular linear combination required is that one which fulfills the desired end conditions. The linearly predicted deviations of the end conditions, as a function of deviations in initial conditions, are used to determine the first partial derivatives of the optimum thrust angle with respect to position and velocity components. Quadratic time approximations to these partial derivatives provide the time-variable gains for a linear feedback system which is analyzed by the same methods employed in the open loop analysis described in Reference 1. To reduce the effect that thrust acceleration variations would have on this system, the weighting functions involved are linearized with respect to changes in the coefficients of the gain polynomials. Changes in these coefficients are determined to minimize the weighted sum of squares of the resulting expressions over a number of time points. Although this is only one step of an iterative procedure, the reduction in magnitude of the weighting functions from this one iteration is sufficient to demonstrate the effectiveness of the method. What little effect of thrust acceleration deviations is left is further reduced by the addition of thrust acceleration feedback with gains which are also quadratic time functions. A polynomial of time, whose coefficients are determined as linear functions of initial conditions, effectively cancels the remaining error due to initial conditions. The resulting guidance function has been determined for an early SA-6 second stage vehicle designed for a 100 n.m. circular orbit. The tabulated results for a number of examples reflect the effectiveness with which linear analysis can be applied to problems of this type.

I. LINEARIZED EULER-LAGRANGE EQUATIONS

An explicit solution to the linearized equations of motion was determined in Reference 1. The deviations in end conditions, Δr and $\Delta\theta$, and the additional burning time, Δt, were determined as functions of deviations in initial conditions, ΔX_o, variations in thrust angle, ΔX, and variations in thrust acceleration, $\Delta f/m$. These terminal deviations were found to be adequately represented by the following expressions:

$$\Delta t = U_o(t_o)\,\Delta X(t_o) + \int_{t_o}^{t_n} h(\Delta X, \Delta f/m, t)\,dt, \tag{1.1}$$

$$\Delta r = U_1(t_o)\,\Delta X(t_o) + \int_{t_o}^{t_n} f(\Delta X, \Delta f/m, t)\,dt, \tag{1.2}$$

$$\Delta\theta = U_2(t_o)\,\Delta X(t_o) + \int_{t_o}^{t_n} g(\Delta X, \Delta f/m, t)\,dt, \tag{1.3}$$

where

$$h(\Delta X,\ \Delta f/m,\ t) = h_o\ \Delta f/m + \left(1 + \frac{\Delta f/m}{f/m}\right)(h_1\ \Delta X + h_2\ \Delta X^2 + \ldots), \tag{1.4}$$

$$f(\Delta X,\ \Delta f/m,\ t) = f_o\ \Delta f/m + \left(1 + \frac{\Delta f/m}{f/m}\right)(f_1\ \Delta X + f_2\ \Delta X^2 + \ldots), \tag{1.5}$$

$$g(\Delta X,\ \Delta f/m,\ t) = g_o\ \Delta f/m + \left(1 + \frac{\Delta f/m}{f/m}\right)(g_1\ \Delta X + g_2\ \Delta X^2 + \ldots), \tag{1.6}$$

Using the Euler-Lagrange equation to minimize Δt under the constraint that Δr and $\Delta\theta$ have specified values leads to the following necessary condition:

$$\left[1 + \frac{\Delta f/m}{f/m}\right]\left[(h_1 + \lambda_1 f_1 + \lambda_2 g_1) + 2(h_2 + \lambda_1 f_2 + \lambda_2 g_2)\ \Delta X + \ldots\right] = 0. \tag{1.7}$$

For the standard calculus of variations solution, this means that

$$h_1 + \bar{\lambda}_1 f_1 + \bar{\lambda}_2 g_1 = 0. \tag{1.8}$$

To simplify further analysis, the following definitions are employed

$$\left.\begin{aligned} \lambda_1 &= \bar{\lambda}_1 + \Delta\lambda_1 \\ \lambda_2 &= \bar{\lambda}_2 + \Delta\lambda_2 \end{aligned}\right\}. \tag{1.9}$$

Substitution of these expressions into equation (1.7), neglecting terms higher than first order and taking cognizance of the relationship defined in equation (1.8), results in the following:

$$f_1\Delta\lambda_1 + g_1\Delta\lambda_2 + 2(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)\ \Delta X = 0. \tag{1.10}$$

For convenience, the following quantities are defined:

$$p_1 = \frac{-f_1}{100(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)}$$

$$p_2 = \frac{-g_1}{(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)}$$

$$\alpha = \frac{100\Delta\lambda_1}{2} \tag{1.11}$$

$$\beta = \frac{\Delta\lambda_2}{2}$$

$$P = [p_1 \quad p_2]$$

$$\Lambda = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

With these definitions, it can be readily verified that the following expression for ΔX satisfies equation (1.10):

$$\Delta X = P\Lambda. \tag{1.12}$$

<u>Any linear combination of the functions p_1 and p_2 will satisfy, to first order, the Euler-Lagrange equation</u>. The functions p_1 and p_2

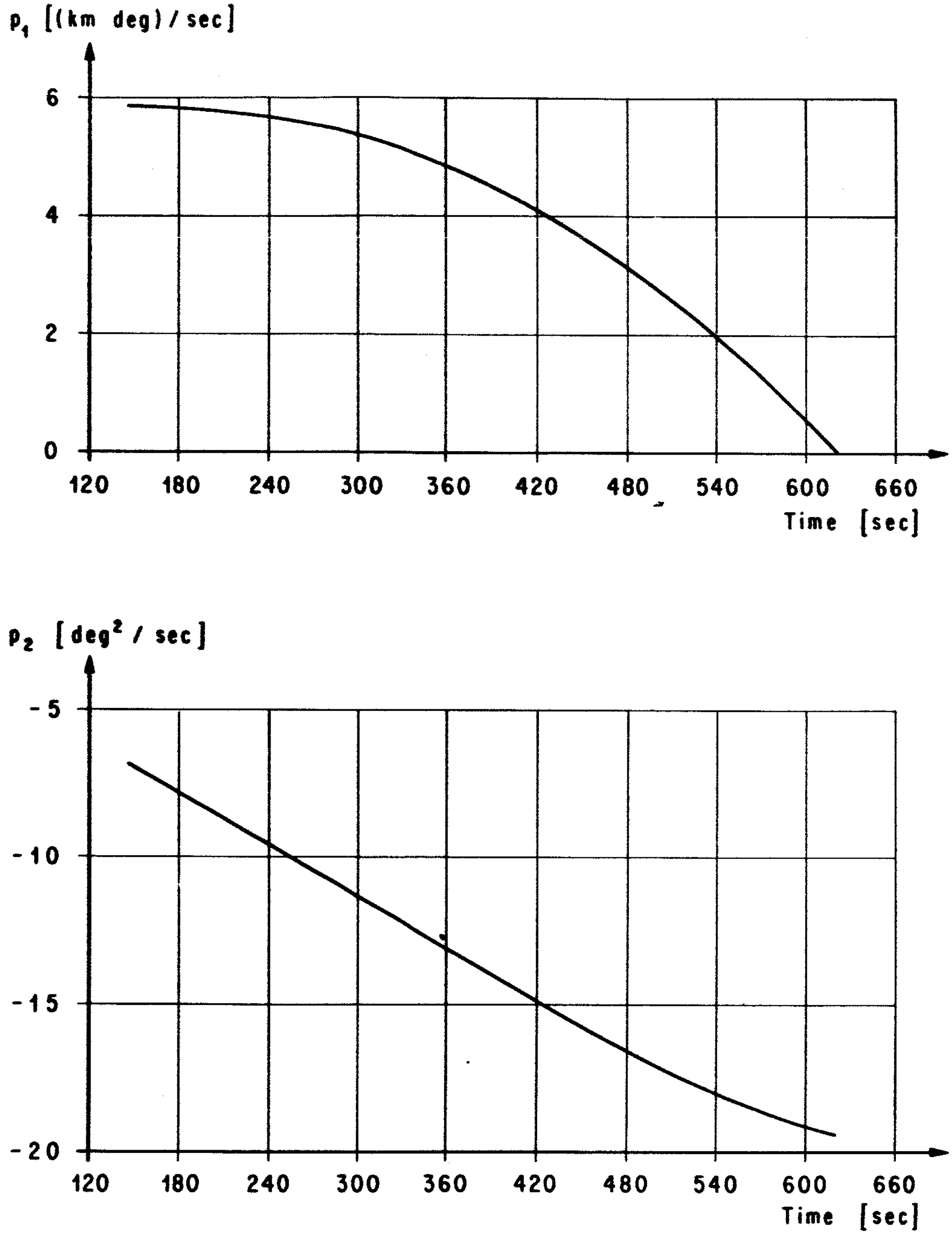

FIG. 1-1. FUNCTIONS USED TO DETERMINE Δp

are illustrated in Figure 1.1. These particular functions can be well represented by the following quadratic functions of time:

$$
\begin{aligned}
p_1 &= 5.8096 + .1734\tau - .2948\tau^2 \\
p_2 &= -6.5992 - 3.2636\tau + .1000\tau^2
\end{aligned}
\tag{1.13}
$$

where

$$\tau = \frac{t - t_o}{100} .$$

It was shown in Reference 1 that for this same mission the value of X on the standard trajectory could be adequately represented by a quadratic in time. Thus, for this example, X can be well represented as a quadratic time function with two arbitrary parameters, α and β. If α and β can be determined in any way so that $\Delta r = \Delta\theta = 0$, the resulting guidance function will satisfy the Euler-Lagrange equation to first order.

In actual practice, a single pair of constants, α and β, to be used throughout the trajectory cannot be determined because of the future uncertainties in the thrust vector. However, if their best approximation is available, a good quadratic approximation to the optimum X is available until such time as later information provides better values for α and β. As the time intervals between corrections in α and β become smaller, α and β become continuous; ΔX then becomes a time variable function of the current state of the system.

II. FIRST PARTIAL DERIVATIVES

In order to attain a first approximation to a linear feedback system, equations (1.2) and (1.3) will be linearized with respect to ΔX, and all other terms will be considered zero except the initial conditions, ΔX_o. Then equation (1.12) can be used to determine the linear effect of ΔX_o on the optimum value of ΔX. To simplify subsequent derivations, the following definitions will be used:

$$\Delta R = \begin{bmatrix} \Delta r \\ \Delta\theta \end{bmatrix} \tag{2.1}$$

$$F_i = \begin{bmatrix} f_i \\ g_i \end{bmatrix}, \quad i = 0,1,2, \ldots \tag{2.2}$$

$$M(t_o) = \int_{t_o}^{t_n} F_1 \, P \, dt \tag{2.3}$$

$$U(t_o) = \begin{bmatrix} u_1(t_o) \\ u_2(t_o) \end{bmatrix}. \tag{2.4}$$

Considering only the effects of ΔX_o and the linear effect of ΔX, equations (1.2) and (1.3) can be written in the following matrix form:

$$\Delta R = U(t_o)\, \Delta X(t_o) + \int_{t_o}^{t_n} F_1 \, \Delta X \, dt. \tag{2.5}$$

Substituting the expression for ΔX obtained in equation (1.12) and using the definition of equation (2.3) yields the following expression:

$$\Delta R = U(t_o)\ \Delta X(t_o) + M(t_o)\ \Lambda . \tag{2.6}$$

Setting $\Delta R = 0$ and solving for Λ gives

$$\Lambda = -M^{-1}(t_o)\ U(t_o)\ \Delta X\ (t_o). \tag{2.7}$$

The expression for ΔX at $t = t_o$ then becomes

$$\Delta X(t_o) = -P(t_o)\ M^{-1}(t_o)\ U(t_o)\ \Delta X(t_o).$$

The uniqueness of an existing series expansion ensures that

$$-P(t_o)\ M^{-1}(t_o)\ U(t_o) = \left[\frac{\partial X}{\partial x}\ \frac{\partial X}{\partial y}\ \frac{\partial X}{\partial \dot{x}}\ \frac{\partial X}{\partial \dot{y}}\right]_{t=t_o},$$

and since the derivation is independent of the particular value of t_o, the following expression is obtained for the first partial derivatives:

$$\left[\frac{\partial X}{\partial x}\ \frac{\partial X}{\partial y}\ \frac{\partial X}{\partial \dot{x}}\ \frac{\partial X}{\partial \dot{y}}\right] = -P(t)\ M^{-1}(t)\ U(t). \tag{2.8}$$

A polynomial fit of these partial derivatives will be used as a first approximation to the desired linear feedback system. Subsequent analysis will determine the acceptability of this system, and methods of improving its performance will be outlined and applied. Figures 2.1 and 2.2 show the values obtained for the first partial derivatives from equation (2.8) where they are compared with their quadratic approximations defined by the following functions:

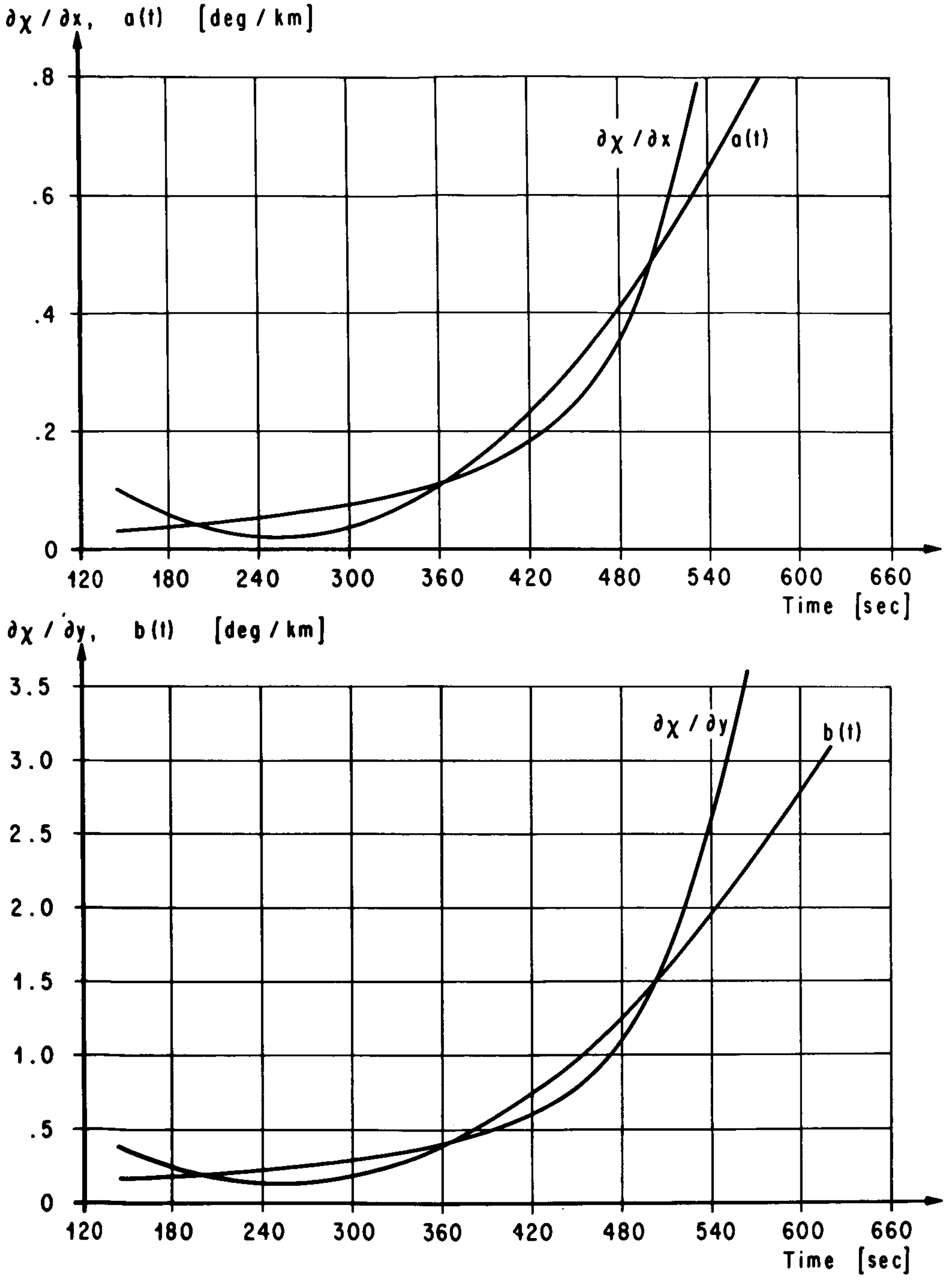

FIG. 2-1. FIRST PARTIALS OF χ WITH RESPECT TO POSITION VARIABLES COMPARED WITH THEIR POLYNOMIAL APPROXIMATION

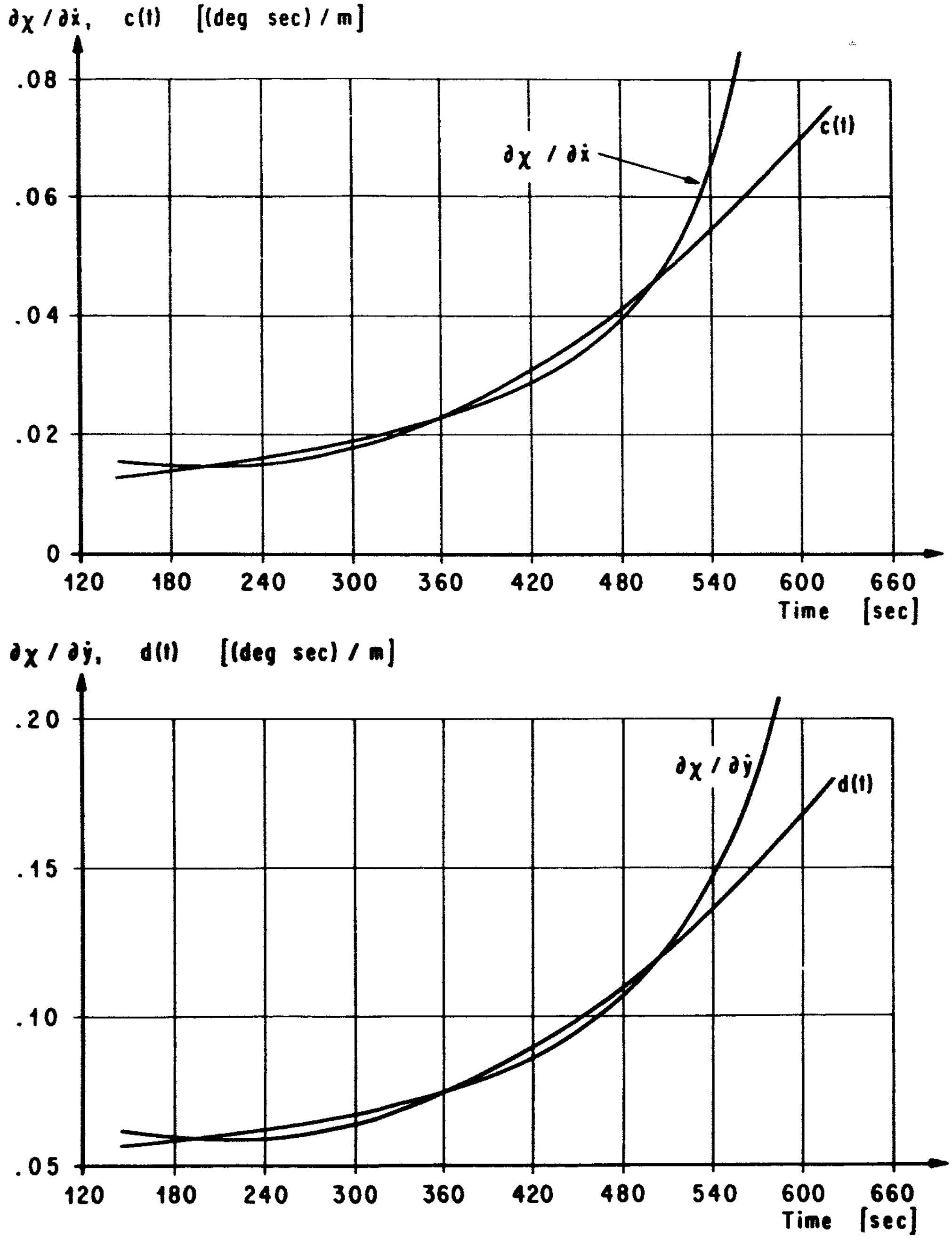

FIG. 2-2. FIRST PARTIALS OF χ WITH RESPECT TO VELOCITY VARIABLES COMPARED WITH THEIR POLYNOMIAL APPROXIMATION

$$\left.\begin{aligned} a(t) &= a_o + a_1\tau + a_2\tau^2 \sim \frac{\partial X}{\partial x} \\ b(t) &= b_o + b_1\tau + b_2\tau^2 \sim \frac{\partial X}{\partial y} \\ c(t) &= c_o + c_1\tau + c_2\tau^2 \sim \frac{\partial X}{\partial \dot{x}} \\ d(t) &= d_o + d_1\tau + d_2\tau^2 \sim \frac{\partial X}{\partial \dot{y}} \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{array}{lll} a_o = .104259 & a_1 = -.160995 & a_2 = .076398 \\ b_o = .37021 & b_1 = -.44893 & b_2 = .21551 \\ c_o = .01551 & c_1 = -.003883 & c_2 = .003477 \\ d_o = .062320 & d_1 = -.010020 & d_2 = .007364 \end{array}\right\} \tag{2.10}$$

$$\tau = \frac{t - t_o}{100} .$$

The deviation in thrust angle defined by this system can be expressed as follows:

$$\Delta X = B\Delta X + \delta X , \tag{2.11}$$

where

$$B = [a(t) \quad b(t) \quad c(t) \quad d(t)]. \tag{2.12}$$

The quantity δX represents any disagreement between the commanded thrust angle and that which actually exists.

The effect on end conditions of deviations in the thrust acceleration vector is reflected by the weighting functions f_i and g_i shown in equations (1.5) and (1.6). A review of the derivation of these equations in Reference 1 reveals the following relationships:

$$\left.\begin{aligned} F_2 &= -\ (\pi/180)^2\,\frac{f/m}{2}\,F_o \\ \text{and} \qquad & \\ F_3 &= -\ (\pi/180)^2\,\frac{1}{6}\,F_1 \end{aligned}\right\} \tag{2.13}$$

where

$$F_i = \begin{bmatrix} f_i \\ g_i \end{bmatrix}, \quad i = 0,1, \ldots .$$

Furthermore, all even ordered vectors are constant multiples of F_2, and odd ordered vectors are constant multiples of F_1. It follows that all the weighting functions can be determined immediately from either F_o or F_1. These two vectors represent the weighting functions associated with the linear terms of $\Delta f/m$ and ΔX, respectively, and can be determined from the solution to the following system:

$$\Delta\dot{X} = A\Delta X + H_o\,\Delta f/m + H_1\,\Delta X. \tag{2.14}$$

The solution is

$$\Delta R = U(t_o)\Delta X(t_o) + \int_{t_o}^{t_n} F_1\,\Delta X\,dt + \int_{t_o}^{t_n} F_o\,\Delta f/m\,dt. \tag{2.15}$$

Substitution of the expression for ΔX shown in equation (2.11) into equation (2.14) results in the following system:

$$\Delta\dot{X} = (A + H_1B)\Delta X + H_o\,\Delta f/m + H_1\,\delta X$$

or

$$\Delta\dot{X} = \bar{A}\,\Delta X + H_o\,\Delta f/m + H_1\,\delta X, \tag{2.16}$$

where

$$\bar{A} = A + H_1 B.$$

This expression is of identical form to that appearing in equation (2.14). The solution can be expressed as follows:

$$\Delta R = \bar{U}(t_o)\,\Delta X(t_o) + \int_{t_o}^{t_n} \bar{F}_o\,\Delta f/m\,dt + \int_{t_o}^{t_n} \bar{F}_1\,\delta X\,dt. \tag{2.17}$$

A comparison of $U(t_o)$ and $\bar{U}(t_o)$ for the mission under consideration yields the following:

$$U(t_o) = \begin{bmatrix} .349481 & 1.25909 & .170980 & .491843 \\ -.008175 & -.007129 & -.005744 & -.008092 \end{bmatrix} \tag{2.18}$$

and

$$\bar{U}(t_o) = \begin{bmatrix} .045516 & .033588 & .017544 & .00957 \\ -.007110 & -.000262 & -.003038 & -.000534 \end{bmatrix}. \tag{2.19}$$

The coefficients above were obtained for variables expressed in the following units:

Δx, Δy and Δr are expressed in kilometers.

$\Delta\dot{x}$ and $\Delta\dot{y}$ are expressed in meters per second.

$\Delta\theta$ is expressed in degrees.

Although the elements of $\bar{U}(t_o)$ are not small enough to be ignored, they provide a good linear prediction of the effect of $\Delta X(t_o)$ on the end conditions. With this knowledge available, it should not be difficult to determine a time function of second stage ignition to cancel these effects. With this in mind, primary consideration will be given to the elements of $\bar{F}_o$ and $\bar{F}_1$. The elements of these two vectors are illustrated in Figures 2.3 and 2.4 where they are compared with the corresponding elements of F_o and F_1. That the magnitude of the weighting functions defined by equations (2.9) - (2.12) is apparent. A better indication of the effect that δX and $\Delta f/m$ might have on Δr and $\Delta\theta$ is reflected in the following values which have been determined.

$$\int_{t_o}^{t_n} \begin{bmatrix} |\bar{f}_1| \\ |\bar{g}_1| \end{bmatrix} dt = \begin{bmatrix} .498 \text{ km/deg} \\ .12 \end{bmatrix} \tag{2.20}$$

$$\int_{t_o}^{t_n} \begin{bmatrix} |\bar{f}_o| \\ |\bar{g}_o| \end{bmatrix} dt = \begin{bmatrix} 4.135 & \text{km sec}^2/\text{m} \\ .550 & \text{deg sec}^2/\text{m} \end{bmatrix}. \tag{2.21}$$

Although the effect of δX represented by equation (2.20) might be acceptable, it is apparent by equation (2.21) that the effect of variations in f/m is not. Even variations as small as .25 m/sec^2 could cause as much as 1 km in Δr. The magnitude of elements of $\bar{F}_o$ are functions of the coefficients defined by equations (2.10). By altering these coefficients, the elements of $\bar{F}_o$ can be altered. A procedure to accomplish this will be outlined in the following section.

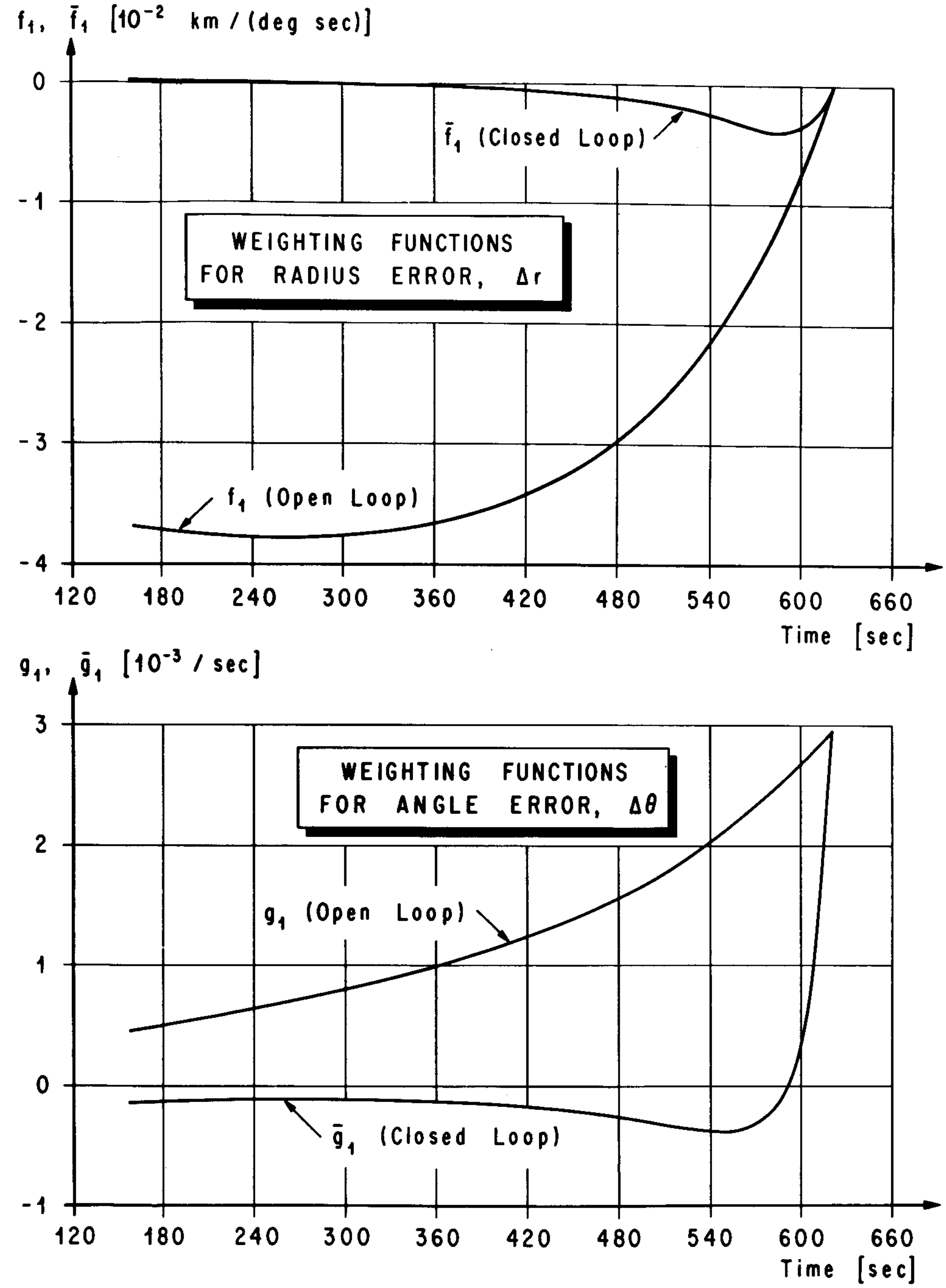

FIG. 2-3. EFFECT OF VARIATIONS IN χ
(Comparison of Closed Loop & Open Loop Weighting Functions)

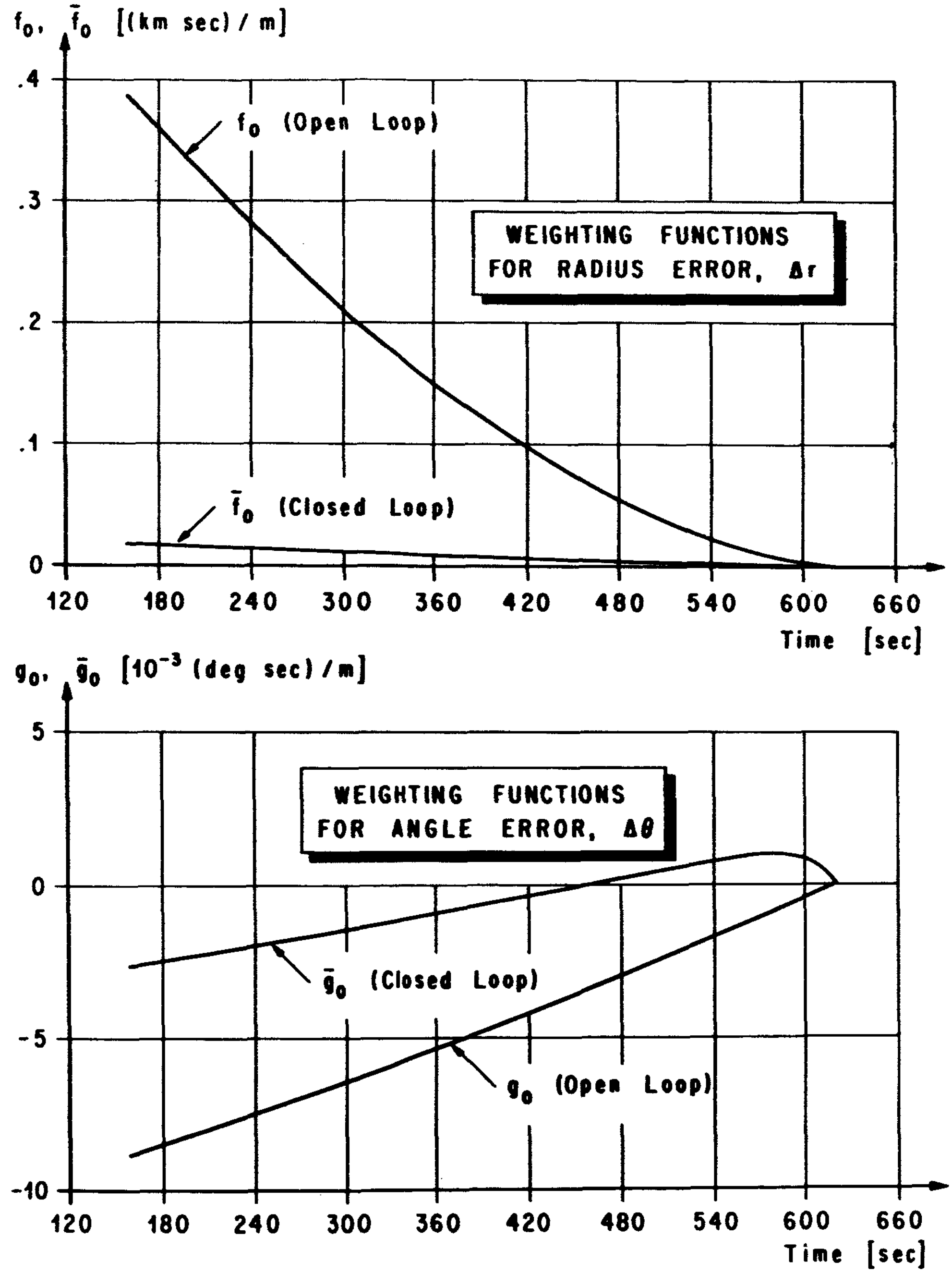

FIG. 2-4. EFFECT OF VARIATIONS IN F/m
(Comparison of Closed Loop & Open Loop Weighting Functions)

III. PERTURBING THE POLYNOMIAL COEFFICIENTS

The weighting functions represented by the vectors $\bar{F}_0$ and $\bar{F}_1$ were determined from the following system of equations:

$$\bar{F}_i(t) = T\ U(t_n,t)\ H_i(t) \quad (i = 0, 1, \ldots), \tag{3.1}$$

$$\dot{\bar{U}}(t,t_1) = \bar{A}(t)\ \bar{U}(t,t_1), \tag{3.2}$$

where

$$\bar{U}(t_1,t_1) = I \tag{3.3}$$

and

$$\bar{A}(t) = A(t) + H_1(t)\ B(t). \tag{3.4}$$

The effect of changing the polynomial coefficients in B(t) can be investigated by defining another weighting function vector $\hat{F}_0(t)$ resulting from another choice of coefficients defining a different function $\hat{B}(t)$ where

$$\hat{F}_0 = \bar{F}_0 + \Delta F_0 \tag{3.5}$$

$$\hat{B} = B + \Delta B. \tag{3.6}$$

Then $\hat{F}_0$ is determined from the following relationships:

$$\hat{F}_0(t) = T\ U(t_n,t)\ H_0(t) \tag{3.7}$$

$$\dot{U}(t,t_1) = \hat{A}(t)\ U(t,t_1) \tag{3.8}$$

where

$$U(t_1,t_1) = I \tag{3.9}$$

and

$$\hat{A}(t) = A(t) + H_1(t)\ \hat{B}(t). \tag{3.10}$$

Defining the new variables in terms of the previous ones yields the following definitions:

$$\hat{A}(t) = \bar{A}(t) + \Delta A \qquad (3.11)$$

$$U(t,t_1) = \bar{U}(t,t_1) + \Delta U(t,t_1). \qquad (3.12)$$

From equations (3.10), (3.6), and (3.4), it can be seen that

$$\Delta A = H_1 \Delta B. \qquad (3.13)$$

Substitution of equations (3.11) and (3.12) into equation (3.8) yields

$$\dot{\bar{U}}(t,t_1) + \Delta\dot{U}(t,t_1) = \bar{A}(t)\ U(t,t_1) + \bar{A}(t)\ \Delta U(t,t_1) + \Delta A(t)\ \bar{U}(t,t_1)$$

$$+ \Delta A(t)\ \Delta U(t,t_1).$$

The relationship defined by equation (3.2) reduces this to

$$\Delta\dot{U}(t,t_1) = \bar{A}(t)\ \Delta U(t,t_1) + \Delta A(t)\ \bar{U}(t,t_1) + \Delta A(t)\ \Delta U(t,t_1). \qquad (3.14)$$

Equations (3.3), (3.9), and (3.12) imply the following constraint on the initial conditions:

$$\Delta U(t_1,t_1) = 0. \qquad (3.15)$$

From equations (3.1), (3.5), and (3.12), it can be determined that

$$\Delta F_O(t_1) = T\ \Delta U(t_n,t_1)\ H_O(t_1), \qquad (3.16)$$

where $\Delta U(t_n,t_1)$ satisfies equation (3.14).

An approximate solution, F_o^* can be determined by linearizing equation (3.14). Then

$$F_o^*(t_1) = \bar{F}_o(t_1) + \Delta F_o^*(t_1) \tag{3.17}$$

and

$$\Delta F_o^*(t_1) = T\ \Delta\bar{U}(t_n,t_1)\ H_o(t_1), \tag{3.18}$$

where

$$\Delta\dot{\bar{U}}(t,t_1) = \bar{A}(t)\ \Delta U(t,t_1) + \Delta A(t)\ \bar{U}(t,t_1) \tag{3.19}$$

and

$$\Delta\bar{U}(t_1,t_1) = 0. \tag{3.20}$$

The solution to equation (3.19) can immediately be determined:

$$\Delta\bar{U}(t_n,t_1) = \bar{U}(t_n,t_1)\ \Delta\bar{U}(t_1,t_1) + \int_{t_1}^{t_n} \bar{U}(t_n,t)\ \Delta A(t)\ \bar{U}(t,t_1)\ dt.$$

From equation (3.13) and (3.20), this can be reduced to

$$\Delta\bar{U}(t_n,t_1) = \int_{t_1}^{t_n} \bar{U}(t_n,t)\ H_1(t)\ \Delta B(t)\ \bar{U}(t,t_1)\ dt.$$

Inserting this expression into equation (3.18) gives the following equation:

$$\Delta F_o^*(t_1) = \int_{t_1}^{t_n} T\ \bar{U}(t_n,t)\ H_1(t)\Delta B(t)\ \bar{U}(t,t_1)\ H_o(t_1)\ dt.$$

Equation (3.1) reduces this to

$$\Delta F_o^*(t_1) = \int_{t_1}^{t_n} \bar{F}_1(t)\ \Delta B(t)\ \bar{U}(t,t_1)\ H_o(t_1)\ dt. \tag{3.21}$$

$\Delta B(t)$ can be expressed in terms of changes in the polynomial coefficients as follows:

$$\left.\begin{aligned}
\Delta B(t) &= [\Delta a(t) \quad \Delta b(t) \quad \Delta c(t) \quad \Delta d(t)] \\
\Delta a(t) &= \Delta a_o + \Delta a_1\tau + \Delta a_2\tau^2 \\
\Delta b(t) &= \Delta b_o + \Delta b_1\tau + \Delta b_2\tau^2 \\
\Delta c(t) &= \Delta c_o + \Delta c_1\tau + \Delta c_2\tau^2 \\
\Delta d(t) &= \Delta d_o + \Delta d_1\tau + \Delta d_2\tau^2
\end{aligned}\right\}. \tag{3.22}$$

Equations (3.17), (3.21), and (3.22) provide an approximation to the vector $\hat{F}_o$ expressed as a linear time-variable function of perturbation in the polynomial gain coefficients. This expression has been evaluated over a number of time points for the application being investigated. The least squares criterion was applied to the function.

$$\sum_{i=1}^{12} [f_o^{*2}(t_i) + 25g_o^{*2}(t_i)]. \tag{3.23}$$

In order to take advantage of an available computer program which could handle only a restricted number of coefficients, Δa_o, Δb_o, Δc_o, and Δd_o were set equal to zero and the remaining eight coefficients were determined to minimize expression (3.23) with the following results.

$$\left.\begin{array}{llll} \Delta a_1 = -.007333 & & \Delta a_2 = .046081 \\ \Delta b_1 = .764653 & & \Delta b_2 = -.093713 \\ \Delta c_1 = .045045 & & \Delta c_2 = -.007476 \\ \Delta d_1 = .077065 & & \Delta d_2 = -.007011 \end{array}\right\} . \tag{3.24}$$

This defines a new feedback system defined below:

$$\Delta X = \hat{B}\Delta X, \tag{3.25}$$

where

$$\hat{B} = B + \Delta B. \tag{3.26}$$

The elements of $\hat{B}(t)$ are compared in Figures 3.1 and 3.2, where they are compared with the corresponding elements of $B(t)$.

The use of the polynomial gains represented by $B(t)$ resulted in the end condition errors expressed by equation (2.17). Similarly, the following expression is obtained for $\hat{B}(t)$:

$$\Delta R = \hat{U}(t_o)\ \Delta X(t_o) + \int_{t_o}^{t_n} \hat{F}\ (t)\ \Delta f/m\ dt + \int_{t_o}^{t_n} \hat{F}_1(t)\ \delta X\ dt. \tag{3.27}$$

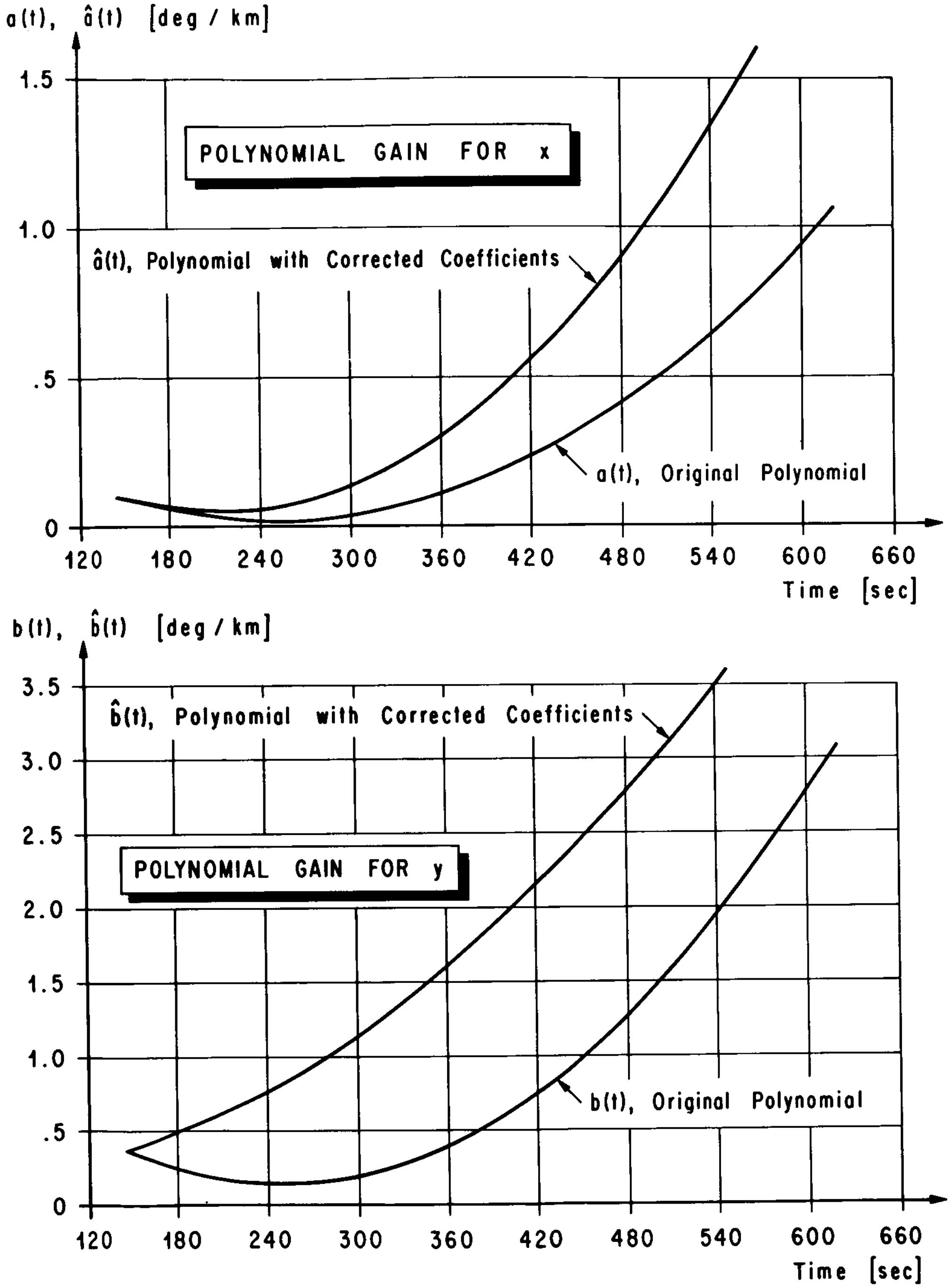

FIG. 3-1. COMPARISON OF GAINS ON THE POSITION VARIABLES (Before & After Coefficient Changes)

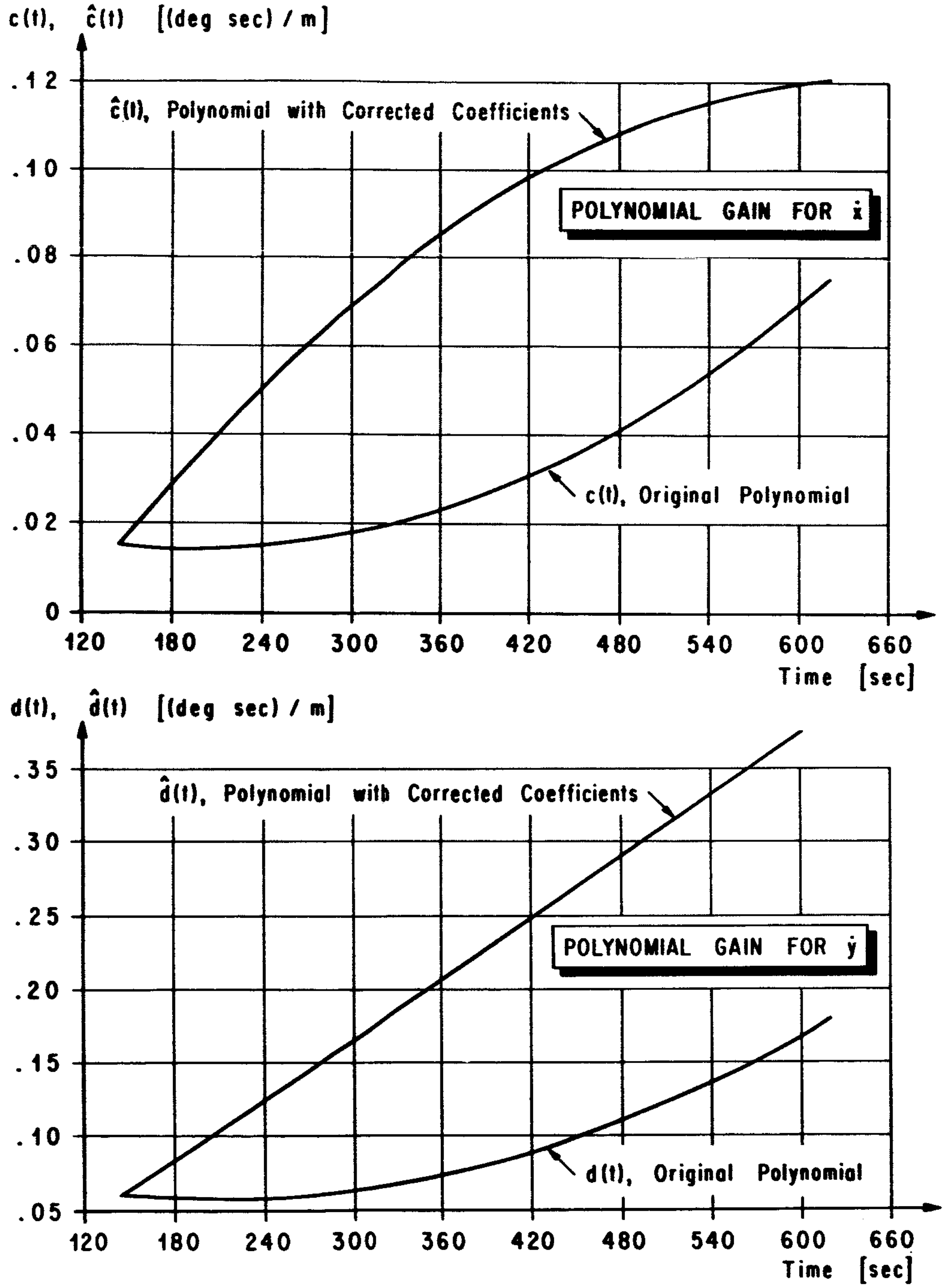

FIG. 3-2. COMPARISON OF GAINS ON THE VELOCITY VARIABLES (Before & After Coefficient Changes)

The reduction in the effect of $\Delta f/m$ on Δr and $\Delta\theta$ as a result of changing the gains from $B(t)$ to $\hat{B}(t)$ is vividly illustrated in Figure 3.3 where the elements of $\hat{F}_o(t)$ are compared with the corresponding elements of $\bar{F}_o(t)$. A comparison of the integrals of the absolute values reflects the following reduction in upper bounds.

$$\int_{t_o}^{t_n} \begin{bmatrix} |\hat{f}_o| \\ |\hat{g}_o| \end{bmatrix} dt = \begin{bmatrix} .666 \text{ km sec}^2/\text{m} \\ .258 \text{ deg sec}^2/\text{m} \end{bmatrix} \tag{3.28}$$

$$\int_{t_o}^{t_n} \begin{bmatrix} |\bar{f}_o| \\ |\bar{g}_o| \end{bmatrix} dt = \begin{bmatrix} 4.135 \text{ km sec}^2/\text{m} \\ .550 \text{ deg sec}^2/\text{m} \end{bmatrix} \tag{3.29}$$

The reduction in the effect of $\Delta f/m$ could be expected because this was the criterion employed to determine the coefficient changes defined in equations (3.24). However, these coefficient changes are also expected to alter the effects of $\Delta X(t_o)$ and ΔX. A comparison of the effect of these variables by using $\hat{B}(t)$ instead of $B(t)$ is reflected in the following comparisons as well as in Figure 3.4 where the elements of $\hat{F}_1(t)$ are illustrated.

$$\hat{U}(t_o) = \begin{bmatrix} .017670 & .019729 & .001184 & .002432 \\ -.001861 & -.000221 & -.001149 & -.000349 \end{bmatrix} \tag{3.30}$$

$$\bar{U}(t_o) = \begin{bmatrix} .045516 & .033588 & .017544 & .009570 \\ -.007110 & -.000262 & -.003038 & -.000534 \end{bmatrix} \tag{3.31}$$

$$\int_{t_o}^{t_n} \begin{bmatrix} |\hat{f}_1| \\ |\hat{g}_1| \end{bmatrix} dt = \begin{bmatrix} .311 \text{ km/deg} \\ .04 \end{bmatrix} \tag{3.32}$$

$$\int_{t_o}^{t_n} \begin{bmatrix} |\bar{f}_1| \\ |\bar{g}_1| \end{bmatrix} dt = \begin{bmatrix} .498 \text{ km/deg} \\ .12 \end{bmatrix}. \tag{3.33}$$

The weighting functions of this system could again be linearized and further iterations performed; however, the results already obtained are sufficient to demonstrate the effectiveness of this operation. The additional improvement afforded by the inclusion of thrust acceleration feedback will be investigated.

The expression for ΔR defined in equation (3.27) was derived for the following function.

$$\Delta X = \hat{B}(t)\ \Delta X + \delta X.$$

The effect of a feedback of $\Delta f/m$ would give the following expression:

$$\Delta X = \hat{B}(t)\ \Delta X + e(t)\ \Delta f/m + \delta X. \tag{3.34}$$

The expression corresponding to equation (3.27) would then be

$$\Delta R = \hat{U}(t_o)\ \Delta X_o + \int_{t_o}^{t_n} [\hat{F}_o(t) + e(t)\ \hat{F}_1(t)]\ \Delta f/m\ dt + \int_{t_o}^{t_n} \hat{F}_1(t)\ \delta X\ dt. \tag{3.35}$$

Choosing $e(t)$ as a quadratic function of time gives

$$e(t) = e_o + e_1\tau + e_2\tau^2. \tag{3.36}$$

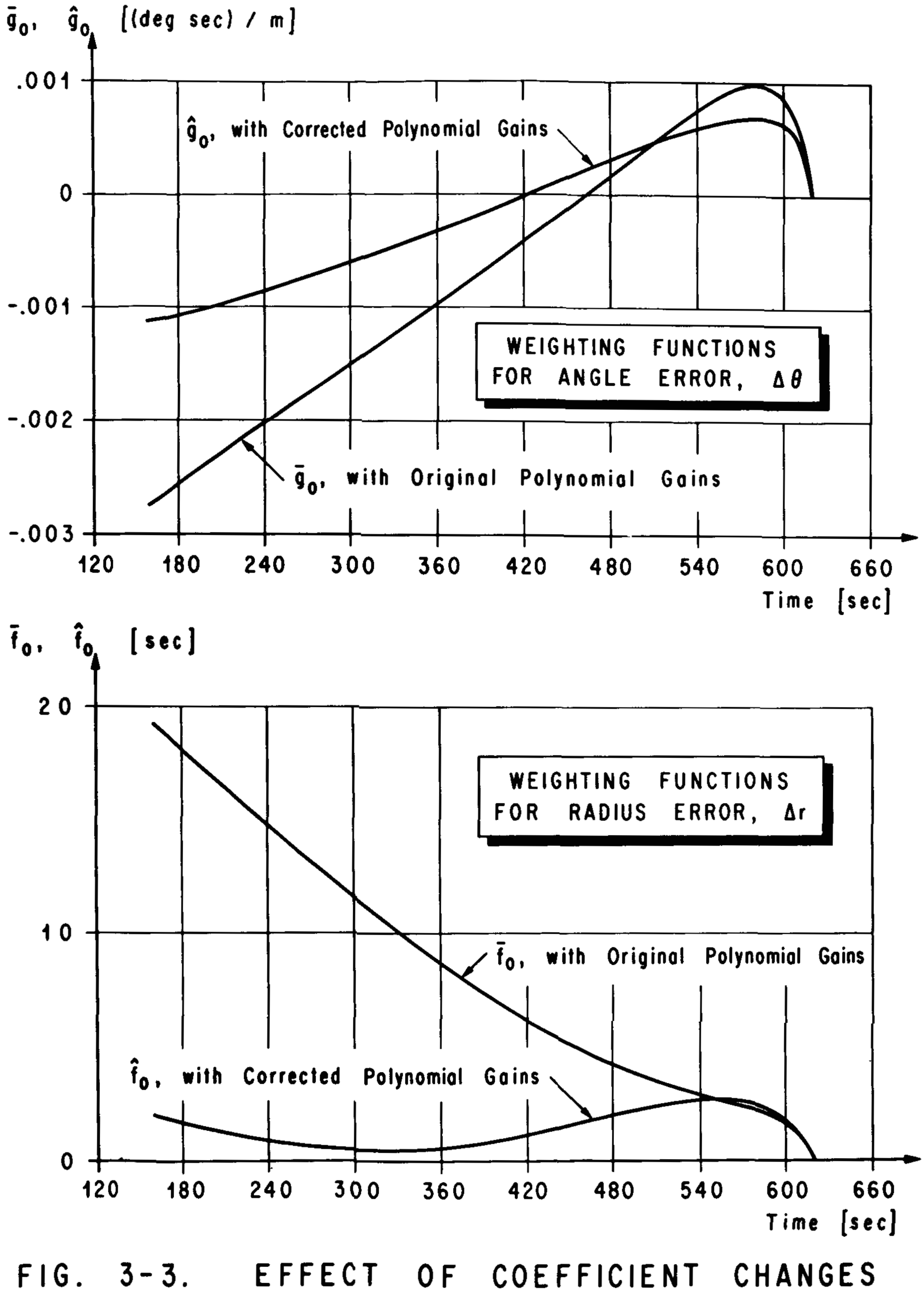

FIG. 3-3. EFFECT OF COEFFICIENT CHANGES ON WEIGHTING FUNCTIONS OF $\Delta F/m$

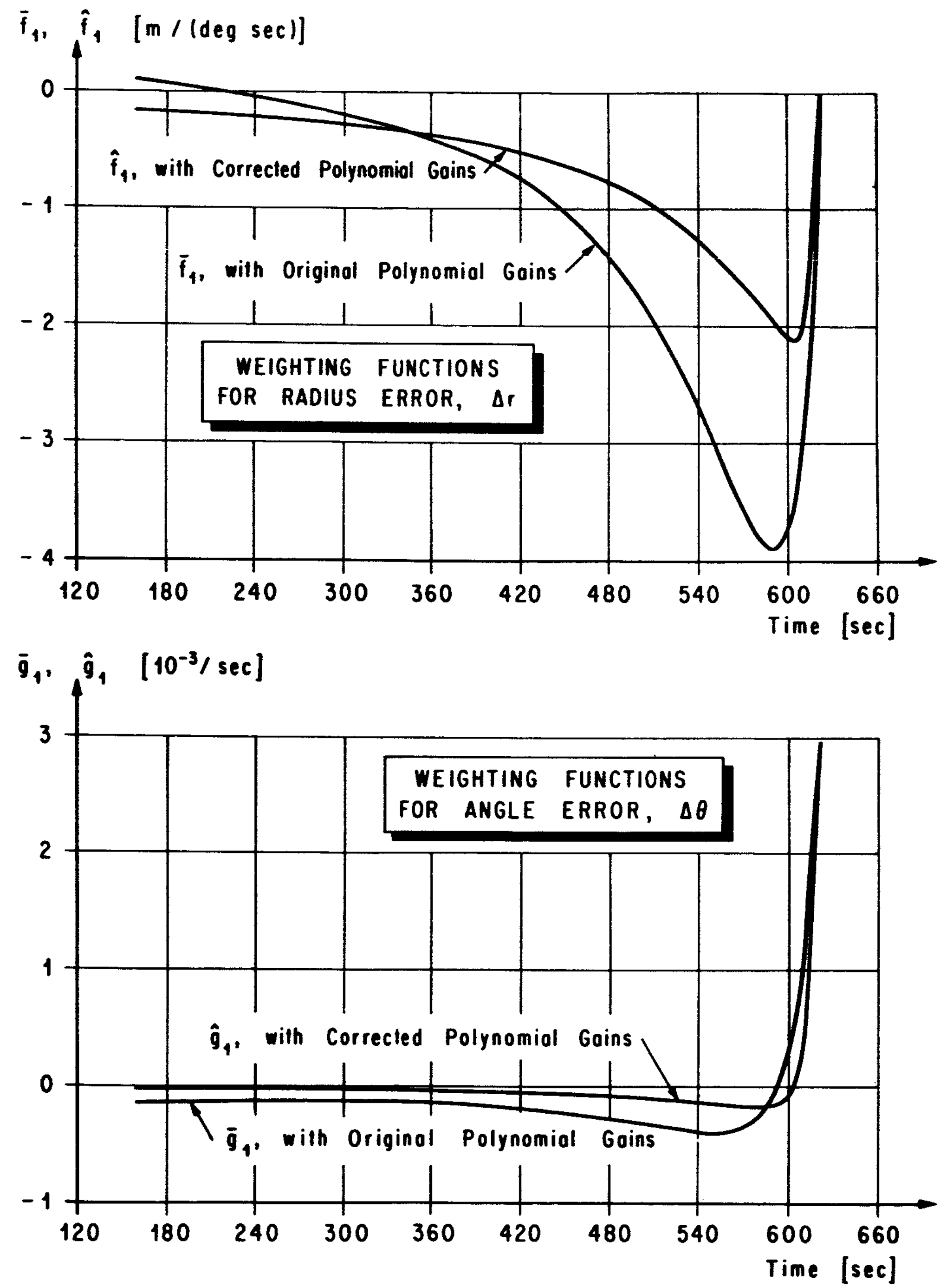

FIG. 3-4. EFFECT OF COEFFICIENT CHANGES ON WEIGHTING FUNCTIONS OF $\Delta\chi$

e_o, e_1 and e_2 can be determined to minimize the weighted sum of squares of $\hat{f}_o + e(t)\,\hat{f}_1$ and $\hat{g}_o + e(t)\,\hat{g}_1$, simultaneously. This was carried out for this example with the following results.

$$\left.\begin{aligned} e_o &= -14.2412 \\ e_1 &= 9.8851 \\ e_2 &= -\ 1.4328 \end{aligned}\right\} . \qquad (3.37)$$

The effect of $\Delta f/m$ is determined by the elements of the vector

$$\hat{F}_o(t) + e(t)\,\hat{F}_1(t).$$

The integral of the absolute value of these functions is not significantly smaller than that for $\hat{F}_o$ as shown below.

$$\left.\begin{aligned} \int_{t_o}^{t_n} \begin{bmatrix} |\hat{f}_o + e(t)\,\hat{f}_1| \\ |\hat{g}_o + e(t)\,\hat{g}_1| \end{bmatrix} dt &= \begin{bmatrix} .543 \text{ km sec}^2/\text{m} \\ .190 \text{ deg sec}^2/\text{m} \end{bmatrix} \\ \int_{t_o}^{t_n} \begin{bmatrix} |\hat{f}_o| \\ |\hat{g}_o| \end{bmatrix} dt &= \begin{bmatrix} .666 \text{ km sec}^2/\text{m} \\ .258 \text{ deg sec}^2/\text{m} \end{bmatrix} \end{aligned}\right\} . \qquad (3.38)$$

Although the improvement reflected by the above comparison is slight, a comparison of the functions themselves which appear in Figure 3.5 shows that each of the elements of $\hat{F}_o(t) + e(t)\,\hat{F}_1(t)$ changes sign which tends to cancel the effect of any bias of $\Delta f/m$. Since a bias in $\Delta f/m$ is a likely occurrence, this will be considered sufficient justification for including the feedback of $\Delta f/m$.

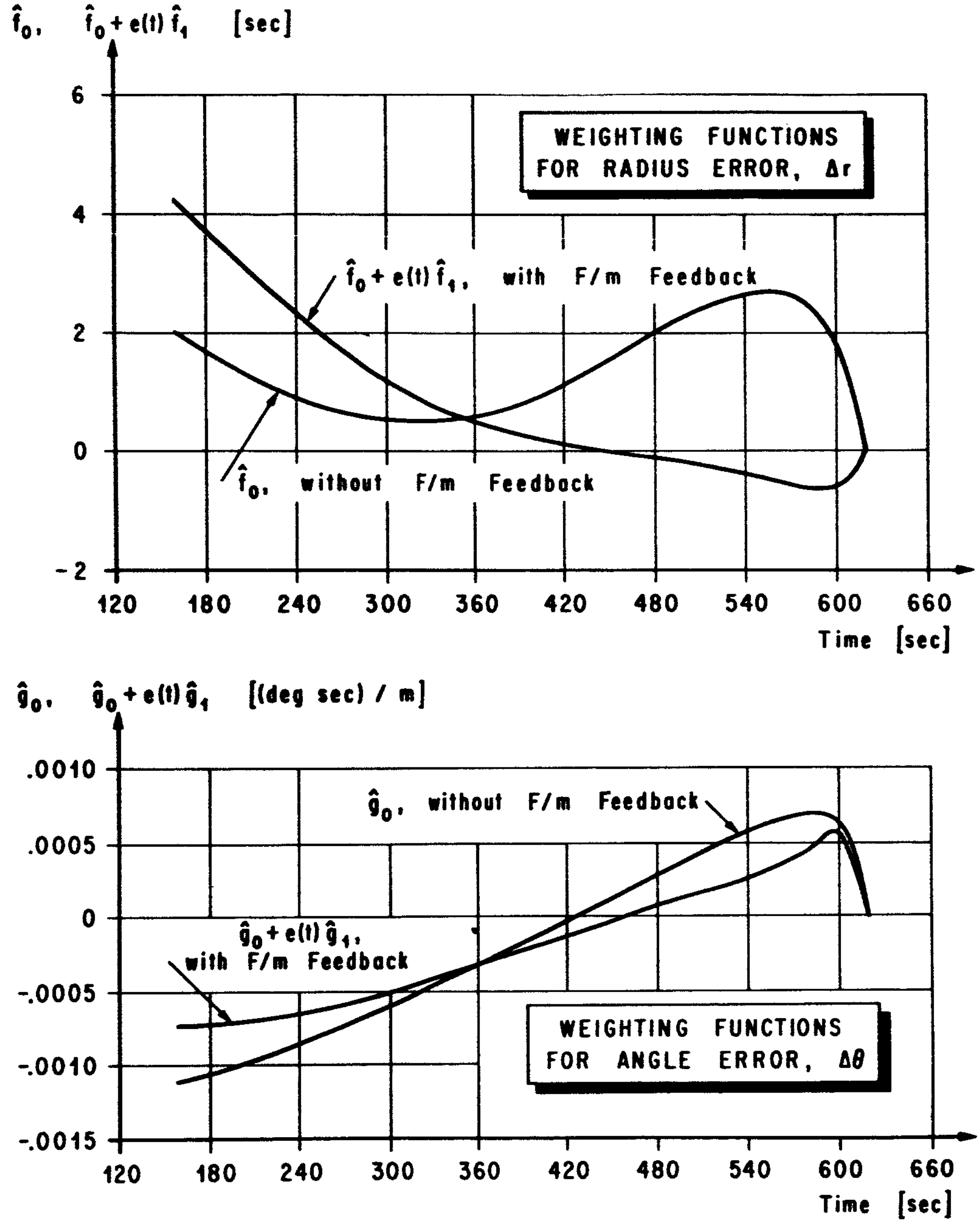

FIG. 3-5. EFFECT OF F/m FEEDBACK
ON WEIGHTING FUNCTIONS OF ΔF/m

IV. CANCELLATION OF THE EFFECT OF INITIAL CONDITIONS

Although the effect of initial conditions on ΔR is not considerable, the linear portion is known at second stage ignition. It is possible to define a time function to cancel this effect. Suppose the thrust angle should be changed by an amount $\Delta X_o(t)$. The expression for ΔR would then include the following two terms in addition to others.

$$\Delta R = \hat{U}(t_o)\,\Delta X(t_o) + \int_{t_o}^{t_n} F_1(t)\,\Delta X_o(t)\,dt + \dots \,. \tag{4.1}$$

Let $\Delta X_o(t)$ be defined as follows:

$$\Delta X_o(t) = -P(t)\,M^{-1}(t_o)\,\hat{U}(t_o)\,\Delta X(t_o). \tag{4.2}$$

Substitution into equation (4.1) gives

$$\Delta R = \hat{U}(t_o)\,\Delta X(t_o) - \int_{t_o}^{t_n} F_1(t)\,P(t)\,dt\,M^{-1}(t_o)\,\hat{U}(t_o)\,\Delta X(t_o). \tag{4.3}$$

Equation (2.3) gives the following definition:

$$M(t_o) = \int_{t_o}^{t_n} F_1(t)\,P(t)\,dt.$$

Substituting this into equation (4.3) gives

$$\Delta R = \hat{U}(t_o)\,\Delta X(t_o) - \hat{U}(t_o)\,\Delta X(t_o) + \dots \,, \tag{4.4}$$

and the effect of initial conditions represented by $\hat{U}(t_o)\ \Delta X(t_o)$ is cancelled. The problem is reduced to determining a time function $\Delta k(t)$ to add to the commanded thrust angle deviation so that the actual thrust angle will be changed by an amount $\Delta\chi_o(t)$ after including the effect of the feedback. This can be expressed as follows:

$$\Delta\chi_o(t) = B(t)\ \Delta X(\Delta\chi_o) + \Delta k(t). \tag{4.5}$$

Thus, $\Delta k(t)$ can be written

$$\Delta k(t) = \Delta\chi_o(t) - B(t)\ \Delta X(\Delta\chi_o). \tag{4.6}$$

The linear expression for $\Delta X(\Delta\chi_o)$ is

$$\Delta X(\Delta\chi_o) = \int_{t_o}^{t} U(t,t_1)\ F_1(t_1)\ \Delta\chi_o(t_1)\ dt_1. \tag{4.7}$$

Substituting this expression into equation (4.6) and including the expression for $\Delta\chi_o(t)$ defined in equation (4.3) gives the following expression for $\Delta k(t)$:

$$\Delta k(t) = \left[-P(t) + B(t) \int_{t_o}^{t} U(t,t_1)\ F_1(t_1)\ P(t_1)\ dt_1\right] M^{-1}(t_o)\ \hat{U}(t_o)\ \Delta X(t_o). \tag{4.8}$$

This can be expressed as

$$\Delta k(t) = K(t)\ \Delta X(t_o), \tag{4.9}$$

where

$$K(t) = \left[-P(t) + B(t) \int_{t_o}^{t} U(t,t_1)\ F_1(t_1)\ P(t_1)\ dt_1\right] M^{-1}(t_o)\ \hat{U}(t_o). \tag{4.10}$$

$K(t)$ is a 1 x 4 matrix of time-variable elements which will be fitted by some polynomial expressions. It will be convenient to use the same degree polynomials as will be required to fit the standard X with the feedback. Consequently, the time function required to allow the state variables to be measured directly, rather than in terms of differences, will be determined and fitted first. $K(t)$ will then be fitted to the same degree polynomials.

Except for the correction for initial conditions, the deviation in X which is predicted by the guidance function is

$$\Delta X = \hat{B} \Delta X + e(t)\ \Delta f/m. \tag{4.11}$$

This can be expressed as

$$X = \hat{B}X + e(t)\ f/m + \left[X_s - \hat{B}X_s - e(t)\ f/m_s \right]. \tag{4.12}$$

Equivalently,

$$X = \hat{B}X + e(t)\ f/m + k(t),$$

where $k(t)$ is a polynomial in time. Since $\hat{B}(t)$ is composed of elements which are quadratic time functions and the variables x and y are expected to be at least quadratics, it appears that a function of at least fourth degree might be required. Since it fits satisfactorily, a sixth-degree polynomial is used in this example:

$$k(t) = k_o + k_1\tau + k_2\tau^2 + k_3\tau^3 + k_4\tau^4 + k_5\tau^5 + k_6\tau^6. \tag{4.13}$$

The following numerical values were determined

$$\left.\begin{aligned} k_o &= -2354.749911 \\ k_1 &= -2267.341539 \\ k_2 &= -738.009535 \\ k_3 &= -39.95716990 \\ k_4 &= 5.34562706 \\ k_5 &= -1.09387216 \\ k_6 &= .10090398 \end{aligned}\right\} . \tag{4.14}$$

These coefficients were determined by the method of least squares with slight adjustments to ensure that $\Delta R = 0$ for the standard trajectory.

The effects of the initial conditions, similarly, were cancelled by fitting the four time functions of K(t) defined in equation (4.9) to sixth-degree polynomials of time. Slight adjustments in these coefficients were also made to ensure the following condition:

$$\int_{t_o}^{t_n} \hat{F}_1(t)\ \hat{K}(t)\ dt = -\hat{U}(t_o), \tag{4.15}$$

where $\hat{K}(t)$ is composed of functions fitted to approximate the elements of K(t). The time function $\hat{k}(t)$ can then be defined as follows:

$$\hat{k}(t) = k(t) + \Delta\hat{k}(t), \tag{4.16}$$

where

$$\Delta\hat{k}(t) = \Delta k_o + \Delta k_1\tau + \Delta k_2\tau^2 + \Delta k_3\tau^3 + \Delta k_4\tau^4 + \Delta k_5\tau^5 + \Delta k_6\tau^6. \tag{4.17}$$

These coefficients can be determined at second stage ignition by the following matrix operation.

$$\begin{bmatrix} \Delta k_o \\ \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \\ \Delta k_4 \\ \Delta k_5 \\ \Delta k_6 \end{bmatrix} = \Delta\hat{K}\ \Delta X(t_o). \tag{4.18}$$

The values of the elements of $\Delta\hat{K}$ which have been determined for this application are shown in the next section where a complete definition of the initial and final conditions and the guidance function are listed with the results obtained from actual integration.

V. RESULTS AND CONCLUSIONS

The guidance function derived in the preceding sections resulted in the following expressions:

$$X = \hat{B}(t)\ X(t) + e(t)\ f/m + \hat{k}(t), \tag{5.1}$$

$$\hat{B}(t) = [\hat{a}(t) \quad \hat{b}(t) \quad \hat{c}(t) \quad \hat{d}(t)]$$

$$\hat{a}(t) = \hat{a}_o + \hat{a}_1\tau + \hat{a}_2\tau^2$$

$$\hat{b}(t) = \hat{b}_o + \hat{b}_1\tau + \hat{b}_2\tau^2$$

$$\hat{c}(t) = \hat{c}_o + \hat{c}_1\tau + \hat{c}_2\tau^2$$

$$\hat{d}(t) = \hat{d}_o + \hat{d}_1\tau + \hat{d}_2\tau^2$$

$$e(t) = e_o + e_1\tau + e_2\tau^2$$

$$\hat{k}(t) = K(t) + \Delta\hat{k}(t),$$

$$k(t) = k_o + k_1\tau + k_2\tau^2 + k_3\tau^3 + k_4\tau^4 + k_5\tau^5 + k_6\tau^6$$

$$\Delta\hat{k}(t) = \Delta k_o + \Delta k_1\tau + \Delta k_2\tau^2 + \Delta k_3\tau^3 + \Delta k_4\tau^4 + \Delta k_5\tau^5 + \Delta k_6\tau^6,$$

where

$$\tau = \frac{t - t_o}{100}$$

and

$$\begin{bmatrix} \Delta k_o \\ \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \\ \Delta k_4 \\ \Delta k_5 \\ \Delta k_6 \end{bmatrix} = \Delta K\ \Delta X(t_o), \qquad X(t_o) = \begin{bmatrix} 153.983 \text{ km} \\ 6435.878 \text{ km} \\ 2818.329 \text{ m/sec} \\ 988.358 \text{ m/sec} \end{bmatrix}.$$

The numerical values obtained for this example are as follows:

$$\left.\begin{array}{lll}
\hat{a}_o = .104259 & \hat{a}_1 = -.168328 & \hat{a}_2 = .122479 \\
\hat{b}_o = .37021 & \hat{b}_1 = .315723 & \hat{b}_2 = .121797 \\
\hat{c}_o = .01551 & \hat{c}_1 = .041162 & \hat{c}_2 = -.003999 \\
\hat{d}_o = .062320 & \hat{d}_1 = .067045 & \hat{d}_2 = .000353 \\
e_o = -14.2412 & e_1 = 9.8851 & e_2 = -1.4328
\end{array}\right\} \qquad (5.2)$$

$$\left.\begin{array}{l}
k_o = -2354.749911 \\
k_1 = -2267.341539 \\
k_2 = -\ 738.009535 \\
k_3 = -\ \ 39.957169900 \\
k_4 = \ \ 5.34562706 \\
k_5 = \ \ -1.09387216 \\
k_6 = \ \ .10090398
\end{array}\right\} \qquad (5.3)$$

$$\Delta\hat{K} = \begin{bmatrix}
-.160198 & .249259 & -.246918 & -.043367 \\
-.072324 & .150760 & -.132564 & -.021482 \\
.229492 & .008411 & .152094 & .043929 \\
-.369459 & .224627 & -.376247 & -.082579 \\
.202657 & -.075912 & .180285 & .042941 \\
-.048929 & .021807 & -.045447 & -.010541 \\
.006095 & -.002447 & .005513 & .001300
\end{bmatrix} \times 10^{-2}. \qquad (5.4)$$

The above coefficients were derived for use with x and y measured in kilometers, $\dot{x}$ and $\dot{y}$ measured in meters per second, and f/m measured in meters per second^2. In addition, they were derived for the following system of differential equations:

$$\ddot{x} = f/m \sin \chi + \ddot{x}_g$$

$$\ddot{y} = f/m \cos \chi + \ddot{y}_g,$$

where

$$\ddot{x}_g = \frac{x}{r} g,$$

$$\ddot{y}_g = \frac{y}{r} g,$$

$$g = -\frac{g_o r_o^2}{r^2}, \quad g_o = 9.81 \text{ m/sec}^2, \quad r_o = 6{,}370 \text{ km.}$$

$$f/m = \frac{8.78065}{1.3751 - .20888\tau}, \quad \tau = \frac{t - t_o}{100} .$$

The following initial conditions existed for the standard trajectory:

x_o = 153.98343 km

y_o =6435.8783 km

$\dot{x}_o$ =2818.3294 m/sec

$\dot{y}_o$ = 988.35767 m/sec

t_o = 146.815 sec.

Cutoff was assumed to occur at the following velocity:

$$V_c = 7792.5746 \text{ m/sec},$$

at which time the following values should exist for r_c and θ_c.

$$r_c = 6555.200 \text{ km}$$

$$\theta_c = 90°.$$

Table 5.1 presents a sample of twelve different second stage initial conditions caused by variations in first stage performance. The initial conditions are measured as deviations from the standard. Tables 5.2 and 5.3 show the resulting deviations in Δr and $\Delta\theta$, respectively. The burning time of these examples were compared with those required for the corresponding calculus of variations solutions. Table 5.4 shows the additional propellant required for this guidance function beyond that which was required by the calculus of variations solution. Negative values are readily explained by the fact that errors in end conditions resulted in a different mission. The effect on burning time of errors in end conditions is readily determined from the following equation derived in Reference 2.

$$\Delta t = -\lambda_1 \Delta r - \lambda_1 \Delta\theta + \int_{t_o}^{t_n} [\lambda_2 + \lambda_1 f_2 + \lambda_2 g_2] \, \Delta X^2 + \dots, \tag{5.5}$$

where

$$\lambda_1 = -.08734 \text{ sec/km}$$

$$\lambda_2 = .8589 \text{ sec/deg.}$$

For Example No. 13, with +1% F, $\dot{W}$ second stage perturbation, the following errors were obtained:

$$\Delta r = -.161 \text{ km}$$

$$\Delta\theta = .013 \text{ deg.}$$

From this, the following variation in burning time should occur. (Only first order terms are considered.)

$$\Delta t = -(-.08734)(-.161) - .8589(.013) = -.0252 \text{ sec.}$$

The flow rate for this case was $\dot{W} = 210$ lb/sec. Thus, in order to meet the end conditions indicated, additional propellant, ΔW, should have been required. In this case,

$$\Delta W = -5.3 \text{ lbs.}$$

The end conditions attained by the guidance function should have required 5.3 lbs of propellant less than that required by the calculus of variations solution to reach the nominal end conditions. However, Table 5.3 shows that the guidance function saved only 4 lbs of propellant, a net increase of 1.3 lbs more than the calculus of variations solution would have required to fulfill the same end conditions actually met by the guidance function. However, since the actual concern is the amount of fuel required to reach cutoff conditions from whatever cause, the comparison of fuel was made with respect to calculus of variations solution to nominal end conditions.

In conclusion, it should be pointed out that the purpose of this report is not to propose a guidance function but rather to demonstrate several numerical methods on a rather intricate problem. These methods are applicable to a wide variety of problems, and it is felt that the results of this particular application demonstrate their effectiveness.

TABLE 5.1

Deviations in Initial Conditions

Example No.	First Stage Deviations	Δx_o (km)	Δy_o (km)	$\Delta \dot{x}_o$ m/sec	$\Delta \dot{y}_o$ m/sec
1	None	0	0	0	0
2	+5000 lbs	- .93152	-1.1623	-30.2692	-22.78749
3	-5000 lbs	.94444	1.1824	30.8335	23.36393
4	Engine #2 out at 100 sec	17.28599	5.5463	- 4.6494	-58.34471
5	Tail Wind	.10183	1.8180	2.7670	25.69483
6	Head Wind	-.34235	- .4055	-5.3173	- 5.40216
7	Left Cross Wind	-.10434	- .1788	-1.5250	- 2.33100
8	Right Cross Wind	.09092	.2155	1.4240	2.87015
9	-1% $\dot{W}$	4.76116	1.1588	48.5244	- 4.54684
10	+1% $\dot{W}$	-3.31692	- .6959	-39.7298	3.74070
11	+1% F	1.19032	1.9245	29.8569	29.73483
12	-1% F	-1.22157	-1.9050	-30.7597	-29.22741
13	This example did not involve any first stage deviations. It is the case where the actual thrust angle exceeded the angle predicted by the guidance function by +1/2°. (δX = + 1/2°). Since it is combined with other second stage perturbations, it is listed with the first stage deviations.				

TABLE 5.2

Δr (meters)

PERTURBATIONS

2nd Stage / 1st Stage	None	-1% f, $\dot{W}$	+1% f, $\dot{W}$	Example No.
None	0	-26	-3	1
+5000 lbs	34	35	4	2
-5000 lbs	43	-11	64	3
Engine #2 Out at 100 sec.	-6	-69	28	4
Tail Wind	8	-28	15	5
Head Wind	0	-20	-8	6
Left Cross Wind	0	-24	-5	7
Right Cross Wind	1	-28	0	8
-1% $\dot{W}$	70	2	102	9
+1% $\dot{W}$	37	44	2	10
+1% f	50	-7	73	11
-1% f	44	49	10	12
$\delta X = +1/2°$ (2nd Stage)	-154	-177	-161	13

TABLE 5.3

$\Delta\theta$ (degrees)

PERTURBATIONS 2nd Stage / 1st Stage	None	-1% f, $\dot{W}$	+1% f, $\dot{W}$	Example No.
None	.000	-.014	.020	1
+5000 lbs	.000	-.015	.021	2
-5000 lbs	.002	-.014	.018	3
Engine #2 Out at 100 seconds	.000	-.014	.021	4
Tail Wind	.002	-.014	.021	5
Head Wind	.000	-.014	.020	6
Left Cross Wind	.000	-.014	.020	7
Right Cross Wind	.000	-.014	.019	8
-1% $\dot{W}$	.000	-.014	.016	9
+1% $\dot{W}$	-.001	-.017	.021	10
+1% f	.002	-.013	.019	11
-1% f	.001	-.014	.021	12
δX = +1/2° (2nd Stage)	-.007	-.020	.013	13

TABLE 5.4

ΔW (pounds)

PERTURBATIONS

1st Stage \ 2nd Stage	None	-1% f, $\dot{W}$	$+1\%$ f, $\dot{W}$	Example No.
None	0	4	-1	1
+5000 lbs	8	8	8	2
-5000 lbs	7	14	2	3
Engine #2 Out at 100 sec	9	12	7	4
Tail Wind	6	14	1	5
Head Wind	1	4	0	6
Left Cross Wind	0	4	-1	7
Right Cross Wind	0	5	-2	8
-1% $\dot{W}$	3	8	0	9
$+1\%$ $\dot{W}$	2	5	0	10
$+1\%$ f	11	19	5	11
-1% f	12	12	14	12
$\delta\chi = 1/2°$ (2nd Stage)	-1	4	-4	13

REFERENCES

1. Dickey, Lyle R., "Guidance Applications of Linear Analysis," NASA TM X-53166, MSFC, November 27, 1964, Unclassified.

2. Dickey, Lyle R., "Near-Optimum Guidance: An Analysis of Fuel Penalty," NASA TM X-53249, MSFC, April 23, 1965, Unclassified.

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LINEAR FEEDBACK GUIDANCE

By Lyle R. Dickey

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