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Optimum Design of Columns Supported by Tension Ties

A thesis submitted in partial satisfaction of the requirements for the degree Master of Science in Engineering

by

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NOMENCLATURE

A_c	cross sectional area of the column
A_j	cross sectional area of a tie in the j^{th} panel
B	as defined in equation (24)
C_1	as defined in equation (88)
C_2	as defined in equation (89)
D	column outside diameter
E_c	modulus of elasticity of the column
E_{ct}	tangent modulus of elasticity of the column
E_j	modulus of elasticity of a tie in the j^{th} panel
F	as defined in equation (146)
I_c	moment of inertia of the column
L	length of the column
ΔL_o	change in length due to a load P_o
ΔL_p	change in length due to a load P_p
P	applied axial load on the column in x-direction
P_o	force in the column due to initial tension
R_j	lateral reaction at joint j
R_{jj}	lateral reaction at joint j due to the ties in the panel j
T_j	tension in a tie in the j^{th} panel when P is acting
T_{oj}	initial tension in a tie in the j^{th} panel (no external load)

NOMENCLATURE

(continued)

T_j^a, T_j^b	if two ties lie between two struts (see Fig. 14)
U	strain energy
U_1	spring strain energy
W	weight of the whole system (column, wires and strut assemblies)
W_1	work done by the external load P
Z	as defined in equation (144)
a_k	factor for the k^{th} term in the Fourier series expressing the deflection of the column
c_j	length of the tie between joints j-1 and j
h_j	length of typical strut in the j^{th} panel
k_2	local buckling factor chosen as 0.4
m	number of tension ties
t	wall thickness of the column
w	specific weight of the column material
w_j	specific weight of the tie in the j^{th} panel
x	coordinate in axial direction
y	coordinate in lateral direction

NOMENCLATURE
(continued)

α_j	$\theta_j - \tau/2$
β_j	angle between the plane of a tie i and the deflection plane
$\bar{\kappa}$	parameter as defined in equation (55)
κ	parameter defined as $\bar{\kappa}/A$
λ	difference between the length of the cord and the deflection curve
θ	angle defined in Fig. 2
Θ	as defined in equation (101)
σ_c	compression stress in the column
τ	defined as E_{ct}/E_c
ξ	coordinate as defined in Fig. 5
η	coordinate as defined in Fig. 5

ABSTRACT OF THE THESIS

Optimum Design of Columns Supported by

Tension Ties

by

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Professor Lewis P. Felton, Chairman

When optimizing simple thin-walled columns on a weight basis, the maximum obtainable stress is found to be that at which local and general buckling failure occur simultaneously. This stress can be expressed as a function of load and distance, allowing the introduction of the

structural index P/L^2 (P = buckling load, L = length of the column), for equal values of which all dimensionally similar columns develop the same stress at failure. At low values of the structural index, the optimum stress is low, indicating that the simple column is not an efficient structure in such circumstances. It has been found that expansion of the cross-section, for example, by using tension ties which serve to provide intermediate elastic support for the column, allows the column to operate at higher stress levels, thereby increasing efficiency.

This thesis summarizes a method of analysis and presents a procedure for optimizing tension-tie supported thin-walled cylindrical columns. The optimized column for a given structural index is defined by a particular diameter, wall thickness, tie prestress, tie cross-sectional area, tie angle, and strut dimensions. For the cases considered it is found that, in the low range of the structural index, the tie supported column offers a potential weight saving of up to 50% over the simple tubular column.

SECTION 1

INTRODUCTION

The primary function of a structure is to transmit forces through space, where, from the designer's point of view, the objective often is to do this with the minimum possible weight. For any structure that fails as a result of instability under compressive loading, the maximum obtainable stress depends, in a complex manner, on the properties of the material and the geometric properties of the structure. To apply the principle of dimensional similarity, the structural index P/L^2 is introduced (P = buckling load, L = column length). This quantity can be considered as a measure of the loading intensity. All dimensionally similar columns having the same value of structural index will develop the same stress at failure. Therefore, for a particular material and a particular type of cross-section, an easily obtainable relationship between optimum stress and structural index constitutes the information needed for the design of the entire family of minimum weight simple columns.

At low values of the structural index, the optimum stress is far below the elastic limit of most structural materials and an expansion of the cross-section will allow the column to operate at a higher stress level, thereby possibly increasing efficiency, from a weight standpoint. In the age of space technology the long column with small compression load becomes more and more interesting and, considering the extremely high cost per pound of orbited load, even the smallest weight savings is appreciated.

The thin-walled circular tube, which is the most efficient simple column, is chosen as a basis for the investigation. The weight of the simple column is a minimum when the allowable stress σ is a maximum. To find the maximum values of σ both primary buckling (Engesser formula) and local buckling are considered and the optimum design is obtained when both failures can occur simultaneously.

Using two equal columns and equipping one with tension ties always results in an increase of maximum stress, hence a decrease in weight of the central tube is possible; however, additional weight is added in conjunction with ties and struts. Once the structural index $P/_{L}2$ is specified, the problem is then to define the parameters associated with the tie supported column like column diameter, wall thickness, tie cross-sectional area and tie angle α to obtain minimum weight. The column supported by tension ties considered herein consists of three parts: a thin walled tube with circular cross-section, tension ties, and struts, as shown in Fig. 1. The theory of analyzing such columns has been developed previously [1] and the improved efficiency is proved in tests. [1] Nevertheless in none of the solutions was optimization of the structure with respect to weight attempted, as has been done for the simple column. [2]

For the actual calculations the following assumptions are made:

a. The effect of the deformation of the struts is negligible.

b. The connection between the struts and the column and the connection between the ties and struts are ideal hinges.

c. There is no initial eccentricity or crookedness in the column.

d. There is no lateral deflection before buckling.

e. The pretension in the wires is of such magnitude that at impending buckling the wires are stress free.

f. For small lateral deflection the axial deformation is negligible.

g. The angles between the planes of the ties are equal.

h. The struts are distributed symmetrically with respect to the midpoint of the column.

The optimum design can again be found by equating the primary and local buckling stresses. The weight of the column is a function of the above mentioned parameters and a minimization yields optimum values of these parameters.

Comparing the optimum weight of the simple column with the optimum weight of the column supported by tension ties for identical values of the structural index will show how much more efficient this supported column can be.

SECTION 2

GENERAL THEORY OF THE COLUMN SUPPORTED BY TENSION TIES

2.1 Force and deformation relationship up to the instant of buckling

The following theory dealing with the mechanical behavior of the column supported by tension ties is based on Ref. 1 and is repeated here in a slightly modified form for completeness. The geometry of the supported column is sketched in Fig. 1 and Fig. 2.

2.1.1 Pretension

Let the pretension in a tie in the j^{th} panel be denoted by T_{oj}. If there is no external load applied in the x-direction, then the force in the column (P_o) induced by pretension is

$$P_o = m\, T_{o1} \sin \theta_1 \tag{1}$$

where m is the number of tension ties.

Due to this load, the column has shortened a distance ΔL_o.

$$\Delta L_o = \frac{m\, T_{o1} \sin\theta_1}{A_c\, E_c}\, L \tag{2}$$

From the equilibrium of forces in x-direction at any joint, and by neglecting the effect of small angle changes $\Delta\theta_j$

$$T_{oj} = T_{o1} \frac{\sin \theta_1}{\sin \theta_j} \tag{3}$$

The elongation of the tie in the j^{th} panel due to T_{oj} is therefore

$$\Delta c_{oj} = T_{oj} \frac{c_j}{A_j E_j} = T_{o1} \frac{\sin \theta_1}{\sin \theta_j} \frac{c_j}{A_j E_j} \tag{4}$$

By neglecting again small angle changes and with the assumption that there is no lateral deflection before buckling starts, then the component of tie deflection in the x-direction is

$$(\Delta c_{oj})_x = \Delta c_{oj} \sin \theta_j = T_{o1} \sin \theta_1 \frac{c_j}{A_j E_j} \tag{5}$$

and the total displacement in the x-direction is

$$(\Delta c_o)_x = \sum_{j=1}^{n} T_{o1} \sin \theta_1 \frac{c_j}{A_j E_j} \tag{6}$$

2.1.2 Relationship between external load and internal forces

Let an external load P be applied to the strut with tension ties that are tightened to a certain value of initial tension. Then the force acting in the column will be increased by the amount P_p (Fig. 2) and the increase in axial deformation will be

$$\Delta L_p = P_p \frac{L}{A_c E_c} \tag{7}$$

The force acting in a tie in the j^{th} panel is decreasing by the amount T_{pj}. Similar to equation (3) this force is

$$T_{pj} = T_{p1} \frac{\sin \theta_1}{\sin \theta_j} \tag{8}$$

The decrease in stretch in the axial direction due to T_{pj} is similar to that given by equation (6).

$$(\Delta c_p)_x = \sum_{j=1}^{n} T_{p1} \sin \theta_1 \frac{c_j}{A_j E_j} \tag{9}$$

Since $\Delta L_p = (\Delta c_p)_x$, equations (7) and (9) give

$$P_p = \frac{A_c E_c}{L} T_{p1} \sin \theta_1 \sum_{j=1}^{n} \frac{c_j}{A_j E_j} \tag{10}$$

By taking the summation of the forces in a section at the end of the strut (Fig. 3) and considering the fact that the pretension forces in the column and tension ties are in equilibrium regardless of the applied force P, it is found that

$$P = m\, T_{p1} \sin \theta_1 + P_p \tag{11}$$

Substituting P_p from equation (10) in equation (11) yields

$$T_{p1} = \frac{P}{\sin \theta_1 \left(m + \frac{A_c E_c}{L} \sum_{j=1}^{n} \frac{c_j}{A_j E_j} \right)} \tag{12}$$

With the assumption that at the onset of buckling the tension ties are stress free, it follows that equations (12) and (3) must be equal, or simply

$$T_{o1} = T_{p1} \tag{13}$$

2.2 Lateral reactions produced by tension ties at buckling load

Change in length and slope of the tension ties due to lateral deflection have to be considered next. Let the displacement of joint j in the direction perpendicular to the column and in the plane of column and tie be denoted by $(\Delta y)_j$. Then, as shown in Fig. 4 for small displacements, changes in length of the ties may be expressed as

$$\Delta c_j = \left[(\Delta y)_{j-1} - (\Delta y)_j\right] \cos \theta_j + \left[(\Delta x)_{j-1} - (\Delta x)_j\right] \sin \theta_j \tag{14}$$

This is only valid if $\Delta \theta_j \rightarrow 0$.

When the lateral deflection starts, the axial deflection $(\Delta x)_{j-1} - (\Delta x)_j$ is negligible and may be ignored. Furthermore, it is assumed that there is no lateral displacement at the end of the column and that the warping of the planes of the ties is negligible. Equation (14) can therefore be simplified to

$$\Delta c_j = \left[(\Delta y)_{j-1} - (\Delta y)_j\right] \cos \theta_j \tag{15}$$

and with

$$\Delta T_j = \frac{A_j E_j}{c_j} \Delta c_j$$

$$\Delta T_j = \frac{A_j E_j}{c_j} [(\Delta y)_{j-1} - (\Delta y)_j] \cos \theta_j \quad (16)$$

Now consider the case in which the deflection of the column occurs in the direction § (Fig. 4) in the x-§ plane. Let $(\Delta §)_j$ be the deflection of joint j in the § direction and $(\Delta y)_{ji}$ be the components of $(\Delta §)_j$ in the plane of column and tie i (i = 1,2,3..m). Let β_i be the angle between the plane of column and tie and the x-§ plane. Then referring to Fig. 4

$$(\Delta y)_{ji} = (\Delta §)_j \cos \beta_i \quad (i = 1,2,3,4...m) \quad (17)$$

Assuming that the angles between the planes of the ties are equal, then if $\beta_1 = \beta$, it follows that

$$\begin{aligned} \beta_2 &= \frac{2\pi}{m} + \beta \\ \beta_3 &= \frac{4\pi}{m} + \beta \\ &\cdots\cdots \\ \beta_m &= \frac{m-1}{m} 2\pi + \beta \end{aligned} \quad (18)$$

Let the change in length of the tie i in the panel j be $(\Delta c_j)_i$, then equation (15) may be written as follows

$$(\Delta c_j)_i = (\Delta §_{j-1} - \Delta §_j) \cos \theta_j \cos \beta_i \quad (19)$$

2.3 Lateral reactions when some ties are relaxed at the instant of buckling

If the ties are stress free at the instant of buckling, some ties are relaxed for an infinitesimal amount of buckling deflection. This gives some constraint to the initial tension which shall be considered later. With this assumption it is clear that ties with an angle β_i defined by

$$\pi/2 < \beta_i < 3\pi/2$$

will be relaxed and the ties with an angle β_i of

$$-\pi/2 < \beta_i < +\pi/2$$

are going to be stretched.

For three or four tension ties only one or two ties, but not more than two ties can possibly lay in the region from $-\pi/2$ to $+\pi/2$. Therefore in those cases only a maximum of two tension ties can be stretched. Assume now first that tension tie $i = 1$ and $i = m$ are stretched. The perpendicular directions to the column axis in each of the planes containing these ties are called y_1 and y_m. (Fig. 5, Fig. 6, Fig. 7).

In the direction of ξ, the lateral components of the respective tensile forces produced by the changes in length of the ties are (Fig. 6)

$$(R_{jj})_{\xi 1} = - \frac{A_j E_j}{c_j} (\Delta c_j)_1 \cos \beta \cos \theta_j$$

$$= - \frac{A_j E_j}{c_j} \left[(\Delta \xi)_{j-1} - (\Delta \xi)_j \right] \cos^2 \theta_j \cos^2 \beta \qquad (20)$$

$$(R_{jj})_{\xi m} = - \frac{A_j E_j}{c_j} \left[(\Delta \xi)_{j-1} - (\Delta \xi)_j \right] \cos^2 \theta_j \cos^2 (\frac{m-1}{m} 2\pi + \beta) \qquad (21)$$

where the angle β_m is replaced by the expression calculated in equation (18).

In order to maintain equilibrium of forces in the η direction, the following condition must be satisfied (Fig. 6)

$$(R_{jj})_{\eta 1} = (R_{jj})_{\eta m} \qquad (22)$$

with

$$(R_{jj})_{\eta 1} = (R_{jj})_{\xi 1} \tan \beta$$

$$(R_{jj})_{\eta m} = (R_{jj})_{\xi m} \tan (\frac{m-1}{m} 2\pi + \beta)$$

or

$$\sin \beta \cos \beta = \sin(\frac{m-1}{m} 2\pi + \beta) \cos(\frac{m-1}{m} 2\pi + \beta) \qquad (23)$$

This equation can only be satisfied for

$$\beta = \pi/m$$

With this result the actual direction of deflection is known and this particular ξ direction is denoted as y. (Fig.5)

The reaction force R_{jj} in this y direction is given by

$$R_{jj} = (R_{jj})_{y1} + (R_{jj})_{ym}$$

$$= -\frac{A_j E_j}{c_j}\left[(\Delta y)_{j-1} - (\Delta y)_j\right] \cos^2\theta_j \; B \qquad (24)$$

where

$$B = 2\cos^2\beta/m = 2\cos^2(\pi/m)$$

If only one tension tie is stretched, there is only one case of equilibrium possible, namely when $\beta = 0$. In this case

$$R_{jj} = (R_{jj})_{y1}$$

$$= -\frac{A_j E_j}{c_j}\left[(\Delta y)_{j-1} - (\Delta y)_j\right] \cos^2\theta_j \; B \qquad (25)$$

where

$$B = 1$$

The lateral reactions are due to the action of the ties in the j^{th} panel on strut plane j. The reaction on strut plane j due to the ties in the $(j+1)^{th}$ panel is given by

$$R_{j(j+1)} = +\frac{A_{j+1}E_{j+1}}{c_{j+1}}\left[(\Delta y)_j - (\Delta y)_{j+1}\right] \cos^2\theta_{j+1} \; B \qquad (26)$$

Therefore the resultant reaction at joint j is obtained by combining equations (24) or (25) with equation (26), as follows:

$$R_j = R_{jj} + R_{j(j+1)} \tag{27}$$

or

$$R_j = \frac{A_{j+1}E_{j+1}}{c_{j+1}} \left[(y)_j - (y)_{j+1} \right] \cos^2 \theta_{j+1}\; B$$

$$- \frac{A_jE_j}{c_j} \left[(y)_{j-1} - (y)_j \right] \cos^2 \theta_j \; B \tag{28}$$

where Δy is replaced by y, for convenience.

2.4 Buckling theory

A strut with tension ties may be considered as a continuous beam on elastic intermediate supports as shown in Fig. 8. In the case of simply supported ends a representation of the deflection curve in the form of a trigonometric series is advantageous.

The deflection curve can be represented as

$$y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} + \ldots$$

$$a_k \sin \frac{k\pi x}{L} \tag{29}$$

Each term satisfies the boundary conditions, since each term together with its second derivative becomes zero at the end of the beam.

For the case of the column supported by tension ties, the coefficients in the above series, and the buckling load, are obtainable from an energy approach. The work done by the longitudinal force P must be equal to the work done by the reaction force R_j plus the strain energy. The strain energy is defined as

$$U = \frac{E_c I_c}{2} \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad (30)$$

Introducing equation (28) and integrating yields

$$U = \frac{\pi^4 E_c I_c}{4 L^3} \sum_{k=1}^{k=\infty} k^4 a_k^2 \qquad (31)$$

Any change in the shape of the deflection curve results in some longitudinal displacement at the hinge B. This displacement is equal to the difference between the length of the deflection curve and the length of the chord AB (Fig. 8). In terms of the chosen coefficients this distance is

$$\lambda = \frac{\pi^2}{4L} \sum_{k=1}^{k=\infty} k^2 a_k^2 \qquad (32)$$

and the work done by the external force P is therefore

$$W_1 = \frac{P\pi^2}{4L} \sum_{k=1}^{k=\infty} k^2 a_k^2 \qquad (33)$$

The displacements at the struts are simply $(y)_j$ and the reaction forces R_j are linear functions of $(y)_j$ [equation (28)]. That means the spring characteristic is linear and the spring strain energy is

$$U_1 = \sum_{j=1}^{n-1} 1/2\ y_j R_j = 1/2 \sum_{j=1}^{n-1} R_j \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x_j}{L} \tag{34}$$

For a conservative system

$$W_1 = U + U_1$$

Therefore with equations (31), (33), and (34)

$$\frac{P\pi^2}{4L} \sum_{k=1}^{\infty} k^2 a_k^{\ 2} = \frac{\pi^4 E_c I_c}{4\ L^3} \sum_{k=1}^{\infty} k^4 a_k^{\ 2} + 1/2 \sum_{j=1}^{n-1} R_j \sum_{k=1}^{\infty} a_k \sin \frac{k\pi\ x_j}{L} \tag{35}$$

SECTION 3

OPTIMUM DESIGN OF THE SIMPLE COLUMN

The theory of optimum design of the simple column is presented in [2] and the most important results are repeated here for convenience.

The Engesser formula for column buckling is

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L/\rho)^2} \tag{36}$$

and may be expressed in terms of the structural index P/L^2, using the proper value of ρ for a thin-walled circular tube

$$\frac{P}{L^2} = \frac{8\,\sigma_{cr}^2}{\pi\,E_t\,\frac{D}{t}} \tag{37}$$

For local buckling

$$\sigma_{cr} = k_2\,\frac{\sqrt{E\,E_t}}{D/t} \tag{38}$$

where $k_2 \doteq 0.4$.

Solving equation (38) for D/t and substituting σ_{cr} (i.e., $\sigma_{cc} = \sigma_{cr}$) from equation (37) yields an optimal value for D/t:

$$\left(\frac{D}{t}\right)_{opt} = 2\left[\frac{k_2^2\,E}{\pi\,\frac{P}{L^2}}\right]^{\frac{1}{3}} \tag{39}$$

Or, in terms of the structural index and the stress σ, these equations can be combined to yield

$$\frac{P}{L^2} = \frac{8\ \sigma^3}{\pi k_2\ E^2\ \tau^{3/2}} \tag{40}$$

where $\tau = \dfrac{E_t}{E}$

For any given value of σ from a particular stress-strain curve, the value of P/L^2 may be calculated. The weight is obtainable from the relation

$$\frac{W}{L^3} = \frac{P/L^2\ w}{\sigma} \tag{41}$$

where w is the specific weight.

SECTION 4

OPTIMUM DESIGN OF A COLUMN WITH ONE STRUT

4.1 Buckling theory for the column with one strut

From the geometry of the system (Fig. 9) it is seen that

$$\theta_1 = \pi/2 + \alpha \qquad \cos^2 \theta_1 = \sin^2 \alpha$$

$$\theta_2 = \pi/2 - \alpha \qquad \cos^2 \theta_2 = \sin^2 \alpha$$

Consider the first three terms of the sine series

$$y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} \tag{42}$$

and

$$U = \frac{\pi^4 E_c I_c}{4 L^3} \left[a_1^2 + 16 a_2^2 + 81 a_3^2\right] \tag{43}$$

$$W_1 = \frac{P\pi^2}{4L} \left[a_1^2 + 4 a_2^2 + 9 a_3^2\right] \tag{44}$$

$$U_1 = \frac{R}{2} \left[a_1 - a_3\right] \tag{45}$$

The reaction force is obtainable from equation (28). With the assumptions that $A_1 = A_2$, $E_1 = E_2$, and $c_1 = c_2 = \frac{L}{2 \cos \alpha}$, then

$$R = \frac{4 A_1 E_1}{L} (a_1 - a_3) \sin^2 \alpha \cos \alpha B \tag{46}$$

Introducing equation (46) in (45)

$$U_1 = \frac{2\ A_1 E_1}{L} \sin^2\alpha \cos\alpha\ B \left[a_1^{\ 2} - 2\ a_1 a_3 + a_3^{\ 2}\right] \qquad (47)$$

If any coefficient a_k in series (39) is given an increase da_k, the term $(a_k + da_k) \sin k\pi_x/L$ replaces the term $a_k \sin k\pi_x/L$. This increase da_k in the coefficient a_k represents an additional small deflection of the beam given by $da_k \sin k\pi_x/L$ superposed upon the original deflection curve.

The change in strain energy of the column, due to the increase da_k is

$$\frac{\partial U}{\partial a_k} da_k = \frac{\pi^4\ E_c I_c}{2\ L^3} k^4\ a_k\ da_k \qquad (48)$$

The change in work done by the compression load is

$$\frac{\partial W_1}{\partial a_k} da_k = P\ \frac{\pi^2\ k^2}{2L}\ a_k\ da_k \qquad (49)$$

and the change in strain energy of the springs is

$$\frac{\partial U_1}{\partial a_k} da_k = \frac{2\ \sin^2\alpha \cos\alpha\ A_1 E_1\ B}{L} \frac{\partial}{\partial a_k} (a_1^{\ 2} - 2a_1 a_3 + a_3^{\ 2}) \qquad (50)$$

For a conservative system

$$\frac{\partial W_1}{\partial a_k} da_k = \frac{\partial U}{\partial a_k} da_k + \frac{\partial U_1}{\partial a_k} da_k$$

For K = 1,2,3 this results in three equations

$$\frac{P\pi^2}{2L} a_1 = \frac{\pi^4 E_c I_c}{2 L^3} a_1 + \frac{4 \sin^2\alpha \cos\alpha A_1 E_1 B}{L} (a_1 - a_3) \tag{51}$$

$$4 \frac{P\pi^2}{2L} a_2 = \frac{\pi^4 E_c I_c}{2 L^3} 16 a_2 \tag{52}$$

$$9 \frac{P\pi^2}{2L} a_3 = \frac{\pi^4 E_c I_c}{2 L^3} 81 a_3 + \frac{4 \sin^2\alpha \cos\alpha A_1 E_1 B}{L} (a_3 - a_1) \tag{53}$$

Equation (52) states that the second buckling mode of the simple column is a solution of this column supported by tension ties. It can be seen that equations (51) and (53) can only be satisfied if the following determinant is zero.

$$\begin{bmatrix} \frac{\pi^4 E_c I_c}{L^2} - P\pi^2 + \bar{\kappa} & -\bar{\kappa} \\ -\bar{\kappa} & 81 \frac{\pi^4 E_c I_c}{L^2} - 9 P\pi^2 + \bar{\kappa} \end{bmatrix} = 0 \tag{54}$$

where

$$\bar{\kappa} = 8 A_1 E_1 B \sin^2\alpha \cos\alpha$$

To extend the theory in the plastic region, the modulus of elasticity of the column is replaced by the tangent modulus. Considering, furthermore, local buckling, it can be shown that the moment of inertia I_c must be replaced by the expression [2]

$$I_c = \delta^2 A = k_1 \frac{P}{\sigma} A = \frac{1}{8\pi} k_2 \frac{\sqrt{E\,E_t}}{2} P\,A \qquad (56)$$

where, as for the simple column, $k_2 \doteq 0.4$.

Introducing equation (56) in the above determinant, and replacing E_c with the tangent modulus E_t yields

$$\begin{bmatrix} \frac{\pi^3 k_2}{8\sigma^2} \tau^{3/2} E^2\left(\frac{P}{L^2}\right) - \sigma\pi^2 + \kappa & -\kappa \\ -\kappa & 81 \frac{\pi^3 k_2}{8\sigma^2} \tau^{3/2} E^2\left(\frac{P}{L^2}\right) - 9\sigma\pi^2 + \kappa \end{bmatrix} = 0$$

(57)

where $\tau = E_t/E$ and $\kappa = \frac{\bar{\kappa}}{A} = 8 \frac{A_1 E_1}{A} B \sin^2\alpha \cos\alpha$

Expanding equation (57) yields

$$\left(\frac{P}{L^2}\right) = \frac{8\sigma^2}{81\,\pi^3 k_2\,\tau^{3/2} E^2} \left(45\sigma\pi^2 - 41\kappa\right) \qquad (58)$$

$$\pm\sqrt{16 \cdot 81\pi^4\sigma^2 - 90 \cdot 32\kappa\sigma\pi^2 + 41^2\kappa^2}$$

This equation should reduce to the solution for the simple column in the limit. For the simple column the cross section of the tension ties is zero, so that κ takes the value zero. Substituting $\kappa = 0$ in equation (58) yields

$$\frac{P}{L^2} = \frac{8\sigma^3\pi}{9\ k_2\tau^{3/2}\ E^2}\ (5 \pm 4) \qquad (59)$$

Considering the positive sign in the bracket, the buckling formula for the first mode of the simple column (equation (40)) is obtained. The negative sign yields the buckling load for the third mode shape which is not critical. The tension ties increase the buckling force in all modes, so that the positive sign in equation (58) has to be chosen to obtain the critical load.

For cylindrical tubes the factor k_2 is chosen as 0.4. κ has the same dimensions as the structural index and can be considered as something like a structural index of the elastic support of the column and serves as a parameter in further calculations.

Equations (51) and (53) indicate that the buckling shape resulting from the influence of the tension ties is a combination of the natural first and third buckling mode shape. This buckling mode shall be called "constrained first buckling mode."

Equation (58) can be written in simplied form as

$$\frac{P}{L^2} = 0.3265 \frac{\sigma^2}{\tau^{3/2}E^2}\left[(10.8325\ \sigma-\kappa)+\sqrt{75.0981\ \sigma^2-16.9097\ \kappa\sigma+\kappa^2}\right] \tag{60}$$

To observe the influence of the tension ties, the "constrained first buckling mode" is calculated for an actual case for some typical values of κ assuming a column constructed of 2024-T4 aluminum alloy with material properties as shown in Fig. 10. The result is sketched in Fig. 11 and indicates that for various values of P/L^2 different values of κ result in curves which intersect the curve plotted for the second buckling mode shape. Since at all of these points two modes of failure occur simultaneously, the optimum design is reached when κ as a function of P/L^2 is chosen such that the graphs for the constrained first buckling mode shape and the second buckling mode shape are identical. This relationship can be found, when the structural index for the second mode is introduced in the left hand side of equation (60). The second buckling mode shape is given in equation (52) for the following structural index

$$\frac{P}{L^2} = \frac{2\sigma^3}{\pi\ k_2\ E^2\ \tau^{3/2}} \tag{61}$$

Combining equations (60) and (61) yields

$$\kappa = \frac{45}{56}\,\sigma\pi^2 = 7.93091\ \sigma \tag{62}$$

This result is independent of the properties of the chosen material.

4.2 Weight assumptions

From Fig. 12 it can be seen that the weight of the column with one strut and m tension ties is the summation of

W_1 - the weight of the column

W_2 - the weight of the supporting wires

W_3 - the weight of the strut assembly (struts and connecting ring)

4.2.1 Weight of the column

As given in equation (41) the weight of the simple column is

$$\frac{W_1}{L^3} = \frac{w\,\frac{P}{L^2}}{\sigma} \tag{63}$$

4.2.2 Weight of the wires

$$W_2 = m\ w_1\ A_1\ L/\cos\alpha \quad \text{or} \quad \frac{W_2}{L^3} = \frac{m\ w_1\ T_1}{\sigma_1\ \cos\alpha\ L^2} \tag{64}$$

T_1 must be taken as the highest possible tie force. From previous calculations the tie forces are given in the form of equation (16)

$$\Delta T_j = T_j = \frac{A_j E_j}{c_j} \left[(\Delta y)_{j-1} - (\Delta y)_j \right] \cos \theta_j \qquad (65)$$

For the case of only one strut, there is only one possible tensile stress, because Δy_o and Δy_2 are zero.

For x = L/2 the sine series for the deflection gives

$$y_1 = (a_1 = a_3)$$

Again Δy is replaced by y and the angle θ by α. The length of tie 1 is expressed in terms of L. Introducing now the value of y_1 in equation (65) yields

$$\sigma_1 = \frac{2E_1 \cos \alpha \sin \alpha}{L} \quad (a_1 - a_3) \qquad (66)$$

This equation gives a constraint for the maximum value of σ in the tension ties. For convenience the weight function for the tension ties is written in the form

$$\frac{W_2}{L^3} = \frac{m\, w_1}{E_1 \cos \alpha} \left(\frac{E_1 A_1}{L^2} \right) \qquad (67)$$

4.2.3 Weight of the strut assembly

The struts are assumed to be simple columns, welded on a ring with a weight W_4, which for simplification is chosen to be the same as the weight of one of the struts. The m struts at the midpoint of the column are assumed to fail as simply supported columns. Assuming furthermore that buckling occurs elastically

$$\frac{R}{h^2} = \frac{8\sigma^3}{\pi\ k_2\ E^2} \text{ and } \sigma^3 = \frac{\pi\ k_2\ E^2\ R}{8\ h^2} \tag{68}$$

with tan $\alpha = \frac{2h}{L}$ it follows that $\frac{1}{h^2} = \frac{4}{L^2\ \tan^2\alpha}$

Therefore

$$\sigma^3 = \frac{\pi\ k_2\ E^2\ R}{2\ \tan^2\alpha\ L^2} \tag{69}$$

The weight of the strut assembly is now

$$W_3 = (m+1)\ w\ A\ h = (m+1)\ w\ \frac{R}{\sigma}\ \frac{L\ \tan\alpha}{2}$$

and

$$\frac{W_3}{L^3} = \frac{m+1}{2}\ w\ \frac{\tan\alpha}{\sigma}\left(\frac{R}{L^2}\right)$$

Introducing equation (69) for the optimum stress yields

$$\frac{W_3}{L^3} = \frac{m+1}{2}\ w(\tan\alpha)^{5/3}\left(\frac{R}{L^2}\right)^{2/3}\left[\frac{2}{\pi\ k_2\ E^2}\right]^{1/3} \tag{70}$$

where R is the reaction force given in equation (46). The specific weight of the strut assembly is assumed to be the same as for the column.

Equations (63), (67), and (69) combined give the total weight as

$$\frac{W}{L^3} = \frac{w\left(\frac{P}{L^2}\right)}{\sigma} + \frac{m\ w_1}{E_1 \cos\alpha}\left(\frac{A_1 E_1}{L^2}\right) + \frac{m+1}{2} w\,(\tan\alpha)^{5/3}\left(\frac{R}{L^2}\right)^{2/3}\left[\frac{2}{\pi k_2 E^2}\right]^{1/3} \tag{71}$$

Rewriting the equation for the reaction force R yields

$$\frac{R}{L^2} = \frac{4\ A_1 E_1}{L^3}\ (a_1 - a_3)\ \sin^2\alpha\ \cos\alpha\ B$$

At the onset of buckling, the theory gives only the buckling shape, but does not specify a fixed magnitude of deflection. It is assumed that the struts buckle when y/L reaches a value given by

$$\frac{y}{L} = \frac{1}{150} \tag{72}$$

After completing the calculation it must be ascertained whether or not the tension ties are stressed to a value which is below the yield stress. To do so, equation (66) must be used.

With the above assumptions for y/L the reaction force is

$$\frac{R}{L^2} = \frac{2}{75}\ \sin^2\alpha\ \cos\alpha\ B\left(\frac{A_1 E_1}{L^2}\right) \tag{73}$$

Equation (66) takes the form

$$\sigma_1 = \frac{E_1\ \cos\alpha\ \sin\alpha}{75} \tag{74}$$

With the definition of κ, and equation (62), the following is obtained:

$$\frac{45}{56} \sigma\pi^2 = 8 \sin^2\alpha \cos\alpha \frac{A_1E_1}{A} B$$

or

$$A_1E_1 = \frac{45 \pi^2 P}{448 \sin^2\alpha \cos\alpha B} \tag{75}$$

Introducing equation (75) in (73)

$$\frac{R}{L^2} = \frac{3\pi^2}{5\cdot 224} \frac{P}{L^2} \tag{76}$$

With equations (75) and (76) the total weight is expressible in terms of P/L^2, σ, and α:

$$\frac{W}{L^3} = \frac{w\left(\frac{P}{L^2}\right)}{\bar{\sigma}} + \frac{m\, w_1}{E_1 \cos\alpha} \frac{45 \pi^2}{448 \sin^2\alpha \cos\alpha B} \left(\frac{P}{L^2}\right)$$

$$+ \frac{m+1}{2} w\left(\frac{3\pi^2}{5\cdot 224} \frac{P}{L^2}\right)^{2/3} \left[\frac{2}{\pi k_2 E^2}\right]^{1/3} (\tan\alpha)^{5/3} \tag{77}$$

4.3 Optimization

The minimum weight can be obtained by setting the derivative of equation (77) with respect to α equal to zero.

$$\frac{\partial}{\partial\alpha} \frac{W}{L^3} = 0 \tag{78}$$

For simplification, it is assumed that in the following calculations the only cases considered are those for which w_1 and w are the same. Differentiating equation (77) with respect to α and simplifying the result in such a way that the left hand side of the equation contains terms in α only, yields

$$\frac{1-2\sin^2\alpha}{\sin^3\alpha\cos\alpha}\,\frac{1}{\tan\alpha^{2/3}} = \frac{(m+1)}{9\pi m}\,E_1\left[\frac{56}{75\;k_2E^2\;P/L^2}\right]^{1/3} B \qquad (79)$$

The term on the left hand side of equation (79) is independent of material properties and the number of tension ties. To find the optimum angle α the function

$$f(\alpha) = \frac{1-2\sin^2\alpha}{\sin^3\alpha\cos\alpha}\,\frac{1}{\tan\alpha^{2/3}} \qquad (80)$$

must be calculated for different angles. The result is plotted in Fig. 13. With the help of this figure the optimum angle α can be obtained for all values of P/L^2.

4.4 Method of solution

For any values of the stress σ the tangent-modulus ratio τ can be found from any tangent-modulus curve for typical materials. From equation (61) the structural index for this particular stress value is obtainable. Choosing now the number of tension ties as 3 or 4, makes it possible to calculate the right hand side of equation (79). The optimum value of α can be found now from Fig. 13, and equation (77) can be solved to obtain the optimum weight of the column supported by tension ties in the form W/L^3.

SECTION 5

OPTIMUM DESIGN OF THE COLUMN WITH THREE STRUTS

In using the theory for more than one strut it can be seen that the second buckling mode is always an independent solution of the "boom problem." This coincides with the solution in Ref. 1, where it was found that, if the intermediate supports are spaced symmetrically with respect to the midpoint, then no matter how many intermediate supporting points are used, the column will always buckle in the second mode.

Nevertheless, it seems to be possible to arrange the tension ties in such a way that the ties influence the load for the second buckling mode also, so that this column could be designed to have buckling occur in the first, second, third, or fourth mode. A geometry as sketched in Fig. 14 results in a constrained second buckling mode, thereby increasing buckling stresses over the previous case.

In this new problem the following additional assumptions are made:

a. The tension ties E-1-C-3-A and 1-2-3 have the same cross-section and material properties.

b. The ties 1-C and 3-C are hinged at point C, but cannot move in the x-direction.

Again it is assumed that the tension ties have a pretension of such a magnitude that at the instant of buckling all ties are stress free, but not relaxed.

Instead of two variables, as in the case of one strut, four variables, α_1, α_2, A_1, and A_2, must now be determined. (Fig. 14)

5.1 Buckling theory for the column with three struts

The deflection curve must be introduced with four terms and reads

$$y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} + a_4 \sin \frac{4\pi x}{L} \tag{81}$$

For x = L/4, L/2, and 3L/4 this series yields

$$\begin{aligned} y\Big|_{x=\frac{L}{4}} &= y_1 = \frac{a_1}{2}\sqrt{2} + a_2 + \frac{a_3}{2}\sqrt{2} \\ y\Big|_{x=\frac{L}{2}} &= y_2 = a_1 - a_3 \\ y\Big|_{x=\frac{3L}{4}} &= y_3 = \frac{a_1}{2}\sqrt{2} - a_2 + \frac{a_3}{2}\sqrt{2} \end{aligned} \tag{82}$$

The changes in length of the tension ties, neglecting displacements in the x-direction and assuming $y_o = y_4 = 0$ are (Fig. 14)

$$\Delta c_1 = y_1 \sin\alpha_1 \qquad \Delta c_3{}^a = (y_3 - y_2) \sin\alpha_1$$

$$\Delta c_2{}^a = (y_1 - y_2) \sin\alpha_1 \qquad \Delta c_3{}^b = (y_2 - y_3) \sin\alpha_2$$

$$\Delta c_2{}^b = (y_2 - y_1) \sin\alpha_2 \qquad \Delta c_4 = y_3 \sin\alpha_1$$

Expressing y_1, y_2, and y_3 in terms of the series as indicated in equation (82) and using

$$\Delta T_j = \frac{A_j E_j}{c_j} \Delta c_j$$

the following tension stresses in the ties are obtained:

$$\Delta T_1 = \frac{4\ A_1 E_1}{L}\left[\frac{a_1}{2}\sqrt{2} + a_2 + \frac{a_3}{2}\sqrt{2}\right] \sin\alpha_1 \cos\alpha_1$$

$$\Delta T_2{}^a = \frac{4\ A_1 E_1}{L}\left[a_1\left(\frac{\sqrt{2}}{2} - 1\right) + a_2 + a_3\left(\frac{\sqrt{2}}{2} + 1\right)\right] \sin\alpha_1 \cos\alpha_1$$

$$\Delta T_2{}^b = -\frac{4\ A_2 E_2}{L}\left[a_1\left(\frac{\sqrt{2}}{2} - 1\right) + a_2 + a_3\left(\frac{\sqrt{2}}{2} + 1\right)\right] \sin\alpha_2 \cos\alpha_2$$

$$\Delta T_3{}^a = \frac{4\ A_1 E_1}{L}\left[a_1\left(\frac{\sqrt{2}}{2} - 1\right) - a_2 + a_3\left(\frac{\sqrt{2}}{2} + 1\right)\right] \sin\alpha_1 \cos\alpha_1 \qquad (83)$$

$$\Delta T_3{}^b = -\frac{4\ A_2 E_2}{L}\left[a_1\left(\frac{\sqrt{2}}{2} - 1\right) - a_2 + a_3\left(\frac{\sqrt{2}}{2} + 1\right)\right] \sin\alpha_2 \cos\alpha_2$$

$$\Delta T_4 = \frac{4\ A_1 E_1}{L}\left[\frac{a_1}{2}\sqrt{2} - a_2 + \frac{a_3}{2}\sqrt{2}\right] \sin\alpha_1 \cos\alpha_1$$

It can be seen, that the forces T^a and T^b have different signs. A negative stress cannot exist in a tension tie, and the above equations must therefore be handled very carefully.

For the simplified theory three different cases, as shown in Fig. 15, must be considered separately.

a. symmetric buckling shape ($y_2 > y_1$)

b. antisymmetric buckling shape

c. symmetric buckling shape ($y_2 < y_1$)

This distinction can be made if the structure has at least one plane of symmetry [5]. For three tension ties this symmetry condition is not fully applicable, because the reaction forces, which are defined as forces in the strut planes and are opposite to the deflection, are dependent on the sign of the deflection. ($B = 1$ for $y < 0$, $B = 1/2$ for $y > 0$.) Nevertheless an approximation is made by assuming that symmetric and antisymmetric modes are independent.

5.1.1 Symmetric mode shape ($y_2 > y_1$)

From Fig. 15 it can be seen, that a reaction force R opposing the deflection in all three struts exists and can be written as

$$
\begin{aligned}
R_1 &= \left[\Delta T_1 \sin\alpha_1 - \Delta T_2^{\,b} \sin\alpha_2\right] B \\
R_2 &= \left[(\Delta T_2^{\,b} + \Delta T_3^{\,b}) \sin\alpha_2\right] B \qquad (84) \\
R_3 &= \left[\Delta T_4 \sin\alpha_1 - \Delta T_3^{\,b} \sin\alpha_2\right] B
\end{aligned}
$$

whereas in the case of a column with one strut, for

$$y > 0 \qquad \begin{cases} B = 1 \text{ for four tension ties} \\ B = 1/2 \text{ for three tension ties} \end{cases}$$

$$y < 0 \qquad B = 1 \text{ for three or four tension ties}$$

Introducing now equation (83) in (84) yields

$$R_1 = \left[\frac{4\ A_1E_1}{L}\left\{ \frac{a_1}{2}\sqrt{2} + a_2 + \frac{a_3}{2}\sqrt{2} \right\} \sin^2\alpha_1\ \cos\alpha_1 - \frac{4\ A_2E_2}{L}\left\{ a_1\left(1 - \frac{\sqrt{2}}{2}\right) - a_2 - a_3\left(1 + \frac{\sqrt{2}}{2}\right) \right\} \sin^2\alpha_2\ \cos\alpha_2 \right] B \qquad (85)$$

$$R_2 = \left[\frac{4\ A_2E_2}{L}\left\{ a_1\left(2 - \sqrt{2}\right) - a_3\left(2 + \sqrt{2}\right) \right\} \sin^2\alpha_2\ \cos\alpha_2 \right] B \qquad (86)$$

$$R_3 = \left[\frac{4\ A_1E_1}{L}\left\{ \frac{a_1}{2}\sqrt{2} - a_2 + \frac{a_3}{2}\sqrt{2} \right\} \sin^2\alpha_1\ \cos\alpha_1 - \frac{4\ A_2E_2}{L}\left\{ a_1\left(1 - \frac{\sqrt{2}}{2}\right) + a_2 - a_3\left(1 + \frac{\sqrt{2}}{2}\right) \right\} \sin^2\alpha_2\ \cos\alpha_2 \right] B \qquad (87)$$

The following notation will be introduced for convenience:

$$C_1 = \frac{4\ A_1E_1}{L}\ \sin^2\alpha_1\ \cos\alpha_1 \tag{88}$$

$$C_2 = \frac{4\ A_2E_2}{L}\ \sin^2\alpha_2\ \cos\alpha_2 \tag{89}$$

Using these equations the spring strain energy can be computed as

$$U_1 = 1/2\ [R_1y_1 + R_2y_2 + R_3y_3]$$

$$= \frac{B}{2}\Big[C_1(a_1{}^2 + 2a_1a_3 + 2a_2{}^2 + a_3{}^2) \tag{90}$$

$$+ C_2\Big\{a_1{}^2(3-2\sqrt{2})+2a_2{}^2-2a_1a_3+a_3{}^2(3+2\sqrt{2})\Big\}\Big]$$

For a virtual displacement of da_k the following equation is again valid

$$\frac{\partial W_1}{\partial a_k}\,da_k = \frac{\partial U}{\partial a_k}\,da_k + \frac{\partial U_1}{\partial a_k}\,da_k \tag{91}$$

As formerly stated there is a clear distinction between symmetric and antisymmetric buckling modes. Therefore in this case only virtual displacements da_k which are symmetric modes can be applied and the following two equations are obtained.

$$\frac{P\pi^2}{2L} a_1 = \frac{\pi^4 E_t I_c}{2\ L^3} a_1 + B\left[C_1(a_1+a_3) + C_2\left\{ a_1(3-2\sqrt{2}) - a_3\right\}\right] \quad (92)$$

$$9\frac{P\pi^2}{2L} a_3 = 81\frac{\pi^4 E_t I_c}{2\ L^3} a_3+B\left[C_1(a_1+a_3)+C_2\left\{a_3(3+2\sqrt{2}) -a_1\right\}\right] \quad (93)$$

5.1.2 <u>Symmetric mode shape ($y_2 < y_1$)</u>

Again, from Fig. 15 the following reaction forces are obtained

$$R_1 = \left[\Delta T_1 + \Delta T_2{}^a\right] \sin\alpha_1\ B + \Delta T_2{}^b \sin\alpha_2$$

$$R_2 = - \left[\Delta T_2{}^b + \Delta T_3{}^b\right] \sin\alpha_2 - \left[\Delta T_2{}^a + \Delta T_3{}^a\right] \sin\alpha_1\ B \quad (94)$$

$$R_3 = \left[\Delta T_4 + \Delta T_3{}^a\right] \sin\alpha_1\ B + \Delta T_3{}^b \sin\alpha_2$$

Following the same procedure as before the following two energy equations are obtained

$$\frac{P\pi^2}{2L} a_1 = \frac{\pi^4 E_t I_c}{2\ L^3} a_1+4C_1\ B\left(1-\frac{\sqrt{2}}{2}\right)a_1+C_2[(2\sqrt{2} -3)a_1+a_3] \quad (95)$$

$$9\frac{P\pi^2}{2L} a_3 = \frac{\pi^4 E_t I_c}{2\ L^3}\ 81\ a_3+4C_1 B\left(1+\frac{\sqrt{2}}{2}\right)a_3+C_2[a_1-(2\sqrt{2} +3)a_3] \quad (96)$$

5.1.3 Antisymmetric mode shape

$$R_1 = B(\Delta T_1 + \Delta T_2{}^a) \sin\alpha_1$$

$$R_2 = (-\Delta T_2{}^a B - \Delta T_3{}^a) \sin\alpha_1 \tag{97}$$

$$R_3 = (\Delta T_4 + \Delta T_3{}^a) \sin\alpha_1$$

For the antisymmetric mode there is only one virtual displacement possible, namely, a displacement in the second mode.

$$4\frac{P\pi^2}{2L} a_2 = \frac{\pi^4 E_t I_c}{2\ L^3} 16\ a_2$$
$$+C_1\left\{B[a_1(\sqrt{2}-1)+2a_2+a_3(\sqrt{2}+1)] - [a_1(\sqrt{2}-1)-2a_2+a_3(\sqrt{2}+1)]\right\} \tag{98}$$

5.1.4 Fourth mode shape

The fourth mode shape is independent of the tention ties and the following energy equation is valid

$$\frac{P\pi^2 16}{2\ L} a_4 = \frac{\pi^4 E_t I_c}{2\ L^3}\ 256\ a_4 \tag{99}$$

5.2 Buckling theory and optimization

For the optimum design the tension ties must be arranged in such a way that the first, second, third, and fourth buckling modes occur at the same buckling force P, which is, in this case, the solution of equation (99). So far, it has not been proved which set of equations, (92) - (93) or (95) - (96), is applicable, but the values of a_1, a_2, and a_3 can be found, and it can be checked easily whether the assumption $y_2 > y_1$ or $y_2 < y_1$ is valid.

5.2.1 Symmetric shape ($y_2 > y_1$)

For the assumption that $y_2 > y_1$ equations (92) and (93) together with equation (98) and (99) give a system of four equations which can be solved. Introducing the solution of equation (99) in equation (92), (93), and (98) yields

$$\frac{15}{2}\,\theta a_1 = B\left\{C_1(a_1+a_3) + C_2[a_1(3-2\sqrt{2}) - a_3]\right\}$$

$$\frac{48}{2}\,\theta a_2 = C_1\left\{B[a_1(\sqrt{2}-1) + 2a_2 + a_3(\sqrt{2}+1)] - [a_1(\sqrt{2}-1) - 2a_2 + a_3(\sqrt{2}+1)]\right\} \tag{100}$$

$$\frac{63}{2}\,\theta a_3 = B\left\{C_1(a_1+a_3) + C_2[a_3(3+2\sqrt{2}) - a_1]\right\}$$

where

$$\theta = \frac{\pi^4 E_t I_c}{L^3} \tag{101}$$

Nontrivial solutions are only obtainable if the following two sub-determinants are zero

$$2\,C_1\,(1 + B) - 24\theta = 0 \tag{102}$$

$$\begin{vmatrix} \frac{15}{2}\theta - B\left[C_1 + C_2(3-2\sqrt{2})\right] & B(C_2 - C_1) \\ B(C_2 - C_1) & \frac{63}{2}\theta - B\left[C_1 + C_2(3+2\sqrt{2})\right] \end{vmatrix} = 0 \tag{103}$$

In solving these two equations the following is obtained

$$C_1 = \frac{12\theta}{1+B} \qquad C_2 = \frac{4.875\,\frac{B}{1+B} - 2.4609375}{\frac{B}{1+B} - 0.511643}\,\theta$$

and, for the two different cases under consideration

$$\underline{m = 4} \qquad C_1 = 6\theta \qquad C_2 = 2.01297\,\theta \tag{104}$$

$$\underline{m = 3} \qquad C_1 = 8\theta \qquad C_2 = 0.93762\,\theta \tag{105}$$

Introducing these results in equations (82) and (100) leads to the following

$$\underline{m = 4} \qquad a_1 = 3.336734\,a_3 \text{ and } y_2 - y_1 = -0.72977\,a_3 \tag{106}$$

$$\underline{m = 3} \qquad a_1 = 1.04897\,a_3 \text{ and } y_2 - y_1 = -1.399857\,a_3 \tag{107}$$

For a positive deflection, a_1 must have a positive sign, so that for both cases $y_2 - y_1 < 0$. That means y_2 cannot be larger than y_1 and the solution is physically impossible.

5.2.2 Symmetric shape ($y_2 < y_1$)

In this case the following two sub-determinants must be zero

$$2\,C_1\,(1 + B) - 24\theta = 0 \tag{108}$$

$$\begin{vmatrix} \frac{15}{2}\theta - 4C_1\,B\left(1-\frac{\sqrt{2}}{2}\right) - C_2(2\sqrt{2}-3) & -C_2 \\ -C_2 & \frac{63}{2}\theta - 4C_1\,B\left(1+\frac{\sqrt{2}}{2}\right) + C_2(2\sqrt{2}+3) \end{vmatrix} = 0 \tag{109}$$

For three and four tension ties this results in

<u>m = 4</u> $C_1 = 6\,\theta$ $C_2 = 13.11419\,\theta$ (110)

<u>m = 3</u> $C_1 = 8\,\theta$ $C_2 = 2.805837\,\theta$ (111)

Introducing these results in the energy equations

<u>m = 4</u> $a_1 = 4.82033\,a_3$ and $y_2-y_1 = -0.29521\,a_3$ (112)

<u>m = 3</u> $a_1 = 0.851514\,a_3$ and $y_2-y_1 = -1.4577\,a_3$ (113)

Again, it is obtained that $y_2-y_1 < 0$, which agrees with the assumptions made. Equations (110) and (111) are therefore valid.

5.3 Reaction forces

To calculate the optimum weight, the maximum values of the reaction forces R_1, R_2, and R_3 are necessary and should be calculated as functions of the buckling load P.

5.3.1 Reaction forces for the symmetric mode

In this case $a_2 = 0$ and the equations for the reaction forces can be written down with the help of equation (94).

$$R_1 = C_1[a_1(\sqrt{2}-1)+a_3(\sqrt{2}+1)]B+C_2\left[a_1\left(1-\frac{\sqrt{2}}{2}\right)-a_3\left(\frac{\sqrt{2}}{2}+1\right)\right]$$

$$R_2 = C_1[a_1(\sqrt{2}-2)+a_3(\sqrt{2}+2)]B+C_2[a_1(\sqrt{2}-2)+a_3(\sqrt{2}+2)$$

$$R_3 = C_1[a_1(\sqrt{2}-1)+a_3(\sqrt{2}+1)]B+C_2\left[a_1\left(1-\frac{\sqrt{2}}{2}\right)-a_3\left(\frac{\sqrt{2}}{2}+1\right)\right]$$

For the two different cases of four or three tension ties the values C_1, C_2, B_1, and a_3/a_1 have been calculated previously and therefore the reaction forces can be found as linear functions of θ and a_1.

<u>m = 4</u> $\quad B = 1 \quad C_1 = 6\,\theta; C_2 = 13.11419\,\theta; a_3 = \frac{1}{4.82033}\,a_1$

$$R_1 = R_3 = 4.687027\,\theta\,a_1$$

$$R_2 = 0.8715523\,\theta\,a_1 \qquad (114)$$

$\underline{m = 3}$ $\quad B = 1/2 \quad C_1 = 8\theta;\ C_2 = 2.805837\ \theta;\ a_3 = 1.174379\ a_1$

$$\left.\begin{aligned} R_1 = R_3 &= 8.194398\ \theta\ a_1 \\ R_2 &= -4.088532\ \theta\ a_1 \end{aligned}\right. \qquad (115)$$

5.3.2 Reaction forces for the antisymmetric mode

From equation (97) the reaction forces are obtainable as

$$R_1 = C_1\ [a_1(\sqrt{2}-1) + 2a_2 + a_3(\sqrt{2}+1)]$$

$$R_2 = -C_1 B\left[a_1\left(\frac{\sqrt{2}}{2} -1\right) + a_2 + a_3\left(\frac{\sqrt{2}}{2} +1\right)\right] + \left[a_1\left(\frac{\sqrt{2}}{2} -1\right) - a_2 + a_3\left(\frac{\sqrt{2}}{2}+1\right)\right]$$

$$R_3 = C_1\ [a_1(\sqrt{2}-1) - 2a_2 + a_3(\sqrt{2}+1)]$$

Independent of the number of tension ties it is obtained that

$$a_3 = -0.171573\ a_1$$

Therefore

$\underline{m = 4}$ $$\left.\begin{aligned} R_1 = -R_3 &= 12\theta\ a_2 \\ R_2 &= 0 \end{aligned}\right. \qquad (116)$$

$\underline{m = 3}$ $$\left.\begin{aligned} R_1 = -R_3 &= 8\theta\ a_2 \\ R_2 &= 4\theta\ a_2 \end{aligned}\right. \qquad (117)$$

5.3.3 Reaction forces as functions of buckling load P

As in equation (72) an assumption for the maximum deflection must be made. Introducing the same value as for the case of a column with one strut results in stresses in the tension ties of nearly three times the yield stress of the chosen material. After some trials it is found that the tension stresses reach the yield stresses of the tie material for values of P/L^2, which are calculated for the highest stresses, if $a_1 = L/400$ for $m = 4$, and $a_1 = L/800$ for $m = 3$.

Assuming, furthermore, that the reaction force R_1 is the same for the symmetric and antisymmetric mode gives a constraint for the magnitude of a_2.

From equation (99)

$$\theta = \frac{P\pi^2}{16\, L}$$

With these assumptions

$$\underline{m = 4} \qquad a_1 = 2.5\ 10^{-3}\ L \qquad a_2 = 0.976463\ 10^{-3}\ L$$

$$R_1 = 7.2279847\ 10^{-3} P \qquad R_2 = 1.3440432\ 10^{-3}\ P \tag{118}$$

$$\underline{m = 3} \qquad a_1 = 1.25\ 10^{-3}\ L \qquad a_2 = 1.279749\ 10^{-3}\ L$$

$$R_1 = 6.318396\ 10^{-3}\ P \qquad R_2 = 3.1525151\ 10^{-3}\ P \tag{119}$$

5.4 Weight assumptions

The weight of the assembled column consists of five parts. A sketch of the parts is shown in Fig. 16.

5.4.1 Weight of the column

$$\frac{W_1}{L^3} = \frac{w \frac{P}{L^2}}{\sigma} \tag{120}$$

5.4.2 Weight of the long wires

$$W_2 = \frac{m\ w_1\ A_1\ L}{\cos\alpha_1}$$

and with equation (88)

$$E_1 A_1 = \frac{C_1\ L}{4\ \sin^2\alpha_1\ \cos\alpha_1} \tag{121}$$

$$\frac{W_2}{L^3} = \frac{m\ w_1\ C_1}{4\ E_1\ L\ \sin^2\alpha_1\ \cos^2\alpha_1} \tag{122}$$

Again this results in a constraint on σ in the tension ties.

$$\sigma_1 = \frac{4E_1}{L}\left[\frac{a_1}{2}\sqrt{2} + a_2 + \frac{a_3}{2}\sqrt{2}\right]\cos\alpha_1\ \sin\alpha_1 \tag{123}$$

$$\sigma_2 = \frac{4E_1}{L}\left[\frac{a_1}{2}\sqrt{2} + a_2 + \frac{a_3}{2}\sqrt{2} - a_1 + a_3\right]\cos\alpha_1\ \sin\alpha_1 \tag{124}$$

$$\sigma_3 = \frac{4E_1}{L}\left[\frac{a_1}{2}\sqrt{2} - a_2 + \frac{a_3}{2}\sqrt{2} - a_1 + a_3\right]\cos\alpha_1\ \sin\alpha_1 \quad (125)$$

$$\sigma_4 = \frac{4E_1}{L}\left[\frac{a_1}{2}\sqrt{2} - a_2 + \frac{a_3}{2}\sqrt{2}\right]\cos\alpha_1\ \sin\alpha_1 \quad (126)$$

5.4.3 Weight of the short wires

$$W_3 = \frac{m\ w_2\ A_2\ L}{2\ \cos\alpha_2}$$

from equation (89)

$$E_2 A_2 = \frac{C_2\ L}{4\ \sin^2\alpha_2\ \cos\alpha_2}$$

$$\frac{W_3}{L^3} = \frac{m\ w_2\ C_2}{8\ E_2\ L\ \sin^2\alpha_2\ \cos^2\alpha_2} \quad (127)$$

The constraints for σ are, in this case

$$\sigma_2 = \frac{4E_2}{L}\left[a_1 - a_3 - \frac{a_1}{2}\sqrt{2} - a_2 - \frac{a_3}{2}\sqrt{2}\right]\sin\alpha_2\ \cos\alpha_2 \quad (128)$$

$$\sigma_3 = \frac{4E_2}{L}\left[a_1 - a_3 - \frac{a_1}{2}\sqrt{2} + a_2 - \frac{a_3}{2}\sqrt{2}\right]\sin\alpha_2\ \cos\alpha_2 \quad (129)$$

5.4.4 Weight of the strut assembly at $x = L/4$ and $x = 3L/4$

As in the case of the column with one strut assembly, the struts are assumed to be simple columns with circular cross section and shall buckle in the elastic region of the material used.

The optimum buckling load for these simple columns is given by

$$\frac{R_1}{h_1^2} = \frac{8\sigma^3}{\pi k_2 E^2} \quad \text{and} \quad \sigma = \left[\frac{\pi k_2 E^2}{8} \quad \frac{R_1}{h_1^2}\right]^{1/3} \tag{130}$$

Introducing

$$h_1 = \frac{L}{4} \tan\alpha_1$$

yields

$$\sigma = \left[\frac{2\pi k_2 E^2}{\tan^2\alpha_1}\left(\frac{R_1}{L^2}\right)\right]^{1/3} \tag{131}$$

The weight of the strut assemblies is now

at $x = L/4$ $$W_4^{I} = (m+1)\, w \,\frac{R_1}{\sigma}\, \frac{L \tan\alpha_1}{4}$$

at $x = 3L/4$ $$W_4^{II} = (m+1)\, w \,\frac{R_3}{\sigma}\, \frac{L \tan\alpha_1}{4}$$

With equation (131) and the consideration that $R_1 = R_3$

$$\frac{W_4}{L^3} = \frac{W_4^{\,I}}{L^3} + \frac{W_4^{\,II}}{L^3} = \frac{m+1}{2}\, w\left(\frac{R_1}{L^2}\right)^{2/3} \left[\frac{1}{2\pi k_2 E^2}\right]^{1/3} \tan\alpha_1^{\,5/3} \tag{132}$$

5.4.5 <u>Weight of the strut assembly at x = L/2</u>

Here

$$\frac{R_2}{h^2} = \frac{8\sigma^3}{\pi k_2 E^2} \quad \text{and} \quad \sigma = \left[\frac{\pi k_2 E^2}{8}\ \frac{R_2}{h^2}\right]^{1/3} \tag{133}$$

Introducing

$$h = \frac{L}{4}\,(\tan\alpha_1 + \tan\alpha_2)$$

yields

$$\sigma = \left[\frac{2\pi k_2 E^2}{(\tan\alpha_1 + \tan\alpha_2)^2}\ \frac{R_2}{L^2}\right]^{1/3} \tag{134}$$

The weight of the strut assembly is therefore

$$\frac{W_5}{L^3} = \frac{(m+1)}{4}\, w\left(\frac{R_2}{L^2}\right)^{2/3} (\tan\alpha_1 + \tan\alpha_2)^{5/3} \left[\frac{1}{2\pi k_2 E^2}\right]^{1/3} \tag{135}$$

5.4.6 Weight of the whole column with three struts

As can be seen from Fig. 16

$$\frac{W}{L^3} = \frac{W_1}{L^3} + \frac{W_2}{L^3} + \frac{W_3}{L^3} + \frac{W_4}{L^3} + \frac{W_5}{L^3}$$

$$= \frac{w\frac{P}{L^2}}{\sigma} + \frac{m\ w_1}{E_1}\ \frac{C_1}{4L\ \sin^2\alpha_1\ \cos^2\alpha_1} + \frac{m\ w_2}{E_2}\ \frac{C_2}{8L\ \sin^2\alpha_2\ \cos^2\alpha_2}$$

$$+ \frac{(m+1)}{2}\ w\left(\frac{R_1}{L^2}\right)^{2/3}\left[\frac{1}{2\pi k_2 E^2}\right]^{1/3}(\tan\alpha_1)^{5/3}$$

$$+ \frac{(m+1)}{4}\ w\left(\frac{R_2}{L^2}\right)^{2/3}\left[\frac{1}{2\pi k_2 E^2}\right]^{1/3}(\tan\alpha_1 + \tan\alpha_2)^{5/3} \quad (136)$$

Assuming again that the tension ties, the column and the struts, are all made from the same material, then the weight equation can be simplified to

$$\frac{W}{L^3} = w\left[\frac{\frac{P}{L^2}}{\sigma} + \frac{m}{8E_1 L}\left\{\frac{2\ C_1}{\sin^2\alpha_1\ \cos^2\alpha_1} + \frac{C_2}{\sin^2\alpha_2\ \cos^2\alpha_2}\right\}\right.$$

$$\left. + \frac{m+1}{4}\left\{\frac{1}{2k_2E^2}\right\}^{1/3}\left[\left(\frac{R_1}{L^2}\right)^{2/3} 2(\tan\alpha_1)^{5/3} + \left(\frac{R_2}{L^2}\right)^{2/3}(\tan\alpha_1 + \tan\alpha_2)^{5/3}\right]\right]$$

(137)

5.5 Optimization

Since C_1, C_2, R_1, and R_2 are functions of P only, the weight equation is again a function of P/L^2 with the angles α_1 and α_2 as parameters. For the optimum design the partial derivatives of equation (137) with respect to α_1 and α_2 must be zero.

$$\frac{\partial}{\partial \alpha 1}\left(\frac{W}{L^3}\right) = \frac{\partial}{\partial \alpha 2}\left(\frac{W}{L^3}\right) = 0 \qquad (138)$$

Differentiating equation (137) the following two equations are obtained.

$$\frac{m\,C_1}{E_1\,L}\,\frac{(2\sin^2\alpha_1 - 1)}{\cos\alpha_1\,\sin^3\alpha_1} + \frac{5(m+1)}{6}\left[\frac{1}{2\pi k_2 E^2}\right]^{1/3}\left\{2\left(\frac{R_1}{L^2}\right)^{2/3}\tan\alpha_1^{\,2/3} + \left(\frac{R_2}{L^3}\right)^{2/3}(\tan\alpha_1 + \tan\alpha_2)^{2/3}\right\} = 0 \qquad (139)$$

$$\frac{m\,C_2}{E_1\,L}\,\frac{(2\sin^2\alpha_2 - 1)}{\cos\alpha_2\sin^3\alpha_2} + \frac{5(m+1)}{3}\left[\frac{1}{2\pi k_2 E^2}\right]^{1/3}\left(\frac{R_2}{L^3}\right)^{2/3}(\tan\alpha_1 + \tan\alpha_2)^{2/3} = 0 \qquad (140)$$

For further calculations all trigonometric functions are expressed as tangent functions. It can be verified that

$$\frac{2\ \sin^2\alpha\ -1}{\sin^3\alpha\ \cos\alpha} = \frac{\tan^4\alpha\ -1}{\tan^3\alpha} \tag{141}$$

Equations (139) and (140) are now

$$C_1 = Z\ \frac{\tan^3\alpha_1}{1-\tan^4\alpha_1}\left[2\left(\frac{R_1}{L^2}\right)^{2/3}\tan\alpha_1^{\ 2/3}+\left(\frac{R_2}{L^2}\right)^{2/3}(\tan\alpha_1+\tan\alpha_2)^{2/3}\right] \tag{142}$$

$$C_2 = 2\ Z\ \frac{\tan^3\alpha_2}{1-\tan^4\alpha_2}\left[\left(\frac{R_2}{L^2}\right)^{2/3}(\tan\alpha_1 + \tan\alpha_2)^{2/3}\right] \tag{143}$$

where Z is defined by

$$Z = \frac{5\,(m+1)}{6}\left[\frac{1}{2\pi k_2 E^2}\right]^{1/3}\frac{E_1\ L}{m} \tag{144}$$

For further calculations the cases m =4 and m = 3 must be considered separately.

<u>m = 4</u>

With m = 4 and k_2 again chosen as 0.4, equation (144) yields

$$Z = 0.7661367\ E^{1/3}\ L$$

From equation (118)

$$R_1 = 7.2279847\ 10^{-3}\ P$$

$$R_2 = 1.3440432\ 10^{-3}\ P$$

and from equation (110)

$$C_1 = 6\ \theta = 3.7011018\ \frac{P}{L}$$

$$C_2 = 13.11419\ \theta = 8.0894920\ \frac{P}{L}$$

With these values equations (142) and (143) can be simplified, and after some algebraic manipulations the following two equations are obtained.

$$F\ \frac{1-\tan^4\alpha_1}{\tan^3\alpha_1} = 1.547667\ \tan\alpha_1^{2/3} + 0.2520965(\tan\alpha_1+\tan\alpha_2)^{2/3} \tag{145}$$

$$\tan\alpha_1 = 9.02438888\ F^{3/2}\left[\frac{1-\tan^4\alpha_2}{\tan^3\alpha_2}\right]^{3/2} - \tan\alpha_2 \tag{146}$$

where

$$F = 10^2\left[\frac{\frac{P}{L^2}}{E}\right]^{1/3}$$

<u>m = 3</u>

For m = 3 equation (144) yields

$$Z = 0.8172125\ \ E^{1/3}\ L$$

From equation (119)

$$R_1 = 6.318396\ 10^{-3}\ \ P$$

$$R_2 = 3.1525151\ 10^{-3}\ \ P$$

and from equation (111)

$$C_1 = 8\,\theta = 4.9351108\ \frac{P}{L}$$

$$C_2 = 2.805837\,\theta = 1.7307814\ \frac{P}{L}$$

With these values, two equations are obtained which must be solved for α_1 and α_2.

$$F\,\frac{1-\tan^4\alpha_1}{\tan^3\alpha_1} = 1.1318870\ \tan\alpha_1^{2/3} + 0.3560137(\tan\alpha_1+\tan\alpha_2)^{2/3} \tag{147}$$

$$\tan\alpha_1 = 0.34563669\ F^{3/2}\left[\frac{1-\tan^4\alpha_2}{\tan^3\alpha_2}\right]^{3/2} - \tan\alpha_2 \tag{148}$$

Equations (145), (146), (147), and (148) must be solved for a relatively large number of values of F, and because equations are not independent the actual calculations would be very involved. It is therefore advisable that a computer be used to solve the problem. The solution can be obtained by using a "Newton Iteration Process."

$$x^{(i+1)} = x^{(i)} - \frac{y^{(i)}}{y'^{(i)}} \tag{149}$$

Introducing now the notation

$$x = \tan\alpha_2$$

$$x_1 = \tan\alpha_1$$

m = 4

(Equation (145) can be written as

$$y = -x_1 + \frac{1}{x_1^{\,3}} - \frac{1.547667}{F} x_1^{\,2/3} - 1.0928455\left(\frac{1}{x^3} - x\right) \quad (150)$$

$$y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial x_1} \; \frac{\partial x_1}{\partial x}$$

$$= -x_1' - \frac{3}{x_1^{\,4}} - \frac{2/3\ 1.547667}{F} x_1^{\,-1/3} x_1' + 1.0928455\left(1 - \frac{3}{x^4}\right)$$

(151)

From equation (146), x_1 and x_1' can be expressed in x.

$$x_1 = 9.025389\ F^{3/2}\left[\frac{1 - x^4}{x^3}\right]^{3/2} - x \quad (152)$$

$$x_1' = 3/2\ 9.024389\ F^{3/2}\left[\frac{1 - x^4}{x^3}\right]^{1/2}\left(1 + \frac{3}{x^4}\right) - 1 \quad (153)$$

m = 3

In the same way equation (147) becomes

$$y = -x_1 + \frac{1}{x_1^{\,3}} - \frac{1.131887}{F} x_1^{\,2/3} - 0.17535375\left(\frac{1}{x^3} - x\right) \quad (154)$$

$$y' = -x_1' - \frac{3}{x_1^{\,4}} - \frac{2/3\ 1.131887}{F} x_1^{\,-1/3} x_1' + 0.17535375\left(1 - \frac{3}{x^4}\right)$$

(155)

And with equation (148)

$$x_1 = 0.34563669\ F^{3/2} \left[\frac{1-x^4}{x^3}\right]^{3/2} - x \tag{156}$$

$$x_1' = 3/2\ \ 0.34563669\ F^{3/2} \left[\frac{1-x^4}{x^3}\right]^{1/2} \left(1 + \frac{3}{x^4}\right) - 1 \tag{157}$$

For an initial assumed value of x the values of x_1 and x_1' are obtainable. The three values x, x_1, and x_1' allow the calculation of y and y'.

With equation (149) a new value of x is obtained. This procedures has to be repeated until a specified accuracy of x is reached.

For the calculation of the cross sectional area, the expressions

$$\frac{1}{\sin^2\alpha_1\ \cos\alpha_1} \quad \text{and} \quad \frac{1}{\sin^2\alpha_2\ \cos\alpha_2}$$

are necessary. Therefore they have also been calculated by the computer. The computer program listed is in Appendix II of this paper.

With the optimum angles α_1 and α_2 calculated, the weight of the column with three struts can be obtained and plotted against the structural index to give the desired results.

SECTION 6

CALCULATIONS AND RESULTS FOR THE COLUMN WITH ONE STRUT

6.1 Material properties

For the application of the theory developed on the previous pages a typical aluminum-alloy and steel are chosen.

Al-alloy	2024-T4-Al-Alloy (material properties, see Fig. 10)
Steel	Stainless steel 3/4 hard (material properties, see Fig. 17)

From Ref. 6 the following material data for the tension ties are found.

Al-alloy	$\sigma_{yield} = 60$ (ksi)	$w = 0.10$ (lb/in.^{3})
	$E = 10.5\times10^6$ (psi)	
Carbon steel	$\sigma_{yield} = 130$ (ksi)	$w = 0.283$ (lb/in.^{3})
	$E = 28.3\times10^6$ (psi)	

6.2 Application for aluminum-alloy

6.2.1 Four tension ties.

With $m = 4$ and $B = 1$ equation (79) takes the form

$$f(\alpha) = \frac{11.9197}{(P/L^2)^{1/3}} \qquad (158)$$

Considering the fact that at the instant of buckling the tension ties are stress free as long as there is no lateral deflection, this allows the definition

$$P = \sigma A \tag{159}$$

where A is the cross section of the column.

With this relation equation (75) can be modified and gives a relation between wire and column cross sections.

$$\frac{A_1}{A} = \frac{8.4554\ 10^{-5}}{\sin^2\alpha\ \cos\alpha}\ \sigma \quad \text{(ksi)} \tag{160}$$

The weight equation for this particular case is obtained from equation (77)

$$\frac{W}{L^3} = \left\{ \frac{P}{\sigma L^2} + \frac{0.37766}{10^6}\ \frac{P}{L^2}\ \frac{1}{\sin^2\alpha\ \cos^2\alpha} + \frac{2.51037}{10^6}\ \frac{P}{L^2}^{2/3} (\tan\alpha)^{5/3} \right\} w \tag{161}$$

Now all equations necessary for the solution are obtained and the steps which are followed in solving the problem are shown by solving for one value of σ.

For a stress of $\sigma = 20$ (ksi) Fig. 10 gives

$$\tau = 1.0 \tag{162}$$

Equation (40) gives now the structural index for the simple column

$$\frac{P}{L^2}^{(I)} = 0.45070 \text{ (psi)} \tag{163}$$

In the same way equation (61) gives the structural index for the optimum column with one strut.

$$\frac{P}{L^2}^{(II)} = 0.11267 \text{ (psi)} \tag{164}$$

Now $f(\alpha)$ can be calculated and Fig. 13 gives an optimum value of

$$\alpha = 22.65° \tag{165}$$

The weight of the simple column can be calculated by introducing equation (163) in (41)

$$\frac{W}{L^3} = 2.2535 \text{ (lb/in.}^3\text{)} \tag{166}$$

With equations (164) and (165) all information is obtained to calculate the weight of the column with one strut. Equation (161) yields

$$\frac{W}{L^3} = 0.6107 \text{ (lb/in.}^3\text{)} \tag{167}$$

Equation (160) gives

$$\frac{A_1}{A} = 1.2355\ 10^{-2} \tag{168}$$

The results for different values of σ are noted in Table 1 in Appendix I.

The constraint for the tension ties is

$$\sigma_1 = \frac{2\ E_1 \cos\alpha \sin\alpha}{150} \tag{169}$$

For the example

$$0.2605 < \cos\alpha \sin\alpha < 0.4655$$

or

$$\sigma_{min} = 36.47 \quad (ksi) \qquad \sigma_{max} = 65.17 \quad (ksi) \qquad (170)$$

σ_{yield} for aluminum wires goes up to 70 (ksi), so that the wires are safe.

6.2.2 Three tension ties

With m = 3 and B = 1/2 equation (79) can be written as

$$f(\alpha) = \frac{6.3572}{P/L^{2\ 1/3}} \qquad (171)$$

Equation (75) in modified form writes

$$\frac{A_1}{A} = \frac{16.9108\ 10^{-5}}{\sin^2\alpha\ \cos\alpha}\ \sigma \quad (ksi) \qquad (172)$$

The weight equation is

$$\frac{W}{L^3} = w\left\{\frac{P}{\sigma L^2} + \frac{0.56649}{10^6}\ \frac{P}{L^2}\ \frac{1}{\sin^2\alpha\cos^2\alpha} + \frac{2.0083}{10^6}\ (\tan\alpha)^{5/3}\ \frac{P^{2/3}}{L^2}\right\} \qquad (173)$$

The calculations are similar to the calculation in Section 6.2.1. The results are noted in Table 2 in the Appendix.

The stresses in the tension ties are

$$\sigma_{min} = 42.77 \ (ksi) \qquad \sigma_{max} = 67.55 \ (ksi)$$

The tension ties are therefore safe.

The various results for a column with one strut, made from aluminum are shown in Fig. (18), (19), and (26).

6.3 Application for steel

6.3.1 Four tension ties

With equation (79)

$$f(\alpha) = \frac{16.588}{(P/L^2)^{1/3}} \tag{174}$$

From equation (75)

$$\frac{A_1}{A} = \frac{3.1018\ 10^{-5}}{\sin^2\alpha\cos\alpha} \quad \text{(ksi)} \tag{175}$$

The weight equation for this case is

$$\frac{W}{L^3} = w\left\{\frac{P}{\sigma L^2} + \frac{0.14012}{10^6}\left(\frac{P}{L^2}\right)\frac{1}{\sin^2\alpha\cos^2\alpha} + \frac{1.2958}{10^6}\left(\frac{P}{L^2}\right)^{2/3}(\tan\alpha)^{5/3}\right\} \tag{176}$$

The results are noted in Table 3 in Appendix I.

The stresses in the tension ties are calculated as

$$\sigma_{min} = 78.635\ 10^3\ \text{(psi)} \qquad \sigma_{max} = 176.485\ 10^3\ \text{(psi)}$$

σ_{yield} for steel goes up to 158 10^3 (psi), so that for higher values of P/L^2 (over 2.0) the deflection should be restricted to a value of L/200.

6.3.2 Three tension ties

The angle α, the cross section relation A_1/A, and the weight W/L^3 can be calculated using the following equations:

$$f(\alpha) = \frac{8.84693}{(P/L^2)^{1/3}} \tag{177}$$

$$\frac{A_1}{A} = \frac{7.0061\ 10^{-5}}{\sin^2\alpha\ \cos\alpha} \quad (\text{ksi}) \tag{178}$$

$$\frac{W}{L^3} = w\left\{\frac{P}{\sigma L^2} + \frac{0.21018}{10^6}\left(\frac{P}{L^2}\right)\frac{1}{\sin^2\alpha\cos^2\alpha} + \frac{1.03695}{10^6}\ (\tan\alpha)^{5/3}\left(\frac{P}{L^2}\right)^{2/3}\right\}$$

The results are noted in Table 4 in Appendix I. (179)

The tension stresses are calculated as

$$\sigma_{min} = 91.178\ 10^3\ (\text{psi}) \qquad \sigma_{max} = 182.737\ 10^3\ (\text{psi})$$

Therefore again the deflections of $P/L^2 > 2.0$ should be restricted to values of L/200.

The various results for a column with one strut, made from steel, are shown in Fig. (20, (21), and (31).

SECTION 7

CALCULATIONS AND RESULTS FOR THE COLUMN WITH THREE STRUTS

7.1 Optimum angles

With the computer program given on previous pages, the angles α_1 and α_2 are obtained for various values of F, which are functions of σ.

The computed results are printed in Appendix II.

7.2 Application for aluminum-alloy

7.2.1 Four tension ties

With the definitions of C_1 and C_2 [equations (88) and (89)] and the results for these variables for the various cases, the cross sections are obtainable. Introducing $E_1 = E_2 = 10.5 \quad 10^6 (\text{lb/in.}^2)$ yields

$$\frac{A_1}{A} = 0.0881215 \; 10^{-3} \; \frac{\sigma \; (\text{ksi})}{\sin^2\alpha_1 \cos\alpha_1} \tag{180}$$

$$\frac{A_2}{A} = 0.1926069 \; 10^{-3} \; \frac{\sigma \; (\text{ksi})}{\sin^2\alpha_2 \cos\alpha_2} \tag{181}$$

The weight can be obtained from equation (137) and is written here in such a form that a computer program can be set up easily.

$$\frac{W}{L^3}\,10^6 = 10^3 \quad 0.105\left\{\frac{P}{L^2}\left[\frac{1}{\sigma} + \frac{0.35248588}{F_1} + \frac{0.38483295}{F_2}\right] + \frac{0.19173222}{10}\,(F_3 + F_4)\right\} \qquad (182)$$

where

$$F_1 = 10^3 \sin^2\alpha_1 \cos^2\alpha_1 \qquad (183)$$

$$F_2 = 10^3 \sin^2\alpha_2 \cos^2\alpha_2 \qquad (184)$$

$$F_3 = 2 \tan\alpha_1^{5/3} \left(\frac{7.2279847}{10^3} \quad \frac{P}{L^2}\right)^{2/3} \qquad (185)$$

$$F_4 = (\tan\alpha_1 + \tan\alpha_2)^{5/3} \left(\frac{1.3440432}{10^3} \quad \frac{P}{L^2}\right)^{2/3} \qquad (186)$$

Again the computing of all necessary values for the solution of the problem are indicated for one example.

Choosing again a stress of $\sigma = 20$ (ksi) the structural index for the optimal column with three struts can be calculated from equation (98).

$$\frac{P}{L^2}^{(IV)} = 0.028168693 \qquad (187)$$

With the help of equations (150), (151), (152), and (153) the computer calculates the optimum angles α_1 and α_2 using the above value of the structural index and the modulus of elasticity for aluminum to calculate the value F. The following is obtained:

$$\alpha_1 = 25.420° \tag{188}$$

$$\alpha_2 = 36.012° \tag{189}$$

The calculation of A_1/A and A_2/A can be done now with equations (180) and (181).

$$A_1/A = 1.05902\ 10^{-2} \tag{190}$$

$$A_2/A = 1.37758\ 10^{-2} \tag{191}$$

With this information given, the computer solves equation (182)

$$\frac{W}{L^3}\ 10^6 = 0.16701 \tag{192}$$

The program for the calculation of the weight is printed in Appendix II for all different cases, discussed in the next three sections.

The results for various values of P/L^2 are noted in Table 5 in Appendix I and in the tables containing the computer results (Appendix II).

7.2.2 Three tension ties

For this case the following two cross section relations are obtained

$$A_1/A = 0.117495\ 10^{-3}\ \frac{\sigma\ (\text{ksi})}{\sin^2\alpha_1\ \cos\alpha_1} \tag{193}$$

$$A_2/A = 0.041209\ 10^{-3} \frac{\sigma\ (ksi)}{\sin^2\alpha_2\ \cos\alpha_2} \qquad (194)$$

The weight equation is written in a similar form as equation (182).

$$\frac{W}{L^3} 10^6 = 10^3\ \ 0.105 \left\{ \frac{P}{L^2}\left[\frac{1}{\sigma} + \frac{0.35250792}{F_1} + \frac{0.618136095}{F_2}\right] + \frac{0.15338577}{10^4}\ (F_3 + F_4) \right\} \qquad (195)$$

where

$$F_1 = 10^3 \sin^2\alpha_1 \cos^2\alpha_1 \qquad (196)$$

$$F_2 = 10^3 \sin^2\alpha_2 \cos^2\alpha_2 \qquad (197)$$

$$F_3 = 2 \tan\alpha_1{}^{5/3} \left(\frac{6.318396}{10^3}\ \frac{P}{L^2}\right)^{2/3} \qquad (198)$$

$$F_4 = (\tan\alpha_1 + \tan\alpha_2)^{5/3} \left(\frac{3.1525151}{10^3}\ \frac{P}{L^2}\right)^{2/3} \qquad (199)$$

The results are noted in Table 5 in Appendix I and in the results of the computer calculations (Appendix II).

The various results for a column with three struts made from aluminum-alloy are shown in Figs. (22), (23), (24), (25), and (26).

7.3 Application for steel

7.3.1 Four tension ties

Introducing $E_1 = E_2 = 28.3 \quad 10^6 (\text{lb/in.}^2)$ yields here for the cross section relations

$$\frac{A_1}{A} = 0.032695 \quad 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2\alpha_1 \cos\alpha_1} \tag{200}$$

$$\frac{A_2}{A} = 0.0714619 \quad 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2\alpha_2 \cos\alpha_2} \tag{201}$$

The weight equation reads

$$\frac{W}{L^3} 10^6 = 10^3 \quad 0.283 \left\{ \frac{P}{L^2}\left[\frac{1}{\sigma} + \frac{0.13078098}{F_1} + \frac{0.14278254}{F_2}\right] + \frac{0.098999204}{10} \quad (F_3 + F_4) \right\} \tag{202}$$

where F_1, F_2, F_3, and F_4 are defined as in equations (183), (184), (185), and (186).

The results for various values of P/L^2 are noted in Table 6 in Appendix I and in the tables containing the computer results (Appendix II).

7.3.2 Three tension ties

The cross section relations are

$$\frac{A_1}{A} = 0.043594 \quad 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2\alpha_1 \cos\alpha_1} \tag{203}$$

$$\frac{A_2}{A} = 0.015289 \quad 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2\alpha_2 \cos\alpha_2} \tag{204}$$

The weight equation is

$$\frac{W}{L^3} 10^6 = 10^3 \quad 0.283 \left\{ \frac{P}{L^2} \left[\frac{1}{\sigma} + \frac{0.13078916}{F_1} + \frac{0.02293438}{F_2} \right] + \frac{0.079197485}{10} \quad (F_3 + F_4) \right\} \tag{205}$$

where F_1, F_2, F_3, and F_4 are defined as in equations (196), (197), (198), and (199).

The results for various values of P/L^2 are noted in Table 7 in Appendix I and in the tables containing the computer results (Appendix II).

The results for a column with three struts are plotted in Figs. (27), (28), (29), (30), and (31).

7.4 Check of the tension stresses in the ties

7.4.1 Tension stresses for a construction with four tension ties

From equation (118) it can be seen that $a_1 = 2.5 \quad 10^{-3}$ L, $a_2 = 0.976463 \quad 10^{-3}$ L, $a_3 = 0.207455\ a_1$.

In looking at equations (123), (124), (125), and (126) it can be seen that only σ_1 is critical. It is obtained

$$\sigma_1 = 12.44385 \quad 10^{-3}\ E_1 \cos\alpha_1 \sin\alpha_1 \tag{206}$$

Equations (128) and (129) again yield only one critical stress σ_2.

$$\sigma_2 = 10.462488 \ 10^{-3} E_2 \cos\alpha_2 \sin\alpha_2 \quad (207)$$

The maximum values of the product $\cos\alpha \ \sin\alpha$ can be calculated from the computer results

Aluminum

$$(\cos\alpha_1 \ \sin\alpha_1)_{max} = 0.48059 \quad \text{and} \quad \sigma_1 = 62.794 \ (\text{ksi}) \quad (208)$$

$$(\cos\alpha_2 \ \sin\alpha_2)_{max} = 0.49867 \quad \text{and} \quad \sigma_2 = 54.782 \ (\text{ksi}) \quad (209)$$

Steel

$$(\cos\alpha_1 \ \sin\alpha_1)_{max} = 0.48223 \quad \text{and} \quad \sigma_1 = 169.822 \ (\text{ksi}) \quad (210)$$

$$(\cos\alpha_2 \ \sin\alpha_2)_{max} = 0.49882 \quad \text{and} \quad \sigma_2 = 147.650 \ (\text{ksi}) \quad (211)$$

7.4.2 Tension stresses for a construction with three tension ties

From equation (119)

$$a_1 = 1.25 \ 10^{-3} L, \quad a_2 = 1.27975 \ 10^{-3} L, \quad a_3 = 0.58789 \ a_1$$

Again there are two critical stresses σ_1 and σ_2

$$\sigma_1 = 10.73055 \ 10^{-3} E_1 \cos\alpha_1 \sin\alpha_1 \quad (212)$$

$$\sigma_2 = 0.41028 \ 10^{-3} E_2 \cos\alpha_2 \sin\alpha_2 \ (\text{not critical}) \quad (213)$$

With the maximum values of $\cos\alpha_1 \sin\alpha_1$ the maximum tension stresses are obtained as

Aluminum

$$(\cos\alpha_1 \sin\alpha_1)_{max} = 0.48566 \text{ and } \sigma_1 = 54.720 \text{ (ksi)} \tag{214}$$

Steel

$$(\cos\alpha_1 \sin\alpha_1)_{max} = 0.48428 \text{ and } \sigma_1 = 147.064 \text{ (ksi)} \tag{215}$$

The results indicate that the maximum deflections are chosen in such a way that the tension ties are never loaded beyond their yield strength.

SECTION 8

DISCUSSION AND CONCLUSIONS

The weight curves Figs. (29) and (31) indicate that for small values of the structural index the column supported by tension ties is up to 50% lighter than the simple column. To demonstrate for which values of the structural index a wire supported column configuration is advisable, a weight savings factor is defined as

$$\frac{\text{weight of the simple column} - \text{weight of the wire supported column}}{\text{weight of the simple column}}$$

Using the weight curves in Figs. (29) and (31), respectively, this weight saving function can be found graphically and is plotted in Figs. (32) and (33) for the two used materials. There is hardly any difference between the weight of the column with three and four ties, so that the weight figures and the weight saving figures are drawn without the distinction between a column with three and four ties.

During the calculations it was found that the angles α are always smaller than 45° and in a very reasonable region. The cross sectional area of a tension tie is usually from one to five percent that of the central column.

Both values (the tie angles and tie cross sections) increase with increasing values of the structural index, where the greatest rate of change occurs for very small values of the structural index. This indicates that the additional weight from the tension ties also increases rapidly for these values of the structural index [Figs. (18),(20),(22),(23),(27),(28)].

If typical design parameters (P and L) are given, a decision must be made whether a column with three or one strut is chosen. When the weight curves indicate significant weight savings and weight savings are important for a particular problem, it seems reasonable to calculate the column with three struts first. Except for large values of P and very large values of L, the wall thickness of the central column usually is too small to be manufactured. In this case the column with one strut must be used.

To indicate how the calculations in this paper can be used, a typical practical example is calculated. Assuming that a load of 1000 pounds has to be carried over a distance of 10 feet and that an aluminum construction with one strut and three tension ties is chosen. This allows the calculation of the structural index:

$$P/L^2 = 0.0695 \qquad \text{and} \qquad \sqrt{P/L^2} = 0.2635$$

Fig. (18) gives immediately

$$\alpha = 25.15°$$

$$\frac{A_1}{A} = 1.73 \quad 10^{-2}$$

From the expression for the second buckling mode, equation (61), the following maximum stress is obtained

$$\sigma = \sqrt[3]{\frac{P}{L^2} \quad \frac{\pi \; k_2 \; E^2 \tau^{3/2}}{2}}$$

for the given values

$$\frac{\sigma}{\tau^{1/2}} = 16.34 \text{ (ksi)}$$

To evaluate σ an iteration is usually required. In this case however Fig. (10) yields $\tau = 1$, so that

$$\sigma = 16.34 \text{ (ksi)}$$

From equation (159) it is obtained

$$A = \frac{P}{\sigma} = \frac{1000}{1.634 \; 10^4} = 0.0612 \text{ (in.}^2\text{)}$$

Now the cross section of the tension ties is

$$A_1 = 1.73 \quad 10^{-2} \quad A = 0.1059 \quad 10^{-2} \text{ (in.}^2\text{)}$$

and for a circular cross section

$$D_1 = 0.0372 \text{ (in.)}$$

From equation (39) it is obtained

$$\frac{D}{t} = 2\left(\frac{k_2^{\,2}\ E}{\pi\ P/L^2}\right)^{1/3} = 394.95$$

With D/t and A given, the wall thickness and column diameter D can be calculated

$$t = 0.715 \quad 10^{-2} \text{ (in.)}$$

$$D = 2.825 \text{ (in.)}$$

The result for t indicates that a column with three struts would be impossible for this particular case.

The required pretension in the tension ties is obtained from equation (12). Introducing in this equation

$$\sin \theta_1 = \cos\alpha$$

$$c_1 = c_2 = \frac{L}{2 \cos\alpha}$$

$$m = 3$$

$$E_1 = E_c$$

the following simple equation is obtained

$$T_{01} = \frac{P}{3 \cos\alpha + \frac{A_c}{A_1}} = 16.53 \text{ (lbs)}$$

The fact that the weight of the optimal wire supported column is nearly independent of the number of tension ties leads to the conclusion that an arrangement of four ties does not bring any advantages, so that from the manufacturing point of view the column with three tension ties is preferred. Figures (29) and (31) show that for small values of the structural index P/L^2 the column supported by tension ties has a significantly smaller weight than the simple column. Nevertheless it should be mentioned that the calculated optimum weight for very small values of P/L^2 cannot be reached in practice because the optimum ratio D/t reaches values which give dimensions of a column which is nearly impossible to fabricate. As is often the case in calculations in optimum design, the results presented in this paper have to be considered as the theoretical limit.

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APPENDIX I

Tables and Figures

TABLE 1 (a)

Column with one strut, four tension ties, made from aluminum-alloy.

σ [ksi]	P/L^2 (I) [psi]	$\sqrt{P/L^2}$ (I)	P/L^2 (II) [psi]	$\sqrt{P/L^2}$ (II)	W/L^3 simple column [$lb/inch^3$]
5	0.00704	0.08391	0.00176	0.0419	0.1408
10	0.05634	0.2374	0.01408	0.1187	0.5634
15	0.19014	0.4365	0.04753	0.2182	1.2676
20	0.45070	0.6713	0.11267	0.3356	2.2535
25	0.88028	0.9382	0.22007	0.4691	3.5211
30	1.6527	1.2855	0.41317	0.6427	5.5090
35	3.5038	1.8718	0.87595	0.9359	10.0109
40	11.1626	3.3410	2.7906	1.6705	27.9065
45	53.2218	7.2953	13.3054	3.6476	118.2706
50	185.1575	13.606	46.2894	6.803	-

TABLE 1 (b)

Column with one strut, four tension ties, made from aluminum-alloy.

σ [ksi]	α°	W/L^3 [lb/inch^3]	$A_1/A \times 10^2$
5	15.70	0.0366	0.5997
10	19.20	0.1488	0.8278
15	21.20	0.3395	1.0403
20	22.65	0.6107	1.2355
25	23.85	0.9645	1.4136
30	24.95	1.5225	1.5726
35	26.38	2.8124	1.6729
40	28.72	7.7527	1.6705
45	31.83	32.709	1.6098
50	34.30	102.356	1.6115

TABLE 2

Column with one strut, three tension ties,

made from aluminum-alloy.

σ [ksi]	$\alpha°$	W/L^3 [lb/inch^3]	$A_1/A \times 10^2$
5	18.83	0.0367	0.8573
10	22.30	0.1499	1.2694
15	24.45	0.3429	1.6266
20	26.12	0.6185	1.9438
25	27.45	0.9794	2.2420
30	28.70	1.5500	2.5080
35	30.22	2.8392	2.6842
40	32.50	7.9345	2.7782
45	35.42	33.5885	2.7804
50	37.40	105.4972	2.8851

σ [ksi]	P/L^2 (I) [psi]	$\sqrt{P/L^2 (I)}$	P/L^2 (II) [psi]	$\sqrt{P/L^2 (II)}$	W/L^3 simple column [lb/inch^3]	α °	W/L^3 [lb/inch^3]	$A_1/A \times 10^2$
5	0.00099	0.00315	0.00025	0.00157	0.0559	12.3	0.01432	0.3481
10	0.00795	0.08915	0.00198	0.04457	0.2238	14.8	0.05787	0.4916
15	0.02683	0.16379	0.00671	0.08189	0.5061	16.25	0.13192	0.6189
20	0.06359	0.25218	0.01590	0.12609	0.8998	17.72	0.23598	0.7032
25	0.12420	0.35242	0.03105	0.17621	1.4059	18.8	0.37073	0.7887
30	0.21462	0.46323	0.05365	0.23161	2.0246	19.77	0.53662	0.8645
35	0.34081	0.58379	0.08520	0.29189	2.7557	20.5	0.73409	0.9450
40	0.50873	0.71325	0.12718	0.35667	3.5992	21.22	0.96326	1.0162
50	0.99361	0.99680	0.24840	0.49840	5.6102	22.35	1.51545	1.1596
60	1.9300	1.38927	0.4825	0.6946	9.1032	23.45	2.4754	1.2810
70	3.6336	1.9062	0.9084	0.9531	14.690	24.60	3.8721	1.3780
80	6.2661	2.5032	1.5665	1.2516	22.166	25.65	6.0951	1.4690
90	10.3344	3.2147	2.5836	1.6073	32.496	26.60	8.9823	1.5572
100	16.6857	4.0848	4.1714	2.0424	47.220	27.65	13.1934	1.6260
110	27.4216	5.2365	6.7554	2.6183	69.519	28.60	19.3767	1.7052
120	47.8794	6.9195	11.9698	3.4597	112.915	29.70	31.5442	1.7456
130	117.872	10.857	29.4542	5.4283	275.635	31.5	70.6791	1.7323
135	618.465	24.869	154.616	12.4344	1259.21	34.65	348.554	1.5746

Table 3. Column with one strut, four tension ties, made from steel.

TABLE 4

Column with one strut, three tension ties made from steel.

σ [ksi]	α°	W/L^3 [lb/inch^3]	$A_1/A \times 10^2$
5	14.45	0.01435	0.5809
10	17.35	0.05808	0.8254
15	19.40	0.13243	1.0099
20	20.80	0.23722	1.1886
25	22.0	0.37319	1.3461
30	22.85	0.54095	1.5125
35	23.70	0.74073	1.6575
40	24.45	0.97304	1.7970
50	25.72	1.53041	2.0650
60	27.0	2.50986	2.2890
70	28.27	4.0809	2.4828
80	29.35	6.2036	2.6766
90	30.35	9.1587	2.8620
100	31.35	13.3948	3.0294
110	32.30	19.8377	3.1932
120	33.43	32.3650	3.3186
130	35.15	72.7204	3.3607
135	37.80	359.551	3.1865

TABLE 5

Cross-sections for a column with three struts made from aluminum-alloy.

σ [ksi]	$A_1/A \times 10^2$	$A_2/A \times 10^2$
a. Aluminum, four tension ties		
5	0.47614	0.49210
10	0.69934	0.80191
15	0.89810	1.10036
20	1.05902	1.37758
25	1.21974	1.65809
30	1.36118	1.92900
35	1.46380	2.18063
40	1.49475	2.39816
45	1.48305	2.60242
50	1.52370	2.83670
b. Aluminum, three tension ties		
5	0.58023	0.28604
10	0.85935	0.42299
15	1.11027	0.52447
20	1.31643	0.61119
25	1.52328	0.69771
30	1.70715	0.77123
35	1.84640	0.81951
40	1.90227	0.81966
45	1.90997	0.78856
50	1.91193	0.78987

TABLE 6

Cross-sections for a column with three struts, four tension ties, made from steel.

σ [ksi]	$A_1/A \times 10^2$	$A_2/A \times 10^2$
5	0.28273	0.26323
10	0.40425	0.40324
15	0.50193	0.52712
20	0.58822	0.64402
25	0.66732	0.74832
30	0.74127	0.86311
35	0.81176	0.97527
40	0.87901	1.08232
50	1.00729	1.29371
60	1.11332	1.48636
70	1.20580	1.68601
80	1.29687	1.88009
90	1.38358	2.08258
100	1.46535	2.26512
110	1.54110	2.45446
120	1.59827	2.63642
130	1.61218	2.79976
135	1.50938	2.83423

TABLE 7

Cross-sections for a column with three struts, three tension ties, made from steel.

σ [ksi]	$A_1/A \times 10^2$	$A_2/A \times 10^2$
5	0.34190	0.17257
10	0.49116	0.24423
15	0.61226	0.30337
20	0.72005	0.35022
25	0.81920	0.39497
30	0.91945	0.43648
35	1.00146	0.47557
40	1.08718	0.51278
50	1.25084	0.58295
60	1.38845	0.63868
70	1.51025	0.68539
80	1.63065	0.73089
90	1.74650	0.77342
100	1.85632	0.81226
110	1.95833	0.84622
120	2.04084	0.86828
130	2.07327	0.86065
135	1.96292	0.77879

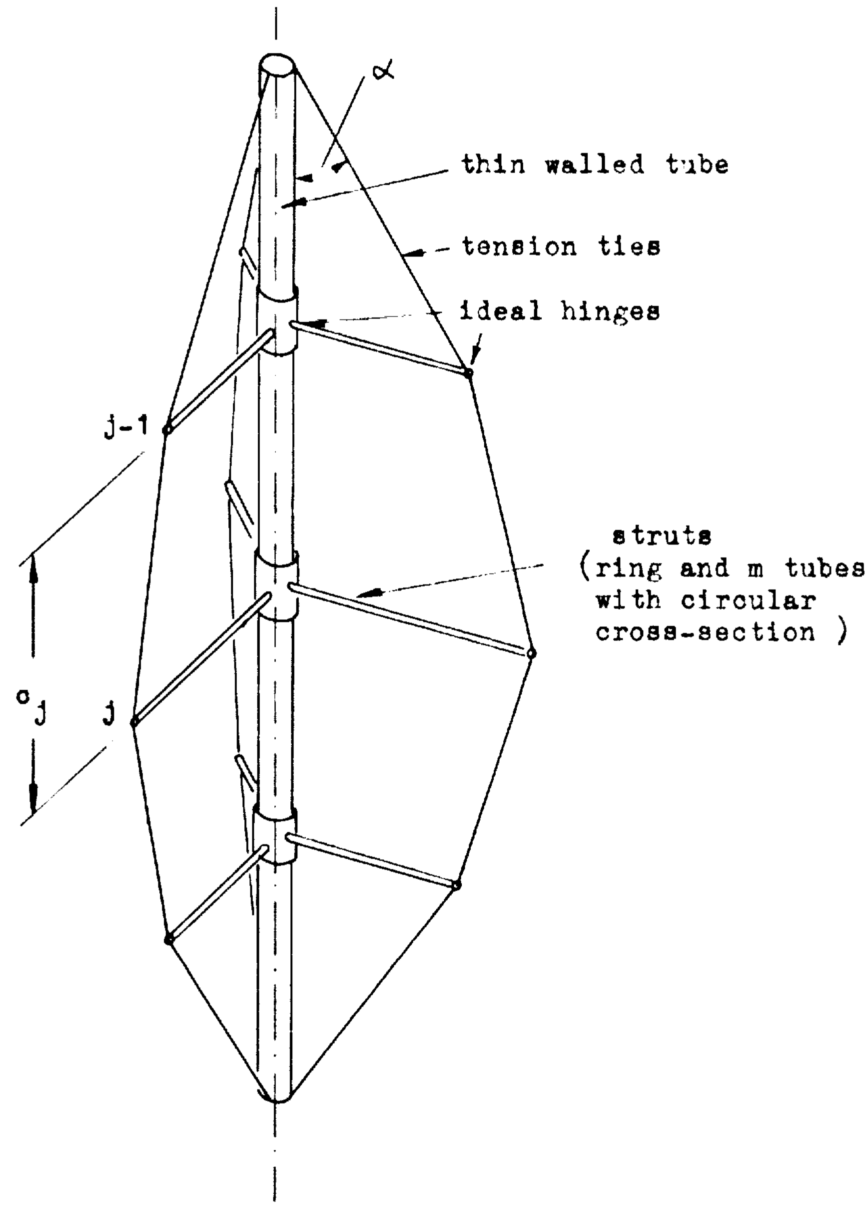


FIG. 1: Structural parts of the column supported by tension ties.

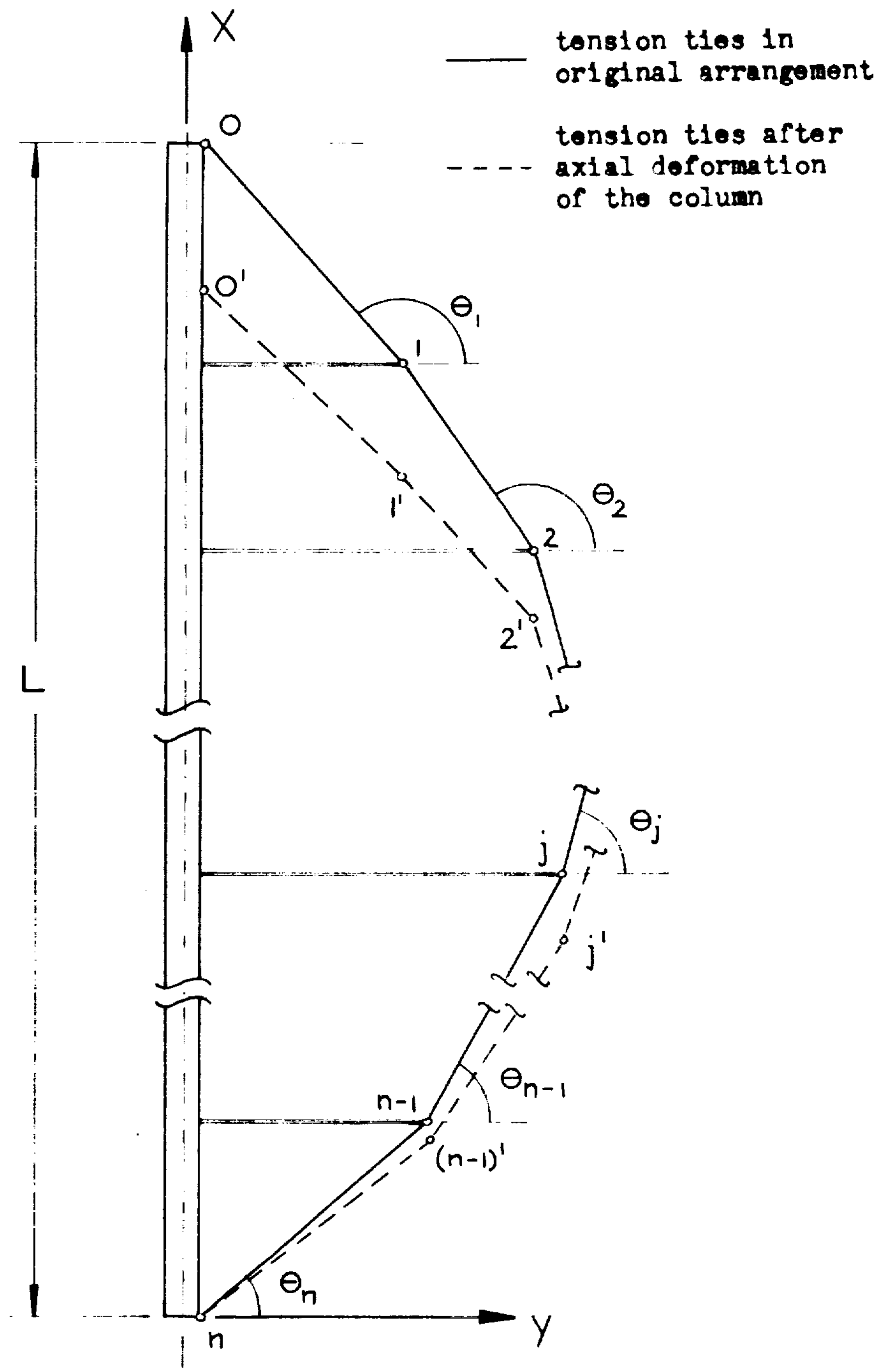


FIG. 2: Geometry of the wire supported column with n-1 struts.

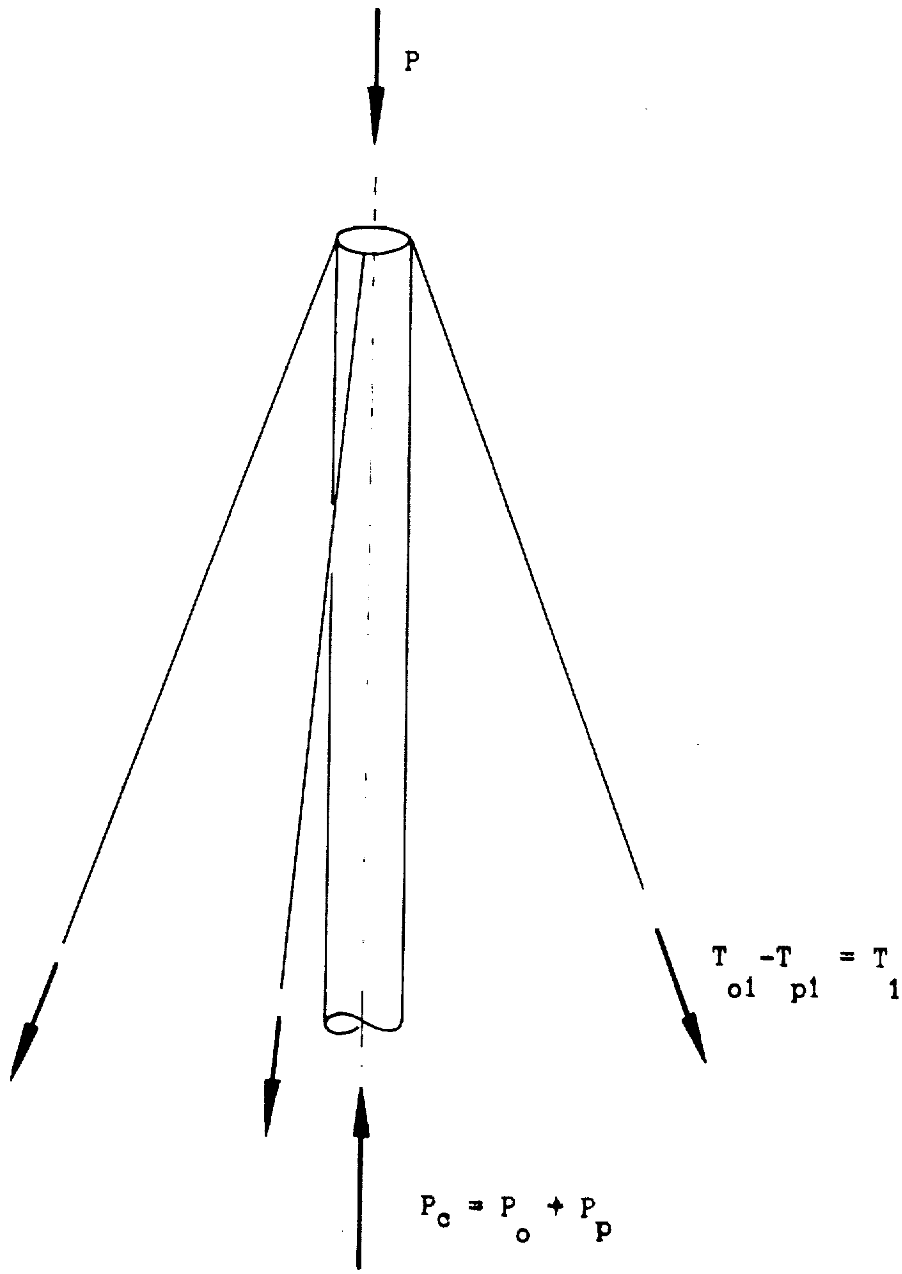


FIG.3: Equilibrium of forces at the end of the column (x=L).

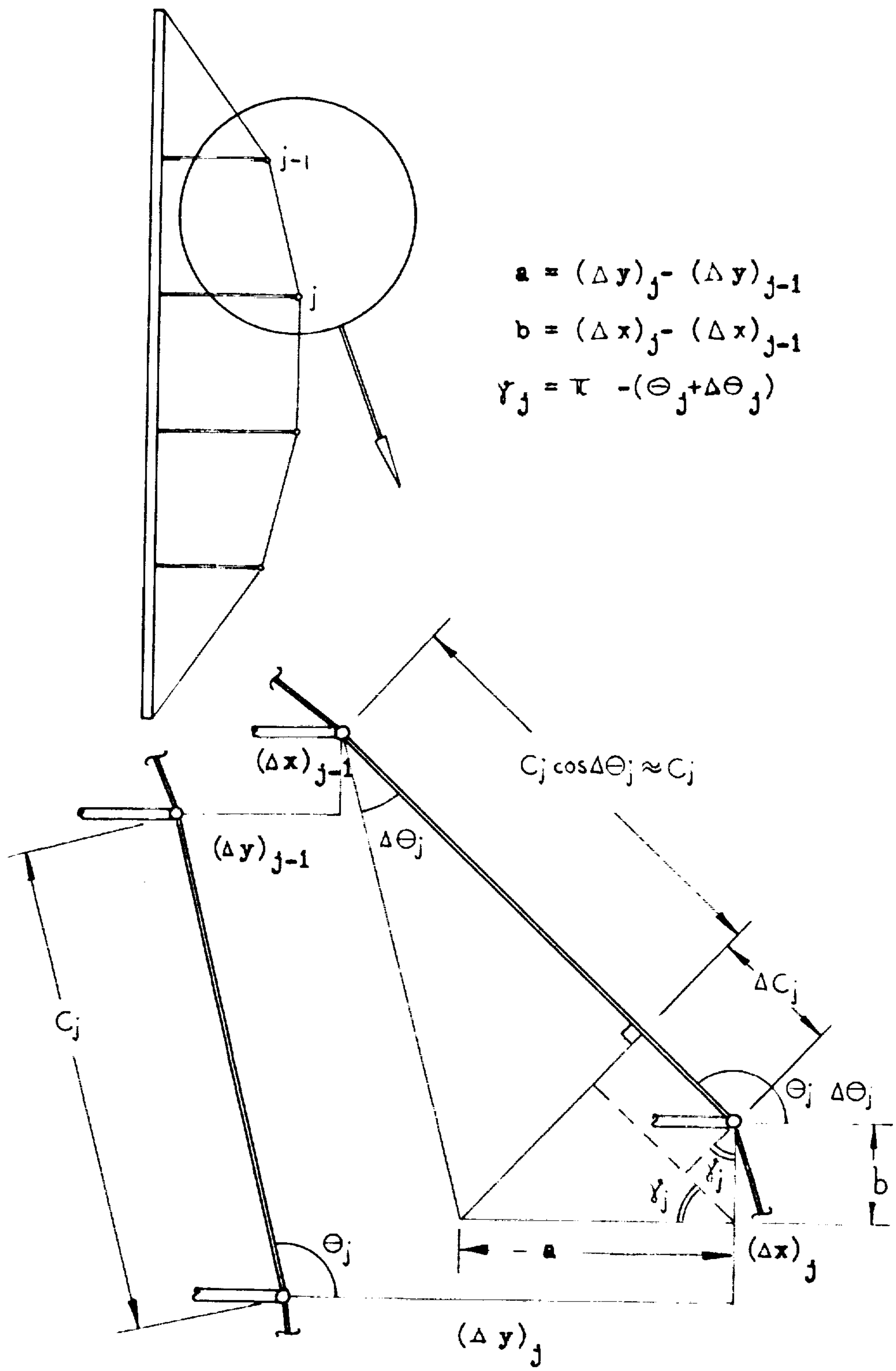
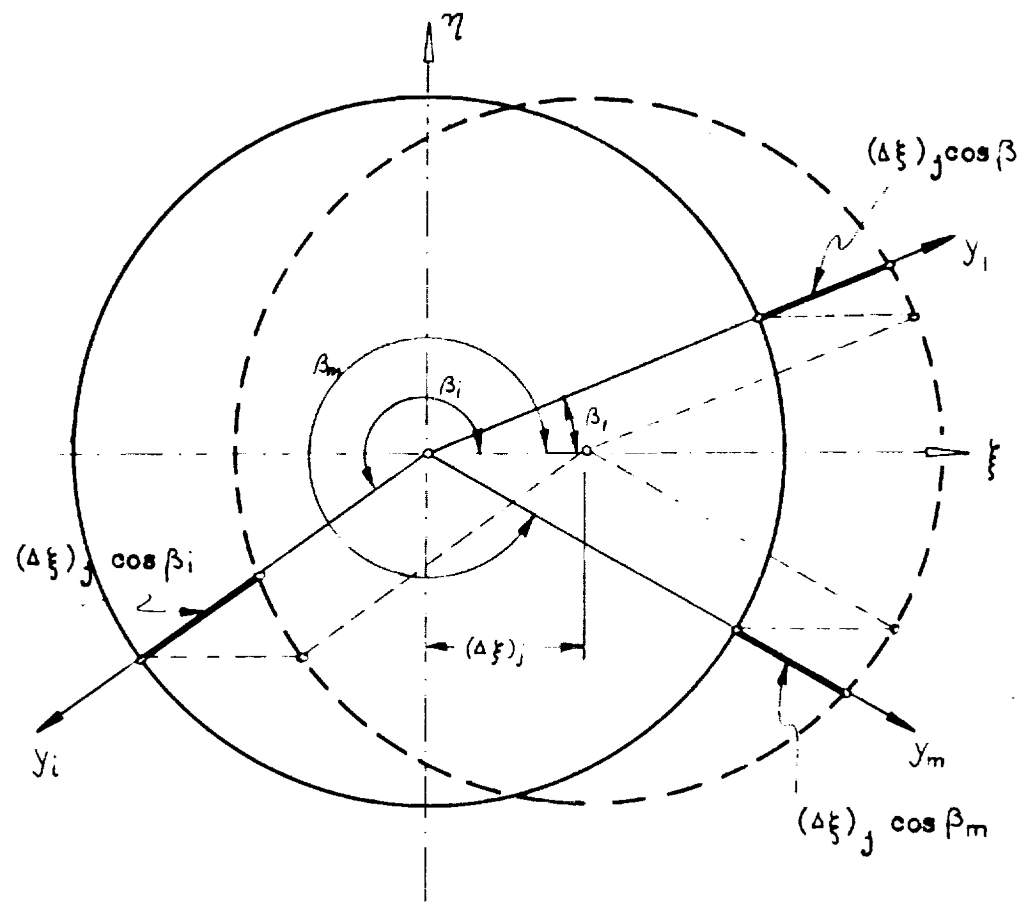


FIG. 4: Geometry of tie j before and after lateral deflection.



———— strut plane j before deflection

– – – – strut plane j after deflection $(\Delta\xi)_j$

FIG.5 : Deflection geometry at the plane of strut j.

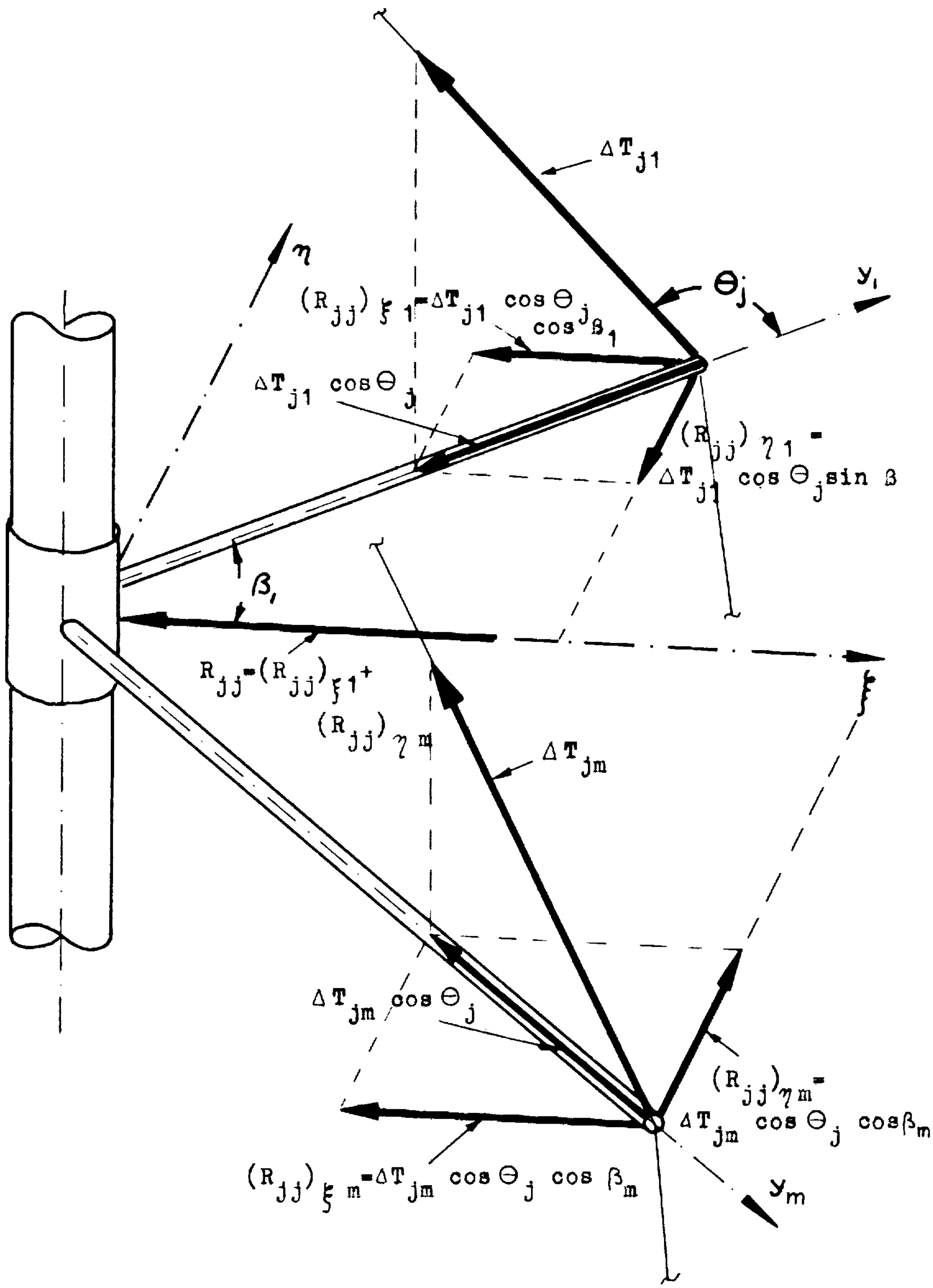


FIG. 6 : Components of tensile forces in direction ξ in the strut panel j.

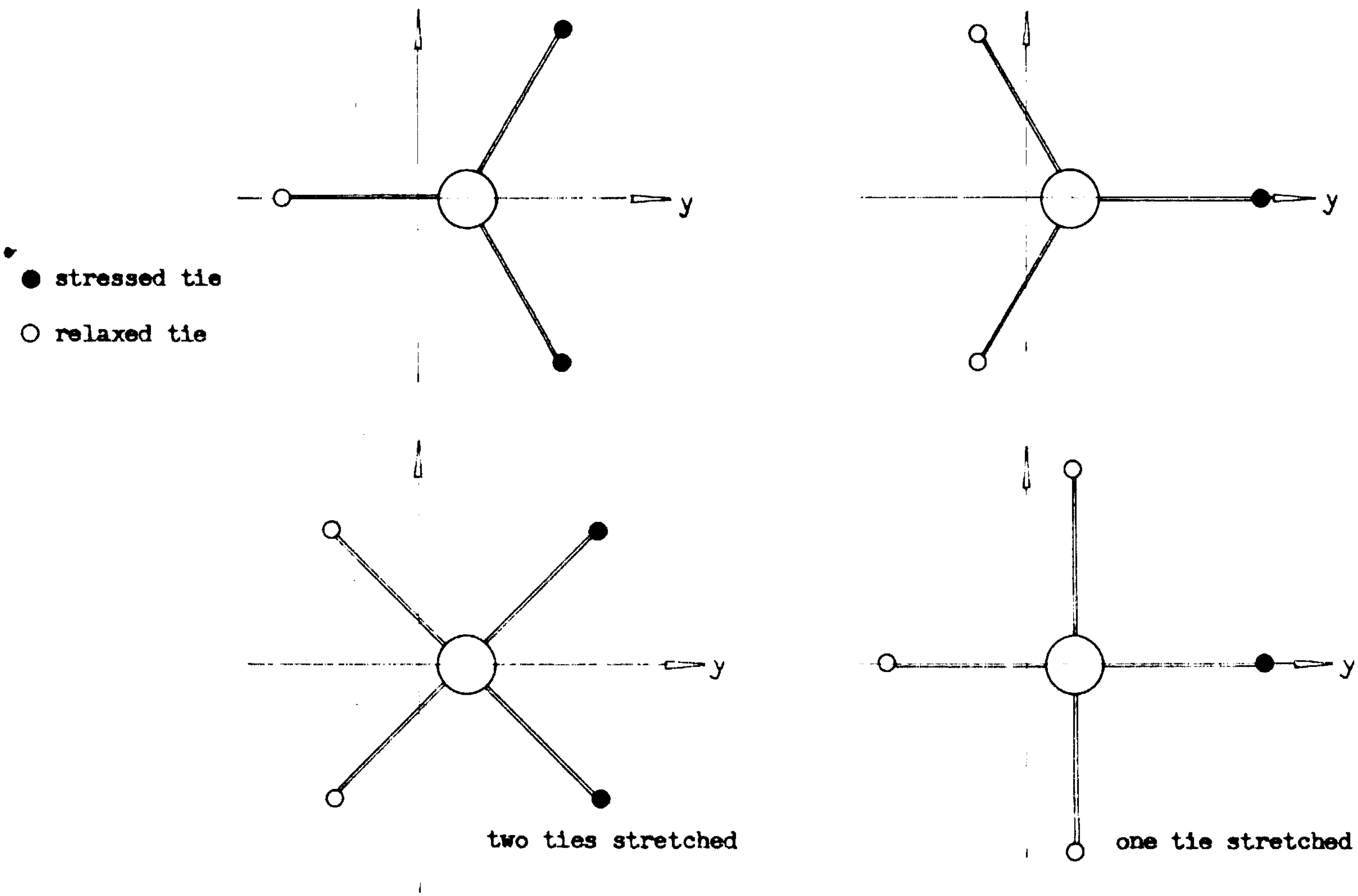


FIG. 7: Stressed and relaxed tension ties for a deflection y.

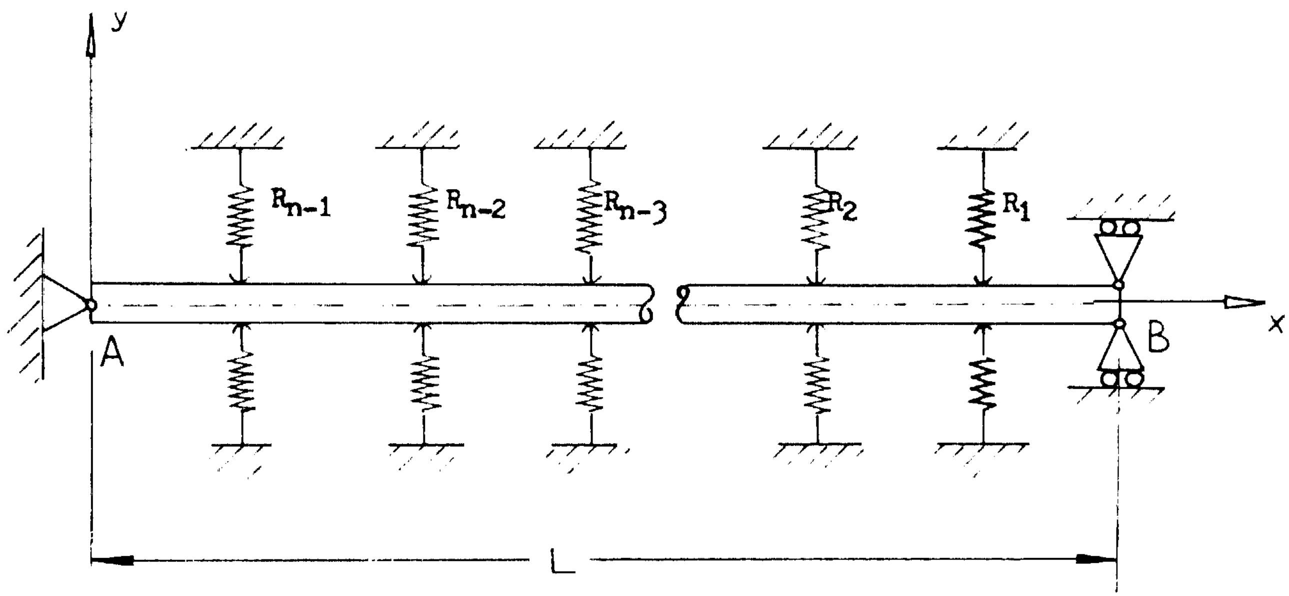


FIG. 8: Simplified buckling system.

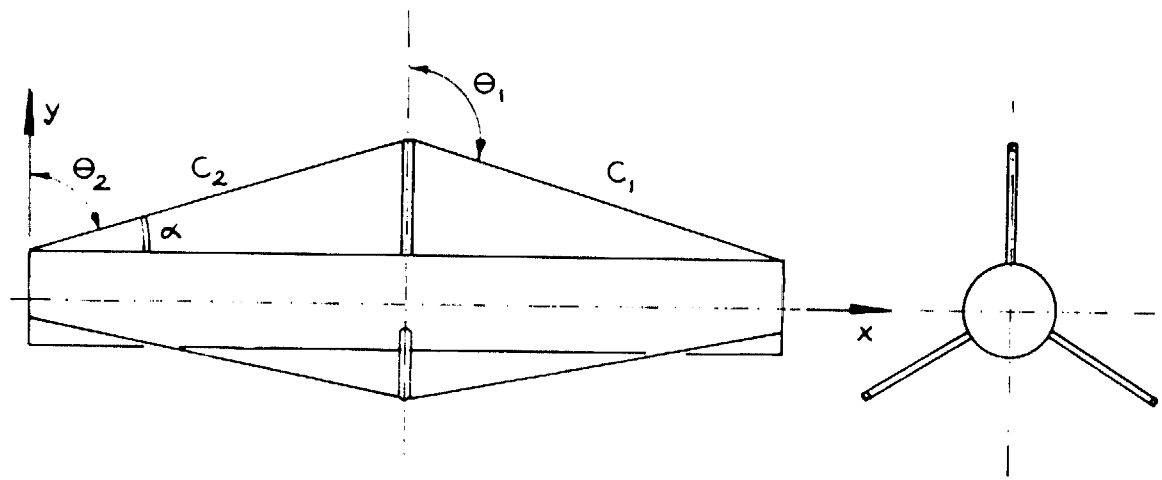


FIG.9: Geometry of the column with one strut.

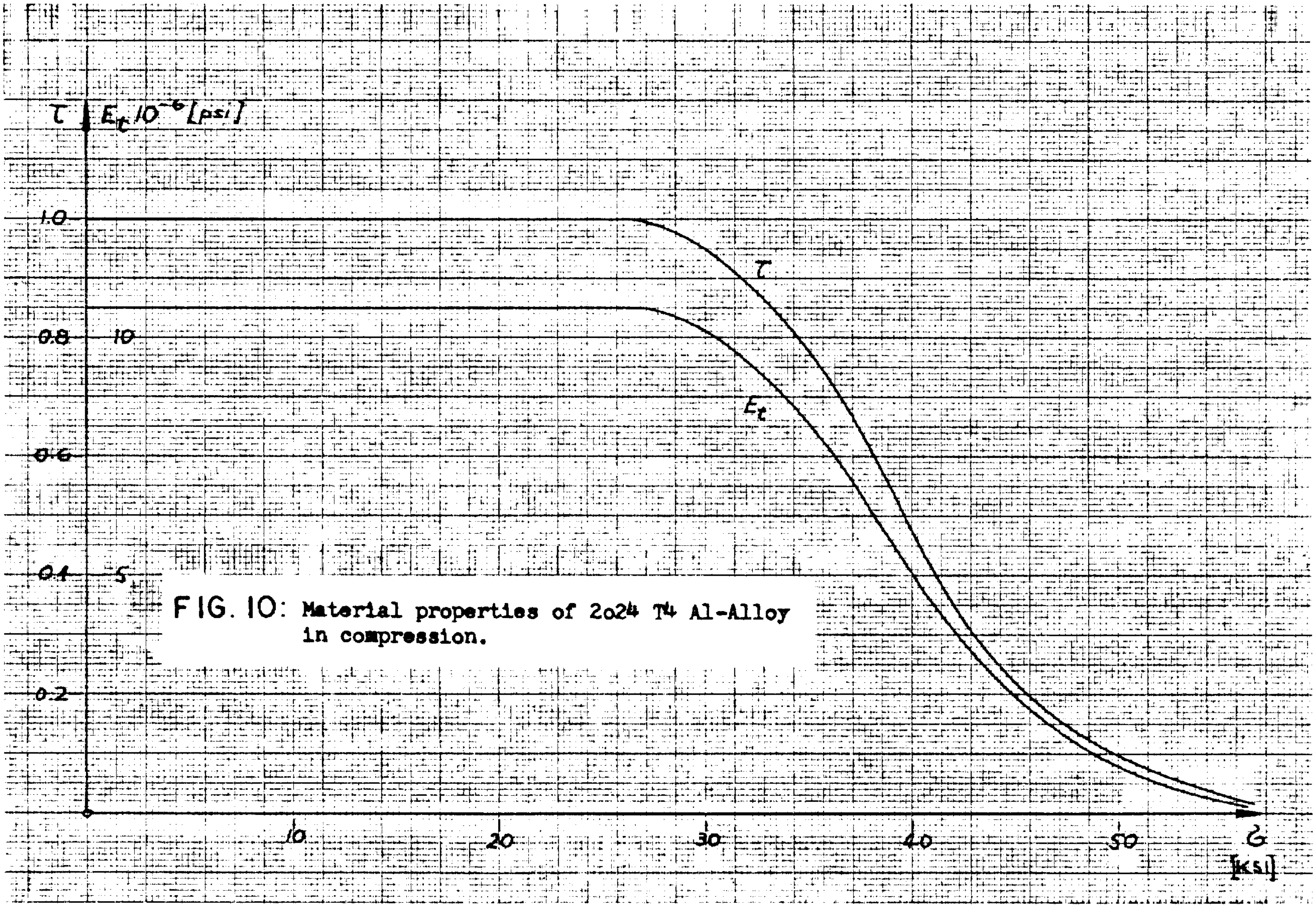


FIG. 10: Material properties of 2024 T4 Al-Alloy in compression.

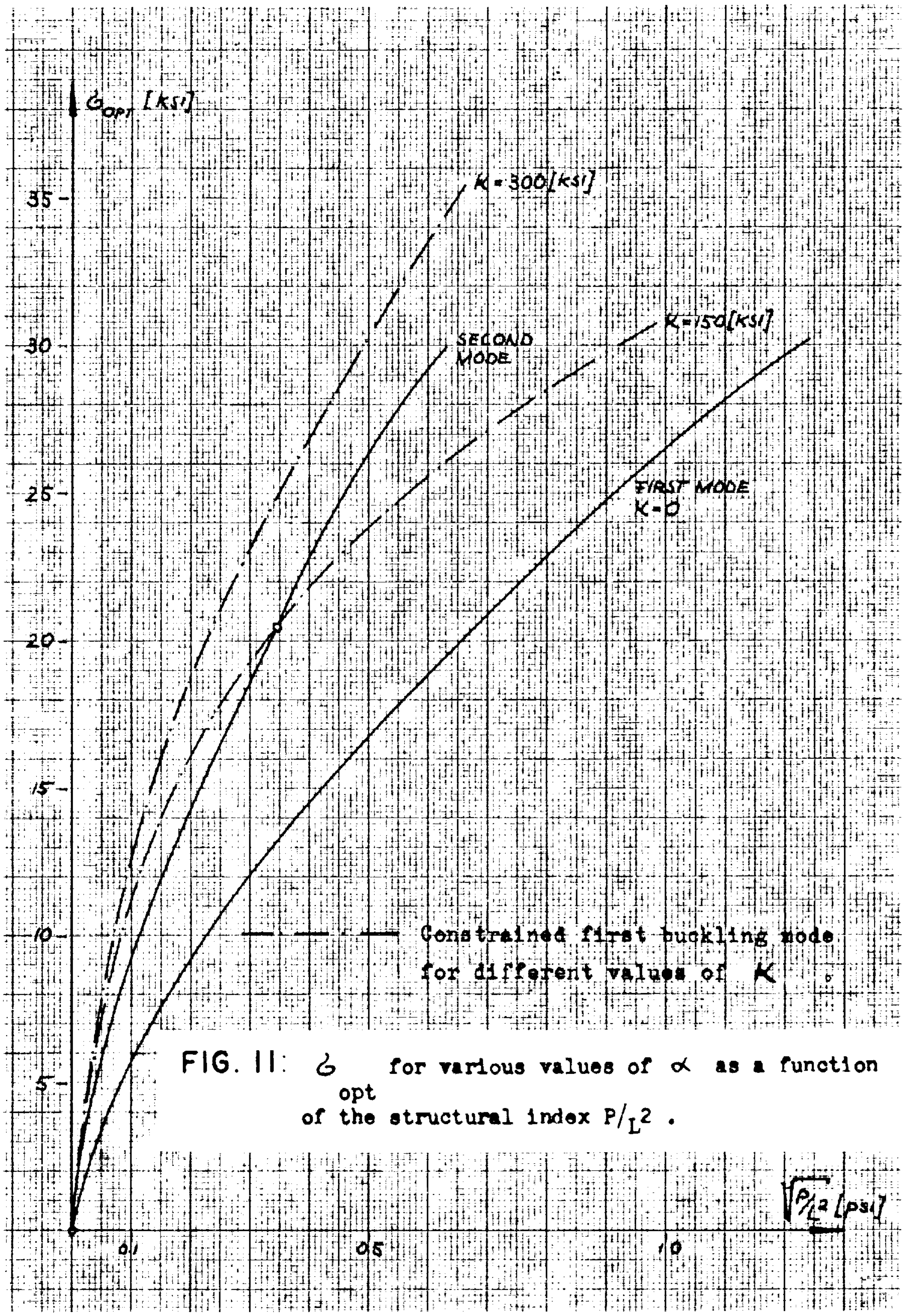


FIG. 11: σ_{opt} for various values of α as a function of the structural index P/L^2 .

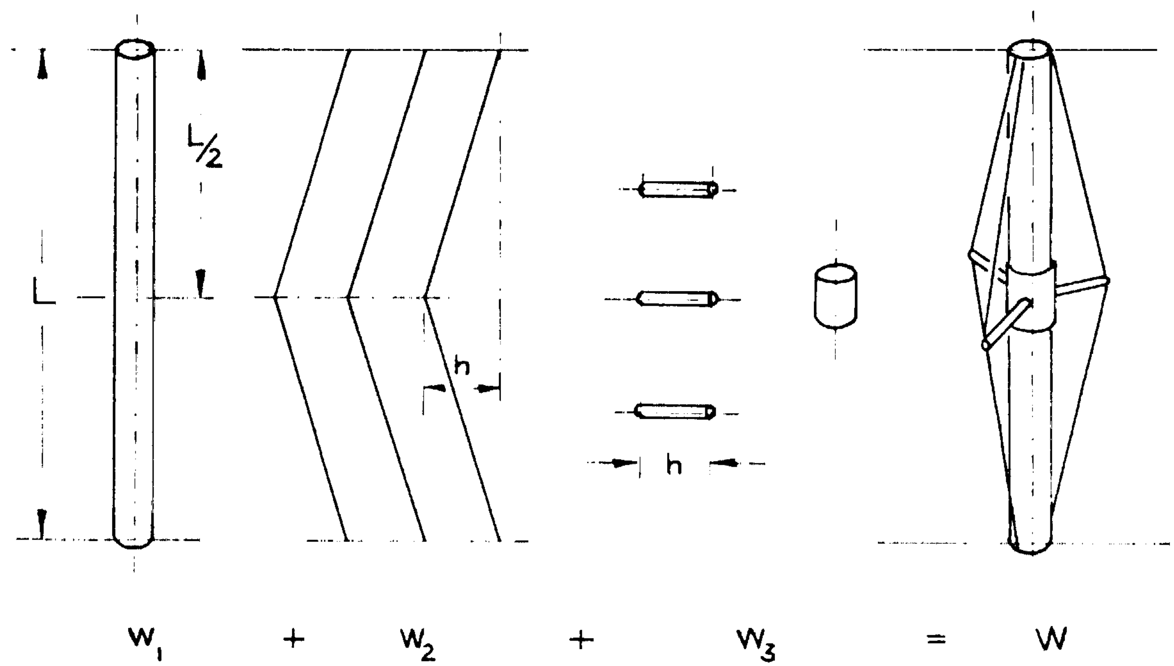


FIG. 12: Contribution to the weight of the wire supported column with one strut.

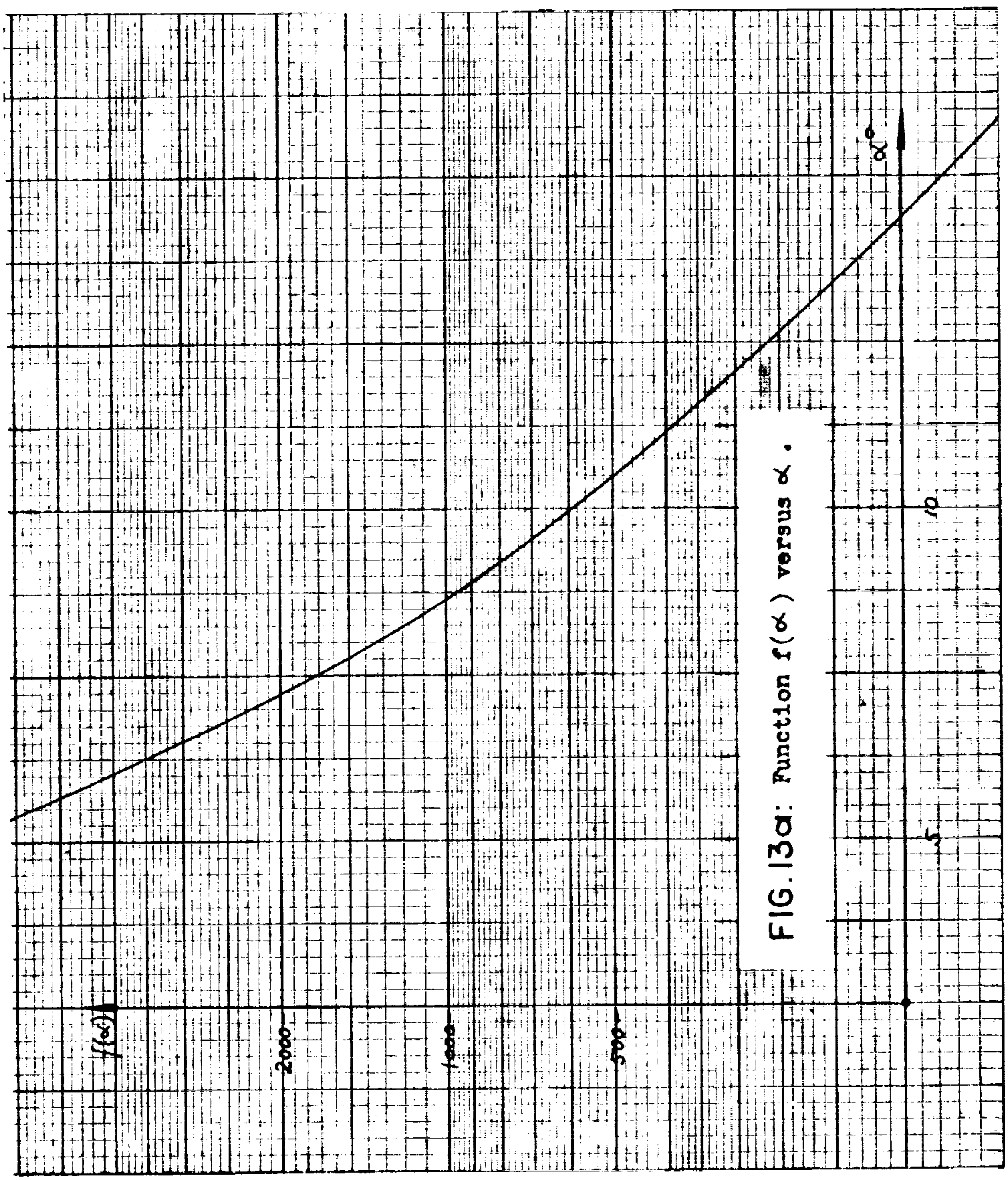


FIG. 13a: Function f(α) versus α.

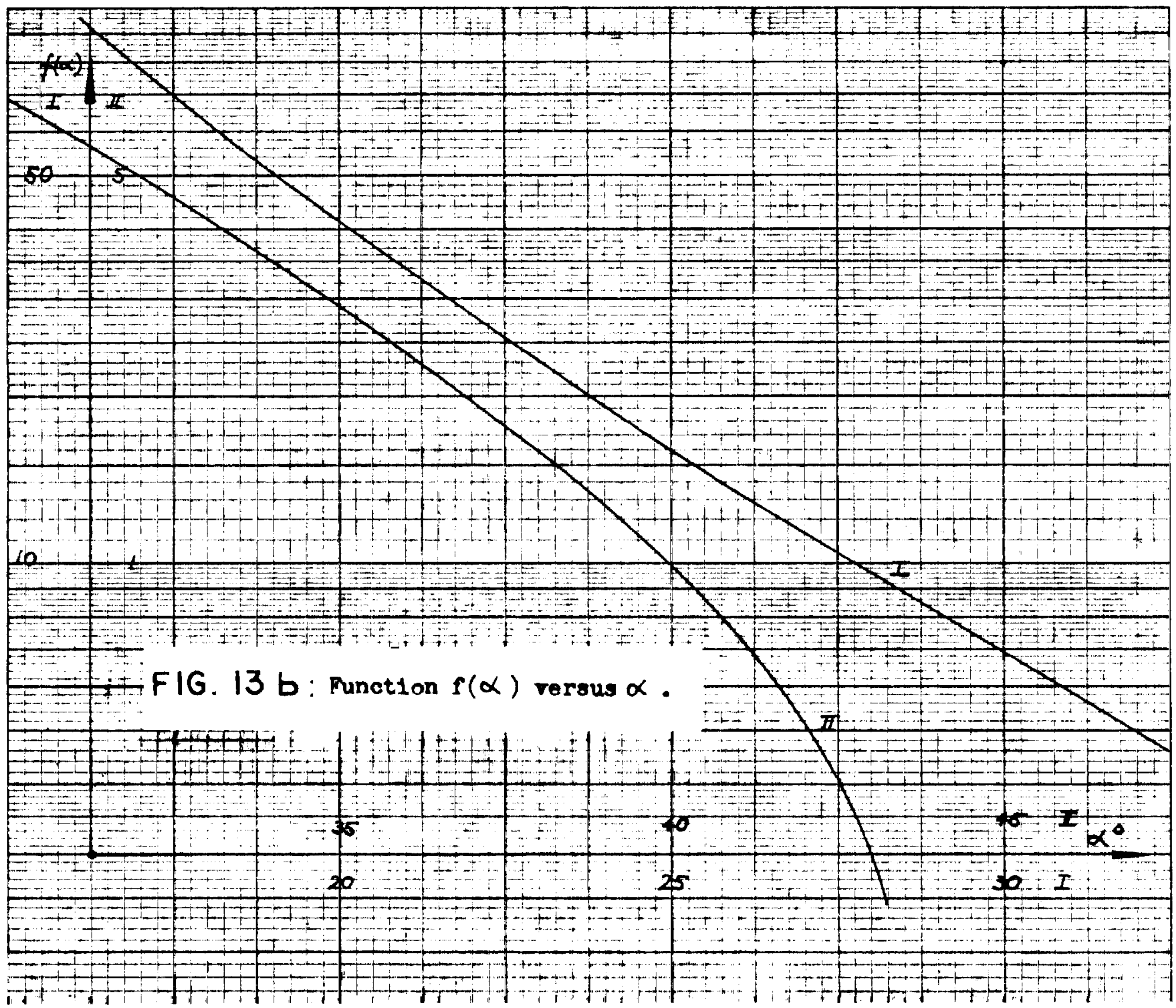


FIG. 13 b : Function f(α) versus α .

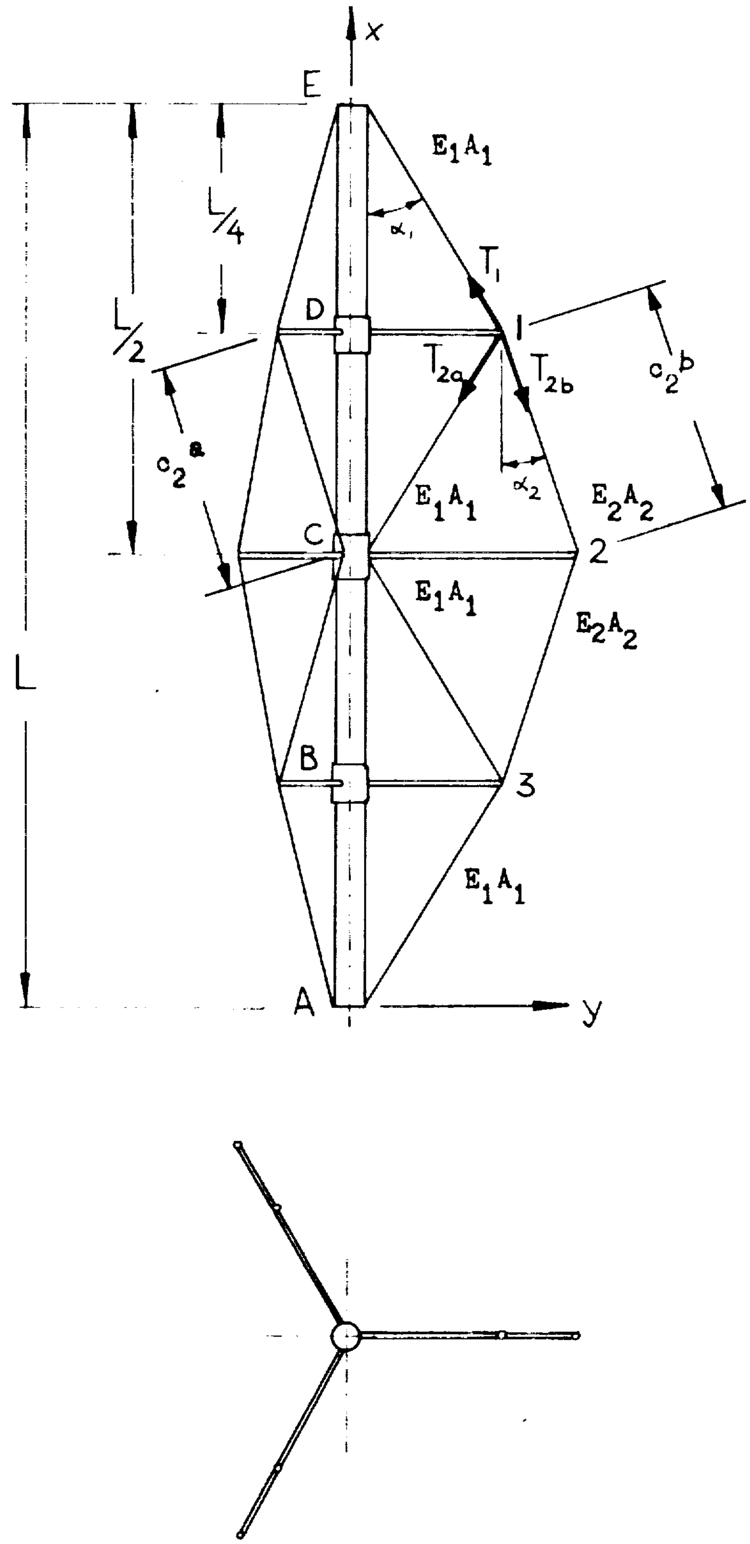


FIG. 14 : Geometry of the system with three struts.

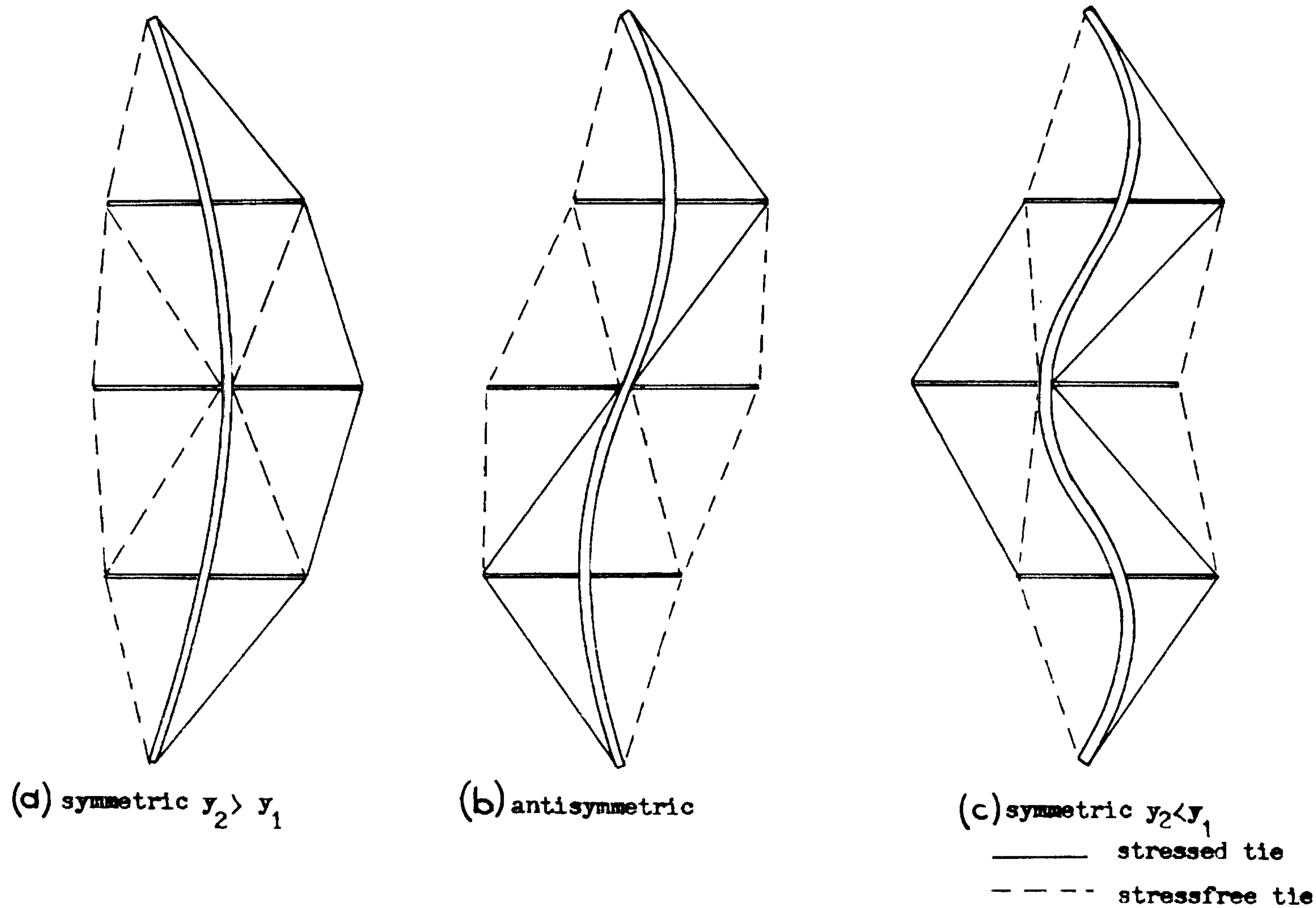


FIG. 15: Tension stresses at different buckling shapes.

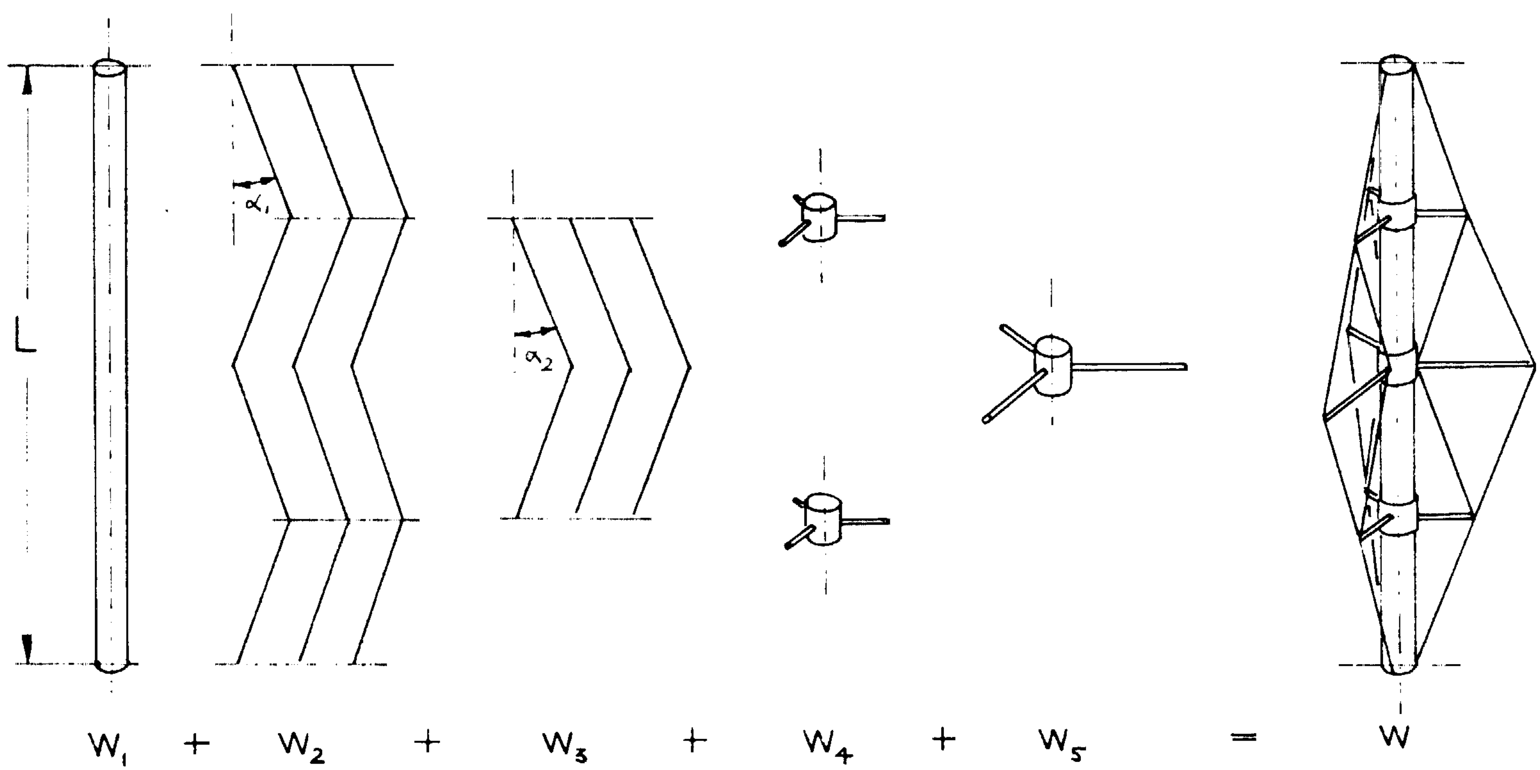


FIG. 16: Contributions to the weight of the wire supported column with three struts.

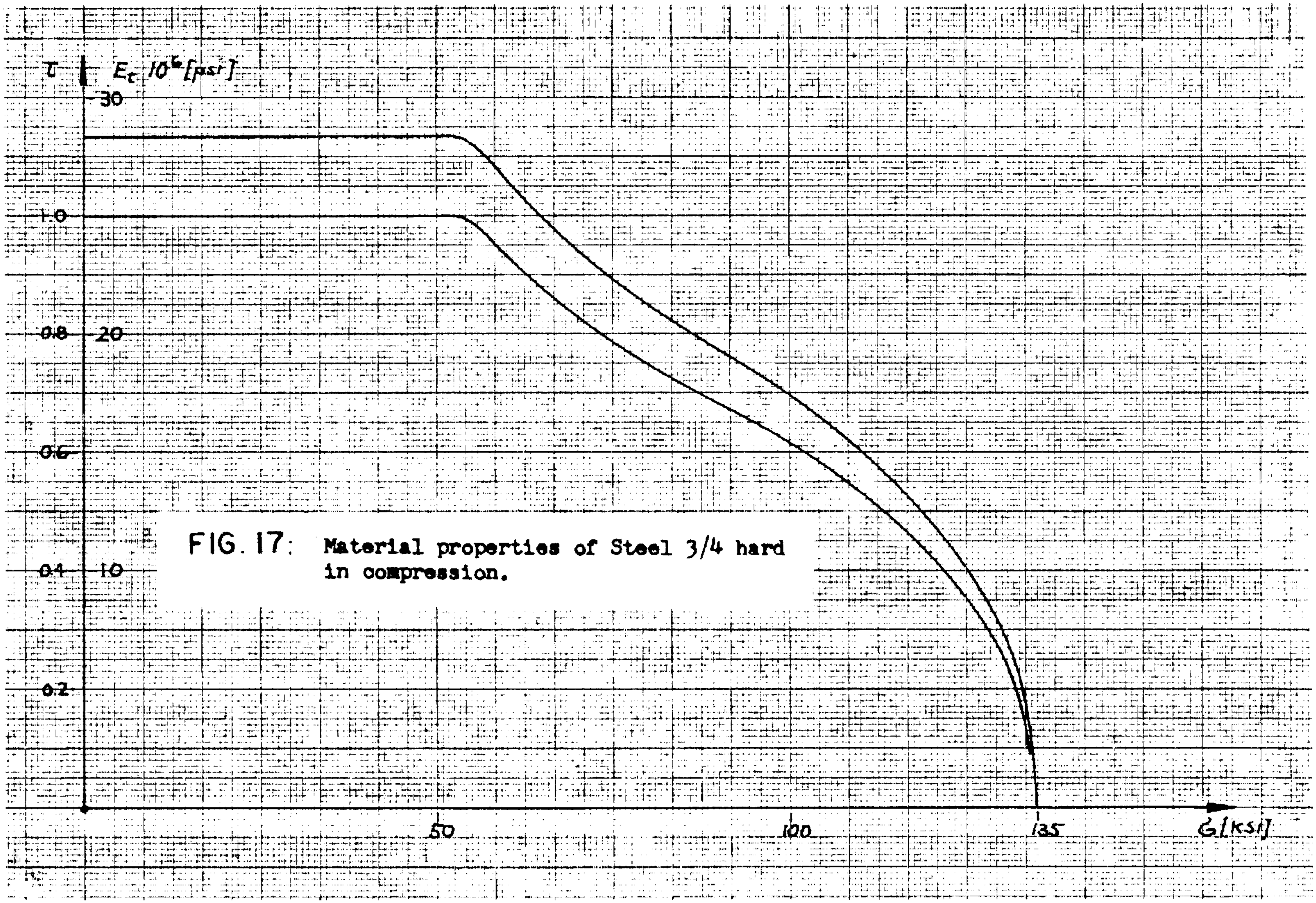


FIG. 17: **Material properties of Steel 3/4 hard in compression.**

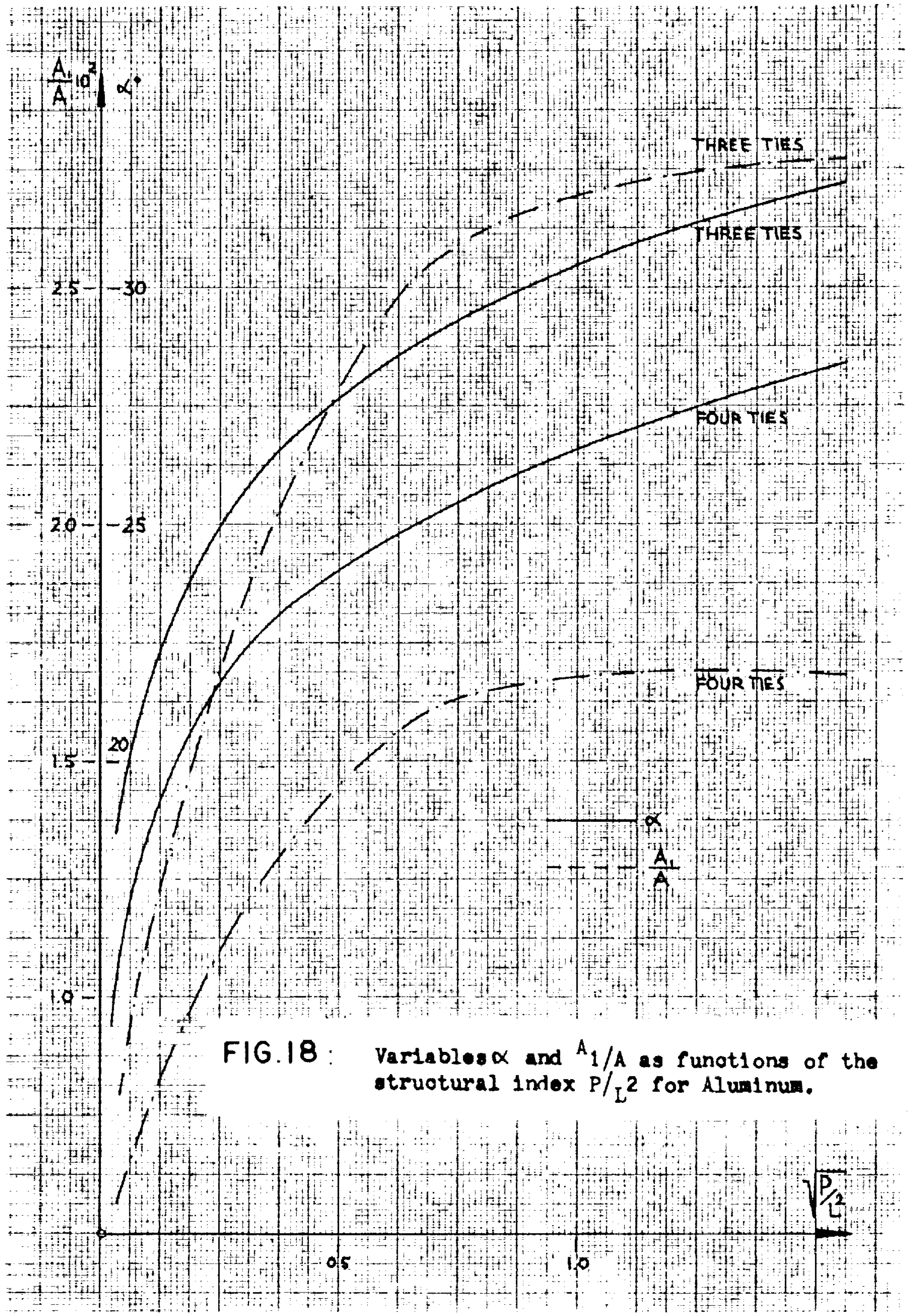


FIG. 18 : Variables α and A_1/A as functions of the structural index P/L^2 for Aluminum.

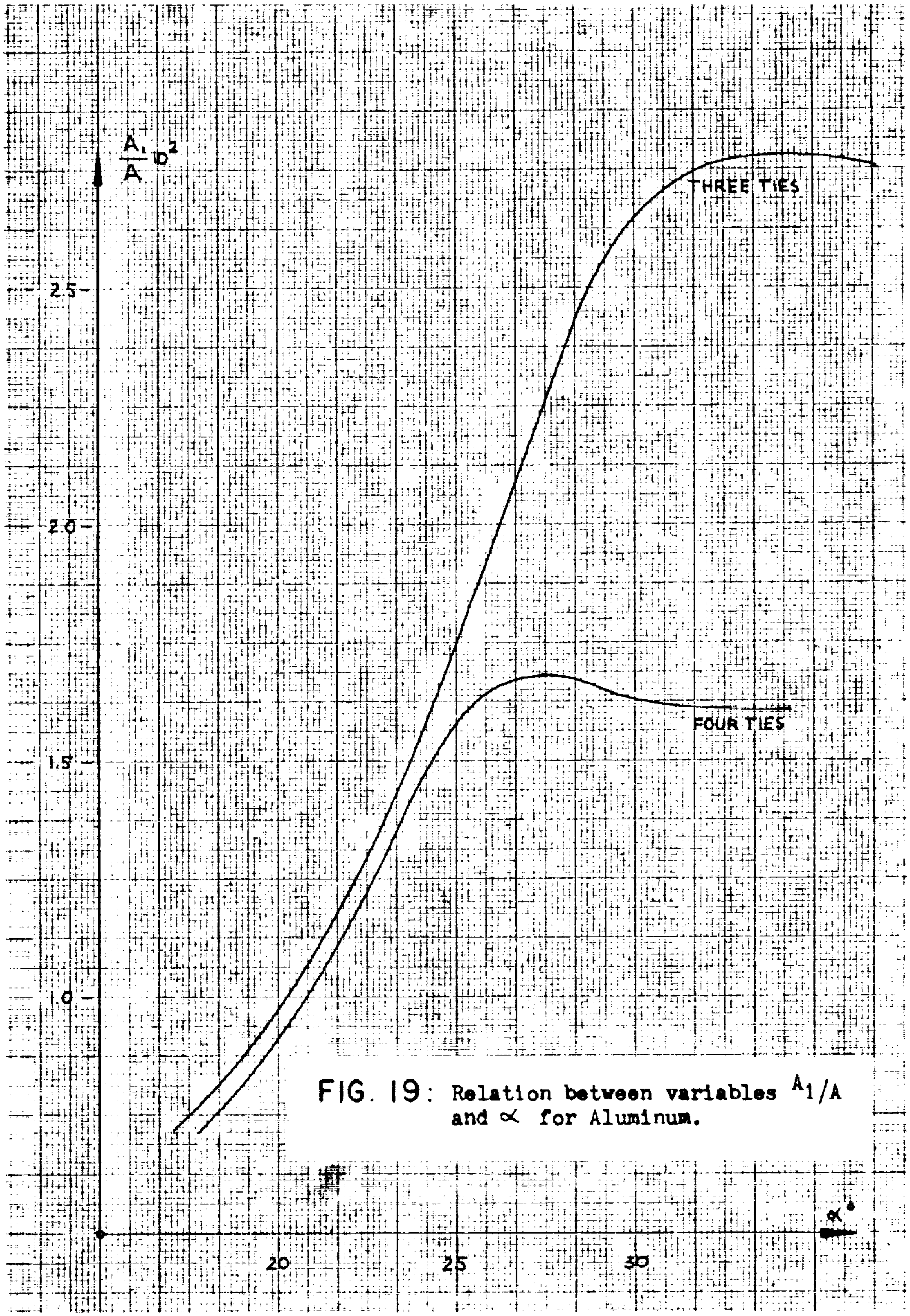


FIG. 19: Relation between variables A_1/A and α for Aluminum.

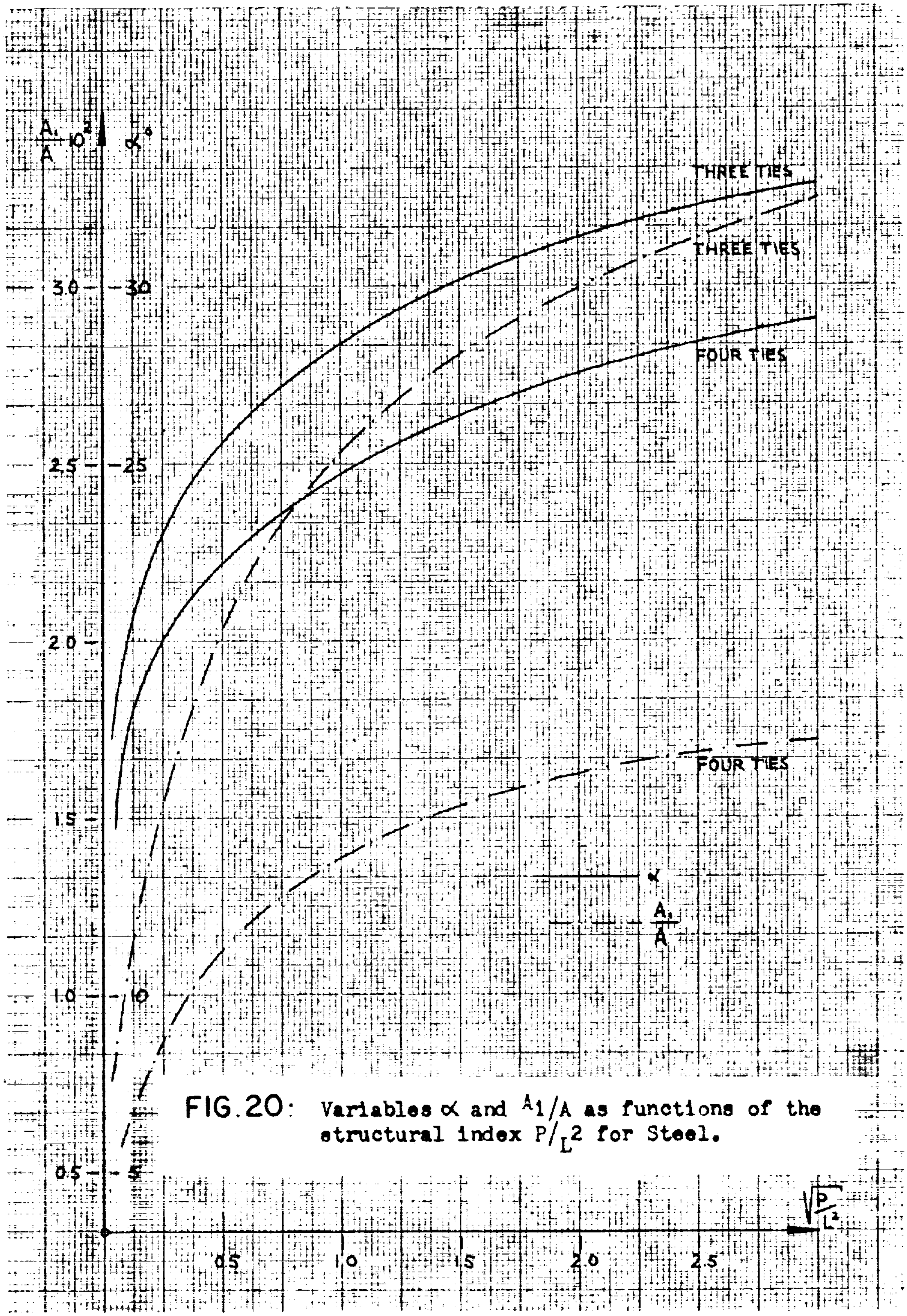


FIG. 20: Variables α and A_1/A as functions of the structural index P/L^2 for Steel.

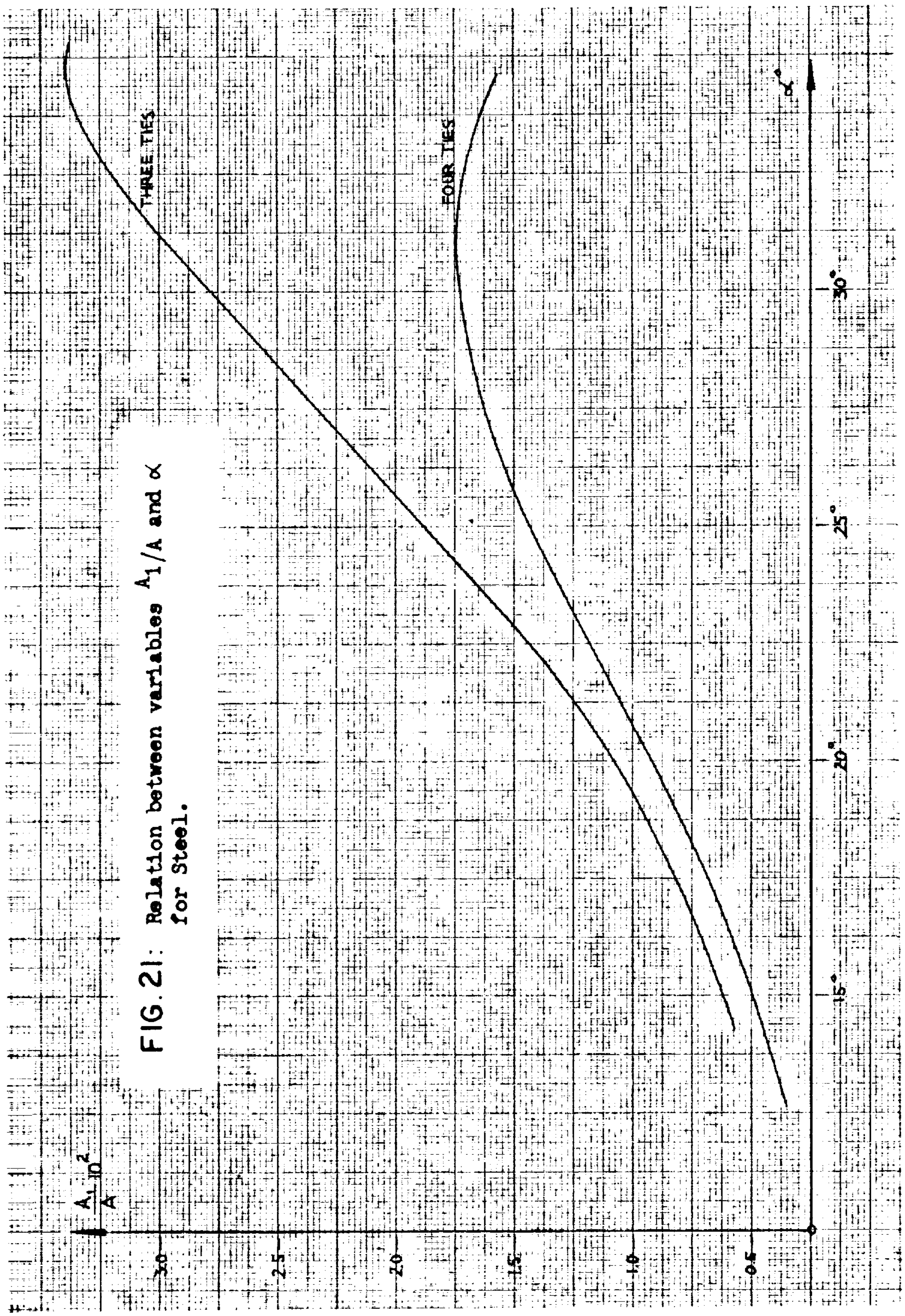


FIG. 21: Relation between variables A_1/A and α for Steel.

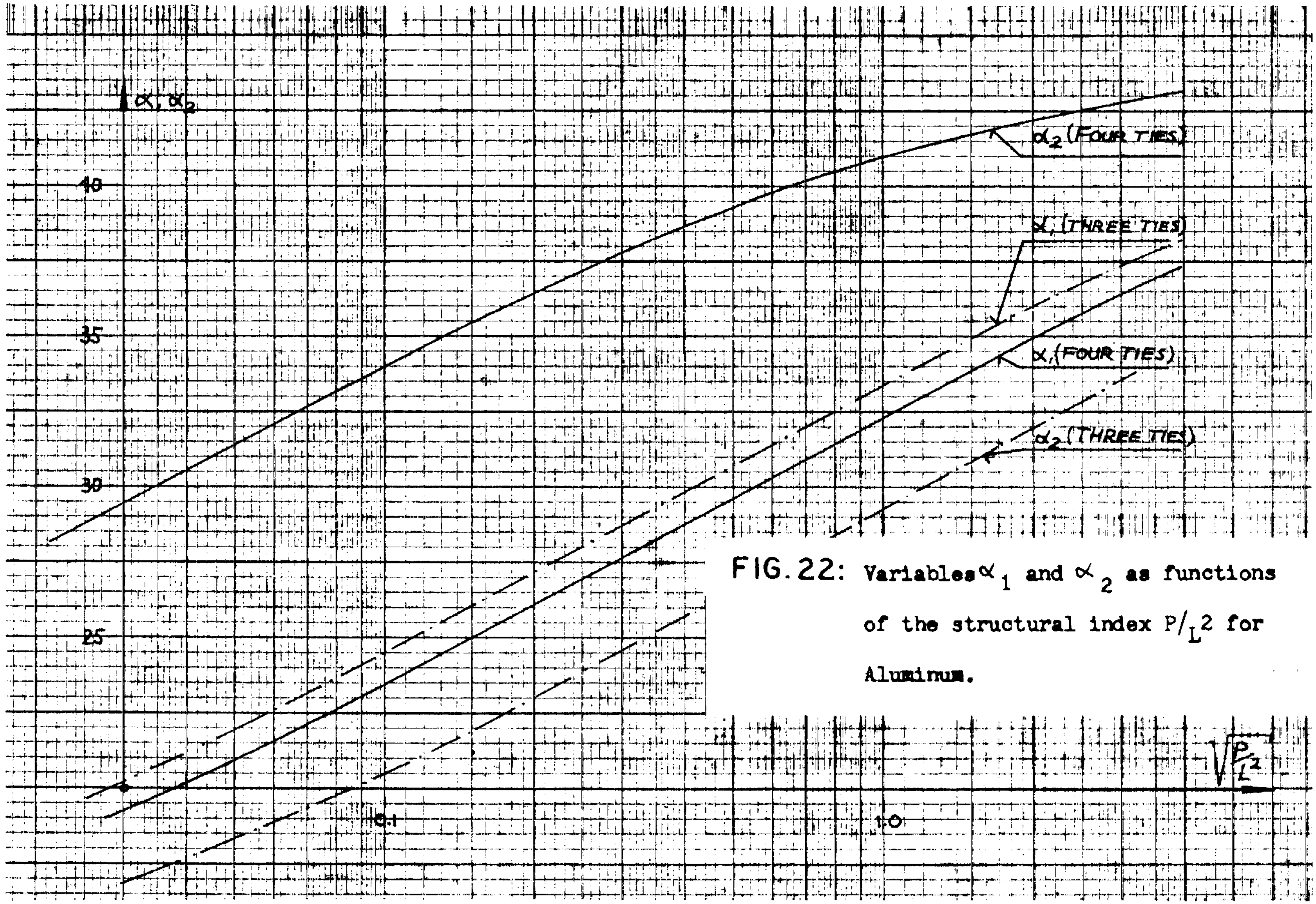


FIG. 22: Variables α_1 and α_2 as functions of the structural index P/L^2 for Aluminum.

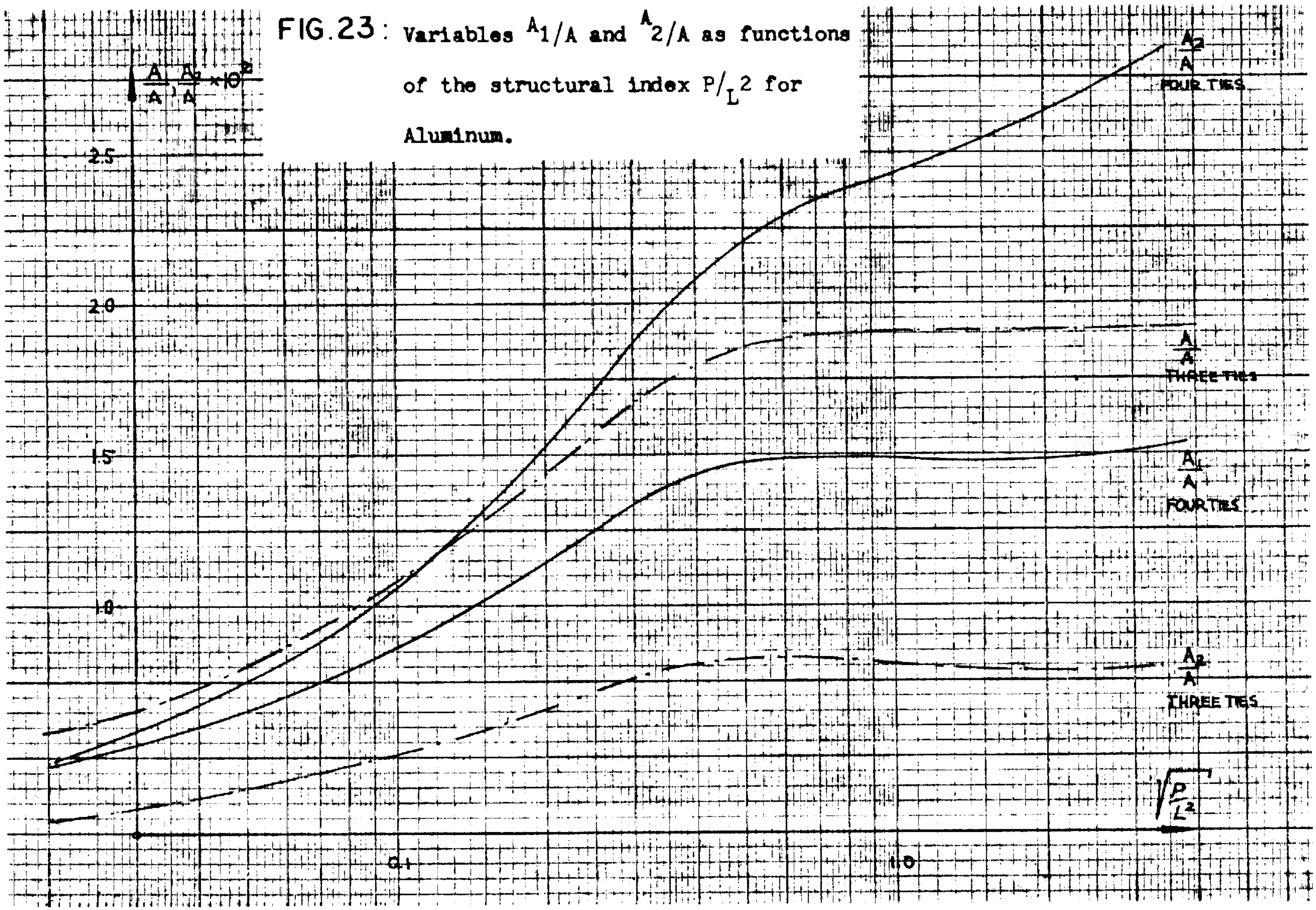


FIG. 23: Variables A_1/A and A_2/A as functions of the structural index P/L^2 for Aluminum.

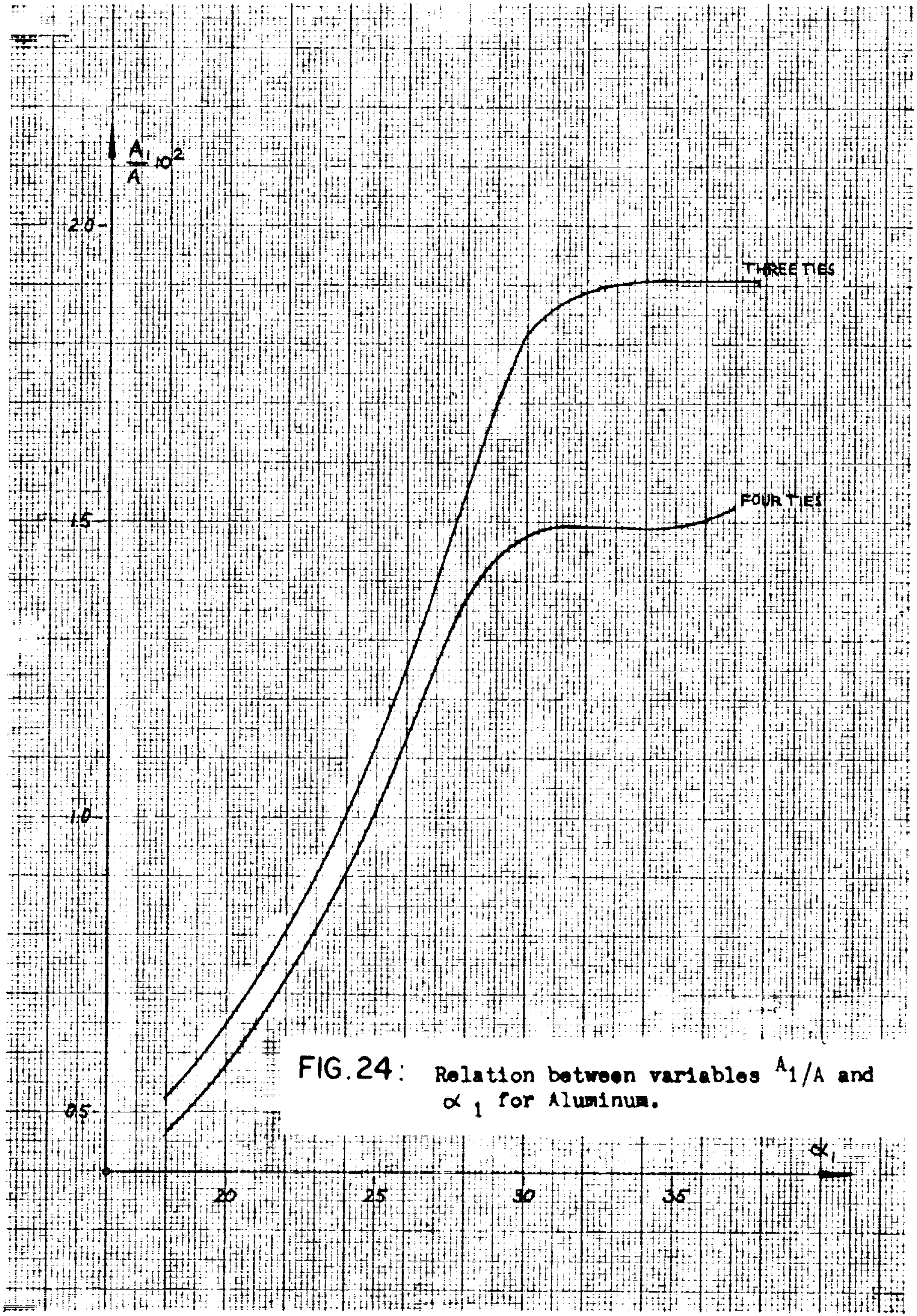


FIG. 24: Relation between variables A_1/A and α_1 for Aluminum.

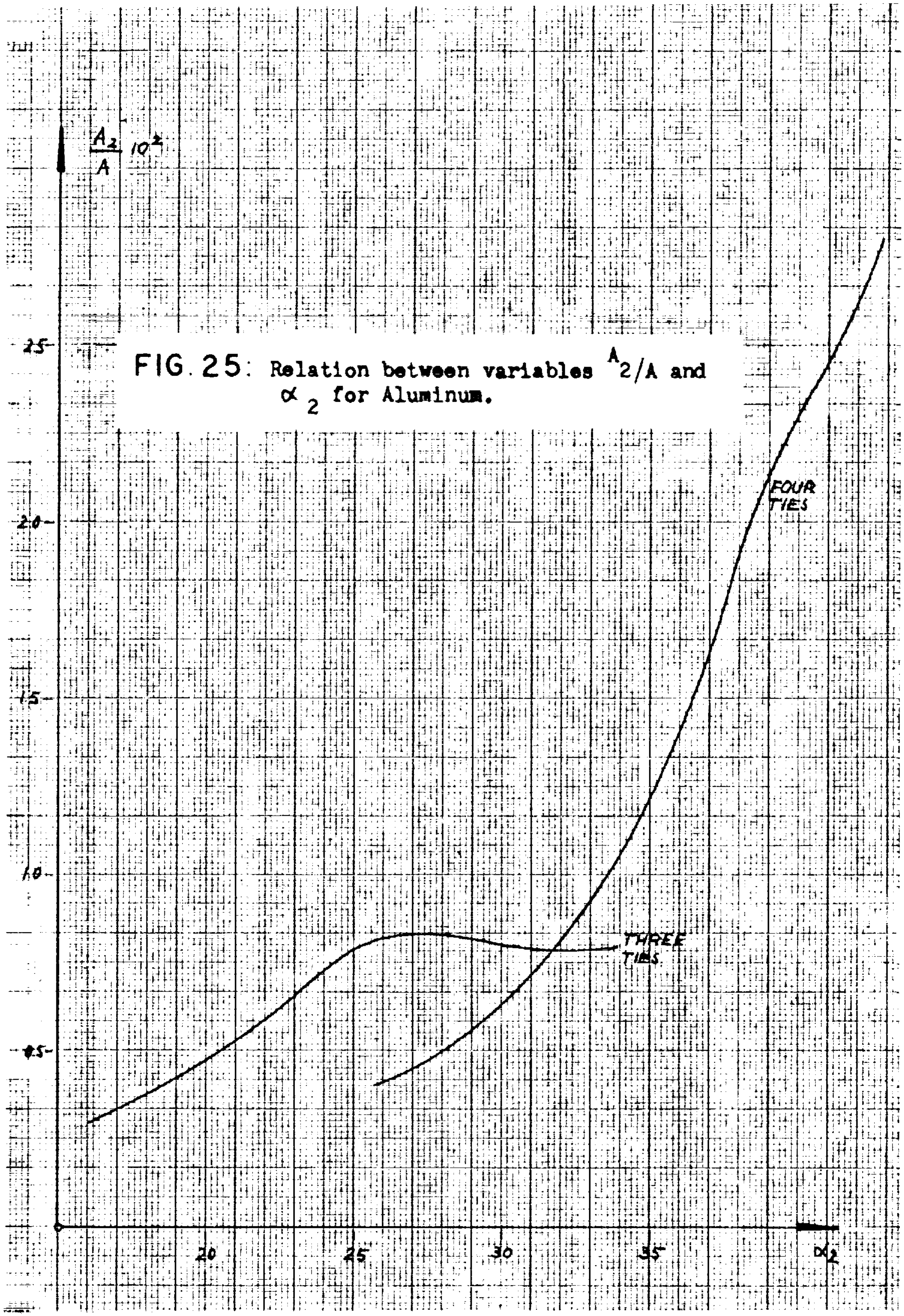


FIG. 25: Relation between variables A_2/A and α_2 for Aluminum.

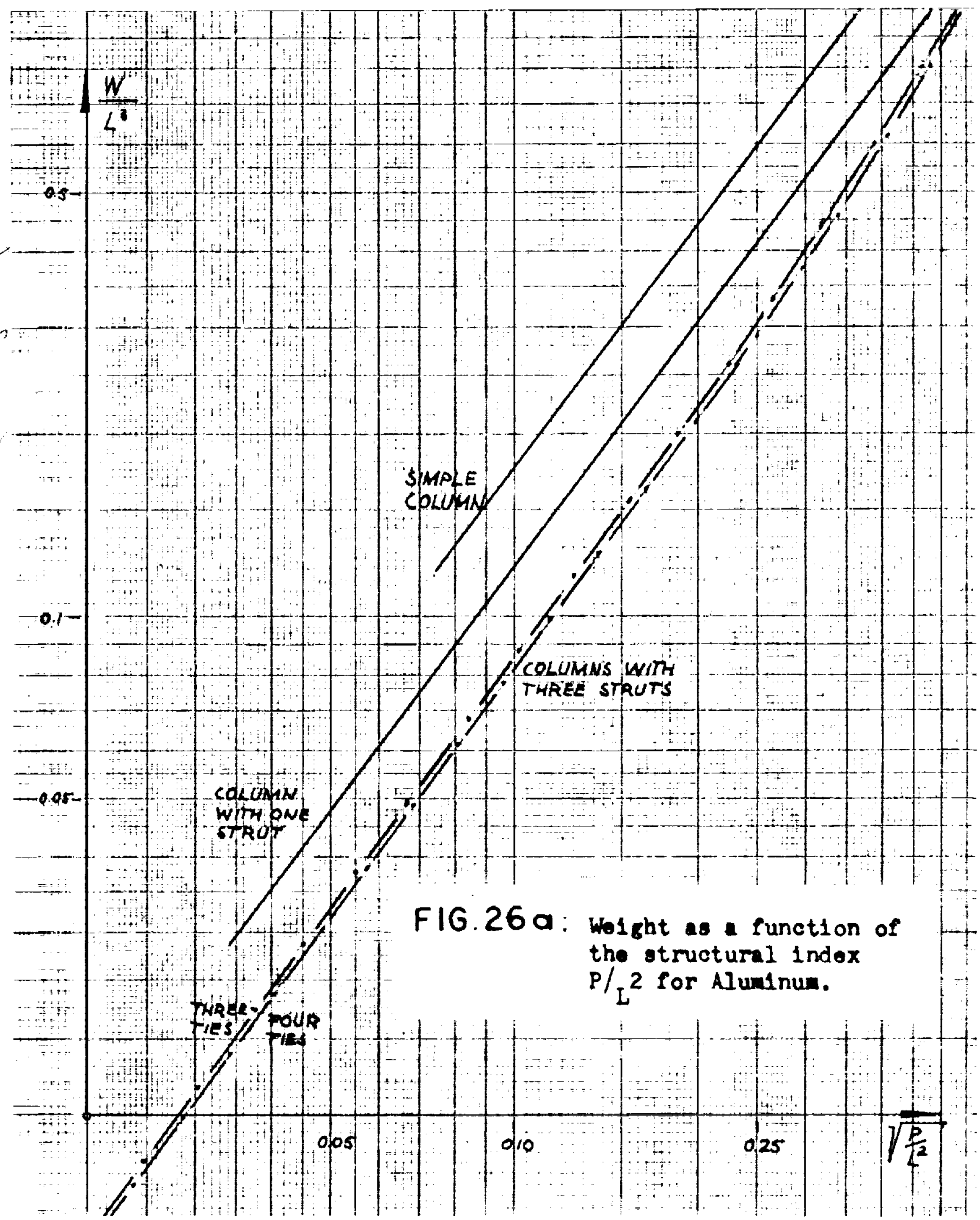


FIG. 26a: Weight as a function of the structural index P/L^2 for Aluminum.

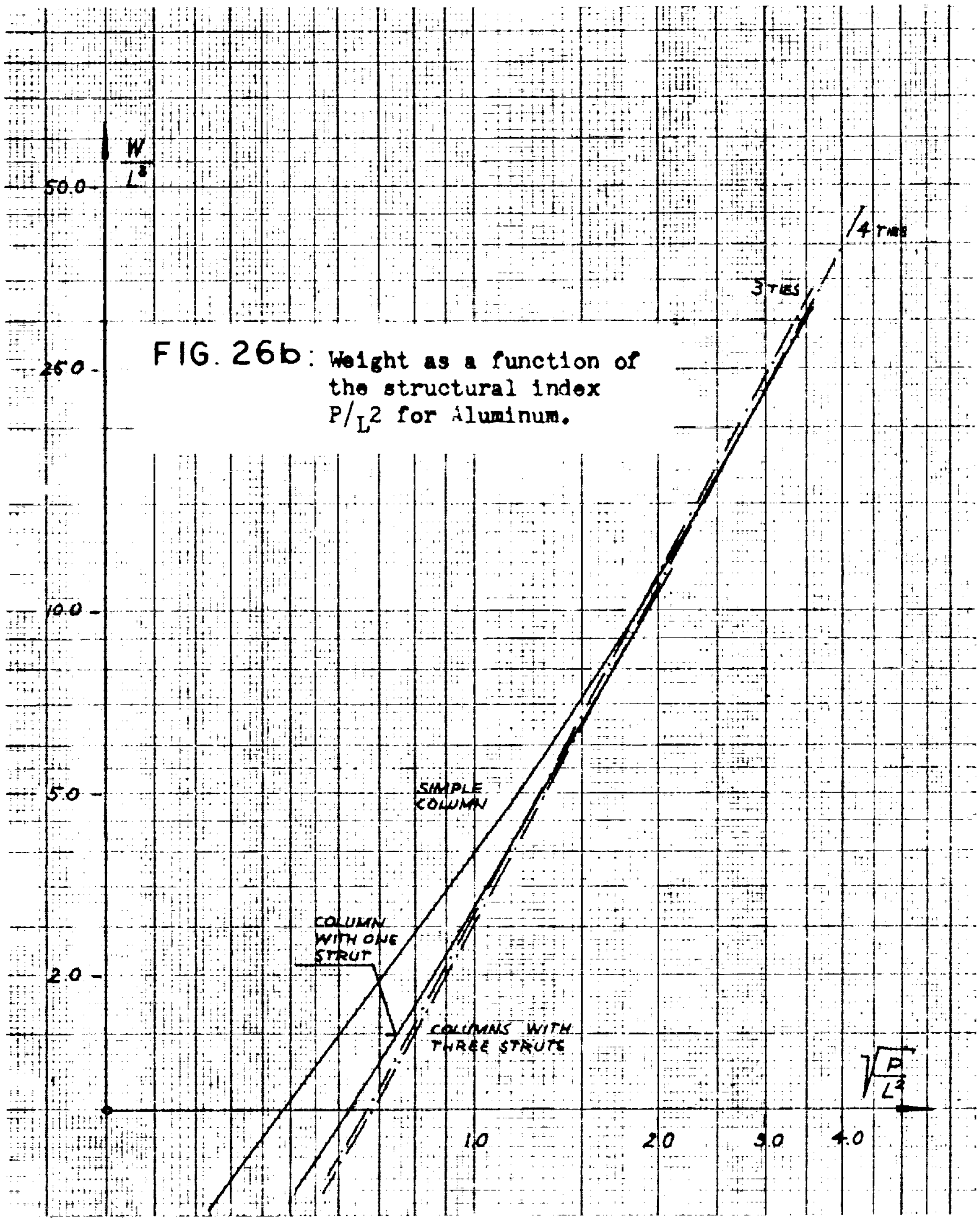


FIG. 26b: Weight as a function of the structural index $P/_{L}2$ for Aluminum.

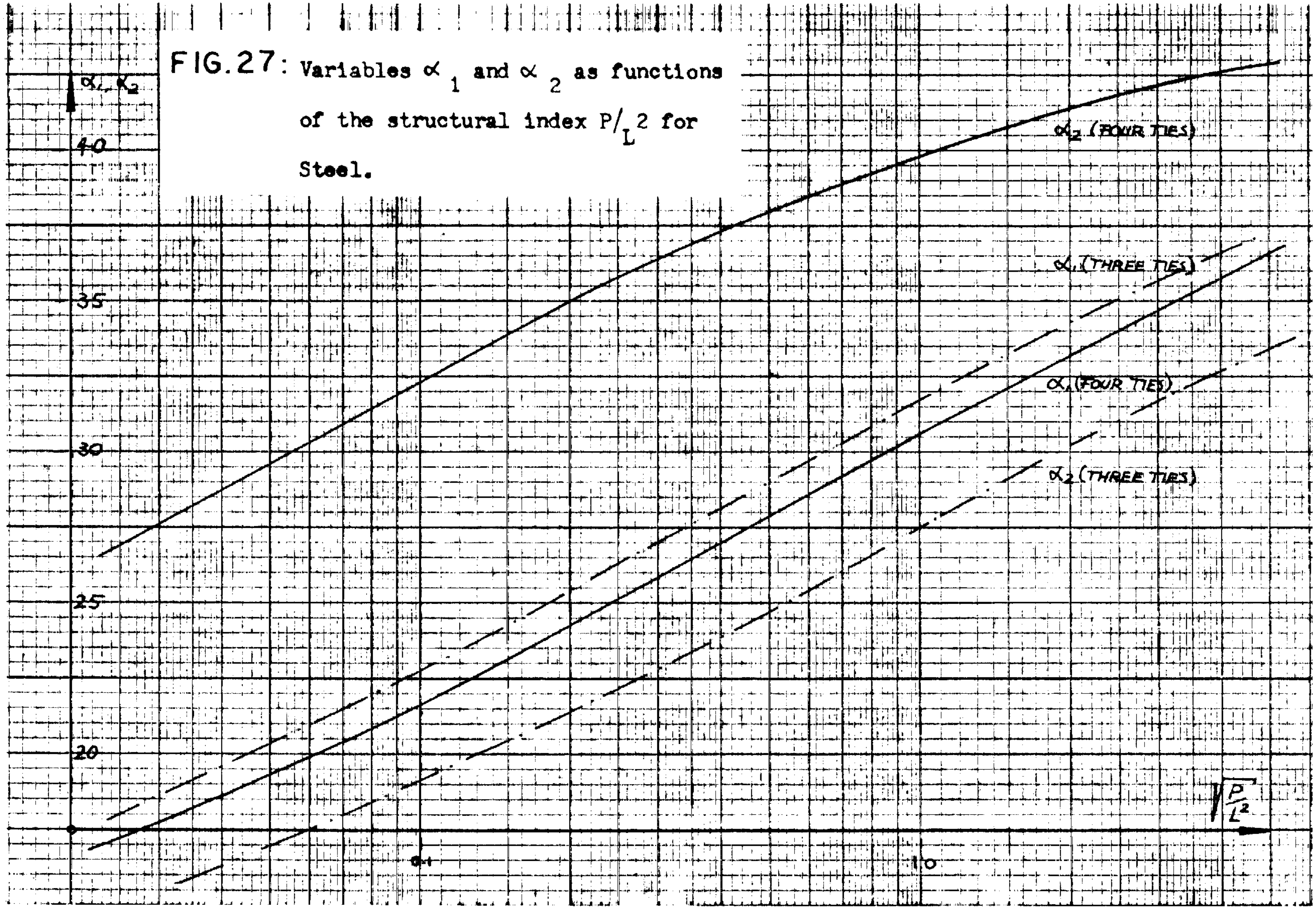


FIG. 27: Variables α_1 and α_2 as functions of the structural index P/L^2 for Steel.

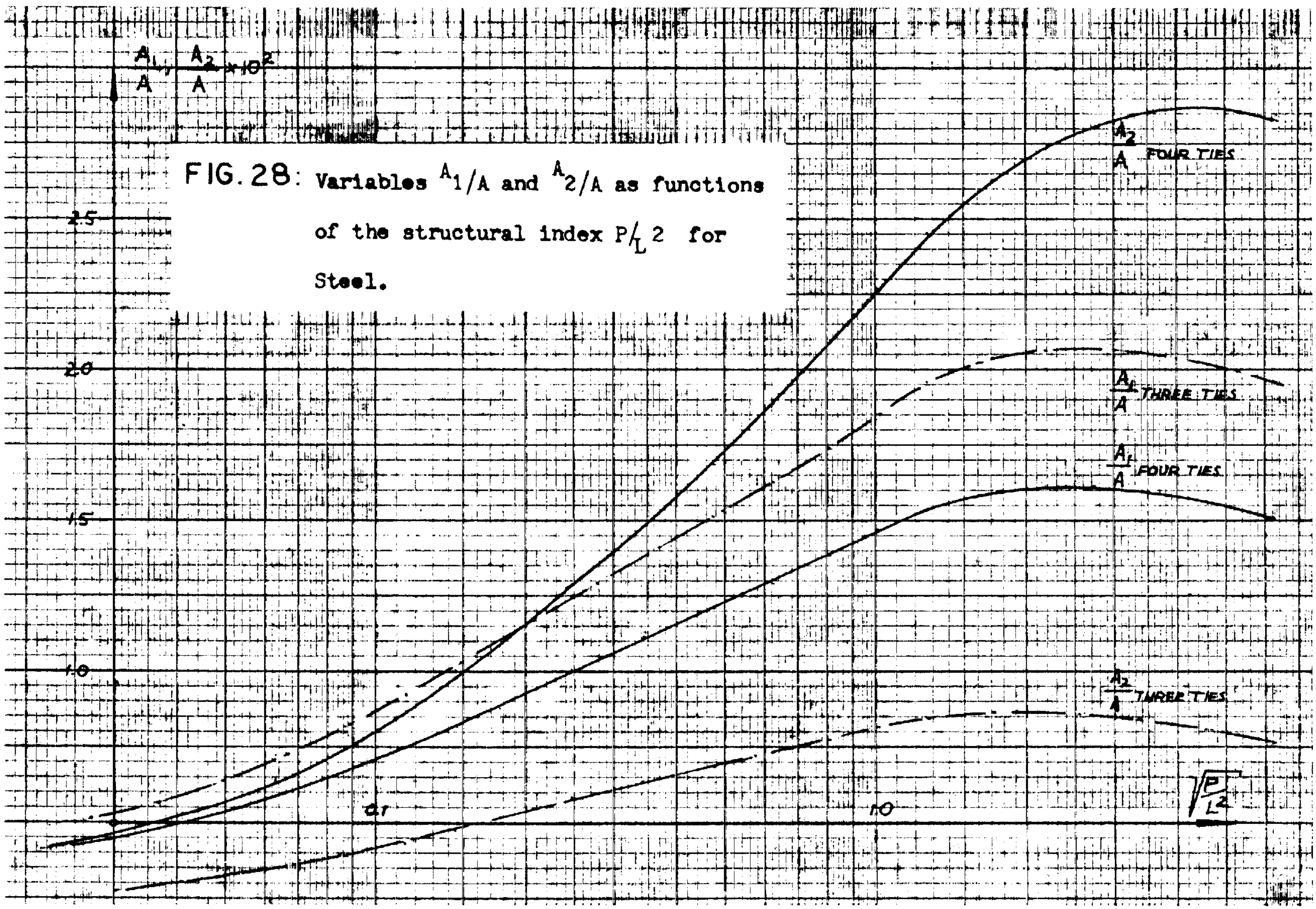


FIG. 28: Variables A_1/A and A_2/A as functions of the structural index P/L^2 for Steel.

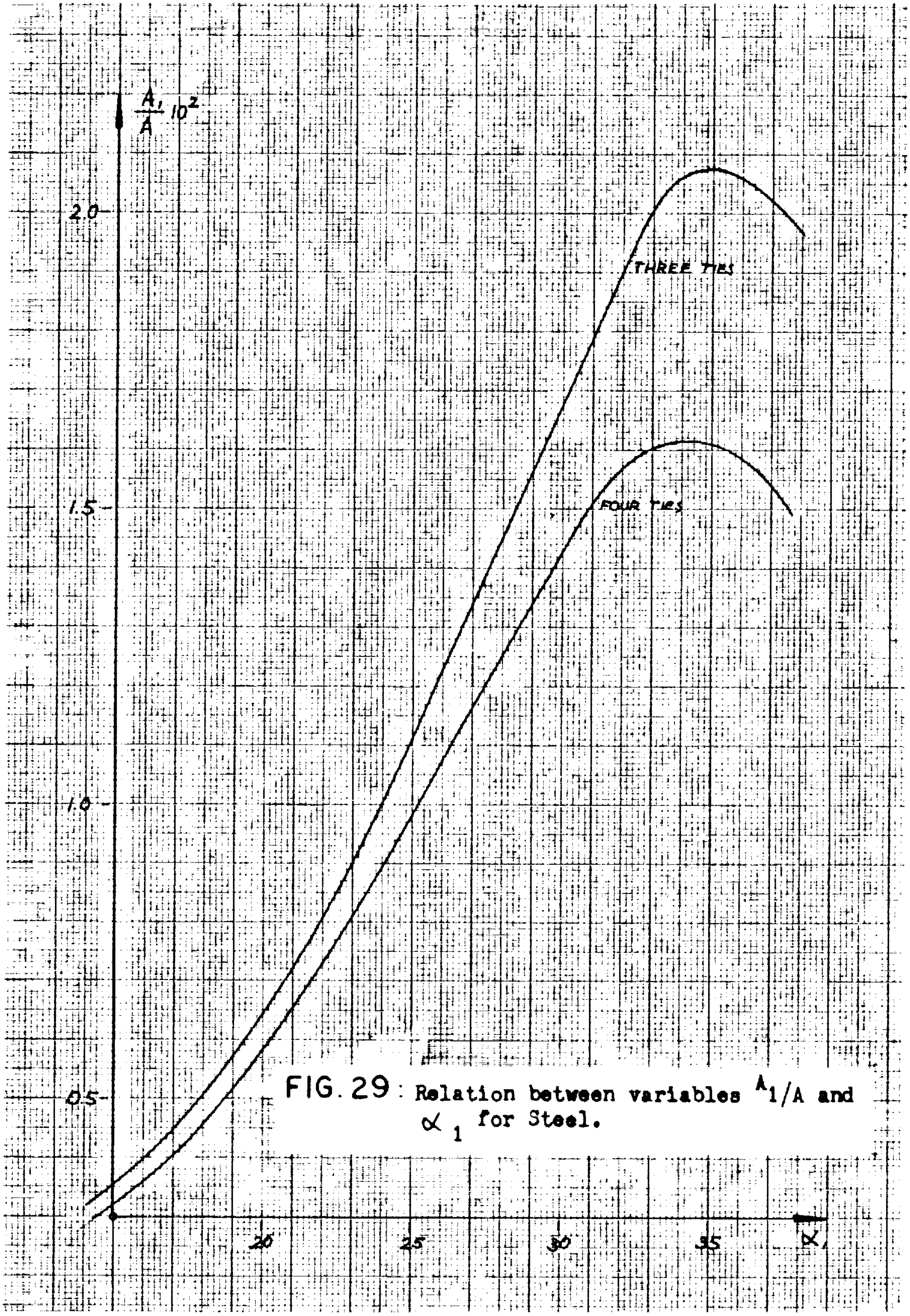


FIG. 29: Relation between variables A_1/A and α_1 for Steel.

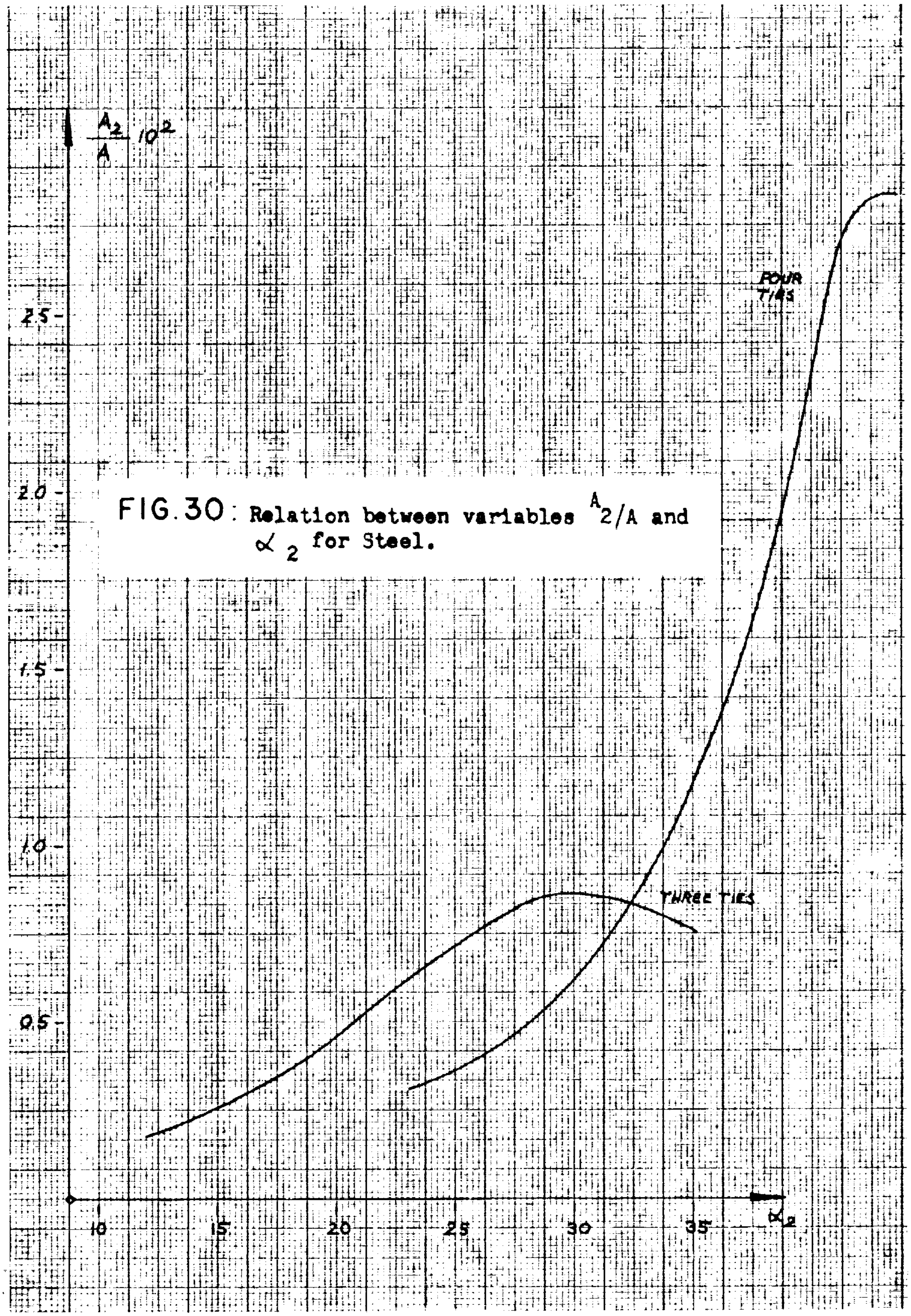


FIG. 30: Relation between variables A_2/A and α_2 for Steel.

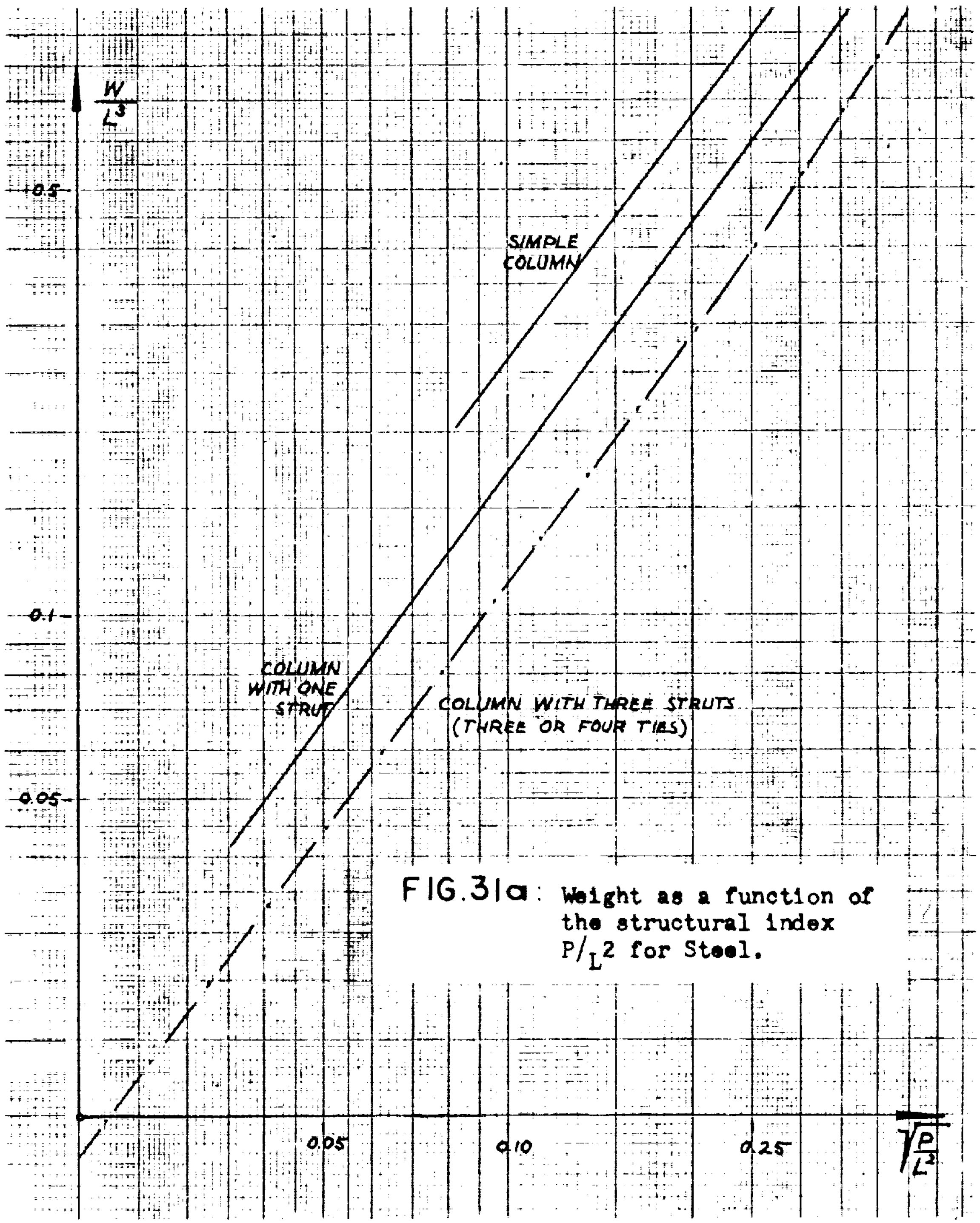


FIG.31a: Weight as a function of the structural index P/L^2 for Steel.

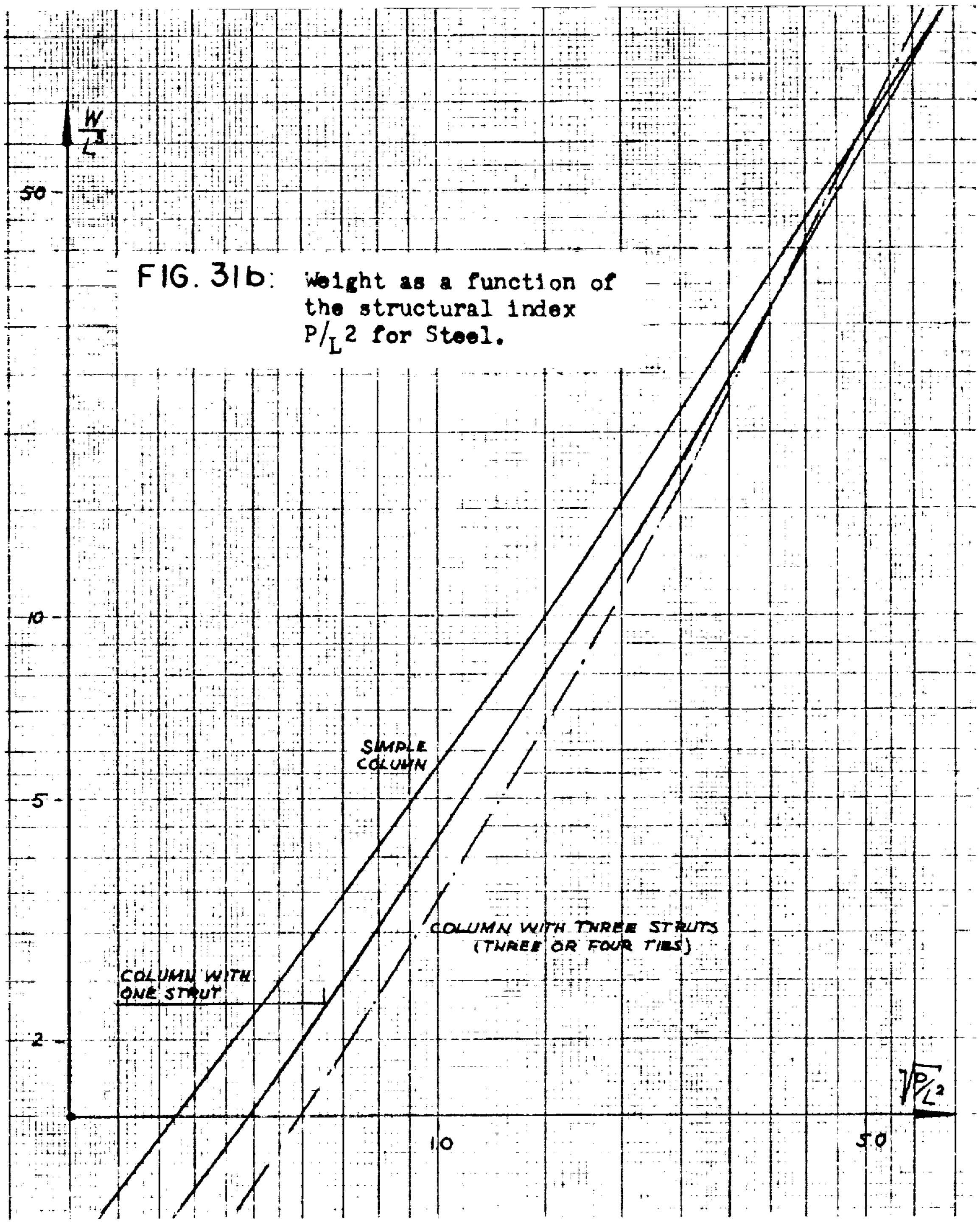


FIG. 31b: Weight as a function of the structural index $P/_{L}2$ for Steel.

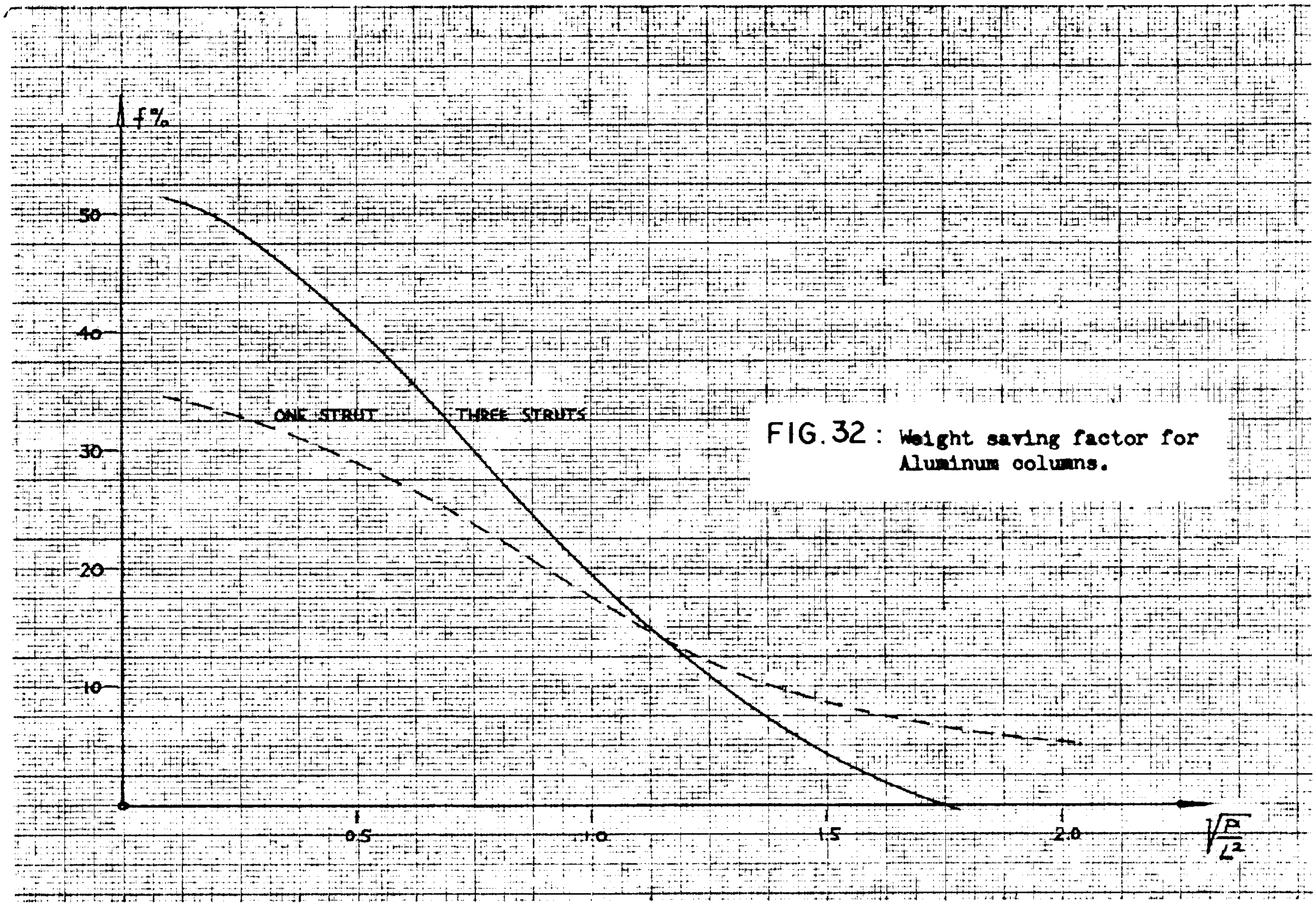


FIG. 32 : **Weight saving factor for Aluminum columns.**

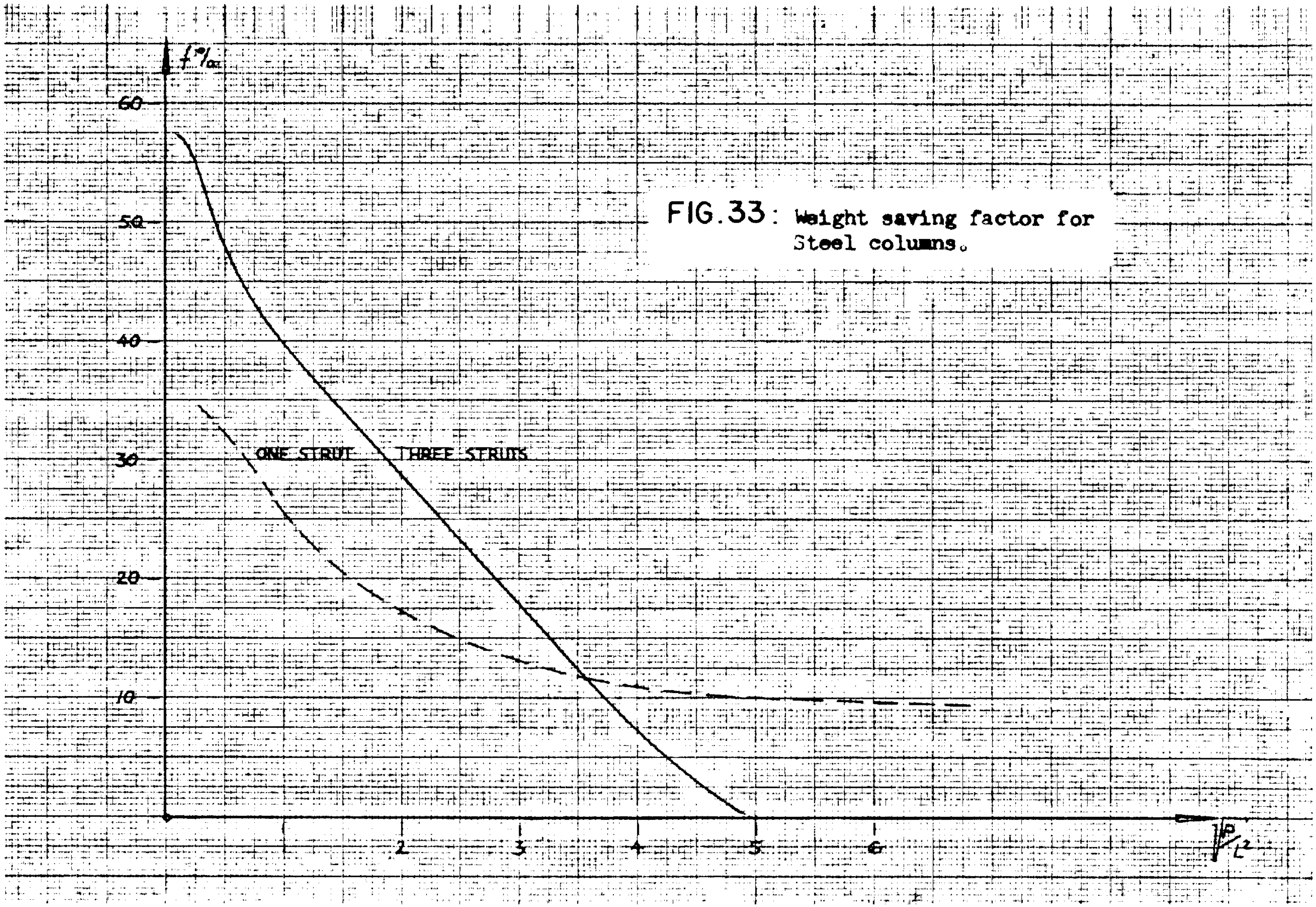


FIG. 33: **Weight saving factor for Steel columns.**

APPENDIX II

Computer Programs and Results

```
     $JOB   CC694G 002 015 HAGEN MAUCH     BOOM PROBLEM  OPTIMUM ANGLE
            $EXECUTE        IBJOB
IBJOB VERSION   4
$IBJOB BOOM
$IBFTC 3TIES
     BEGIN COMPILATION     0 M 11 S
   98 FORMAT (1H1)
   99 FORMAT (F10.9)
  100 FORMAT (1H0,10X,F10.7,10X,F6.3,10X,F6.3,10X,F10.5,10X,F10.5 )
      TOL=0.00001
      WRITE (6,98)
    1 READ (5,99)F
       B1= 0.3456367
      B2=3.0*B1/2.0
       B3=1.131887
      B4= 0.17535375
      B5=2.0*B3/3.0
      A=2.0/3.0
      B=1.0/3.0
      X=0.1
   10 XA= (1.0-X*X*X*X)/(X*X*X)
      X1= B1*F*SQRT(F)*XA*SQRT(XA)-X
      X2=1.0+B2*F*SQRT(F)*SQRT(XA)*(1.0+3.0/(X*X*X*X))
      Y=-X1+1.0/(X1*X1*X1)-B3*X1**A/F-B4*(1.0/(X*X*X)-X)
      Y1=X2+3.0*X2/(X1*X1*X1*X1)+B5*X2/(F*X1**B)+B4*(1.0+3.0/(X*X*X*X))
      XA1=X-Y/Y1
      IF (ABS(XA1-X).LT.TOL) GO TO 500
      X=XA1
      GO TO 10
  500 A1=ATAN(X1)*180.0/3.14159
      A2=ATAN(X)*180.0/3.14159
      A3= ATAN(X1)
      A4= ATAN(X)
      C1=1.0/(SIN(A3)*SIN(A3)*COS(A3))
      C2=1.0/(SIN(A4)*SIN(A4)*COS(A4))
      WRITE (6,100) F,A1,A2,C1,C2
      GO TO 1
      END
     BEGIN ASSEMBLING   3TIES     0 M 17 S
```

```
     $JOB  CC694H 002 015 HAGEN MAUCH     BOOM PROBLEM  OPTIMUM ANGLE
$EXECUTE        IBJOB
$IBJOB BOOM
$IBFTC 4TIES
     BEGIN COMPILATION     0 M 09 S
   98 FORMAT (1H1)
   99 FORMAT (F10.9)
  100 FORMAT (1H0,10X,F10.7,10X,F6.3,10X,F6.3,10X,F10.5,10X,F10.5 )
      TOL=0.00001
      WRITE (6,98)
    1 READ (5,99)F
      B1= 9.025389
      B2=3.0*B1/2.0
      B3= 1.547667
      B4= 1.092845
      B5=2.0*B3/3.0
      A=2.0/3.0
      B=1.0/3.0
      X=0.1
   10 XA= (1.0-X*X*X*X)/(X*X*X)
      X1= B1*F*SQRT(F)*XA*SQRT(XA)-X
      X2=1.0+B2*F*SQRT(F)*SQRT(XA)*(1.0+3.0/(X*X*X*X))
      Y=-X1+1.0/(X1*X1*X1)-B3*X1**A/F-B4*(1.0/(X*X*X)-X)
      Y1=X2+3.0*X2/(X1*X1*X1*X1)+B5*X2/(F*X1**B)+B4*(1.0+3.0/(X*X*X*X))
      XA1=X-Y/Y1
      IF (ABS(XA1-X).LT.TOL) GO TO 500
      X=XA1
      GO TO 10
  500 A1=ATAN(X1)*180.0/3.14159
      A2=ATAN(X)*180.0/3.14159
      A3= ATAN(X1)
      A4= ATAN(X)
      C1=1.0/(SIN(A3)*SIN(A3)*COS(A3))
      C2=1.0/(SIN(A4)*SIN(A4)*COS(A4))
      WRITE (6,100) F,A1,A2,C1,C2
      GO TO 1
      END
     BEGIN ASSEMBLING  4TIES     0 M 15 S
```

OPTIMUM ANGLES
4 TIES

F	α_1	α_2	C_1	C_2
0.0129780	14.134	22.542	17.29486	7.36697
0.0259470	16.902	26.412	12.36414	5.64280
0.0389840	18.736	28.798	10.23445	4.91754
0.0519790	20.126	30.497	8.99554	4.50605
0.0649740	21.256	31.806	8.16417	4.23608
0.0779690	22.214	32.859	7.55734	4.04385
0.0909640	23.041	33.733	7.09375	3.89926
0.1039580	23.780	34.472	6.72125	3.78635
0.1299440	25.039	35.666	6.16167	3.62071
0.1621370	26.319	36.787	5.67525	3.48194
0.2002050	27.564	37.786	5.26818	3.37045
0.2400850	28.646	38.588	4.95817	3.28862
0.2836560	29.649	39.272	4.70196	3.22373
0.3327650	30.608	39.880	4.48184	3.16969
0.3907780	31.565	40.445	4.28308	3.12240
0.4728760	32.698	41.053	4.07197	3.07439
0.6385090	34.427	41.881	3.79303	3.01372
1.1095184	37.344	43.030	3.41837	2.93783
0.0346330	18.185	28.100	10.80642	5.10986
0.0694240	21.601	32.191	7.93614	4.16344
0.1011460	23.628	34.322	6.79474	3.80867
0.1389500	25.420	36.012	6.00889	3.57614
0.1736860	26.723	37.120	5.53665	3.44347
0.2142690	27.965	38.093	5.14888	3.33841
0.2752590	29.469	39.153	4.74604	3.23476
0.4050280	31.783	40.564	4.24061	3.11276
0.6817000	34.793	42.041	3.73992	3.00257
1.0329250	36.992	42.906	3.45819	2.94559

OPTIMUM ANGLES
3 TIES

F	α_1	α_2	C_1	C_2
0.0129780	14.882	12.294	15.68581	22.57367
0.0259470	17.776	14.740	11.26676	15.97356
0.0389840	19.682	16.372	9.36313	13.11759
0.0519790	21.117	17.618	8.25863	11.45301
0.0649740	22.283	18.638	7.51672	10.33301
0.0779690	23.265	19.505	6.97716	9.51580
0.0909640	24.114	20.263	6.56364	8.88696
0.1039580	24.863	20.938	6.23475	8.38456
0.1299440	26.141	22.099	5.73864	7.62548
0.1621370	27.433	23.294	5.30829	6.96200
0.2002050	28.679	24.469	4.94910	6.40393
0.2400850	29.759	25.506	4.67571	5.97537
0.2836560	30.748	26.477	4.45147	5.62049
0.3327650	31.692	27.419	4.25824	5.31254
0.3907780	32.631	28.378	4.08385	5.03149
0.4728760	33.720	29.524	3.90126	4.73244
0.6385090	35.382	31.330	3.65838	4.33001
1.1095184	38.122	34.563	3.33539	3.77305
0.0346330	19.107	15.883	9.87672	13.88219
0.0694240	22.637	18.949	7.31393	10.02645
0.1011460	24.709	20.798	6.29970	8.48476
0.1389500	26.530	22.457	5.60207	7.41568
0.1736860	27.838	23.674	5.18584	6.77239
0.2142690	29.083	24.854	4.84320	6.23838
0.2752590	30.571	26.301	4.48990	5.68188
0.4050280	32.838	28.592	4.04754	4.97257
0.6817000	35.730	31.722	3.61239	4.25237
1.0329250	37.798	34.157	3.36919	3.83347

```
     $JCB   CC694G CC2 015   HAGEN MAUCH          BCCM PROBLEM     WEIGHT
$EXECUTE          IBJCB
$IBJCB BCCM
$IBFTC MTWB
     BEGIN COMPILATION      0 M 05 S
   98 FCRMAT (1H1)
   99 FCRMAT (F14.0,F16.C,F1C.C,I5,F12.0)
  100 FCRMAT (1H0,1CX,F5.1,F18.9,F1C.3,F10.3,F15.5)
      C1=5.0/3.0
      C2=2.0/3.0
      WRITE (6,98)
   10 READ (5,99) A6,A5,A1,N,A2
      GC TC (20,21,22,23),N
   20 B1=0.13076098
      B2=0.14278254
      B3=0.098999204
      B4=7.2279847
      B5=1.3440432
      B6=0.283
       GC TC 30
   21 B1=0.35248588
      B2=0.38483295
      B3=0.1927460
      B4=7.2279847
      B5=1.3440432
      B6=0.105
       GC TC 30
   22 B1=0.13078916
      B2=0.02293438
      B3=0.079197485
      B4=6.318396
      B5=3.1525151
      B6=0.283
       GC TC 30
   23 B1=0.35250792
      B2=0.618136095
      B3=0.15338577
      B4=6.318396
      B5=3.1525151
      B6=0.105
   30 A3=A1*3.14159/180.0
      A4=A2*3.14159/180.0
       G1= SIN(A3)
       G2= SIN (A4)
      G3=SQRT(1.0-G1*G1)
      G4=SQRT(1.0-G2*G2)
       F1=1000.0*G1*G1*(1.0-G1*G1)
       F2=1000.0*G2*G2*(1.0-G2*G2)
      F3=2.0*(G1/G3)**C1*(B4*A5/1000.0)**C2
      F4=(G1/G3+G2/G4)**C1*(B5*A5/1000.0)**C2
       XA=1000.0*B6*(A5*(1.0/A6+B1/F1+B2/F2)+B3*(F3+F4)/10.0)
       WRITE (6,100)A6,A5,A1,A2,XA
       GC TC 10
       END
    BEGIN ASSEMBLING  MTWB     0 M 11 S
```

WEIGHT

3 TIES

	σ [KSI]	P/L^2	α_1	α_2	$W/L^3\ 10^6$
STEEL	5.0	0.000061870	14.882	12.294	0.00360
	10.0	0.000494309	17.776	14.740	0.01453
	15.0	0.001676686	19.682	16.372	0.03312
	20.0	0.003974363	21.117	17.618	0.05931
	25.0	0.007762499	22.283	18.636	0.09327
	30.0	0.013413750	23.265	19.505	0.13512
	35.0	0.021300599	24.114	20.263	0.18494
	40.0	0.031795613	24.863	20.938	0.24282
	50.0	0.062100619	26.141	22.099	0.38314
	60.0	0.120624997	27.433	23.294	0.62492
	70.0	0.227100000	28.679	24.469	1.01479
	80.0	0.391629994	29.759	25.506	1.54052
	90.0	0.645901203	30.748	26.477	2.27108
	100.0	1.042856202	31.692	27.419	3.31680
	110.0	1.688849986	32.531	28.378	4.90421
	120.0	2.992462486	33.720	29.524	7.98654
	130.0	7.366980970	35.382	31.330	18.12055
	135.0	38.654049873	38.122	34.563	90.46618
ALUMINUM	5.0	0.000437498	19.107	15.683	0.00997
	10.0	0.003521100	22.637	18.949	0.04153
	15.0	0.011883698	24.709	20.798	0.09616
	20.0	0.028168693	26.530	22.457	0.17504
	25.0	0.055017494	27.838	23.674	0.27765
	30.0	0.103293695	29.083	24.854	0.44580
	35.0	0.218987495	30.571	26.301	0.82036
	40.0	0.697662495	32.838	28.592	2.29036
	45.0	3.326362491	35.730	31.722	9.65565
	50.0	11.572343588	37.798	34.157	30.30718

4 TIES

	σ [KSI]	P/L^2	α_1	α_2	$W/L^3\ 10^6$
STEEL	5.0	0.000061870	14.134	22.542	0.00367
	10.0	0.000494309	16.902	26.412	0.01462
	15.0	0.001676686	18.736	28.798	0.03357
	20.0	0.003974363	20.126	30.497	0.06027
	25.0	0.007762499	21.256	31.806	0.09501
	30.0	0.013413750	22.214	32.859	0.13797
	35.0	0.021300599	23.041	33.733	0.18927
	40.0	0.031795613	23.780	34.472	0.24904
	50.0	0.062100619	25.039	35.666	0.39464
	60.0	0.120624997	26.319	36.787	0.64619
	70.0	0.227100000	27.564	37.786	1.05320
	80.0	0.391629994	28.646	38.588	1.60469
	90.0	0.645901203	29.649	39.272	2.37419
	100.0	1.042856202	30.608	39.880	3.47961
	110.0	1.688849986	31.565	40.445	5.16267
	120.0	2.992462486	32.698	41.053	8.43517
	130.0	7.366980970	34.427	41.661	19.19617
	135.0	38.654049873	37.344	43.030	95.92616
ALUMINUM	5.0	0.000437498	18.185	28.100	0.00972
	10.0	0.003521100	21.601	32.191	0.04013
	15.0	0.011883698	23.628	34.322	0.09220
	20.0	0.028168693	25.420	36.012	0.16701
	25.0	0.055017494	26.723	37.120	0.26551
	30.0	0.103293695	27.965	38.093	0.42168
	35.0	0.218987495	29.469	39.153	0.77459
	40.0	0.697662495	31.783	40.564	2.16637
	45.0	3.326362491	34.793	42.041	9.16917
	50.0	11.572343588	36.997	42.906	28.81984