UNIVERSITY OF CALIFORNIA
Los Angeles

Optimum Design of Columns Supported by Tension Ties

A thesis submitted in partial satisfaction of the requirements for the degree Master of Science in Engineering

by

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NOMENCLATURE

$A_c$  cross sectional area of the column

$A_j$  cross sectional area of a tie in the $j^{th}$ panel

$B$  as defined in equation (24)

$C_1$  as defined in equation (88)

$C_2$  as defined in equation (89)

$D$  column outside diameter

$E_c$  modulus of elasticity of the column

$E_{ct}$  tangent modulus of elasticity of the column

$E_j$  modulus of elasticity of a tie in the $j^{th}$ panel

$F$  as defined in equation (146)

$I_c$  moment of inertia of the column

$L$  length of the column

$\Delta L_o$  change in length due to a load $P_o$

$\Delta L_p$  change in length due to a load $P_p$

$P$  applied axial load on the column in $x$-direction

$P_o$  force in the column due to initial tension

$R_j$  lateral reaction at joint $j$

$R_{jj}$  lateral reaction at joint $j$ due to the ties in the panel $j$

$T_j$  tension in a tie in the $j^{th}$ panel when $P$ is acting

$T_{0j}$  initial tension in a tie in the $j^{th}$ panel

(no external load)
NOMENCLATURE
(continued)

$t_j^a, t_j^b$ if two ties lie between two struts
(see Fig. 14)

$U$ strain energy

$U_1$ spring strain energy

$W$ weight of the whole system (column, wires and strut assemblies)

$W_1$ work done by the external load $P$

$Z$ as defined in equation (144)

$a_k$ factor for the $k^{th}$ term in the Fourier series expressing the deflection of the column

$c_j$ length of the tie between joints $j-1$ and $j$

$h_j$ length of typical strut in the $j^{th}$ panel

$k_2$ local buckling factor chosen as 0.4

$m$ number of tension ties

$t$ wall thickness of the column

$w$ specific weight of the column material

$w_j$ specific weight of the tie in the $j^{th}$ panel

$x$ coordinate in axial direction

$y$ coordinate in lateral direction
NOMENCLATURE
(continued)

\( a_j \) \hspace{1cm} \theta_j - \tau/2

\( \beta_j \) \hspace{1cm} angle between the plane of a tie i and the deflection plane

\( \bar{\kappa} \) \hspace{1cm} parameter as defined in equation (55)

\( \kappa \) \hspace{1cm} parameter defined as \( \bar{\kappa}/A \)

\( \lambda \) \hspace{1cm} difference between the length of the cord and the deflection curve

\( \theta \) \hspace{1cm} angle defined in Fig. 2

\( \theta \) \hspace{1cm} as defined in equation (101)

\( \sigma_c \) \hspace{1cm} compression stress in the column

\( \tau \) \hspace{1cm} defined as \( E_{ct}/E_c \)

\( \xi \) \hspace{1cm} coordinate as defined in Fig. 5

\( \eta \) \hspace{1cm} coordinate as defined in Fig. 5
When optimizing simple thin-walled columns on a weight basis, the maximum obtainable stress is found to be that at which local and general buckling failure occur simultaneously. This stress can be expressed as a function of load and distance, allowing the introduction of the
structural index $P/L^2$ ($P =$ buckling load, $L =$ length of the column), for equal values of which all dimensionally similar columns develop the same stress at failure. At low values of the structural index, the optimum stress is low, indicating that the simple column is not an efficient structure in such circumstances. It has been found that expansion of the cross-section, for example, by using tension ties which serve to provide intermediate elastic support for the column, allows the column to operate at higher stress levels, thereby increasing efficiency.

This thesis summarizes a method of analysis and presents a procedure for optimizing tension-tie supported thin-walled cylindrical columns. The optimized column for a given structural index is defined by a particular diameter, wall thickness, tie prestress, tie cross-sectional area, tie angle, and strut dimensions. For the cases considered it is found that, in the low range of the structural index, the tie supported column offers a potential weight saving of up to 50% over the simple tubular column.
SECTION 1
INTRODUCTION

The primary function of a structure is to transmit forces through space, where, from the designer's point of view, the objective often is to do this with the minimum possible weight. For any structure that fails as a result of instability under compressive loading, the maximum obtainable stress depends, in a complex manner, on the properties of the material and the geometric properties of the structure. To apply the principle of dimensional similarity, the structural index \( P/L^2 \) is introduced (\( P \) = buckling load, \( L \) = column length). This quantity can be considered as a measure of the loading intensity. All dimensionally similar columns having the same value of structural index will develop the same stress at failure. Therefore, for a particular material and a particular type of cross-section, an easily obtainable relationship between optimum stress and structural index constitutes the information needed for the design of the entire family of minimum weight simple columns.

At low values of the structural index, the optimum stress is far below the elastic limit of most structural materials and an expansion of the cross-section will allow the column to operate at a higher stress level, thereby possibly increasing efficiency, from a weight standpoint. In the age of space technology the long column with small
compression load becomes more and more interesting and, considering the extremely high cost per pound of orbited load, even the smallest weight savings is appreciated.

The thin-walled circular tube, which is the most efficient simple column, is chosen as a basis for the investigation. The weight of the simple column is a minimum when the allowable stress \( \sigma \) is a maximum. To find the maximum values of \( \sigma \) both primary buckling (Engesser formula) and local buckling are considered and the optimum design is obtained when both failures can occur simultaneously.

Using two equal columns and equipping one with tension ties always results in an increase of maximum stress, hence a decrease in weight of the central tube is possible; however, additional weight is added in conjunction with ties and struts. Once the structural index \( P/L^2 \) is specified, the problem is then to define the parameters associated with the tie supported column like column diameter, wall thickness, tie cross-sectional area and tie angle \( \alpha \) to obtain minimum weight. The column supported by tension ties considered herein consists of three parts: a thin walled tube with circular cross-section, tension ties, and struts, as shown in Fig. 1. The theory of analyzing such columns has been developed previously [1] and the improved efficiency is proved in tests. [1] Nevertheless in none of the solutions was optimization of the structure with respect to weight attempted, as has been done for the simple
column. [2]

For the actual calculations the following assumptions are made:

a. The effect of the deformation of the struts is negligible.

b. The connection between the struts and the column and the connection between the ties and struts are ideal hinges.

c. There is no initial eccentricity or crookedness in the column.

d. There is no lateral deflection before buckling.

e. The pretension in the wires is of such magnitude that at impending buckling the wires are stress free.

f. For small lateral deflection the axial deformation is negligible.

g. The angles between the planes of the ties are equal.

h. The struts are distributed symmetrically with respect to the midpoint of the column.

The optimum design can again be found by equating the primary and local buckling stresses. The weight of the column is a function of the above mentioned parameters and a minimization yields optimum values of these parameters.

Comparing the optimum weight of the simple column with the optimum weight of the column supported by tension
ties for identical values of the structural index will show how much more efficient this supported column can be.
SECTION 2

GENERAL THEORY OF THE COLUMN SUPPORTED BY TENSION TIES

2.1 Force and deformation relationship up to the instant of buckling

The following theory dealing with the mechanical behavior of the column supported by tension ties is based on Ref. 1 and is repeated here in a slightly modified form for completeness. The geometry of the supported column is sketched in Fig. 1 and Fig. 2.

2.1.1 Pretension

Let the pretension in a tie in the jth panel be denoted by \( T_{oj} \). If there is no external load applied in the x-direction, then the force in the column (\( P_0 \)) induced by pretension is

\[
P_0 = m T_{ol} \sin \theta_1
\]

(1)

where \( m \) is the number of tension ties.

Due to this load, the column has shortened a distance \( \Delta L_0 \).

\[
\Delta L_0 = \frac{m T_{ol} \sin \theta_1}{A_c E_c} L
\]

(2)

From the equilibrium of forces in x-direction at any joint, and by neglecting the effect of small angle changes \( \Delta \theta_j \),

\[
T_{oj} = T_{ol} \frac{\sin \theta_1}{\sin \theta_j}
\]

(3)
The elongation of the tie in the \(j\)th panel due to \(T_{oj}\) is therefore

\[
\Delta c_{oj} = T_{oj} \frac{c_j}{A_j E_j} = T_{ol} \frac{\sin \theta_1}{\sin \theta_j} \frac{c_j}{A_j E_j}
\]  

(4)

By neglecting again small angle changes and with the assumption that there is no lateral deflection before buckling starts, then the component of tie deflection in the \(x\)-direction is

\[
(\Delta c_{oj})_x = \Delta c_{oj} \sin \theta_j = T_{ol} \frac{\sin \theta_1}{A_j E_j} \frac{c_j}{A_j E_j}
\]  

(5)

and the total displacement in the \(x\)-direction is

\[
(\Delta c_o)_x = \sum_{j=1}^{n} \frac{T_{ol} \sin \theta_1}{A_j E_j} \frac{c_j}{A_j E_j}
\]  

(6)

2.1.2 Relationship between external load and internal forces

Let an external load \(P\) be applied to the strut with tension ties that are tightened to a certain value of initial tension. Then the force acting in the column will be increased by the amount \(P_p\) (Fig. 2) and the increase in axial deformation will be

\[
\Delta L_p = P_p \frac{L}{A_c E_c}
\]  

(7)

The force acting in a tie in the \(j\)th panel is decreasing by the amount \(T_{pj}\). Similar to equation (3) this force is
\[ T_{pj} = T_{pl} \frac{\sin \theta_1}{\sin \theta_j} \] (8)

The decrease in stretch in the axial direction due to \( T_{pj} \) is similar to that given by equation (6).

\[ (\Delta \alpha_p)_x = \sum_{j=1}^{n} T_{pl} \sin \theta_1 \frac{C_j}{A_j E_j} \] (9)

Since \( \Delta L_p = (\Delta \alpha_p)_x \), equations (7) and (9) give

\[ P_p = \frac{A_c E_c}{L} T_{pl} \sin \theta_1 \sum_{j=1}^{n} \frac{C_j}{A_j E_j} \] (10)

By taking the summation of the forces in a section at the end of the strut (Fig. 3) and considering the fact that the pretension forces in the column and tension ties are in equilibrium regardless of the applied force \( P \), it is found that

\[ P = m T_{pl} \sin \theta_1 + P_p \] (11)

Substituting \( P_p \) from equation (10) in equation (11) yields

\[ T_{pl} = \frac{P}{\sin \theta_1 \left( m + \frac{A_c E_c}{L} \sum_{j=1}^{n} \frac{C_j}{A_j E_j} \right)} \] (12)

With the assumption that at the onset of buckling the tension ties are stress free, it follows that equations (12)
and (3) must be equal, or simply

\[ T_{ol} = T_{pl} \]  \hfill (13)

### 2.2 Lateral reactions produced by tension ties at buckling load

Change in length and slope of the tension ties due to lateral deflection have to be considered next. Let the displacement of joint \( j \) in the direction perpendicular to the column and in the plane of column and tie be denoted by \( (\Delta y)_j \). Then, as shown in Fig. 4 for small displacements, changes in length of the ties may be expressed as

\[
\Delta c_j = \left[ (\Delta y)_{j-1} - (\Delta y)_j \right] \cos \theta_j + \left[ (\Delta x)_{j-1} - (\Delta x)_j \right] \sin \theta_j \]  \hfill (14)

This is only valid if \( \Delta \theta_j \neq 0 \).

When the lateral deflection starts, the axial deflection \( (\Delta x)_{j-1} - (\Delta x)_j \) is negligible and may be ignored. Furthermore, it is assumed that there is no lateral displacement at the end of the column and that the warping of the planes of the ties is negligible. Equation (14) can therefore be simplified to

\[
\Delta c_j = \left[ (\Delta y)_{j-1} - (\Delta y)_j \right] \cos \theta_j \]  \hfill (15)

and with

\[
\Delta T_j = \frac{A_i E_j}{c_j} \Delta c_j
\]
\[ \Delta T_j = \frac{A_j E_j}{C_j} [(\Delta y)_{j-1} - (\Delta y)_j] \cos \theta_j \] (16)

Now consider the case in which the deflection of the column occurs in the direction \( \delta \) (Fig. 4) in the \( x-\delta \) plane. Let \( (\Delta \delta)_j \) be the deflection of joint \( j \) in the \( \delta \) direction and \( (\Delta y)_{ji} \) be the components of \( (\Delta \delta)_j \) in the plane of column and tie \( i \) \( (i = 1, 2, 3, \ldots m) \). Let \( \beta_i \) be the angle between the plane of column and tie and the \( x-\delta \) plane. Then referring to Fig. 4

\[ (\Delta y)_{ji} = (\Delta \delta)_j \cos \beta_i \quad (i = 1, 2, 3, 4 \ldots m) \] (17)

Assuming that the angles between the planes of the ties are equal, then if \( \beta_1 = \beta \), it follows that

\[ \beta_2 = \frac{2\pi}{m} + \beta \] (18)
\[ \beta_3 = \frac{4\pi}{m} + \beta \]
\[ \ldots \]
\[ \beta_m = \frac{m-1}{m} 2\pi + \beta \]

Let the change in length of the tie \( i \) in the panel \( j \) be \( (\Delta c_j)_i \), then equation (15) may be written as follows

\[ (\Delta c_j)_i = (\Delta \delta_{j-1} - \Delta \delta_j) \cos \theta_j \cos \beta_i \] (19)
2.3 Lateral reactions when some ties are relaxed at the instant of buckling

If the ties are stress free at the instant of buckling, some ties are relaxed for an infinitesimal amount of buckling deflection. This gives some constraint to the initial tension which shall be considered later. With this assumption it is clear that ties with an angle $\beta_i$ defined by

$$\frac{\pi}{2} < \beta_i < \frac{3\pi}{2}$$

will be relaxed and the ties with an angle $\beta_i$ of

$$-\frac{\pi}{2} < \beta_i < +\frac{\pi}{2}$$

are going to be stretched.

For three or four tension ties only one or two ties, but not more than two ties can possibly lay in the region from $-\pi/2$ to $+\pi/2$. Therefore in those cases only a maximum of two tension ties can be stretched. Assume now first that tension tie $i = 1$ and $i = m$ are stretched. The perpendicular directions to the column axis in each of the planes containing these ties are called $y_1$ and $y_m$. (Fig. 5, Fig. 6, Fig. 7).

In the direction of $i$, the lateral components of the respective tensile forces produced by the changes in length of the ties are (Fig. 6)
\[(R_{jj})_{s1} = -\frac{A_j E_j}{c_j} (\Delta c_j)_{s1} \cos \beta \cos \theta_j\]

\[= -\frac{A_j E_j}{c_j} [(\Delta \delta)_{s1} - (\Delta \delta)_{j}] \cos^2 \theta_j \cos^2 \beta \quad (20)\]

\[(R_{jj})_{sm} = -\frac{A_j E_j}{c_j} [(\Delta \delta)_{s1} - (\Delta \delta)_{j}] \cos^2 \theta_j \cos^2 (\frac{m-1}{m} 2\pi + \beta) \quad (21)\]

where the angle $\beta_m$ is replaced by the expression calculated in equation (18).

In order to maintain equilibrium of forces in the $\eta$ direction, the following condition must be satisfied (Fig. 6)

\[(R_{jj})_{\eta l} = (R_{jj})_{\eta m} \quad (22)\]

with

\[(R_{jj})_{\eta l} = (R_{jj})_{s1} \tan \beta\]

\[(R_{jj})_{\eta m} = (R_{jj})_{sm} \tan (\frac{m-1}{m} 2\pi + \beta)\]

or

\[\sin \beta \cos \beta = \sin (\frac{m-1}{m} 2\pi + \beta) \cos (\frac{m-1}{m} 2\pi + \beta) \quad (23)\]

This equation can only be satisfied for

\[\beta = \pi/m\]

With this result the actual direction of deflection is known and this particular $s$ direction is denoted as $y$. (Fig. 5)
The reaction force $R_{jj}$ in this $y$ direction is given by

$$R_{jj} = (R_{jj})_y l + (R_{jj})_y m$$

$$= \frac{A_j E_j}{C_j} \left[ (\Delta y)_j - (\Delta y)_{j-1} \right] \cos^2 \theta_j B \quad (24)$$

where

$$B = 2 \cos^2 \beta/m = 2 \cos^2 (\pi/m)$$

If only one tension tie is stretched, there is only one case of equilibrium possible, namely when $\beta = 0$. In this case

$$R_{jj} = (R_{jj})_y l$$

$$= \frac{A_j E_j}{C_j} \left[ (\Delta y)_j - (\Delta y)_{j-1} \right] \cos^2 \theta_j B \quad (25)$$

where

$$B = 1$$

The lateral reactions are due to the action of the ties in the $j^{th}$ panel on strut plane $j$. The reaction on strut plane $j$ due to the ties in the $(j+1)^{th}$ panel is given by

$$R_{j(j+1)} = \frac{A_{j+1} E_{j+1}}{C_{j+1}} \left[ (\Delta y)_j - (\Delta y)_{j+1} \right] \cos^2 \theta_{j+1} B \quad (26)$$

Therefore the resultant reaction at joint $j$ is obtained by combining equations (24) or (25) with equation (26), as follows:
\[ R_j = R_{jj} + R_j(j+1) \]  \hspace{1cm} (27)

or

\[ R_j = \frac{A_j+1E_j+1}{C_j+1} \left[ (y)_j - (y)_{j+1} \right] \cos^2 \theta_j+1 \] 

\[ - \frac{A_jE_j}{C_j} \left[ (y)_{j-1} - (y)_j \right] \cos^2 \theta_j \] \hspace{1cm} (28)

where \( Ay \) is replaced by \( y \), for convenience.

2.4 Buckling theory

A strut with tension ties may be considered as a continuous beam on elastic intermediate supports as shown in Fig. 8. In the case of simply supported ends a representation of the deflection curve in the form of a trigonometric series is advantageous.

The deflection curve can be represented as

\[ y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} + \ldots \]

\[ a_k \sin \frac{k\pi x}{L} \] \hspace{1cm} (29)

Each term satisfies the boundary conditions, since each term together with its second derivative becomes zero at the end of the beam.

For the case of the column supported by tension ties, the coefficients in the above series, and the buckling load, are obtainable from an energy approach. The work done by the longitudinal force \( P \) must be equal to the work done
by the reaction force $R_j$ plus the strain energy. The strain energy is defined as

$$U = \frac{EcI_c}{2} \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$$

(30)

Introducing equation (28) and integrating yields

$$U = \frac{\pi^4 EcI_c}{4L^3} \sum_{k=1}^{k=\infty} k^4 a_k^2$$

(31)

Any change in the shape of the deflection curve results in some longitudinal displacement at the hinge $B$. This displacement is equal to the difference between the length of the deflection curve and the length of the chord $AB$ (Fig. 8). In terms of the chosen coefficients this distance is

$$\lambda = \frac{\pi^2}{4L} \sum_{k=1}^{k=\infty} k^2 a_k^2$$

(32)

and the work done by the external force $P$ is therefore

$$W_1 = \frac{P\pi^2}{4L} \sum_{k=1}^{k=\infty} k^2 a_k^2$$

(33)
The displacements at the struts are simply \((y)_j\) and the reaction forces \(R_j\) are linear functions of \((y)_j\) [equation (28)]. That means the spring characteristic is linear and the spring strain energy is

\[
U_1 = \sum_{j=1}^{n-1} \frac{1}{2} y_j R_j = \frac{1}{2} \sum_{j=1}^{n-1} R_j \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x_j}{L}
\]

For a conservative system

\[
W_1 = U + U_1
\]

Therefore with equations (31), (33), and (34)

\[
\frac{P\pi^2}{4L} \sum_{k=1}^{\infty} k^2 a_k^2 = \frac{4EIC}{4L^3} \sum_{k=1}^{\infty} k^4 a_k^{2+1/2} \sum_{j=1}^{n-1} R_j \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x_j}{L}
\]

(35)
SECTION 3

OPTIMUM DESIGN OF THE SIMPLE COLUMN

The theory of optimum design of the simple column is presented in [2] and the most important results are repeated here for convenience.

The Engesser formula for column buckling is

\[
\sigma_{cr} = \frac{\pi^2 E_t}{(L/\rho)^2}
\]

and may be expressed in terms of the structural index \(P/L^2\), using the proper value of \(\rho\) for a thin-walled circular tube

\[
\frac{P}{L^2} = \frac{8 \sigma_{cr}^2}{\pi E_t D/t}
\]

For local buckling

\[
\sigma_{cr} = k_2 \frac{\sqrt{E E_t}}{D/t}
\]

where \(k_2 \approx 0.4\).

Solving equation (38) for \(D/t\) and substituting \(\sigma_{cr}\) (i.e., \(\sigma_{cc} = \sigma_{cr}\)) from equation (37) yields an optimal value for \(D/t\):

\[
(D/t)_{opt} = 2 \left[ \frac{k_2}{\pi} \frac{2 E}{\rho} \right]^{\frac{1}{2}} \left[ \frac{P}{L^2} \right]
\]
Or, in terms of the structural index and the stress \( \sigma \), these equations can be combined to yield

\[
\frac{P}{L^2} = \frac{8 \sigma^3}{\pi k_2 E^2 \tau^{3/2}}
\]

(40)

where \( \tau = \frac{E_t}{E} \).

For any given value of \( \sigma \) from a particular stress-strain curve, the value of \( \frac{P}{L^2} \) may be calculated. The weight is obtainable from the relation

\[
\frac{W}{L^3} = \frac{P/L^2}{\sigma} w
\]

(41)

where \( w \) is the specific weight.
SECTION 4
OPTIMUM DESIGN OF A COLUMN WITH ONE STRUT

4.1 Buckling theory for the column with one strut

From the geometry of the system (Fig. 9) it is seen that

\[ \begin{align*}
\theta_1 &= \frac{\pi}{2} + \alpha \\
\cos^2 \theta_1 &= \sin^2 \alpha \\
\theta_2 &= \frac{\pi}{2} - \alpha \\
\cos^2 \theta_2 &= \sin^2 \alpha
\end{align*} \]

Consider the first three terms of the sine series

\[ y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} \quad (42) \]

and

\[ U = \frac{\tau^4 E_C I_C}{4 L^3} \left[ a_1^2 + 16 a_2^2 + 81 a_3^2 \right] \quad (43) \]

\[ W_1 = \frac{P \pi^2}{4 L} \left[ a_1^2 + 4 a_2^2 + 9 a_3^2 \right] \quad (44) \]

\[ U_1 = \frac{R}{2} \left[ a_1 - a_3 \right] \quad (45) \]

The reaction force is obtainable from equation (28). With the assumptions that \( A_1 = A_2, E_1 = E_2, \) and \( c_1 = c_2 = \frac{L}{2 \cos \alpha}, \) then

\[ R = \frac{4 A_1 E_1}{L} \left( a_1 - a_3 \right) \sin^2 \alpha \cos \alpha \quad (46) \]
Introducing equation (46) in (45)

\[
U_1 = \frac{2 A_1 E_1}{L} \sin^2 \alpha \cos \alpha \left[ a_1^2 - 2 a_1 a_3 + a_3^2 \right]
\]  

(47)

If any coefficient \( a_k \) in series (39) is given an increase \( \delta a_k \), the term \( (a_k + \delta a_k) \sin k x/L \) replaces the term \( a_k \sin k x/L \). This increase \( \delta a_k \) in the coefficient \( a_k \) represents an additional small deflection of the beam given by \( \delta a_k \sin k x/L \) superposed upon the original deflection curve.

The change in strain energy of the column, due to the increase \( \delta a_k \) is

\[
\frac{\partial U}{\partial a_k} \delta a_k = \frac{\pi^4 E I}{2 L^3} k^4 a_k \delta a_k
\]  

(48)

The change in work done by the compression load is

\[
\frac{\partial W_1}{\partial a_k} \delta a_k = \frac{p}{2L} k^2 a_k \delta a_k
\]  

(49)

and the change in strain energy of the springs is

\[
\frac{\partial U_1}{\partial a_k} \delta a_k = \frac{2 \sin^2 \alpha \cos \alpha A_1 E_1 B}{L} \frac{\partial}{\partial a_k} \left( a_1^2 - 2a_1 a_3 + a_3^2 \right)
\]  

(50)

For a conservative system

\[
\frac{\partial W_1}{\partial a_k} \delta a_k = \frac{\partial U}{\partial a_k} \delta a_k + \frac{\partial U_1}{\partial a_k} \delta a_k
\]
For $K = 1, 2, 3$ this results in three equations

\[
\frac{P\pi^2}{2L} a_1 = \frac{\pi^4 E IC}{2L^3} a_1 + \frac{4\sin^2 \alpha \cos \alpha A_1 E_1 B}{L} (a_1 - a_3)
\]

(51)

\[
4 \frac{P\pi^2}{2L} a_2 = \frac{\pi^4 E IC}{2L^3} 16 a_2
\]

(52)

\[
9 \frac{P\pi^2}{2L} a_3 = \frac{\pi^4 E IC}{2L^3} 81 a_3 + \frac{4\sin^2 \alpha \cos \alpha A_1 E_1 B}{L} (a_3 - a_1)
\]

(53)

Equation (52) states that the second buckling mode of the simple column is a solution of this column supported by tension ties. It can be seen that equations (51) and (53) can only be satisfied if the following determinant is zero.

\[
\begin{bmatrix}
\frac{\pi^4 E IC}{L^2} - P\pi^2 + \tilde{\kappa} & - \tilde{\kappa} \\
- \tilde{\kappa} & 81 \frac{\pi^4 E IC}{L^2} - 9 P\pi^2 + \tilde{\kappa}
\end{bmatrix}
= 0
\]

(54)

where

\[
\tilde{\kappa} = 8 A_1 E_1 B \sin^2 \alpha \cos \alpha
\]
To extend the theory in the plastic region, the modulus of elasticity of the column is replaced by the tangent modulus. Considering, furthermore, local buckling, it can be shown that the moment of inertia $I_c$ must be replaced by the expression [2]

$$I_c = \frac{E}{A} = k_1 \frac{P}{\sigma} A = \frac{1}{8\pi} k_2 \frac{\sqrt{E t}}{2} P A \quad (56)$$

where, as for the simple column, $k_2 = 0.4$.

Introducing equation (56) in the above determinant, and replacing $E_c$ with the tangent modulus $E_t$ yields

$$\begin{bmatrix}
\frac{\pi^3 k_2}{8\sigma^2} \tau^{3/2} E^2 \left(\frac{P}{L^2}\right) - \sigma^2 + \kappa & -\kappa \\
-\kappa & 81 \frac{\pi^3 k_2}{8\sigma^2} \tau^{3/2} E^2 \left(\frac{P}{L^2}\right) - 9\sigma^2 + \kappa
\end{bmatrix} = 0 \quad (57)
$$

where $\tau = E_t/E$ and $\kappa = \frac{\kappa}{A} = \frac{A_1 E_1}{A} B \sin^2 \alpha \cos \alpha$.

Expanding equation (57) yields

$$\left(\frac{P}{L^2}\right) = \frac{8\sigma^2}{81 \pi^3 k_2 \tau^{3/2} E^2} (45\sigma^2 - 41\kappa) \quad (58)$$

$$\pm \sqrt{16 \cdot 81 \pi^4 \sigma^2 - 90 \cdot 32\kappa \sigma^2 + 41\kappa^2}$$
This equation should reduce to the solution for the simple column in the limit. For the simple column the cross section of the tension ties is zero, so that $\kappa$ takes the value zero. Substituting $\kappa = 0$ in equation (58) yields

$$\frac{P}{L^2} = \frac{8\sigma^3}{9k_2\tau^{3/2}E^2} (5 \pm 4)$$  \hspace{1cm} (59)

Considering the positive sign in the bracket, the buckling formula for the first mode of the simple column (equation (40)) is obtained. The negative sign yields the buckling load for the third mode shape which is not critical. The tension ties increase the buckling force in all modes, so that the positive sign in equation (58) has to be chosen to obtain the critical load.

For cylindrical tubes the factor $k_2$ is chosen as 0.4. $\kappa$ has the same dimensions as the structural index and can be considered as something like a structural index of the elastic support of the column and serves as a parameter in further calculations.

Equations (51) and (53) indicate that the buckling shape resulting from the influence of the tension ties is a combination of the natural first and third buckling mode shape. This buckling mode shall be called "constrained first buckling mode."

Equation (58) can be written in simplified form as
To observe the influence of the tension ties, the "constrained first buckling mode" is calculated for an actual case for some typical values of \( \kappa \) assuming a column constructed of 2024-T4 aluminum alloy with material properties as shown in Fig. 10. The result is sketched in Fig. 11 and indicates that for various values of \( P/L^2 \) different values of \( \kappa \) result in curves which intersect the curve plotted for the second buckling mode shape. Since at all of these points two modes of failure occur simultaneously, the optimum design is reached when \( \kappa \) as a function of \( P/L^2 \) is chosen such that the graphs for the constrained first buckling mode shape and the second buckling mode shape are identical. This relationship can be found, when the structural index for the second mode is introduced in the left hand side of equation (60). The second buckling mode shape is given in equation (52) for the following structural index

\[
\frac{P}{L^2} = \frac{2\sigma^3}{\pi k_2 E^2 \tau^{3/2}}
\]  
(61)
Combining equations (60) and (61) yields

\[ \kappa = \frac{45}{56} \sigma \pi^2 = 7.93091 \sigma \] (62)

This result is independent of the properties of the chosen material.

4.2 Weight assumptions

From Fig. 12 it can be seen that the weight of the column with one strut and \(m\) tension ties is the summation of

- \(W_1\) - the weight of the column
- \(W_2\) - the weight of the supporting wires
- \(W_3\) - the weight of the strut assembly (struts and connecting ring)

4.2.1 Weight of the column

As given in equation (41) the weight of the simple column is

\[ \frac{W_1}{L^3} = \frac{w P}{L^3} \] (63)

4.2.2 Weight of the wires

\[ W_2 = m w_1 A_1 L / \cos \alpha \text{ or } \frac{W_2}{L^3} = \frac{m w_1 T_1}{\sigma_1 \cos \alpha L^2} \] (64)
\( T_1 \) must be taken as the highest possible tie force. From previous calculations the tie forces are given in the form of equation (16)

\[
\Delta T_j = T_j = \frac{A_j E_j}{c_j} \left[ (\Delta y)_j - (\Delta y)_{j-1} \right] \cos \theta_j
\]  

For the case of only one strut, there is only one possible tensile stress, because \( \Delta y_0 \) and \( \Delta y_2 \) are zero.

For \( x = L/2 \) the sine series for the deflection gives

\[
y_1 = (a_1 = a_3)
\]

Again \( \Delta y \) is replaced by \( y \) and the angle \( \theta \) by \( \alpha \). The length of tie 1 is expressed in terms of \( L \). Introducing now the value of \( y_1 \) in equation (65) yields

\[
\sigma_1 = \frac{2E_1 \cos \alpha \sin \alpha}{L} (a_1 - a_3)
\]  

This equation gives a constraint for the maximum value of \( \sigma \) in the tension ties. For convenience the weight function for the tension ties is written in the form

\[
\frac{W_2}{L^3} = \frac{m w_1}{E_1 \cos \alpha} \left( \frac{E_1 A_1}{L^2} \right)
\]  

4.2.3 **Weight of the strut assembly**

The struts are assumed to be simple columns, welded on a ring with a weight \( W_4 \), which for simplification is chosen to be the same as the weight of one of the struts. The \( m \) struts at the midpoint of the column are
assumed to fail as simply supported columns. Assuming furthermore that buckling occurs elastically

$$\frac{R}{h^2} = \frac{8\sigma^3}{\pi k_2 E^2} \quad \text{and} \quad \sigma^3 = \frac{\pi k_2 E^2 R}{8 h^2} \quad (68)$$

with $\tan \alpha = \frac{2h}{L}$ it follows that $\frac{1}{h^2} = \frac{4}{L^2 \tan^2 \alpha}$

Therefore

$$\sigma^3 = \frac{\pi k_2 E^2 R}{2 \tan^2 \alpha L^2} \quad (69)$$

The weight of the strut assembly is now

$$W_3 = (m+1) w A h = (m+1) w \frac{R}{\sigma} \frac{L \tan \alpha}{2}$$

and

$$\frac{W_3}{L^3} = \frac{m+1}{2} w \frac{\tan \alpha}{\sigma} \left( \frac{R}{L^2} \right)^{2/3} \quad (70)$$

Introducing equation (69) for the optimum stress yields

$$\frac{W_3}{L^3} = \frac{m+1}{2} w (\tan \alpha)^{5/3} \left( \frac{R}{L^2} \right)^{2/3} \left[ \frac{2}{\pi k_2 E^2} \right]^{1/3}$$

where $R$ is the reaction force given in equation (46). The specific weight of the strut assembly is assumed to be the same as for the column.

Equations (63), (67), and (69) combined give the total weight as
\[
\frac{w}{L^3} = \frac{w(P)}{\sigma L^2} + \frac{m}{E_1 \cos \alpha} \left( \frac{A_1 E_1}{L^2} \right) + \frac{m+1}{2} w(tan \alpha)^{5/3} \left( \frac{R}{L^2} \right)^{2/3} \left[ \frac{2}{\pi k_x E^2} \right]^{1/3}
\]  
(71)

Rewriting the equation for the reaction force \( R \) yields

\[
\frac{R}{L^2} = \frac{4}{L^3} \frac{A_1 E_1}{L^3} (a_1 - a_3) \sin^2 \alpha \cos \alpha B
\]

At the onset of buckling, the theory gives only the buckling shape, but does not specify a fixed magnitude of deflection. It is assumed that the struts buckle when \( y/L \) reaches a value given by

\[
\frac{y}{L} = \frac{1}{150}
\]  
(72)

After completing the calculation it must be ascertained whether or not the tension ties are stressed to a value which is below the yield stress. To do so, equation (66) must be used.

With the above assumptions for \( y/L \) the reaction force is

\[
\frac{R}{L^2} = \frac{2}{75} \sin^2 \alpha \cos \alpha B \left( \frac{A_1 E_1}{L^2} \right)
\]  
(73)

Equation (66) takes the form

\[
\sigma_1 = \frac{E_1 \cos \alpha \sin \alpha}{75}
\]  
(74)
With the definition of κ, and equation (62), the following is obtained:

\[
\frac{45}{56} \sigma^2 = 8 \sin^2 a \cos a \frac{A_1 E_1}{A} B
\]

or

\[
A_1 E_1 = \frac{45 \pi^2 P}{448 \sin^2 a \cos a B}
\] (75)

Introducing equation (75) in (73)

\[
\frac{R}{L^2} = \frac{3 \pi^2}{5.224} \frac{P}{L^2}
\] (76)

With equations (75) and (76) the total weight is expressible in terms of \( P/L^2, \sigma, \) and \( a \):

\[
\frac{W}{L^3} = \frac{W(P/L^2)}{\sigma} + \frac{m w_1}{E_1 \cos a} \frac{45 \pi^2}{448 \sin^2 a \cos a B} \left( \frac{P}{L^2} \right) + \frac{m+1}{2} w \left( \frac{3 \pi^2}{5.224} \frac{P}{L^2} \right)^{2/3} \left[ \frac{2}{\pi k_2 E^2} \right]^{1/3} (\tan a)^{5/3}
\] (77)

4.3 Optimization

The minimum weight can be obtained by setting the derivative of equation (77) with respect to \( a \) equal to zero.

\[
\frac{3}{3a} \frac{W}{L^3} = 0
\] (78)
For simplification, it is assumed that in the following calculations the only cases considered are those for which $w_1$ and $w$ are the same. Differentiating equation (77) with respect to $a$ and simplifying the result in such a way that the left hand side of the equation contains terms in $a$ only, yields

$$\frac{1 - 2 \sin^2 a}{\sin^3 a \cos a \tan a^{2/3}} = \frac{1}{\sin^3 a \cos a \tan a^{2/3}} \left( \frac{(m+1)}{9 \pi m} E_1 \left[ \frac{56}{75 k_2 E^2 P/L^2} \right] \right)^{1/3}$$

The term on the left hand side of equation (79) is independent of material properties and the number of tension ties. To find the optimum angle $a$ the function

$$f(a) = \frac{1 - 2 \sin^2 a}{\sin^3 a \cos a \tan a^{2/3}}$$

must be calculated for different angles. The result is plotted in Fig. 13. With the help of this figure the optimum angle $a$ can be obtained for all values of $P/L^2$.

4.4 Method of solution

For any values of the stress $\sigma$ the tangent-modulus ratio $\tau$ can be found from any tangent-modulus curve for typical materials. From equation (61) the structural index for this particular stress value is obtainable. Choosing now the number of tension ties as 3 or 4, makes it possible to calculate the right hand side of equation (79). The
optimum value of $a$ can be found now from Fig. 13, and equation (77) can be solved to obtain the optimum weight of the column supported by tension ties in the form $W/L^3$. 

OPTIMUM DESIGN OF THE COLUMN WITH THREE STRUTS

In using the theory for more than one strut it can be seen that the second buckling mode is always an independent solution of the "boom problem." This coincides with the solution in Ref. 1, where it was found that, if the intermediate supports are spaced symmetrically with respect to the midpoint, then no matter how many intermediate supporting points are used, the column will always buckle in the second mode.

Nevertheless, it seems to be possible to arrange the tension ties in such a way that the ties influence the load for the second buckling mode also, so that this column could be designed to have buckling occur in the first, second, third, or fourth mode. A geometry as sketched in Fig. 14 results in a constrained second buckling mode, thereby increasing buckling stresses over the previous case.

In this new problem the following additional assumptions are made:

a. The tension ties E-1-C-3-A and 1-2-3 have the same cross-section and material properties.

b. The ties 1-C and 3-C are hinged at point C, but cannot move in the x-direction.
Again it is assumed that the tension ties have a pretension of such a magnitude that at the instant of buckling all ties are stress free, but not relaxed.

Instead of two variables, as in the case of one strut, four variables, \( a_1, a_2, A_1, \) and \( A_2 \), must now be determined. (Fig. 14)

5.1 Buckling theory for the column with three struts

The deflection curve must be introduced with four terms and reads

\[
y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} + a_4 \sin \frac{4\pi x}{L}
\]

\[(81)\]

For \( x = L/4, L/2, \) and \( 3L/4 \) this series yields

\[
y \bigg|_{x = \frac{L}{4}} = y_1 = \frac{a_1}{2} \sqrt{2} + a_2 + \frac{a_3}{2} \sqrt{2}
\]

\[
y \bigg|_{x = \frac{L}{2}} = y_2 = a_1 - a_3
\]

\[
y \bigg|_{x = \frac{3L}{4}} = y_3 = \frac{a_1}{2} \sqrt{2} - a_2 + \frac{a_3}{2} \sqrt{2}
\]

The changes in length of the tension ties, neglecting displacements in the x-direction and assuming \( y_0 = y_4 = 0 \) are (Fig. 14)
\[ \Delta c_1 = y_1 \sin a \]
\[ \Delta c_2^a = (y_1 - y_2) \sin a \]
\[ \Delta c_2^b = (y_2 - y_1) \sin a \]
\[ \Delta c_3 = (y_3 - y_2) \sin a \]
\[ \Delta c_3^b = (y_2 - y_3) \sin a \]
\[ \Delta c_4 = y_3 \sin a \]

Expressing \( y_1, y_2, \) and \( y_3 \) in terms of the series as indicated in equation (82) and using

\[ \Delta T_j = \frac{A_j E_j}{c_j} \Delta c_j \]

the following tension stresses in the ties are obtained:

\[ \Delta T_1 = \frac{4 A_1 E_1}{L} \left[ \frac{a_1}{2} \frac{\sqrt{2}}{a_2} + \frac{a_2}{2} \frac{\sqrt{2}}{a_3} \right] \sin a_1 \cos a_1 \]

\[ \Delta T_2^a = \frac{4 A_1 E_1}{L} \left[ a_1 \left( \frac{\sqrt{2}}{2} - 1 \right) + a_2 + a_3 \left( \frac{\sqrt{2}}{2} + 1 \right) \right] \sin a_1 \cos a_1 \]

\[ \Delta T_2^b = -\frac{4 A_2 E_2}{L} \left[ a_1 \left( \frac{\sqrt{2}}{2} - 1 \right) + a_2 + a_3 \left( \frac{\sqrt{2}}{2} + 1 \right) \right] \sin a_2 \cos a_2 \]

\[ \Delta T_3^a = \frac{4 A_1 E_1}{L} \left[ a_1 \left( \frac{\sqrt{2}}{2} - 1 \right) - a_2 + a_3 \left( \frac{\sqrt{2}}{2} + 1 \right) \right] \sin a_1 \cos a_1 \]

\[ \Delta T_3^b = -\frac{4 A_2 E_2}{L} \left[ a_1 \left( \frac{\sqrt{2}}{2} - 1 \right) - a_2 + a_3 \left( \frac{\sqrt{2}}{2} + 1 \right) \right] \sin a_2 \cos a_2 \]

\[ \Delta T_4 = \frac{4 A_1 E_1}{L} \left[ \frac{a_1}{2} + \frac{a_2}{2} \frac{\sqrt{2}}{a_3} \right] \sin a_1 \cos a_1 \]

It can be seen, that the forces \( T\) and \( T^b \) have different signs. A negative stress cannot exist in a tension tie, and the above equations must therefore be handled very carefully.
For the simplified theory three different cases, as shown in Fig. 15, must be considered separately.

a. symmetric buckling shape \( y_2 > y_1 \)
b. antisymmetric buckling shape
c. symmetric buckling shape \( y_2 < y_1 \)

This distinction can be made if the structure has at least one plane of symmetry [5]. For three tension ties this symmetry condition is not fully applicable, because the reaction forces, which are defined as forces in the strut planes and are opposite to the deflection, are dependent on the sign of the deflection. \( (B = 1 \text{ for } y < 0, \ B = 1/2 \text{ for } y > 0. \) Nevertheless an approximation is made by assuming that symmetric and antisymmetric modes are independent.

5.1.1 Symmetric mode shape \( (y_2 > y_1) \)

From Fig. 15 it can be seen, that a reaction force \( R \) opposing the deflection in all three struts exists and can be written as

\[
R_1 = \left[ \Delta T_1 \sin \alpha_1 - \Delta T_2^b \sin \alpha_2 \right] B \\
R_2 = \left[ \left( \Delta T_2^b + \Delta T_3^b \right) \sin \alpha_2 \right] B \\
R_3 = \left[ \Delta T_4 \sin \alpha_1 - \Delta T_3^b \sin \alpha_2 \right] B
\]

(84)
whereas in the case of a column with one strut, for

\[
y > 0 \quad \begin{cases} 
B = 1 & \text{for four tension ties} \\
B = 1/2 & \text{for three tension ties}
\end{cases}
\]

\[
y < 0 \quad B = 1 \text{ for three or four tension ties}
\]

Introducing now equation (83) in (84) yields

\[
R_1 = \left[ \frac{4 A_1 E_1}{L} \left( \frac{a_1}{2} \sqrt{2} + a_2 + \frac{a_3}{2} \sqrt{2} \right) \sin^2 \alpha_1 \cos \alpha_1 \right] B
\]

\[
R_2 = \left[ \frac{4 A_2 E_2}{L} \left( a_1 \left( 1 - \frac{\sqrt{2}}{2} \right) - a_2 - a_3 \left( 1 + \frac{\sqrt{2}}{2} \right) \right) \sin^2 \alpha_2 \cos \alpha_2 \right] B
\]

\[
R_3 = \left[ \frac{4 A_1 E_1}{L} \left( \frac{a_1}{2} \sqrt{2} - a_2 + \frac{a_3}{2} \sqrt{2} \right) \sin^2 \alpha_1 \cos \alpha_1 \right] B
\]

\[
\left. \frac{4 A_2 E_2}{L} \left( a_1 \left( 1 - \frac{\sqrt{2}}{2} \right) + a_2 - a_3 \left( 1 + \frac{\sqrt{2}}{2} \right) \right) \sin^2 \alpha_2 \cos \alpha_2 \right] B
\]
The following notation will be introduced for convenience:

\[
C_1 = \frac{4}{L} A_1 E_1 \sin^2 \alpha_1 \cos \alpha_1 \quad (88)
\]

\[
C_2 = \frac{4}{L} A_2 E_2 \sin^2 \alpha_2 \cos \alpha_2 \quad (89)
\]

Using these equations the spring strain energy can be computed as

\[
U_1 = \frac{1}{2} \left[ R_1 y_1 + R_2 y_2 + R_3 y_3 \right]
\]

\[
= \frac{B}{2} \left[ C_1 (a_1^2 + 2a_1a_3 + 2a_2^2 + a_3^2) \right. \\
+ \left. C_2 \left( a_1^2 (3-2\sqrt{2}) + 2a_2^2 - 2a_1a_3 + a_3^2 (3+2\sqrt{2}) \right) \right] \quad (90)
\]

For a virtual displacement of \( \delta a_k \) the following equation is again valid

\[
\frac{\partial W_1}{\partial a_k} \delta a_k = \frac{\partial U_1}{\partial a_k} \delta a_k + \frac{\partial U_1}{\partial a_k} \delta a_k \quad (91)
\]

As formerly stated there is a clear distinction between symmetric and antisymmetric buckling modes. Therefore in this case only virtual displacements \( \delta a_k \) which are symmetric modes can be applied and the following two equations are obtained.
\[
\frac{P\pi^2}{2L} a_1 = \frac{4EIc}{2L^3} a_1 + B \left[ C_1 (a_1 + a_3) + C_2 \left( a_1 (3-2\sqrt{2}) - a_3 \right) \right] \tag{92}
\]
\[
9 \frac{P\pi^2}{2L} a_3 = \frac{4EIc}{2L^3} a_3 + B \left[ C_1 (a_1 + a_3) + C_2 \left( a_3 (3+2\sqrt{2}) - a_1 \right) \right] \tag{93}
\]

### 5.1.2 Symmetric mode shape \((y_2 < y_1)\)

Again, from Fig. 15 the following reaction forces are obtained

\[
R_1 = \left[ \Delta T_1 + \Delta T_2 \right] \sin a_1 B + \Delta T_2 \sin a_2
\]
\[
R_2 = - \left[ \Delta T_2 \sin a_2 - \Delta T_3 \sin a_1 \right] B
\]
\[
R_3 = \left[ \Delta T_4 + \Delta T_3 \right] \sin a_1 B + \Delta T_3 \sin a_2
\]

Following the same procedure as before the following two energy equations are obtained

\[
\frac{P\pi^2}{2L} a_1 = \frac{4EIc}{2L^3} a_1 + 4C_1 B (1-\sqrt{2}) a_1 + C_2 [(2\sqrt{2} - 3) a_1 + a_3] \tag{95}
\]
\[
9\frac{P\pi^2}{2L} a_3 = \frac{4EIc}{2L^3} a_3 + 4C_1 B (1+\sqrt{2}) a_3 + C_2 [a_1 - (2\sqrt{2} + 3) a_3] \tag{96}
\]
5.1.3 Antisymmetric mode shape

\[ R_1 = B(\Delta T_1 + \Delta T_2^a) \sin \alpha \]
\[ R_2 = (-\Delta T_2^a B - \Delta T_3^a) \sin \alpha \]  \hspace{1cm} (97)
\[ R_3 = (\Delta T_4 + \Delta T_3^a) \sin \alpha \]

For the antisymmetric mode there is only one virtual displacement possible, namely, a displacement in the second mode.

\[ \frac{4P\pi^2}{2L} a_2 = \frac{\pi^4 E_t I_c}{2L^3} 16 a_2 \]  \hspace{1cm} (98)
\[ + C_1 \left\{ B[a_1(\sqrt{2}-1)+2a_2+a_3(\sqrt{2}+1)] - [a_1(\sqrt{2}-1)-2a_2+a_3(\sqrt{2}+1)] \right\} \]

5.1.4 Fourth mode shape

The fourth mode shape is independent of the tension ties and the following energy equation is valid

\[ \frac{P\pi^2 16}{2L} a_4 = \frac{\pi^4 E_t I_c}{2L^3} 256 a_4 \]  \hspace{1cm} (99)

5.2 Buckling theory and optimization

For the optimum design the tension ties must be arranged in such a way that the first, second, third, and fourth buckling modes occur at the same buckling force P,
which is, in this case, the solution of equation (99). So far, it has not been proved which set of equations, (92) - (93) or (95) - (96), is applicable, but the values of \( a_1 \), \( a_2 \), and \( a_3 \) can be found, and it can be checked easily whether the assumption \( y_2 > y_1 \) or \( y_2 < y_1 \) is valid.

5.2.1 Symmetric shape \((y_2 > y_1)\)

For the assumption that \( y_2 > y_1 \) equations (92) and (93) together with equation (98) and (99) give a system of four equations which can be solved. Introducing the solution of equation (99) in equation (92), (93), and (98) yields

\[
\frac{15}{2} \theta a_1 = B \left\{ C_1 (a_1 + a_3) + C_2 [a_1 (3 - 2\sqrt{2}) - a_3] \right\}
\]

\[
\frac{48}{2} \theta a_2 = C_1 \left\{ B[a_1 (\sqrt{2} - 1) + 2a_2 + a_3 (\sqrt{2} + 1)] - [a_1 (\sqrt{2} - 1) - 2a_2 + a_3 (\sqrt{2} + 1)] \right\}
\]

\[
\frac{63}{2} \theta a_3 = B \left\{ C_1 (a_1 + a_3) + C_2 [a_3 (3 + 2\sqrt{2}) - a_1] \right\}
\]

where

\[
\theta = \frac{\pi^4 E \tau I c}{L^3}
\]
Nontrivial solutions are only obtainable if the following two sub-determinants are zero

\[ 2C_1 (1 + B) - 240 = 0 \]  
\[ \frac{15}{2} \theta - B \left[ C_1 + C_2 (3 - 2\sqrt{2}) \right] \quad \begin{vmatrix} B(C_2 - C_1) \\ B(C_2 - C_1) \end{vmatrix} = 0 \]

In solving these two equations the following is obtained

\[ C_1 = \frac{12\theta}{1+B} \quad C_2 = \frac{4.875 \frac{B}{1+B} - 2.4609375}{\frac{B}{1+B} - 0.511643} \theta \]

and, for the two different cases under consideration

\[ m = 4 \quad C_1 = 6\theta \quad C_2 = 2.01297 \theta \]  
\[ m = 3 \quad C_1 = 8\theta \quad C_2 = 0.93762 \theta \]

Introducing these results in equations (82) and (100) leads to the following

\[ m = 4 \quad a_1 = 3.336734 \quad a_3 \text{ and } y_2 - y_1 = -0.72977 a_3 \]

\[ m = 3 \quad a_1 = 1.04897 \quad a_3 \text{ and } y_2 - y_1 = -1.399857 a_3 \]
For a positive deflection, $a_1$ must have a positive sign, so that for both cases $y_2 - y_1 < 0$. That means $y_2$ cannot be larger than $y_1$ and the solution is physically impossible.

5.2.2 Symmetric shape ($y_2 < y_1$)

In this case the following two sub-determinants must be zero

$$
\begin{vmatrix}
2 C_1 (1 + B) - 24 \theta & 0 \\
\frac{15}{2} \theta - 4 C_1 B \left(1 - \frac{3}{2} \sqrt{2}\right) - C_2 (2 \sqrt{2} - 3) & -C_2 \\
-C_2 & \frac{63}{2} \theta - 4 C_1 B \left(1 + \frac{3}{2} \sqrt{2}\right) + C_2 (2 \sqrt{2} + 3)
\end{vmatrix} = 0
$$

For three and four tension ties this results in

$$
\begin{align*}
&\text{m = 4} & C_1 &= 6 \theta & C_2 &= 13.11419 \theta \\
&\text{m = 3} & C_1 &= 8 \theta & C_2 &= 2.805837 \theta
\end{align*}
$$

Introducing these results in the energy equations

$$
\begin{align*}
&\text{m = 4} & a_1 &= 4.82033 a_3 & y_2 - y_1 &= -0.29521 a_3 \\
&\text{m = 3} & a_1 &= 0.851514 a_3 & y_2 - y_1 &= -1.4577 a_3
\end{align*}
$$

Again, it is obtained that $y_2 - y_1 < 0$, which agrees with the assumptions made. Equations (110) and (111) are
therefore valid.

5.3 Reaction forces

To calculate the optimum weight, the maximum values of the reaction forces $R_1$, $R_2$, and $R_3$ are necessary and should be calculated as functions of the buckling load $P$.

5.3.1 Reaction forces for the symmetric mode

In this case $a_2 = 0$ and the equations for the reaction forces can be written down with the help of equation (94).

$$R_1 = C_1 [a_1 (\sqrt{2}-1)+a_3 (\sqrt{2}+1)]B+C_2 [a_1 (1-\frac{\sqrt{2}}{2})-a_3 (\frac{\sqrt{2}}{2} +1)]$$

$$R_2 = C_1 [a_1 (\sqrt{2}-2)+a_3 (\sqrt{2}+2)]B+C_2 [a_1 (\sqrt{2}-2)+a_3 (\sqrt{2}+2)]$$

$$R_3 = C_1 [a_1 (\sqrt{2}-1)+a_3 (\sqrt{2}+1)]B+C_2 [a_1 (1-\frac{\sqrt{2}}{2})-a_3 (\frac{\sqrt{2}}{2} +1)]$$

For the two different cases of four or three tension ties the values $C_1$, $C_2$, $B_1$, and $a_3/a_1$ have been calculated previously and therefore the reaction forces can be found as linear functions of $\theta$ and $a_1$.

$$m = 4 \quad B = 1 \quad C_1 = 6 \quad C_2 = 13.11419 \quad a_3 = \frac{1}{4.82033} a_1$$

$$R_1 = R_3 = 4.687027 \theta a_1$$

$$R_2 = 0.8715523 \theta a_1$$

(114)
\(m = 3\) \(B = 1/2\) \(C_1 = 8\theta; C_2 = 2.805837 \theta; a_3 = 1.174379 a_1\)

\[
R_1 = R_3 = 8.194398 \theta a_1
\]

\[
R_2 = -4.088532 \theta a_1
\]  

(115)

5.3.2 Reaction forces for the antisymmetric mode

From equation (97) the reaction forces are obtainable as

\[
R_1 = C_1 [a_1(\sqrt{2} - 1) + 2a_2 + a_3(\sqrt{2} + 1)]
\]

\[
R_2 = -C_1B [a_1(\frac{\sqrt{2}}{2} - 1) + a_2 + a_3(\frac{\sqrt{2}}{2} + 1)] + [a_1(\frac{\sqrt{2}}{2} - 1) - a_2 + a_3(\frac{\sqrt{2}}{2} + 1)]
\]

\[
R_3 = C_1 [a_1(\sqrt{2} - 1) - 2a_2 + a_3(\sqrt{2} + 1)]
\]

Independent of the number of tension ties it is obtained that

\[
a_3 = -0.171573 a_1
\]

Therefore

\[
m = 4\]

\[
R_1 = - R_3 = 12\theta a_2
\]

\[
R_2 = 0
\]  

(116)

\[
m = 3\]

\[
R_1 = - R_3 = 8\theta a_2
\]

\[
R_2 = 4\theta a_2
\]  

(117)
5.3.3 Reaction forces as functions of buckling load $P$

As in equation (72) an assumption for the maximum deflection must be made. Introducing the same value as for the case of a column with one strut results in stresses in the tension ties of nearly three times the yield stress of the chosen material. After some trials it is found that the tension stresses reach the yield stresses of the tie material for values of $P/L^2$, which are calculated for the highest stresses, if $a_1 = L/400$ for $m = 4$, and $a_1 = L/800$ for $m = 3$.

Assuming, furthermore, that the reaction force $R_1$ is the same for the symmetric and antisymmetric mode gives a constraint for the magnitude of $a_2$.

From equation (99)

$$\theta = \frac{P L^2}{16 L}$$

With these assumptions

$$m = 4 \quad a_1 = 2.5 \times 10^{-3} L \quad a_2 = 0.976463 \times 10^{-3} L$$
$$R_1 = 7.2279847 \times 10^{-3} P \quad R_2 = 1.3440432 \times 10^{-3} P$$

$$m = 3 \quad a_1 = 1.25 \times 10^{-3} L \quad a_2 = 1.279749 \times 10^{-3} L$$
$$R_1 = 6.318396 \times 10^{-3} P \quad R_2 = 3.1525151 \times 10^{-3} P$$
5.4 Weight assumptions

The weight of the assembled column consists of five parts. A sketch of the parts is shown in Fig. 16.

5.4.1 Weight of the column

\[ \frac{W_1}{L^3} = \frac{w}{\frac{L^2}{\sigma}} \]  
(120)

5.4.2 Weight of the long wires

\[ W_2 = \frac{m w_1 A_1 L}{\cos \alpha_1} \]

and with equation (88)

\[ E_1 A_1 = \frac{C_1 L}{4 \sin^2 \alpha_1 \cos \alpha_1} \]  
(121)

\[ \frac{W_2}{L^3} = \frac{m w_1 C_1}{4 E_1 L \sin^2 \alpha_1 \cos^2 \alpha_1} \]  
(122)

Again this results in a constraint on \( \sigma \) in the tension ties.

\[ \sigma_1 = \frac{4E_1}{L} \left[ \frac{a_1 \sqrt{2}}{2} + a_2 + \frac{a_3}{2} \sqrt{2} \right] \cos \alpha_1 \sin \alpha_1 \]  
(123)

\[ \sigma_2 = \frac{4E_1}{L} \left[ \frac{a_1 \sqrt{2}}{2} + a_2 + \frac{a_3}{2} \sqrt{2} - a_1 + a_3 \right] \cos \alpha_1 \sin \alpha_1 \]  
(124)
\( \sigma_3 = \frac{4E_1}{L} \left[ \frac{a_1}{2} \sqrt{2} - a_2 + \frac{a_3}{2} \sqrt{2} - a_1 + a_3 \right] \cos a_1 \sin a_1 \)  \hspace{1cm} (125)

\( \sigma_4 = \frac{4E_1}{L} \left[ \frac{a_1}{2} \sqrt{2} - a_2 + \frac{a_3}{2} \sqrt{2} \right] \cos a_1 \sin a_1 \)  \hspace{1cm} (126)

5.4.3 Weight of the short wires

\[
W_3 = \frac{m w_2 A_2 L}{2 \cos a_2}
\]

from equation (89)

\[
E_2 A_2 = \frac{C_2 L}{4 \sin^2 a_2 \cos a_2}
\]

\[
W_3 = \frac{m w_2 C_2}{8 E_2 L \sin^2 a_2 \cos^2 a_2}
\]  \hspace{1cm} (127)

The constraints for \( \sigma \) are, in this case

\[
\sigma_2 = \frac{4E_2}{L} \left[ a_1 - a_3 - \frac{a_1}{2} \sqrt{2} - a_2 - \frac{a_3}{2} \sqrt{2} \right] \sin a_2 \cos a_2
\]  \hspace{1cm} (128)

\[
\sigma_3 = \frac{4E_2}{L} \left[ a_1 - a_3 - \frac{a_1}{2} \sqrt{2} + a_2 - \frac{a_3}{2} \sqrt{2} \right] \sin a_2 \cos a_2
\]  \hspace{1cm} (129)
5.4.4 Weight of the strut assembly at \( x = L/4 \) and \( x = 3L/4 \)

As in the case of the column with one strut assembly, the struts are assumed to be simple columns with circular cross section and shall buckle in the elastic region of the material used.

The optimum buckling load for these simple columns is given by

\[
\frac{R_1}{h_1^2} = \frac{8\sigma^3}{\pi k_2 E^2} \quad \text{and} \quad \sigma = \left[ \frac{\pi k_2 E^2}{8} \left( \frac{R_1}{h_1^2} \right) \right]^{1/3}
\]

Introducing

\[ h_1 = \frac{L}{4} \tan \alpha_1 \]

yields

\[
\sigma = \left[ \frac{2\pi k_2 E^2}{\tan \alpha_1} \left( \frac{R_1}{L^2} \right) \right]^{1/3}
\]

The weight of the strut assemblies is now

at \( x = L/4 \)

\[ W_4^I = (m+1) \ w \ \frac{R_1}{\sigma} \ \frac{L \ tan \alpha_1}{4} \]

at \( x = 3L/4 \)

\[ W_4^{II} = (m+1) \ w \ \frac{R_3}{\sigma} \ \frac{L \ tan \alpha_1}{4} \]
With equation (131) and the consideration that \( R_1 = R_3 \)

\[
\frac{W_4}{L^3} = \frac{W_{4I}}{L^3} + \frac{W_{4II}}{L^3} = \frac{m+1}{2} w \left( \frac{R_1}{L^2} \right)^{2/3} \left[ \frac{1}{2 \pi k_2 E^2} \right]^{1/3} \tan \alpha_{1}^{5/3}
\]

(132)

5.4.5 Weight of the strut assembly at \( x = L/2 \)

Here

\[
\frac{R_2}{h^2} = \frac{8 \sigma^3}{\pi k_2 E^2} \quad \text{and} \quad \sigma = \left[ \frac{\pi k_2 E^2}{8} \frac{R_2}{h^2} \right]^{1/3}
\]

(133)

Introducing

\[ h = \frac{L}{4} (\tan \alpha_{1} + \tan \alpha_{2}) \]

yields

\[
\sigma = \left[ \frac{2 \pi k_2 E^2 R_2}{(\tan \alpha_{1} + \tan \alpha_{2})^2 \frac{L^2}{2}} \right]^{1/3}
\]

(134)

The weight of the strut assembly is therefore

\[
\frac{W_5}{L^3} = \frac{(m+1)}{4} w \left( \frac{R_2}{L^2} \right)^{2/3} (\tan \alpha_{1} + \tan \alpha_{2})^{5/3} \left[ \frac{1}{2 \pi k_2 E^2} \right]^{1/3}
\]

(135)
5.4.6 **Weight of the whole column with three struts**

As can be seen from Fig. 16

\[
\frac{W}{L^3} = \frac{W_1}{L^3} + \frac{W_2}{L^3} + \frac{W_3}{L^3} + \frac{W_4}{L^3} + \frac{W_5}{L^3}
\]

\[
= \frac{w}{L^2} \frac{P}{\sigma} + \frac{m w_1}{E_1} \frac{C_1}{4L \sin^2 a_1 \cos^2 a_1} + \frac{m w_2}{E_2} \frac{C_2}{8L \sin^2 a_2 \cos^2 a_2}
\]

\[
+ \frac{(m+1)}{2} w \left( \frac{R_1}{L^2} \right)^{2/3} \left[ \frac{1}{2 \pi k_2 E_2^2} \right]^{1/3} (\tan a_1)^{5/3}
\]

\[
+ \frac{(m+1)}{4} w \left( \frac{R_2}{L^2} \right)^{2/3} \left[ \frac{1}{2 \pi k_2 E_2^2} \right]^{1/3} (\tan a_1 + \tan a_2)^{5/3}
\] (136)

Assuming again that the tension ties, the column and the struts, are all made from the same material, then the weight equation can be simplified to

\[
\frac{W}{L^3} = \frac{w}{L^2} \left\{ \frac{P}{\sigma} + \frac{m}{8E_1} \left\{ \frac{2 C_1}{\sin^2 a_1 \cos^2 a_1} + \frac{C_2}{\sin^2 a_2 \cos^2 a_2} \right\} \right\}
\]

\[
+ \frac{(m+1)}{4} \left\{ \frac{1}{2 \pi k_2 E_2^2} \right\}^{1/3} \left[ \left( \frac{R_1}{L^2} \right)^{2/3} + \left( \frac{R_2}{L^2} \right)^{2/3} \right] (\tan a_1 + \tan a_2)^{5/3}
\]

(137)
5.5 Optimization

Since \( C_1, C_2, R_1, \) and \( R_2 \) are functions of \( P \) only, the weight equation is again a function of \( P/L^2 \) with the angles \( \alpha_1 \) and \( \alpha_2 \) as parameters. For the optimum design the partial derivatives of equation (137) with respect to \( \alpha_1 \) and \( \alpha_2 \) must be zero.

\[
\frac{3}{\alpha_1} \left( \frac{W}{L^3} \right) = \frac{3}{\alpha_2} \left( \frac{W}{L^3} \right) = 0 \tag{138}
\]

Differentiating equation (137) the following two equations are obtained.

\[
\frac{m C_1}{E_1 L} \frac{(2 \sin^2 \alpha_1 - 1)}{\cos \alpha_1 \sin^3 \alpha_1} + \frac{5(m+1)}{6} \left[ \frac{1}{2 \pi k_2 E^2} \right]^{1/3} \left\{ \frac{R_1^{2/3}}{L^3} \tan \alpha_1^{2/3}
\right\}
\]

\[\left. \left. + \left[ \frac{R_2^{2/3}}{L^3} \right] (\tan \alpha_1 + \tan \alpha_2) \right\} = 0 \tag{139}\]

\[\frac{m C_2}{E_1 L} \frac{(2 \sin^2 \alpha_2 - 1)}{\cos \alpha_2 \sin^3 \alpha_2} + \frac{5(m+1)}{3} \left[ \frac{1}{2 \pi k_2 E^2} \right]^{1/3} \left( \frac{R_2^{2/3}}{L^3} \right)^{2/3} (\tan \alpha_1 + \tan \alpha_2) = 0 \tag{140}\]

For further calculations all trigonometric functions are expressed as tangent functions. It can be verified that
\[
\frac{2 \sin^2 \alpha - 1}{\sin^3 \alpha \cos \alpha} = \frac{\tan^4 \alpha - 1}{\tan^3 \alpha} \quad (141)
\]

Equations (139) and (140) are now

\[
C_1 = Z \frac{\tan^3 \alpha_1}{1 - \tan^4 \alpha_1} \left[ 2 \left( \frac{R_1}{L^2} \right)^{2/3} \left( \frac{R_2}{L^2} \right)^{2/3} (\tan \alpha_1 + \tan \alpha_2) \right]^{2/3} \quad (142)
\]

\[
C_2 = 2Z \frac{\tan^3 \alpha_2}{1 - \tan^4 \alpha_2} \left[ \left( \frac{R_2}{L^2} \right)^{2/3} (\tan \alpha_1 + \tan \alpha_2) \right]^{2/3} \quad (143)
\]

where \( Z \) is defined by

\[
Z = \frac{5(m+1)}{6} \left[ \frac{1}{2 + k_2 E^2} \right]^{1/3} \frac{E_1 L}{m} \quad (144)
\]

For further calculations the cases \( m = 4 \) and \( m = 3 \) must be considered separately.

\( m = 4 \)

With \( m = 4 \) and \( k_2 \) again chosen as 0.4, equation (144) yields

\[
Z = 0.7661367 E^{1/3} L
\]

From equation (118)

\[
R_1 = 7.2279847 \times 10^{-3} \text{ Pa}
\]

\[
R_2 = 1.3440432 \times 10^{-3} \text{ Pa}
\]
and from equation (110)

\[ C_1 = 6 \Theta = 3.7011018 \frac{P}{L} \]

\[ C_2 = 13.11419 \Theta = 8.0894920 \frac{P}{L} \]

With these values equations (142) and (143) can be simplified, and after some algebraic manipulations the following two equations are obtained.

\[ F \frac{1 - \tan^4 \alpha_1}{\tan^3 \alpha_1} = 1.547667 \tan \alpha_1^{2/3} + 0.2520965 (\tan \alpha_1 + \tan \alpha_2) \quad (145) \]

\[ \tan \alpha_1 = 9.02438888 F \left[ \frac{1 - \tan^4 \alpha_2}{\tan^3 \alpha_2} \right]^{3/2} - \tan \alpha_2 \quad (146) \]

where

\[ F = 10^2 \left[ \frac{\frac{P^2}{L^2}}{E} \right] \]

\[ m = 3 \]

For \( m = 3 \) equation (144) yields

\[ Z = 0.8172125 E^{1/3} L \]

From equation (119)

\[ R_1 = 6.318396 10^{-3} P \]

\[ R_2 = 3.1525151 10^{-3} P \]
and from equation (111)

\[ C_1 = 8 \theta = 4.9351108 \frac{P}{L} \]

\[ C_2 = 2.805837 \theta = 1.7307814 \frac{P}{L} \]

With these values, two equations are obtained which must be solved for \( a_1 \) and \( a_2 \).

\[
F \frac{1-\tan^4 a_1}{\tan^3 a_1} = 1.1318870 \tan a_1 + 0.3560137(\tan a_1 + \tan a_2)^{2/3} \quad (147)
\]

\[
\tan a_1 = 0.34563669 F \left[ \frac{1-\tan^4 a_2}{\tan^3 a_2} \right]^{3/2} - \tan a_2 \quad (148)
\]

Equations (145), (146), (147), and (148) must be solved for a relatively large number of values of \( F \), and because equations are not independent the actual calculations would be very involved. It is therefore advisable that a computer be used to solve the problem. The solution can be obtained by using a "Newton Iteration Process."

\[
x^{(i+1)} = x^{(i)} - \frac{y^{(i)}}{y'(i)} \quad (149)
\]

Introducing now the notation

\[ x = \tan a_2 \]

\[ x_1 = \tan a_1 \]
\( m = 4 \)

(Equation (145) can be written as

\[
y = -x_1 + \frac{1}{3} x_1^{2/3} - \frac{1.547667}{F} x_1^{2/3} - 1.0928455 \left( \frac{1}{x^3} - x \right) \quad (150)
\]

\[
y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial x}
\]

\[
= -x_1' - \frac{3}{x_1^{4/3}} \frac{2/3}{F} 1.547667 \quad 1^{1/3} x_1' + 1.0928455 \left(1 - \frac{3}{x^4} \right)
\quad (151)
\]

From equation (146), \( x_1 \) and \( x_1' \) can be expressed in \( x \).

\[
x_1 = 9.025389 F^{3/2} \left[ \frac{1 - x^4}{x^3} \right]^{3/2} - x \quad (152)
\]

\[
x_1' = 3/2 9.024389 F^{3/2} \left[ \frac{1 - x^4}{x^3} \right]^{1/2} \left(1 + \frac{3}{x^4} \right) - 1 \quad (153)
\]

\( m = 3 \)

In the same way equation (147) becomes

\[
y = -x_1 + \frac{1}{3} x_1^{2/3} - \frac{1.131887}{F} x_1^{2/3} - 0.17535375 \left( \frac{1}{x^3} - x \right) \quad (154)
\]

\[
y' = -x_1' - \frac{3}{x_1^{4/3}} \frac{2/3}{F} 1.131887 \quad 1^{1/3} x_1' + 0.17535375 \left(1 - \frac{3}{x^4} \right)
\quad (155)
\]
And with equation (148)

\[
x_1 = 0.34563669 F^{3/2} \left[ \frac{1-x^4}{x^3} \right]^{3/2} - x
\]

(156)

\[
x_1' = 3/2 \ 0.34563669 F^{3/2} \left[ \frac{1-x^4}{x^3} \right]^{1/2} \left( 1 + \frac{3}{x^4} \right) - 1
\]

(157)

For an initial assumed value of \( x \) the values of \( x_1 \) and \( x_1' \) are obtainable. The three values \( x, x_1, \) and \( x_1' \) allow the calculation of \( y \) and \( y' \).

With equation (149) a new value of \( x \) is obtained. This procedure has to be repeated until a specified accuracy of \( x \) is reached.

For the calculation of the cross sectional area, the expressions

\[
\frac{1}{\sin^2 \alpha_1 \cos \alpha_1} \quad \text{and} \quad \frac{1}{\sin^2 \alpha_2 \cos \alpha_2}
\]

are necessary. Therefore they have also been calculated by the computer. The computer program listed is in Appendix II of this paper.

With the optimum angles \( \alpha_1 \) and \( \alpha_2 \) calculated, the weight of the column with three struts can be obtained and plotted against the structural index to give the desired results.
SECTION 6
CALCULATIONS AND RESULTS FOR THE COLUMN WITH ONE STRUT

6.1 Material properties

For the application of the theory developed on the previous pages a typical aluminum-alloy and steel are chosen.

Al-alloy 2024-T4-Al-Alloy
(material properties, see Fig. 10)

Steel Stainless steel 3/4 hard
(material properties, see Fig. 17)

From Ref. 6 the following material data for the tension ties are found.

Al-alloy \( \sigma_{\text{yield}} = 60 \text{ (ksi)} \quad w = 0.10 \text{ (lb/in.}^3 \text{)} \)
\[ E = 10.5 \times 10^6 \text{ (psi)} \]

Carbon steel \( \sigma_{\text{yield}} = 130 \text{ (ksi)} \quad w = 0.283 \text{ (lb/in.}^3 \text{)} \)
\[ E = 28.3 \times 10^6 \text{ (psi)} \]

6.2 Application for aluminum-alloy

6.2.1 Four tension ties.

With \( m = 4 \) and \( B = 1 \) equation (79) takes the form

\[ f(a) = \frac{11.9197}{(P/L^2)^{1/3}} \quad (158) \]
Considering the fact that at the instant of buckling the tension ties are stress free as long as there is no lateral deflection, this allows the definition

\[ P = \sigma A \]  \hspace{1cm} (159)

where \( A \) is the cross section of the column.

With this relation equation (75) can be modified and gives a relation between wire and column cross sections.

\[ \frac{A}{A_1} = \frac{8.4554 \times 10^{-5}}{\sin^2 \alpha \cos \alpha} \sigma \hspace{1cm} \text{(ksi)} \]  \hspace{1cm} (160)

The weight equation for this particular case is obtained from equation (77)

\[
\frac{W}{L^3} = \left\{ \frac{P}{\sigma L^2} + \frac{0.37766}{10^6} \frac{P}{L^2} \frac{1}{\sin^2 \alpha \cos \alpha} + \frac{2.51037}{10^6} \frac{P}{L^2} \left( \tan \alpha \right) \right\} w^{2/3} \hspace{1cm} (161)
\]

Now all equations necessary for the solution are obtained and the steps which are followed in solving the problem are shown by solving for one value of \( \sigma \).

For a stress of \( \sigma = 20 \) (ksi) Fig. 10 gives

\[ \tau = 1.0 \]  \hspace{1cm} (162)

Equation (40) gives now the structural index for the simple column

\[ \frac{P}{L^2} = 0.45070 \text{ (psi)} \]  \hspace{1cm} (163)
In the same way equation (61) gives the structural index for the optimum column with one strut.

\[
\frac{P^{(II)}}{L^2} = 0.11267 \text{ (psi)}
\] (164)

Now \( f(a) \) can be calculated and Fig. 13 gives an optimum value of

\[
\alpha = 22.65^\circ
\] (165)

The weight of the simple column can be calculated by introducing equation (163) in (41)

\[
\frac{W}{L^3} = 2.2535 \text{ (lb/in.}^3)\]
(166)

With equations (164) and (165) all information is obtained to calculate the weight of the column with one strut. Equation (161) yields

\[
\frac{W}{L^3} = 0.6107 \text{ (lb/in.}^3)\]
(167)

Equation (160) gives

\[
\frac{A_1}{A} = 1.2355 \times 10^{-2}
\] (168)

The results for different values of \( \sigma \) are noted in Table 1 in Appendix I.

The constraint for the tension ties is

\[
\sigma_1 = \frac{2 E_1 \cos \alpha \sin \alpha}{150}
\] (169)
For the example

\[ 0.2605 < \cos \alpha \sin \alpha < 0.4655 \]

or

\[ \sigma_{\text{min}} = 36.47 \text{ (ksi)} \quad \sigma_{\text{max}} = 65.17 \text{ (ksi)} \quad (170) \]

\( \sigma_{\text{yield}} \) for aluminum wires goes up to 70 (ksi), so that the wires are safe.

### 6.2.2 Three tension ties

With \( m = 3 \) and \( B = 1/2 \) equation (79) can be written as

\[
f(\alpha) = \frac{6.3572}{P/L^2}^{1/3}
\]

Equation (75) in modified form writes

\[
\frac{A_1}{A} = \frac{16.9108 \times 10^{-5}}{\sin^2 \alpha \cos \alpha} \sigma \text{ (ksi)}
\]

The weight equation is

\[
\frac{W}{L^3} = w\left\{\frac{P}{\sigma L^2} + \frac{0.56649}{10^6} \frac{P}{L^2} \frac{1}{\sin^2 \alpha \cos^2 \alpha} + \frac{2.0083}{10^6} \frac{(\tan \alpha)^{5/3} P^{2/3}}{L^2}\right\}
\]

The calculations are similar to the calculation in Section 6.2.1. The results are noted in Table 2 in the Appendix.

The stresses in the tension ties are

\[ \sigma_{\text{min}} = 42.77 \text{ (ksi)} \quad \sigma_{\text{max}} = 67.55 \text{ (ksi)} \]
The tension ties are therefore safe. 

The various results for a column with one strut, made from aluminum are shown in Fig. (18), (19), and (26).

6.3 **Application for steel**

6.3.1 **Four tension ties**

With equation (79)

$$f(\alpha) = \frac{16.588}{(P/L^2)^{1/3}}$$  \hspace{1cm} (174)

From equation (75)

$$\frac{A_1}{A} = \frac{3.1018 \times 10^{-5}}{\sin^2\alpha \cos\alpha}$$  \hspace{1cm} (175)

The weight equation for this case is

$$\frac{W}{L^3} = w \left[ \frac{P}{\sigma L^2} + \frac{0.14012}{10^6} \frac{P}{(L^2)^2} \frac{1}{\sin^2 \alpha \cos^2 \alpha} + \frac{1.2958}{10^6} \frac{P}{L^2} \frac{2/3 5/3}{\tan \alpha} \right]$$  \hspace{1cm} (176)

The results are noted in Table 3 in Appendix I.

The stresses in the tension ties are calculated as

$$\sigma_{\text{min}} = 78.635 \times 10^3 \text{ (psi)} \quad \sigma_{\text{max}} = 176.485 \times 10^3 \text{ (psi)}$$

$\sigma_{\text{yield}}$ for steel goes up to 158 $10^3$ (psi), so that for higher values of $P/L^2$ (over 2.0) the deflection should be restricted to a value of $L/200$. 
6.3.2 Three tension ties

The angle $\alpha$, the cross section relation $A_1/A$, and the weight $W/L^3$ can be calculated using the following equations:

$$f(\alpha) = \frac{8.84693}{(P/L^2)^{1/3}}$$  \hspace{1cm} \text{(177)}

$$\frac{A_1}{A} = \frac{7.0061 \times 10^{-5}}{\sin^2 \alpha \cos \alpha} \text{ (ksi)}$$  \hspace{1cm} \text{(178)}

$$\frac{W}{L^3} = w \left\{ \frac{P}{\sigma L^2} + \frac{0.21018 (P)}{10^6 (L^2) \sin^2 \alpha \cos^2 \alpha} + \frac{1.03695}{10^6} (\tan \alpha) \left( \frac{P}{L^2} \right)^{5/3} \right\}$$

The results are noted in Table 4 in Appendix I. \hspace{1cm} \text{(179)}

The tension stresses are calculated as

$$\sigma_{\text{min}} = 91.178 \times 10^3 \text{ (psi)} \quad \sigma_{\text{max}} = 182.737 \times 10^3 \text{ (psi)}$$

Therefore again the deflections of $P/L^2 > 2.0$ should be restricted to values of $L/200$.

The various results for a column with one strut, made from steel, are shown in Fig. (20, (21), and (31).
SECTION 7
CALCULATIONS AND RESULTS FOR THE COLUMN WITH THREE STRUTS

7.1 Optimum angles

With the computer program given on previous pages, the angles $a_1$ and $s_2$ are obtained for various values of $F$, which are functions of $\sigma$.

The computed results are printed in Appendix II.

7.2 Application for aluminum-alloy

7.2.1 Four tension ties

With the definitions of $C_1$ and $C_2$ [equations (88) and (89)] and the results for these variables for the various cases, the cross sections are obtainable. Introducing $E_1 = E_2 = 10.5 \times 10^6$ (lb/in.$^2$) yields

$$\frac{A_1}{A} = 0.0881215 \times 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2 a_1 \cos a_1}$$ (180)

$$\frac{A_2}{A} = 0.1926069 \times 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2 a_2 \cos a_2}$$ (181)

The weight can be obtained from equation (137) and is written here in such a form that a computer program can be set up easily.
\[
\frac{W}{L^3} \cdot 10^6 = 10^3 \left\{ \frac{P}{L^2} \left[ \frac{1}{\sigma} + \frac{0.35248588}{F_1} + \frac{0.38483295}{F_2} \right] + \frac{0.19173222}{10} (F_3 + F_4) \right\} 
\]

where

\[
F_1 = 10^3 \sin^2 \alpha_1 \cos^2 \alpha_1 
\]

\[
F_2 = 10^3 \sin^2 \alpha_2 \cos^2 \alpha_2 
\]

\[
F_3 = 2 \tan \alpha_1 \cdot \left( \frac{7.2279847}{10^3} \cdot \frac{P}{L^2} \right)^{2/3} 
\]

\[
F_4 = (\tan \alpha_1 + \tan \alpha_2) \cdot \left( \frac{1.3440432}{10^3} \cdot \frac{P}{L^2} \right)^{2/3} 
\]

Again the computing of all necessary values for the solution of the problem are indicated for one example.

Choosing again a stress of \( \sigma = 20 \) (ksi) the structural index for the optimal column with three struts can be calculated from equation (98).

\[
\frac{P}{L^2}^{(IV)} = 0.028168693 
\]

With the help of equations (150), (151), (152), and (153) the computer calculates the optimum angles \( \alpha_1 \) and \( \alpha_2 \) using the above value of the structural index and the modulus of elasticity for aluminum to calculate the value \( F \). The following is obtained:
\[ \alpha_1 = 25.420^\circ \] (188)

\[ \alpha_2 = 36.012^\circ \] (189)

The calculation of \( A_1/A \) and \( A_2/A \) can be done now with equations (180) and (181).

\[ A_1/A = 1.05902 \times 10^{-2} \] (190)

\[ A_2/A = 1.37758 \times 10^{-2} \] (191)

With this information given, the computer solves equation (182)

\[ \frac{W}{L^3} 10^6 = 0.16701 \] (192)

The program for the calculation of the weight is printed in Appendix II for all different cases, discussed in the next three sections.

The results for various values of \( P/L^2 \) are noted in Table 5 in Appendix I and in the tables containing the computer results (Appendix II).

7.2.2 Three tension ties

For this case the following two cross section relations are obtained

\[ A_1/A = 0.117495 \times 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_1 \cos \alpha_1} \] (193)
\[
A_2/A = 0.041209 \times 10^{-3} \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_2 \cos \alpha_2} \tag{194}
\]

The weight equation is written in a similar form as equation (182).

\[
\frac{W}{L^3} 10^6 = 10^3 \times 0.105 \left\{ \frac{P}{L^2} \left[ \frac{1}{\sigma} + \frac{0.35250792}{F_1} + \frac{0.618136095}{F_2} \right] + \frac{0.15338577}{10^4} (F_3 + F_4) \right\} \tag{195}
\]

where

\[
F_1 = 10^3 \sin^2 \alpha_1 \cos^2 \alpha_1 \tag{196}
\]
\[
F_2 = 10^3 \sin^2 \alpha_2 \cos^2 \alpha_2 \tag{197}
\]
\[
F_3 = 2 \tan \alpha_1^{5/3} \left( \frac{6.318396}{10^3} \frac{P}{L^2} \right)^{2/3} \tag{198}
\]
\[
F_4 = (\tan \alpha_1 + \tan \alpha_2)^{5/3} \left( \frac{3.1525151}{10^3} \frac{P}{L^2} \right)^{2/3} \tag{199}
\]

The results are noted in Table 5 in Appendix I and in the results of the computer calculations (Appendix II).

The various results for a column with three struts made from aluminum-alloy are shown in Figs. (22), (23), (24), (25), and (26).

7.3 Application for steel

7.3.1 Four tension ties
Introducing \( E_1 = E_2 = 28.3 \times 10^6 \text{ (lb/in.}^2) \)
yields here for the cross section relations

\[
\frac{A_1}{A} = 0.032695 \ 10^{-3} \ \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_1 \cos \alpha_1} \quad (200)
\]

\[
\frac{A_2}{A} = 0.0714619 \ 10^{-3} \ \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_2 \cos \alpha_2} \quad (201)
\]

The weight equation reads

\[
\frac{W}{L^3} = 10^6 = 10^3 \ 0.283 \left[ \frac{P}{L^2} \left( \frac{1}{\sigma} + \frac{0.13078098}{F_1} + \frac{0.14278254}{F_2} \right) \right.
\]

\[
+ \left. \frac{0.098999204}{10} (F_3 + F_4) \right] \quad (202)
\]

where \( F_1, F_2, F_3, \) and \( F_4 \) are defined as in equations (183), (184), (185), and (186).

The results for various values of \( P/L^2 \) are noted in Table 6 in Appendix I and in the tables containing the computer results (Appendix II).

7.3.2 Three tension ties

The cross section relations are

\[
\frac{A_1}{A} = 0.043594 \ 10^{-3} \ \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_1 \cos \alpha_1} \quad (203)
\]

\[
\frac{A_2}{A} = 0.015289 \ 10^{-3} \ \frac{\sigma \text{ (ksi)}}{\sin^2 \alpha_2 \cos \alpha_2} \quad (204)
\]
The weight equation is

\[ \frac{W}{L^3} \times 10^6 = 0.283 \left\{ \frac{P}{L^2} \left[ \frac{1}{\sigma} + \frac{0.13078916}{F_1} + \frac{0.02293438}{F_2} \right] + \frac{0.079197485}{10} (F_3 + F_4) \right\} \quad (205) \]

where \( F_1, F_2, F_3, \) and \( F_4 \) are defined as in equations (196), (197), (198), and (199).

The results for various values of \( P/L^2 \) are noted in Table 7 in Appendix I and in the tables containing the computer results (Appendix II).

The results for a column with three struts are plotted in Figs. (27), (28), (29), (30), and (31).

7.4 Check of the tension stresses in the ties

7.4.1 Tension stresses for a construction with four tension ties

From equation (118) it can be seen that

\[ a_1 = 2.5 \times 10^{-3} L, \quad a_2 = 0.976463 \times 10^{-3} L, \quad a_3 = 0.207455 a_1. \]

In looking at equations (123), (124), (125), and (126) it can be seen that only \( \sigma_1 \) is critical. It is obtained

\[ \sigma_1 = 12.44385 \times 10^{-3} E_1 \cos \alpha_1 \sin \alpha_1 \quad (206) \]

Equations (128) and (129) again yield only one critical stress \( \sigma_2 \).
The maximum values of the product $\cos \alpha \sin \alpha$ can be calculated from the computer results.

### Aluminum

$$(\cos \alpha_1 \sin \alpha_1)_{\text{max}} = 0.48059 \quad \text{and} \quad \sigma_1 = 62.794 \text{ (ksi)} \quad (208)$$

$$(\cos \alpha_2 \sin \alpha_2)_{\text{max}} = 0.49867 \quad \text{and} \quad \sigma_2 = 54.782 \text{ (ksi)} \quad (209)$$

### Steel

$$(\cos \alpha_1 \sin \alpha_1)_{\text{max}} = 0.48223 \quad \text{and} \quad \sigma_1 = 169.822 \text{ (ksi)} \quad (210)$$

$$(\cos \alpha_2 \sin \alpha_2)_{\text{max}} = 0.49882 \quad \text{and} \quad \sigma_2 = 147.650 \text{ (ksi)} \quad (211)$$

#### 7.4.2 Tension stresses for a construction with three tension ties

From equation (119)

$$a_1 = 1.25 \times 10^{-3} L, \quad a_2 = 1.27975 \times 10^{-3} L, \quad a_3 = 0.58789 a_1$$

Again there are two critical stresses $\sigma_1$ and $\sigma_2$

$$\sigma_1 = 10.73055 \times 10^{-3} E_1 \cos \alpha_1 \sin \alpha_1 \quad (212)$$

$$\sigma_2 = 0.41028 \times 10^{-3} E_2 \cos \alpha_2 \sin \alpha_2 \quad \text{(not critical)} \quad (213)$$
With the maximum values of \( \cos \alpha \sin \alpha \) the maximum tension stresses are obtained as:

**Aluminum**

\[
(\cos \alpha \sin \alpha)_{\text{max}} = 0.48566 \text{ and } \sigma_1 = 54.720 \text{ (ksi)}
\]  
(214)

**Steel**

\[
(\cos \alpha \sin \alpha)_{\text{max}} = 0.48428 \text{ and } \sigma_1 = 147.064 \text{ (ksi)}
\]  
(215)

The results indicate that the maximum deflections are chosen in such a way that the tension ties are never loaded beyond their yield strength.
SECTION 8
DISCUSSION AND CONCLUSIONS

The weight curves Figs. (29) and (31) indicate that for small values of the structural index the column supported by tension ties is up to 50% lighter than the simple column. To demonstrate for which values of the structural index a wire supported column configuration is advisable, a weight savings factor is defined as

$$\text{weight of the simple column - weight of the wire supported column} \over \text{weight of the simple column}$$

Using the weight curves in Figs. (29) and (31), respectively, this weight saving function can be found graphically and is plotted in Figs. (32) and (33) for the two used materials. There is hardly any difference between the weight of the column with three and four ties, so that the weight figures and the weight saving figures are drawn without the distinction between a column with three and four ties.

During the calculations it was found that the angles $\alpha$ are always smaller than 45° and in a very reasonable region. The cross sectional area of a tension tie is usually from one to five percent that of the central column.
Both values (the tie angles and tie cross sections) increase with increasing values of the structural index, where the greatest rate of change occurs for very small values of the structural index. This indicates that the additional weight from the tension ties also increases rapidly for these values of the structural index [Figs. (18), (20), (22), (23), (27), (28)].

If typical design parameters (P and L) are given, a decision must be made whether a column with three or one strut is chosen. When the weight curves indicate significant weight savings and weight savings are important for a particular problem, it seems reasonable to calculate the column with three struts first. Except for large values of P and very large values of L, the wall thickness of the central column usually is too small to be manufactured. In this case the column with one strut must be used.

To indicate how the calculations in this paper can be used, a typical practical example is calculated. Assuming that a load of 1000 pounds has to be carried over a distance of 10 feet and that an aluminum construction with one strut and three tension ties is chosen. This allows the calculation of the structural index:

\[ \frac{P}{L^2} = 0.0695 \quad \text{and} \quad \sqrt{\frac{P}{L^2}} = 0.2635 \]
Fig. (18) gives immediately

\[ \alpha = 25.15^\circ \]

\[ \frac{A_1}{A} = 1.73 \times 10^{-2} \]

From the expression for the second buckling mode, equation (61), the following maximum stress is obtained

\[ \sigma = 3 \sqrt{\frac{p}{L^2} \frac{k_2 E^2 \tau^{3/2}}{2}} \]

for the given values

\[ \sigma_{\tau^{1/2}} = 16.34 \text{ (ksi)} \]

To evaluate \( \sigma \) an iteration is usually required. In this case however Fig. (10) yields \( \tau = 1 \), so that

\[ \sigma = 16.34 \text{ (ksi)} \]

From equation (159) it is obtained

\[ A = \frac{p}{\sigma} = \frac{1000}{1.634 \times 10^4} = 0.0612 \text{ (in.}^2) \]

Now the cross section of the tension ties is

\[ A_1 = 1.73 \times 10^{-2} \quad A = 0.1059 \times 10^{-2} \text{ (in.}^2) \]

and for a circular cross section

\[ D_1 = 0.0372 \text{ (in.)} \]
From equation (39) it is obtained

\[ \frac{D}{t} = 2 \left( \frac{k^2 E}{\pi \frac{E}{P/L^2}} \right)^{1/3} = 394.95 \]

With \( D/t \) and \( A \) given, the wall thickness and column diameter \( D \) can be calculated

\[ t = 0.715 \times 10^{-2} \text{ (in.)} \]
\[ D = 2.825 \text{ (in.)} \]

The result for \( t \) indicates that a column with three struts would be impossible for this particular case.

The required pretension in the tension ties is obtained from equation (12). Introducing in this equation

\[ \sin \theta_1 = \cos \alpha \]
\[ c_1 = c_2 = \frac{L}{2 \cos \alpha} \]
\[ m = 3 \]
\[ E_1 = E_c \]

the following simple equation is obtained

\[ T_{01} = \frac{P}{3 \cos \alpha + \frac{A_c}{A_1}} = 16.53 \text{ (lbs)} \]

The fact that the weight of the optimal wire supported column is nearly independent of the number of tension ties leads to the conclusion that an arrangement
of four ties does not bring any advantages, so that from the manufacturing point of view the column with three tension ties is preferred. Figures (29) and (31) show that for small values of the structural index $P/L^2$ the column supported by tension ties has a significantly smaller weight than the simple column. Nevertheless it should be mentioned that the calculated optimum weight for very small values of $P/L^2$ cannot be reached in practice because the optimum ratio $D/t$ reaches values which give dimensions of a column which is nearly impossible to fabricate. As is often the case in calculations in optimum design, the results presented in this paper have to be considered as the theoretical limit.
REFERENCES


APPENDIX I

Tables and Figures
TABLE 1 (a)

Column with one strut, four tension ties, made from aluminum-alloy.

<table>
<thead>
<tr>
<th>$\sigma$ [ksi]</th>
<th>$\frac{P}{L^2}$ (I) [psi]</th>
<th>$\sqrt[2]{\frac{P}{L^2}}$ (I) [psi]</th>
<th>$\frac{P}{L^2}$ (II) [psi]</th>
<th>$\sqrt[2]{\frac{P}{L^2}}$ (II) [psi]</th>
<th>$\frac{W}{L^3}$ simple column [lb/inch$^3$]</th>
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<td>46.2894</td>
<td>6.803</td>
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</tr>
</tbody>
</table>
TABLE 1 (b)

Column with one strut, four tension ties, made from aluminum-alloy.

<table>
<thead>
<tr>
<th>$\sigma$ [ksi]</th>
<th>$a^0$</th>
<th>$\frac{W}{L^3}$ [lb/inch$^3$]</th>
<th>$\frac{A_1}{A x 10^2}$</th>
</tr>
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<tbody>
<tr>
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<td>0.5997</td>
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<tr>
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<td>19.20</td>
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<td>21.20</td>
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<td>1.0403</td>
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<td>0.9645</td>
<td>1.4136</td>
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<td>1.5225</td>
<td>1.5726</td>
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<td>28.72</td>
<td>7.7527</td>
<td>1.6705</td>
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<td>50</td>
<td>34.30</td>
<td>102.356</td>
<td>1.6115</td>
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</tbody>
</table>
TABLE 2
Column with one strut, three tension ties, made from aluminum-alloy.

<table>
<thead>
<tr>
<th>$\sigma$ [ksi]</th>
<th>$\alpha^o$</th>
<th>$W/L^3$ [lb/inch$^3$]</th>
<th>$A_1/Ax10^2$</th>
</tr>
</thead>
<tbody>
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<td>0.8573</td>
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<td>22.30</td>
<td>0.1499</td>
<td>1.2694</td>
</tr>
<tr>
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<td>24.45</td>
<td>0.3429</td>
<td>1.6266</td>
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<td>20</td>
<td>26.12</td>
<td>0.6185</td>
<td>1.9438</td>
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<tr>
<td>25</td>
<td>27.45</td>
<td>0.9794</td>
<td>2.2420</td>
</tr>
<tr>
<td>30</td>
<td>28.70</td>
<td>1.5500</td>
<td>2.5080</td>
</tr>
<tr>
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<td>30.22</td>
<td>2.8392</td>
<td>2.6842</td>
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<tr>
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<td>32.50</td>
<td>7.9345</td>
<td>2.7782</td>
</tr>
<tr>
<td>45</td>
<td>35.42</td>
<td>33.5885</td>
<td>2.7804</td>
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<tr>
<td>50</td>
<td>37.40</td>
<td>105.4972</td>
<td>2.8851</td>
</tr>
<tr>
<td>$\delta$ [ksi]</td>
<td>$P/L^2$ (I) [psi]</td>
<td>$\sqrt{P/L^2}$ (I) [psi]</td>
<td>$P/L^2$ (II) [psi]</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
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<td>0.00315</td>
<td>0.00025</td>
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<tr>
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<td>0.008915</td>
<td>0.00319</td>
</tr>
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<td>0.016379</td>
<td>0.00671</td>
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<td>0.025218</td>
<td>0.01590</td>
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<td>0.035242</td>
<td>0.03105</td>
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<td>0.058379</td>
<td>0.08520</td>
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<td>0.071325</td>
<td>0.12718</td>
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<td>0.99361</td>
<td>0.09680</td>
<td>0.24840</td>
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<td>0.138927</td>
<td>0.4825</td>
</tr>
<tr>
<td>70</td>
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<td>0.19062</td>
<td>0.9084</td>
</tr>
<tr>
<td>80</td>
<td>6.2661</td>
<td>0.25032</td>
<td>1.5665</td>
</tr>
<tr>
<td>90</td>
<td>10.3344</td>
<td>0.32147</td>
<td>2.5836</td>
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<td>4.1714</td>
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<td>27.4216</td>
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<td>6.7554</td>
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<td>11.9698</td>
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<td>135</td>
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</tr>
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</table>

*Table 3.* Column with one strut, four tension ties, made from steel.
TABLE 4

Column with one strut, three tension ties
made from steel.

<table>
<thead>
<tr>
<th>(\sigma) [ksi]</th>
<th>(a^o)</th>
<th>(W/L^3) [lb/inch^3]</th>
<th>(A_1/Ax10^2)</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>14.45</td>
<td>0.01435</td>
<td>0.5809</td>
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<tr>
<td>10</td>
<td>17.35</td>
<td>0.05808</td>
<td>0.8254</td>
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<td>0.13243</td>
<td>1.0099</td>
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<tr>
<td>20</td>
<td>20.80</td>
<td>0.23722</td>
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<tr>
<td>25</td>
<td>22.0</td>
<td>0.37319</td>
<td>1.3461</td>
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<tr>
<td>30</td>
<td>22.85</td>
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<td>1.5125</td>
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<td>23.70</td>
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<td>1.6575</td>
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<td>24.45</td>
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<td>1.7970</td>
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<td>25.72</td>
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<td>27.0</td>
<td>2.50986</td>
<td>2.2890</td>
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<td>28.27</td>
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<td>2.4828</td>
</tr>
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<td>80</td>
<td>29.35</td>
<td>6.2036</td>
<td>2.6766</td>
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<td>30.35</td>
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<tr>
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<td>13.3948</td>
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<td>110</td>
<td>32.30</td>
<td>19.8377</td>
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<tr>
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<td>33.43</td>
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<td>3.3186</td>
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<td>35.15</td>
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<tr>
<td>135</td>
<td>37.80</td>
<td>359.551</td>
<td>3.1865</td>
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</table>
TABLE 5
Cross-sections for a column with three struts
made from aluminum-alloy.

<table>
<thead>
<tr>
<th>$\sigma$ [ksi]</th>
<th>$A_1/\text{Ax}10^2$</th>
<th>$A_2/\text{Ax}10^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Aluminum, four tension ties</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.47614</td>
<td>0.49210</td>
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<td>10</td>
<td>0.69934</td>
<td>0.80191</td>
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<td>0.89810</td>
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<td>1.21974</td>
<td>1.65809</td>
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<td>2.60242</td>
</tr>
<tr>
<td>50</td>
<td>1.52370</td>
<td>2.83670</td>
</tr>
<tr>
<td>b. Aluminum, three tension ties</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.58023</td>
<td>0.28604</td>
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<td>10</td>
<td>0.85935</td>
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<td>1.11027</td>
<td>0.52447</td>
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<td>1.31643</td>
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<tr>
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<td>1.52328</td>
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<td>1.90227</td>
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<tr>
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<td>0.78987</td>
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</table>
TABLE 6
Cross-sections for a column with three struts, four tension ties, made from steel.

<table>
<thead>
<tr>
<th>$\sigma$ [ksi]</th>
<th>$A_1/Ax10^2$</th>
<th>$A_2/Ax10^2$</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>0.28273</td>
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<td>1.59827</td>
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<td>130</td>
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<tr>
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<td>2.83423</td>
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</table>
TABLE 7
Cross-sections for a column with three struts, three tension ties, made from steel.

<table>
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<th>$\sigma$ [ksi]</th>
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<th>$A_2/\text{Ax}10^2$</th>
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</thead>
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</tbody>
</table>
Thin walled tube

Tension ties

Ideal hinges

Struts (ring and m tubes with circular cross-section)

FIG. I: Structural parts of the column supported by tension ties.
FIG. 2: Geometry of the wire supported column with n-1 struts.
FIG. 3: Equilibrium of forces at the end of the column \((x=L)\).
\[ a = (\Delta y)_j - (\Delta y)_{j-1} \]
\[ b = (\Delta x)_j - (\Delta x)_{j-1} \]
\[ r_j = r - (\Theta_j + \Delta \Theta_j) \]

**FIG. 4:** Geometry of tie \( j \) before and after lateral deflection.
FIG. 5: Deflection geometry at the plane of strut \( j \).
FIG. 6: Components of tensile forces in direction $j$ in the strut panel $j$. 
FIG. 8: Simplified buckling system.
FIG. 9: Geometry of the column with one strut.
FIG. 10: Material properties of 2024 T4 Al-Alloy in compression.
FIG. 11: $\phi$ for various values of $\alpha$ as a function of the structural index $P/12$. 

Constrained first buckling mode for different values of $k$. 

$\phi_2 [\text{kips}]$ 

$k=300 [\text{kips}]$ 

SECOND MODE 

$k=150 [\text{kips}]$ 

FIRST MODE 

$K=2$ 

$\sqrt{P/12} [\text{psi}]$
FIG. 12: Contribution to the weight of the wire supported column with one strut.
FIG. 14: Geometry of the system with three struts.
FIG. 15: Tension stresses at different buckling shapes.
FIG. 16: Contributions to the weight of the wire supported column with three struts.
FIG. 17: Material properties of Steel 3/4 hard in compression.
FIG. 18: Variables $\alpha$ and $A_1/A$ as functions of the structural index $P/L$ for Aluminum.
FIG. 19: Relation between variables $\frac{A_1}{A}$ and $\propto$ for Aluminum.
FIG. 20: Variables $\alpha$ and $A_1/A$ as functions of the structural index $P/L^2$ for steel.
FIG. 21: Relation between variables $A_1/A$ and $\alpha$
for Steel.
FIG. 23: Variables $A_1/A$ and $A_2/A$ as functions of the structural index $P/L^2$ for Aluminum.
FIG. 24: Relation between variables \( A_1/A \) and \( \alpha_1 \) for Aluminum.
FIG. 25: Relation between variables $\frac{A_2}{A}$ and $\alpha_2$ for Aluminum.
FIG. 26a: Weight as a function of the structural index $P/L$ for Aluminum.
FIG. 26b: Weight as a function of the structural index $P/L^2$ for Aluminum.
FIG. 27. Variables $\alpha$ and $\omega$ as functions of the structural index $P/L$ for steel.
FIG. 29: Relation between variables $A_1/A$ and $\kappa_1$ for Steel.
FIG. 30: Relation between variables $\frac{A_2}{A}$ and $\alpha_2$ for Steel.
FIG. 31a: Weight as a function of the structural index $P/L$ for steel.
FIG. 31b: Weight as a function of the structural index $P/L^2$ for Steel.
FIG. 33: Weight saving factor for steel columns.
APPENDIX II

Computer Programs and Results
$JOB CC694G 002 015 HAGEN MAUCH BOOM PROBLEM OPTIMUM ANGLE
$EXECUTE IBJOB

IBJOB VERSION 4
$1BJOB BOOM
$1BFTC 3TIES

BEGIN COMPILATION 0 M 11 S
98 FORMAT (1H1)
99 FORMAT (F10.9)
100 FORMAT (1H0,10X,F10.7,10X,F10.3,10X,F10.5,10X,F10.5)

TOL=0.00001

WRITE (6,98)
1 READ (5,99)F
B1= 0.3456367
B2=3.0*B1/2.0
B3=1.131887
B4= 0.17535375
B5=2.0*B3/3.0
A=2.0/3.0
B=1.0/3.0
X=0.1

10 XA= (1.0-X*X*X)/(X*X*X)
X1= B1*F*SQRT(F)*XA*SQRT(XA)-X
X2=1.0+B2*F*SQRT(F)*SQRT(XA)*(1.0+3.0/(X*X*X))
Y=X1+1.0/(X1*X1*X1)-B3*X1*A/F-B4*(1.0/(X*X*X)-X)
Y1=X2+3.0*X2/(X1*X1*X1)+B5*X2/(F*X1*B)+B4*(1.0+3.0/(X*X*X))
XA1=X-Y/Y1

IF (ABS(XA1-X).LT.TCL) GC TO 500

X=XA1
GO TO 10

500 A1=ATAN(X1)*180.0/3.14159
A2=ATAN(X2)*180.0/3.14159
A3= ATAN(X1)
A4= ATAN(X)
C1=1.0/(SIN(A3)*SIN(A3)*COS(A3))
C2=1.0/(SIN(A4)*SIN(A4)*COS(A4))
WRITE (6,100) F, A1, A2, C1, C2
GO TO 1

END

BEGIN ASSEMBLING 3TIES 0 M 17 S
BEGIN COMPILATION 0 M 09 S
98 FORMAT (1H1)
99 FORMAT (F10.9)
100 FORMAT (1H0, 10X, F10.7, 10X, F6.3, 10X, F6.3, 10X, F10.5, 10X, F10.5)
TOL=0.00001
WRITE (6,98)
1 READ (5,99) F
B1= 9.025389
B2=3.0*B1/2.0
B3= 1.547667
B4= 1.092845
B5=2.0*B3/3.0
A=2.0/3.0
B=1.0/3.0
X=0.1
10 XA= (1.0-X*X*X)/(X*X*X)
X1= B1*SQRT(F)*XA*SQRT(XA-X)
X2=1.0+B2*F*SQRT(F)*SQRT(XA)*(1.0+3.0/(X*X*X*X))
Y=X1+1.0/(X1*X1*X1)-B3*X1*A/B4*(1.0/(X*X*X)-X)
Y1=X2+3.0*X2/(X1*X1*X1*X1)+B5*X2/(F*X1*B1)*B4*(1.0+3.0/(X*X*X*X))
XAI=X-Y/Y1
IF (ABS(XA1-X).LY.TOL) GO TO 500
X=XAI
GO TO 10
500 A1=ATAN(X1)*180.0/3.14159
A2=ATAN(X)*180.0/3.14159
A3= ATAN(X1)
A4= ATAN(X)
C1=1.0/(SIN(A3)*SIN(A3)*COS(A3))
C2=1.0/(SIN(A4)*SIN(A4)*COS(A4))
WRITE (6,100) F, A1, A2, C1, C2
GO TO 1
END
BEGIN ASSEMBLING 4TIES 0 M 15 S
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**Optimum Angles**

3 TIES
BEGIN COMPILETION
C M 05 5
98 FCHAR (IH1) 59 FCHAR (F14.0,F16.C,F1C,C,15,F12.0) 100 FCHAR (1H0,1C,H,F5.1,F19.9,F1C.3,F1G.3,F15.5)
  C1=5.0/3.0
  C2=2.0/3.0
WRITE (6,98)
10 READ (5,99) A6, A5, A1, N, A2
GC TC (20,21,22,23), X
20 B1=0.1307E098
   B2=0.1427E254
   B3=0.0985992C4
   B4=7.2279847
   B5=1.3440E32
   B6=0.283
   GC TC 30
21 B1=0.3524E58E
   B2=0.3848E3295
   B3=0.1927E460
   B4=7.2279847
   B5=1.3440E32
   B6=0.105
   GC TC 30
22 B1=0.1307E916
   B2=0.0229E3438
   B3=0.5791E7485
   B4=6.218E96
   B5=3.1525E51
   B6=0.283
   GC TC 30
23 B1=0.3524E752
   B2=0.141E36095
   B3=0.1533E577
   B4=6.218E96
   B5=3.1525E51
   B6=0.105
30 A3=A1*3.14159/180.0
   A4=A2*3.14159/180.0
   G1= SIN (A3)
   G2= SIN (A4)
   G3=SQR (1.0-G1*G1)
   G4=SQR (1.0-G2*G2)
   F1=1CC0.0*G1*G1*(1.0-G1*G1)
   F2=1CC0.0*G2*G2*(1.0-G2*G2)
   F3=2.0*(G1/G3)**C1*(F4*A5/100C.0)**C2
   F4=(G1/G3+G2/G4)**C1*(B5*A5/100C.0)**C2
   XA=1CC0.0*F6*(A5*(1.0/A6+B1/F1+B2/F2)+A3*(F3+F4)/10.0)
WRITE (6,100) A6, A5, A1, A2, XA
GC TC 10
END
BEGIN ASSEMBLING MTWB 0 M 11 5
### WEIGHT

#### 3 TIES

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#### STEEL

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