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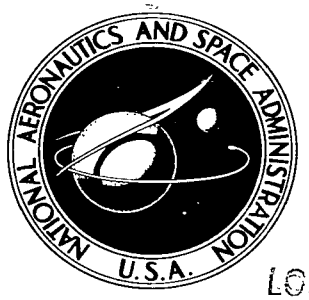

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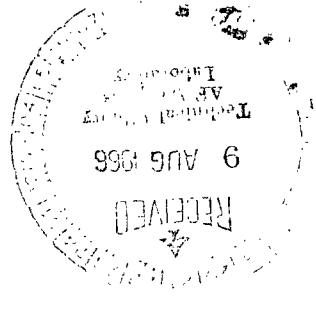

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SUMMARY

The integral equation approach to the solution of the Boltzmann equation has been applied to the flow of electrons through a rarefied gas between plane parallel electrodes. A perfect Lorentz gas model is assumed. Self-consistent-field solutions of the Boltzmann equation with Poisson's equation have been obtained for ratios of electrode spacing to mean free path of 0, 0.5, 1, and 2. Density and potential curves together with current-voltage characteristics are presented.

INTRODUCTION

The flow of charged particles between an electrode and an ionized gas has attracted the attention of several investigators since the early work of Langmuir. Most of these studies have been concerned with the theory of electrostatic probes (refs. 1 to 9). Some work, however, has been done in connection with the analysis of thermionic converters (refs. 10 and 11). All these treatments have involved one of two approaches. In the first, either collisionless or diffusion equations (depending on the density range of interest) have been applied continuously from the electrode to the undisturbed portion of the gas or to another electrode. In the second, the space has been divided into two or more distinct regions; appropriate equations, either collisionless or diffusion, have been applied in each region, and more or less arbitrary conditions have been used to match solutions at the boundaries between regions.

In situations where collisions are important, both of these approaches are questionable. There will inevitably be a transition region, however brief in extent, in which neither collisions nor particle dynamics may be considered as dominant. From the experiences of investigators studying this regime in ordinary gas dynamics, it would seem that no simple solvable model is generally valid here. Hence, in the present work, an

attack is made on the problem by means of the Boltzmann equation. Rather than treating the problem in all its complexity, however, a simpler problem was treated as completely as possible, namely, the flow of electrons through a rarefied gas between parallel electrodes. Taking one of the electrodes to be thermionically emitting and the other cold allows some of the aspects of flow between an ionized gas and a cold electrode to be retained. The next steps would be the inclusion of positive ions and then the substitution of an ionized gas for the hot electrode.

Possible methods for solving the Boltzmann equation for such a boundary value problem are provided by the half-range polynomial method of Gross, Jackson, and Ziering (ref. 12) and the discrete ordinate method of Chandrasekhar (ref. 13). The inclusion of a spatially varying electric field, however, makes these methods intractible. Instead, the original integro-differential equation is converted to a pure integral equation and is solved numerically. The method is similar to that used by Willis to solve the Krook kinetic model for shear flow (ref. 14).

The particular problem considered herein has been treated recently by Goldstein, using Monte Carlo techniques (ref. 15). While the present method is perhaps somewhat more difficult to apply, it offers the advantage of obtaining information about the distribution function itself.

FORMULATION

The problem under consideration is that of the steady flow of electrons between infinite plane-parallel electrodes. The Boltzmann equation for the electron distribution function $\bar{f}$ may be written as (ref. 16)

$$\bar{v} \cos\theta \frac{\partial \bar{f}}{\partial \bar{x}} + \bar{a}_x \left(\cos\theta \frac{\partial \bar{f}}{\partial \bar{v}} + \frac{\sin^2\theta}{\bar{v}} \frac{\partial \bar{f}}{\partial \cos\theta} \right) = \left(\frac{\delta \bar{f}}{\delta t} \right)_c \tag{1}$$

where $\bar{x}$, $\bar{v} \cos\theta$, and $\bar{a}_x$ are, respectively, distance, velocity, and acceleration normal to the electrodes. For electrons interacting with the space charge field

$$\bar{a}_x = \frac{e}{m} \frac{dV}{d\bar{x}} \tag{2}$$

where V is the electrostatic potential. Bars have been used to designate dimensional quantities.

For electrode separations that are not too large compared with the mean free path,

the loss of energy by electrons due to elastic collisions with atoms is negligible. In addition, all inelastic and electron-electron collisions are neglected. The collision term in equation (1) thus reduces to that for a perfect Lorentz gas (ref. 17):

$$\left(\frac{\delta \bar{f}}{\delta t}\right)_c = N\bar{v}\int \left[\bar{f}(\bar{x}, \bar{v}, \theta') - \bar{f}(\bar{x}, \bar{v}, \theta)\right] \sigma(\bar{v}, \chi)\, d\omega' \tag{3}$$

where N is the atom density, σ the differential scattering cross section, χ the scattering angle, and $d\omega'$ the element of solid angle. Primes refer to values after a collision.

The number density and the particle current are given in terms of $\bar{f}$ by

$$\bar{n} = 2\pi \int_0^\infty \int_0^\pi \bar{f} \sin\theta \, d\theta \, \bar{v}^2 \, d\bar{v} \tag{4}$$

and

$$\bar{J} = 2\pi \int_0^\infty \int_0^\pi \bar{f} \cos\theta \sin\theta \, d\theta \, \bar{v}^3 \, d\bar{v} \tag{5}$$

Equations (1) to (4) are to be solved self-consistently with Poisson's equation

$$\frac{d^2V}{d\bar{x}^2} = 4\pi e\bar{n} \tag{6}$$

The boundary conditions on the distribution function $\bar{f}$ are of a rather peculiar nature; $\bar{f}$ is not completely known on one surface in phase space, such as that specified by $\bar{x} = 0$. The number of particles of a given velocity returned to a surface by back scattering from the gas is not known a priori. Instead, the emergent half of $\bar{f}$ is known at each of two bounding surfaces, $\bar{x} = 0$ and $\bar{x} = L$. Hence, with half-Maxwellian emission of electrons at $\bar{x} = 0$ and complete absorption of electrons at both $\bar{x} = 0$ and $\bar{x} = L$, the boundary conditions on $\bar{f}$ and v are

$$\left.\begin{aligned} \bar{x} = 0,\ \cos\theta > 0,\ \bar{f} &= 2\bar{n}_o\left(\frac{m}{2\pi k T_o}\right)^{3/2} e^{-m\bar{v}^2/2kT_o} \\ \bar{x} = L,\ \cos\theta < 0,\ \bar{f} &= 0 \\ V(0) = 0,\ V(L) &= V_c \end{aligned}\right\} \tag{7}$$

where $\bar{n}_o$ and T_o are the density and temperature of the emitted electrons, L the electrode spacing, and V_c the potential of the cold collector.

Before proceeding further, it is necessary to assume that the differential scattering cross section remains finite for small scattering angles χ. Then the collision integral, equation (3), may be separated into two finite integrals. The cross section can then be approximated by a finite sum of Legendre polynomials in $\cos \chi$, and then in turn, expressed in terms of $\cos \theta$ and $\cos \theta'$ by means of the addition theorem for spherical harmonics. This procedure would assume more information than is generally available on electron-atom cross sections. Hence, in the present work, σ is assumed independent of χ. Equation (3) is rewritten as

$$\left(\frac{\delta \bar{f}}{\delta t}\right)_c = -\lambda^{-1}\bar{v}\bar{f} + \frac{1}{2}\lambda^{-1}\bar{v}\int_0^{\pi} \bar{f}(\bar{x}, \bar{v}, \theta') \sin \theta' \, d\theta' \tag{8}$$

where the mean free path λ is given by

$$\lambda^{-1} = 4\pi N\sigma \tag{9}$$

and the element of solid angle $d\omega'$ has been expressed as

$$d\omega' = 2\pi \sin \theta' \, d\theta' \tag{10}$$

Dimensionless variables are now introduced:

$$x = \frac{\bar{x}}{L}, \quad v = \bar{v}\left(\frac{m}{2kT_o}\right)^{1/2}, \quad \mu = \cos \theta, \quad \varphi = \frac{eV}{kT_o}$$

$$f = \frac{\bar{f}}{2\bar{n}_o}\left(\frac{2\pi kT_o}{m}\right)^{3/2}, \quad n = \frac{\bar{n}}{\bar{n}_o}, \quad J = \frac{\bar{J}}{\bar{n}_o}\left(\frac{\pi m}{2kT_o}\right)^{1/2} \equiv \frac{\bar{J}}{\bar{J}_o}$$

$$\kappa = \frac{L}{\lambda}, \quad C = \frac{4\pi e^2 n_o L^2}{kT_o} = 8\left(\frac{\pi}{2kT_o}\right)^{3/2} m^{1/2} e^2 \bar{J}_o L^2$$

Equations (1), (2), and (8) are combined as

$$v\mu \frac{\partial f}{\partial x} + \frac{1}{2} \frac{d\varphi}{dx} \left(\mu \frac{\partial f}{\partial v} + \frac{1 - \mu^2}{v} \frac{\partial f}{\partial \mu} \right) = -\kappa v f + \frac{1}{2} \kappa v \int_{-1}^{1} f(x, v, \mu') d\mu' \tag{11}$$

and equations (4) to (7) become

$$n = \frac{4}{\sqrt{\pi}} \int_0^\infty \int_{-1}^{1} f \, d\mu \, v^2 \, dv \tag{12}$$

$$J = 4 \int_0^\infty \int_{-1}^{1} f\mu \, d\mu \, v^3 \, dv \tag{13}$$

$$\frac{d^2\varphi}{dx^2} = Cn \tag{14}$$

$$\left.\begin{aligned} x = 0, \ \mu > 0, \ f = e^{-v^2} \\ x = 1, \ \mu < 0, \ f = 0 \\ \varphi(0) = 0, \ \varphi(1) = \varphi_c \end{aligned}\right\} \tag{15}$$

The transformation of equation (11) to an integral equation is accomplished by means of a change of variables from (x, v, μ) to (x, w, u) where

$$\left.\begin{aligned} w &= v^2 - \varphi(x) \\ u &= \mu^2 v^2 - \varphi(x) \end{aligned}\right\} \tag{16}$$

The nondimensional total energy of an electron is denoted by w, and u is the nondimensional portion of the total energy associated with the motion in the x-direction; thus, w and u are constants of the motion between collisions. Since f is a double valued function of u, f is denoted by $f^{\pm}$ for $\mu \gtrless 0$. Finally, with note taken of the boundary conditions (15), $g^{\pm}$ is defined by

$$f^{\pm} = e^{-w} g^{\pm} \tag{17}$$

Then equations (11) to (13) become

$$\frac{\partial g^{\pm}}{\partial x} = \mp\kappa\alpha(x)g^{\pm} \pm \frac{1}{2}\frac{\kappa}{\sqrt{u + \varphi(x)}} S(x, w) \tag{18}$$

$$n = \frac{2}{\sqrt{\pi}} \int_{-\varphi(x)}^{\infty} e^{-w} S(x, w)\, dw \tag{19}$$

$$J = \int_{-\varphi(x)}^{\infty} e^{-w} \int_{-\varphi(x)}^{w} (g^{+} - g^{-})\, du\, dw \tag{20}$$

with

$$\alpha(x) = \sqrt{\frac{w + \varphi(x)}{u + \varphi(x)}} \tag{21}$$

$$S(x, w) = \frac{1}{2} \int_{-\varphi(x)}^{w} (g^{+} + g^{-}) \frac{du}{\sqrt{u + \varphi(x)}} \tag{22}$$

The boundary conditions (15) become

$$\left.\begin{aligned} x = 0,\ g^{+} = 1 \\ x = 1,\ g^{-} = 0 \\ \varphi(0) = 0,\ \varphi(1) = \varphi_c \end{aligned}\right\} \tag{23}$$

Equation (18) is formally integrated to give

$$g^{\pm} = A^{\pm} \exp\left[\mp \int_{x_o^{\pm}}^{x} \kappa\alpha(z)dz\right] \pm \frac{1}{2} \int_{x_o^{\pm}}^{x} \frac{\kappa}{\sqrt{u + \varphi(y)}} S(y, w) \exp\left[\mp \int_{y}^{x} \kappa\alpha(z)dz\right] dy \tag{24}$$

The first term in equation (24) is the contribution to the distribution function from particles that have not suffered collisions; it decays exponentially as the ratio of the path length to the mean free path. The second term represents the contribution from particles scattered into a particular "volume" element in phase space. The $A^{\pm}$ and $x_o^{\pm}$

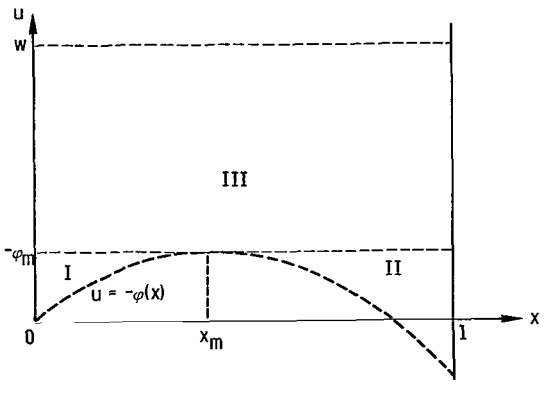

Figure 1. - Diagram of u, x plane.

are arbitrary functions of w and u. They are to be determined from the boundary conditions (23) together with continuity considerations at turning points for electrons in the electric field (see following section). Note that, in general, κ will be a function of $(w + \varphi(x))$.

Equation (24) is an integral equation for $g^{\pm}$, as $S(x, w)$ is functionally dependent on g. A single linear integral equation for $S(x, w)$ may be obtained by eliminating $g^{\pm}$ between equations (22) and (24).

ELECTRON FLOW WITH POTENTIAL MINIMUM

The equations of the previous section are now applied to a situation in which a potential minimum occurs within the interelectrode space. All other real situations can be obtained as special cases of this one by defining the potential minimum to be the point of lowest potential in the system even when this occurs at one of the electrodes.

Consider now the diagram (fig. 1) of the u, x plane for a given value of w. From equation (16), it is seen that $u \geq -\varphi(x)$, and thus the surface $u = -\varphi(x)$ is a lower boundary for the phase space. At this surface, electrons in region I are reflected and transfer from g^+ to g^-; the opposite is true for region II. Conservation of electrons requires that $g^+ = g^-$ on this surface. The complete conditions on $g^{\pm}$ are as follows:

For $0 \leq x \leq x_w$, $-\varphi(x) \leq w \leq -\varphi_m$, $-\varphi(x) \leq u \leq w$ (region I),

$$\left.\begin{aligned} x = 0,\ g^+ &= 1 \\ x = x_t,\ g^- &= g^+ \end{aligned}\right\} \tag{25}$$

For $0 \leq x \leq x_m$, $-\varphi_m \leq w < \infty$, $-\varphi(x) \leq u \leq -\varphi_m$ (region I),

$$\left.\begin{aligned} x = 0,\ g^+ &= 1 \\ x = x_t,\ g^- &= g^+ \end{aligned}\right\} \tag{26}$$

For $x_m \le x \le 1$, $-\varphi_m \le w < \infty$, $-\varphi(x) \le u \le -\varphi_m$ (region II),

$$\left.\begin{aligned} x = x_t,\ g^+ = g^- \\ x = 1,\ g^- = 0 \end{aligned}\right\} \tag{27}$$

For $0 \le x \le 1$, $-\varphi_m \le w < \infty$, $-\varphi_m \le u \le w$ (region III),

$$\left.\begin{aligned} x = 0,\ g^+ = 1 \\ x = 1,\ g^- = 0 \end{aligned}\right\} \tag{28}$$

where

$$\left.\begin{aligned} w + \varphi(x_w) = 0 \\ u + \varphi(x_t) = 0 \end{aligned}\right\} \tag{29}$$

The points $x_t(u)$ are turning points for electrons of "energy" u in the electric field.

Condition (25) is the same as condition (26) and is not considered separately. Application of the conditions to equation (24) gives the following equations:

For $0 \le x \le x_m$, $-\varphi(x) \le u \le -\varphi_m$ (region I),

$$\left.\begin{aligned} g^+ &= \exp\left[-\int_0^x \kappa\alpha(z)dz\right] + \frac{1}{2}\int_0^x \frac{\kappa}{\sqrt{u+\varphi(y)}} S(y,w)\exp\left[-\int_y^x \kappa\alpha(z)dz\right]dy \\ g^- &= \exp\left[-\int_0^{x_t} \kappa\alpha(z)dz - \int_x^{x_t} \kappa\alpha(z)dz\right] \\ &\quad + \frac{1}{2}\int_0^{x_t} \frac{\kappa}{\sqrt{u+\varphi(y)}} S(y,w)\exp\left[-\int_y^{x_t} \kappa\alpha(z)dz - \int_x^{x_t} \kappa\alpha(z)dz\right]dy \\ &\quad + \frac{1}{2}\int_x^{x_t} \frac{\kappa}{\sqrt{u+\varphi(y)}} S(y,w)\exp\left[-\int_x^{y} \kappa\alpha(z)dz\right]dy \end{aligned}\right\} \tag{30}$$

For $x_m \le x \le 1$, $-\varphi(x) \le u \le -\varphi_m$ (region II),

$$\left.\begin{aligned} g^+ &= \frac{1}{2}\int_{x_t}^{1} \frac{\kappa}{\sqrt{u+\varphi(y)}}\, S(y,w) \exp\left[-\int_{x_t}^{y} \kappa\alpha(z)dz - \int_{x_t}^{x} \kappa\alpha(z)dz\right] dy \\ &\quad + \frac{1}{2}\int_{x_t}^{x} \frac{\kappa}{\sqrt{u+\varphi(y)}}\, S(y,w) \exp\left[-\int_{y}^{x} \kappa\alpha(z)dz\right] dy \\ g^- &= \frac{1}{2}\int_{x}^{1} \frac{\kappa}{\sqrt{u+\varphi(y)}}\, S(y,w) \exp\left[-\int_{x}^{y} \kappa\alpha(z)dz\right] dy \end{aligned}\right\} \quad (31)$$

For $0 \le x \le 1$, $-\varphi_m \le u \le w$ (region III),

$$\left.\begin{aligned} g^+ &= \exp\left[-\int_{0}^{x} \kappa\alpha(z)dz\right] + \frac{1}{2}\int_{0}^{x} \frac{\kappa}{\sqrt{u+\varphi(y)}}\, S(y,w) \exp\left[-\int_{y}^{x} \kappa\alpha(z)dz\right] dy \\ g^- &= \frac{1}{2}\int_{x}^{1} \frac{\kappa}{\sqrt{u+\varphi(y)}}\, S(y,w) \exp\left[-\int_{x}^{y} \kappa\alpha(z)dz\right] dy \end{aligned}\right\} \quad (32)$$

Examination of equations (30) to (32) shows that there are discontinuities in g^- between regions I and III and in g^+ between regions II and III. These discontinuities are permissible, however, as $u = -\varphi_m$ is a characteristic surface for the original integro-differential equation (11).

Substitution of equations (30) to (32) into equation (22), which defines $S(x,w)$, and interchanging the order of the u and y integrations gives a Fredholm integral equation for $S(x,w)$. Care must be taken in interchanging the order of integration, as x_t is a function of u. For example,

$$\int_{-\varphi(x)}^{b}\int_{x}^{x_t} dy\, du = \int_{x}^{x_b}\int_{-\varphi(y)}^{b} du\, dy$$

where

$$u + \varphi(x_t) = 0, \quad b + \varphi(x_b) = 0$$

The integral equation may be written as

$$S(x, w) = F(x, w) + \frac{1}{2}\int_0^{x_1} \kappa H(x, y, w) S(y, w)\, dy \tag{33}$$

where

$$x_1 = \begin{cases} x_w, & -\varphi(x) \le w \le -\varphi_m \\ 1, & -\varphi_m \le w < \infty \end{cases} \tag{34}$$

$$F(x, w) = \frac{1}{2}\int_{a_f}^{w} \exp\left[-\int_0^x \kappa\alpha(z)\, dz\right] \frac{du}{\sqrt{u + \varphi(x)}}$$

$$+ \frac{1}{2}\int_{a_f}^{b} \exp\left[-\int_0^{x_t} \kappa\alpha(z)\, dz - \int_x^{x_t} \kappa\alpha(z)\, dz\right] \frac{du}{\sqrt{u + \varphi(x)}} \tag{35}$$

$$H(x, y, w) = \frac{1}{2}\int_{a_h}^{w} \exp\left[-\left|\int_y^x \kappa\alpha(z)\, dz\right|\right] \frac{du}{\sqrt{u + \varphi(x)}\,\sqrt{u + \varphi(y)}}$$

$$+ \frac{1}{2}\int_{a_h}^{b} \exp\left[-\left|\int_{x_t}^x \kappa\alpha(z)\, dz\right| - \left|\int_{x_t}^y \kappa\alpha(z)\, dz\right|\right] \frac{du}{\sqrt{u + \varphi(x)}\,\sqrt{u + \varphi(y)}} \tag{36}$$

and

$$b = \begin{cases} w, & -\varphi(x) \le w \le -\varphi_m \\ -\varphi_m, & -\varphi_m \le w < \infty \end{cases} \tag{37}$$

$$a_f = \begin{cases} -\varphi(x), & 0 \le x \le x_m \\ -\varphi_m, & x_m \le x \le 1 \end{cases} \tag{38}$$

$$a_h = \begin{cases} -\varphi(x), & \varphi(x) \le \varphi(y) \\ -\varphi(y), & \varphi(y) \le \varphi(x) \\ -b, & y \le x_m \le x \text{ or } x \le x_m \le y \end{cases} \tag{39}$$

From equations (36) and (39), it may be seen that $H(x, y, w)$ is symmetric in x and y.

For $x > x_m$, the distribution function vanishes for $w < -\varphi_m$. Hence, equation (19) for the density may be rewritten as

$$n = \frac{2}{\sqrt{\pi}} \int_{a_f}^{\infty} e^{-w} S(x, w)\, dw \tag{40}$$

The current is most conveniently evaluated at the emitter ($x = 0$). Only those electrons whose energy $w > -\varphi_m$ can contribute to the current; hence, substitution of equations (30) and (32) for $g^{\pm}$ into equation (20) and interchanging the order of the u and y integrations give

$$J = e^{\varphi_m}(1 - \varphi_m) - \int_{-\varphi_m}^{\infty} e^{-w} \int_0^{-\varphi_m} \exp\left[-2 \int_0^{x_t} \kappa \alpha(z)\, dz\right] du\, dw$$

$$- \int_{-\varphi_m}^{\infty} e^{-w} \int_0^1 \kappa S(y, w) F(y, w)\, dy\, dw \tag{41}$$

NUMERICAL INTEGRATION

The solution of equation (33) presents certain difficulties as the kernel $H(x, y, w)$ is singular for $y = x$. The following method, however, proved to be quite satisfactory.

On deleting the w, equation (33) may be written as

$$S(x) = F(x) + \frac{1}{2} \int_0^{x_1} \kappa(y) H(x, y) S(y)\, dy \tag{42}$$

Now as $y \to x$ for $x \neq x_m$

$$H(x, y) \to -\log\left(|x - y|\right) + \text{const} + \cdots \tag{43}$$

This fact is utilized to replace equation (42) by a system of linear algebraic equations. Equation (42) is rewritten as

$$S(x) = F(x) + \frac{1}{2} L(x)S(x) + \frac{1}{2} \int_0^{x_1} \kappa(y)\left[\log\left(|x - y|\right)S(x) + H(x, y)S(y)\right]dy \tag{44}$$

with

$$L(x) = -\int_0^{x_1} \kappa(y)\log\left(|x - y|\right)dy \tag{45}$$

The integral in equation (44), whose integrand is everywhere finite, is replaced by a quadrature sum

$$\int_0^{x_1} f(x)dx \approx \sum_{j=1}^{n} a_j f(x_j) \tag{46}$$

and the equation is evaluated at each of the quadrature points to give the desired set of algebraic equations

$$S_i = F_i + \frac{1}{2} \sum_{j=1}^{n} G_{ij}S_j \qquad (i = 1, \cdots, n) \tag{47}$$

where

$$S_i = S(x_i)$$

$$F_i = F(x_i)$$

$$G_{ij} = \begin{cases} a_j \kappa(x_j) H(x_i, x_j) & (j \neq i) \\ L(x_i) + a_i \kappa(x_i) H_\ell(x_i) + \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \kappa(x_k) \log\left(|x_i - x_k|\right) & (j = i) \end{cases} \tag{48}$$

and

$$H_\ell(x) = \lim_{y \to x} \left[\log\left(|x - y|\right) + H(x, y) \right] \tag{49}$$

This limit can be found numerically.

The set of equations (47) can be solved by standard techniques. As $H(x, y)$ is symmetric in x and y, the set may be rearranged into a form with a symmetric matrix. Care must be taken in the choice of the quadrature formula (46). The function $F(x)$ has a large negative derivative (in some cases infinite) at $x = 0$. To achieve sufficient accuracy from the approximation (eq. (46)), the quadrature points must be grouped closely together near $x = 0$. In the present calculations, this was accomplished by subdividing the range of integration into two parts and applying Gaussian quadrature formulas in each section. The point of subdivision was taken quite close to $x = 0$.

The complete calculational procedure can be outlined as follows. An initial potential distribution was assumed. Equations (47) were solved for $S(x, w)$ for selected values of w, and the density $n(x)$ was obtained from equation (40). By means of various curve-fitting techniques, a Chebyshev series was fitted to $n(x)$, and the potential $\varphi(x)$ was obtained from Poisson's equation (14) by analytically integrating the Chebyshev series. This Picard iteration sequence was continued with the new potential for $\varphi(x)$ until convergence was achieved. The current was then obtained from equation (41) with values from the final iteration. Solutions from the collisionless case were used as initial iterates for cases with low values of κ. As collisional results became available, they were used as initial iterates for cases with higher values of κ.

RESULTS

The equations have been solved on a high-speed computer for the case of rigid sphere scattering (κ constant). The particular solutions were for $\kappa = 0,\ 0.5,\ 1,$ and 2 with C (eq. (14)) = 50. An emitter temperature of 1900° K, a spacing of 10 microns,

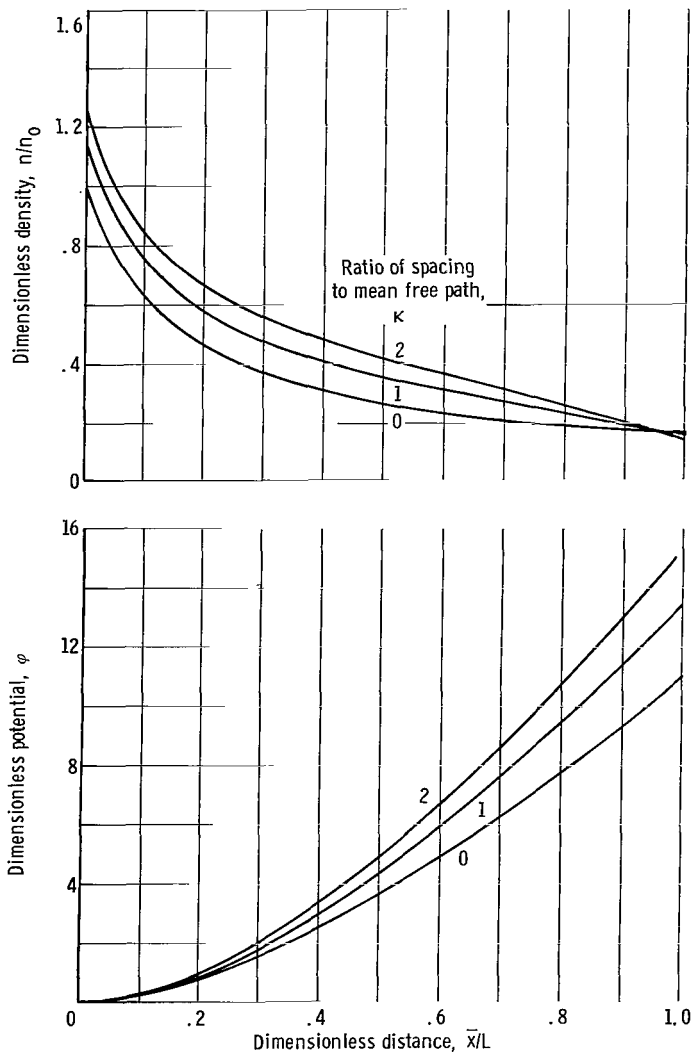

Figure 2. - Variation of density and potential in interelectrode space. Initial slope, $\varphi'(0) = 0$; space charge parameter, $C = 50$ (eq. (14)).

and an emission current of 10 amperes per square centimeter would give a value for C of about 50.

Figure 2 shows the variations of density and potential, respectively, with x and κ for the potential with an initial slope $\varphi'(0)$, of 0. The steep initial slopes in the density curves reflect the similar behavior in the function F(x, w) of equation (35).

Figure 3 shows the current-voltage characteristics obtained as the result of many solutions of the combined equations. The agreement between the present results and those of Goldstein (ref. 15) is of a peculiar form. While the points appear to lie on the same curve, attention must be given to the manner in which these points were obtained. In both calculations, φ and φ' were specified at $x = 0$; nevertheless, Goldstein's points lie somewhat to the left of those obtained in this work for the same conditions.

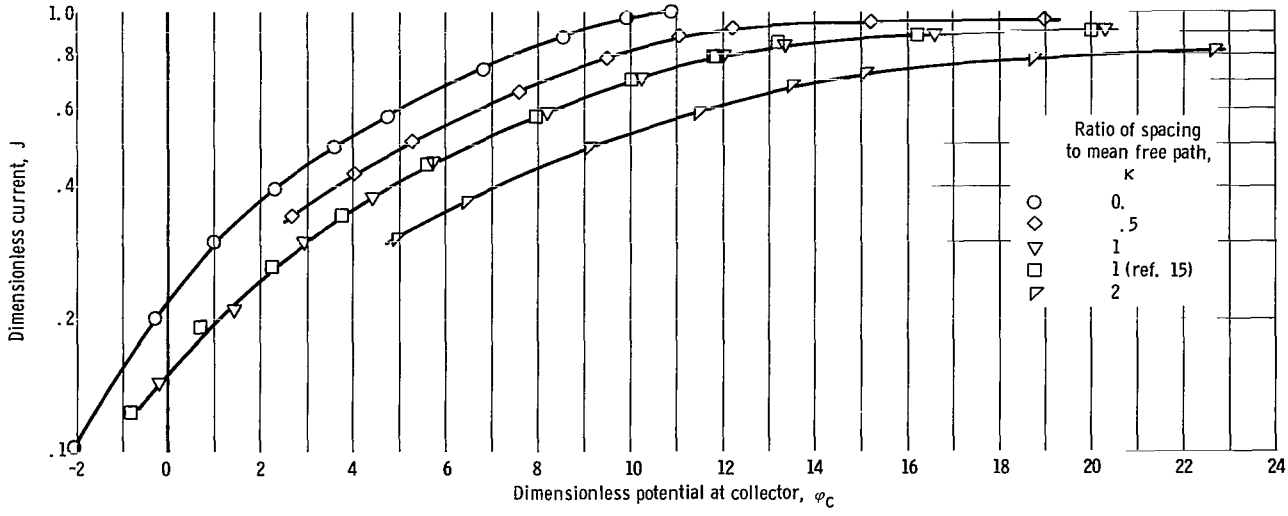

Figure 3. - Effect of electron scattering on current-voltage characteristics. Space charge parameter, C = 50 (eq. (14)).

This discrepancy is well outside the standard deviation of the Monte Carlo calculation at the low end of the current-voltage characteristic. The difference is probably due to numerical difficulties in one or both of the treatments.

CONCLUSIONS

The integral equation approach to the solution of the Boltzmann equation has been demonstrated for a problem of electron flow through a rarefied gas over distances comparable to the electron mean free path. The method should be capable of extension to more complicated problems including the flow of charged particles between an electrode and an ionized gas. In addition, as the method provides information about the distribution function itself, it could be used as the basis of an investigation into approximate methods of solving these types of problems.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, March 7, 1966.

APPENDIX - SYMBOLS

a_f	integration limit (eq. (38))
a_h	integration limit (eq. (39))
a_j	quadrature weight (eq. (46))
a_x	acceleration
b	integration limit (eq. (37))
C	space charge parameter (eq. (14))
e	electronic charge
F(x, w)	inhomogeneous function if integral equation
$f(x, v, \theta)$	distribution function
G_{ij}	matrix (eq. (48))
g(x, w, u)	transformed distribution function
$H_\ell(x)$	limit function (eq. (49))
H(x, y, w)	kernel of integral equation
J	dimensionless current
L	electrode spacing
L(x)	logarithmic integral (eq. (45))
m	electronic mass
N	atom density
n(x)	electron density
$\bar{n}_o$	electron emission density
S(x, w)	source function (eq. (22))
T_o	electron emission temperature
u	dimensionless energy variable (eq. (16))
V(x)	potential
v	velocity
w	dimensionless energy variable (eq. (16))
x	distance
x_j	quadrature point
x_t, x_w	turning points (eq. (29))
x_1	integration limit (eq. (42))
$\alpha(x)$	function (eq. (21))
θ	polar angle in velocity space
κ	ratio of spacing to mean free path
λ	mean free path
μ	cos θ
$\sigma(\bar{v}, x)$	collision cross section
$\varphi(x)$	dimensionless potential
χ	scattering angle
ω	solid angle

Subscripts:

c	at collector
m	at potential minimum

Superscripts:

+	flow toward collector
-	flow toward emitter

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