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RESEARCH IN STABILITY OF PERIODIC MOTIONS

QUARTERLY PROGRESS

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FOREWORD

The principal objective of Contract NAS8-20323, Research in Stability of Periodic Motions, is to derive exact analytical results concerning the degree of instability of certain periodic orbits in the restricted and reduced three body problems. However, before this problem can be formulated, considerable background material must be assimilated. This report summarizes the background material studied during this period of contract performance.

This investigation is being performed by LMSC/HREC for the Aero-Astrodynamics Laboratory of the George C. Marshall Space Flight Center.

SUMMARY

For this investigation, the book by Dr. Carl Ludwig Siegel, Vorlesungen über Himmelsmechanik, is the basic reference. Consequently, during the initial period, this reference work was emphasized and chapters 1 through 16 and 19 were read. A brief summary of the material from these chapters deemed most important to the contractual objective is presented.

CONTENTS

INTRODUCTION

A discovery of particular significance to astronautics is the existence, within the context of the restricted and reduced three body problems, of periodic orbits enclosing the Earth and Moon. These orbits have great potential application for future space missions. However, before these orbits can be exploited, their dynamical characteristics, in particular their stability properties, must be completely understood.

The purpose of this investigation is to develop procedures for evaluating the stability of approximately periodic motion for a specified, finite time. The primary emphasis will be upon the area-preserving mapping near a fixed point developed by Birkhoff and extended by Moser and Arnold. In this context, the fixed point is the true periodic orbit which closes exactly upon itself after a time $t = T =$ period. Using this method, a small neighborhood of the fixed point at $t = 0$ is mapped into the corresponding neighborhood at $t = T$. In this way, rigorous analytical results concerning the stability of the approximate periodic motions can be obtained.

TECHNICAL DISCUSSION

For most problems in theoretical celestial mechanics, a Hamiltonian formulation is standard since, by suitable canonical transformations, the invariants of the motion will emerge and, also, the form of the differential equations will be preserved.

Consider the column vector z composed of n position and momentum components x_k and y_k, respectively. Then the Hamiltonian equations can be written

$$\dot{z} = J \frac{\partial H}{\partial z} = J H_z \tag{1}$$

where J is a 2n x 2n matrix composed of two identity matrices

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{2}$$

Introducing new variables ξ_k, η_k the necessary and sufficient condition for the preservation of the Hamiltonian form of the equations is sought. If ζ is the column vector of the ξ_k, η_k, the substitution can be written

$$z_k = z_k(\zeta_1, \zeta_2, \cdots, \zeta_{2n}, t) \quad k = 1, 2, \cdots, 2n \tag{3}$$

The functional matrix of the substitution is written

$$M = z_\zeta = \left(\frac{\partial z_k}{\partial \zeta_\ell} \right) \quad k, \ell = 1, 2, \cdots, 2n \tag{4}$$

and it is required that the transformation of variables is reversible. Thus $|M| \neq 0$ (M^{-1} then exists). Now, the appropriate condition satisfied by all the desired reversible transformations is

$$M^T J M = \lambda J \tag{5}$$

where $\lambda \neq 0$ is a scalar constant. The matrix M is termed symplectic (if $\lambda = 1$) and, in general, the substitution (3) is termed canonical.

The solution of the Hamiltonian equations of motion can be transformed to the solution of the Hamilton-Jacobi partial differential equation. This rather elegant technique is based upon the determination of a generating function $W = W(x, \eta, t)$ such that the old Hamiltonian system

$$\dot{x}_k = H_{y_k} \qquad \dot{y}_k = -H_{x_k} \quad (k = 1, 2, \cdots, n) \tag{6}$$

is transformed to a new Hamiltonian system

$$\dot{\xi}_k = \mathrm{H}_{\eta_k} \qquad \dot{\eta}_k = -\mathrm{H}_{\xi_k} \tag{7}$$

This generating function in fact constructs the desired canonical transformation of variables by the relationships

$$\begin{aligned} y_k &= W_{x_k} \qquad \xi_k = -W_{\eta_k} \qquad (k = 1, 2, \ldots n) \\ \mathrm{H} &= H + \frac{\partial W}{\partial t} \end{aligned} \tag{8}$$

with the assumption, which must be confirmed, that

$$\left| W_{x_k \eta_\ell} \right| \neq 0 \tag{9}$$

The simplest case for the transformed system (7) would be $\mathrm{H}(\xi, \eta, t) = 0$ for then we would have

$$\dot{\xi}_k = 0 \quad , \qquad \dot{\eta}_k = 0 \tag{10}$$

the so-called normal form which is immediately integrable with the ξ_k, η_k entering as 2n integration constants. The last equation in (8) then is

$$\mathcal{H} = H(x, y, t) + W_t = H(x, W_x, t) + W_t = 0 \tag{11}$$

the Hamilton-Jacobi partial differential equation for the generating function $W = W(x, \eta, t)$.

In order to demonstrate this technique, the following simple example is presented. Consider

$$H = \frac{1}{2} y_1^2 + x_1 y_1$$

with

$$\dot{x}_1 = H_{y_1} = y_1 + x_1$$

$$\dot{y}_1 = -H_{x_1} = -y_1 \tag{12}$$

which has the solution

$$y_1 = y_{10} e^{-t}$$

$$x_1 = c e^{t} - \frac{y_{10}}{2} e^{-t} \qquad c = x_{10} + \frac{y_{10}}{2}$$

$$x_{10} = x_1(0), \qquad y_{10} = y_1(0) \tag{13}$$

Now the canonical integration constants ξ_1, η_1 will be derived and related to the initial conditions. The Hamilton-Jacobi equation is

$$\frac{1}{2} W_{x_1}^2 + x_1 W_{x_1} + W_t = 0 \tag{14}$$

and, since the explicit time dependence is only in the last term, we put

$$W(x_1, \eta_1, t) = \omega(x_1, \eta_1) - \eta_1 t \qquad (15)$$

Then

$$\omega_{x_1}^2 + 2x_1\omega_{x_1} - 2\eta_1 = 0 \qquad (16)$$

$$\omega_{x_1} = -x_1 \pm \sqrt{x_1^2 + 2\eta_1} = y_1$$

$$W = -\frac{x_1^2}{2} \pm \int \sqrt{x_1^2 + 2\eta_1}\, dx_1 - \eta_1 t$$

for the generating function. Now, the assumption $W_{x_1 \eta_1} \neq 0$ yields

$$\left(x_1^2 + 2\eta_1\right)^{-\frac{1}{2}} \neq 0 \qquad (17)$$

and

$$\xi_1 = W_{\eta_1} = \pm \int \left(x_1^2 + 2\eta_1\right)^{-\frac{1}{2}} dx_1 - t$$

$$t + \xi_1 = \pm \log\left(x_1 + \sqrt{x_1^2 + 2\eta_1}\right) \qquad (18)$$

Finally

$$\eta_1 = x_{10}\, y_{10} + \frac{y_{10}^2}{2} = H$$

$$\xi_1 = \pm \log\left(2x_{10} + y_{10}\right) \qquad (19)$$

It should be emphasized that these results have only a local character on account of the requirement that certain functional determinants do not vanish. In general, only certain domains of the variables are allowed.

The Cauchy Existence Theorem is next applied for a precise statement of the allowable domains of the variables. If the Hamiltonian function H(x,y) is analytic in each of the 2n variables x_k, y_k in a region $|x_k - \xi_k| < 2\rho$, $|y_k - \eta_k| < 2\rho (k = 1, \ldots, n)$ for the initial conditions

$$\begin{aligned} x_k(\tau) &= \xi_k \\ y_k(\tau) &= \eta_k \end{aligned} \tag{20}$$

and satisfies the estimate $|H(x,y)| \leq m$ in that region, then the solutions $x_k(t)$, $y_k(t)$ of the Hamiltonian system are regular for the time duration

$$|t - \tau| < \frac{\rho^2}{(2n+1)m} \tag{21}$$

and lie within the circles

$$\begin{aligned} |x_k(t) - \xi_k| &< \rho \\ |y_k(t) - \eta_k| &< \rho \end{aligned} \tag{22}$$

This result is derived by constructing a majorizing power series in such a way that the general system of equations

$$\dot{x}_k = f(x) \qquad (k = 1, \cdots, m) \tag{23}$$

is majorized (dominated) by

$$\dot{y}_k = g(y) \qquad (k = 1, \cdots, m) \tag{24}$$

and, furthermore, such that the majorant can be integrated directly. These results are then extended to the Hamiltonian system of equations.

In the next chapters, it is shown that, for the three body problem, a triple collision can only occur if all three components of the angular momentum of the system are zero. Further, for a two body collision, it is shown that the collision occurs at a specific point in space q_k $(t = t_1)$, $k = 1, 2, 3$, (q_k denotes either x_k, y_k, z_k) while at least one $\dot{q}_k$ is unbounded as $t \longrightarrow t_1$. Then it is shown, by calculation of a suitable canonical transformation, that the equations of motion can be regularized for binary collisions. In this way, the solution can be analytically continued over the singularity at $t = t_1$. In the mathematical statement of the problem, the two physical bodies collide at $t = t_1$ and rebound elastically from each other.

Chapters 9 and 10 present the two Sundman lemmas. The first lemma states that if the three components of the angular momentum of the system are not all zero, then the sum of the mutual distances between the three bodies is higher than a positive constant for all time. The second lemma states, if the same assumptions are observed, that the velocity of the particle lying farthest from the other two bodies will remain bounded for all time by a positive constant. Then, with these two lemmas, Sundman's theorem is presented. This theorem states:

If the three angular momentum components referred to the center of mass are not all zero, then the Cartesian coordinates of the three bodies and the time t can be expanded in power series in ω, which converge for $|\omega| < 1$, and which represent the motion for all time. Here

$$\omega = \frac{e^{\frac{\pi}{2\delta}s} - 1}{e^{\frac{\pi}{2\delta}s} + 1} \qquad s = \int_T^t (\mathcal{V} + 1)\, dt \tag{25}$$

and

$$\mathcal{V} = \text{potential energy} = \mathcal{V}(m, q)$$

$\delta = \delta$ (A, m) is a positive number based upon the masses and the initial conditions.

Next, in Chapter 14, an important existence theorem for periodic solutions of Hamiltonian systems is presented. We consider the system (1) with H(z) regular in the neighborhood of the equilibrium solution at z = 0. H(z) is next expanded in a power series beginning with quadratic terms

$$H(z) = \tfrac{1}{2} z^T S z + \cdots \tag{26}$$

where S is a symmetrical matrix of dimension (2n x 2n).

This series is to converge in some neighborhood of z = 0, and the 2n eigenvalues $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n$ of JS are to be mutually distinct. The existence theorem now states:

Let λ_1 be purely imaginary and none of the n-1 quotients $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$ be equal to an integer. Then there is a family of real periodic solutions which depend analytically upon a real parameter ρ and, for $\rho = 0$, transform to the equilibrium solution. Moreover, the period $\tau(\rho)$ is analytic in ρ with

$$\tau(0) = \frac{2\pi}{|\lambda_1|}$$

To illustrate this theorem, the example

$$H = \tfrac{1}{2}\rho(x_1^2 + y_1^2) - \rho(x_2^2 + y_2^2) + \tfrac{1}{2}(x_1^2 x_2 - x_2 y_1^2 - 2x_1 y_1 y_2) \tag{27}$$

is presented. The equations of motion are

$$\dot{x}_1 = \rho y_1 - x_2 y_1 - x_1 y_2 \qquad \dot{y}_1 = -\rho x_1 - x_1 x_2 + y_1 y_2$$

$$\dot{x}_2 = -2\rho y_2 - x_1 y_1 \qquad \dot{y}_2 = 2\rho x_2 - \tfrac{1}{2}(x_1^2 - y_1^2)$$

(28)

This system possesses the equilibrium solution $x_1 = x_2 = y_1 = y_2 = 0$ with the obvious period $\tau(0) = \left.\frac{2\pi}{2\rho}\right|_{\rho=0} = \infty$.

From inspection

$$S = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -2\rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & -2\rho \end{bmatrix}$$

(29)

and the eigenvalues of the linear system are

$$|JS - \lambda I| = |A - \lambda I|$$

$$= \begin{vmatrix} -\lambda & 0 & \rho & 0 \\ 0 & -\lambda & 0 & -2\rho \\ -\rho & 0 & -\lambda & 0 \\ 0 & -2\rho & 0 & -\lambda \end{vmatrix} = -\lambda(-\lambda^3 - 4\rho^2\lambda) + \rho(+4\rho^3 + \rho\lambda^2)$$

$$= \lambda^4 + 5\rho^2\lambda^2 + 4\rho^4 = 0$$

(30)

Thus $\lambda_{1,3} = \pm 2\rho i \qquad \lambda_{2,4} = \pm \rho i$

Now λ_1 is purely imaginary and the quotient

$$\frac{\lambda_2}{\lambda_1} = \frac{1}{2} \tag{31}$$

is not an integer. Thus, the conditions of the existence theorem are satisfied and we are able to conclude the existence of periodic solutions of the system (28) from the obvious periodic solutions of the linearized system (32).

$$\begin{aligned} \dot{x}_1 &= \rho y_1 & \dot{y}_1 &= - \rho x_1 \\ \dot{x}_2 &= -2 \rho y_2 & \dot{y}_2 &= 2 \rho x_2 \end{aligned} \tag{32}$$

These periodic solutions of (28) have the approximate period

$$T = \frac{2\pi i}{\lambda_1} = \pi/\rho$$

In Chapter 19 Siegel presents the continuity method of Poincaré. Commencing with a set of autonomous first order differential equations, with some parametric dependence, and with sufficient conditions to ensure a solution in some domain (Figure 1) a method of analytically continuing the solution in the independent variable is given. Supposing that a periodic solution is known for some set of initial conditions and parameters, the solution is extended to other values of the parameter upon application of the implicit function theorem.

Now, consider a set of m first order autonomous differential equations

$$\dot{x} = f_k(x, \alpha) \qquad (k = 1, \ldots, m) \tag{33}$$

where x is a vector of the m dependent variables and α is a parameter. Regularity of the $f_k(x, \alpha)$ in $|x_\ell - \xi_\ell^*| < r$, $\ell = 1, \ldots, m$ and $\alpha \in G$ is sufficient to ensure a solution of (33), which can be written formally as

$$X = X(t, \xi, \alpha) \tag{34}$$

where ξ is a vector of the initial conditions.

Now it is assumed for $\xi = \xi^*$ and $\alpha = \alpha^*$ that the solution is analytically continuable to some time point $t = t_1$, and, because of the regularity of the functions involved, $x = x(t, \xi, \alpha)$ is continuable to $t = t_1$ for values of ξ, α sufficiently close to ξ^* and α^* respectively.

To ensure the existence of solutions of $\dot{x}_k = f_k(x, \alpha)$ for variations in the initial conditions suppose that the $f_k(x, \alpha)$ are regular for $\left| \xi_\ell^* - x_\ell \right| < r$, $\ell = 1, \ldots, m$ (see Figure 1). Now for $\left| \xi_\ell - \xi_\ell^* \right| < r/2$ the $f_k(x, \alpha)$ are regular in $\left| \xi_\ell - x_\ell \right| < r/2$ since this region is completely contained in $\left| \xi_\ell^* - x_\ell \right| < r$. The parameter α must also be restricted to some neighborhood of α^*, $\alpha \in G$ say, to ensure regularity of the $f_k(x, \alpha)$ when α is varied. These restrictions on the extent of the choice of the initial conditions within the domain in which the $f_k(x, \alpha)$ are regular ensures the existence of a solution for each ξ. Further, if the solution $x = x(t, \xi^*, \alpha^*)$ is continuable from t_0 to t_1 then the regularity conditions insure that $x = x(t, \xi, \alpha)$ is continuable to t_1 for ξ sufficiently close to ξ^*, $\alpha \in G$.

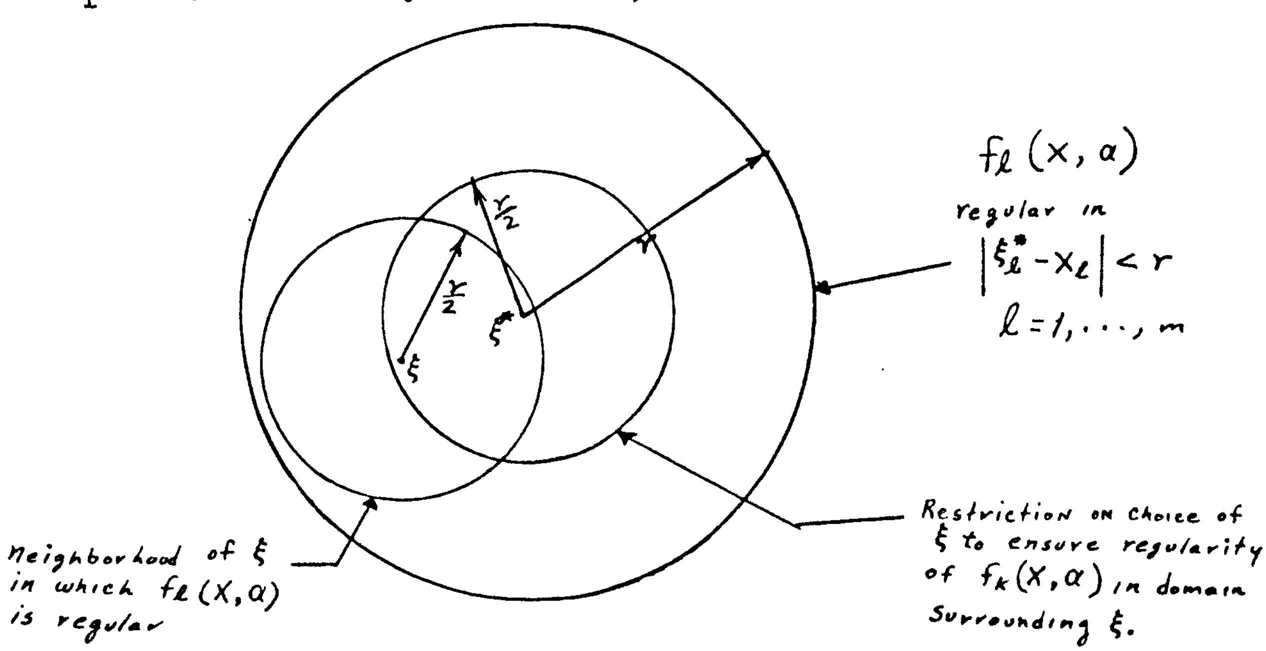

Figure 1 - Domain of Regularity of the $f_k(x, \alpha)$

Before proceeding further consider the specific case of the restricted three body problem, in Hamiltonian form the equations of motion are:

$$\dot{X}_1 = X_3, \qquad \dot{X}_2 = X_4, \qquad \dot{X}_3 = 2X_4 + X_1 + F_{X_1},$$

$$\dot{X}_4 = -2X_3 + X_2 + F_{X_2} \tag{35}$$

where $F = (1-\mu)\left((X_1+\mu)^2 + X_2^2\right)^{-\frac{1}{2}} + \mu\left((X_1+\mu-1)^2 + X_2^2\right)^{-\frac{1}{2}}$

These are of the form $\dot{x} = f_k(x, \alpha)$ where μ corresponds to α, a parameter. The f_k's ($k = 1, \ldots, 4$) are autonomous and regular except for $x_2 = 0$, and either $x_1 = -\mu$ or $x_1 = 1 - \mu$ so that in general a solution exists. In particular, for $\mu = 0$, a periodic solution exists and it is this solution that we wish to continue for $\mu \neq 0$.

For the more general case, suppose that for $\xi = \xi^*$, $\alpha = \alpha^*$ a non-equilibrium but periodic solution exists, denoted formally by

$$X = X(t, \xi^*, \alpha^*) \tag{36}$$

which is contained entirely within some domain of regularity of the functions involved. This solution holds for all real t, since

$$\chi(t + \tau^*, \xi^*, \alpha^*) = \chi(t, \xi^*, \alpha^*), \tag{37}$$

where τ^* is not necessarily the smallest positive period of the motion.

Siegel then demonstrates that continuation of this solution for $\mu \neq 0$ holding the period τ^* constant is not possible. To see this, set

$$\varphi_k(\xi, \alpha) = X_k(\tau^*; \xi, \alpha) - \xi_k \tag{38}$$

which gives, on putting $t = 0$ in (37), $\varphi_k(\xi, \alpha) = 0$ for a periodic solution. We have then m equations in the m + 1 unknowns $\xi_1, \ldots, \xi_m, \alpha$. As a consequence of the implicit function theorem and the fact that $x = x(t, \xi^*, \alpha^*)$ is a periodic solution, the m equations

$$\varphi_k(\xi, \alpha) = 0 \tag{39}$$

can be solved for the ξ as functions of α provided

$$\left| \varphi_{k_{\xi_l}} (\xi^*, \alpha^*) \right| \neq 0. \tag{40}$$

But this determinant is necessarily zero since essentially fixing the period and allowing the initial conditions to vary just selects a different starting point on the same periodic solution, that is, the selection of the ξ is limited to points on the same trajectory.

There are two means of overcoming this problem. The first method is to allow the period τ to vary and fixing some other quantity, such as a coordinate. It is no restriction to suppose that ξ_m may be fixed thus giving the extra condition $\xi_m = \xi_m^*$.

By choosing $\xi_m = \xi_m^*$ we are restricting the variation of the initial conditions $\xi_1, \ldots, \xi_{m-1}$ to the m-1 dimensional plane $x_m = \xi_m^*$. On requiring further that the solution $\alpha = \alpha^*$, $\xi = \xi^*$ is not tangential to this plane, but truly intersects it, we ensure that the solution of $\varphi_k(\tau, \xi, \alpha) = 0$ does not give the same orbit as the known one with $\xi = \xi^*$, $\alpha = \alpha^*$. The situation is depicted in Figure 2.

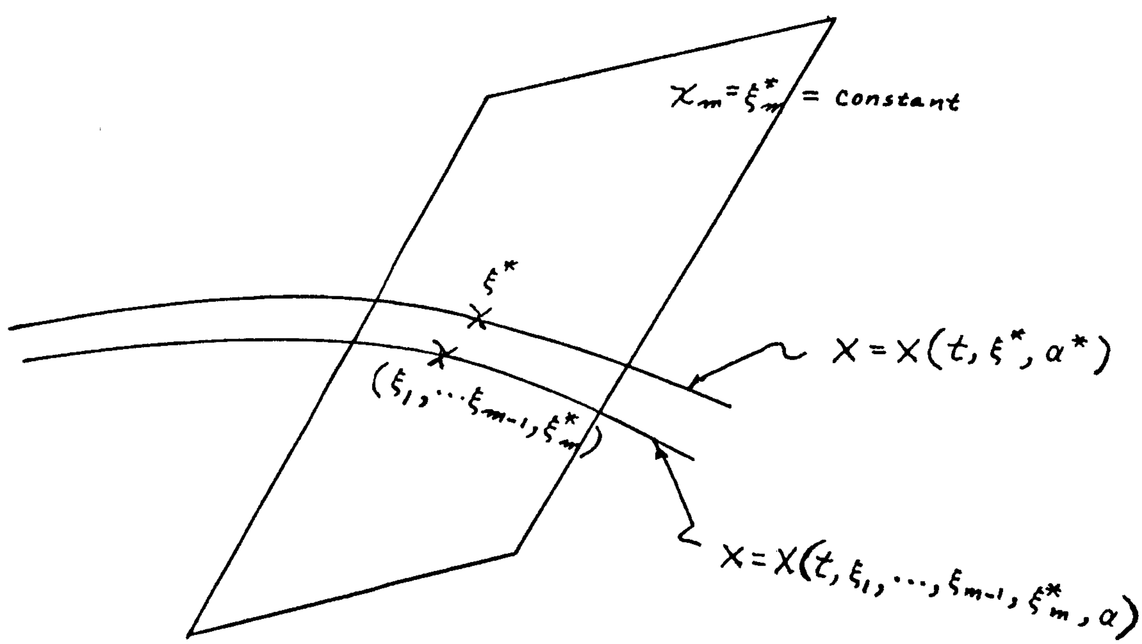

Figure 2

However in the case of the restricted three body problem the Jacobi integral

$$\psi(X, \mu) = \frac{1}{2}\left(X_3^2 + X_4^2 - X_1^2 - X_2^2\right) - F \tag{41}$$

causes the value of the associated functional determinant to be zero. In the second method, a sufficient requirement for the existence of periodic orbits near $\mu = 0$ with period $\tau = \tau^*$ is just the existence of the Jacobi integral. This method is used by Siegel to demonstrate the existence of certain periodic solutions for the restricted three body problem.

FUTURE WORK

During the next phase, emphasis will be concentrated upon:

Chapter 19 Poincare Continuity Method

Chapter 20 Fixed Point Method

Chapter 21 Content - Preserving Analytic Transformations

Chapter 22 Birkhoff Fixed Point Theorem

Chapter 26 Lyapunov Theorem

Chapter 27 Dirichlet Theorem

Chapter 28 Normal Form of Hamiltonian Systems

Chapter 29 Content - Preserving Mappings

These chapters are directly related to the contractual scope of work. The theorems and concepts presented here will be applied in developing the techniques for analyzing stability of periodic motions.

REFERENCES

1. Siegel, C. L., Vorlesungen über Himmelsmechanik, Springer-Verlag, Berlin, 1956
2. Pars, L. A., A Treatise on Analytical Dynamics, John Wiley and Sons, Inc., New York, 1965