Stevens Inst. Tech.

FINAL REPORT

Nasa Research Contract Ns G-403

Application of Formal Power Series to Some Classical Problems of Mechanics

GPO PRICE	$
CFSTI PRICE(S)	$
Hard copy (HC)	2.00
Microfiche (MF)	.50

ff 653 July 65

by

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FACILITY FORM 602

(ACCESSION NUMBER)	(THRU)
28 (PAGES)	1 (CODE)
CR-76433 (NASA CR OR TMX OR AD NUMBER)	23 (CATEGORY)

Introduction

One of the oldest problems in the history of mechanics is to find functions of the coordinates which satisfy a simpler (lower order) system of differential equations. In particular, to find functions of the coordinates which satisfy a first order differential equation. The energy integral and the integrals of angular momentum are examples of such functions: they are algebraic functions of the coordinates and their derivatives which satisfy, the simplest of all first order differential equations, $Y' = 0$.

In 1887 Bruns [1] proved that in the three body problem the only algebraic functions of the coordinates and their derivatives which satisfy the equation $Y' = 0$ are those generated by the known integrals. These results were later extended to the n-body problem $(n > 2)$ by Painlevé [2] and to the restricted three body problem, for small values of a parameter μ, which appears in the problem, by Poincaré [3, vol. 1 ch. v].

In this paper we extend these negative results of Bruns and Poincaré. One consequence of our results, for the n-body problem, is that no algebraic function of the coordinates and their derivatives is an exponential. Similar results are obtained for a large class of problems in mechanics.

We use the tools and ideas of differential algebra developed by Ritt [4]. The following is a summary of definitions, notations and results that we use frequently.

A differential field, differential ring, differential ideal is a field, ring, ideal which is closed under a given derivation. In this paper the characteristic of the field is zero and the derivation is the derivative with respect to the time t. By $K\langle x\rangle$, $K\{x\}$ we mean the differential field, differential ring respectively of K and the elements $x = (x_1, \ldots, x_n)$. $K(x)$, $K[x]$ is used for field and ring adjunction, respectively.

Let K be a differential field and let S be a set of differential polynomials $P(X) \in K\{X\}$. The perfect (radical) differential ideal I generated by S, in the differential ring $K\{X\}$, is the intersection of a finite number of prime differential ideals $\pi_1, \ldots, \pi_\ell$. Each π_i has an irreducible manifold of zeros and the manifold of zeros of S is the union of the manifolds of π_i, $i = 1, \ldots, \ell$. A zero x of π_i is called generic if every differential polynomial $P(X) \in K\{X\}$, which vanishes for $X = x$, belongs to π_i. Every prime differential ideal has a generic zero. Let y be any zero of S, then the degree of transcendency of $K\langle y\rangle$ over K is called the order of y. The order of a prime differential ideal π is the order of its generic zero. We shall call the order of S the $\max.(\text{order } \pi_i)_{1\leq i\leq \ell}$. If S is a system of differential polynomials defining a mechanical system of n degrees of freedom, then order of S is $2n$ which agrees with the usual definition of order of S.

The problem that concerns us may now be restated as follows: Let x be a solution of max. order of a mechanical system S, what are the differential subfields of $K\langle x\rangle$ which are of order one over K? For the n-body problem, the theorems of Bruns and Painlevé give the subfield of constants of $K\langle x\rangle$, where $K = Q(M_1, \ldots, M_n)$, Q stands for the reals and $M_{i, i=1,\ldots n}$ are the masses.

These results are possible to obtain without finding the decomposition of S into its prime components, because all the irreducible manifolds of S, of maximal order, are isomorphic (Theorem three).

Theorems one and two prepare the background for the application of formal power series, in an arbitrary parameter, to our problem. In [4] Ritt proved that if a is any zero of the highest or lowest degree terms P^* of a single differential polynomial $P(X)$, then there exists a formal fractional power series of the form

$$x = ac^j + \sum_{i=j+1}^{\infty} a_i c^{k_i} \qquad j = \begin{cases} -1 & \text{if } P^* \text{ is highest degree} \\ +1 & \text{if } P^* \text{ is lowest degree} \end{cases}$$

such that $P(x) = 0$. That this theorem does not hold if S consists of more than one differential polynomial can be shown by the following example which arises in problems of coupled springs:

$$
(S) \qquad
\begin{aligned}
X_1'' &= \alpha_1 X_1 + \beta_1 X_2 + \lambda_1 (X_2 - X_1)^3 \\
X_2'' &= \alpha_2 X_1 + \beta_2 X_2 + \lambda_2 (X_2 - X_1)^3
\end{aligned}
$$

$X_1 = X_2 = a \neq 0$ is a solution of the highest degree terms $(X_2 - X_1)^3 = 0$. This solution cannot be extended to a formal fractional power series solution of S except in the case where $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$. However, any solution of the lowest degree terms i.e. of

$$
\begin{aligned}
X_1'' &= \alpha_1 X_1 + \beta_1 X_2 \\
X_2'' &= \alpha_2 X_1 + \beta_2 X_2
\end{aligned}
$$

can be extended to a formal power series solution of S. The fact that the highest order terms appear to the first degree, among the highest degree terms or lowest degree terms of the system, makes it possible to find a formal power series solution in an arbitrary parameter, where the first term of the series is a solution of a linear system of the same order. This is, precisely what occurs in a large class of mechanics problems which includes practically all the problems arising in celestial mechanics.

I. Formal Power Series for Algebraic Systems

Let K be a field. Let

$$x_i = \sum_{j = s_i}^{\infty} x_{ij} c^j \quad ; \quad i = 1, \ldots, n; \; s_i \text{ integers, and}$$

$$(x_{ij})_{\substack{1 \leq i \leq n \\ s_i \leq j \leq \infty}} \quad \text{are algebraically independent over } K,$$

c transcendental over the field $K(x_{ij})$. Any element $u \in K(x) = K(x_1, \ldots, x_n)$ has the form

$$u = \sum_{k = \ell}^{\infty} u_k c^k \; , \quad \ell \text{ an integer and}$$

$u_k \in K(x_{ij})$ for $\ell \leq k < \infty$.

Lemma 1. Let $u = \sum_{k = \ell}^{\infty} u_k c^k \in K[x]$. Then

$$\text{(A)} \quad \frac{\partial u_k}{\partial x_{i, s_i + r}} = \frac{\partial u_{k-r}}{\partial x_{i, s_i}} \qquad \text{for all } r \geq 0 ,$$

where we set $\frac{\partial u_k}{\partial x_{i, s_i}} = 0$ if $k < \ell$.

Proof: Let S be the set of all elements $u \in K[x]$ for which (A) holds. S is not empty, since $x_i \in S$, $i = 1, \ldots, n$. Also, let (A) hold for $u, v \in K[x]$ and let

$$u = \sum_{k = \ell_1}^{\infty} u_k c^k \; , \quad v = \sum_{k = \ell_2}^{\infty} v_k c^k, \quad \text{then } w = u + v =$$

$$= \sum_{k = \ell_1}^{\infty} (u_k + v_k) c^k \; .$$ Without loss of generality we assumed

$\ell_2 \geq \ell_1$ and set $v_k = 0$ for $k < \ell_2$.

Now, $$\frac{\partial w_k}{\partial x_{i,s_i+r}} = \frac{\partial u_k}{\partial x_{i,s_i+r}} + \frac{\partial v_k}{\partial x_{i,s_i+r}}, \qquad r \geq 0$$

$$= \frac{\partial u_{k-r}}{\partial x_{i,s_i}} + \frac{\partial v_{k-r}}{\partial x_{i,s_i}}$$

$$= \frac{\partial w_{k-r}}{\partial x_{i,s_i}}$$

So that (A) holds for $u + v$. Similarly, let $w = uv =$

$$= \sum_{k=\ell_1+\ell_2}^{\infty} w_k c^k = \sum_{k=\ell_1+\ell_2}^{\infty} \Big(\sum_{j=\ell_1}^{k-\ell_2} u_j v_{k-j} \Big) c^k$$

$$\frac{\partial w_k}{\partial x_{i,s_i+r}} = \sum_{j=\ell_1}^{k-\ell_2} v_{k-j} \frac{\partial u_j}{\partial x_{i,s_i+r}} + u_j \frac{\partial v_{k-j}}{\partial x_{i,s_i+r}}$$

$$= \sum_{j=\ell_1}^{k-\ell_2} v_{k-j} \frac{\partial u_{j-r}}{\partial x_{i,s_i}} + u_j \frac{\partial v_{k-j-r}}{\partial x_{i,s_i}}$$

$$= \sum_{j=\ell_1+r}^{k-\ell_2} v_{k-j} \frac{\partial u_{j-r}}{\partial x_{i,s_i}} + \sum_{j=\ell_1}^{k-r-\ell_2} u_j \frac{\partial v_{k-j-r}}{\partial x_{i,s_i}}$$

$$= \sum_{j=\ell_1}^{k-r-\ell_2} v_{k-r-j} \frac{\partial u_j}{\partial x_{i,s_i}} + u_j \frac{\partial v_{k-r-j}}{\partial x_{i,s_i}}$$

$$= \frac{\partial w_{k-r}}{\partial x_{i,s_i}}$$

So (A) holds for uv and hence S is a ring containing $x_1, \ldots, x_n$ and $S = K[x]$.

Lemma 2. Let u_k be as in Lemma 1, then $u_k \in K(x_{ij})_{\substack{1 \leq i \leq n \\ s_i \leq j \leq s_i + k - \ell}}$ for all $u \in K(x)$.

Proof. For any $u \in K[x]$, by Lemma 1, $\dfrac{\partial u_k}{\partial x_{i,s_i+r}} = 0$ for all $r > k - \ell$. Let $v = \dfrac{1}{u} = \sum_{k=-\ell}^{\infty} v_k c^k = c^{-\ell} \sum_{j=0}^{\infty} w_j c^j$ where $w_j \in K(u_\ell, \ldots, u_{\ell+j}) \in K(x_{im})_{\substack{1 \leq i \leq n \\ s_i \leq m \leq s_i + j + \ell - \ell = s_i + j}}$

Since $w_{j+\ell} = v_j$, we have $v_j \in K(x_{im})_{\substack{1 \leq i \leq n \\ s_i \leq m \leq s_i + j + \ell = s_i + j - (-\ell)}}$

Hence Lemma 2 holds for all $u \in K(x)$.

Theorem 1. Let $P(X, Y) \in K[X, Y]$.
let r, s be integers such that
$P(Xc^s, Yc^r) = P^*(X, Y)c^\ell$ + higher power of c, where $P^* \neq kY$; $k \in K$, and the discriminant of P^*, considered as a polynomial in Y with coefficients in $K(X)$, is not zero. Then there exists

$$x_i = \sum_{k=s}^{\infty} x_{ik} c^k, \quad i = 1, \ldots, n; \qquad y = \sum_{k=r}^{\infty} y_k c^k$$

such that $P(x, y) = 0$ and the set $(x_{ik})_{\substack{1 \leq i \leq n \\ s \leq k < \infty}}$ are algebraically independent while y_k is algebraic over $(x_{im})_{\substack{1 \leq i \leq n \\ s \leq m \leq k+s-r}}$.

Proof. Let (x_{ik}) be a set of algebraically independent elements over K in some field extension of K and let

$Y = \overline{Y} = \sum_{k=r}^{\infty} \overline{Y}_k c^k$, then

$P(x, \overline{Y}) = \sum_{k=\ell}^{\infty} p_k c^k$. By Lemma 1

$$\frac{\partial p_j}{\partial \overline{Y}_k} = \frac{\partial p_j}{\partial \overline{Y}_{r+(k-r)}} = \frac{\partial p_{j-k+r}}{\partial \overline{Y}_r} \quad \text{for } j \geq \ell + k - r$$

so that $\dfrac{\partial p_j}{\partial \overline{Y}_k} = \dfrac{\partial p_j}{\partial \overline{Y}_r}$ if $j = \ell + k - r$

and $\dfrac{\partial p_j}{\partial \overline{Y}_k} = 0$ if $j < \ell + k - r$. Hence $\overline{Y}_k$ first appears in $p_{\ell+k-r}$. Also, $\dfrac{\partial p_j}{\partial x_{i,k}} = 0$, if $j < s + k - r$. Now, let y_r be a zero of $p_\ell(x_{is}, \overline{Y}_r)$, $y_r \neq 0$, (such zeros exist because P^* is not a power of Y). Since

$$\left(\frac{\partial p_\ell}{\partial \overline{Y}_r}\right)_{\substack{x_i = x_{ir} \\ \overline{Y}_r = y_r}} \neq 0 \quad \text{we have} \quad \frac{\partial p_{\ell+k-r}}{\partial \overline{Y}_k} \neq 0$$

Let y_k be a zero of

$$p_{\ell+k-r} \in K[x_{im}, y]_{\substack{1 \leq i \leq n \\ s \leq m \leq k+s-r \\ r \leq j \leq k}} ,$$

then $X = x$, $Y = y = \sum_{i=r}^{\infty} y_i c^i$ is a zero of $P(X, Y)$ with the stated properties.

Def. A polynomial $P(X, Y)$ satisfying the conditions of Theorem 1 will be called non-singular.

II. Algebraic Mechanical Systems

A mechanical system of n-degrees of freedom is given by a system of second order differential equations of the form:

$$X'' = F(X, X')$$

where X, X', X'' are n-vectors and the derivatives are with respect to the time - t. If the system is algebraic, the functions $(F_i) = F$ are algebraic functions of X, X'. More precisely, the system is of the form:

$$U_i \equiv Q_i(X, X', Y)\, X_i'' + P_i(X, X', Y) = 0$$

$$\text{(S)} \qquad A_j(X, X', Y_j) = 0 \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, m \end{array}$$

$Y = (Y_1, \ldots, Y_m)$ and

$P_i, Q_i, A_j \in K[X, X', Y]$, where K is some differential field of functions of t.

Let $A_j(Xc^s, X'c^s, Y_j c^{r_j}) = A_j^*(X, X', Y_j)c^{\ell_j} + \ldots +$

and let A_j^* be non-singular, $j = 1, \ldots, m$. Then, by Theorem 1, there exist formal power series solution of the form:

$$X_i = x_i = \sum_{j = s}^{\infty} x_{ij}c^j \qquad i = 1, \ldots, n$$

$$X_i' = x_i' = \sum_{j = s}^{\infty} x_{ij}'c^j$$

$$Y_j = y_j = \sum_{k = r_j}^{\infty} y_{jk}c^k \qquad j = 1, \ldots, m$$

such that $A_j(x, x', y_j) = 0$. Since the $(x_{ij}), (x_{ij}')$ are indeterminates, we may set x_{ij}' to the time-derivative of x_{ij}. Then

$$U_i(x, x', x'', y) = \sum_{j = f_i}^{\infty} u_{ij}c^j = 0, \qquad i = 1, \ldots, n$$

$$u_{ij} \in K[(x_{\ell\mu}), (x_{\ell\mu}'), (x_{\ell\mu}''), (y_{g\nu})]_{\substack{1 \leq \ell \leq n \\ s \leq \mu \leq s + j - f_i \\ 1 \leq g \leq m \\ r_g \leq \nu \leq r_g + j - f_i}}$$

and the $(y_{g\nu})$ are algebraic over $(x_{\ell\alpha}), (x_{\ell\alpha}')$ $\quad 1 \leq \ell \leq n$;

$$s \leq \alpha \leq \nu + s - r_g \leq r_g + j - f_i + s - r_g = j - f_i + s$$

Thus the u_{ij} are rational integral in $(x_{\ell\mu}'')$, algebraic over $(x_{\ell\mu}), (x_{\ell\mu}')$ $1 \leq \ell \leq n$, $s \leq \mu \leq s + j - f_i$. If we let degree of $y_j = r_j/s$; $j = 1, \ldots, m$ then every element of $K\{x, y\}$ has highest and lowest degree terms. (x_{is}); $1 \leq i \leq n$ is of order $2n$ if and only if u_{if_i} is of order 2; that is if x_i'' appears among the lowest degree terms of U_i when $s \geq 0$, or among the highest degree terms when $s < 0$.

Therefore, the relative degree of the terms, in which x_i'' appears, is determined by the degree assigned to the y_j. It may, of course, happen that the system $A_j = 0$; $j = 1, \ldots, m$ has more than one formal power series solution and x_i'' appears among the lowest (highest) degree terms for one determination of the y_i but not for the other. It may, also, happen that x_i'' appears among the lowest degree terms for one determination of the y_i and among the highest degree terms for another determination of y_j , $j = 1, \ldots, m$, as the following example illustrates.

Example:

$$(S) \qquad \begin{aligned} YX_1'' + \lambda_1 X_1 + \lambda_2 X_2^2 &= 0 \\ YX_2'' + \lambda_2 X_2 + \lambda_4 X_1^2 &= 0 \end{aligned} \qquad \begin{aligned} X^2 &= (X_1^2 + X_2^2)^2 + 1 \\ \lambda_i \in K \, ; & \quad i = 1, \ldots, 4 \end{aligned}$$

We may let $X_i = x_i = \sum_{j=1}^{\infty} x_{ij} c^j$, $Y = y = 1 + \sum_{j=1}^{\infty} y_j c^j$

the degree of y is zero, and x_{11}, x_{21} is a solution of the linear system

$$X_1'' + \lambda_1 X_1 = 0 \quad , \quad X_2'' + \lambda_3 X_1 = 0$$

or we may set $X_i = x_i = \sum_{j=-1}^{\infty} x_{ij} c^j \quad Y = y = \sum_{j=-2}^{\infty} y_j c^j$

the degree of y is two and x_{11}, x_{21} is a solution of the linear system

$$X_1'' = 0 \qquad X_2'' = 0 \quad .$$

Theorem 2. Let (S) be an algebraic mechanical system. Let $A_j(X, X', Y_j) = 0$; $j = 1, \ldots, m$ have a formal power series solution

$$X_i = x_i = \sum_{k=s}^{\infty} x_{ik} c^k \quad ; \qquad i = 1, \ldots, n,$$

$$X_i' = x_i' = \sum_{k=s}^{\infty} x_{ik}' c^k$$

$$Y_j = y_j = \sum_{k=r_j}^{\infty} y_{jk} c^k \quad ; \qquad j = 1, \ldots, m,$$

as described in Theorem 1.

Let $\quad U_i = \sum_{k=f_i}^{\infty} u_{ik} c^k \qquad$ where

$$u_{if_i} = q_i(x_{\ell s}, x_{\ell s}', y_{jr_j})[x_{is}'' + L_i(x_{\ell s}, x_{\ell s}')] \quad \begin{matrix} 1 \le \ell \le n \\ 1 \le j \le m \end{matrix}$$

and L_i is linear homogeneous in $(x_{\ell s})$, $(x_{\ell s}')$ with coefficients in some algebraic extension of K. Then the system (S) has a formal power series solution where the x_{ik} are generic solutions of

$$x_{ik}'' + L_i(x_{\ell k}, x_{\ell k}') = p_i \quad ; \qquad i = 1, \ldots, n$$

and the p_i belong to an algebraic extension of $K\langle(x_{\ell\mu})\rangle$; $1 \le \ell \le n$, $\quad s \le \mu < k$.

Proof. By Theorem 1 the x_{ik}, x_{ik}' are indeterminates. Hence we may specialize x_{is}, x_{is}' to be generic solutions of the homogeneous linear system

$$X_i'' + L_i(X, X') = 0 \qquad i = 1, \ldots, n$$

We may extend this specialization to y_{jr_j} $\quad j = 1, \ldots, n$

Since $q_i(x_{\ell s}, x'_{\ell s}, y_{jr_j}) \neq 0$ and

$$u_{i,f_i+k} = q_i(x_{\ell s}, x'_{\ell s}, y_{jr_j})[x''_{i,s+k} + L_i(x_{\ell,s+k}, x'_{\ell,s+k})] + A_i$$

where $A_i \in K[x_{\ell\mu}, x'_{\ell\mu}, y_{j\nu}]$; $1 \leq \ell \leq n, \quad x \leq \mu < s + k,$

$$y \leq j \leq m, \quad r_j \leq \nu < r_j + k$$

we may successively specialize $x_{i,s+k}$ to be a generic solution of the linear system

$$x''_{i,s+k} + L_i(x_{\ell,s+k}, x'_{s+k}) = -A_{i/q_i} = p_i$$

and then extend the specialization to y_{j,r_j+k}; This is possible, for by Theorem 1 y_{j,r_j+k} is algebraic over

$$K(x_{\ell\mu}, x'_{\ell\mu}) \qquad \ell = 1, \ldots, n$$
$$\mu = s, s+1, \ldots, s + k - 1$$

and the set $x_{\ell\mu}, x'_{\ell\mu}$ are algebraically independent over K. This proves our assertion.

Theorem 3. Let (S) be an algebraic mechanical system with n degrees of freedom. Let $K(X, X', Y)$ be a field extension of degree r over $K(X, X')$. Then (S) has r irreducible manifolds of order $2n$. Any two such manifolds are isomorphic.

Proof. Since (X, X') is a transcendence base for $K(X, X', Y)$ over K we may extend any derivation of K by defining X' to be the derivative of X and $(X')' = \dfrac{P_i(X, X', Y)}{Q_i(X, X', Y)}$, so that (S) has manifolds of order $2n$. Since $K(X, X', Y)$ has r distinct isomorphism in some differential extension field of $K\langle X, X'\rangle$ it follows that (S) has r distinct irreducible manifolds of order $2n$. Now, let M_1, M_2 be any two such manifolds and let (x, y), $(\bar{x}, \bar{y})$ be a generic point of M_1, M_2 respectively. The isomorphism

$$\begin{aligned} \sigma\colon x &\longrightarrow \bar{x} \\ x' &\longrightarrow \bar{x}' \end{aligned}$$

can be extended to an isomorphism of $K(x, x', y)$ onto $K(\bar{x}, \bar{x}', \bar{y})$ by letting $\bar{y} = \sigma(y)$. This clearly defines an isomorphism of $K\langle x, y\rangle$ onto $K\langle \bar{x}, \bar{y}\rangle$; for the successive derivatives of $\bar{x}, \bar{y}$ are given by the same equations as those given for x, y with the substitution of $\bar{x}, \bar{y}$ for x, y.

Remark 1. The statement that (S) has r irreducible manifolds of order $2n$ means that the field $K(x, x')$ belongs to a differential field T such that T contains r differential subfields $T_1, \ldots, T_r$; $\bigcap_{i=1}^{r} T_i = K(x, x')$ and each T_i is generated by a solution of the system (S) with

$$X = x \qquad X' = x'$$

Remark 2. For the n-body problem, this theorem implies that it is immaterial whether the forces between the bodies are attractive or repulsive; the algebraic properties of the general solution are the same.

Theorem 4. Let (S) be an algebraic mechanical system with n degrees of freedom. Let $A_j(X, X', Y_j)$ $j = 1, \ldots, m$ be non-singular. Let $U_i = \sum_{k = f_i}^{\infty} u_{ik}c^k$ (as in Theorem 2) and let $u_{if_i} = q_i(x_{\ell s}, x'_{\ell s}, y_{jr_j})[x''_i + L_i(x_{\ell s}, x'_{\ell s})]$. Let $\theta \in K < \xi, \eta >$ where $x = \xi$, $y = \eta$ is a solution of order $2n$ of (S). Let θ be a solution of a differential equation $B(\Theta) = 0$ where $B(\Theta) \in K\{\Theta\}$. Then $\theta^*(x_{\ell s}, y_{jr_j})$, $1 \leq \ell \leq n$; $1 \leq j \leq m$, is a solution of $B^*(\Theta) = 0$, where $\theta(\xi c^s, \eta_j c^{r_j}) = \theta^*(\xi, \quad)c$ + higher powers of c and B^* is the lowest (highest) degree terms of B if $\alpha \geq 0$ $(\alpha < 0)$.

Proof. Since the A_j; $j = 1, \ldots, n$, are non-singular, by Theorem 1, $X = x$, $Y = y$ is a solution of the algebraic system $A_j = 0$ and by Theorem 2 $X = x$, $Y = y$ is a solution of the system (S) with $K\langle(x_{\ell s})\rangle$ of order $2n$ over K. Therefore $K < x, y >$ is of order $2n$. By Theorem 3 there exists an isomorphism

$$\sigma: \begin{array}{l} x \longleftrightarrow \xi \\ y \longleftrightarrow \eta \end{array}$$

Therefore, $\theta(x, y)$ is a solution of $B(\Theta) = 0$. Now,

$$B(\theta(x, y)) = B^*(\theta^*(x_{\ell s}, y_{jr_j}))c^\beta + \text{higher powers of } c.$$

Hence θ^* is a solution of $B^* = 0$.

III. Applications

1. The n-Body Problem

$$(S)\qquad X_i'' = \sum_{j \neq i} \frac{M_j(X_j - X_i)}{R_{ij}^3} \qquad i = 1, \ldots, n$$

$$Y_i'' = \sum_{j \neq i} \frac{M_j(Y_j - Y_i)}{R_{ij}^3}$$

$$Z_i'' = \Sigma \frac{M_j(Z_j - Z_i)}{R_{ij}^3}$$

$$R_{ij}^2 = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2$$

This is a system with $3n$ degrees of freedom. The differential field K is $C(M_1, \ldots, M_n)$; C is the field of complex numbers and the (M_i) are n transcendental constants.

This system, clearly, satisfies the conditions of Theorem 1, 2. Therefore:

$$X_i = x_i = \sum_{k=-1}^{\infty} x_{ik}c^k$$

$$Y_i = y_i = \sum_{k=-1}^{\infty} y_{ik}c^k \qquad R_{ij} = \sum_{k=-1}^{\infty} r_{ijk}c^k$$

$$Z_i = z_i = \sum_{k=-0}^{k=-1} z_{ik}c^k$$

is a solution of (S), where $(x_{i,-1})$, $(y_{i,-1})$, $(z_{i,-1})$ is a generic solution of the homogeneous linear system

$$X_i'' = 0,\ Y_i'' = 0,\ Z_i'' = 0; \qquad i = 1, \ldots, n.$$

The field of constants of $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})>$ is $D = K(<(x_{i,-1}'), (y_{i,-1}'), (z_{i,-1}'), (u_k u_\ell' - u_k' u_\ell)>$, where u_k spans the set $\{(x_{i,-1}), (y_{i,-1}), (z_{i,-1})\}$.

Let $X = \xi$, $Y = \eta$, $Z = \zeta$ be any solution of order $6n$ of the n-body problem. Let θ be a polynomial in ξ', η', ζ', with coefficients in the field $K(\xi, \eta, \zeta, (r_{ij}))$ such that $\theta' = 0$. By Theorem 4, $\theta^*((x_{i,-1}), (y_{i,-1}), (z_{i,-1})) \in D$. Hence the highest degree terms of θ are polynomials in ξ', η', ζ', $(u_k u_\ell' - u_k' u_\ell)$ with coefficients in K. Thus, the highest degree terms of the energy integral is a polynomial in ξ', η', ζ', and the integrals of angular momentum are of the form $u_k u_\ell' - u_k' u_\ell$.

Theorem 5. Let $X = \xi$, $Y = \eta$. $Z = \zeta$ be any solution (real or complex), of order 6n, of the n-body problem. Let $\theta \in K<\xi, \eta, \zeta>$ and let θ be a solution of $P(\Theta) = 0$; $P(\Theta) \in K\{\Theta\}$, P or order 1. Then $P^*(\Theta)$ (the highest degree terms of $P(\Theta)$ of degree if $\theta > 0$, the lowest degree terms of $P(\Theta)$ if degree $\theta < 0$) is divisible by Θ'.

Proof. By theorem 4, $\theta^*(x_{i,-1}), (y_{i,-1}), (z_{i,-1})$ is a solution of $P^*(\Theta) = 0$ which is homogeneous and of order 1. If $P^*(\Theta)$ is not divisibel by Θ', every zero of, $P^*(\Theta) = \Theta' - k_i\Theta$ (k_i ϵ to an algebraic extension of K), is an exponential, but $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})>$ has no exponential elements for $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})> = D<t>$. Hence $P^*(\Theta)$ is divisible by Θ'.

Cor. Let ξ, η, ζ, θ be as in Theorem 5. Then θ cannot be an exponential.

2. Restricted Three-Body Problem.

$$X'' - 2Y' - X = -\frac{(X+\mu)(1-\mu)}{R_1^3} - \frac{(X+\mu-1)\mu}{R_2^3}$$

(S) $$Y'' + 2X' - Y = -\frac{Y(1-\mu)}{R_1^3} - \frac{Y\mu}{R_2^3}$$

$$R_1^2 = (X+\mu)^2 + Y^2 \ , \ R_2^2 = (X+\mu-1)^2 + Y^2$$

Let $K = C(\mu)$ where C is the field of complex numbers and μ is a transcendental constant over C.

It follows from Theorems 2, 3 that (S) has a solution

$$X = x = \sum_{k=-1}^{\infty} x_k c^k \quad , \quad Y = y = \sum_{k=-1}^{\infty} y_k c^k$$

$$R_1 = \sum_{k=-1}^{\infty} r_{1k} c^k \qquad R_2 = \sum_{k=-1}^{\infty} r_{2k} c^k$$

x_{-1}, y_{-1} is a generic solution of the homogeneous linear system:

$$X'' - 2Y' - X = 0$$
$$Y'' + 2X' - Y = 0$$

$$r^2_{1,-1} = x^2_{1,-1} + y^2_{1,-1} = r^2_{2,-1}$$

Note that if $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ is any solution of (S) then

$$K< \xi, \eta, \rho_1, \rho_2 > = K< \xi, \eta >$$

Lemma 3. Let $X = a$, $Y = b$, $R = \rho$ be a generic solution of the system:

$$X'' - 2Y' - X = 0 \qquad R^2 = X^2 + Y^2$$
$$Y'' + 2X' - Y = 0$$

Then the field of constants of $K<a, b, \rho> = K(C_1, C_2)$ where

$$C_1 = a'^2 + b'^2 - (a^2 + b^2)$$

$$C_2 = ab' - a'b + a^2 + b^2$$

Proof. $C_1' = 2(a'a'' + b'b'' - aa' - bb')$

$$= 2(a'[2b' + a] + b'[b - 2a'] - aa' - bb') = 0$$

$$C_2' = ab'' - a''b + 2(aa' + bb')$$

$$= a[b - 2a'] - [2b' + a]b + 2(aa' + bb') = 0$$

Let $u = a' - b$ then

$$u' = a'' - b' = 2b' + a - b' = b' + a$$

and $$u^2 + u'^2 = (a'^2 + b'^2) + (a^2 + b^2) + 2(ab' - a'b)$$
$$= C_1 + 2C_2$$

Also, $$a'u - au' = a'(a' - b) - a(b' + a)$$
$$= a'^2 - a^2 - (a'b + ab')$$
$$= u^2 - C_2$$

Therefore, $$\left(\frac{a}{u}\right)' = \frac{u^2 - C_2}{u^2}$$

Also, $u^2 + u'^2 = C_1 + 2C_2$ implies

$$2u'(u + u'') = 0 \quad \text{but} \quad u' = a + b' \neq 0$$

since a, b is generic, so that $u'' = - u$

let $$v = \frac{a}{u} - \frac{C_2}{C_1 + 2C_2}\frac{u'}{u}$$ then

$$v' = \frac{u^2 - C_2}{u^2} + \frac{C_2}{C_1 + 2C_2} + \frac{C_2' \, u'^2}{(C_1 + 2C_2) \, u^2}$$
$$= 1 - \frac{C_2}{u^2} + \frac{C_2}{C_1 + 2C_2} + \frac{C_2(C_1 + 2C_2 - u^2)}{(C_1 + 2C_2)u^2}$$
$$= 1$$

Since $K\langle a, b\rangle = K(C_1, C_2')\langle u, v\rangle$,

for $$a = \left(v + \frac{C_2 \, u'}{(C_1 + 2C_2)u}\right) u$$

$$b = a' - u ,$$

and $K(C_1, C_2) < u, v >$ is isomorphic to

$K(C_1, C_2) < \sqrt{C_1 + 2C_2}$ sine $t, t >$, we have field of constants of $K< a, b >$ equals field of constants of

$K(C_1, C_2) < \sqrt{C_1 + 2C_2}$ sine $t, t >$ equals $K(C_1, C_2)$.

Now let $\alpha \varepsilon K < a. b, \rho >$; $\rho^2 = a^2 + b^2$ and let $\alpha' = 0$.

Let $\alpha = \gamma + \delta\rho$; $\gamma, \delta \varepsilon K < a, b >$

$$\alpha' = \gamma' + \delta'\rho + \delta\rho' = 0, \quad \gamma'\rho + \delta'(a^2 + b^2) + \delta(aa' + bb') = 0$$

$\therefore$ $\gamma' = 0$ and $(\delta^2\rho^2)' = 0$ but $\delta^2\rho^2 \varepsilon K <a, b>$

$\therefore$ $\delta^2\rho^2 \varepsilon K(C_1, C_2') = K(a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2)$

It is clear that a rational function in the polynomials $a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2$ cannot be divisible by $a^2 + b^2$. Therefore $\delta = 0$ and $\alpha = \nu \varepsilon K(C_1, C_2)$.

Theorem 6. Let $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ be any (real or complex) solution, of order four, of the restricted three body problem. Let $\theta \varepsilon K(\xi, \eta, \rho_1, \rho_2)[\xi', \eta']$ and let $\theta' = 0$. Then θ^* is a polynomial in $P_1 = \xi'^2 + \eta'^2 - \xi^2 - \eta^2$, $P_2 = \xi\eta' - \xi'\eta + \xi^2 + \eta^2$.

Proof. There is no loss in generality in assuming degree of $\theta \geq 0$, since degree of $\frac{1}{\theta} = -$ degree of θ and $(\frac{1}{\theta})' = 0$. By theorem 4, $\theta^*(a, b) = 0$, so that

$$\theta^*(a, b) \varepsilon K(a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2) .$$

But θ is a polynomial in $\xi' \; \eta'$ so that $\theta^*(a, b)$ is a polynomial in

$$P_1(a, b) = a'^2 + b'^2 - a^2 - b^2,$$

$$P_2(a, b) = ab' - a' b + a^2 + b^2$$

Therefore $\theta^*(\xi, \eta)$ is a polynomial in $P_1(\xi, \eta), P_2(\xi, \eta)$.

Theorem 7. Let $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ be any real solution, of order four, of the restricted three-body problem. Let $\theta \in K < \xi, \eta >$; $K = C(\mu)$ where C is the field of real numbers. Then θ is not a real exponential.

Proof. Same argument as for the n-body problem. For, the differential field $K < a, b >$ does not contain any real exponentials.

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