OPTIMAL CONTROL AND CONVEX PROGRAMMING

by

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ABSTRACT

A general type of discrete optimal control problem with both control and state constraints is considered. Necessary conditions for a relative minimum are given (assuming only differentiability) based on the Kuhn-Tucker theory. For a convex function and linear system of differential equations it is shown that these conditions are also sufficient for a global minimum. A computational scheme is described for the state constrained problem where the conditions are sufficient. The scheme is based on a convex programming method and determines first if any admissible control exists, and if so, finds an optimal control. The solution of a four-dimensional system with state constraints is presented in order to illustrate this computational scheme.

OPTIMAL CONTROL AND CONVEX PROGRAMMING

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1. Introduction

The problems arising in optimal control theory are similar mathematically to those met in the calculus of variations with additional requirements in the form of inequality constraints which must be satisfied. The subject received its initial impetus from problems arising in the area of guidance and control, and the basic results of Pontryagin [11] are developed from this point of view, as is much of the subsequent work on this subject [10]. However, as emphasized by Bellman [2] a continuous spectrum of problems encountered by systems analysts and operations researchers, by economists and management consultants, in various phases of industrial, scientific and military activity, can be included in an appropriate formulation of control theory. Two such potentially important applications are dynamic economic models [16] and long range capital investment studies.

As a greatly simplified example of the latter application suppose the control $u(t)$ is the rate of investment at time t. The state of the system $x(t)$ is described by the quantity of the $i^{\underline{th}}$ product $x_i(t)$, produced by time t. The $x_i(t)$ are determined for any given $u(t)$ by the system of differential equations

$$\dot{x} = f(x, u, t) \ , \ x(o) = x_o \ , \quad t \in [o, T]$$

The rate of investment is, of course, nonnegative and also may not exceed a specified upper bound, so that $O \leq u(t) \leq \alpha$. Furthermore, it is required that the production schedule satisfy the state constraints $p_i(t) \leq x_i(t) \leq q_i(t)$;

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where the $p_i(t)$ and $q_i(t)$ are specified, and that this be done so as to minimize the total discounted investment $\rho[u] = \int_0^T e^{-\beta t} u(t)\,dt$, over a finite time T . Because of the presence of the state constraints this problem is of a type which is difficult both theoretically and computationally (see [4] and Chapter 6 of [11]). In actual practice the investment decisions would not be made continuously but rather would be made at discrete intervals, say once a month. This is typical of a dynamic process which can be formulated as continuous but which is more usefully considered as discrete, since this gives both a more realistic model and a computational method of solution.

The two important questions to be answered are:

1. Will any admissible $(0 \leq u(t) \leq \alpha)$ investment program satisfy the production constraints? That is, does an admissible control exist?

2. If there are admissible controls, how do we find one which is optimal?

The remainder of this paper is devoted to answering these two questions for a general class of discrete optimal control problems.

Some of the material here is based on parts of an earlier report [13]. The author has also had the benefit of several discussions with J. Abadie whose work in this area [1] has been most stimulating. The theory of optimal control for discrete systems with bounded control only, has recently been investigated by Jordan and Polak [19] and by Halkin [18] .

2. Discrete Problem with State Constraints

It will be useful to give a further motivation for the approach taken here for the solution of optimal control problems. Such problems fall naturally into two classes depending on their initial formulation, namely continuous and discrete. In general, we will solve the continuous problems on a digital computer and this will require the numerical integration of systems of differential equations, which is in fact a discrete approximation to the continuous process. We may therefore assume, at least for computational purposes, that we will always be dealing with discrete problems.

To be specific, we will consider a discrete problem as follows. Let $x_i \in E^n$ represent the state vector at time t_i, $i = 0, 1, \ldots n$, and $u_i \in E^r$, the corresponding control vector for $i = 0, 1, \ldots m-1$. The initial value x_0 is specified and we wish to determine the vectors x_i and u_i so as to minimize

$$\sum_{i=0}^{m-1} \sigma(x_i, u_i) \tag{2.1}$$

where the x_i and u_i must satisfy the recursion relation

$$x_{i+1} - x_i = f(x_i, u_i), \quad i = 0, 1, \ldots \quad m-1 \tag{2.2}$$

and each u_i must be selected from a convex, compact subset $U \subset E^r$, and x_m must lie in a convex, compact subset $X_m \subset E^n$. We assume* that $\sigma(x, u)$ is a function from $E^n \times U$ to E^1 with $\sigma \in C^1$ on $E^n \times U$, and that $f(x, u)$ is a function from $E^n \times U$ to E^n with $f \in C^1$ on $E^n \times U$. We assume further that the sets U and X_m are each specified by a system of inequality constraints. That is

$$X_m = \{x \mid g(x) \leq 0\} \tag{2.3}$$

*The results obtained actually hold for the more general case where $\sigma = \sigma(x, u, t)$ and $f = f(x, u, t)$. However, in order to simplify the presentation we will assume that σ and f do not depend explicitly on t.

and

$$U = \{ u \mid h(u) \leq 0 \} \tag{2.4}$$

where $g(x)$ is a function from E^n to E^k with $g \in C^1$ and convex on E^n, and $h(u)$ is a function from E^r to E^ℓ with $h \in C^1$ and convex on E^r. The sets X_m and U are assumed nonempty, and by the convexity of $g(x)$ and $h(x)$ they are convex.

We may think of this discrete problem as arising from a finite difference approximation to the continuous problem

$$\min \int_0^T \bar{\sigma} \ (x(t), \ u(t) \ ) \ dt \tag{2.5}$$

where

$$\dot{x} = \bar{f}(x,u) , \quad t \in [0,T] \tag{2.6}$$

$$x(0) = x_0, \quad x(T) \in X_m$$

$$u(t) \in U , \quad t \in [0,T]$$

The sum (2.1) is the simplest approximation to the integral (2.5) with $\Delta t = T/m$, $t_i = i \Delta t$, and $\sigma = \Delta t \ \bar{\sigma}$. The recursion relation (2.2) is the simplest finite difference approximation to the differential equation (2.6) with $f = \Delta t \ \bar{f}$.

We may now consider the discrete problem as the minimization of a convex function on a finite dimensional Euclidean space subject to the equality constraints (2.2) and the inequality constraints (2.3) and (2.4). For problems of this type the appropriate theory is that developed by Kuhn and Tucker [9,8,3]. For our purposes the most convenient statement of this theory is essentially that given by Berge [3].

We let $s = m(n+r)$ and denote by $z \in E^s$ the vector $z' = (x_1', \ \cdots\cdots x_m', u_0', \ \ldots \ u_{m-1}')$, where unprimed vectors are column vectors and the prime denotes transpose. We will call z an <u>admissible point</u> if the x_i, $i = 1, \dots m$, satisfy (2.2), $x_m \epsilon X_m$, and $u_i \epsilon U$, $i = 0, 1, \dots m-1$. Suppose we have an admissible point z^*, determined by x_i^*, $i = 1, \dots m$, u_i^*, $i = 0, \dots m-1$. We will denote by $g_x(x_m^*)$ the $k \times n$ Jacobian matrix of $g(x)$ evaluated at x_m^*, and by $h_u(u_i^*)$ the $\ell \times r$ Jacobian matrix of $h(u)$ evaluated at u_i^*. We will also denote by $\hat{g}_x(x_m^*)$ the matrix in which we have replaced by zeros the $j^{\underline{th}}$ row of $g_x(x_m^*)$ if the $j^{\underline{th}}$ element of $g(x_m^*) < 0$. Thus we have $g'(x_m^*)\, \hat{g}_x(x_m^*) = 0$. The matrix $\hat{h}_u(u_i^*)$ is defined similarly for $i = 0, 1, \dots m-1$, so that $h'(u_i^*)\, \hat{h}_u(u_i^*) = 0$, $i = 0, 1, \dots m-1$. We also let $f_x(x, u)$ and $f_u(x, u)$ denote the $n \times n$ and $n \times r$ Jacobian matrices of f.

An <u>admissible direction</u> $\bar{z}$ at z^* is given by vectors $\bar{x}_i$, $i = 1, \dots m$, and $\bar{u}_i$, $i = 0, \dots m-1$, such that

$$\bar{x}_{i+1} - \bar{x}_i = f_x(x_i^*, u_i^*)\, \bar{x}_i + f_u(x_i^*, u_i^*)\, \bar{u}_i, \quad i = 0, \dots m-1$$

$$\bar{x}_0 = 0$$

and

$$\hat{g}_x(x_m^*)\, \bar{x}_m \leq 0$$

$$\hat{h}_u(u_i^*)\, \bar{u}_i \leq 0, \quad i = 0, 1, \dots m-1$$

It follows that if $y \epsilon E^s$ is not an admissible direction at z^*, then it points outward from the set of admissible points at z^*; that is, $z^* + \alpha y$ is not an admissible point for every sufficiently small $\alpha > 0$.

The sum (2.1) to be minimized is given in terms of z by letting

$$\varphi(z) = \sum_{i=0}^{m-1} \sigma\,(x_i, u_i) \tag{2.7}$$

We will say that an admissible point z^* is a relative minimum if

$$\varphi(z^*) \leq \varphi(z^* + \alpha\bar{z}) \qquad (2.8)$$

for every admissible direction $\bar{z}$ at z^* and sufficiently small $\alpha > 0$.

We can now state the necessary Kuhn-Tucker conditions for a relative minimum.

Theorem 1A

If an admissible point z^* is a relative minimum, then there exist vectors $\lambda_i \in E^n$, $i = 1, \ldots m$, a vector $\nu_m \geq 0$, $\nu_m \in E^k$ and vectors $\eta_i \geq 0$, $\eta_i \in E^\ell$, $i = 0, 1, \ldots m-1$, such that

$$\nu'_m \; g(x^*_m) = 0 \qquad (2.9)$$

$$\eta'_i \; h(u^*_i) = 0 , \quad i = 0, 1, \ldots m-1 \qquad (2.10)$$

and the Lagrangian function

$$\Phi(z) = \varphi(z) + \sum_{i=0}^{m-1} \lambda'_{i+1} [x_{i+1} - x_i - f(x_i, u_i)] + \nu'_m \; g(x_m) + \sum_{i=0}^{m-1} \eta'_i \; h(u_i) \qquad (2.11)$$

has a stationary point at $z = z^*$; that is

$$\Phi_z(z^*) = 0 \qquad (2.12)$$

Proof:

The proof is essentially that given in [9] or [3] and is based on the Farkas lemma.

Corollary

At a relative minimum z^* the value of the Lagrangian functi[illegible] $\Phi(z)$ and the sum (2.1) are equal. Furthermore, the vectors λ_i, ν_m and η_i must satisfy the following system of equations:

$$\lambda_{i+1} - \lambda_i = -f'_x (x^*_i, u^*_i) \lambda_{i+1} + \sigma_x(x^*_i , u^*_i) \quad i = 1, \ldots m-1 \qquad (2.13)$$

$$\lambda_m = -g'_x(x^*_m) \; \nu_m \qquad (2.14)$$

and

$$h'_u(u^*_i)\ \eta_i = f'_u(x^*_i, u^*_i)\ \lambda_{i+1} - \sigma_u(x^*_i, u^*_i) \quad i = 0,1,\dots m-1 \tag{2.15}$$

Proof:

Because of the complementarity requirements (2.9) and (2.10) and the fact that the admissible point z^* satisfies (2.2), we have $\Phi(z^*) = \varphi(z^*) = \sum_{i=0}^{m-1} \sigma(x^*_i, u^*_i)$. The system (2.13) to (2.15) is equivalent to (2.12) and is obtained by setting to zero the partial derivative of Φ with respect to each component of z.

It is clear from the form of (2.13) that this recursion relation for the λ_i is closely related to the usual adjoint equation for the continuous problem. The terminal value λ_m for the adjoint vector is specified by (2.14). The relationship between the system (2.13) to (2.15) and the continuous problem will be considered in another paper [14].

In Theorem 1A necessary conditions for a relative minimum were given with no conditions on $\sigma(x,u)$ and $f(x,u)$ other than differentiability. We now show that if $\sigma(x,u)$ is convex on $E^n \times U$, and $f(x,u)$ is linear on $E^n \times U$ then the conditions are also <u>sufficient</u> for a <u>global</u> minimum.

<u>Theorem 2A</u>

Let $\sigma(x,u)$ be convex and $f(x,u)$ be linear on $E^n \times U$. If z^* is an admissible point and there exist vectors λ_i, and nonnegative vectors ν_m and η_i, such that (2.9), (2.10) and (2.13) to (2.15) are satisfied, then z^* is a global minimum.

Proof:

We will denote by $Z \subset E^s$ the direct product of the sets $x_i \in E^n$, $i = 1,\dots m$ and $u_i \in U$, $i = 0,1,\dots m-1$. Then the function $\varphi(z)$ is convex on Z since each term is convex on $E^n \times U$. The first summation in (2.11) is linear in z and therefore also convex on Z. The remaining two terms are convex by assumption and the fact that ν_m and the η_i are nonnegative. Therefore $\Phi(z)$ is onvex on Z. Now a stationary point of a convex function is a global minimum, so that

$$\Phi(z^*) = \min_{z \in Z} \Phi(z) \tag{2.16}$$

As above, we have $\varphi(z^*) = \Phi(z^*)$. Furthermore for every admissible point z we have from (2.2), (2.3) and (2.4) that

$$\Phi(z) \leq \varphi(z) \tag{2.17}$$

Then from (2.16) and (2.17), $\varphi(z^*) \leq \varphi(z)$ for every admissible point z, so that z^* is a global minimum.

By means of a straightforward modification the previous results can be extended to include the case of constraints on the state vectors x_i, $i = 1, \ldots m-1$, in addition to the constraint $x_m \in X_m$. To show this let us require that

$$x_i \in X_i \ , \ i = 1, \ldots m \tag{2.18}$$

where each X_i is a convex subset of E^n, specified in terms of convex functions $g^i(x)$ from E^n to E^{k_i} for $i = 1, \ldots m$, with $g^m(x) = g(x)$. We therefore have

$$X_i = \{ x \mid g^i(x) \leq 0 \} \ , \ i = 1, \ldots m \tag{2.19}$$

Note that X_m is identical to that given by (2.3). An admissible point is now one which satisfies (2.2), (2.18) and $u_i \in U$.

The extension of Theorem 1A to this state bounded problem is given by

Theorem 1B

If an admissible point z^* is a relative minimum, then there exist vectors $\lambda_i \in E^n$, $i = 1, \ldots m$, vectors $\nu_i \geq 0$, $\nu_i \in E^{k_i}$, $i = 1, \ldots m$, and vectors $\eta_i \geq 0$, $\eta_i \in E^\ell$, $i = 0,1,\ldots m-1$, such that

$$\nu_i' g^i(x_i^*) = 0 \ , \quad i = 1,\ldots m \tag{2.20}$$

$$\eta_i' \, h(u_i^*) = 0 \ , \quad i = 0, \ldots m-1 \tag{2.21}$$

and the Lagrangian function

$$\Phi(z) = \varphi(z) + \sum_{i=0}^{m-1} \lambda_{i+1}' [x_{i+1} - x_i - f(x_i, u_i)]$$
$$\sum_{i=1}^{m} \nu_i' \, g^i(x_i) + \sum_{i=0}^{m-1} \eta_i' h(u_i) \tag{2.22}$$

has a stationary point at $z = z^*$.

Similarly the corollary to Theorem 1A now becomes

Corollary

At a relative minimum z^* , we have $\Phi(z^*) = \varphi(z^*) = \sum_{i=0}^{m-1} \sigma(x_i^*, u_i^*)$. Furthermore, the vectors λ_i, ν_i , and η_i must satisfy (2.14), (2.15) and

$$\lambda_{i+1} - \lambda_i = f_x'(x_i^*, u_i^*) \, \lambda_{i+1} + \sigma_x(x_i^*, u_i^*) + g_x^{i'}(x_i^*)\nu_i \quad i = 1,\ldots m-1 \tag{2.23}$$

Finally, the extension of Theorem 2A gives

Theorem 2B

Let $\sigma(x,u)$ be convex and $f(x,u)$ be linear on $E^n \times U$. If z^* is an admissible point and there exist vectors λ_i, and nonnegative vectors ν_i and η_i , such that (2.20), (2.21), (2.23) , (2.14) and 2.15) are satisfied, then z^* is a global minimum.

3. Convex Programming Solution

We are now in a position to consider the computational solution of the discrete optimal control problem with state constraints. We limit our discussion here to problems for which the optimality conditions are sufficient, namely, $\sigma(x, u)$ convex and $f(x, u)$ linear. In the interest of simplicity we will also assume that the constraint sets U and X_i are defined by linear inequalities, that is, the functions $h(u)$ and $g^i(x)$ are linear. The method for $f(x, u)$ linear discussed here is the basis for a convergent iterative procedure for solving the more general case where $f(x, u)$ is convex. This more general case is described in another paper [14].

The general,variable coefficient linear case will be considered, that is, a discrete approximation to the differential equation

$$\dot{x} = A(t)x + B(t)u \qquad t \in [0, T] \tag{3.1}$$

We will let $A_i = A(t_i)$ and $B_i = B(t_i)$, $i = 0, \ldots m$, and use the finite ... ence approximation

$$x_{i+1} - x_i = \Delta t[\theta A_{i+1} x_{i+1} + (1-\theta) A_i x_i] + \Delta t B_i u_i \tag{3.2}$$
$$i = 0, 1, \ldots m - 1$$

where $0 \leq \theta \leq 1$. For $\theta = 0$, this gives the explicit (forward) scheme (2.2), while for $\theta = 1$ it gives the fully implicit (backward) scheme. The value $\theta = 1/2$ gives a numerically stable method with minimum truncation error.

The relation (3.2) may be solved for x_{i+1} to give

$$x_{i+1} = K_i x_i + \bar{B}_i u_i \ , \quad i = 0, 1, \dots m-1 \tag{3.3}$$

where

$$K_i = [I - \frac{\theta}{m} A_{i+1}]^{-1} [I + \frac{1-\theta}{m} A_i] \tag{3.4}$$

and

$$\bar{B}_i = \frac{1}{m} [I - \frac{\theta}{m} A_{i+1}]^{-1} B_i \tag{3.5}$$

The solution to the finite difference equation (3.3) is given by

$$x_i = Y_i x_0 + Y_i \sum_{j=0}^{i-1} \Lambda'_{j+1} \bar{B}_j u_j \ , \quad i = 1, \dots m \tag{3.6}$$

where the matrices Y_i satisfy the homogeneous equation

$$Y_{i+1} = K_i Y_i \ , \quad Y_0 = I \ , \quad i = 0, 1 \dots m-1 \tag{3.7}$$

and the matrices Λ_i satisfy the homogeneous adjoint equation

$$\Lambda_i = K'_i \Lambda_{i+1} \ , \quad Y_m \Lambda'_m = I \ , \quad i = m-1, \dots 1 \tag{3.8}$$

It follows from (3.7) and (3.8) that $Y_i \Lambda'_i = Y_{i+1} \Lambda'_{i+1} = I$, so that $\Lambda'_i = Y_i^{-1}$, $i = 1, \dots m$. Furthermore, the actual calculation of $Y_i x_o$ and the coefficients of the u_j in (3.6) requires only the inversion of an $n \times n$ matrix to get each K_i , and multiplication of $n \times r$ matrices. These quantities are therefore readily calculated from the specified values of x_o, $A(t)$, $B(t)$, m and θ.

Because of the linearity of (3.6) we can use these relations to map the original problem into the control space, the product space of the u_i. This reduces the original problem to one of minimizing a convex function subject to linear inequality constraints in the space E^{mr}. Since the original problem involved $s = m(n + r)$ variables, and since $r \leq n$ (often with $r = 1$ or $r = 2$), this may effect a considerable reduction in the number of variables. To accomplish this reduction we replace each x_i in the sum (2.1) and in the linear inequalities $g^i(x_i) \leq 0$ which define the X_i, by the corresponding right hand side of (3.8). Each vector $g^i(x_i)$ thus gives rise to k_i linear inequalities on the u_i. We also have the original set of ℓ linear inequalities $h(u_j) \leq 0$, which insures that each $u_j \in U$. We therefore have a system of

$$m\ell + \sum_{i=1}^{m} k_i = m(\ell + \bar{k})$$

linear inequalities which must be satisfied by any admissible set of vectors u_j. Because of the way in which these inequalities arise they have a special structure which can be used to advantage. We will represent the inequalities obtained from the $g^i(x_i)$ by

$$\sum_{j=0}^{m-1} D_j' u_j - p \leq 0 \tag{3.9}$$

where each D_j is an $r \times m\bar{k}$ matrix and $p \in E^{m\bar{k}}$. Because x_{i+1} involves only values of u_j for $j \leq i$, the matrix $D' = [D_0' D_1' \ldots D_{m-1}']$ has a lower triangular structure. The matrices D_j will depend on the matrices A_i, B_i and the matrices which define the linear transformations $g^i(x_i)$, as well as θ and m . The vector p will also depend on x_o , as well as these other quantities. The important point, however, is that the matrices D_j and the vector p can be explicitly computed with a reasonable amount of computation.

In order to simplify the discussion we will denote by $w \in E^{mr}$, a vector which specifies the control for $i = 0, 1, \ldots m-1$. That is, $w' = (u'_0, u'_1, \ldots u'_{m-1})$. Two subsets of E^{mr} are then given by

$$W_1 = \{w \mid \sum_{j=0}^{m-1} D'_j u_j - p \leq 0\} \tag{3.10}$$

and

$$W_2 = \{w \mid h(u_j) \leq 0 \text{ , } j = 0 \text{ , } \ldots m-1 \} \tag{3.11}$$

Since it is determined by linear inequalities W_1 is closed and convex, if it is not empty. Since W_2 is the direct product of compact convex sets, it is compact and convex. Then

$$W = W_1 \cap W_2 \tag{3.12}$$

is compact and convex, if it is not empty.

The first important question about the discrete problem can now be answered, namely does there exist any admissible control? This is equivalent to the question, is W an empty set? Good computational methods are available for determining if a solution to a system of linear inequalities exists and if so, finding such a solution. Since the inequalities of (3.10) and (3.11) have the natural form of constraints for a dual linear programming problem, a dual simplex procedure can be used for this purpose [5]. The starting procedure for the gradient projection method [12] is equivalent to this and may conveniently be used for this purpose. Another approach would be to use the duality theory of linear programming and consider the primal problem corresponding to the dual constraints (3.10) and (3.11) and an arbitrary linear dual objective function. This objective function can always be chosen so as to give an initial primal feasible solution. The duality theory then says that if the primal problem has a finite maximum the corresponding dual solution is dual feasible (i.e., an admissible control), and if the primal has an infinite solution then no dual feasible solution exists (i.e., there is no admissible control). Any suitable linear programming code can therefore be used for this purpose.

Once we have determined that an admissible control exists, and in fact have actually determined such a control, we can proceed to find an optimal control. We do this by once again using (3.8) to eliminate the x_i; this time in the sum (2.1), to get a function $\rho(w)$ to be minimized. Since convexity is preserved by a linear transformation the function $\rho(w)$ is convex. We have now reduced the original discrete problem to that of finding $\rho(w^*) = \min_{w \in W} \rho(w)$; the minimization of a convex function subject to linear inequality constraints. Furthermore, we have an admissible control w^0 (determined as discussed above) with which to start the minimization procedure. A number of computationally tested methods are available for the solution of such convex nonlinear programming problems [12, 6]. In the special case where $\sigma(x, u)$ is linear on $E^n \times U$, the problem can be solved in the dual form by a dual simplex method or in its primal form by any primal simplex code. The possibility of formulating a discrete linear optimal control problem as a linear programming problem has previously been considered by Zadeh [17]. An efficient method of solution for linear problems with large values of m has been proposed by Dantzig, based on his generalized upper bounding technique [5].

Once the optimal control $w^{*\prime} = (u_0^{*\prime}, u_1^{*\prime}, \ldots u_{m-1}^{*\prime})$ has been calculated in this way, the optimal state vectors x_i^*, $i = 1, \ldots m$, are immediately given by (3.6). The Lagrange multipliers (or shadow price vectors) ν_i, $i = 1, \ldots m$, and η_i, $i = 0, \ldots m-1$, corresponding to the state and control constraints are also available as part of the convex programming solution. These quantities may be of considerable interest since they give the rate of decrease in the function value with relaxation of each constraint. The influence of parameter changes on the optimal solution can also be obtained by use of the parametric solution features of many codes. Finally, if desired, the optimal adjoint vectors satisfying (2.23) with $f(x, u)$ linear can be calculated from

$$\lambda_i = K_i' \lambda_{i+1} - \sigma_x(x_i^*, u_i^*) - g_x^{i'}(x_i^*) \nu_i \ , \quad i = m-1, \ldots 1$$

starting with $\lambda_m = -g_x^{m'}(x_m^*) \nu_m$.

4. Computational Example

The previous discussion will now be illustrated by means of a variable coefficient linear problem with four state variables and a scalar control. In addition to bounded control we also impose state constraints on one of the state variables. The system considered is in the form (3.1), with

$$TA(t) = tA_f + (T - t)A_0$$

and

$$A_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -10 & -10 & -5 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -16 & -12 & -5 \end{bmatrix}, \quad B(t) = b = \begin{bmatrix} 0 \\ 0 \\ 1.0 \\ -4.5 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The control must satisfy $|u(t)| \le 1$, so that $h(u) = \binom{u-1}{-u-1}$. We also impose the terminal constraints $x_3(T) = x_4(T) = 0$ and the state constraint $|x_4(t)| \le 0.5$ for $0 \le t < T$. These give

$$g^i(x(t_i)) = \begin{bmatrix} x_4(t_i)-0.5 \\ -x_4(t_i)-0.5 \end{bmatrix}, \quad i = 1, \ldots m-1, \quad g^m(x(T)) = \begin{bmatrix} x_3(T) \\ -x_3(T) \\ x_4(T) \\ -x_4(T) \end{bmatrix}$$

We wish to minimize the terminal Euclidean norm $||x(T)||$. This system is similar to a constant coefficient system for which a numerical solution has previously been obtained [7].

The optimum solution to this problem using the finite difference scheme (3.2) with $\theta = 1/2$, is shown in Figures 1 and 2. The values $T = 2.5$ and $m = 25$

were used so that $\Delta t = 0.1$. The optimum control is shown in Fig. 1 and the trajectory as given by the four state variables $x_j(t_i)$, $j = 1, \ldots 4$, $i = 0, \ldots 25$, is shown in Fig. 2 for the case with no state constraint on $x_4(t_i)$. The minimum value of the objective function attained is $\| x(T) \|^2 = 0.0085$. The optimal solution to the same problem with the state constraint $|x_4(t_i)| \leq 0.5$ is shown in Figs. 3 and 4. The distinct change in the control required to satisfy the state bound should be noted, as well as the increase in the terminal norm squared to 0.0269, due to the fact that the admissible control set W is smaller because of the state bound.

These solutions were obtained using a program based on the scheme described by (3.2) thru (3.11). The convex programming problem obtained in this way was solved using the gradient projection computer program [20]. The solution time required for each of these problems on the IBM 7090 was approximately two minutes. The program and its use to obtain the optimal solution to a variety of typical problems will be described elsewhere [15].

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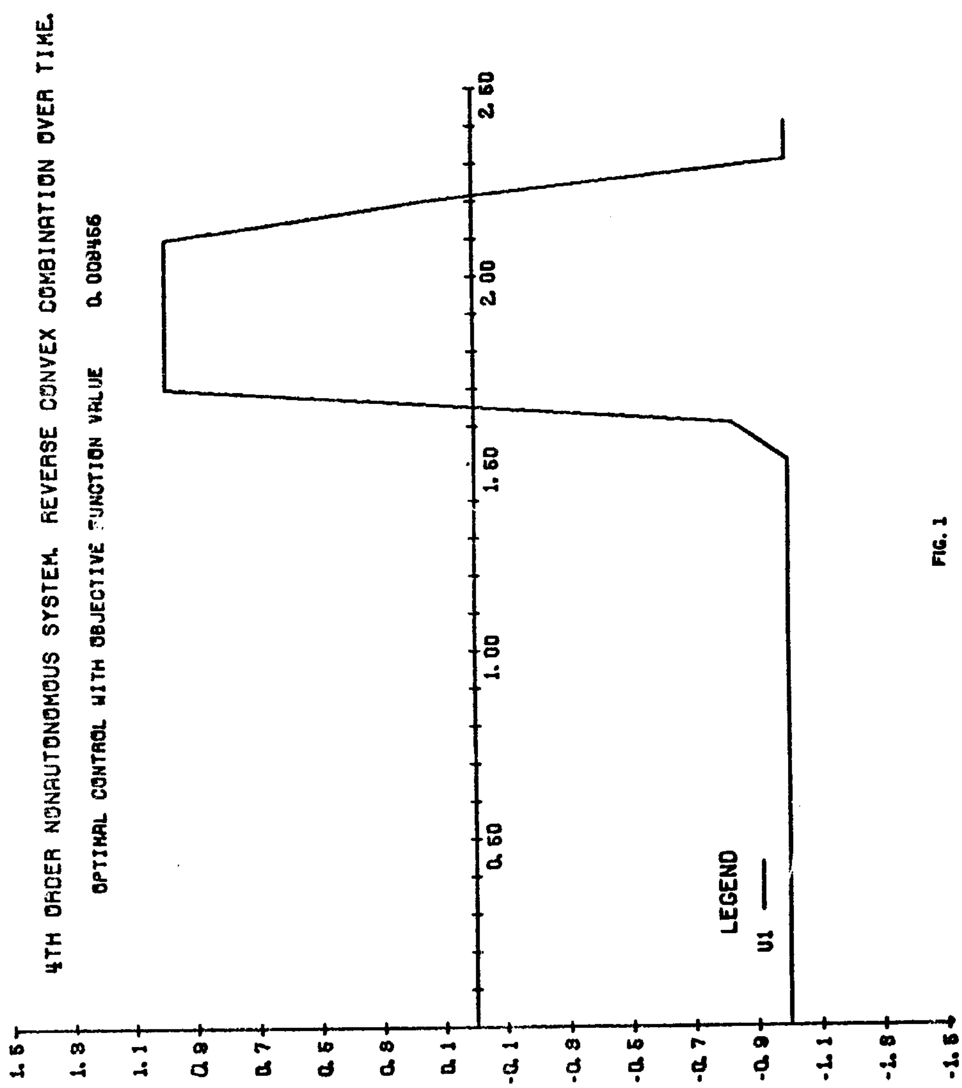
4TH ORDER NONAUTONOMOUS SYSTEM. REVERSE CONVEX COMBINATION OVER TIME.
OPTIMAL CONTROL WITH OBJECTIVE FUNCTION VALUE 0.008456
1.5
1.3
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
-0.3
-0.5
-0.7
-0.9
-1.1
-1.3
-1.5
0.50
1.00
1.50
2.00
2.50
LEGEND
U1

FIG. 1

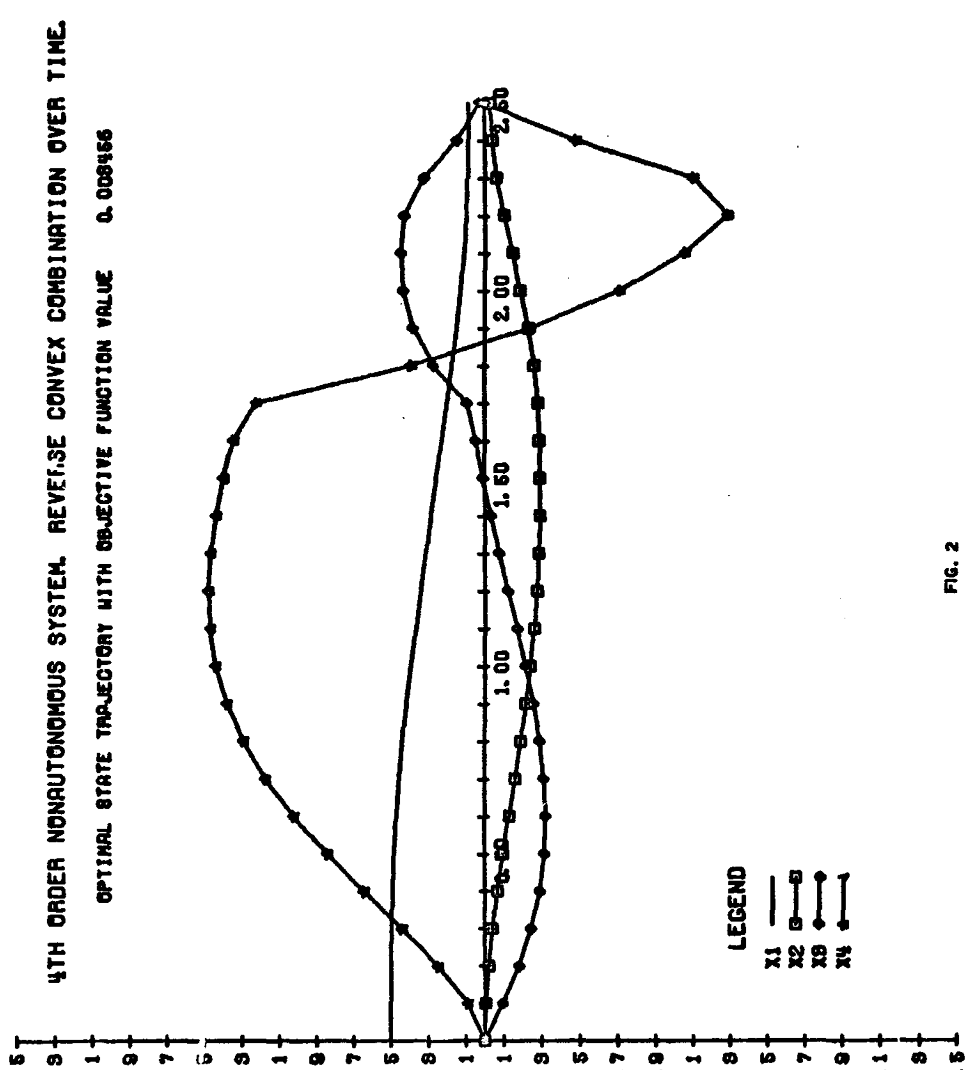
4TH ORDER NONAUTONOMOUS SYSTEM. REVERSE CONVEX COMBINATION OVER TIME.
OPTIMAL STATE TRAJECTORY WITH OBJECTIVE FUNCTION VALUE 0.008456
LEGEND
X1
X2
X3
X4
0.50
1.00
1.50
2.00
2.50
2.5
2.3
2.1
1.9
1.7
1.5
1.3
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
-0.3
-0.5
-0.7
-0.9
-1.1
-1.3
-1.5
-1.7
-1.9
-2.1
-2.3
-2.5

FIG. 2

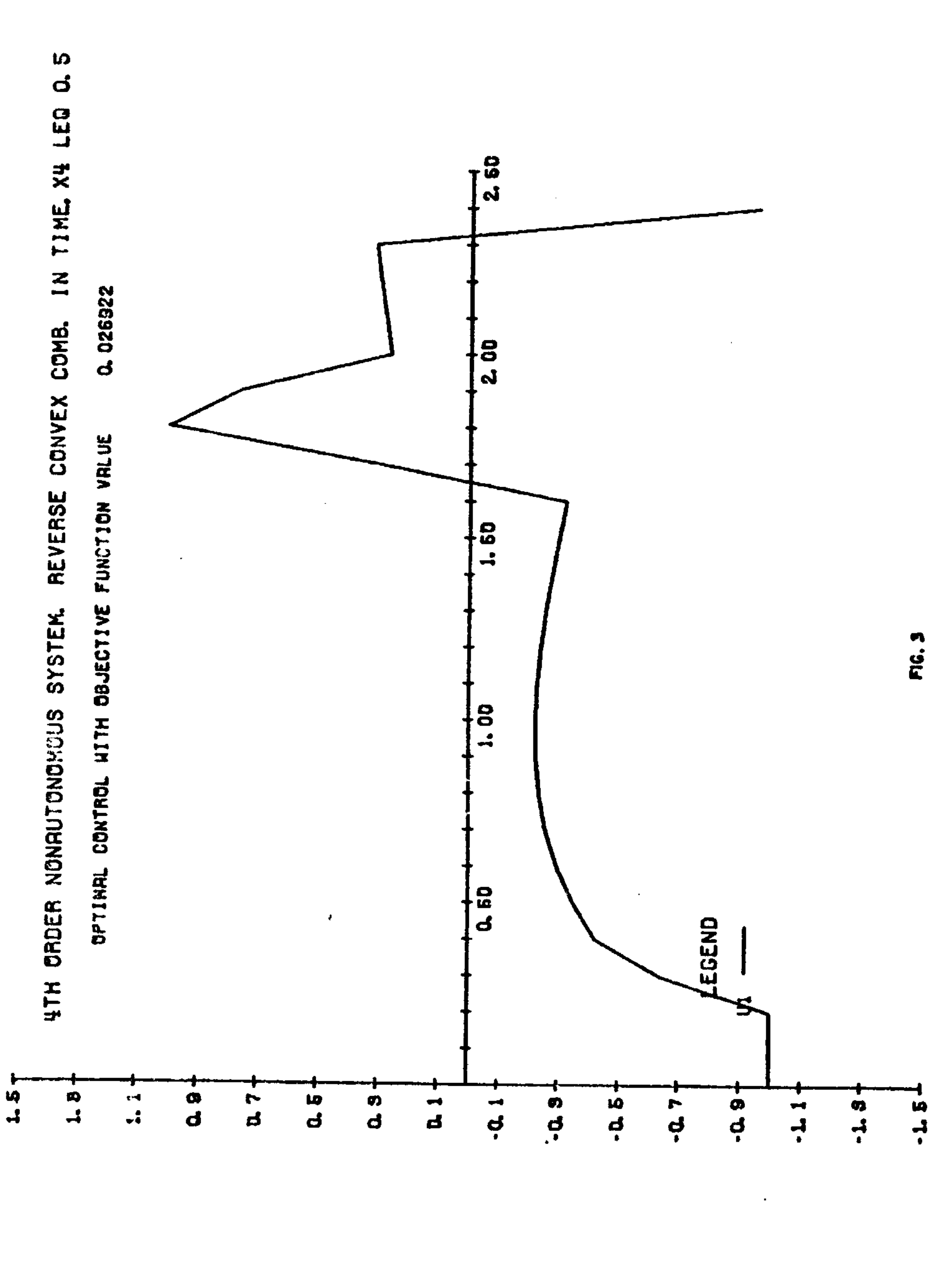
4TH ORDER NONAUTONOMOUS SYSTEM. REVERSE CONVEX COMB. IN TIME, X4 LEQ 0.5
OPTIMAL CONTROL WITH OBJECTIVE FUNCTION VALUE 0.026922
LEGEND
U1
0.50
1.00
1.50
2.00
2.50
1.5
1.3
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
-0.3
-0.5
-0.7
-0.9
-1.1
-1.3
-1.5

FIG. 3

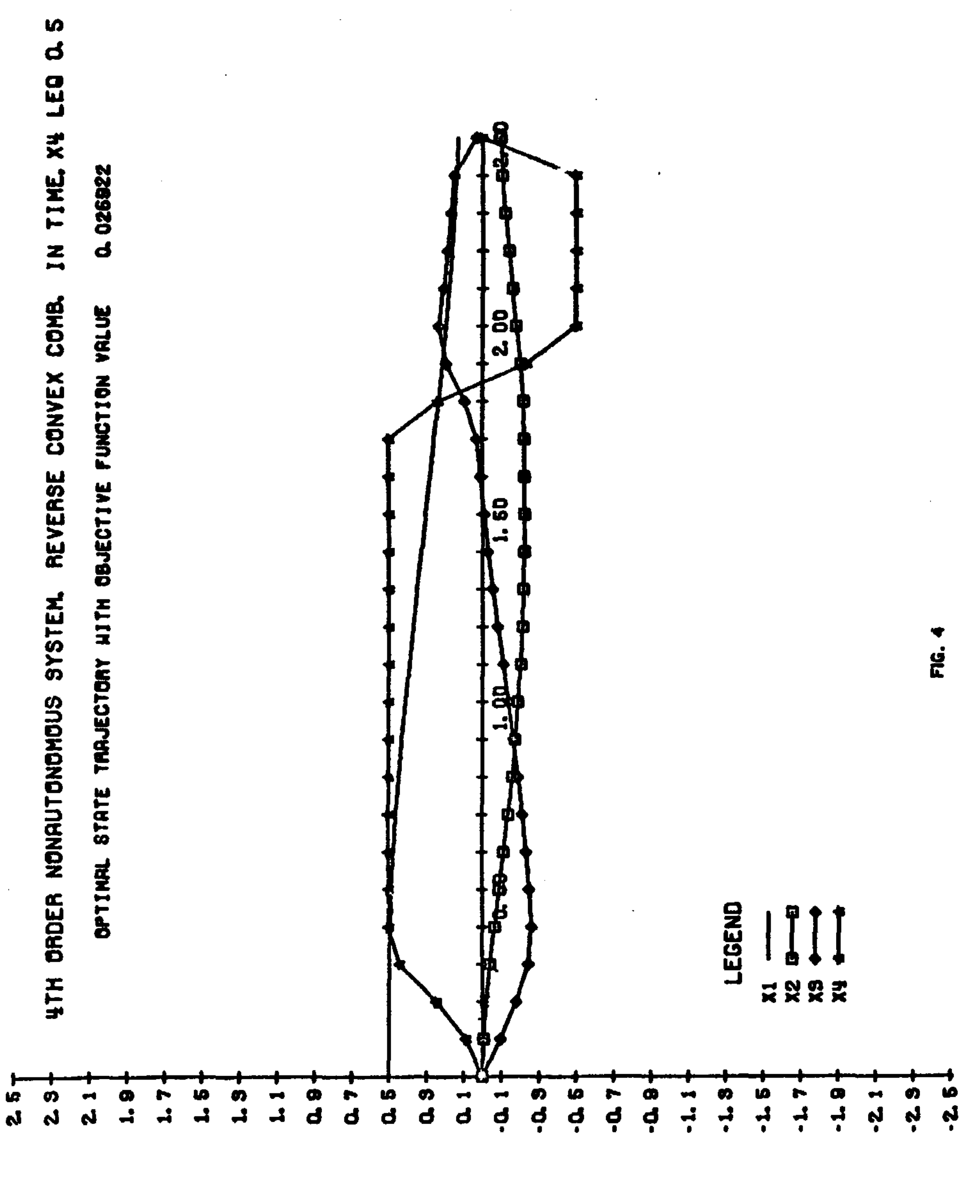
4TH ORDER NONAUTONOMOUS SYSTEM. REVERSE CONVEX COMB. IN TIME, X4 LEQ 0.5
OPTIMAL STATE TRAJECTORY WITH OBJECTIVE FUNCTION VALUE 0.026922
0.50
1.00
1.50
2.00
2.50
2.5
2.3
2.1
1.9
1.7
1.5
1.3
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
-0.3
-0.5
-0.7
-0.9
-1.1
-1.3
-1.5
-1.7
-1.9
-2.1
-2.3
-2.5
LEGEND
X1
X2
X3
X4

FIG. 4