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SECOND ANNUAL REPORT

For Period

15 September 1965 through 30 June 1966

INVESTIGATION OF OPTIMIZATION OF ATTITUDE CONTROL SYSTEMS

For

Jet Propulsion Laboratory

4800 Oak Grove Drive

Pasadena, California

on

Contract No. 950670

NAS 7-100

School of Electrical Engineering

Purdue University

Lafayette, Indiana 47907

J. Y. S. LUH, Principal Investigator

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FOREWORD

This is the second annual report submitted in accordance with the provisions of Contract No. 950670, "Investigation of Optimization of Attitude Control Systems". It summarizes the research activities of the period September 15, 1965 through June 30, 1966.

This report is in three parts. The first part outlines the progress during the reporting year. The technical discussions are given in Parts B and C, in which the conclusions of the results and the plan of future work are included.

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PART A

GENERAL DISCUSSION

1. INTRODUCTION

This annual report summarizes the results of the research achieved during the period September 15, 1965 through June 30, 1966. Some of the material included in this report was presented in the two previous quarterly progress reports. The repetition makes this annual report an independent document so that no references to the previous progress reports are necessary.

2. TECHNICAL PROGRESS

In the design of an autopilot for a space vehicle which is capable of performing the task of soft landing, the problem of optimal control with bounded phase-coordinate and bounded control is relevant. By using the necessary and sufficient conditions, the general theory for linear autonomous systems was developed. A method of determining the optimal control, which is a direct application of the theory, was derived. As an illustration, two particular systems were studied in detail. The first example deals with the time-optimal control of an unstable booster with actuator position and rate limits. The results, when evaluated with numerical data, agree with those that have been published by other authors using different methods. The second example considers a flexible vehicle subject to wind disturbances.

This problem is more complicated than the first one since the ratio of actuator position to its rate plays an important role in the extremal control law. The control variable is found to enter and exit from its bound as often as the time duration permits, which is a natural result of the oscillatory behavior of the system. In both examples the optimal controls are expressed as explicit time functions. These results and their conclusions together with the future research plan are presented in Part B.

The optimal control of antenna pointing direction was investigated. The problem is formulated in such a manner that the pointing direction is kept within an accepted region with maximum probability all the time. Essentially the controller forces the antenna to point in a desired direction by minimizing the error rate of transmission of information during the entire flight journey of the space vehicle. In the study, an assumption of the Markovian property of the random jittering of the antenna is made. In addition, the disturbances in any two small consecutive time intervals are assumed to be statistically independent. Thus the probability distribution satisfies the backward diffusion equation, and the problem reduces to the determination of a controller which maximizes the probability. A computational scheme based on an iteration procedure was developed. The technical discussion and the conclusions of the results, as well as the plan of future work, are given in Part C.

3. PROFESSIONAL CONTRIBUTORS

Professional personnel contributing to the progress during the reporting year are as follows:

J.Y.S. Luh, Principal Investigator

G.E. O'Connor, Staff Researcher

J.S. Shafran, Staff Researcher

PART B

OPTIMAL CONTROL IN BOUNDED PHASE-COORDINATE PROCESSES

1. INTRODUCTION

In recent years, much effort has been applied to optimal control problems with bounded phase-coordinates. Among the published literature, Gamkrelidze [1,2] treated the problem based on Pontryagin's maximum principle. Berkovitz [3], however, showed that Gamkrelidze's results can be achieved by solving the relevant calculus of variations problem. Dreyfus [4] studied the same problem by means of the dynamic programming formulation. His results are in agreement with that of Berkovitz [5]. Among all the studies, sufficiency conditions were virtually ignored. For practical applications, even when solutions do exist, the necessary conditions derived by the various authors are difficult to apply.

For a more restricted class of problems, Chang derived a simpler necessary condition [6], and the existence theorems based on an extension of Ascoli's Theorem [7]. For linear time-optimal control systems with a convex restraint set, the necessary condition is also sufficient. An elegant proof of the necessity of the condition can be deduced from Neustadt's recent work [8] while a rigorous proof of the sufficiency is given by Russell [9]. This condition is an improvement on Gamkrelidze's result. It establishes the fact that the normal vector appearing in the modified adjoint differential equation is always outward with respect to the set of attainability, and hence the necessary and sufficient condition is relatively easy to apply.

As to the computational aspects of the problem, there are essentially two classes of methods. One class includes the method of the gradient, steepest-descent or its equivalent, which was studied by Dreyfus [4], Denham [10, 11] and Bryson [12] using the necessary conditions of the optimal control, and by Paiewonsky, et al. [13] using conditions both of the optimal control and from the calculus of variations. The other class was discussed by Kahne [14], Ho and Brentani [15], and Nagata, et al. [16]. Because of the nature of the problem, each computational procedure requires either an iterative solution or a simulation on a sizable computer. Since a new computation is required for each different initial state, the possibility of on-line operation using currently available facilities is out of the question.

An ideal approach is to synthesize a so-called closed-loop optimal controller so that the control input is a function of the current state. This problem, however, is too difficult to solve. An alternative approach is to obtain the so-called open-loop optimal control as an explicit time function for each initial state. This problem, although not so difficult as the closed-loop optimal control problem, is complicated enough that no published results are known. This report presents a new method of solving the open-loop control problem with a bounded phase-coordinate.

In the following sections, a discussion of the problem is presented. Section 2 defines the problem and outlines the background results. Section 3 discusses the method of solving the problem through a reformulation. The analysis is based on the necessary and sufficient condition of the optimal control. The method is then applied to the time-optimal control of an unstable booster. The complete solution is given in Section 4. The extremal control problem for an oscillatory plant is presented in Section 5. The study of this problem is exhaustive since it includes almost all possible ratios of control amplitude to its rate. Section 6 gives the conclusions of the results while Section 7 outlines the plan of future work.

2. PROBLEM STATEMENT AND BACKGROUND RESULTS

Consider a linear autonomous control process as described by the differential system

$$\dot{x} = Ax + Bu(t) \tag{1}$$

in R^n on the interval $[o,t_1]$. A and B are n by n and n by m constant matrices, respectively. Let G be a closed convex subset of R^n and Ω be a non-empty restraint set in R^m given by $|u_i| \leq c_i$, $i=1,2,\ldots,m$. It is further assumed that the system (1) is normal, i.e. the vectors $Bw, ABw, \ldots, A^{n-1}Bw$ are linearly independent where w is a vector having the direction of an edge of the polyhedron Ω. The problem is to choose an admissible control $u(t) \subset \Omega$ on $[0,t_1]$ which steers the system (1) from a given initial state $x(0) = x_o$ to $x(t_1) = 0$, such that the response $x(t) \subset G$ for all $t \epsilon [0,t_1]$ and t_1 is minimal.

Gamkrelidze [1,2] and others have given necessary conditions that the extremal controls must satisfy. These necessary conditions imply that an extremal control corresponds to a solution of a set of adjoint equations. The adjoint solution is allowed certain jump discontinuities and hence depends on a number of parameters representing:

(a) The magnitudes of the possible jumps that appear in the adjoint solution, and

(b) The time lengths of the arcs of the corresponding trajectory which lie on ∂G, the boundary of the phase coordinate restraint set G.

The discontinuities are allowed at points where the trajectory (corresponding to an extremal control) enters upon or exits from an arc on ∂G.

These are the general results. They do not, however, indicate specifically at which points the trajectory enters upon the arc, and when the trajectory must exit from it. This paper attempts to investigate these questions. In the following section, a reformulation of the problem is introduced which will lead to a method that determines extremal controls as explicit time functions. Then these functions can be represented in terms of adjoint solutions. A sufficiency condition given by Russell [9] shows that the solutions so obtained are optimal controls.

3. REFORMULATION OF THE PROBLEM

For a linear autonomous process, the calculation of trajectories by the "backing out of the target" procedure is valid. To reverse the time sense, define $\tau = t_1 - t$ and $\tau \in [0, t_1]$. Then the system (1) becomes

$$dx/d\tau = -Ax - Bu(\tau), \tag{2}$$

with initial condition $x(\tau) = 0$ at $\tau = 0$. Let

$$K(\tau) = \{x(\tau) \,|\, x(\tau) = -\int_0^\tau e^{-A(\tau-s)} B\, u(s)\, ds,\ x(\tau) \subset G \text{ for all } \tau \in [0, t_1],$$
$$u(s) \subset \Omega \text{ for all } s \in [0, \tau]\}$$

be a set of attainability at τ, then $K(\tau)$ is the set of all points that can be attained in time τ from $x(0) = 0$ using admissible controls.

If τ is small enough then $K(\tau)$ is within the interior of G, and it is known that $K(\tau)$ is compact, convex, and continuous in τ. Moreover, the transversality condition applies at $\partial K(\tau)$, the boundary of $K(\tau)$; and for each point on $\partial K(\tau)$, there is a corresponding unique and admissible extremal control [17].

When τ is large, some segments of $\partial K(\tau)$ may coincide with ∂G. Since G is convex by hypothesis, then $K(\tau)$ is again convex; and Russell

[9, pp. 22-53] showed that:

(a) at $\partial K(\tau)$, the transversality condition is still valid if the corresponding adjoint system is modified, and

(b) corresponding to each point on $\partial K(\tau)$, there is a unique admissible extremal control.

Thus, by (a), for every unit vector η in R^n there is a state vector x corresponding to a point on $\partial K(\tau)$ for a fixed τ such that the projection P of x onto η:

$$P = (\eta, x) = -\int_0^\tau \eta' e^{A(\tau-s)} B u(s) ds$$

is a maximum, where $()' =$ transpose of (), and $u(s) \subset \Omega$ for all $s \epsilon [0,\tau]$. By (a) and (b), the corresponding unique admissible extremal control $u^*(s)$, which maximizes P, steers the linear, autonomous, normal system (2) from the origin to the furthest point x in the direction η in a fixed time τ.

This is equivalent to the case that, with the time sense reversed once more, the same extremal control will steer the system from x to the origin in a fixed time τ where τ is minimal. Russell's sufficiency condition [9] shows that the unit vector η is the adjoint vector at time τ, and the extremal control so obtained is the time-optimal control.

Thus, the problem of determining a time-optimal controller is now reduced to obtaining an admissible extremal control that maximizes the projection P of a state vector x at a fixed time τ (in the sense of "backing out of the target") onto a unit vector η. In so doing, it is possible to find an extremal control for every fixed finite time τ and for every unit vector η, and hence to express the extremal controls as explicit time functions. Once this is completed, the state vector $x = x(\tau)$ can be computed from the variation of parameters formula with the corresponding extremal control.

The domain of controllability of the system can be determined by considering the limit of $x(\tau)$ as τ approaches infinity. If all the components of $x(\tau)$ approach $\pm\infty$ as τ approaches ∞ then the domain of controllability is the entire state space. If some components of $x(\tau)$ approach finite limiting values, the domain of controllability is a proper subset of the state space, and the boundary of this domain can be determined from the limits of $x(\tau)$.

4. THE UNSTABLE BOOSTER CONTROL PROBLEM

Friedland [18] and Toohey [19] have studied the optimal autopilot design problem of an unstable booster with actuator position and rate limits. Their simplified plant transfer function consists of three poles in the frequency domain: one at the origin and two on the real axis with equal magnitude but opposite signs. They simplified the problem further by cancelling the pole at the origin through physical design. Essentially the simplified and normalized unstable booster is described by a second order differential equation

$$\ddot{x}_1 - x_1 = u(t)$$

or, in matrix notation

$$\dot{\hat{x}} = \hat{A}\,\hat{x} + \hat{b}\,u(t) \tag{3}$$

in R^2 with

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The problem is: (for a fixed value of $u(0)$ which satisfies $|u(0)| \leq 1$)

(a) To determine the domain of controllability (in R^2) in which every point can be steered to the origin by a scalar control $u(t)$ subject to the constraints $|u(t)| \leq 1$ and $|\dot{u}(t)| \leq D$ on $[0,\infty)$, and

(b) To find a time-optimal control function for each initial state in the domain of controllability.

This problem will be formulated as a bounded phase-coordinate problem and solved by the method described above.

4.1 Bounded Phase-Coordinate Formulation of the Booster Problem

The system is augmented by defining $x_3(t) = u(t)$ and $v(t) = \dot{u}(t)$. Then the system (3) can be rewritten as

$$\dot{x} = Ax + b\, v(t) \tag{4}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ u(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is a bounded phase-coordinate problem (in the sense $|x_3| = |u| \leq 1$) in which the scalar variable $v(t)$ is required, subject to the constraint $|v(t)| \leq D$ on $[0, t_1]$, to steer system (4) from an initial state $x(0) = x_o$ to $x(t_1) = 0$ with minimal t_1.

To proceed by the method of "backing out of the target $x = 0$" we write the system (4) with time sense reversed (by defining $\tau = -t$),

$$dx/d\tau = -A\, x(\tau) - b\, v(\tau) \tag{5}$$

with $x(0) = 0$. By the variation of parameters formula, the system (5) has a solution

$$x(\tau) = \begin{bmatrix} \int_o^\tau [1 - \cosh(\tau - s)]\, v(s)ds \\ \int_o^\tau \sinh(\tau - s)\, v(s)ds \\ -\int_o^\tau v(s)ds \end{bmatrix} \tag{6}$$

where $|v(s)| \leq D$ is admissible on $[0, \tau]$. The adjoint system for the system (5) is

$$d\psi/d\tau = -(-A)'\,\psi(\tau) = A'\psi(\tau)$$

Gamkrelidze [1,2] showed that, in order to represent the extremal v as a multiple of the signum of an adjoint solution for the bounded phase-coordinate control problem, the adjoint system must be modified. Thus a "total adjoint vector" $p(\tau)$ must satisfy the relation

$$dp/d\tau = \begin{cases} A'\,p(\tau), & \text{if } |x_3(\tau)| < 1 \\ \tilde{A}'\,p(\tau), & \text{if } |x_3(\tau)| = 1 \end{cases} \tag{7}$$

in which $\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In so doing, the necessary conditions for v to be extremal can be expressed as

$$v(\tau) = D \operatorname{sgn} [p(\tau)'\ (-b)]$$

or

$$-v(\tau) = D \operatorname{sgn} [p_3(\tau)] \tag{8}$$

where:

(a) $p(\tau)$ satisfies the system (7),

(b) $p_3(\tau) = 0$ if $|x_3(\tau)| = 1$,

(c) $p(\tau)$ is allowed certain jump discontinuities at endpoints of intervals where $|x_3(\tau)| = 1$ (for this problem, p_1 and p_2 are required to be continuous and jumps can occur only in p_3 since only x_3 is restrained), and

(d)

$$\operatorname{sgn} p_3 = \begin{cases} +1, & \text{if } p_3 > 0 \\ 0, & \text{if } p_3 = 0 \\ -1, & \text{if } p_3 < 0 \end{cases} \tag{9}$$

Thus, the solution of the system (7) can be written as

$$p_1(\tau) = p_1(0) \cosh \tau + p_2(0) \sinh \tau,$$

$$p_2(\tau) = p_1(0) \sinh \tau + p_2(0) \cosh t, \tag{10}$$

$$p_3(\tau) = \begin{cases} p_1(0) \cosh \tau + p_2(0) \sinh \tau + k, & \text{if } |x_3(\tau)| < 1, \\ 0 & \text{, if } |x_3(\tau)| = 1, \end{cases}$$

where the value of the constant k in $p_3(\tau)$ depends upon the interval in which $|x_3(\tau)| < 1$, and upon $p_1(0)$ and $p_2(0)$.

4.2 The Extremal Controls

To determine extremal controls as explicit time functions, form the projection P as defined previously. Let the unit adjoint vector at time τ be

$$\eta = \begin{cases} \cos \theta \cos \phi \\ \sin \theta \cos \phi \\ \sin \phi \end{cases} , \quad |\theta| \leq \pi, \ |\phi| < \pi/2.$$

Then, by equations (6) and the definition of P,

$$P = \int_0^\tau g(s; \tau, \theta, \phi) \, v(s) \, ds$$

in which

$$g(s; \tau, \theta, \phi) = \cos \phi \, [\cos \theta - \cos \theta \cosh (\tau - s) + \sin \theta \sinh (\tau - s)] - \sin \phi, \tag{11}$$

and $|v(s)| \leq D$ is admissible on $[0, \tau]$. By the transversality condition at $x(\tau)$ on $\partial K(\tau)$, $v(s)$ is extremal on $[0, \tau]$ if it maximizes P. By equation (8), the only possible values for $v(s)$ are $\pm$ D and zero. When $|x_3| < 1$, the system (4) is normal and hence the value of v can only be either + D or -D. If v is zero on an interval then by equations (8), (9) and (10) the value of $|x_3|$ is one. This conclusion is in agreement with Chang's statement [20] that if the system is time-optimally controlled, then either u is extremal or $du/d\tau$ is extremal.

The function $g(s;\ \tau,\ \theta,\ \emptyset)$ given by equation (11) has the property that

$$g(s;\ \tau,\ \theta,\ \emptyset) = -g(s;\ \tau,\ \pi + \theta,\ -\ \emptyset);$$

hence it suffices to consider only half of the range of θ. For convenience, choose $-\pi \leq \theta \leq 0$. Then P may be written as

$$P = \cos \emptyset \int_0^\tau f(s;\ \tau,\ \theta,\ \emptyset)\ [-v(s)]\ ds$$

where

$$f(s;\ \tau,\ \theta,\ \emptyset) = \cos\theta \cosh(\tau-s) - \cos\theta - \sin\theta \sinh(\tau-s) + \tan\emptyset \tag{12}$$

To determine the form of the extremal $v(s)$ that maximizes P, the method given by Schmaedeke and Russell [21] can be used. For this particular problem, however, $v(s)$ can be obtained by inspection from geometrical reasoning. On the interval $0 < s < \tau$ the function $f(s;\ \tau,\ \theta,\ \emptyset)$ is either monotone or has one maximum and no minima. In fact, for $0 < s < \tau$ and $|\varphi| < \pi/2$ there are two cases of interest. These are: (a) $-3\pi/4 \leq \theta \leq 0$ and (b) $-\pi \leq \theta < -3\pi/4$.

In the case (a) the derivative $df/ds < 0$ so that f is monotone decreasing in s.

In the case (b) f has a maximum at $s_m = \tau - \tanh^{-1}(\tan\theta)$. However for $\tan^{-1}(\tanh\tau) \leq \theta < -3\pi/4$ where $\tan^{-1}(\tanh\tau) > -\pi$ the value of s_m is negative.

Thus, for $|\emptyset| < \pi/2$ and $0 \leq s \leq \tau < \infty$, f is monotone decreasing in s if $-\pi < \tan^{-1}(\tanh\tau) \leq \theta < 0$; or f has a maximum at $s_m = \tau - \tanh^{-1}(\tan\theta)$ if $-\pi \leq \theta < \tan^{-1}(\tanh\tau) < -3\pi/4$. Furthermore, for any real k,

$$f(s_m + k;\ \tau,\ \theta,\ \emptyset) = f(s_m - k;\ \tau,\ \theta,\ \emptyset) \text{ if } -\pi \leq \theta < -3\pi/4,$$

hence f is symmetric with respect to s_m. Therefore, for a fixed τ, a fixed θ and a fixed $\emptyset$, $f(s;\ \tau,\ \theta,\ \emptyset)$ can be sketched on the interval

$0 \leq s \leq \tau$. Two typical cases are shown below, one corresponds to f being monotone decreasing in s and the other to f having a maximum at some $s_m > 0$.

In the case shown in Fig. 1, the ranges are $-3\pi/4 < \theta \leq 0$ and $1/D < \tau \leq 3/D$; hence f is monotone decreasing in s. The form of extremal v(s) is

$$-v(s) = \begin{Bmatrix} D \text{ for } 0 \leq s < 1/D \\ 0 \text{ for } 1/D \leq s \leq \tau \end{Bmatrix} \quad \text{if } \pi/2 > \phi \geq \phi_1; \tag{13}$$

or

$$-v(s) = \begin{Bmatrix} D \text{ for } 0 \leq s < 1/D \\ 0 \text{ for } 1/D \leq s < \tau - \ln(\alpha+\beta) \\ -D \text{ for } \tau - \ln(\alpha+\beta) \leq s \leq \tau \end{Bmatrix} \quad \text{if } \phi_1 > \phi > \phi_2; \tag{14}$$

or

$$-v(s) = \begin{Bmatrix} D \text{ for } 0 \leq s < \tau - \ln(\alpha+\beta) \\ -D \text{ for } \tau - \ln(\alpha+\beta) \leq s \leq \tau \end{Bmatrix} \quad \text{if } \phi_2 \geq \phi > \phi_3; \tag{15}$$

or

$$-v(s) = \begin{Bmatrix} D \text{ for } 0 \leq s < (\tau - 1/D)/2 \\ -D \text{ for } (\tau - 1/D)/2 \leq s \leq \tau \end{Bmatrix} \quad \text{if } \phi_3 \geq \phi > -\pi/2; \tag{16}$$

where

$$\phi_1 = 0$$

$$\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-1/D) - 1] - \sin\theta \sinh(\tau-1/D)\},$$

$$\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\tau/2-1/2D) - 1] - \sin\theta \sinh(\tau/2-1/2D)\},$$

$$\alpha = (\cos\theta - \tan\phi)/(\cos\theta - \sin\theta), \text{ and}$$

$$\beta = \sqrt{\alpha^2 - (\cos\theta + \sin\theta)/(\cos\theta - \sin\theta)}.$$

By an inspection of the sketches in Fig. 1 with the basic requirement in mind that either $|v(s)| = D$ or $|u(s)| = 1$ on the entire interval $0 \leq s \leq \tau$,

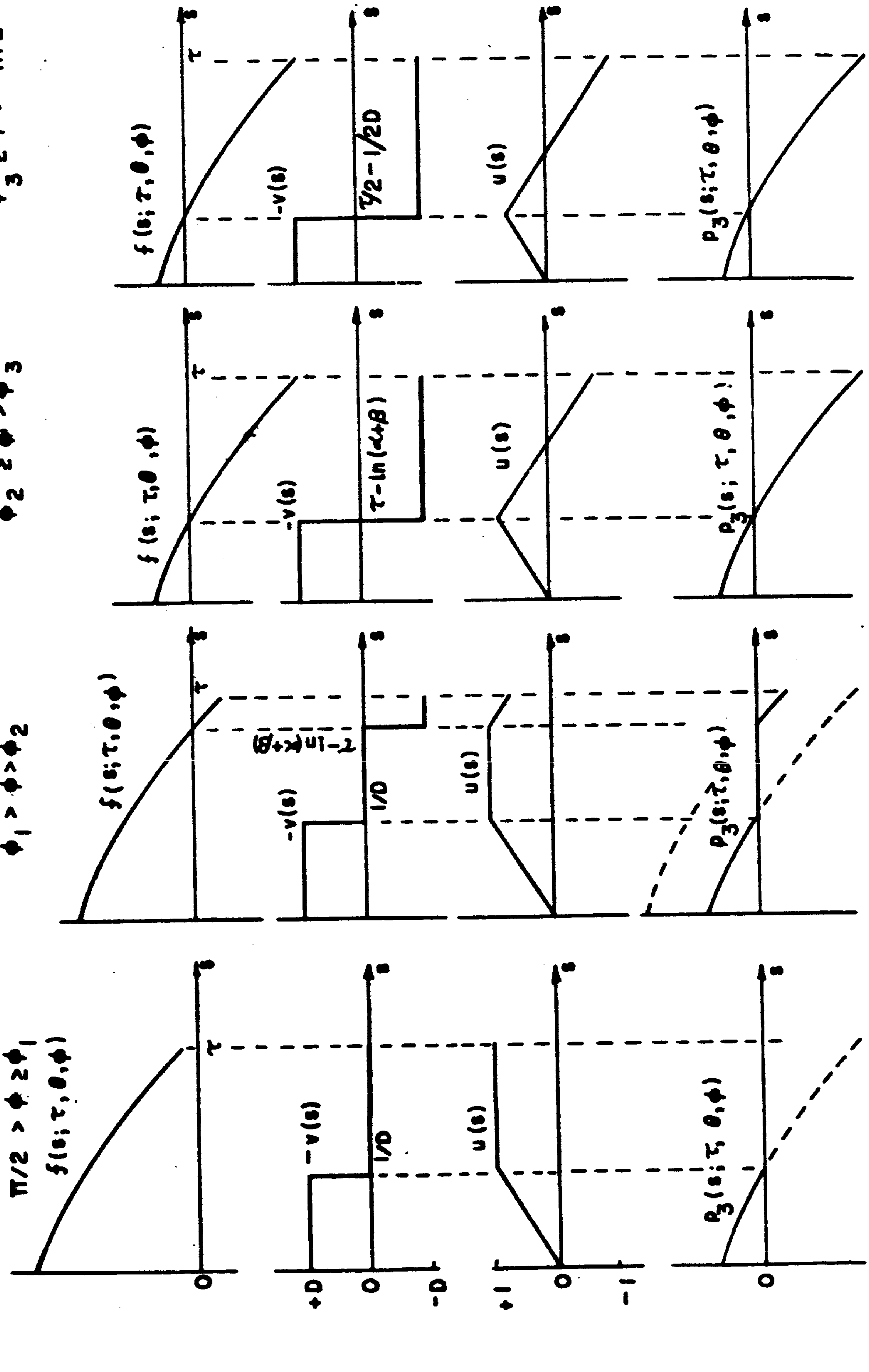

π/2 > φ ≥ φ1
φ1 > φ > φ2
φ2 ≥ φ > φ3
φ3 ≥ φ > −π/2
f(s; τ, θ, φ)
−v(s)
1/D
u(s)
p3(s; τ, θ, φ)
τ − ln(α+β)
τ/2 − 1/2D
τ
s
0
+D
−D
+1
−1

FIGURE 1

it is easy to show that any deviation from the v(s) given above would decrease the value of P.

For the case shown in Fig. 2, the ranges are

$$-\pi < \tan^{-1}(\tanh 5/2D) < \theta \leq \tan^{-1}(\tanh 3/D) < -3\pi/4$$

and

$$\tanh^{-1}(\tan\theta) + 1/2D < \tau < 4\tanh^{-1}(\tan\theta) - 7/D;$$

hence f has a maximum at $s_m = \tau - \tanh^{-1}(\tan\theta)$. The form of the extremal v(s) is

$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s \leq \tau \end{cases} \text{ if } \pi/2 > \phi \geq \phi_1;$$

or

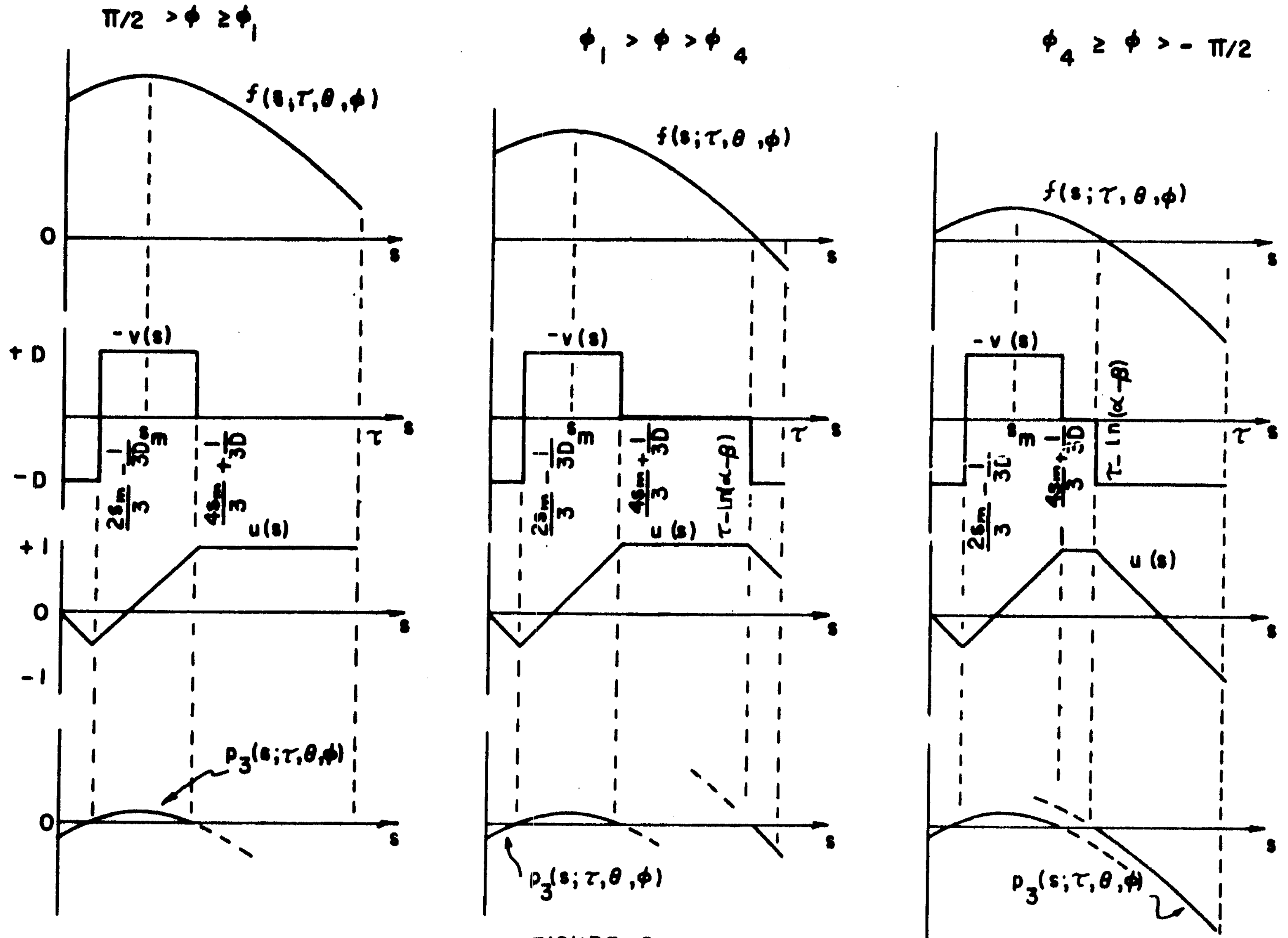
$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s < \tau - \ln(\alpha + \beta) \\ -D \text{ for } \tau - \ln(\alpha + \beta) \leq s \leq \tau \end{cases} \text{ if } \phi_1 > \phi > \phi_4;$$

or

$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s < \tau - (2/D) \\ -D \text{ for } \tau - (2/D) \leq s \leq \tau \end{cases} \text{ if } \phi_4 \geq \phi > -\pi/2;$$

where $\phi_4 = -\tan^{-1}\{\cos\theta\,[\cosh 2/D - 1] - \sin\theta \sinh 2/D\}$, and all other parameters were defined previously. By an inspection of Fig. 2 with the same argument given in the previous case, the extremal v(s) must have the present form.

This procedure was carried out for all the possible cases. It was found that the extremal v(s) reaches zero and leaves zero as many as four



π/2 > φ ≥ φ₁
φ₁ > φ > φ₄
φ₄ ≥ φ > - π/2
f(s;τ,θ,φ)
-v(s)
u(s)
p₃(s;τ,θ,φ)
0
+D
-D
+1
-1
s
τ
s_m
2s_m/3 - 1/3D
4s_m/3 + 1/3D
τ - ln(α-β)

FIGURE 2

times. Denote the time s at which such events occur by τ_i, $i=1,\ldots,4$, and let $\tau_0 = 0$ and $\tau_5 = \tau$. Supposing the values of $x_3(s) = u(s)$ are such that

$$|u(s)| < 1, \text{ if } \tau_{2i} \leq s < \tau_{2i+1}, \; i = 0, 1, 2;$$

$$|u(s)| = 1, \text{ if } \tau_{2j+1} \leq s < \tau_{2j+2}, \; j = 0, 1.$$

Then

$$dp_3/ds = d\psi_3/ds \text{ for } \tau_{2i} \leq s < \tau_{2i+1}, \; i = 0, 1, 2;$$

$$p_3(s) = 0 \quad \text{for } \tau_{2j+1} \leq s < \tau_{2j+2}, \; j = 0, 1.$$

It follows that choosing $p_3(s)$ to be continuous at τ_{2i+1}, $(i = 0,1,2)$ requires $p_3(\tau_{2i+1}) = 0$, and hence

$$p_3(s) = \psi_3(s) - \psi_3(\tau_{2i+1}) \text{ for } \tau_{2i} \leq s < \tau_{2i+1}, \; i = 0, 1, 2.$$

With $p_3(s)$ so defined, the jump conditions have to be satisfied at τ_{2i}, $i = 0, 1, 2$. Since η is the unit adjoint vector at $\partial K(\tau)$, $p_3(\tau) = \psi_3(\tau) = \eta_3 = \sin \phi$.

Thus

$$\frac{p_3(s;\tau,\theta,\phi)}{\cos \phi} = \begin{cases} \cos\theta\,[\cosh(\tau-s) - \cosh(\tau-\tau_1)] - \sinh\theta\,[\sinh(\tau-s) - \sinh(\tau-\tau_1)], \\ \qquad \text{if } 0 \leq s < \tau_1, \\ 0, \qquad \text{if } \tau_1 \leq s < \tau_2, \\ \cos\theta\,[\cosh(\tau-s) - \cosh(\tau-\tau_3)] - \sin\theta\,[\sinh(\tau-s) - \sinh(\tau-\tau_3)], \\ \qquad \text{if } \tau_2 \leq s < \tau_3, \\ 0, \qquad \text{if } \tau_3 \leq s < \tau_4 \\ \cos\theta\,[\cosh(\tau-s) - 1] - \sin\theta \sinh(\tau-s) + \tan\phi + \delta\,(\tan\phi_\circ - \tan\phi), \\ \qquad \text{if } \tau_4 \leq s < \tau; \end{cases}$$

where

$$\delta = \begin{cases} 0, \text{ if } |x_3(\tau)| = |u(\tau)| < 1 \\ 1, \text{ if } |x_3(\tau)| = |u(\tau)| = 1 \end{cases}$$

Using this expression for $p_3(s;\tau,\theta,\phi)$, it has at most one jump discontinuity at $s = \tau$ (equivalently at $\partial K(\tau)$), and this happens only when $|x_3(t)| = |u(\tau)| = 1$. Furthermore, the explicit form of the extremal $v(s)$ can be expressed as

$$\begin{aligned} -v(s) &= D \operatorname{sgn} [p_3(s;\ \tau,\ \theta,\ \phi)] \\ &= D \operatorname{sgn} [p_3(s;\ \tau,\ \theta,\ \phi/\cos \phi] \end{aligned}$$

since $\cos \phi$ is positive on $-\pi/2 < \phi < \pi/2$. Finally, by Russell's sufficiency condition [9], the extremal $v(s)$ is also the time-optimal $v(s)$.

The function $p_3(s;\ \tau,\ \theta,\ \phi)$ for the two typical cases discussed previously are also sketched in Figs. 1 and 2. The formulas for parameters τ_i, $i=1,\ldots,4$, δ and $\phi_\circ$ are obtained for all possible cases in the ranges $-\pi \leq \theta \leq 0$, $-\pi/2 < \phi < \pi/2$ and $0 \leq \tau < \infty$. The results are listed in Tables I to VI.

4.3 Time-optimal Controls for the Booster

The state vector $x(\tau)$ can be readily computed from equations (6). Take a typical case as an example:

$$-3\pi/4 < \theta \leq 0,\ 1/D < \tau \leq 3/D,\ \phi_2 < \phi < \phi_1 \text{ (see Fig. 1).}$$

For this case, the extremal $v(s)$ is given in equation (14), hence by integration over $[0,\tau]$,

$$\begin{cases} x_1(\tau) = D \sinh (1/D-\tau) + D \sinh \tau\text{-}1 + D\ln (\alpha + \beta) - [\alpha + \beta - 1/(\alpha+\beta)]D/2 \\ x_2(\tau) = D \cosh (\tau-1/D) - D \cosh \tau\text{-}D + [\alpha + \beta - 1/(\alpha+\beta)]\ D/2 \\ x_3(\tau) = 1 - D \ln (\alpha + \beta). \end{cases}$$

Let $\alpha + \beta = e^{1/D}$ so that $x_3(\tau) = u(\tau) = 0$, then

$$\begin{cases} x_1(\tau) = D [\sinh (1/D - \tau) + \sinh \tau - \sinh 1/D] \\ x_2(\tau) = D [\cosh (\tau - 1/D) - \cosh \tau - 1 + \sinh 1/D] \end{cases}$$

for $1/D < \tau \leq 1/D$. A further choice of $\tau = 2.5/D$ reduces the above to

TABLE I

<table>
<tr><td>(I) $0 \geqslant \theta \geqslant -\frac{3\pi}{4}$
ϕ / τ_i's δ & ϕ_0 / τ</td><td>$\frac{\pi}{2} > \phi \geqslant \phi_1$
$\phi_1 = 0$</td><td>$\phi_1 > \phi > \phi_2$
$\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$</td><td colspan="2">$\phi_2 \geqq \phi > \phi_3$
$\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$</td><td>$\phi_3 \geqslant \phi > -\frac{\pi}{2}$</td></tr>
<tr><td>$0 \leq \tau \leq \frac{1}{D}$
f is monotone</td><td colspan="5">$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0 \quad , \quad \delta = 0$</td></tr>
<tr><td>$\frac{1}{D} < \tau \leq \frac{3}{D}$
f is monotone</td><td>$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \tau$
$\delta = 1 \; , \; \phi_0 = \phi_1$</td><td>$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \begin{cases} \tau - \ln[\alpha+\beta], \text{ if } 0 \leq \theta < \frac{3}{4}\pi \\ \tau + \ln[1+\sqrt{2}\tan\phi], \text{ if } \theta = \frac{3}{4}\pi \end{cases}$
$\delta = 0$</td><td colspan="2">$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$
$\delta = 0$</td><td>$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$
$\delta = 1 \; , \; \phi_0 = \phi_3$</td></tr>
<tr><td>$\frac{3}{D} < \tau < \infty$
f is monotone</td><td>" "</td><td colspan="2">$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \begin{cases} \tau - \ln[\alpha+\beta], \text{ if } 0 \leq \theta < \frac{3}{4}\pi \\ \tau + \ln[1+\sqrt{2}\tan\phi], \text{ if } \theta = \frac{3}{4}\pi \end{cases}$
$\delta = 0$</td><td colspan="2">$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \tau - \frac{2}{D}$
$\delta = 1 \; , \; \phi_0 = \phi_4$</td></tr>
<tr><td>τ / τ_i's δ & ϕ_0 / ϕ
$0 \geqslant \theta \geqslant -\frac{3\pi}{4}$</td><td>$\frac{\pi}{2} > \phi \geqslant \phi_1$
$\phi_1 = 0$</td><td colspan="2">$\phi_1 > \phi > \phi_4$
$\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{2}{D}-1] - \sin\theta\sinh\frac{2}{D}\}$</td><td colspan="2">$\phi_4 \geqslant \phi > -\frac{\pi}{2}$</td></tr>
</table>

TABLE II

(ii) $-\frac{3}{4}\pi > \theta > \tan^{-1}(\tanh\frac{3}{D}) > -\pi$; τ ; τ_i's δ & ϕ_0 ; ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$	$\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0 \;, \quad \delta = 0$			
$\frac{1}{D} < \tau \leq \frac{3}{D}$ f is monotone	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1 \;, \quad \phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1 \;, \quad \phi_0 = \phi_3$

$\frac{3}{D} < \tau \leq \tanh^{-1}(\tan\theta)$	" "	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \frac{2}{D}$ $\delta = 1 \;, \quad \phi_0 = \phi_4$
$\tanh^{-1}(\tan\theta) < \tau \leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1 \;, \quad \phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \frac{2}{D}$ $\delta = 1 \;, \quad \phi_0 = \phi_4$
$\tanh^{-1}(\tan\theta) + \frac{2}{D} < \tau < \infty$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - 1/D$ $\tau_3 = \tau_2 + 2/D$ $\tau_4 = \tau$ $\delta = 1 \;, \quad \phi_0 = \phi_1$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - 2/D$ $\delta = 1 \;, \quad \phi_0 = \phi_4$
τ ; τ_i's δ & ϕ_0 ; ϕ ; $-\frac{3}{4}\pi > \theta > \tan^{-1}(\tanh\frac{3}{D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_4$ $\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{2}{D}-1] - \sin\theta\sinh\frac{2}{D}\}$	$\phi_4 \geq \phi > -\frac{\pi}{2}$

TABLE III

(III) $\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{3}{D}) \geq \theta > \tan^{-1}(\tanh\frac{3}{D}) > -\pi$; τ ; τ_i's ; δ & ϕ_0 ; ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_4$ $\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{\xi}{3}-1]+\sin\theta\sinh\frac{\xi}{3}\}$ $\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1]+\sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_4 \geq \phi > -\frac{\pi}{2}$	$\phi_2 \geq \phi > \phi_5$, $\zeta = \tanh^{-1}(\ldots)$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh\zeta-1]+\sin\theta\sinh\zeta\}$ $\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\xi}{3}+\frac{1}{3D})-1]+\sin\theta\sinh(\frac{\xi}{3}+\frac{1}{3D})\}$	$\phi_5 \geq \phi > -\frac{\pi}{2}$ $\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$				
$\frac{1}{D} < \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D}$ f is monotone on $\tau \in (\frac{1}{D}, \frac{5}{3D}]$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$
$\frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D} < \tau \leq \frac{3}{D}$	,,	,,		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\frac{3}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	,,	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$		
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau < 4\tanh^{-1}(\tan\theta) - \frac{7}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(-\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$		
$4\tanh^{-1}(\tan\theta) - \frac{7}{D} \leq \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	,,	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	,,	,,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D} < \tau < \infty$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + 2/D$ $\tau_4 = \tau$, $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + 2/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$, $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$		,,
$-\frac{1}{4}\pi > \tan^{-1}(\tanh\frac{3}{D}) \geq \theta > \tan^{-1}(\tanh\frac{5}{D}) > -\pi$; τ ; τ_i's ; δ & ϕ_0 ; ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_7$, $\rho = \tanh^{-1}(\tan\theta) + \frac{1}{D}$ $\phi_7 = -\tan^{-1}\{\cos\theta[\cosh\rho - 1] - \sin\theta\sinh\rho\}$ $\phi_1 > \phi > \phi_6$ $\phi_6 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\xi}{3}+\frac{1}{3D})) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\xi}{3}+\frac{1}{3D}))\}$	$\phi_7 \geq \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta)) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_6 \geq \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau - \frac{1}{D}) - 1] - \sin\theta\sinh(\tau - \frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$

TABLE IV

(IV) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{5}{2D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{D}) > -\pi$; τ / τ_i's δ & ϕ_0 / ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta\,[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$		$\phi_2 \geq \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-(\frac{\pi}{3}-\frac{1}{2D}))-1] + \sin\theta\sinh[\tanh^{-1}(\tan\theta)-\frac{4}{3}(\tau-\frac{1}{D})]\}$ $\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$	$\phi_5 \geq \phi > -\frac{\pi}{2}$ $\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$				
$\frac{1}{D} < \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha-\beta]$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$
$\frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	, ,	, ,		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha-\beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		, ,
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	, ,	, ,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D} < \tau < \infty$	$\tau_1 = \frac{1}{D}$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau$, $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \frac{1}{D}$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - \ln[\alpha-\beta]$, $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$		, ,
τ / τ_i's δ & ϕ_0 / ϕ ; $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{5}{2D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_7$, $\rho = \tanh^{-1}(\tan\theta) + 1/D$ $\phi_7 = -\tan^{-1}\{\cos\theta[\cosh\rho - 1] - \sin\theta\sinh\rho\}$ $\phi_1 > \phi > \phi_6$ $\phi_6 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D})) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D}))\}$	$\phi_7 \geq \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta)) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_6 \geq \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D}) - 1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$

TABLE V

(V) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{2D}) > -\pi$; τ \ τ_i's, δ & ϕ_0 \ ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_6$ $\phi_6 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta)-(\frac{\tau}{3}+\frac{1}{3D}))-1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta)-(\frac{\tau}{3}+\frac{1}{3D}))\}$ $\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_6 \geq \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-\frac{4}{3}(\tau-\frac{1}{D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-\frac{4}{3}(\tau-\frac{1}{D}))\}$ $\phi_2 \geq \phi > \phi_5$	$\phi_5 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone on $\tau \in [0, \frac{1}{2D}]$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$			
$\frac{1}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau$ $< 4\tanh^{-1}(\tan\theta) - \frac{1}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	„
$4\tanh^{-1}(\tan\theta) - \frac{1}{D} \leq \tau$ $\leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		„
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau$ $\leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	„	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D}$ $< \tau < \infty$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$		„
τ \ τ_i's, δ & ϕ_0 \ ϕ ; $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{2D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_4$ $\phi_4 = \tan^{-1}\{\cos\theta[1-\cosh\frac{2}{D}] + \sin\theta\sinh\frac{2}{D}\} \geq 0$ $\frac{\pi}{2} > \phi \geq \phi_3$ $\phi_3 = \tan^{-1}\{\cos\theta[1-\cosh(\frac{\tau}{3}+\frac{1}{3D})] + \sin\theta\sinh(\frac{\tau}{3}+\frac{1}{3D})\} \geq 0$	$\phi_4 > \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_3 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$ $\phi_3 > \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-\frac{4}{3}(\tau-\frac{1}{D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-\frac{4}{3}(\tau-\frac{1}{D}))\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$ $\phi_5 \geq \phi > -\frac{\pi}{2}$

TABLE VI

(VI) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{2D}) \geq \theta > -\pi$; ϕ ; τ_i's δ & ϕ_o ; τ	$\frac{\pi}{2} > \phi \geq \phi_3$ $\phi_3 = \tan^{-1}\{\cos\theta[1-\cosh(\frac{\tau}{2}+\frac{1}{2D})] + \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$ ≥ 0	$\phi_3 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$ $\phi_3 > \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))\}$	$\phi_2 \geq \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{2}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{2}{3}\tanh^{-1}(\tan\theta))\}$	$\phi_8 \geq \phi > -\frac{\pi}{2}$ $\phi_5 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0,$		$\delta = 0$	
$\frac{1}{D} < \tau \leq 2\tanh^{-1}(\tan\theta)+\frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1, \quad \phi_o = \phi_3$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1, \quad \phi_o = \phi_5$
$2\tanh^{-1}(\tan\theta)+\frac{1}{D} < \tau \leq \frac{3}{D}$	, ,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1, \quad \phi_o = \phi_8$
$\frac{3}{D} < \tau < \infty$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1, \quad \phi_o = \phi_4$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$		, ,
$-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{2D}) \geq \theta > -\pi$; τ ; τ_i's δ & ϕ_o ; ϕ	$\frac{\pi}{2} > \phi \geq \phi_4$ $\phi_4 = \tan^{-1}\{\cos\theta[1-\cosh\frac{2}{D}] + \sin\theta\sinh\frac{2}{D}\}$ ≥ 0	$\phi_4 > \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{2}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{2}{3}\tanh^{-1}(\tan\theta))\}$		$\phi_8 \geq \phi > -\frac{\pi}{2}$

$$\begin{cases} x_1(2.5/D) = D[-\sinh(1.5/D) + \sinh(2.5/D) - \sinh(1/D)] \\ x_2(2.5/D) = D[\cosh(1.5/D) - \cosh(2.5/D) - 1 + \sinh(1/D)] \\ x_3(2.5/D) = 0 \end{cases}$$

Using the results so obtained to solve the original booster problem stated in equation (3), reverse the time sense once again. Thus the extremal $v(s)$ now starts from $s = \tau$ and backs up to $s = 0$. Since $\tau = -\tau$, it follows that equation (14) is now replaced by

$$-v(s) = \begin{cases} D \text{ for } t \geq s > t - 1/D \\ 0 \text{ for } t - 1/D \geq s > \ln(\alpha + \beta) \\ -D \text{ for } \ln(\alpha + \beta) \geq s \geq 0 \end{cases}$$

Since $dx_3/dt = v(t)$ in equation (4) replaces $dx_3/d\tau = -v(t)$ in equation (5), hence $x_3 = u$ (shown in Fig. 1) now reverses its sign. Thus, the above example (now $t = 2.5/D$ instead) can be interpreted as follows: The control

$$u(s) = \begin{cases} Ds, \text{ if } 2.5/D \geq s > 1.5/D \\ -1, \text{ if } 1.5/D \geq s > 1/D \\ -Ds, \text{ if } 1/D \geq s \geq 0 \end{cases}$$

will steer the original booster control system (3) from the initial state

$$\begin{cases} x_1(0) = D[-\sinh(1.5/D) + \sinh(2.5/D) - \sinh(1/D)] \\ x_2(0) = D[\cosh(1.5/D) - \cosh(2.5/D) - 1 + \sinh(1/D)] \end{cases}$$

with $u(0) = 0$ to the origin in the minimum time $t_1 = 2.5/D$ and $u(2.5/D) = 0$.

This example also illustrates the fact that the parameters θ and ϕ introduced in the adjoint vector η serve as an aid to derive the extremal $v(s)$ only; they disappear in the final solution of the time-optimal control problem.

4.4 Maximum Controllable Region

The maximum controllable region is determined by examining the values of $x(\tau)$ as $t \to \infty$. Among the total of twenty different cases for large τ in Tables I - VI, the boundary of the region for $u = 1$ can be determined from the cases of (a) $\pi/2 > \phi \geq \phi_1$, $3/D < \tau < \infty$ in Table I, and (b) $\pi/2 > \phi \geq \phi_4$, $3/D < \tau < \infty$ in Table VI as follows:

(a) By equations (6), this case yields

$$\begin{cases} x_1(\tau) = D \sinh(1/D - \tau) + D \sinh \tau - 1 \\ x_2(\tau) = D \cosh(\tau - 1/D) - D \cosh \tau \\ x_3(\tau) = 1 \end{cases}$$

Thus $\dfrac{x_1 + 1}{x_2} \to -1$ as $\tau \to \infty$ which gives the equation

$$x_1 + x_2 = -1 \text{ for } u = 1 \tag{17}$$

(b) This case yields

$$\begin{cases} x_1(\tau) = -D \sinh(1/D - \tau) - D \sinh \tau + 1 + D \sinh(2/D) - 2 \\ x_2(\tau) = -D \cosh(\tau - 1/D) + D \cosh \tau + D - D \cosh(2/D) \\ x_3(\tau) = -1 + 2 = 1 \end{cases}$$

Thus $$\frac{x_1 + 1 - D \sinh(2/D)}{x_2 - D + D \cosh(2/D)} \to -1 \text{ as } \tau \to \infty.$$

or $$x_1 + x_2 = -1 + D\,[1 - \exp(-2/D)] \text{ for } u = 1 \tag{18}$$

The boundary of the region for $u = -1$ can be obtained from other cases, such as the case of $\phi_4 \geq \phi > -\pi/2$, $3/D < \tau < \infty$ in Table I. However, since

$$g(s;\, \tau,\, \theta,\, \phi) = g(s;\, \tau,\, \pi + \theta,\, \pi - \phi), \tag{19}$$

known relations will hold if all the signs of x_1, x_2 and $u(= x_3)$ are changed simultaneously. Therefore, corresponding to equations (17) and (18), the boundary for $u = -1$ is given by

$$-x_1 - x_2 = -1 \qquad \text{for } u = -1 \tag{20}$$

$$-x_1 - x_2 = -1 + D\,[1 - \exp(-2/D)] \text{ for } u = -1 \tag{21}$$

The boundary of the region for $-1 \leq u \leq 1$ can be found from the case of $\phi_1 > \phi > \phi_4$, $3/D < \tau < \infty$ in Table I, which yields

$$\begin{cases} x_1(\tau) = D \sinh(1/D-\tau) + D \sinh\tau + D[1/(\alpha+\beta)-(\alpha+\beta)]/2 + D\ln(\alpha+\beta) - x_3(\tau) \\ x_2(\tau) = D \cosh(\tau-1/D) - D \sinh\tau - D + D[1/(\alpha+\beta) - (\alpha+\beta)]/2 \\ x_3(\tau) = 1 - D\ln(\alpha+\beta) \end{cases}$$

Since $u = x_3$ and $\alpha + \beta = \exp[(1-u)/D]$, the limit as $\tau \to \infty$ yields

$$x_1 + x_2 = -u - D\{1 - \exp[-(1-u)/D]\} \text{ for } u = 1 - D\ln(\alpha+\beta) \quad (22)$$

which reduces to equation (17) if $u = 1$, and to (21) if $u = -1$. By the property of equation (19) and the same argument, the other boundary equation for $-1 \leq u \leq 1$ can be deduced from (22) as

$$-x_1 - x_2 = u - D\{1 - \exp[-(1+u)/D]\} \text{ for } -u = 1 - D\ln(\alpha+\beta). \quad (23)$$

Equation (23) reduces to (18) if $u = 1$, and to (20) if $u = -1$. Consequently, equations (22) and (23) determine the maximum controllable region (Fig. 3) for $-1 \leq u \leq 1$. Fig. 4 shows the regions for $1/D = 0.709$, which agree with those* given in Friedland's paper [18] when a scale factor of 0.709 for the x_1 and x_2 axes are considered.

C. THE OSCILLATORY SPACE VEHICLE PROBLEM

In the design of an autopilot for a large flexible space vehicle the problem of bending moments related to the wind disturbances is of relative importance. When the motion-controlling actuator has saturation limits on both position and rate, the design problem is quite involved. As a rule, the autopilot should be capable of maneuvering the actuator in a most efficient manner while encountering the worst wind disturbance.

* In a private communication with Dr. B. Friedland of General Precision, Inc., Little Falls, New Jersey, we agreed that in Fig. 6 of his paper [18], the scale of the q_1-axis should carry negative signs.

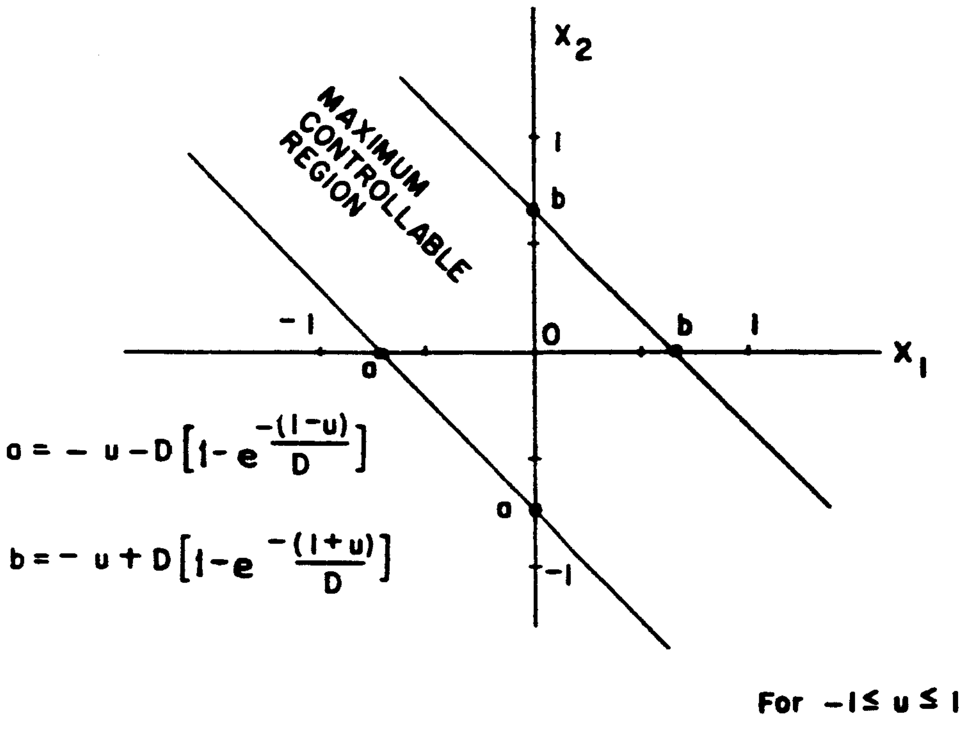


FIGURE 3

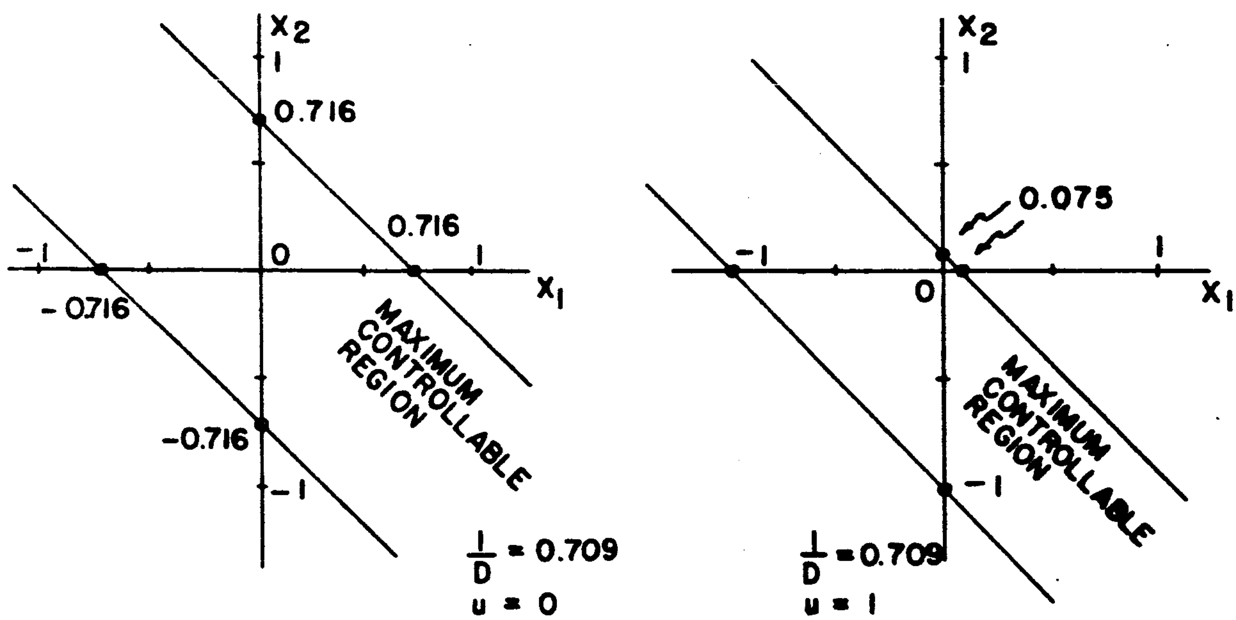


FIGURE 4

Therefore, an initial investigation is the determination of the worst disturbance that can be handled by the available actuator in a fixed time interval. The control inputs generated by the actuator to serve this purpose are called extremal inputs.

5.1 Problem Formulation

When the angle of attack is small, the longitudinal motion of the vehicle can be described by a system of linear differential equations. It is assumed that (a) a pole of the plant transfer function at the origin in the frequency domain is cancelled by means of some compensating device, and (b) the damping is neglibible, and the plant transfer function is essentially dominated by a pair of almost purely imaginary poles. Thus the approximate vehicle can be represented by a second order undamped oscillatory system. This approximation is allowed for many flexible vehicle systems. For the convenience of analysis, the control variable is treated as an augmented-system state variable, and the equation of motion is normalized as follows:

$$dx/dt = \dot{x}(t) = Ax(t) + bv(t) \text{ on } 0 \leq t \leq t_1 \tag{24}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ u(t) \end{bmatrix}, \; v = \dot{u}, \; A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

x_1 = normalized plant position, $u(t)$ = normalized thrust deflection with $|u| \leq 1$, $v(t)$ = normalized thrust deflection rate with $|v| \leq \pi/k$. The value of k is greater than or equal to 2, which allows the extremal control to enter upon and exit from its bound once every half cycle of the oscillation. The three-dimensional coordinate system is so chosen that the origin is an equilibrium state. It is required to determine an extremal $v(t)$ which steers the system (24) from the origin to a furthest point x in a given direction in a fixed time t. This is again a bounded phase-coordinate control problem and all known results on this type of problem apply.

By the same argument, which was based on Gamkrelidze's [1,2] result, the "total adjoint vector" $p(t)$ must satisfy the relation

$$\dot{p}(t) = \begin{cases} -A' \, p(t) , & \text{if } |x_3| < 1, \\ -\tilde{A}' \, p(t), & \text{if } |x_3| = 1, \end{cases}$$

where $()'$ = transpose of $()$ and

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Thus

$$v(t) = \frac{k}{\pi} \operatorname{sgn} [p(t)' \, b]$$

$$= \frac{k}{\pi} \operatorname{sgn} [p_3 (t)]$$

in which

$$\operatorname{sgn} p_3 = \begin{cases} +1, & \text{if } p_3 > 0, \\ 0, & \text{if } p_3 = 0, \\ -1 & \text{if } p_3 < 0. \end{cases}$$

Furthermore, p_3 is allowed certain jump discontinuities at endpoints of intervals where $|x_3| = 1$; and $p_3 = 0$ whenever $|x_3| = 1$.

5.2 The Extremal Controls

Let $K(t) = \{x(t) | x(t) = \int_0^t e^{A(t-s)} bv(s) \, ds, \ |x_3| \leq 1, \ |v| \leq \pi/k\}$

be a set of attainability at time t. The transversality condition implies that for every unit vector η in R^3 there is a state vector x corresponding to a point on the boundary of $K(t)$ for a fixed t such that the projection, P, of x onto η is a maximum. Moreover, the corresponding unique admissible extremal $v(s)$, which maximizes P, steers the system (24) from the origin to the furthest point x in the direction η in a fixed time t. Russell's sufficiency condition [9] shows that η is the adjoint vector at time t, and v is the minimal-time control.

Let

$$\eta = \begin{bmatrix} \cos\theta & \cos\phi \\ \sin\theta & \cos\phi \\ \sin\phi & \end{bmatrix}, \quad |\theta| \leq \pi, \ |\phi| < \pi/2,$$

then

$$P = \cos\phi \int_0^t [\cos\theta - \cos(t+\theta-s) + \tan\phi]\, v(s)\, ds$$

The extremal v which maximizes P can be determined by inspection for every fixed θ, ϕ and t. Since $P(s;\, t,\, \theta,\, \phi) = -P(s;\, t,\, \pi+\theta,\, -\phi)$, it suffices to consider only half of the range of θ. Figure 5 shows a typical case of $k = 2.5$, $3\pi/5 \leq \theta \leq 4\pi/5$, with $11\pi/5 < t + \theta < 14\pi/5$. The form of extremal $v(s)$ is

(a) for $\pi/2 > \phi \geq \tan^{-1}[\cos(6\pi/5 - \theta) - \cos\theta] \geq 0$

$$-v(s) = \begin{cases} 2.5/\pi, & \text{if } 0 < s < 2(t+\theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, & \text{if } s_1 < s < 4(t+\theta)/3 - 38\pi/15 = \tau_1 \\ 0, & \text{if } \tau_1 < s < t - 4\pi/5 = \tau_2 \\ 2.5/\pi, & \text{if } \tau_2 < s < t, \end{cases}$$

or

(b) for $\tan^{-1}[\cos(6\pi/5 - \theta) - \cos\theta] > \phi$
$\geq -\tan^{-1}[\cos\theta - \cos((2\pi+\theta)/3)]$

$$-v(s) = \begin{cases} 2.5/\pi, & \text{if } 0 < s < 2(t+\theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, & \text{if } s_1 < s < 4(t+\theta)/3 - 38\pi/15 = \tau_1 \\ 0, & \text{if } \tau_1 < s < t + \theta + \cos^{-1}[\cos\theta + \tan\phi] - 2\pi = \tau_2 \\ 2.5/\pi, & \text{if } \tau_2 < s < t + \theta - \cos^{-1}[\cos\theta + \tan\phi] = s_2 \\ -2.5/\pi, & \text{if } s_2 < s < t, \end{cases}$$

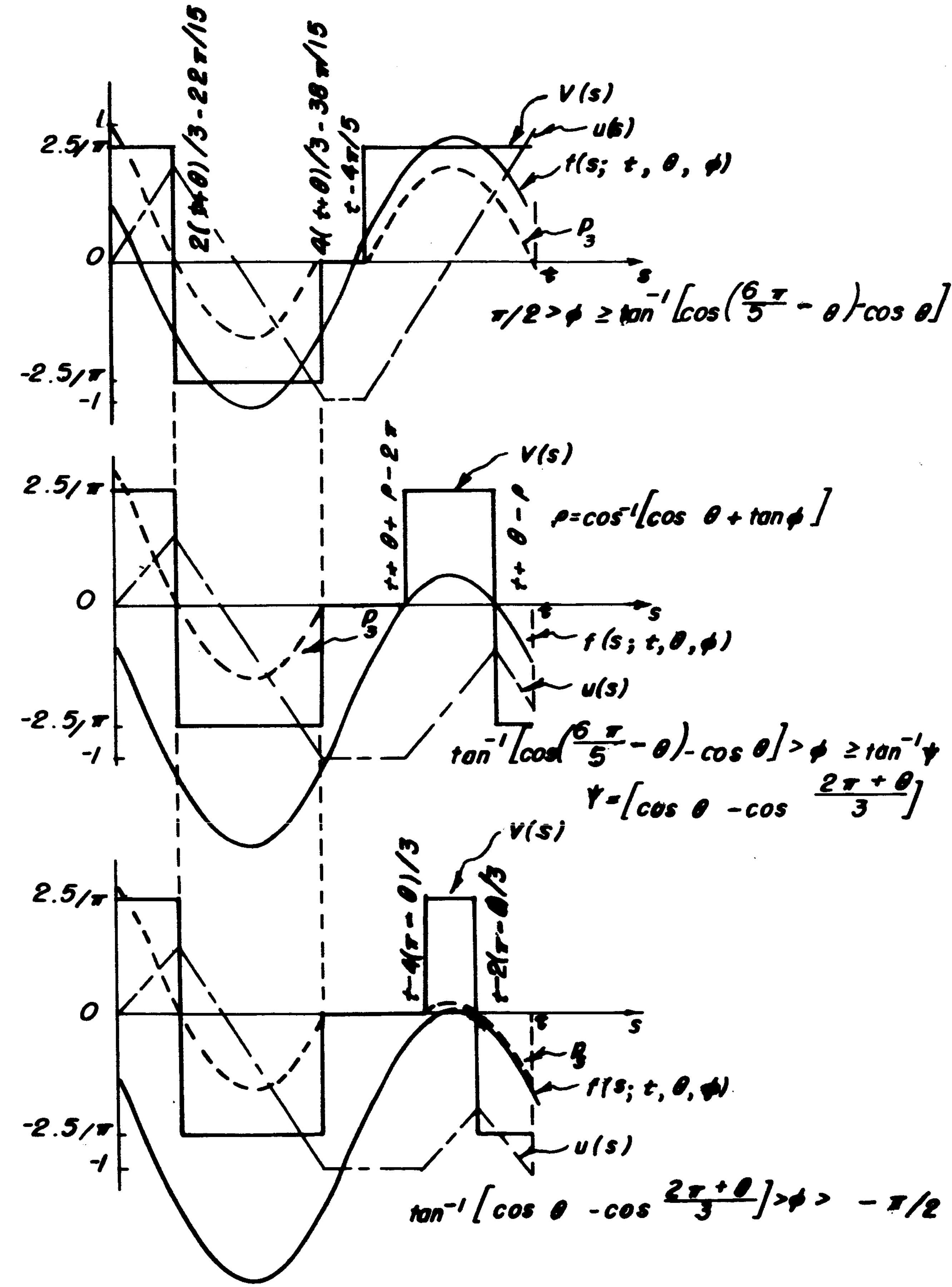

V(s)
u(s)
f(s; t, θ, φ)
P_3
2.5/π
0
-2.5/π
-1
1
s
t
2(t+θ)/3 - 22π/15
4(t+θ)/3 - 38π/15
t - 4π/5
π/2 > φ ≥ tan⁻¹[cos(6π/5 - θ) - cos θ]
ρ = cos⁻¹[cos θ + tan φ]
t + θ + ρ - 2π
t + θ - ρ
tan⁻¹[cos(6π/5 - θ) - cos θ] > φ ≥ tan⁻¹ψ
ψ = [cos θ - cos (2π + θ)/3]
t - 4(π - θ)/3
t - 2(π - θ)/3
tan⁻¹[cos θ - cos (2π + θ)/3] > φ > - π/2

FIGURE 5

or

(c) for $-\tan^{-1}[\cos\theta - \cos((2\pi + \theta)/3)] > \phi > -\pi/2$

$$-v(s) = \begin{cases} 2.5/\pi, \text{ if } 0 < s < 2(t+\theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, \text{ if } s_1 < s < 4(t+\theta)/3 - 38\pi/15 = \tau_1 \\ 0, \text{ if } \tau_1 < s < t - 4(\pi - \theta)/3 = \tau_2 \\ 2.5/\pi, \text{ if } \tau_2 < s < t - 2(\pi - \theta)/3 = s_2 \\ -2.5/\pi, \text{ if } s_2 < s < t. \end{cases}$$

The same procedure was carried out for all possible ranges of θ and t. It was found that, in general, the extremal v(s) reaches zero and leaves zero as often as the length of t permits. Denote the time at which such events occur by τ_i, i=1, 2, ..., 2N, and let $\tau_o = 0$, $\tau_{2N+1} = t$. Furthermore, let

$$|x_3(s)| < 1, \text{ if } \tau_{2i} \leq s < \tau_{2i+1},\ i=0, 1, \ldots, N;$$

and

$$|x_3(s)| = 1, \text{ if } \tau_{2j+1} \leq s < \tau_{2j+2},\ j=0, \ldots, N-1.$$

Then

$$dp_3/ds = d\psi_3/ds \text{ for } \tau_{2i} \leq s < \tau_{2i+1},\ i=0, \ldots, N;$$

and

$$p_3(s) = 0 \text{ for } \tau_{2j+1} \leq s < \tau_{2j+2},\ j=0, \ldots, N-1;$$

where ψ_3 is a component of ψ satisfying $d\psi/ds = -A'\psi$. As indicated in Section 4.2, choosing $p_3(s) = \psi_3(s) - \psi_3(\tau_{2i+1})$ for $\tau_{2i} \leq s < \tau_{2i+1}$, i=0, ..., N, yields $p_3(s)$ being zero and continuous at τ_{2i+1}, and consequently the jump conditions must be satisfied at τ_{2i}, i=0, ..., N-1. Since η is the unit adjoint vector at time t, $p_3(t) = \psi_3(t) = \eta_3 = \sin\phi$.

Therefore

$$\frac{p_3(s;t,\theta,\phi)}{\cos\phi} = \begin{cases} \cos(t-\tau_{2i+1}+\theta)-\cos(t-s+\theta), \text{ if } \tau_{2i} \le s < \tau_{2i+1}, \\ 0, \text{ if } \tau_{2i+1} \le s < \tau_{2i+2}, \\ \cos\theta - \cos(t+\theta-s) + \tan\phi + \delta(\tan\phi_o - \tan\phi), \\ \qquad\qquad \text{if } \tau_{2N} \le s < t, \end{cases}$$

where

$$i=0,\ 1,\ \ldots,\ N-1,$$

$$\delta = \begin{cases} 0, \text{ if } |x_3(t)| = |u(t)| < 1 \\ 1, \text{ if } |x_3(t)| = |u(t)| = 1, \end{cases}$$

and

ϕ_o = direction limit for η at which p_3 has a jump discontinuity (ϕ_o is a real number). Thus $p_3(s;\ t,\ \theta,\ \phi)$ has at most one jump discontinuity at $s = t$ which happens only when $|x_3(t)| = 1$. The explicit form of extremal $v(s)$ can now be expressed as

$$v(s) = \frac{k}{\pi}\,\mathrm{sgn}\,[\ p_3(s;\ t,\ \theta,\ \phi)/\cos\phi]$$

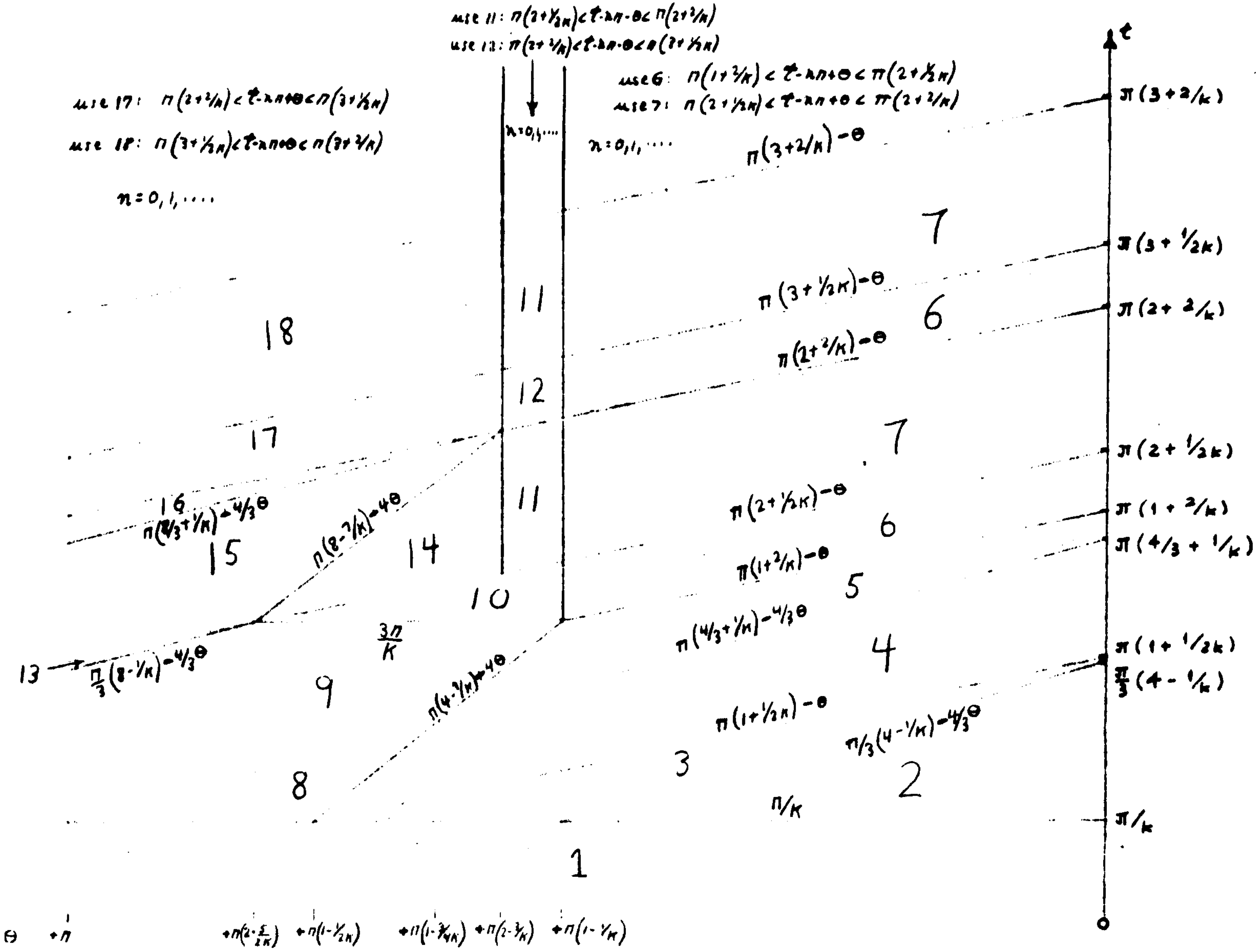
$$= \frac{k}{\pi}\,\mathrm{sgn}\,[\ p_3(s;\ t,\ \theta,\ \phi)]$$

In Figure 5, the function p_3 for the typical case is also sketched. The formulas for parameters τ_i, $i=1,\ 2,\ \ldots,\ 2N$, δ, and ϕ_o are determined for all possible choices of $k \ge 2$, $0 \le \theta \le \pi$, $|\phi| < \pi/2$, and $0 \le t < \infty$. All the results are tabulated in Tables VII and VIII. To use these tables, first locate the Case Number from Tables VII for the appropriate ranges of k, t and θ. Then on Tables VIII, for every Case Number and every range of ϕ, a set of parameters of τ_i, $i=1,\ 2,\ \ldots,\ 2N$, δ, and ϕ_o are given.

6. CONCLUSIONS

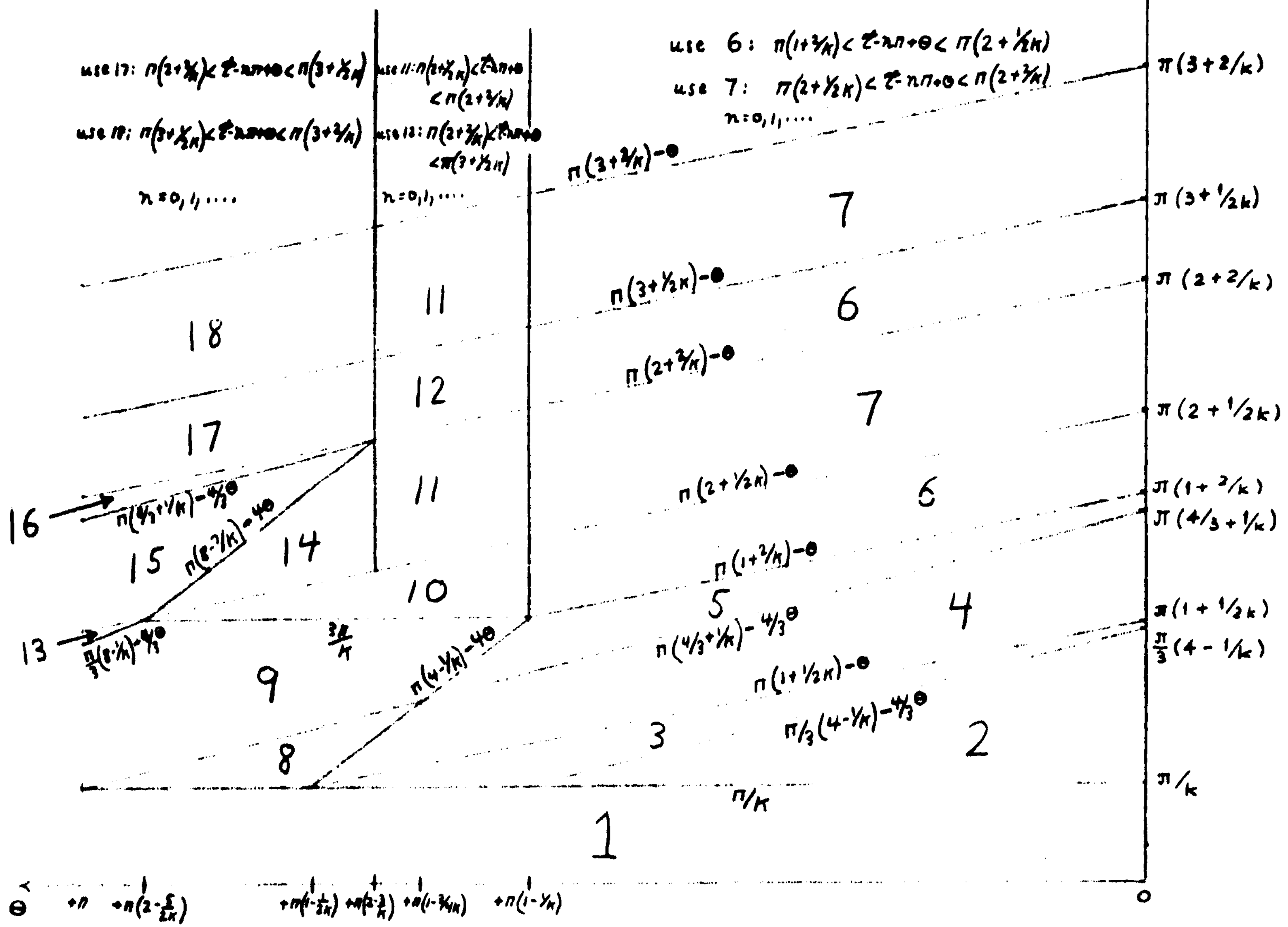
The analysis presented in this report is a direct application of the optimal control theory. The scheme, which is based on the necessary and sufficient conditions of the optimal control with bounded phase-coordinate

TABLE VII A



$(2 \leq K < 2\ldots)$

TABLE VII B



$(2.25 \le k \le 2.5)$

TABLE VII c

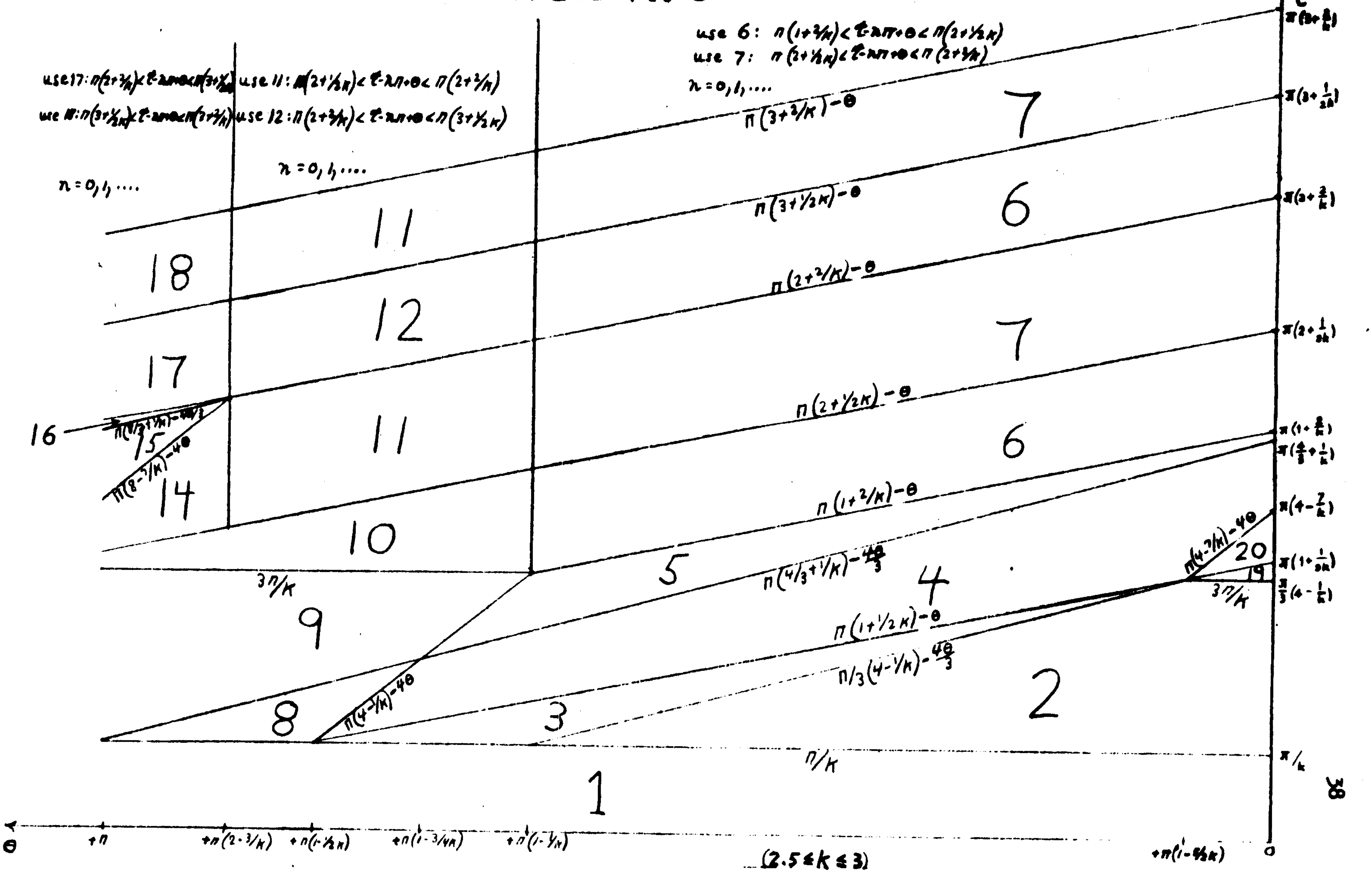
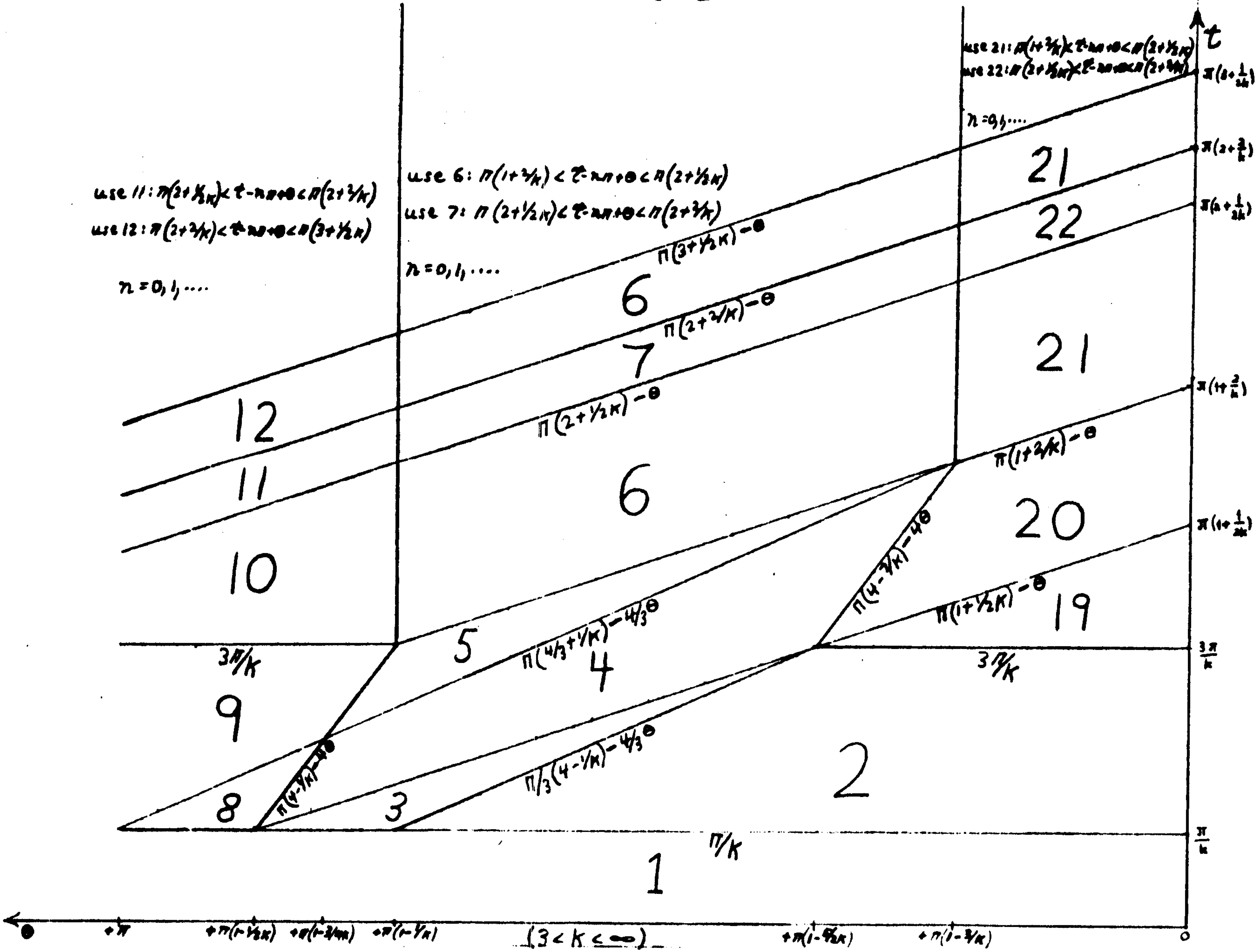

use 6: $\pi(1+3/k) < t - 2n\pi + \theta < \pi(2+1/2k)$
use 7: $\pi(2+1/2k) < t - 2n\pi + \theta < \pi(2+3/k)$
$n = 0, 1, \ldots$
use 11: $\pi(2+1/2k) < t - 2n\pi + \theta < \pi(2+2/k)$
use 12: $\pi(2+2/k) < t - 2n\pi + \theta < \pi(3+1/2k)$
$\pi(3+3/k) - \theta$
$\pi(3+1/2k) - \theta$
$\pi(2+2/k) - \theta$
$\pi(2+1/2k) - \theta$
$\pi(1+2/k) - \theta$
$\pi(1+1/2k) - \theta$
π/k
$3\pi/k$
$(2.5 \le k \le 3)$

TABLE VII D



t
use 21: $\pi(1+3/k) < t - n\pi + \theta < \pi(2+1/2k)$
use 22: $\pi(2+1/2k) < t - n\pi + \theta < \pi(2+3/k)$
n = 0, 1, ...
use 6: $\pi(1+3/k) < t - n\pi + \theta < \pi(2+1/2k)$
use 7: $\pi(2+1/2k) < t - n\pi + \theta < \pi(2+3/k)$
n = 0, 1, ...
use 11: $\pi(2+1/2k) < t - n\pi + \theta < \pi(2+3/k)$
use 12: $\pi(2+3/k) < t - n\pi + \theta < \pi(3+1/2k)$
n = 0, 1, ...
$\pi(3+1/2k) - \theta$
$\pi(2+3/k) - \theta$
$\pi(2+1/2k) - \theta$
$\pi(1+3/k) - \theta$
$\pi(1+1/2k) - \theta$
$\pi(4-3/k) - 4\theta$
$\pi(4/3+1/k) - 4/3\theta$
$\pi/3(4-1/k) - 4/3\theta$
$\pi(4-3/k) - 4\theta$
$3\pi/k$
$3\pi/k$
π/k
1
2
3
4
5
6
7
6
8
9
10
11
12
19
20
21
22
21
$\pi(2+\frac{3}{k})$
$\pi(1+\frac{3}{k})$
$\frac{3\pi}{k}$
$\frac{\pi}{k}$
0
$(3 < k \leq \infty)$
$+\pi(1-3/k)$
$+\pi(1-1/k)$
$+\pi$

TABLE VIIIA

<table>
<tr><td rowspan="3">$2\le K<\infty$ / ϕ / τ_i', δ, ϕ_0 / CASE</td><td rowspan="3">$\pi/2>\phi>0$</td><td>$0>\phi>\phi_2$
$\phi_2=-\tan^{-1}[\cos\theta-\cos(\frac{\pi}{3}(4-\frac{1}{K})-\frac{2\theta}{3})]$</td><td colspan="5">$\phi_2>\phi>\phi_6$
$\phi_6=-\tan^{-1}[\cos\theta-\cos(\pi(1+\frac{1}{3K})-\frac{\theta}{4})]$</td><td rowspan="2">$\phi_6>\phi>-\pi/2$</td></tr>
<tr><td colspan="3" rowspan="2">$0>\phi>\phi_1$
$\phi_1=-\tan^{-1}[\cos\theta-\cos(\theta+\tau-\pi/K)]$</td><td colspan="3">$\phi_1>\phi>\phi_6$</td></tr>
<tr><td colspan="2">$\phi_1>\phi>\phi_4$
$\phi_4=-\tan^{-1}[\cos\theta-\cos(\theta+\frac{\tau}{2}+\frac{\pi}{K})]$</td><td colspan="2">$\phi_4>\phi>-\pi/2$</td></tr>
<tr><td>1</td><td colspan="8">$\tau_1=\tau_{2N}=0$, $N=1$, $\delta=0$</td></tr>
<tr><td>2</td><td>$\tau_1=\pi/K$
$\tau_{2N}=\tau$, $N=1$
$\delta=1$, $\phi_0=0$</td><td colspan="3">$\tau_1=\pi/K$
$\tau_{2N}=\theta+\tau-\cos^{-1}[\cos\theta+\tan\phi]$
$N=1$, $\delta=0$</td><td colspan="2">$\tau_1=\tau_{2N}=0$
$N=1$
$\delta=0$</td><td colspan="2">$\tau_1=\tau_{2N}=0$
$N=1$
$\delta=1$, $\phi_0=\phi_4$</td></tr>
<tr><td>3</td><td>" "</td><td>" "</td><td colspan="2"></td><td colspan="3">$\tau_1=\tau_{2N}=0$
$N=1$
$\delta=0$</td><td>$\tau_1=\tau_{2N}=0$
$N=1$
$\delta=1$, $\phi_0=\phi_6$</td></tr>
<tr><td>4</td><td>$\tau_1=\frac{4}{3}(\tau+\theta)-\frac{\pi}{3}(4-\frac{1}{K})$
$\tau_{2N}=\tau$, $N=1$
$\delta=1$, $\phi_0=0$</td><td>$\tau_1=\frac{4}{3}(\tau+\theta)-\frac{\pi}{3}(4-\frac{1}{K})$
$\tau_{2N}=\theta+\tau-\cos^{-1}[\cos\theta+\tan\phi]$
$N=1$, $\delta=0$</td><td colspan="5">$\tau_1=\tau_{2N}=0$
$N=1$, $\delta=0$</td><td>" "</td></tr>
<tr><td>5</td><td>" "</td><td>" "</td><td colspan="2">$\tau_1=\tau_{2N}=0$
$N=1$, $\delta=0$</td><td>$\tau_1=\pi/K$
$\tau_{2N}=\theta+\tau+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$
$N=1$, $\delta=0$</td><td colspan="3">$\tau_1=\pi/K$
$\tau_{2N}=\tau-\frac{4}{3}(\pi-\theta)$
$N=1$, $\delta=1$, $\phi_0=\phi_5$</td></tr>
<tr><td>6
$n=0,1,\dots$</td><td>$\tau_1=\pi/K$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+2$
$\tau_{2N}=\tau$, $\delta=1$, $\phi_0=0$</td><td colspan="2">$\tau_1=\pi/K$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+2$
$\tau_{2N}=\theta+\tau-\cos^{-1}[\cos\theta+\tan\phi]$, $\delta=0$</td><td colspan="2">$\tau_1=\pi/K$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+1$
$\tau_{2N}=\theta+\tau+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$, $\delta=0$</td><td colspan="3">$\tau_1=\pi/K$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+1$
$\tau_{2N}=\tau-\frac{4}{3}(\pi-\theta)$, $\delta=1$, $\phi_0=\phi_5$</td></tr>
<tr><td>7
$n=0,1,\dots$</td><td>$\tau_1=\frac{4}{3}(\tau+\theta)-(8-1/K+4n)\pi/3$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+2$
$\tau_{2N}=\tau$, $\delta=1$, $\phi_0=0$</td><td colspan="2">$\tau_1=\frac{4}{3}(\tau+\theta)-(8-1/K+4n)\pi/3$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+2$
$\tau_{2N}=\theta+\tau-\cos^{-1}[\cos\theta+\tan\phi]$, $\delta=0$</td><td colspan="2">$\tau_1=\frac{4}{3}(\tau+\theta)-(8-1/K+4n)\pi/3$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+1$
$\tau_{2N}=\theta+\tau+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$, $\delta=0$</td><td colspan="3">$\tau_1=\frac{4}{3}(\tau+\theta)-(8-1/K+4n)\pi/3$
$\tau_{2i}=\theta+\tau-(2+1/K+n-i)\pi$, $i=1,\dots,N-1$
$\tau_{2i+1}=\theta+\tau-(2-1/K+n-i)\pi$, $N=n+1$
$\tau_{2N}=\tau-\frac{4}{3}(\pi-\theta)$, $\delta=1$, $\phi_0=\phi_5$</td></tr>
<tr><td>CASE / τ_i', δ, ϕ_0 / ϕ / $2\le K<\infty$</td><td>$\pi/2>\phi>0$</td><td colspan="2">$0>\phi>\phi_3$
$\phi_3=-\tan^{-1}[\cos\theta-\cos(\pi(1-1/K))]$</td><td colspan="2">$\phi_3>\phi>\phi_5$
$\phi_5=-\tan^{-1}[\cos\theta-\cos(\frac{2\pi\theta}{3})]$</td><td colspan="3">$\phi_5>\phi>-\pi/2$</td></tr>
</table>

TABLE VIII B

$2\le K<\infty$ / ϕ / τ_i's, δ, ϕ_0 / CASE	$\pi/2>\phi>\phi_4\ (>0)$ $\phi_4=\tan^{-1}\left[\cos\left(\theta+\frac{t}{2}-\pi(2-\frac{1}{2K})\right)-\cos\theta\right]$	$\phi_4>\phi>\phi_6$ $\phi_6=-\tan^{-1}\left[\cos\theta-\cos\left(\pi(1+\frac{1}{2K})-\frac{t}{4}\right)\right]$ $\phi_4>\phi>\phi_1$ $\phi_1=-\tan^{-1}\left[\cos\theta-\cos(\theta+t-\pi/K)\right]$		$\phi_6>\phi>-\pi/2$
8	$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=1,\ \phi_0=\phi_4$	$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=0$		$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=1,\ \phi_0=\phi_6$
9	" "	$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=0$	$\tau_1=\pi/K$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1$ $\delta=0$	$\tau_1=\pi/K$ $\tau_{2N}=t-\frac{4}{3}(\pi-\theta)$ $N=1$ $\delta=1,\ \phi_0=\phi_5$

CASE / τ_i's, δ, ϕ_0 / ϕ / $2\le K<\infty$	$\pi/2>\phi>\phi_{13}\ (>0)$ $\phi_{13}=\tan^{-1}\left[\cos\left(2\pi(1-\frac{1}{K})-\theta\right)-\cos\theta\right]$	$\phi_{13}>\phi>\phi_5$ $\phi_5=-\tan^{-1}\left[\cos\theta-\cos\left(\frac{2\pi+\theta}{3}\right)\right]$	$\phi_5>\phi>-\pi/2$
10	$\tau_1=\pi/K$ $\tau_{2N}=t-\frac{2\pi}{K}$, $N=1$ $\delta=1,\ \phi_0=\phi_{13}$	$\tau_1=\pi/K$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1,\ \delta=0$	" "
11 $n=0,1,\dots$	$\tau_1=\frac{4}{3}(2\pi+\theta)-(8-\frac{1}{K}+4n)\pi/3$ $\tau_{2i}=\theta+t-(2+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(2-\frac{1}{K}+n-i)\pi$, $N=n+1$ $\tau_{2N}=t-2\pi/K$, $\delta=1$, $\phi_0=\phi_{13}$	$\tau_1=\frac{4}{3}(2\pi+\theta)-(8-\frac{1}{K}+4n)\pi/3$ $\tau_{2i}=\theta+t-(2+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(2-\frac{1}{K}+n-i)\pi$, $N=n+1$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$, $\delta=0$	$\tau_1=\frac{4}{3}(2\pi+\theta)-(8-\frac{1}{K}+4n)\pi/3$ $\tau_{2i}=\theta+t-(2+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(2-\frac{1}{K}+n-i)\pi$, $N=n+1$ $\tau_{2N}=t-\frac{4}{3}(\pi-\theta)$, $\delta=1$, $\phi_0=\phi_5$
12 $n=0,1,\dots$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-\frac{1}{K}+n-i)\pi$, $N=n+2$ $\tau_{2N}=t-2\pi/K$, $\delta=1$, $\phi_0=\phi_{13}$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-\frac{1}{K}+n-i)\pi$, $N=n+2$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$, $\delta=0$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+\frac{1}{K}+n-i)\pi$, $i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-\frac{1}{K}+n-i)\pi$, $N=n+2$ $\tau_{2N}=t-\frac{4}{3}(\pi-\theta)$, $\delta=1$, $\phi_0=\phi_5$

TABLE VIIIc

$2\le K<\infty$ / ϕ / τ_i / δ, ϕ_0 / Case						
	$\pi/2>\phi>\phi_{13}\ (>0)$ $\phi_{13}=\tan^{-1}[\cos(2\pi(1-1/K)-\theta)-\cos\theta]$			$\phi_{13}>\phi>\phi_5$		$\phi_5>\phi>-\pi/2$
	$\pi/2>\phi>\phi_{12}\ (>0)$ $\phi_{12}=+\tan^{-1}[\cos(\theta/K-\pi/K)-\cos\theta]$	$\phi_{12}>\phi>\phi_{24}\ (>0)$ $\phi_{24}=+\tan^{-1}[\cos(\theta-\pi/3(\theta-1/K))-\cos\theta]$			$\phi_{24}>\phi>\phi_5$ $\phi_5=-\tan^{-1}[\cos\theta-\cos(\frac{2\pi+\theta}{3})]$	
	$\pi/2>\phi>\phi_{23}\ (>0)$ $\phi_{23}=+\tan^{-1}[\cos(\frac{2\pi-\theta}{3})-\cos\theta]$	$\phi_{23}>\phi>\phi_1\ (>0)$ $\phi_1=+\tan^{-1}[\cos(\theta t-\pi/K)-\cos\theta]$		$\phi_1>\phi>\phi_{24}$		
13	$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=1,\ \phi_0=\phi_{12}$		$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=0$	$\tau_1=\pi/K$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1,\ \delta=0$		$\tau_1=\pi/K$ $\tau_{2N}=t-4/3(\pi+\theta)$ $N=1,\ \delta=1,\ \phi_0=\phi_5$
14	$\tau_1=4/3(t+\theta)-\pi/3(\delta-1/K)$ $\tau_{2N}=t-3\pi/K,\ N=1$ $\delta=1,\ \phi_0=\phi_{13}$			$\tau_1=4/3(t+\theta)-\pi/3(\delta-1/K)$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1,\ \delta=0$		$\tau_1=4/3(t+\theta)-\pi/3(\delta-1/K)$ $\tau_{2N}=t-4/3(\pi-\theta)$ $N=1,\ \delta=1,\ \phi_0=\phi_5$
15	$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=1,\ \phi_0=\phi_{12}$		$\tau_1=\tau_{2N}=0$ $N=1,\ \delta=0$		$\tau_1=4/3(t+\theta)-\pi/3(\delta-1/K)$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1,\ \delta=0$	"
16	$\tau_1=\pi/K$ $\tau_{2N}=t-4/3(2\pi-\theta)$ $N=1,\ \delta=1,\ \phi_0=\phi_{23}$	$\tau_1=\pi/K$ $\tau_{2N}=\theta+t-\cos^{-1}[\cos\theta+\tan\phi]-2\pi$ $N=1,\ \delta=0$		$\tau_1=\tau_{2N}=0$ $N=1$ $\delta=0$	"	"

Case				
17 $n=0,1,\dots$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+1$ $\tau_{2N}=t-4/3(2\pi-\theta),\ \delta=1,\ \phi_0=\phi_{23}$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+1$ $\tau_{2N}=\theta+t-\cos^{-1}[\cos\theta+\tan\phi]-2\pi,\ \delta=0$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+2$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi,\ \delta=0$	$\tau_1=\pi/K$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+2$ $\tau_{2N}=t-4/3(\pi-\theta),\ \delta=1,\ \phi_0=\phi_5$
18 $n=0,1,\dots$	$\tau_1=4/3(t+\theta)-(12-1/K+4n)\pi/3$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+1$ $\tau_{2N}=t-4/3(2\pi-\theta),\ \delta=1,\ \phi_0=\phi_{23}$	$\tau_1=4/3(t+\theta)-(12-1/K+4n)\pi/3$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+1$ $\tau_{2N}=\theta+t-\cos^{-1}[\cos\theta+\tan\phi]-2\pi,\ \delta=0$	$\tau_1=4/3(t+\theta)-(12-1/K+4n)\pi/3$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+2$ $\tau_{2N}=\theta+t+\cos^{-1}[\cos\theta+\tan\phi]-2\pi,\ \delta=0$	$\tau_1=4/3(t+\theta)-(12+1/K+4n)\pi/3$ $\tau_{2i}=\theta+t-(3+1/K+n-i)\pi,\ i=1,\dots,N-1$ $\tau_{2i+1}=\theta+t-(3-1/K+n-i)\pi,\ N=n+2$ $\tau_{2N}=t-4/3(\pi-\theta),\ \delta=1,\ \phi_0=\phi_5$
Case / τ_i / δ, ϕ_0 / ϕ / $2\le K<\infty$	$\pi/2>\phi>\phi_{23}\ (>0)$ $\phi_{23}=+\tan^{-1}[\cos(\frac{2\pi-\theta}{3})-\cos\theta]$	$\phi_{23}>\phi>\phi_{25}\ (>0)$ $\phi_{25}=+\tan^{-1}[\cos(\pi/K)-\cos\theta]$	$\phi_{25}>\phi>\phi_5$ $\phi_5=-\tan^{-1}[\cos\theta-\cos(\frac{2\pi+\theta}{3})]$	$\phi_5>\phi>-\pi/2$

TABLE VIII D

$2\le k\le\infty$ / ϕ / τ_i / δ, ϕ_0 / Case	$\pi/2 > \phi > 0$	$0 > \phi > \phi_{13}$ $\phi_{13} = -\tan^{-1}\left[\cos\theta - \cos\left(\theta + \frac{2\pi}{k}\right)\right]$	$\phi_{13} > \phi > -\pi/2$
19	$\tau_1 = \pi/k$ $\tau_{2N} = t, \ N=1,$ $\delta = 1, \ \phi_0 = 0$	$\tau_1 = \pi/k$ $\tau_{2N} = \theta + t - \cos^{-1}[\cos\theta + \tan\phi]$ $N=1, \ \delta = 0$	$\tau_1 = \pi/k$ $\tau_{2N} = t - 2\pi/k$ $N=1, \ \delta=1, \ \phi_0 = \phi_{13}$
20	$\tau_1 = 4/3(t+\theta) - \pi/3(4-1/k)$ $\tau_{2N} = t, \ N=1,$ $\delta = 1, \ \phi_0 = 0$	$\tau_1 = 4/3(t+\theta) - \pi/3(4-1/k)$ $\tau_{2N} = \theta + t - \cos^{-1}[\cos\theta + \tan\phi]$ $N=1, \ \delta = 0$	$\tau_1 = 4/3(t+\theta) - \pi/3(4-1/k)$ $\tau_{2N} = t - 2\pi/k, \ N=1,$ $\delta = 1, \ \phi_0 = \phi_{13}$
21 $n=0,1,\ldots$	$\tau_1 = \pi/k$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = t, \ \delta = 1, \ \phi_0 = 0$	$\tau_1 = \pi/k$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = \theta + t - \cos^{-1}[\cos\theta + \tan\phi], \ \delta = 0$	$\tau_1 = \pi/k$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = t - 2\pi/k, \ \delta = 1, \ \phi_0 = \phi_{13}$
22 $n=0,1,\ldots$	$\tau_1 = 4/3(t+\theta) - \pi/3(8 - 1/k + 4n)$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = t, \ \delta = 1, \ \phi_0 = 0$	$\tau_1 = 4/3(t+\theta) - \pi/3(8 - 1/k + 4n)$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = \theta + t - \cos^{-1}[\cos\theta + \tan\phi], \ \delta = 0$	$\tau_1 = 4/3(t+\theta) - \pi/3(8 - 1/k + 4n)$ $\tau_{2i} = \theta + t - (2 + 1/k + n - i)\pi, \ i=1,\ldots,N-1$ $\tau_{2i+1} = \theta + t - (2 - 1/k + n - i)\pi, \ N = n+2$ $\tau_{2N} = t - 2\pi/k, \ \delta = 1, \ \phi_0 = \phi_{13}$
Case / τ_i / δ, ϕ_0 / ϕ / $2\le k<\infty$	$\pi/2 > \phi > 0$	$0 > \phi > \phi_{13}$	$\phi_{13} > \phi > -\pi/2$

makes it possible to express the control as an explicit time function. In the example of the unstable booster control problem, the results are tabulated and sketched for a comparison with those in the published literature [18, 19]. The investigation of the oscillatory space vehicle reveals the structure of the extremal control variable, which oscillates in accordance with the oscillation of the controlled vehicle.

7. PLAN OF FUTURE WORK

The immediate step will be a study of an underdamped oscillatory plant with bounded amplitude and rate control. The investigation will yield the nature of the time-optimal control function for a process with a pair of complex conjugate characteristic roots.

At this point, the research can be divided into three parallel paths: (a) extend the study to the same processes but with integral quadratic cost criteria, (b) independently simulate the same problems on a computer and compare the data so obtained against those from analytical results, and (c) study the same time-optimal control problems analytically except that one of the state variables be also bounded (so far the bound is only applied to the augmented state variable, viz. $u = x_3$). Gamkrelidze's [1,2] necessary conditions imply that the adjoint solution has certain jump discontinuities. However, his results do not indicate how many discontinuities will occur. So far in all our investigations, only one bounded state variable is involved. The results indicate that there is at most one discontinuity which can be arranged at either the beginning or the end of the time interval. It is therefore conjectured that the number of jump discontinuities in the adjoint solution is the same as the number of bounded state variables. This conjecture remains to be shown in the above study (part (c)).

Next, the investigation of a bounded phase-coordinate problem having one real and a pair of complex conjugate characteristic roots will be started. It is intended to develop an algorithm for the time-optimal control problem first, and then an algorithm for the problems with integral quadratic cost criteria. These algorithms will be programmed on a computer, and the results evaluated.

The simulation will again be carried out in the following order: (a) construct analog simulation of plant and controllers, (b) develop block diagrams of controllers suitable for future mechanization, (c) develop simulation, analog and/or digital, suitable for testing of practical control systems, (d) compare with the results from analytical expressions, and (e) test various ideas for simplifying and approximating the controller.

Finally, the same steps of investigation will be applied to the same class of control problems for linear time-varying processes. If data are available for practical systems, these systems will first be approximated by third-order systems, then computed and simulated by the methods developed in this research. A careful check of these results will determine the relative merit of this research.

PART C

OPTIMAL CONTROL OF ANTENNA POINTING DIRECTION SUBJECT TO RANDOM DISTURBANCE

1. INTRODUCTION

After the target is located, the general control problem for the antenna pointing system is the problem of direction lock-in. For the antenna system which is mounted on a space vehicle, the effect caused by random disturbances is significant. Basically, the purpose of the antenna system is the transmission of information. Since the error rate or transmission is directly related to the direction pointing error, the controller should be designed to minimize the error rate. Graphically, a typical relation between the error rate and the pointing error is sketched in Figure 6. The measure of performance of the controller is arbitrarily classified into four zones, viz., good performance, fair performance, transition, and poor performance. The corresponding zones for pointing angle in the coordinate system of pointing direction is shown in Figure 7. With the measure of performance so defined, the controller is assigned to operate in two modes as follows:

Mode 1--Mode 1 is initiated by the entry of the pointing angle into the good performance zone. In Mode 1, the controller generates a control input which minimizes the probability of entering the transition zone at any instant during some fixed time interval T_1. Mode 1 is terminated by the entry into the poor performance zone.

Mode 2--Mode 2 is in effect whenever Mode 1 is not.

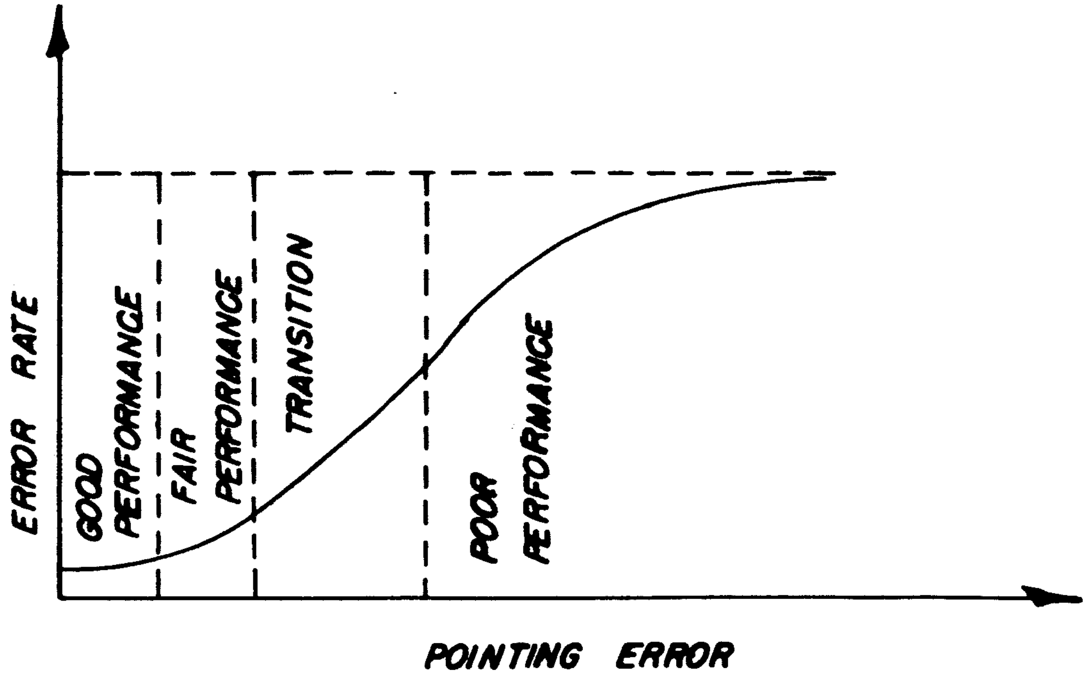


FIGURE 6

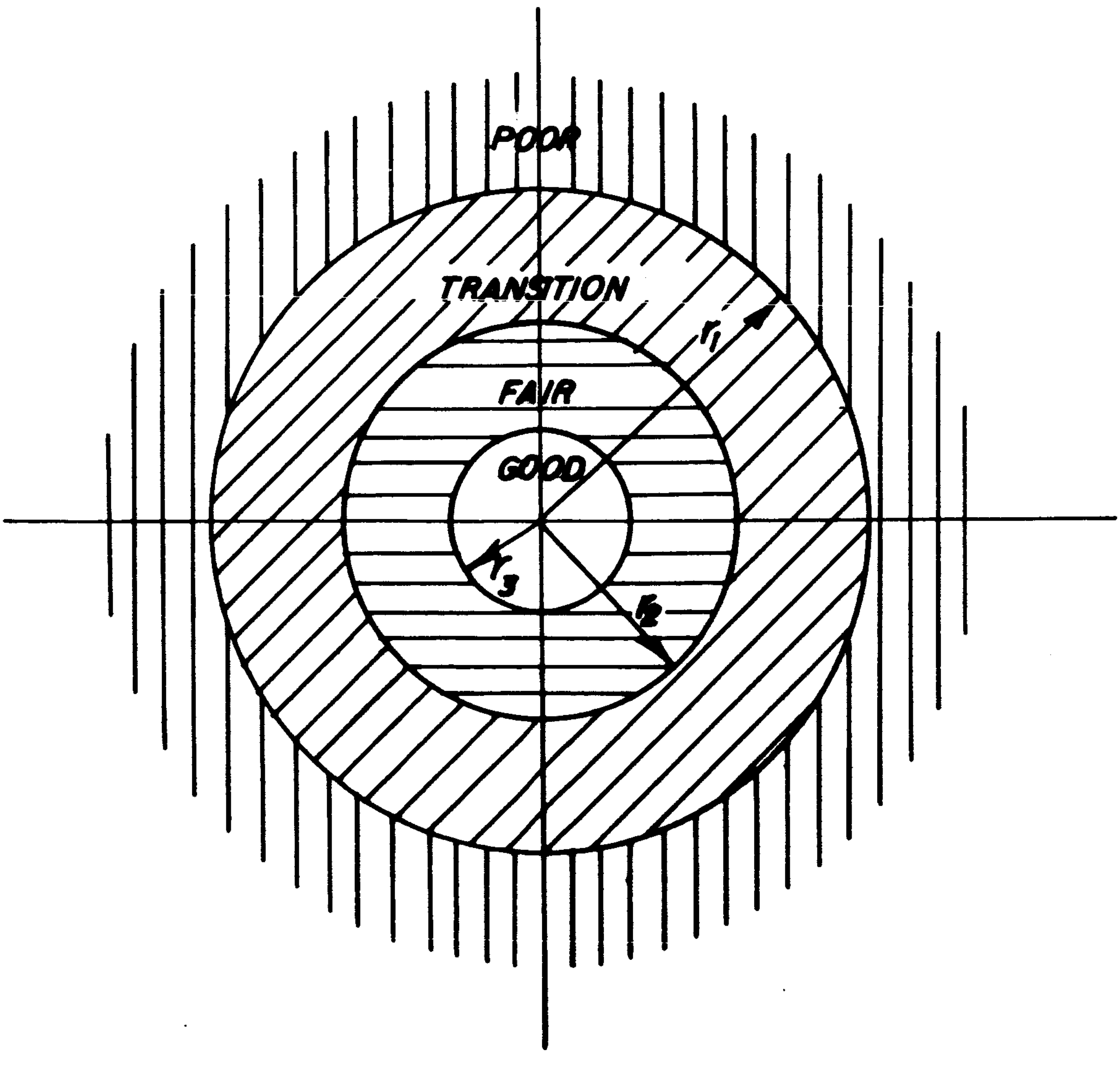


FIGURE 7

In Mode 2 the controller generates a control input which maximizes the probability of entering the good performance zone at some instant during some fixed time interval T_2.

Thus the optimization procedure can be carried out in two seperate parts:

(a) Determine the time intervals T_1 and T_2, and the radii r_1, r_2, and r_3 of boundary circles of the four zones (see Figure 7) by minimizing the error rate with respect to T_1 and T_2, and to r_1, r_2, and r_3.

(b) In each mode of operation, determine the control input that minimizes (or maximizes) the appropriate probability.

Part (b) is defined as the optimal control problem of the antenna pointing direction. An analysis of this problem, which is a direct application of the results by Pontryagin and Mishchenko [1], is presented in the following sections. In the analysis, the random disturbances in any two small consecutive time intervals are assumed statistically independent and hence the response is a Markov process [22]. The probability distribution, therefore, satisfies the Kolmogorov's backward equation. The derivation of the general form of the equation is included in the Section 2, which also serves the purpose of review. Section 3 outlines the Mishchenko's pursuit problem [1]. The material is not new but an edited summary of Mishchenko's work. It is intended to help the reader to understand the problem, as well as the method of solving the problem. Section 4 gives the problem statement. The procedure of investigation is arranged in such a manner that the method used in the pursuit problem can be applied provided certain conditions are met. The technical development and discussion on the antenna pointing direction and the height of vehicle control problem are given in Section 5 and the conclusion of the investigation in Section 6. The flow charts of computer programs for evaluating surface integrals and solving Fredholm equation of the second kind are included in the Appendix. Section 7 outlines the plan of future work.

2. GENERAL FORM OF KOLMOGOROV'S BACKWARD EQUATION

In this section the general form of Kolmogorov's backward equation and its derivation are discussed. The purpose of this section is to outline the essential properties of random processes that are governed by the equation. With additional conditions imposed on the random disturbances, the general form reduces to the familiar backward diffusion equation.

Consider the process whose dynamics are described by the differential system

$$dx = f(x,t)\ dt + dn \tag{25}$$

where x and n are the m-dimensional state and disturbance vectors, respectively; $f(x,t)$ is assumed differentiable with respect to both x and t almost everywhere. In a small time interval δt, the change of state can be written as [22]

$$\delta x = f(x,t)\ \delta t + \delta n + O(\delta t) \tag{26}$$

in which $O(\delta t)$ is such that $\lim_{\delta t \to 0} \frac{O(\delta t)}{\delta t} = 0$. It is assumed that the disturbances $\delta_1 n$ and $\delta_2 n$ in any two small consecutive time intervals $\delta_1 t$ and $\delta_2 t$ are statistically independent. Let $\delta n = \eta \delta t$, then the assumption is equivalent to the condition that the η-process has independent increments. Intuitively the x-process is Markovian since the δx is affected by the value of x at the end of the previous time interval but not the value at any instant prior to that end point. For linear systems, the necessary condition for x-process being Markovian is given in the following theorem.

Theorem 1

Consider the linear process in which the variation of the state in a small time interval is

$$\delta x = A(t)\, x\, \delta t + C(t)\, \eta(t)\, \delta t + O(\delta t) \tag{27}$$

where $A(t)$ and $C(t)$ are measurable matrices on $[t_o, t_1]$ with appropriate dimensions. If the x-process is Markovian then the disturbance η-process has independent increments.

The proof is sketched as follows*. Let $X(t,\tau)$ be an n by n matrix satisfying the relations

$$\begin{cases} \dfrac{d}{dt} X(t,t_o) = A(t)\, X(t,t_o) & \text{for } t_1 \geq t \geq t_o. \\ X(t_o,t_o) = I & \end{cases}$$

where I is an identity matrix. Then

$$x(t) = X(t,t_o)\, x(t_o) + \int_{t_o}^{t} X(t,\tau)\, C(\tau)\eta(\tau)\, d\tau \tag{28}$$

from which the covariance matrix of the x-process at a sequence of time instants can be computed. By applying Doob's theorem on Markov process in the wide sense [23, Theorem 8.1, p. 233], the Theorem 1 can thus be proved.

Let $\Phi_n(iv) = E[e^{iv'n}]$

be the characteristic function of n where v is an arbitrary vector having the same dimension as that of n, E denotes the expection value and $(\,)'$ = transpose of $(\,)$. Let $F = F(x,t,iv,\delta n)$ be a functional satisfying $e^{\delta tF} = \Phi_{\delta n}(iv)$, then F is called a disturbance functional.

---

* The proof is suggested by Dr. Glenn E. Baxter, Professor of Mathematical Sciences and Statistics, Purdue University.

Lemma 1

The disturbance functional is additive if the noise has independent increments.

Proof [22]

$$\Phi_{\Sigma\delta_k n}(iv) = E\, e^{iv' \Sigma \delta_k n}$$

$$= \prod_k E\, e^{iv' \delta_k n}$$

$$= \prod_k e^{\delta t\, F_k}$$

$$= e^{\delta t\, \Sigma F_k}$$

Lemma 2

Let $\phi(z)$ be some arbitrary functional having all its partial derivatives where z is an arbitrary vector. Let $\nabla_z = d/dz$. Then $E\phi(z+n) = \Phi_n(\nabla_z)\phi(z)$.

Proof [22]

Only the case of scalar z and scalar n will be shown. The vector case can be extended by a similar procedure. Compute

$$\Phi_n(iv) = E\, e^{ivn} = \sum_{k=0}^{\infty} E\left[n^k\, (iv)^k/k\,!\right]$$

The Taylor series of $\phi(z+n)$ about z is

$$\phi(z+n) = \sum_{k=0}^{\infty} (n^k/k\,!)\, d^k\, \phi(z)/dz^k$$

$$= \sum_{k=0}^{\infty} n^k\, \nabla_z^k\, \phi(z)/k\,!$$

$$= e^{n\nabla z}\, \phi(z)$$

hence $E\, \phi(z+n) = \Phi_n(\nabla_z)\, \phi(z)$.

A linear differential operator will now be defined which describes the general form of Kolmogorov's backward equation. Let $t = s + \delta s > s$, $y = x(s + \delta s)$, G = a fixed Borel set, $G \subset R^m$. Let $P(G,t|\hat{x},s)$ be the transition probability function, i.e., $\Pr[y \in G|x(s) = \hat{x}]$, and $p(y,t|\hat{x},s)$ be the corresponding density. For a small time interval $[s, s + \delta s)$, $y = x(s + \delta s) = x(s) + \delta\hat{x} + O(\delta s)$ and hence by equation (26),

$$\begin{aligned} \phi(y) &= \phi\,(x(s) + \delta\hat{x} + O(\delta s)) \\ &= \phi\,(\hat{x} + f(\hat{x},s)\delta s + \delta n + O(\delta s)). \end{aligned} \tag{29}$$

Since $E\,\phi(y) = \int_{R^m} \phi(y)\; p\,(y,t|\hat{x},s)dy$

Hence by (29), $$\int_{R^m} \phi(y)\; p\,(y,t|\hat{x},s)dy = E\,\phi(\hat{x} + f(\hat{x},s)\;\delta s + \delta n + O(\delta s)). \tag{30}$$

By Lemma 2 and the definition of the disturbance functional F,

$$\begin{aligned} E\,\phi(z + \delta n) &= \Phi_{\delta n}(\nabla_z)\;\phi(z) \\ &= e^{\delta s\, F(\hat{x},s,\nabla_z,\delta n)}\;\phi(z) \\ &= [1 + \delta s\; F(\hat{x},s,\nabla_z,\delta n) + O(\delta s)]\;\phi(z). \end{aligned}$$

Let $z = \hat{x} + f(\hat{x},s)\delta s$, then (30) becomes

$$\int_{R^m}\phi(y)p(y,t|\hat{x},s)dy = [1 + \delta s\;F(\hat{x},s,\nabla_{\hat{x}},\delta n) + O(\delta s^2)]\;\phi\,(\hat{x} + f(\hat{x},s)\delta s).$$

But by the Taylor series expansion about x,

$$\phi(\hat{x} + f(\hat{x},s)\delta s) = \phi(\hat{x}) + \delta s\; f(\hat{x},s)'\;\nabla_{\hat{x}}\;\phi(\hat{x}) + O(\delta s^2)$$

hence

$$\begin{aligned} \int_{R^m}\phi(y)p(y,t|\hat{x},s)dy &= \phi(\hat{x}) + \delta s[f(\hat{x},s)'\nabla_{\hat{x}} + F(\hat{x},s,\nabla_{\hat{x}},\delta n]\;\phi(\hat{x}) + O(\delta s) \\ &= \phi(\hat{x}) + \delta s\; U_{(\hat{x},s)}\;\phi(\hat{x}) + O(\delta s) \end{aligned} \tag{31}$$

where $U_{(\hat{x},s)} = f(\hat{x},s)'\;\nabla_{\hat{x}} + F(\hat{x},s,\nabla_{\hat{x}},\delta n)$

is the linear differential operator, $f(\hat{x},s)'\nabla_{\hat{x}}$ is the system operator and $F(\hat{x},s,\nabla_{\hat{x}},\delta n)$ is the disturbance operator.

Theorem 2

If $\delta n = \sum_k \delta_k n$ where $\delta_k n$'s are statistically independent, then

$$U_{(\hat{x},s)} = f(\hat{x},s)'\nabla_{\hat{x}} + \sum_k F(\hat{x},s,\nabla_{\hat{x}},\delta_k n).$$

Proof This is a direct consequence of Lemma 1 and the definition of F.

Theorem 3

The Kolmogorov's backward equation

$$-\frac{\partial P(G,t|\hat{x},s)}{\partial s} = U_{(\hat{x},s)} P(G,t|\hat{x},s)$$

holds for $s \leq t$ with terminal conditions

$$P(G,t|\hat{x},s) = \begin{cases} 1, & \text{if } y \in G \text{ as } s \to t, \\ 0, & \text{if } y \notin G \text{ as } s \to t. \end{cases}$$

Proof [22]

Let $\hat{x} = x(s) \in H \subset R^m$ and $z = x(s-\delta s)$. Then

$$P(G,t|z,s-\delta s) = \int_{R^m} P(G,t|\hat{x},s)\ p(\hat{x},s|z,s-\delta s)\ d\hat{x}.$$

By (31), $P(G,t|z,s-\delta s) = P(G,t|\hat{x},s) - \delta s\ U_{(z,s-\delta s)}P(G,t|\hat{x},s) + O(\delta s)$

or $$-\frac{P(G,t|\hat{x},s) - P(G,t|z,s-\delta s)}{\delta s} = U_{(z,s-\delta s)}\ P(G,t|\hat{x},s) + \frac{O(\delta s)}{\delta s}.$$

Since $z \to \hat{x}$ and $O(\delta s)/\delta s \to 0$ as $\delta s \to 0$, hence

$$-\frac{\partial P(G,t|\hat{x},s)}{\partial s} = U_{(\hat{x},s)}\ P(G,t|\hat{x},s).$$

The terminal conditions are satisfied by trivial reasons since in zero time interval, the probability of change of state is zero.

Theorem 4

Let $P = P(G,t|\hat{x},s)$, $\delta\hat{x}_j$ = j th component of vector $\delta\hat{x}$, and

$b_{jk}(\hat{x},s) = \lim_{\delta s \to 0} (E\,\delta\hat{x}_j\,\delta\hat{x}_k)/\delta s$. If the disturbance δn in any small time interval is statistically independent and gaussian distributed with zero mean, then the Kolmogorov's equation becomes a backward diffusion equation

$$-\frac{\partial P}{\partial s} = \sum_i f_i(\hat{x},s)\frac{\partial P}{\partial \hat{x}_i} + \frac{1}{2}\sum_j\sum_k b_{jk}(\hat{x},s)\frac{\partial^2 P}{\partial\hat{x}_j\partial\hat{x}_k}$$

with the same terminal conditions.

<u>Proof</u>

By equation (26),

$$\frac{\partial\hat{x}_j\partial\hat{x}_k}{\delta s} = \frac{[f_j\,\delta s + \delta n_j + O(\delta s)][f_k\,\delta s + \delta n_k + O(\delta s)]}{\delta s}$$

$$= f_j\,\delta n_k + f_k\,\delta n_j + \delta n_j\delta n_k/\delta s + O(\delta s)/\delta s$$

In a small time interval δs, $f(\hat{x},s)\,\delta s$ is the mean of $\delta\hat{x}$ [23, p.273], hence $E\,f_j\,\delta n_k = f_j\,E\,\delta n_k$. Therefore

$$E\,\delta\hat{x}_j\delta\hat{x}_k/\delta s = f_j E\delta n_k + f_k\,E\delta n_j + E\delta n_j\delta n_k/\delta s + O(\delta s)/\delta s$$

$$= E\,\delta n_j\delta n_k/\delta s + O(\delta s)/\delta s.$$

Consequently, $b_{jk}(\hat{x},s) = \lim_{\delta s \to 0} (E\,\delta\hat{x}_j\delta\hat{x}_k)/\delta s$

$$= \lim_{\delta s \to 0} (E\,\delta n_j\delta n_k)/\delta s$$

Since δn is gaussian distributed with zero mean, then

$$\Phi_{\delta n}(iv) = E\,e^{iv'\delta n}$$

$$= E\exp[\sum_h (iv_h)\,\delta n_h]$$

$$= E\{\sum_{r=0}^{\infty}[\sum_h (iv_h)\delta n_h]^r/r\,!\}$$

$$= 1 + \sum_h iv_h\,E\,\delta n_h + \frac{1}{2}\sum_j\sum_k (iv_j)(iv_k)\,E\,\delta n_j\delta n_k + \ldots$$

$$= 1 + \tfrac{1}{2} \sum_j \sum_k (iv_j)(iv_k)\, E\, \delta n_j \delta n_k$$

But $\log(1 + w) = w - \tfrac{1}{2} w^2 + 1/3\, w^3 - \dots$ for $|w| < 1$

Let $\Phi_{\delta n}(iv) = 1+w$

Then $w = \tfrac{1}{2} \sum_j \sum_k (iv_j)(iv_k) E\, \delta n_j \delta n_k + \dots$

Therefore $\log \Phi_{\delta n}(iv) = \tfrac{1}{2} \sum_j \sum_k (iv_j)(iv_k)\, E\, \delta n_j\, \delta n_k + \dots$

By definition, $\log \Phi_{\delta n}(iv) = \delta s\, F(\hat{x}, s, iv, \delta n)$

hence, by replacing iv by $\nabla_{\hat{x}}$ and take $\delta s \to 0$ as a limit,

$$F(\hat{x}, s, \nabla_{\hat{x}}, \partial n) = \lim_{\delta s \to 0} \tfrac{1}{2} \sum_j \sum_k E(\delta n_j \delta n_k / \delta s) \frac{\partial^2}{\partial \hat{x}_j \partial \hat{x}_k}$$

$$= \tfrac{1}{2} \sum_j \sum_k b_{jk}(\hat{x}, s) \frac{\partial^2}{\partial \hat{x}_j \partial \hat{x}_k}$$

Thus, $-\frac{\partial P}{\partial s} = U_{(\hat{x},s)} P$

$$= [f(\hat{x}, s)' \nabla_{\hat{x}} + F(\hat{x}, s, \nabla_{\hat{x}}, \partial n)]\, P$$

$$= \sum_i f_i(\hat{x}, s) \frac{\partial P}{\partial \hat{x}_i} + \tfrac{1}{2} \sum_j \sum_k b_{jk}(\hat{x}, s) \frac{\partial^2 P}{\partial \hat{x}_j \partial \hat{x}_k}.$$

<u>Corollary</u>

Let $\delta n = \eta\, \delta s$. If η-process is a Wiener-Leoy process, then the backward diffusion equation holds.

<u>Proof</u> Since η-process has stationary independent increments, and $\eta(s)$ is gaussian distributed with zero mean, it is equivalent to the condition that δn in any small time interval is statistically independent and gaussian distributed with zero mean. Hence the proof follows Theorem 4.

3. THE PURSUIT PROBLEM [1]

The pursuit problem can be stated as follows. Let y be the m-dimensional state vector of a system defined by

$$\begin{cases} dy/dt = f(y,t,u) \\ y(0) = \bar{y} \end{cases}$$

where u is the k-dimensional control vector and f the m-dimensional measurable vector with $k \leq m$. Let z be the state vector of a randomly moving point. Given that z is a sample function of a Markov process with transition density

$$p(\zeta,\tau|\xi,\sigma) = p_{z(\tau)|z(\sigma)} \quad (\zeta|\xi)$$

where the right side of the equation is the conditional probability density associated with the event $z(\tau) = \zeta$ given the event $z(\sigma) = \xi$ ($z(\tau)$ and $z(\sigma)$ are random variables while ζ and ξ are numbers). It is assumed that the Markov process is continuous with probability 1, and sufficient partial derivatives exist. The problem is to find u which maximizes the probability that

$$\| Z(\tau) - y(\tau) \| < \epsilon$$

for a given $\epsilon > 0$ and for some $\tau \in [0, T]$ where T is given.

This problem is solved as follows. The functional $\psi_u(\sigma,\xi,\tau)$ is defined as the probability that the randomly moving point is captured between times σ and τ given that $z(\sigma) = \xi$, and that the control function is u. If the functional ψ_u were available, it would be straightforward to apply the maximum principle, and thus solve the problem. The following is an outline of Pontryagin's approximation to $\psi_u(\sigma,\xi,\tau)$.

The first step is to show that $\psi_u(\sigma,\xi,\tau)$ is a solution to

$$\frac{\partial \psi_u}{\partial \sigma} + \sum_{i,j} b_{ij} \frac{\partial^2 \psi_u}{\partial \xi_i \partial \xi_j} + \sum_i a_i \frac{\partial \psi_u}{\partial \xi_i} = 0$$

subject to the boundary conditions

$$\psi_u(\tau, \xi, \tau) = 0$$

$$\psi_u(\sigma, \xi, \tau) \Big|_{S\sigma} = 1$$

where $S\sigma$ = surface defined by $\| \xi - y(\sigma) \| = \epsilon$,

$$\xi = z(\sigma),$$

$$a_i(\sigma,\eta) = \lim_{\Delta\sigma \to 0} \frac{1}{\Delta\sigma} \int_{\|\zeta-\eta\| < \delta} (\zeta_i - \eta_i) p(\zeta, \sigma | \eta, \sigma-\Delta\sigma) d\zeta,$$

$$b_{ij}(\sigma,\eta) = \lim_{\Delta\sigma \to 0} \frac{1}{\Delta\sigma} \int_{\|\zeta-\eta\| < \delta} (\zeta_i - \eta_i)(\zeta_j - \eta_j) p(\zeta, \sigma | \eta, \sigma-\Delta\sigma) d\zeta$$

for all $\delta > 0$,

$[b_{ij}(\sigma,\eta)]$ is continuous, bounded and positive definite,

$a_i(\sigma,\eta) = O(\exp\|\eta\|)$ is continuous.

A solution is then obtained in the form, with $z(0) = \bar{z}$,

$$\psi_u(0, \bar{Z}, T) = \epsilon^{m-2}\Gamma(0, \bar{Z}, T) + o(\epsilon^{m-2})$$

where m is the dimension of the state space. For the case where b_{ij} is independent of σ or ξ, Γ is given by

$$\Gamma(0, \bar{Z}, T) = \Gamma_o(0, \bar{Z}, T) + \Gamma_1(0, \bar{Z}, T)$$

where

$$\Gamma_o(\sigma,\xi,\tau) = \frac{\alpha}{\{[\xi-y(\sigma)]'[\beta_{ij}][\xi-y(\sigma)]\}^{(m-2)/2}}$$

$$- \int_{R^m} \frac{\alpha(\prod_1^m \lambda_i)^{\frac{1}{2}} \exp\{-[\zeta-\xi+y(\sigma)]'[\beta_{ij}][\zeta-\xi+y(\sigma)]/4(\tau-\sigma)\}}{[2\pi(\tau-\sigma)]^{m/2} \{\zeta'[\beta_{ij}]\zeta\}^{(m-2)/2}} d\zeta$$

$$\Gamma_1(\sigma,\xi,\tau) = \int_\sigma^\tau \left\{ \int_{R^m} p(v,s|\xi,\sigma) \sum_i^m \left\{a(s,v) - f[y(s),s,u]\right\}_i \frac{\partial\Gamma_o(s,v,\tau)}{\partial v_i} dv\right\} ds$$

λ_i = eigenvalues of $[b_{ij}]$,

$$\alpha = \frac{\int_S \nu_o(\overline{\eta})\, dS}{\int_S \frac{\nu_o(\overline{\eta})}{r^{m-2}(\overline{\eta})} dS}$$

$$[\beta_{ij}] = [b_{ij}]^{-1},$$

$\nu_o(\tilde{\eta})$ = eigenfunction satisfying

$$\nu(\tilde{\eta}) = \int_S \frac{1}{2\pi} \frac{\cos\theta}{\rho^{m-1}(\overline{\eta}\cdot\tilde{\eta})} \nu(\overline{\eta})\, dS, \quad (\tilde{\eta}, \overline{\eta} \in S)$$

$$r(\overline{\eta}) = [\Sigma\, \overline{\eta}_i^{\,2}]^{1/2},$$

S = a continuous closed surface defined by

$$\sum \lambda_i (\overline{\eta}_i)^2 = \epsilon^2,$$

θ = angle between the vector ρ from $\overline{\eta}$ to $\tilde{\eta}$ and the normal to S at $\overline{\eta} \in S$.

4. PROBLEM STATEMENT AND METHOD OF INVESTIGATION

This section states the general problem of interest. The method of investigation and the preparatory computation of transition densities are also presented.

4.1 Problem Statement

Let the motion of the vehicle be described by a system of differential equations. It is convenient to normalize the equations as

$$dx = f(t,x)\, dt + B(t)\, u(t)\, dt + C(t,x)\, dn \tag{33}$$

$$x(t_o) = \overline{x}$$

where x is the system state vector; u is the control input vector; n is a sample vector of a random process; f is the system-parameter vector which is assumed to be differentiable with respect to both t and x almost everywhere; B and C are matrices with appropriate dimensions. As a rule, the dimensions of vectors u and n are lower than that of state vector x. It is known that if the disturbances dn in any two small consecutive time intervals are statistically independent, then the response x to the system (33) is a Markov process [22]. As shown in Section 2, for Gaussian disturbance Δn in any small time interval Δs, the transition probability function $P(G,t|\gamma,s)$ where $\gamma = x(s)$, satisfies the backward diffusion equation

$$\frac{\partial P}{\partial s} + \sum_i \{f(s,\gamma) + B(s)u(s)\}_i \frac{\partial P}{\partial \gamma_i} + \frac{1}{2}\sum_i \sum_j b_{ij}(s,\gamma) \frac{\partial^2 P}{\partial \gamma_i \partial \gamma_j} = 0 \quad (34)$$

for $t > s \geq 0$, in which

$$b_{ij}(s,\gamma) = \lim_{\Delta s \to 0} \frac{E \sum_k \sum_h C_{ik}(s,\gamma)\, C_{hj}(s,\gamma)\, \Delta n_k\, \Delta n_h}{\Delta s}$$

for a fixed t and a fixed Borel set G.

For the purpose of discussion, assume the system state is in a situation such that the control input u is in Mode 2 of operation as defined in Section 1. Let $\psi_u(t,x,t_o + T_2)$ be the probability of entering the good performance zone in the finite time interval $[t, t_o + T_2]$. As outlined in Section 3, Mishchenko [1] showed that if the statistics of the x-process are described by equation (34), then $\psi = \psi_u(t,x,t_o + T_2)$ satisfies the same equation

$$\frac{\partial \psi}{\partial t} + \sum_i \{f(t,x) + B(t)\, u(t)\}_i \frac{\partial \psi}{\partial x_i} + \frac{1}{2}\sum_i \sum_j b_{ij}(t,x) \frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \quad (35)$$

with boundary conditions $\psi_u(t,\tilde{x},t_o + T_2) = 1$ for all t, and $\psi_u(t_o + T_2, \hat{x}, t_o + T_2) = 0$, where $\|\tilde{x}\| = r_3$ and $\|\hat{x}\| > r_3$.

The problem is to determine ψ, which is a functional dependent on u, and choose a control u that maximizes ψ.

4.2 Method of Investigation

In order to apply the results of the pursuit problem (Section 3), the method of investigation is outlined as follows. First of all, the system is assumed to be linear. To be more precise, it is desired to determine the control vector u which maximizes

$$\Pr [\| x(t) \| \leq r_3] \quad \text{for some } t \in [t_o, t_o + T]$$

subject to

$$\begin{cases} dx = [A(t)\, x + B(t)\, u(t)]dt + C\,(t,x)\; dn \\ x(0) = \bar{x} \end{cases}$$

where x is an m dimensional vector.

A is an m by m measurable matrix,

u is an $h < m$ dimensional vector,

B is an m by h measurable matrix,

n is a $k \leq m$ dimensional sample vector of a random process.

C is an m by k measurable matrix.

T, and $\bar{x}$ are given as part of the problem.

For this system, compute the transition densities of the z-process defined by

$$\begin{cases} dz = A(t)\; z\; dt + C\,(t,x)\; dn \\ z(0) = \bar{x} \end{cases} \tag{36}$$

Next, compute the control vector u to maximize

$$\Pr [\| z(t) - y(t) \| \leq r_3] \quad \text{for some } t \in [t_o, t_o + T]$$

subject to

$$\begin{cases} dy = A(t)\; y\; dt - B(t)\; u\; dt \\ y(0) = 0 \end{cases} \tag{37}$$

The method of approach is motivated by the advantage of the superposition property of linear systems. A proper translation of the coordinate-system reduces the present problem to Mishchenko-Pontryagin's pursuit problem which is summarized in Section 3. Thus, if the statistics of the z-process is in agreement with the hypotheses for the pursuit problem, then the known results can be used to complete the solution.

The computation of the transition densities of z-process, which is required for the evaluation of Pr $[\|z-y\| \leq r_3]$, is presented in Section 4.3

4.3 Computation of Transition Densities

Consider the stochastic differential system

$$\begin{cases} dz = A(t)\, z\, dt + C(t)\, dn \\ z(0) = 0 \end{cases} \tag{38}$$

where z is an m dimensional vector,

A is an m by m measurable matrix

C is an m by h measurable matrix,

n is an h dimensional ($h \leq m$) sample vector

of a random process with independent and orthogonal increments. According to Doob [23], the integral $\int C(t)\, dn$ in the usual Stieltjes sense does not exist with probability one because the sample functions of processes with independent increments are of unbounded variation with probability one. This integral, however, can be redefined as a stochastic integral [23] so that it does exist. Under this definition, the limit of the sequence of Stieltjes sums exists in a "limit in the mean" sense.

The solution of system (38) is known as [23]

$$z(t) = \int_0^t \Phi(t, \tau)\, C(\tau)\, dn(\tau) \tag{39}$$

where $\Phi(t, \tau)$ is the m by m continuous matrix satisfying

$$\begin{cases} \dfrac{d}{dt} \Phi(t, \tau) = A(t)\, \Phi(t, \tau), \\ \Phi(\tau, \tau) = \text{identity matrix} \end{cases}$$

and the integral in (39) is a stochastic integral.

To facilitate the discussion let

$$I_{\xi\eta} = \int_{\xi}^{\eta} \Phi(\eta, \tau)\, C(\tau)\, dn(\tau)$$

and

$$I_{\xi\eta}^{i} = \sum_{k=0}^{i-1} \Phi(\eta, \tau_k)\, C(\tau_k)\, [n(\tau_{k+1}) - n(\tau_k)].$$

where

$$\tau_i = \eta \quad \text{and} \quad \tau_0 = \xi,$$

are random variables. The transition density

$$p_{z(\tau_2) \mid z(\tau_1)}(z_2 \mid z_1) = p_{I_{\tau_1\tau_2} \mid I_0,\, \tau_1}(z_2 - z_1 \mid z_1) \qquad (40)$$

where the p's are defined in Section 3.

The sequence $\{ I_{\xi\eta}^{i} \}$ converges to $I_{\xi\eta}$ in a l.i.m. sense as described by Doob [23]. A question arises as to the conditions upon which the convergence of $p_{I_{\xi\eta}^{i}} \to p_{I_{\xi\eta}}$ in a suitable sense as $i \to \infty$. Once the convergence is established, then, (40) implies that the z-process transition densities can be approximated by the conditional densities $p_{I_{\tau_1\tau_2}^{i} \mid I_0^{i},\, \tau_1}$.

The investigation of the convergence problem will be deferred for the future study. The computation of the conditional densities, however, will be discussed in the following.

In order to facilitate the discussion, the problem will be restated in the following notation. Let Y_{kq} be a random vector of dimension m defined by

$$Y_{kq} = \Phi(\tau_q, \tau_k)\, C(\tau_k)\, [n(\tau_{k+1}) - n(\tau_k)] \qquad (41)$$

and S_{qr} be a random vector defined as

$$S_{qr} = \sum_{k=q}^{r} Y_{kq} \tag{42}$$

To express the conditional density

$$p_{S_{qr} | S_{0,\ q-1}} (s_{qr} | s_{0,\ q-1})$$

in terms of n statistics the following two steps are required:

(1) The Y_{kq} statistics will be written in terms of n statistics, and

(2) the desired S distributions will be written in terms of Y_{kq} statistics.

For the first of these two steps, consider the dimension of the elements of (41):

Y_{kq} is an m-dimensional vector,

$n(\tau_{k+1}) - n(\tau_k)$ is an $h \leq m$ dimensional vector,

$\Phi(\tau_q, \tau_k)\ C(\tau_k)$ is an m by h matrix, and is assumed to have rank h.

To facilitate the discussion, let

$$\Delta n_k = n(\tau_{k+1}) - n(\tau_k)$$

$$D_{qk} = \Phi(\tau_q, \tau_k) C(\tau_k)$$

Also, superscripts will be used to denote vector elements, e.g. the $i^{\underline{th}}$ element of Δn_k is Δn_k^i. Thus (41) becomes

$$Y_{kq} = D_{qk} \Delta n_k \tag{43}$$

From the dimensional considerations stated earlier, (43) represents a mapping of E^h into a subspace, Ψ, of E^m. The next step is to construct a suitable "coordinate" system as follows. Let $v_{1kq}, \ldots, V_{hkq}$ be an orthonormal basis for Ψ. Let $v_{1kq}, \ldots, v_{hkq}, \ldots, v_{mkq}$ be an orthonormal basis for E^m. Let V_{hkq} be the m by h matrix whose columns are $v_{1kq}, \ldots, v_{hkq}$, and let V_{mkq} be the m by m matrix whose columns are $v_{1kq}, \ldots, v_{mkq}$. For every Y_{kq}, there is a unique m-dimensional vector α such that

$$Y_{kq} = V_{mkq}\, \alpha_{Y_{kq}}.$$

This is true because the columns of V_{mkq} form a basis for E^m. Moreover, since the first h columns of V_{mkq} form a basis for $\mathfrak{V}$, then

$$\alpha^i_{Y_{kq}} = 0, \; i = h + 1, \ldots, m,$$

if $Y_{kq} \in \mathfrak{V}$. Thus $Y_{kq} \in \mathfrak{V}$ is equivalent to

$$\left(V^{-1}_{mkq} \;\; Y_{kq}\right)^i = 0, \; i = h + 1, \ldots, m. \tag{44}$$

From (44),

$$\Delta n_k = \left(V'_{hkq}\, D_{qk}\right)^{-1} V'_{hkq}\, Y_{kq} \tag{45}$$

where $'$ denotes transpose. From (44) and (45)

$$p_{Y_{kq}}(y) = p_{\Delta n_k}\left[(V'_{hkq}\, D_{qk})^{-1} V'_{hkq} y\right] \delta\left[(V^{-1}_{mkq} Y_{kq})^i\right], \; i = h + 1, \ldots m$$

where δ is the Dirac delta. This completes the first of the two steps.

For the computation of $p_{S_{qr}|S_{0,\; q-1}}$ it is noted that the condition-ing variable is a linear combination of those Δn's which do not appear in S_{qr}. Since the Δn's are independent, hence

$$p_{S_{qr}|S_{0,\; q-1}}\left(s_{qr}|s_{0,\; q-1}\right) = p_{S_{qr}}(s_{qr}). \tag{46}$$

From (46) and (42),

$$p_{S_{qr}|S_{0,\; q-1}}\left(s_{qr}|s_{0,\; q-1}\right) =$$

$$\int ds_{q,\; r-1}\, p_{S_{q,\; r-1},\; Y_{rq}}\left(s_{q,\; r-1}, s_{qr} - s_{q,\; r-1}\right) \tag{47}$$

Since the Δn's are independent,

$$p_{S_{q,\ r-1}\ Y_{rq}}(s_{q,\ r-1},\ s_{qr} - s_{q,\ r-1}) =$$

$$p_{S_{q,\ r-1}}(s_{q,\ r-1})\ p_{Y_{rq}}(s_{qr} - s_{q,\ r-1}). \tag{48}$$

Thus (4) becomes

$$p_{S_{qr}|S_{0,\ q-1}}(s_{qr}|s_{0,\ q-1}) =$$

$$\int ds_{q,\ r-1}\ p_{Y_{rq}}(s_{qr} - s_{q,\ r-1})\ p_{S_{q,\ r-1}}(s_{q,\ r-1}).$$

By applying the same procedure repeatedly, one obtaines

$$p_{S_{qr}|S_{0,\ q-1}}(s_{qr}|s_{0,\ q-1}) =$$

$$\int ds_{q,\ r-1} p_{Y_{rq}}(s_{qr} - s_{q,\ r-1}) \int ds_{q,\ r-2} p_{Y_{r-1,\ q}}(s_{q,\ r-1} - s_{q,\ r-2})$$

$$---\int ds_{qq} p_{Y_{q,\ q}}(s_{qq})$$

which gives the conditional density in terms of Y_{kq} statistics.

5 PROBLEM OF ANTENNA POINTING DIRECTION

In the preceding sections, Mishchenko's pursuit problem and the relevant subject were discussed. The results will now be applied to the problem of antenna pointing direction.

5.1 Problem Statement

As indicated in Section 4, the motion of the vehicle is assumed to be governed by a system of linear stochastic differential equation

$$\begin{cases} dx = [Ax + B\ u(t)]\ dt + C\ dn \\ x(0) = \bar{x} \end{cases} \tag{49}$$

where x is a 3-dimensional state vector; n is a 3-dimensional sample vector of a random process such that the variation δn in any two small consecutive time intervals is statistically independent and gaussian distributed with zero mean; $u(t)$ is a control vector with a dimension ≤ 3; A, B and C are constant matrices with appropriate dimensions. The 3-dimensional coordinate system is so chosen that x_1 and x_2 define the antenna pointing direction and x_3 the distance between the space vehicle and some given reference point. The origin of the coordinate system represents an equilibrium state at which the exact pointing direction and the height of the vehicle is obtained. The relation between the performance of the controller for the pointing direction and the pointing angle in the coordinate system is shown in Figure 7, in which the radii r_1, r_2, and r_3 are given. For convenience, the performance of the controller for the vehicle height is defined in a similar way, i.e., $|x_3| \leq r_3$, $r_3 < |x_3| \leq r_2$, $r_2 < |x_3| \leq r_1$ and $r_1 < |x_3|$ define the four different performance regions. Thus Figure 2 is also a graphical representation in the 3-dimensional space of the performance zones, which are defined by the concentrical spheres, of the controller.

It is required to synthesize a controller which is capable to perform Modes 1 and 2 of operation according to the status of the 3-dimensional state vector x. As indicated before, the problem will be solved by an application of Mishchenko's pursuit problem [1], which is outlined in Section 3. In order to apply Mishchenko's results, a reformulation of the antenna problem is necessary.

5.2 Reformulation of the Antenna Problem

Let $x = z - y$ such that

$$\begin{cases} dz = Az\,dt + C\,dn \\ z(0) = \bar{x} \end{cases} \qquad , \tag{50}$$

and

$$\begin{cases} dy = Ay\,dt - B\,u(t)\,dt \\ y(0) = 0 \end{cases} \qquad , \tag{51}$$

then the x-process (49) is divided into two parts. The y-system yields an ordinary deterministic control problem while the z-process is stochastic.

For the purpose of discussion, the controller is assumed in Mode 2 of operation. Then the problem is to find a control input u which maximizes the probability that $\| x(t) \| = \| z(t) - y(t) \| < r_3$ for some $t \in [0, T_2]$ given that $z(0) - y(0) = \bar{x}$, $\| \bar{x} \| > r_1$.

Since, by assumption, δn in any two small consecutive time intervals is statistically independent and gaussian distributed, it is known [22, 23] that the z-process is Markovian and its transition probability function satisfies the backward diffusion equation

$$\frac{\partial P}{\partial \sigma} + \sum_{i=1}^{3} [A\xi]_i \frac{\partial P}{\partial \xi_i} + \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} b_{jk} \frac{\partial^2 P}{\partial \xi_j \partial \xi_k} = 0 \qquad (52)$$

where $P = \Pr[z(\tau) \epsilon G \mid z(\sigma) = \xi]$ for $\tau > \sigma \geq 0$.

G = a fixed Borel set,

$$b_{ij} = \lim_{\delta t \to 0} E \sum_{k} \sum_{h} C_{ik} C_{hj} \delta n_k \delta n_h / \delta t. \qquad (53)$$

Assume that all the eigenvalues of the symmetric covariance matrix $[b_{ij}]$ are positive and bounded, and the Markov process is continuous in the sense that for all $\delta > 0$,

$$\lim_{\Delta\sigma \to 0} \frac{1}{\Delta\sigma} \int_{\| \xi - \eta \| \geq \delta} p(\xi, \sigma | \eta, \sigma - \Delta\sigma) d\xi = 0$$

where $p(\xi, \sigma | \eta, \sigma - \Delta\sigma)$ is the transition density function as defined in Section 3. Under these conditions, the results of Mishchenko's pursuit problem, which is summarized in Section 3, can be applied.

5.3 Procedure of Obtaining Probability Function

Let $\phi_u(\sigma, \xi, T_2)$ be the probability of entering the good performance zone at some $t \in [\sigma, T_2]$, $0 \leq \sigma < T_2$, given $z(\sigma) = \xi$. According to Mishchenko, ψ_u also satisfies the backward diffusion equation

$$\frac{\partial \psi_u}{\partial \sigma} + \sum_i [A\xi]_i \frac{\partial \psi_u}{\partial \xi_i} + \frac{1}{2} \sum_j \sum_k b_{jk} \frac{\partial^2 \psi_u}{\partial \xi_j \partial \xi_k} = 0 \tag{54}$$

$$\lim_{\|\xi\| \to r_3} \psi_u(\sigma, \xi, T_2) = 1 \quad \text{for } \sigma \in [0, T_2]$$

$$\lim_{\sigma \to T_2} \psi_u(\sigma, \xi, T_2) = 0 \quad \text{for } \|\xi\| > r_3.$$

Then ψ_u can be obtained as follows:

(1) Determine the transition density function of the z-process. In the present case, the sample vector dn has special properties by hypothesis. Since the system is linear, it is known that the state vector z is gaussian distributed. As discussed in Section 5.2, the hypothetical conditions on the n-process in the present also lead to the conclusion that the z-process is Markovian. Thus the transition density of z-process can be written as

$$p(\zeta, \tau | \xi, \sigma) = \frac{1}{(2\pi)^{3/2} (\det|Q|)^{1/2}} e^{-\frac{1}{2}(\zeta-\mu)' Q^{-1} (\zeta-\mu)} \tag{55}$$

where $\zeta = z(\tau)$, $\xi = z(\sigma)$, $\tau > \sigma$

$\mu = E[\zeta|\xi] = E[z(\tau)|z(\sigma)] = \Phi(\tau,\sigma) z(\sigma)$,

$\Phi(\tau,\sigma)$ = 3 by 3 matrix satisfying

$d\,\Phi(\tau,\sigma)/d\tau = A\,\Phi(\tau,\sigma)$ and $\Phi(\sigma,\sigma) = I$,

$$Q = \int_\sigma^\tau \Phi(\tau,t)\, C\, W\, C'\, \Phi(\tau,t)'\, dt,$$

$$W = \lim_{\delta t \to 0} \frac{E(\delta n)(\delta n)'}{\delta t}$$

Note that as $\sigma \to \tau$, Q approaches $C\,W\,C'$ which is equal to $\delta t[b_{ij}]$, where b_{ij} is defined by equation (53).

(2) Determine the eigenvalues λ_1, λ_2, and λ_3 for the 3 by 3 symmetric positive definite covariance matrix $[b_{ij}]$. These eigenvalues are required for the determination of the ellipsoid

$$\sum_{i=1}^{3} \lambda_i \, w_i^{\,2} = r_3^{\,2} \qquad , \tag{56}$$

where $w_i = \sum_{j=1}^{3} M_{ij}\, \xi_j / \sqrt{\lambda_i}$, which is a mapping of the sphere $\sum_{i=1}^{3} \xi_i^{\,2} = r_3^{\,2}$ when $\sum_i \sum_j b_{ij}\, \partial^2 \psi_u / \partial \xi_i \partial \xi_j = 0$ is transformed into the Laplace equation $\sum_i \partial^2 \psi_u / \partial w_i^{\,2} = 0$. Here M is a rotation matrix whose i-th column is the orthonormalized eigenvector associated with λ_i.

(3) Determine the eigenfunction $\nu_o(\tilde{w})$ satisfying

$$\nu(\tilde{w}) = \frac{1}{2\pi} \int_S \frac{\cos\theta}{\rho^2(\hat{w},\tilde{w})} \, \nu(\hat{w}) \, ds \tag{57}$$

where S = closed surface of the ellipsoid defined by equation (56),

$\hat{w}$, $\tilde{w}$ = any points on S,

$\rho(\hat{w},\tilde{w})$ = distance between two points $\tilde{w}$ and $\hat{w}$.

θ = angle between the vector ρ from $\hat{w}$ to $\tilde{w}$ and the outward normal to S at $\hat{w}$.

This is a Fredholm equation of the second kind. No analytical solution is known. To determine $\nu(\tilde{w})$ numerically, the surface integral must be expressed as ordinary double integrals. From equation (56), the ellipsoid can be written as

$$w_3 = \pm \sqrt{r_3^{\,2} - \lambda_1 w_1^{\,2} - \lambda_2 w_2^{\,2}} \tag{58}$$

Then, equation (57) can be reduced to

$$\nu(\tilde{w}) = \pm \frac{1}{2\pi} \iint_D \frac{\cos\theta}{\rho^2(\hat{w},\tilde{w})} \nu(\hat{w}) \sqrt{1 + \left(\frac{\partial \hat{w}_3}{\partial \hat{w}_1}\right)^2 + \left(\frac{\partial \hat{w}_3}{\partial \hat{w}_2}\right)^2}\, d\hat{w}_1\, d\hat{w}_2 \tag{59}$$

where the sign is determined by the projection of the outward normal to S at $\hat{w}$ onto the w_3-axis, and D is the projection of S onto (w_1,w_2)-plane. It is known [1, 24] that there exists a unique eigenfunction which satisfies (57), and hence (59). Thus a numerical solution of $\nu(\tilde{w})$ gives the desired $\nu_o(\tilde{w})$.

(4) Determine the constant α from the formula

$$\alpha = \int_S \nu_o(\tilde{w})\, dS \Big/ \int_S \frac{\nu_o(\tilde{w})}{r(\tilde{w})}\, dS \tag{60}$$

where $r(\tilde{w})$ is the distance from $\tilde{w}$ to the origin. For the purpose of numerical evaluation, equation (60) is written in terms of ordinary double integrals:

$$\alpha = \frac{\pm \iint_D \nu_o(\tilde{w}) \sqrt{1 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_1}\right)^2 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_2}\right)^2}\, d\tilde{w}_1\, d\tilde{w}_2}{\pm \iint_D \frac{\nu_o(\tilde{w})}{r(\tilde{w})} \sqrt{1 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_1}\right)^2 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_2}\right)^2}\, d\tilde{w}_1\, d\tilde{w}_2}$$

(5) Determine the system probability function $\Gamma_o(\sigma,\xi,\tau)$ from the formula

$$\Gamma_0(\sigma,\xi,\tau) = \frac{\alpha}{\sqrt{[\xi-y(\sigma)]'[\beta_{ij}][\xi-y(\sigma)]}} - \int_R \frac{\alpha\sqrt{\lambda_1\lambda_2\lambda_3}}{[\pi(\tau-\sigma)^{3/2}\sqrt{\zeta'[\beta_{ij}]\zeta}} e^{g}\, d\zeta$$

$$g = -\frac{[\zeta-\xi+y(\sigma)]'[\beta_{ij}][\zeta-\xi+y(\sigma)]}{2(\tau-\sigma)} \tag{61}$$

for $0 \leq \sigma < \tau$ where

$\xi = z(\sigma)$ as defined before,

$\zeta = z(\tau)$

$[\beta_{ij}] = [b_{ij}]^{-1}$,

R = 3-dimensional Euclidean space,

$()'$ = transpose of $()$.

(6) Determine the partial derivatives

$$\frac{\partial\Gamma_0(\sigma,\xi,\tau)}{\partial\xi_i} \tag{62}$$

from equation (61).

(7) Determine the controlled probability function

$$\Gamma_1(\sigma,\xi,\tau) = \int_\sigma^\tau \left\{ \int_R p(v,s|\xi,\sigma)[Av-Ay(s)+Bu(s)]' \frac{\partial\Gamma_0(s,v,\tau)}{\partial v} dv\right\} ds \tag{63}$$

where $v = z(s)$ for $s\epsilon[\sigma,\tau]$,

$y(s)$ = state vector satisfying equation (51),

$u(s)$ = control vector appearing in equation (51),

$\Gamma_0(s,v,\tau)$ = system probability function defined by equation (61),

$p(v,s|\xi,\sigma)$ = transition density given in equation (55).

(8) Determine the desired probability function for Mode 2 of operation:

$$\psi_u(\sigma,\xi,T_2) = r_3[\Gamma_0(\sigma,\xi,T_2) + \Gamma_1(\sigma,\xi,T_2)] + O(r_3) \tag{64}$$

where $\psi_u(\sigma,\xi,T_2)$ was defined in Section 5.3,

$O(r_3)$ is such that $\lim_{r_3 \to 0} O(r_3)/r_3 = 0$, $\xi = z(\sigma)$.

Thus equation (64) gives an approximate solution for the backward diffusion equation (54) within an error of the order of r_3.

5.4 The Optimal Control

Once the probability function $\psi_u(\sigma,\xi,T_2)$ is available, the optimization of the control vector u can be achieved by an application of the maximal principle [1, Chap. I]. Since $\Gamma_o(\sigma,\xi,T_2)$ does not depend on u and r_3 is a positive constant, then from equation (64),

$$\max_{u \in \Omega} \psi_u(0, \bar{x}, T_2) = \max_{u \in \Omega} \Gamma_1(0, \bar{x}, T_2)$$

where $\bar{x} = x(0) = z(0)$,

Ω = a restraint set with $\Omega \subset \Delta$,

Δ = class of admissible controls.

The maximization process is subject to the constraint of the deterministic system (51). Let

$$y_o(t) = -\int_o^t \left\{ \int_R p(v,s|\bar{x},0)[Av-Ay(s) + Bu(s)]' \frac{\partial \Gamma_o}{\partial v} dv \right\} ds$$

where $$\frac{\partial \Gamma_o}{\partial v} = \frac{\partial \Gamma_o(s,v,T_2)}{\partial v},$$

then $$\dot{y}_o = - F(y,t) - [B\,u(t)]' \int_R p(v,t|\bar{x},0) \frac{\partial \Gamma_o}{\partial v} dv$$

where $$F(y,t) = \int_R p(v,t|\bar{x},0)[Av-Ay(t)]' \frac{\partial \Gamma_o}{\partial v} dv \qquad (65)$$

The Hamiltonian is

$$H = F(y,t) + [B\,u(t)]' \int_R p(v,t|\bar{x},0) \frac{\partial \Gamma_o}{\partial v} dv + \phi(t)'[Ay(t) - Bu(t)]. \qquad (66)$$

where $\phi(t)$ is an adjoint vector associated with the deterministic system (51).

For the system with fixed terminal time and free end point, it is known that $\phi(T_2) = 0$ from the transversality condition [1]. The adjoint vector plays an important role in the optimal control theory and hence it must be determined. With $y_0(0) = 0$, $y(0) = \phi(T_2) = 0$ and $\phi_0(T_2) = -1$, $\phi(t)$ can be obtained by solving the two-point boundary value problem. A great variety of computational schemes have been proposed for solving this type of problems with various classes of performance indices (the functional to be either maximized or minimized) and different Ω. Most recent text books on the relevant subject are those by Fel' Dbaum [25], by Athans and Falb [26], and by Lee and Markus [27].

Two different cases of optimal controls are considered as follows:

(1) If Ω : $|u_i(t)| \leq 1$, i=1, ..., k for $k \leq 3$, then from equation (19), max H can be achieved if

$$u^*(t) = -\operatorname{sgn}\left\{B'[\phi(t) - \int_R p(v,t|\bar{x},0) \frac{\partial \Gamma_0}{\partial v} dv]\right\} \tag{67}$$

for $t \in [0, T_2]$ where

$$\operatorname{sgn} x = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ if } x < 0, \end{cases}$$

and $u^*(t)$ is an optimal controller which yields max $\Phi_u(0, \bar{x}, T_2)$. However, when

$B'[\phi(t) - \int_R p \frac{\partial \Gamma_0}{\partial v} dv] = 0$ on some subinterval of $[0, T_2]$, $u^*(t)$

is not defined and a singular control problem results. In equation (63), the formula for Γ_1, the integrand contains a linear term of u. Singular solutions in this type of problems were discussed by various authors [28, 29].

(2) If $\Omega = \Delta$ and if control energy $\int_0^{T_2} u' U u \, dt$, where U is a k by k positive definite matrix with $k \leq 3$, must be minimized simultaneously, then the Hamiltonian is formed as $\tilde{H} = H + u' U u$ where H is defined by equation (66). Thus $d \tilde{H}/dt = 0$ yields

$$u^*(t) = [U + U']^{-1} B' [\phi(t) - \int_R p(v,t|\bar{x},0) \frac{\partial \Gamma_o}{\partial v} dv] \qquad (68)$$

for $t \in [0, T_2]$, where $u^*(t)$ is an optimal controller which yields

$$\max [\psi_u(0, \bar{x}, T_2) - \int_0^{T_2} u' U u \, dt].$$

6. CONCLUSIONS

The analysis presented in this report leads to a scheme of synthesizing the optimal controller for the antenna pointing direction and the height of the space vehicle subject to random disturbance. The analysis is an application of Mishchenko's pursuit problem. The computational formulas which are required for the synthesis are presented. Digital computer programs and their flow charts for the evaluation of double integrals and eigenfunctions satisfying the Fredholm equation of the second kind are included in the Appendix.

7. PLAN OF FUTURE WORK

The immediate step will be a simulation study of the antenna pointing system on a digital computer with some data package related to existing vehicles. The convergence problem of the computational scheme will also be investigated.

Problems of time-varying systems with state-dependent noise will also be studied. The computation for these systems is an extension of the results in this report but is much more involved.

The extension of the method to higher order systems, especially those in which the target manifold has a dimension less than that of the system manifold [30], will be examined closely. Attention will be focused on any hidden pitfalls. Once this is completed, numerical data for a physical space vehicle will be used as a test model for the computational method. A comparison of the results so obtained against those from existing control systems is also planned.

The future theoretical studies include the following items:

(1) The sufficiency condition on the noise that guarantees the response of the dynamical system being Markovian and satisfying the hypothesis of the pursuit problem will be determined.

(2) The convergence of sequence $\{p_{I^i_{\xi\eta}}\}$, which was discussed briefly in Section 4.3, will be established regorously.

(3) Along the same line, an error estimate for

$p_{I^i_{\xi\eta}} - p_{I_{\xi\eta}}$ will be developed.

(4) Methods of evaluating (m-1)-dimensional surface integrals will be investigated by means of the tensor analysis.

9. APPENDIX-NUMERICAL EVALUATION OF SURFACE INTEGRALS

In the following, a numerical method for the evaluation of surface integrals in the 3-dimensional phase-coordinate system is presented. The method is developed for the evaluation of equation (60), Section 5.3,

$$\alpha = \frac{\pm \iint_D v_o(\tilde{w})\sqrt{1 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_1}\right)^2 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_2}\right)^2} \; d\tilde{w}_1 \, d\tilde{w}_2}{\pm \iint_D \frac{v_o(\tilde{w})}{r(\tilde{w})}\sqrt{1 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_1}\right)^2 + \left(\frac{\partial \tilde{w}_3}{\partial \tilde{w}_2}\right)} \; d\tilde{w}_1 \, d\tilde{w}_2} \tag{69}$$

where $\tilde{w} \in S$,

S is the surface defined by

$$\lambda_1 w_1^{\,2} + \lambda_2 w_2^{\,2} + \lambda_3 w_3^{\,2} = r_3^{\,2} \tag{70}$$

D is the projection of S onto the (w_1, w_2)-plane,

$$r(\tilde{w}) = \left[\tilde{w}_1^{\,2} + \tilde{w}_2^{\,2} + \tilde{w}_3^{\,2} \right]^{1/2} , \tag{71}$$

$\nu_o(\tilde{w})$ is the eigenfunction satisfying

$$\nu(\tilde{w}) = \frac{1}{2\pi} \int_S \nu(\hat{w}) \; \frac{\rho(\hat{w},\tilde{w}) \cdot dS(\hat{w})}{\| \rho(\hat{w},\tilde{w}) \|^3} \tag{72}$$

in which $\hat{w}$ and $\tilde{w} \in S$,

$$\rho = \hat{w} - \tilde{w}, \text{ (a distance vector)} \tag{73}$$

$dS(\hat{w})$ is the surface element vector with a direction pointing outward at $\hat{w} \in S$.

It is known that

$$dS(\hat{w}) = (a_1, a_2, 1)\; d\hat{w}_1 \; d\hat{w}_2 \tag{74}$$

where

$$a_i = \frac{(\lambda_i/\lambda_3)\,\hat{w}_i}{\left[\left(r_3^{\,2} - \lambda_1\hat{w}_1^2 - \lambda_2\hat{w}_2^{\,2}\right) / \lambda_3 \right]^{1/2}} , \qquad i=1,2. \tag{75}$$

Combine equations (70), and (73) through (75), equation (72) may be written as a double integral

$$\nu(\tilde{w}_1,\tilde{w}_2) = \frac{1}{\pi} \int_{-1/\sqrt{\lambda_1}}^{1/\sqrt{\lambda_1}} \left\{ \int_{-\mu}^{\mu} F(\hat{w}_1,\hat{w}_2,\tilde{w}_1,\tilde{w}_2)\,\nu(\hat{w}_1,\hat{w}_2)\,d\hat{w}_2 \right\} d\hat{w}_1 \tag{76}$$

where $\mu = \left[(r_3^{\,2} - \lambda_1\hat{w}_1^{\,2})/\lambda_2\right]^{1/2}$, (77)

$$F = \frac{(\hat{w}_1 - \tilde{w}_1)a_1 + (\hat{w}_2 - \tilde{w}_2)a_2 + (\hat{w}_3 - \tilde{w}_2)}{\left[(\hat{w}_1 - \tilde{w}_1)^2 + (\hat{w}_2 - \tilde{w}_2)^2 + (\hat{w}_3 - \tilde{w}_3)^2\right]^{3/2}} \tag{78}$$

A numerical method for the computation of the eigenfunction $\nu_o(\tilde{w}_1, \tilde{w}_2)$ associated with equation (76) is as follows:

(a) Discretization of $\hat{w}_1$, $\hat{w}_2$, $\tilde{w}_1$, and $\tilde{w}_2$.

$$\hat{w}_j^i = -r_3/\sqrt{\lambda_j} + (i-1)\Delta w_j, \tag{79}$$

$$\tilde{w}_j^i = -r_3/\sqrt{\lambda_j}\ (i-1)\Delta w_j, \tag{80}$$

for $i=1,2, \ldots, M$, and $j=1,2$, where

$$\Delta w_j = \left[2/(M-1)\sqrt{\lambda_j}\right] r_3 . \tag{81}$$

(b) Discretization of F and ν.

$$\text{Let } F^{khij} = F(\hat{w}_1{}^k, \hat{w}_2{}^h, \tilde{w}_1{}^i, \tilde{w}_2{}^j) \tag{82}$$

$$\nu^{ij} = \nu(\tilde{w}_1{}^i, \tilde{w}_2{}^j) \tag{83}$$

Then by (79) through (81), equation (76) yields

$$\nu^{ij} = \Delta w_1 \Delta w_2 \sum_{k=1}^{M-1} \sum_{h=h^\bullet(k)}^{h^f(k)} F^{k\,h\,i\,j}\ \nu^{kh} \tag{84}$$

where

$$h^\bullet(k) = 1 + \frac{M-1}{2}\left\{1 - \sqrt{\lambda_2\left[1 - \frac{1}{\lambda_1}\left(-1 + \frac{2k-2}{M-1}\right)^2\right]}\right\}, \tag{85}$$

$$h^f(k) = 1 + \frac{M-1}{2}\left\{1 + \sqrt{\lambda_2\left[1 - \frac{1}{\lambda_1}\left(-1 + \frac{2k-2}{M-1}\right)^2\right]}\right\}. \tag{86}$$

(c) Rearrangement of ν^{ij}

The following is a rearrangement of the two-dimensional array ν^{ij}

into the one-dimensional array $\bar{\nu}^p$ so that it can be computed as an eigenvector:

$$\bar{\nu}^1 = \nu^{1,h^\circ(1)}$$

$$\bar{\nu}^2 = \nu^{1,h^\circ(1) + 1}$$

$$\vdots$$

$$\bar{\nu}^{h^f(1) - h^\circ(1) + 1} = \nu^{1,h^f(1)}$$

$$\bar{\nu}^{h^f(1) - h^\circ(1) + 2} = \nu^{2,h^\circ(2)}$$

$$\vdots$$

$$\bar{\nu}^{h^f(1) + h^f(2) - h^\circ(1) - h^\circ(2) + 2} = \nu^{2,h^f(2)}$$

$$\vdots$$

$$\bar{\nu}^Q = \nu^{q + 1,h^\circ(q + 1) + s - 1}$$

$$\vdots$$

where $Q = \sum_{i=1}^{q} \left[h^f(i) - h^\circ(i)\right] + q + s$; $q = 0,1,\ldots,M-2$; $1 \leq s \leq h^f(q+1) - h^o(q+1) + 1$.

Thus the mapping $\nu^{ij} \to \bar{\nu}^p$ is defined by $(i,j) \xrightarrow{g} p$ with

$$g : p = \sum_{\beta=1}^{i-1} \left[h^f(\beta) - h^\circ(\beta)\right] + i + j - h^\circ(i) \tag{87}$$

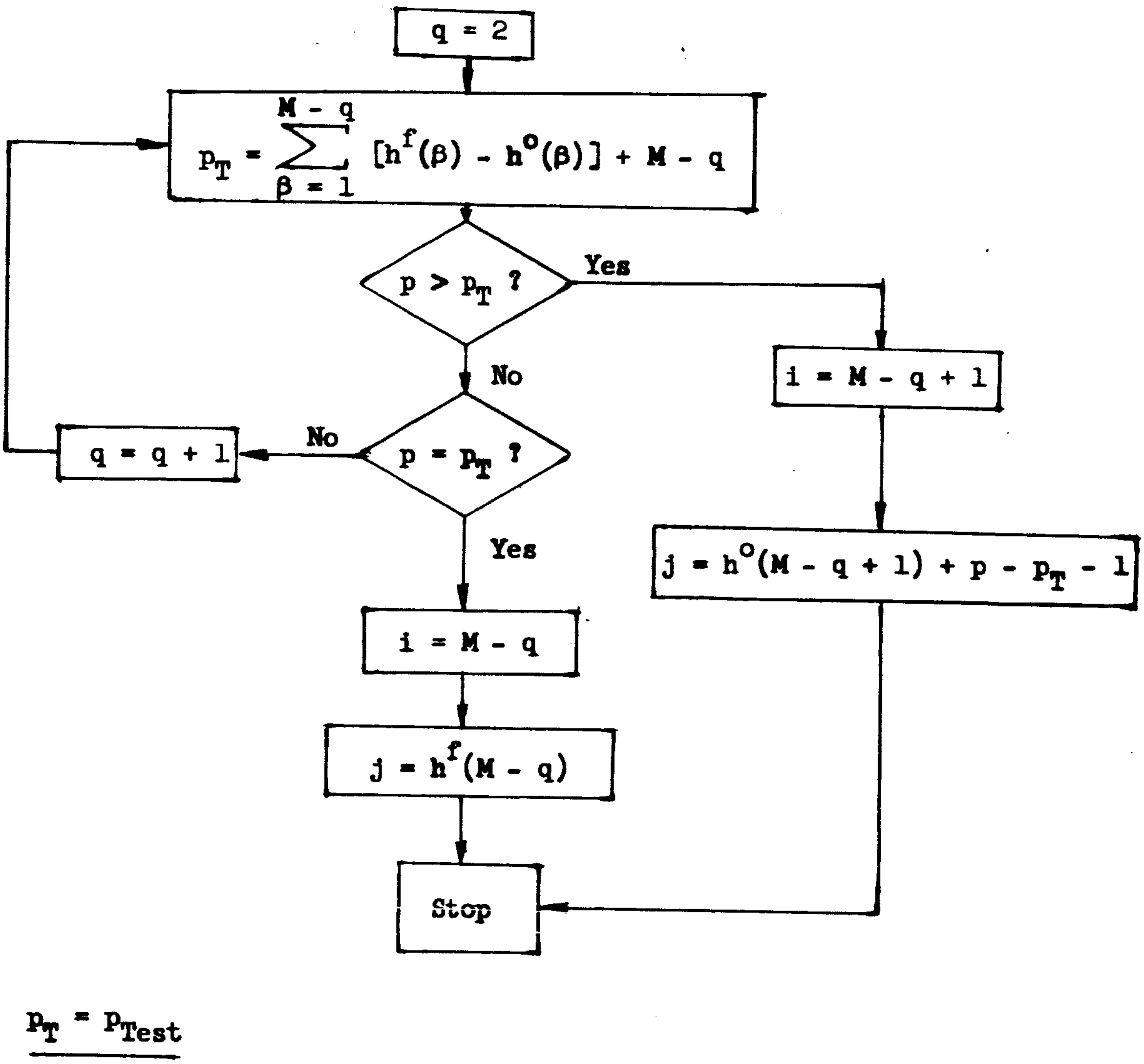
(d) The inverse mapping $p \xrightarrow{g^{-1}} (i,j)$ is given by the algorithm represented by the flow chart in Figure 8. Under the mapping g, equation (84) becomes

$$\bar{\nu}^q = \sum_{\beta=1}^{\gamma} G^{q\beta} \bar{\nu}^\beta \tag{88}$$

where

$$(i,j) \xrightarrow{g} q \tag{89}$$

$$(k,h) \xrightarrow{g} \beta \tag{90}$$



q = 2
$p_T = \sum_{\beta=1}^{M-q} [h^f(\beta) - h^o(\beta)] + M - q$
$p > p_T$?
Yes
No
$i = M - q + 1$
$q = q + 1$
No
$p = p_T$?
Yes
$j = h^o(M - q + 1) + p - p_T - 1$
$i = M - q$
$j = h^f(M - q)$
Stop
$p_T = p_{Test}$

FIGURE 8

$$G^{q\,\beta} = F^{khij}\,\Delta w_1\,\Delta w_2 \tag{91}$$

$$\gamma = \sum_{\beta=1}^{M-1}\left[h^f(\beta) - h^*(\beta)\right] + M - 1 \tag{92}$$

Thus the eigenvector $\left[\bar{v}_o{}^q\right]$ associated with the matrix $\left[G^{q\,\beta}\right]$ represents a discrete approximation to the eigenfunction $v_o(\tilde{w}_1, \tilde{w}_2)$.

(e) The eigenvector mentioned above can be obtained as follows. A prime consideration in the selection of an algorithm for this task is the dimensionality of the matrix $\left[G^{q\,\beta}\right]$. The total number of elements of the matrix is $(M-1)^4$. Thus, if the w_1, w_2 axes are discretized into 20 segments, the matrix contains 160,000 elements. This makes it impractical to attempt the storage of the matrix. For this reason, the elements will be computed as they are needed, using the inverse mapping with the explicit expressions for $G^{q\,\beta}$ given by (91) with associated equations (79) through (90).

(f) The computational algorithm is based on an iterative technique for determining the maximum eigenvalue and associated eigenvector of a given matrix [31]. Figure 9 shows a flow chart which describes the algorithm. After the execution of the iterative cycle four times, the computed data are checked against a specified accuracy requirement (ACC). If the requirement is met, the process is terminated. Otherwise the process enters another iterative cycle. The process is also terminated if the required accuracy has not been attained after a predetermined number of iterations (LSTOP).

The above describes the procedure of determining v^{ij}. The computation of α given by equation (69) is straightforward. By a similar procedure, (69) can be discretized as

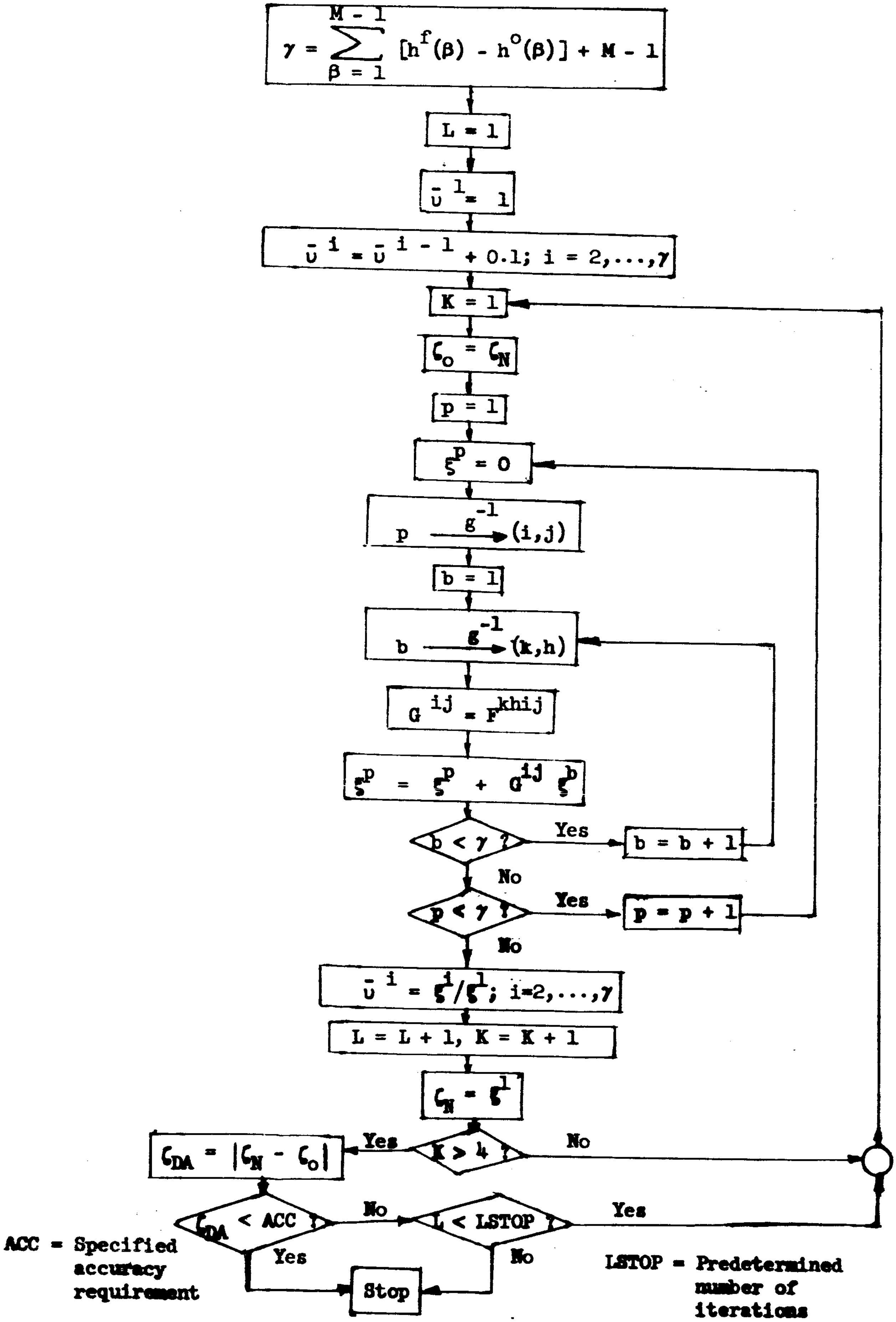

L = 1
K = 1
p = 1
b = 1
Yes
No
b = b + 1
p = p + 1
L = L + 1, K = K + 1
Stop
ACC = Specified accuracy requirement
LSTOP = Predetermined number of iterations

FIGURE 9

$$\alpha = \frac{\displaystyle\sum_{k=1}^{M-1} \sum_{h=h^{*}(k)}^{h^{f}(k)} \nu^{kh} \sqrt{E}}{\displaystyle\sum_{k=1}^{M-1} \sum_{h=h^{*}(k)}^{h^{f}(k)} \nu^{kh} \sqrt{E} \Big/ r(\tilde{w}_1{}^{k}, \tilde{w}_2{}^{h}, \tilde{w}_3)} \tag{93}$$

where

$$E = 1 + \left[a_1(\tilde{w}_1{}^{k}, \tilde{w}_2{}^{h})\right]^2 + \left[a_2(\tilde{w}_1{}^{k}, \tilde{w}_2{}^{h})\right]^2 , \tag{94}$$

$$\tilde{w}_3 = \sqrt{\left[r_3{}^2 - \lambda_1(\tilde{w}_1{}^{k})^2 - \lambda_2(\tilde{w}_2{}^{h})^2\right] / \lambda_3}. \tag{95}$$

Figure 10 shows a flow chart which describes the algorithm for computing α by equation (93).

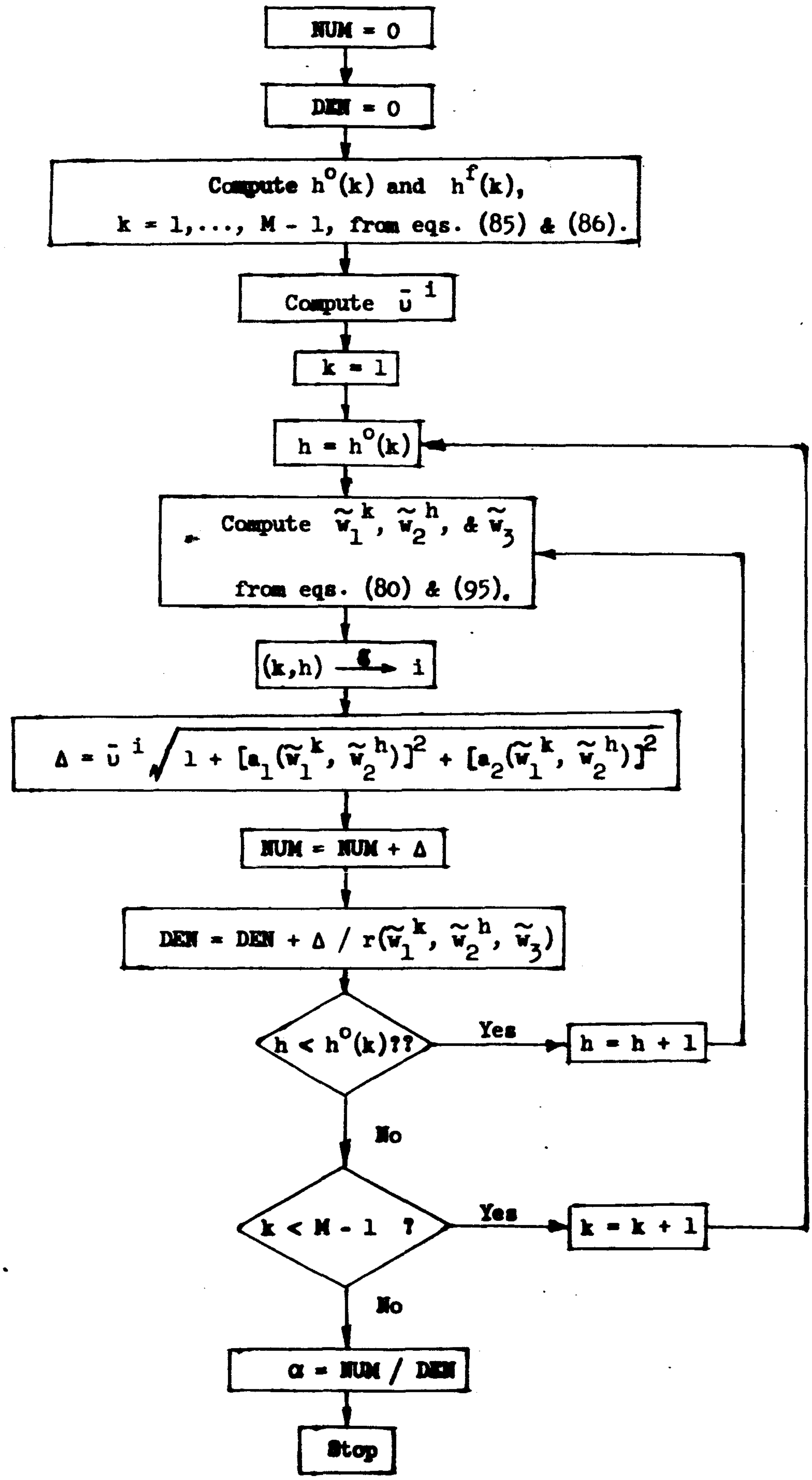

NUM = 0
DEN = 0
Compute h^o(k) and h^f(k),
k = 1,..., M - 1, from eqs. (85) & (86).
Compute ῡ^i
k = 1
h = h^o(k)
Compute w̃_1^k, w̃_2^h, & w̃_3
from eqs. (80) & (95).
(k,h) —ξ→ i
Δ = ῡ^i √(1 + [a_1(w̃_1^k, w̃_2^h)]^2 + [a_2(w̃_1^k, w̃_2^h)]^2)
NUM = NUM + Δ
DEN = DEN + Δ / r(w̃_1^k, w̃_2^h, w̃_3)
h < h^o(k)??
Yes
h = h + 1
No
k < M - 1 ?
Yes
k = k + 1
No
α = NUM / DEN
Stop

FIGURE 10

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