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Large Deflection and Stability Analysis by the Direct Stiffness Method

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PREFACE

This Report summarizes research conducted by the author as a consultant to the Jet Propulsion Laboratory. The author is Professor of Aeronautical and Astronautical Engineering at the University of Washington, Seattle, Washington. This work was originally prepared while the author was on leave of absence (1962–63) as Visiting Professor of Structural Mechanics in the Department of Civil Engineering, University of Hawaii, Honolulu, Hawaii.

CONTENTS

I. Introduction 1

II. Some Comments on the Nonlinear Theory of Elasticity 2

A. General Discussion 2
B. Deformation of Volume Element 2
C. Strain–Displacement Equations 3
D. Physical Strains 4
E. Rotations 4
F. Theory of Small Deformations 5
G. Small Deformations and Small Angles of Rotation 5
H. Reduction to Classical Theory 6
I. Equilibrium Equations 6
J. Concluding Comments 7

III. The Axial Force Member 8

A. Examples—Nonlinear Behavior 8
B. Stiffness Matrix—Axial Force Member 9
C. Application to Simple Problems 12
D. Concluding Comments 15

IV. The Piecewise Linear Calculation Procedure 15

A. Incremental Step Procedure—Discussion 15
B. Simple Truss—Exact Solution 16
C. Simple Truss—Incremental Step Solution 16

V. The Beam-Column 19

A. Present State of Development 19
B. Stiffness Matrix—Beam-Column 19
C. Column Stability—Example 22
D. Tie Rod Deflection—Example 23
E. Comments 24

VI. Stability Calculations 25

A. Preliminary Comments 25
B. Initial Loading Intensity Factor 25
C. Plotting the Determinant of the Stiffness Matrix 25
D. Adaptation of Southwell's Method 26
E. Stability of Rectangular Plate—Example 26
F. Matrix Iterative Procedure 28
G. Stability of Tapered Column 28
H. Additional Cases 29

CONTENTS (Cont'd)

VII. The Thin Plate Element 29

A. Introductory Remarks 29

B. Membrane Stiffness Matrices 29

C. Bending Stiffness Matrices 29

D. Initial Force Stiffness Matrices 30

Nomenclature 31

References 31

Table 1. Exact solution data—truss of Fig. 1 16

Table 2. Computer results—rectangular plate stability problem 27

FIGURES

1. Extensible rod–spring system 8
2. Load–displacement curves 8
3. Heated beam-column 9
4. Beam-column 9
5. Tension -compression member (stringer) 10
6. Arbitrarily oriented tension–compression member 12
7. Nonlinear truss 12
8. Truss stability problem 14
9. Solutions for truss of Fig. 1 16
10. Beam-column nodal forces and displacements 19
11. Loading for simple beam-column 23
12. Extrapolation for critical load 26
13. Application of Southwell's method 26
14. Rectangular plate stability problem 26
15. Idealized plate quadrant containing 32 triangular elements 27
16. Southwell type plot for plate stability problem 28
17. Tapered column 28

ABSTRACT

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The application of the direct stiffness method in the solution of large deflection and stability problems is demonstrated. Discussed first are the basic elasticity equations that contain higher order terms to account for the nonlinear character of large deflections and large rotations. Various simplifications of these equations are made, and the conditions and limitations of the resulting expressions are discussed.

Next, examples of geometrically nonlinear systems are presented to illustrate the importance of considering the nonlinear behavior in many problems.

Finally, the modifications to the stiffness matrix, and the method of formal solution are derived for some of the nonlinear examples that have been discussed.

Author

I. INTRODUCTION

The basic purpose of this Report is to show how the direct stiffness method may be extended to apply to geometrically nonlinear structural problems. Of particular interest are problems involving the deformation of bodies having initial stresses (such as those due to heating) and problems in structural stability. As will be seen subsequently, the conventional stiffness procedure is incapable of handling such problems. Not only must new stiffness matrices be derived, but the method for using these must also be established.

A piecewise linear (incremental step) procedure will be introduced for determining displacements in the large deflection problem. Stability of complex structural systems will be put into a mathematical form such that the widely used matrix iterative procedure may be applied in determining critical loadings and corresponding mode shapes.

The basic element stiffness matrices, plus the piecewise linear usage of these, will yield the overall instantaneous structural stiffness matrix. This in turn will enable the structural dynamicist to determine the natural frequencies and principal modes of the prestressed, and subsequently loaded, structure. A case in point is a body subjected to nonuniform thermal gradients and then acted on by external static and inertia forces. In this instance the self-equilibrating, initial (thermal) stresses must be taken into account when calculating structural stiffness against the subsequently applied external loadings.

It must be emphasized that the direct stiffness method is essentially a matrix numerical procedure intended to be carried out on high speed digital computing equipment. It has inherent characteristics which make it highly suited to such implementation. By the same token, it is only suitable for the simplest problems when hand calculations are to be used. It is nevertheless very important that carefully chosen problems, sufficiently simple for hand calculation, be devised to illustrate the calculation procedure. Such calculations are included in this Report.

II. SOME COMMENTS ON THE NONLINEAR THEORY OF ELASTICITY

A. General Discussion

This section of the Report presents a brief outline of the classical nonlinear theory of elasticity. It has been abstracted from Ref. 1.

A thorough familiarity with the basic nonlinear theory is not essential for reading subsequent sections of this Report. Some of the concepts and resulting equations from the nonlinear theory will prove to be useful, however.

Nonlinearity is introduced into the theory of elasticity in three ways:

1. Through the strain–displacement equations
2. Through the equilibrium equations
3. Through the stress–strain equations

In the stress–strain equations, the nonlinear terms appear when the strains exceed the proportional limit of the material. This condition is therefore termed physical nonlinearity. It is not treated in this Report.

In the first two sets of equations listed above, the nonlinear terms arise from geometrical considerations. The fundamental factor is this: the angles of rotation must be taken into account in determining, (a) changes in length of line elements, and (b) formulating the conditions of equilibrium of the volume element. These are termed geometric nonlinearities and underlie the basic problems to be treated in this Report.

Physical and geometric nonlinearities are independent of each other. For example, smallness of angles of rotation does not imply smallness of elongations and shears and vice versa. As a result, geometric linearity can exist in the presence of physical nonlinearity. The converse is also true.

With these facts in mind, the several types of elasticity problems may be listed as follows:

1. Problems which are both physically and geometrically linear
2. Problems which are physically nonlinear but geometrically linear
3. Problems which are physically linear but geometrically nonlinear
4. Problems which are both physically and geometrically nonlinear

The first type represents the classical elasticity problem. It exists when angles of rotation are of the same order of magnitude as elongations and shears, and when the latter are small compared to unity. Also, strains do not exceed the limit of proportionality.

The third type is the one of interest here. Again, strains are assumed to not exceed the proportional limit, but rotations may be large. This means that rotations may not be neglected in either the strain–displacement equations, or in writing the equilibrium equations. A thin, flexible, steel strip bent into a full circle is an example of this type of problem. Less obvious cases will be mentioned shortly.

The general problem types which can only be investigated by using the nonlinear (geometric) theory of elasticity are as follows:

1. Stability of elastic equilibrium
2. Deformation of bodies having initial stresses
3. Large deflection of rods
4. Torsion and bending in presence of axial forces
5. Bending of plates and shells under deflections of the order of magnitude of the thickness

B. Deformation of Volume Element

An infinitesimal volume element at a point in a body deforms according to a linear relationship having constant coefficients. Without writing the mathematical expressions it is merely stated here that, as a consequence of this, a sphere in the undeformed body will become an ellipsoid in the strained body. Also, a rectangular parallelepiped will deform into a skewed parallelepiped. In other words, the faces of the parallelepiped will change area and also rotate as deformation occurs. This deformation behavior has important consequences when the equilibrium equations are being formed. Translation and rotation of the infinitesimal volume element are not the governing characteristics defining its behavior. Instead, this depends on the strains–elongations and shears. The relationships will be given subsequently.

On the other hand, for the total body the displacement and rotation are the conventional terms describing its deformation behavior. As a result, a duality exists concerning the description of the deformed state. A consequence is that two different interpretations may be given for "small deflections." It can mean either smallness of elongations and shears (compared to unity), or smallness of displacement (compared to the linear dimensions of the body) and smallness of angles of rotation (compared to unity). The latter definition is the more restrictive; when it holds, so does the former. The converse is not true.

In nonlinear elasticity, when "small deformations" are specified, it is understood that this applies to the volume element. In other words, the term applies to the microscopic, rather than the macroscopic, element. If, on the other hand, displacements and rotations are also meant to be small, this will be explicitly stated. Clarity of meaning can thereby be preserved.

C. *Strain–Displacement Equations*

To illustrate terms and the nonlinear form of these relations, the equations will be derived. Let

- x, y, z be the coordinates of a point in the undeformed body
- u, v, w be the displacement components of the point x, y, z
- ξ, η, ζ be the coordinates of the point after deformation

The causes bringing about the deformation need not be specified. It follows that

$$\left.\begin{aligned} \xi &= x + u(x,y,z) \\ \eta &= y + v(x,y,z) \\ \zeta &= z + w(x,y,z) \end{aligned}\right\} \tag{1}$$

To a first order of magnitude, the increments in ξ, η, and ζ are given by

$$\left.\begin{aligned} d\xi &= dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\ d\eta &= dy + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\ d\zeta &= dz + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \end{aligned}\right\} \tag{2}$$

It is now convenient to define partial derivatives in Eqs. (2) in terms of symbols e_{ij} and ω_i as follows:

$$\begin{aligned} &e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{zz} = \frac{\partial w}{\partial z} \\ &e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ &e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \tag{3a}$$

and also,

$$\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned} \tag{3b}$$

Substituting these last expressions into Eqs. (2),

$$\begin{aligned} d\xi &= (1 + e_{xx})\,dx + \left(\frac{1}{2}e_{xy} - \omega_z\right)dy + \left(\frac{1}{2}e_{xz} + \omega_y\right)dz \\ d\eta &= \left(\frac{1}{2}e_{xy} + \omega_z\right)dx + (1 + e_{yy})\,dy + \left(\frac{1}{2}e_{yz} - \omega_x\right)dz \\ d\zeta &= \left(\frac{1}{2}e_{xz} - \omega_y\right)dx + \left(\frac{1}{2}e_{yz} + \omega_x\right)dy + (1 + e_{zz})\,dz \end{aligned} \tag{4}$$

As a special case, consider a line element ds_i parallel to the x-axis before deformation. Then,

$$ds_i = dx \qquad dy = dz = 0$$

After deformation, this line element becomes ds_f where

$$\begin{aligned} ds_f &= (d\xi^2 + d\eta^2 + d\zeta^2)^{1/2} \\ &= \left[(1 + e_{xx})^2\,dx^2 + \left(\frac{1}{2}e_{xy} + \omega_z\right)^2 dx^2 + \left(\frac{1}{2}e_{xz} - \omega_y\right)^2 dx^2\right]^{1/2} \end{aligned}$$

At this point, the strain E_x of line element dx due to deformation can be defined as

$$\begin{aligned} E_x &= \frac{ds_f - ds_i}{ds_i} \\ &= \left[(1 + e_{xx})^2 + \left(\frac{1}{2}e_{xy} + \omega_z\right)^2 + \left(\frac{1}{2}e_{xz} - \omega_y\right)^2\right]^{1/2} - 1 \end{aligned}$$

E_x is a true definition of strain. Alternatively we may express the above equation as

$$1 + E_x = \left[(1 + e_{xx})^2 + \left(\frac{1}{2} e_{xy} + \omega_z \right)^2 + \left(\frac{1}{2} e_{xz} - \omega_y \right)^2 \right]^{1/2} \tag{5}$$

Similar equations may be found for $1 + E_y$ and $1 + E_z$.

Returning to the arbitrarily oriented line element ds_i, it follows that

$$ds_i^2 = dx^2 + dy^2 + dz^2$$

and

$$ds_f^2 = d\xi^2 + d\eta^2 + d\zeta^2$$

Substituting into the last expression from Eqs. (4) and subtracting ds_i^2 gives

$$\begin{aligned} ds_f^2 - ds_i^2 = 2\Bigg[e_{xx} &+ \frac{1}{2} \left\{ e_{xx}^2 + \left(\frac{1}{2} e_{xy} + \omega_z \right)^2 + \left(\frac{1}{2} e_{xz} - \omega_y \right)^2 \right\} \Bigg] dx^2 \\ &+ \text{similar terms in } dy^2 \text{ and } dz^2 \qquad (6a) \\ &+ 2\left[e_{xy} + e_{xx}\left(\frac{1}{2} e_{xy} - \omega_z \right) + e_{yy}\left(\frac{1}{2} e_{xy} + \omega_z \right) + \left(\frac{1}{2} e_{xz} - \omega_y \right)\left(\frac{1}{2} e_{yz} + \omega_x \right) \right] dx\,dy \\ &+ \text{similar terms in } dy\,dz \text{ and } dx\,dz. \end{aligned}$$

This last equation is also expressible as

$$\begin{aligned} ds_f^2 - ds_i^2 = 2\,(&\varepsilon_{xx}\,dx^2 + \varepsilon_{yy}\,dy^2 + \varepsilon_{zz}\,dz^2 \\ &+ \varepsilon_{xy}\,dx\,dy + \varepsilon_{yz}\,dy\,dz + \varepsilon_{zx}\,dz\,dx) \end{aligned} \tag{6b}$$

where, ε_{xx}, etc. are implicitly defined in the previous equation, and on substituting from Eqs. (3a) and (3b), these terms may be written in the form

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \tag{7}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

These may be regarded as the nonlinear strain–displacement equations. From Eqs. (7) it is a simple matter to obtain $\varepsilon_{yy}, \varepsilon_{yz}, \cdots$, etc. by cyclical change of variables u, v, w and x, y, z. The resulting set of six equations are of great importance in deriving stiffness matrices for large deflection problems.

D. Physical Strains

Equations (6) do not define the physical strains. These are represented, for example, by E_x. On comparing Eq. (5) with Eqs. (6), it is observed that

$$(1 + E_x)^2 = 2\,\varepsilon_{xx} + 1$$

or

$$E_x = (2\,\varepsilon_{xx} + 1)^{1/2} - 1 \tag{8}$$

Equation (8) reveals the nonlinear dependence of E_x on the displacements. The specific dependence is, however, masked by the complexity of the relationship.

Up to this point, shears have not been discussed. A derivation will not be presented. However, if $\phi_{xy}, \phi_{yz}, \phi_{zx}$ are the shears, these are given by

$$\sin \phi_{xy} = \frac{\varepsilon_{xy}}{(1 + E_x)(1 + E_y)} \tag{9}$$

with similar expressions for the other two terms.

Equations (8) and (9) indicate that although $\varepsilon_{xx} \cdots$, $\varepsilon_{xy} \cdots$ are not strains, these terms do characterize the actual physical strains. In fact, elongations are characterized by $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ and shears by $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$. For example, when ε_{xy} vanishes, so does the shear ϕ_{xy}.

E. Rotations

As yet the ω terms defined in Eqs. (3b) have not been given a physical interpretation. They are associated with rotation of the volume element.

To clarify the picture it is again helpful to think of a volume element as it is displaced from its initial to its final configuration. In so doing the infinitesimal element undergoes rotation, as well as deformation. Furthermore, the rotation of the element is defined as the mean value of the rotations experienced by the totality of line elements belonging to the given volume element.

The mathematical formulation of the above definition will not be detailed here. Results of such an investigation will, however, be given.

Suppose ψ_z is the angle of rotation of a fiber normal to the z-axis, about that axis, as deformation takes place. It they are of the same order of magnitude. It is for this reason that Eqs. (12) retain rotations to the second power, as well as terms of the type e_{xx}, e_{xy}, etc. to the first power. Equations (12) are still nonlinear.

H. Reduction to Classical Theory

Inspection of Eqs. (12) reveals that the linear strain–displacement expressions of classical elasticity are obtained if all terms in the rotations are dropped. The end result is then the familiar equations,

$$\begin{gathered}\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\end{gathered} \tag{13}$$

As a consequence it is seen that the linearized relations for the strain components depend on the two sets of conditions:

a. Elongations, shears, and angles of rotation must be small compared to unity

b. Terms of the second power in angles of rotation must be small compared to the corresponding strain components (Eqs. 12)

For a massive body, i.e., one having its three linear dimensions of the same order of magnitude, condition (a) above also implies (b). However, this is not true if the body is flexible (rod, plate, or shell). Nevertheless, Eqs. (13) are applicable to some slender body problems, e.g., bending or torsion of rods in the absence of axial forces.

Therefore, for some problems (compression of a thin rod, or the bending of a thin plate), the use of the linear strain–displacement equations is inadmissible even for very small elongations and shears. On the other hand, for certain other problems (tension of a thin rod or bending of a thick plate), the linear equations are applicable even though elongations and shears are relatively much larger than in the first case. It is therefore seen that both the linear and nonlinear theories deal with finite deformations of the same order of smallness. The essential difference in the two theories, as pointed out previously, lies in the fact that the linear theory neglects the influence of rotation on elongations and shears.

I. Equilibrium Equations

In considering equilibrium it is helpful to fix attention on an elementary rectangular parallelepiped before deformation. As pointed out previously, this volume element becomes an oblique parallelepiped after deformation, with different edge lengths and volume from the original, and rotated in space relative to the original. These factors must be taken into account in establishing the equilibrium equations.

When written with this general point of view in mind, the equilibrium equations become very complex and as a result will not be given here. The equations contain terms in rotations, ratios of volume before and after deformation, and ratios of areas of faces before and after deformation.

As in the case of the strain–displacement equations, the general equilibrium equations are first simplified by imposing the condition that elongations and shears are small compared to unity. In addition, E_x, E_y, and E_z are neglected compared to one. The first equilibrium equation then reduces to the following form:

$$\begin{aligned}&\frac{\partial}{\partial x}\left[(1 + e_{xx})\,\sigma_{xx} + \left(\frac{1}{2}e_{xy} - \omega_z\right)\sigma_{xy} + \left(\frac{1}{2}e_{xz} + \omega_y\right)\sigma_{xz}\right] \\ &+ \frac{\partial}{\partial y}\left[(1 + e_{xx})\,\sigma_{yx} + \left(\frac{1}{2}e_{xy} - \omega_z\right)\sigma_{yy} + \left(\frac{1}{2}e_{xz} + \omega_y\right)\sigma_{yz}\right] \\ &+ \frac{\partial}{\partial z}\left[(1 + e_{xx})\,\sigma_{zx} + \left(\frac{1}{2}e_{xy} - \omega_z\right)\sigma_{zy} + \left(\frac{1}{2}e_{xz} + \omega_y\right)\sigma_{zz}\right] + F_x = 0\end{aligned} \tag{14}$$

where F_x is the body force. Two additional equilibrium equations exist representing balance of forces in the y- and z-directions.

The equations of equilibrium given above are valid for the infinitesimal volume element subjected to small deformations, but with arbitrary rotations.

is then possible to show that the mean value of $\tan \psi_z$, which will be writen as $\overline{\tan \psi_z}$, is given by

$$\overline{\tan \psi_z} = \frac{\omega_z}{\left[(1+e_{xx})(1+e_{yy}) - \frac{1}{4}e_{xy}^2\right]^{1/2}} \tag{10a}$$

$$= \frac{\omega_z}{\left[\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right) - \frac{1}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]^{1/2}} \tag{10b}$$

Similarly, expressions can be written for $\overline{\tan \psi_x}$ and $\overline{\tan \psi_y}$.

The three parameters $\overline{\tan \psi_x}$, $\overline{\tan \psi_y}$, and $\overline{\tan \psi_z}$ characterize the rotation of the infinitesimal volume element. These parameters are proportional to ω_x, ω_y, ω_z and vanish when the ω's vanish. If $\omega_x = \omega_y = \omega_z = 0$ at some point in the body, then in the mean, the line elements passing through this point will not experience a rotation relative to any axis passing through the point.

Deformation in the absence of rotation is called a pure strain. In this case fibers may elongate but will preserve their initial directions.

Equations (6), (8), and (9) show that the true physical strains are dependent on rotations. In fact, this is the crucial difference between the linear and the nonlinear theories. The latter takes this dependence into account while the former does not.

F. Theory of Small Deformations

By small deformations, it is to be understood that elongations and shears are small compared to unity. This can be introduced into the strain expression, Eq. (8), as follows:

$$E_x(E_x + 2) = 2\,\varepsilon_{xx}$$

Here E_x is negligible compared to 2. This yields

$$E_x \approx \varepsilon_{xx}$$

Similarly, $E_y \approx \varepsilon_{yy}$ and $E_z \approx \varepsilon_{zz}$.

Treating Eq. (9), the shear expression, in the same way by making E_x and E_y negligible compared to 1 and $\sin \phi_{xy} \approx \phi_{xy}$ gives

$$\phi_{xy} \approx \varepsilon_{xy}$$

Similarly $\phi_{yz} \approx \varepsilon_{yz}$ and $\phi_{zx} \approx \varepsilon_{zx}$.

For small deformations, the components ε_{xx}, ε_{yy} and ε_{zz} become equivalent to the corresponding actual physical strains E_x, E_y, and E_z respectively. Likewise ε_{xy}, ε_{xz}, and ε_{yz} become equivalent to the corresponding shears. In view of this it is seen that for small physical strains Eqs. (7) define the nonlinear strain–displacement relations. This is of importance to subsequent developments in this Report.

G. Small Deformations and Small Angles of Rotation

The detailed reduction of Eqs. (10) will not be discussed here. However, it is possible to show that if the squares of the angle of rotation are neglected (compared to unity), the result is

$$\overline{\psi_x} \approx \omega_x \qquad \overline{\psi_y} \approx \omega_y \qquad \overline{\psi_z} = \omega_z \tag{11}$$

It is now possible to impose the conditions of small deformations and small angles of rotation on the strain–displacement equations. This reduction of Eqs. (7) must be carried out with caution. Dropping all terms higher than the first powers in the strains and squares in the rotations can be shown to leave

$$\varepsilon_{xx} \approx e_{xx} + \frac{1}{2}(\omega_y^2 + \omega_z^2) = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right\}^2 + \left\{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\}^2\right]$$

and similarly for ε_{yy} and ε_{zz}. Likewise,

$$\varepsilon_{xy} \approx e_{xy} - \omega_x\,\omega_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left\{\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right)\right\}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial z} - \frac{\partial w}{\partial x}\right)\right\} \tag{12}$$

Equations (12) are correct to within the accuracy obtainable by neglecting the angles of rotation and the strain components as compared to unity.

The interesting feature about Eqs. (12) is that the simplified form appearing in the classical theory of elasticity has not yet been reached. Strains as represented by Eqs. (12) depend on rotations as well as displacements. It may be inferred from the developments leading to Eqs. (12) that even though strains and rotations have been taken as small compared to unity, this does not imply that

Next, the condition of small rotations is imposed on Eqs. (14). If rotations are small compared to unity, the result of the simplification is to reduce Eqs. (14) to

$$\frac{\partial}{\partial x}(\sigma_{xx} - \omega_z \sigma_{xy} + \omega_y \sigma_{xz})$$
$$+ \frac{\partial}{\partial y}(\sigma_{yx} - \omega_z \sigma_{yy} + \omega_y \sigma_{yz}) \tag{15}$$
$$+ \frac{\partial}{\partial z}(\sigma_{zx} - \omega_z \sigma_{zy} + \omega_y \sigma_{zz}) + F_x = 0$$

Finally, the transition to the classical equations is made by neglecting rotations in Eqs. (15). If rotations are sufficiently small so that they may be neglected, the result is

$$\begin{aligned}
\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx} + F_x &= 0 \\
\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\sigma_{zy} + F_y &= 0 \\
\frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} + \frac{\partial}{\partial z}\sigma_{zz} + F_z &= 0
\end{aligned} \tag{16}$$

It is instructive to examine terms such as $\sigma_{xx} - \omega_z\sigma_{xy} + \omega_y\sigma_{xz}$ appearing in Eqs. (15). The nonlinear terms are $\omega_z\sigma_{xy}$, etc. Whether or not these may be neglected depends not only on the magnitude of the ω's, but also on the magnitude of the σ's. For example, if σ_{xx} is small compared to σ_{xy} and σ_{xz}, dropping the nonlinear terms may be invalid even though the rotations are small compared to unity. In other words, smallness of the angles of rotation compared to unity is not a sufficient condition for linearizing the equilibrium equations.

It is therefore important to keep in mind that in this instance linearization depends on whether the stresses which multiply the rotations are small compared to those which enter linearly into the equations. Problems in elastic stability are cases in point.

J. Concluding Comments

Although not intended to be complete in either detail or scope, the preceding account of nonlinear theory does point out some very important facts.

First, the retention or omission of nonlinear terms must be examined carefully. Particularly, in the equilibrium equations, rotations may be small. Yet if these are multiplied by large stresses, it may not be admissible to drop them. Not only must these terms be retained in stability problems, but also in problems having large self-equilibrating initial stresses due to any source. Manufacturing processes may produce such stresses; so may thermal gradients. A further fact of practical importance is that such residual or initial stresses may have a considerable bearing on buckling and deformation behavior in general.

Second, it is clear that a consistent nonlinear theory is not necessarily an easy matter to define. Care must be exercised when terms are dropped as being negligible, and, if rotations are retained in the equilibrium equations, they should also be retained in the strain–displacement expressions.

It is well known that the classical equations of elasticity, which are based on Hooke's Law and omit nonlinear terms in the equations for the strain components and in the equilibrium equations, lead to a unique solution. In other words, a unique condition of elastic equilibrium is obtained.

Actually, the given physical problem may not possess a unique solution. Under the given loads and constraints, several possible equilibrium conditions may exist. To investigate multiple equilibrium positions of elastic bodies, it is absolutely essential that the effect of rotations be taken into account.

If several equilibrium states may exist, it is often true that not all of them need be stable. An important fact in this connection is that when there are several possible equilibrium states under a given condition of loading, that position which is obtained from the classical linear equations is ordinarily unstable.

Geometrically nonlinear problems as discussed here can only arise in the cases of bodies having some dimensions which are small compared to the others. Rods, thin plates, and thin shells are the outstanding examples. In these cases, nonlinear effects can have real significance.

Finally, the mathematical solution of the nonlinear partial differential equations, which apply to thin plates and shells, presents a task beyond the capability of present day knowledge. Only the simplest problems can be treated in this manner. Structural designs of engineering significance must therefore be investigated by other means. The direct stiffness method presents a useful approach to the analysis of such problems.

III. THE AXIAL FORCE MEMBER

A. Examples—Nonlinear Behavior

A very useful example is afforded by the simple structure of Fig. 1. Under small displacements, vertical member 1–2 is ineffective in resisting applied load, **X**. All the stiffness in this instance is provided by spring, k. The corresponding linear force–displacement curve is shown as OA in Fig. 2. The slope of OA is $k = \mathbf{X}/u$.

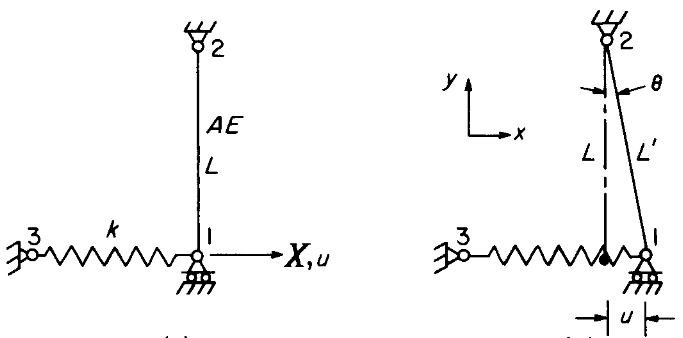


Fig. 1. Extensible rod–spring system

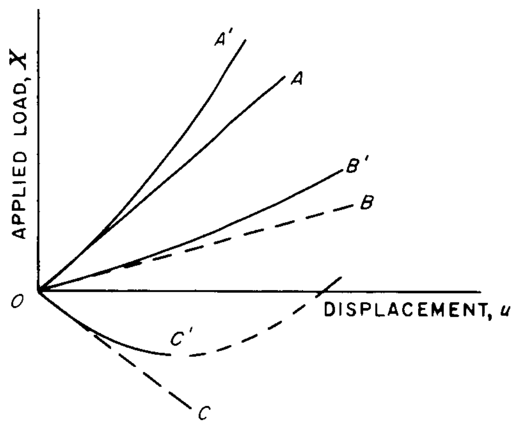


Fig. 2. Load–displacement curves

For this linear case, if mass m is located at node 1, the frequency of free vibrations about the equilibrium position of Fig. 1a is simply $\omega = (k/m)^{1/2}$. As displacements u become appreciable, member 1–2 begins to resist load **X**. The force–deflection curve OA now becomes OA'. The instantaneous stiffness K is then given by

$$K = \frac{d\mathbf{X}}{du} = k + \frac{AE}{L}(1 - \cos^3\theta) \tag{17}$$

This result is no longer constant, reflecting the nonlinear curve OA'.

At some equilibrium state defined by **X** and u, the frequency of free vibration is given by

$$\omega = \left(\left[k + \frac{AE}{L}(1 - \cos^3\theta)\right]/m\right)^{1/2}$$

If, for example, $\theta = 30°$ at the equilibrium position, then ω is approximately

$$\omega = \left[\left(k + 0.35\frac{AE}{L}\right)/m\right]^{1/2}$$

The exact curve OA' can be shown to be given by

$$\mathbf{X} = ku + AE(\tan\theta - \sin\theta) \tag{18}$$

For small θ, $\sin\theta \approx \tan\theta$, so that Eq. (18) degenerates to the usual linear expression. Differentiating Eq. (18) leads to Eq. (17).

A significant fact emerges if the structure of Fig. 1b is subjected to a thermal gradient. For example let member 1–2 be heated T°F above normal temperature. Then a compressive load P_0 is developed in member 1–2; at the same time in the absence of load **X**, no deflection will take place provided that $P_0 < P_{crit}$ or $T_0 < T_{crit}$. The magnitude of P_0 is given by

$$P_0 = A\sigma = AE\varepsilon = AE_T\alpha T \tag{19}$$

where

α = coefficient of thermal expansion

T = temperature increase of member 1–2, °F

E_T = modulus of elasticity of member 1–2 at elevated temperature state

Suppose that load **X** is applied after heating. As soon as a small displacement Δu occurs, the initial force P_0 in member 1–2 will tend to augment this displacement. As a result, the stiffness of the structure against external loading **X** is decreased due to the presence of initial force P_0. The corresponding linear and nonlinear force–deflection curves are shown as OB and OB' respectively in Fig. 2.

It is therefore seen that if the frequency of vibration, ω, is now calculated about the unloaded equilibrium state, a lower stiffness (slope of OB) is involved due to the presence of P_0. Furthermore, P_0 need not be due to heating; it could arise from fabrication with member 1–2 being initially overlength.

As T increases, P_0 increases. At $T = T_{crit}$, curve OB becomes horizontal. The heated structure then has no stiffness against external loading **X**. Frequency ω goes to zero. When $T > T_{crit}$, a slight disturbance of node 1 ($\Delta u > 0$) will cause mass m to accelerate due to the negative stiffness. This is illustrated in Fig. 2 by curve OC. Eventually, member 1–2 becomes effective and results in equilibrium being established at some displaced position as illustrated by the intersection of curve OC' and the u-axis.

Throughout this discussion, member 1–2 (and 1–3 also) is assumed to behave elastically. The significant fact is that the presence of P_0 can drastically affect the stiffness, not only at arbitrary **X** and u, but also at $\mathbf{X} = 0$. Furthermore, the conventional stiffness matrix is entirely independent of initial, internal forces such as P_0. Therefore, the decrease in slope from curve OA to OB cannot be accounted for by conventional stiffness matrices.

The foregoing arguments apply equally well to the heated beam-column of Fig. 3, or the axially loaded but unheated beam-column of Fig. 4. In either case, an initial axial force is assumed to exist. The presence of this force will markedly influence the stiffness of the member in resisting bending loading as due to P. Again the conventional stiffness matrices for a beam in bending are unable to account for the effect of the axial loading.

It becomes clear, therefore, that in addition to the usual elastic stiffness matrix, a new stiffness matrix reflecting the presence of initial internal forces must be found.

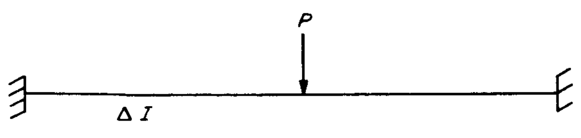


Fig. 3. Heated beam-column

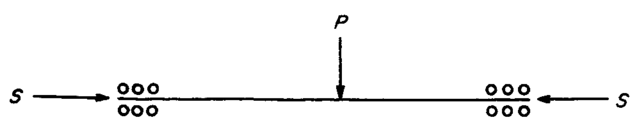


Fig. 4. Beam-column

B. Stiffness Matrix—Axial Force Member

The conventional stiffness matrix characterizing the elastic properties of the member has always been represented by $[K]$ or $\mathbf{K}^*$. It will now be designated by $\mathbf{K}^{(0)}$. The additional stiffness matrix to be found, reflecting the effect of initial axial loading, will then be specified by $\mathbf{K}^{(1)}$.

The first derivation of $\mathbf{K}^{(0)}$ appeared in Ref. 2. In addition, this reference contains some applications of the stiffness method which may assist the reader who is not familiar with the basic features of the method. Extension of the theory to the determination of $\mathbf{K}^{(1)}$ was first given in Ref. 3. For the case of the uniform axial force member the derivation was based on a Taylor's expansion of the forces as the member moved through an incremental displacement consisting of an elongation plus rotation.

A new derivation will be presented here, making more direct use of the fundamental theory appearing in Section II. For this purpose, it will also be necessary to employ a form of Castigliano's Theorem in deriving the elements of the stiffness matrix. For example, since it can be shown that the strain energy is expressible as a homogeneous, quadratic form in the displacements, it is convenient to write

$$U = \frac{1}{2}\mathbf{u}'\mathbf{K}\mathbf{u}$$

where

U = Strain energy

$\mathbf{u}'$ = Transpose of matrix column of displacements

$\mathbf{K}$ = Square matrix of stiffness influence coefficients

Applying Castigliano's Theorem, an element k_{ij} of **K** is given by

$$k_{ij} = \frac{\partial^2 U}{\partial u_i \partial u_j} \tag{20}$$

where, k_{ij} = element in ith row and jth column of **K**.

The procedure is to express U as a function of the nodal displacements. This is established by using the theory of Section II, together with some basic structural principles

*Bold face type will be used to represent matrices. The stiffness matrix will always be symbolized by **K** and will always be a symmetric, square matrix. The general column matrix of nodal forces will be represented by **X**. The general column matrix of nodal displacements will be designated by **u**. A prime will be used to represent the transpose of a matrix—as **X**′ for the row matrix of column **X**.

and reasoning. Once U is known, the application of Eq. (20) for obtaining the elements of the stiffness matrix becomes a routine matter.

First, consider the stringer of Fig. 5. In some initial state let the strain in the member be ε^0. Then the initial member loading is given by $P^0 = AE\varepsilon^0$. This could arise from previous elastic straining, from thermal gradients, or from other sources.

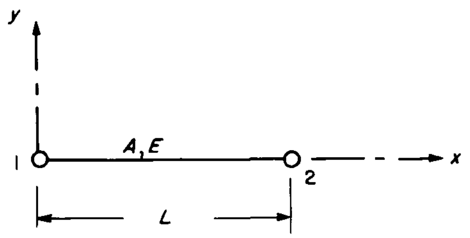


Fig. 5. Tension–compression member (stringer)

Now assume the member to undergo an additional small but finite deformation. This small deformation may be regarded as an incremental step. A finite sum of these incremental steps will be assumed to represent the total deformation. Each step is elastic in the usual sense; hence, the linear stiffness procedure may be applied to each step. Total as well as incremental strains are small; that is, elastic. However, total displacements may be large due to rotation. In the stiffness procedure the actual geometry of the deformed system is taken into account at the start of each step. In this way the effect of previous deformations is not overlooked when writing the incremental stiffness equation—which reflects the equilibrium conditions.

During the incremental step assume an additional strain ε^a to be developed. The total strain ε is then the sum of the initial and additional strains or

$$\varepsilon = \varepsilon^0 + \varepsilon^a \tag{21}$$

Additional strain ε^a must represent the effect of both u and v components of deformation (two dimensional case). For the axial force member of Fig. 5, strain ε^a is given by ε_{xx} of Eqs. (7). However, for the present discussion $w \equiv 0$ and $(\partial u/\partial x)^2$ can be dropped in comparison with $\partial u/\partial x$. On the other hand we cannot omit $(\partial v/\partial x)^2$ since this term represents the lowest order contribution of deformation $v(x)$ to ε^a. With these points in mind ε^a can now be written as

$$\varepsilon^a = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \tag{22}$$

This last equation can also be easily derived from basic geometrical considerations.

This completes the first important step in carrying out the derivation. Next, it is necessary to assume displacement functions $u(x)$ and $v(x)$. The reason for this is that it is ultimately necessary to write U in terms of nodal displacements (u_1, v_1, u_2, v_2). In choosing u and v, the following guiding principle is strictly adhered to: displacements are chosen to be the simplest, nontrivial, polynomial forms consistent with the problem being investigated. In the case of the problem at hand, this is achieved by selecting

$$u = a_0 + a_1 x, \qquad v = b_0 + b_1 x \tag{23}$$

These functions represent the only possible deformations for the axial force member. In Eq. (22), ε^a is then the sum of two constants; this represents the simplest nontrivial form. Had u and v been chosen as constants, ε^a would vanish, representing a trivial state. On the other hand, had u and v been chosen as quadratic functions, U would be linear in x which is inconsistent with the assumption of a constant strain member. An additional check on the forms chosen for u and v will appear subsequently.

This completes the second important step in the derivation. The remaining steps are largely routine in nature.

The strain energy is given by

$$U = \iiint \left[\int_{\varepsilon^0}^{\varepsilon^0+\varepsilon^a} \sigma d\varepsilon \right] dx\,dy\,dz$$

$$U = E \iiint \left[\int_{\varepsilon^0}^{\varepsilon^0+\varepsilon^a} \varepsilon d\varepsilon \right] dx\,dy\,dz$$

which leads to

$$U = E\varepsilon^0 \iiint \varepsilon^a\,dx\,dy\,dz + \frac{E}{2} \iiint (\varepsilon^a)^2\,dx\,dy\,dz \tag{24}$$

At this point, ε^a will be expressed in terms of nodal displacements. Using Eqs. (22) and (23), plus Fig. 5, it is found that

$$u_1 = u(x = 0) = a_0$$

$$u_2 = u(x = L) = a_0 + a_1 L$$

$$v_1 = v(x = 0) = b_0$$

$$v_2 = v(x = L) = b_0 + b_1 L$$

from which

$$a_0 = u_1 \qquad a_1 = \frac{u_2 - u_1}{L}$$
$$b_0 = v_1 \qquad b_1 = \frac{v_2 - v_1}{L} \tag{25}$$

It should be noticed that the number of algebraic equations in the nodal displacements are just sufficient to provide solution for the constants a_0, a_1, b_0, and b_1. Therefore, unless the number of such equations precisely equals the number of constants used in writing $u(x)$ and $v(x)$, the solution fails at this point.

It now follows that since

$$\frac{du}{dx} = a_1 = \frac{u_2 - u_1}{L}$$
$$\frac{dv}{dx} = b_1 = \frac{v_2 - v_1}{L}$$

the additional strain ε^a may be written as

$$\varepsilon^a = \frac{u_2 - u_1}{L} + \frac{1}{2}\left(\frac{v_2 - v_1}{L}\right)^2 \tag{26}$$

Since ε^a is independent of x, the strain energy (Eq. 24) may be expressed as

$$U = AEL \cdot \varepsilon^0 \cdot \varepsilon^a + \frac{AEL}{2}(\varepsilon^a)^2 \tag{27}$$

It is now a direct matter to apply Eq. (20) to Eq. (27). For example

$$k_{ij}^{u_i u_j} = \frac{\partial^2 U}{\partial u_i \partial u_j} = AEL\left[\varepsilon^0 \frac{\partial^2 \varepsilon^a}{\partial u_i \partial u_j} + \varepsilon^a \frac{\partial^2 \varepsilon^a}{\partial u_i \partial u_j} + \frac{\partial \varepsilon^a}{\partial u_i}\frac{\partial \varepsilon^a}{\partial u_j}\right]$$

where u_i, $u_j = u_1$, v_1, u_2, or v_2.

1. Let $u_i = u_j = u_1$

$$k_{11}^{u_1 u_1} = AEL\left[(\varepsilon^0 + \varepsilon^a)\frac{\partial^2 \varepsilon^a}{\partial u_1^2} + \left(\frac{\partial \varepsilon^a}{\partial u_1}\right)^2\right]$$

where $\partial \varepsilon^a / \partial u_1 = -1/L$, $\partial^2 \varepsilon^a / \partial u_1^2 = 0$

$$k_{11}^{u_1 u_1} = \frac{AE}{L}$$

2. Let $u_i = u_j = v_1$

$$k_{11}^{v_1 v_1} = AEL\left[(\varepsilon^0 + \varepsilon^a)\frac{\partial^2 \varepsilon^a}{\partial v_1^2} + \left(\frac{\partial \varepsilon^a}{\partial v_1}\right)^2\right]$$

where $\partial \varepsilon^a / \partial u_1 = (v_1 - v_2)/L^2$, $\partial^2 \varepsilon^a / \partial v_1^2 = 1/L^2$

$$k_{11}^{v_1 v_1} = AEL\left[(\varepsilon^0 + \varepsilon^a)\frac{1}{L^2} + \left(\frac{v_1 - v_2}{L}\right)^2\right]$$
$$= \frac{AE}{L}\varepsilon^0 + \frac{AE}{L^2}[\varepsilon^a + (v_1 - v_2)^2]$$

The term in brackets is a function of displacements; hence it is dropped since it is of higher order than $AE\varepsilon^0/L$. Also, retaining it would be inconsistent with Eq. (20). Note that Eq. (20) is based on strain energy U being a homogeneous quadratic function of the displacements. As a result k_{ij} should not itself be a function of displacements, or their derivatives. Consequently we retain

$$k_{11}^{v_1 v_1} = \frac{AE}{L}\varepsilon^0 = \frac{A\sigma^0}{L} = \frac{P^0}{L}$$

The other elements of $\mathbf{K}$ can be calculated in a similar manner. If this is done and results are collected, it is found that

$$\mathbf{K} = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} + \frac{P^0}{L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \tag{28}$$

$$= \mathbf{K}^{(0)} + \mathbf{K}^{(1)}$$

It is seen that $\mathbf{K}^{(1)}$ depends on initial load P^0. It will be termed the initial stress stiffness matrix.

The stiffness matrix given above is for the member oriented as shown in Fig. 5. For the arbitrarily oriented

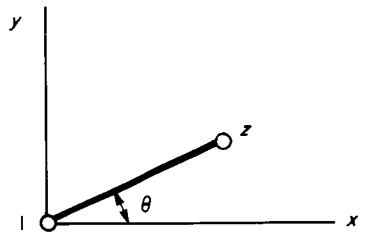


Fig. 6. Arbitrarily oriented tension–compression member

member (Fig. 6) the stiffness matrix relative to x, y axes is given by $\mathbf{T}'\mathbf{K}\mathbf{T}$ where

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{29}$$

The matrix transformation is discussed in Refs. 2 and 12. After transformation, it is found that

$$\mathbf{K}^{(0)} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} \lambda^2 & & SYM & \\ \lambda\mu & \mu^2 & & \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \end{array} \tag{30a}$$

$$\mathbf{K}^{(1)} = \frac{P^0}{L} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1-\lambda^2 & & SYM & \\ -\lambda\mu & 1-\mu^2 & & \\ -(1-\lambda^2) & \lambda\mu & 1-\lambda^2 & \\ \lambda\mu & -(1-\mu^2) & -\lambda\mu & 1-\mu^2 \end{bmatrix} \end{array} \tag{30b}$$

where $\lambda = \cos\theta$, $\mu = \sin\theta$.

The derivation for $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ was based on deformations occurring in an incremental step. Since the member is assumed to remain elastic over the total sum of such steps—that is, the total strains remain small—length L in $\mathbf{K}^{(0)}$, Eq. (30a) can be taken as the initial value existing prior to any deformation of the member. On the other hand for $\mathbf{K}^{(1)}$, length L should be taken as the value in effect at the beginning of the step. This is necessary if the deformed geometry is to be taken into account in establishing $\mathbf{K}^{(1)}$. For both $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ direction cosines (λ, μ) are the values in effect at the start of the step.

During the incremental step, applied force increment $\Delta\mathbf{X}$ produces displacement increments $\Delta\mathbf{u}$. As a result the stiffness equation may be written in the following form:

$$\Delta\mathbf{X} = (\mathbf{K}^{(0)} + \mathbf{K}^{(1)})\,\Delta\mathbf{u} \tag{31}$$

An alternative form may be written for Eq. (31). It relates total applied forces $\mathbf{X}$ to incremental displacements $\Delta\mathbf{u}$. This latter equation takes cognizance of the following fact: at the end of any step forces $\mathbf{X}$ must be in equilibrium with the total internal member forces $\mathbf{P}^0$. For the axial force member this equilibrium condition may be expressed as

$$X_1 = -P^0\lambda = -X_2 \qquad Y_1 = -P^0\mu = -Y_2$$

Here P^0 is the internal tension in the member. In matrix form,

$$\begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix} = P^0 \begin{Bmatrix} -\lambda \\ -\mu \\ \lambda \\ \mu \end{Bmatrix} \tag{32}$$

Equation (32) can now be added to Eq. (31). The resulting equation can, of course, be used in place of Eq. (31). However, there is little reason for doing so; hence in this Report Eq. (31) will be favored.

If $\mathbf{K}^{(1)}$ is required due to the nonnegligible presence of initial loading, and if only a single step is required in representing the problem, Eq. (31) can be used without the incremental designation on forces and displacements. Several problems of this type are considered later in the Report.

C. Application to Simple Problems

1. Nonlinear Truss

As a very simple example of bringing out the role of the $K^{(1)}$ matrix, the truss of Fig. 7 will be considered. The

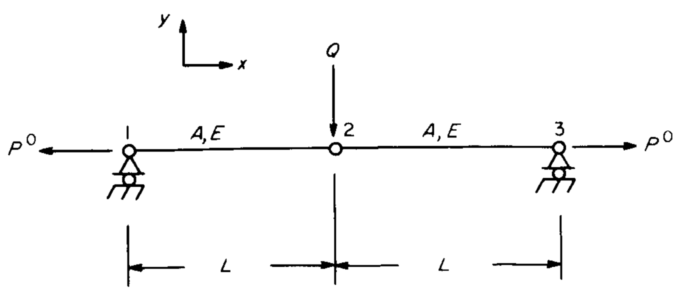


Fig. 7. Nonlinear truss

applied load is Q. The initial axial force P^0 is available for resisting Q. In the absence of P^0 the system is unstable as a linear, small deflection, problem. This is also the problem of the string under initial tension P^0.

The boundary conditions are $u_1 = v_1 = u_3 = v_3 = 0$. Due to symmetry $u_2 = 0$. Hence the stiffness equation degenerates into a scalar equation with v_2 as the only unknown. We recognize that $\lambda = 1$, $\mu = 0$ for both members (1–2 and 2–3); Eqs. (30a) and (30b) give

$$\mathbf{K} = \mathbf{K}^{(0)} + \mathbf{K}^{(1)}$$

$$= \frac{AE}{L}(2\mu^2) + \frac{P^0}{L}[2(1-\mu^2)]$$

$$= 2\frac{P^0}{L}$$

For this problem we do not require the incremental notation in the stiffness equation, Eq. (31). We obtain

$$-Q = 2\frac{P^0}{L}v_2$$

from which $v_2 = -QL/2P^0$. This can easily be checked by imposing equilibrium on the forces of Fig. 7. It should be noted that the stiffness solution fails if $\mathbf{K}^{(1)}$ is not introduced.

2. Truss of Fig. 1

There are several calculations of interest which can be made for this simple, indeterminate structure. First, the total structure stiffness matrix will be determined. Since only u at node 1 is nonvanishing, the matrices reduce to a single element after imposing boundary conditions ($v_1 = u_2 = v_2 = u_3 = v_3 = 0$). In view of this,

$$\mathbf{K}^{(0)}_{1-2} = \frac{AE}{L}\lambda^2_{1-2} \qquad \mathbf{K}^{(0)}_{1-3} = k$$

$$\mathbf{K}^{(1)}_{1-2} = \frac{P^0}{L'}(1-\lambda^2_{1-2}) \qquad \mathbf{K}^{(1)}_{1-3} \equiv 0$$

$$(\lambda_{1-3} = 1 \text{ for all values of } u)$$

It should be noted that actual member length is used in writing $K^{(1)}$, while original member length is used in writing $K^{(0)}$.

Some additional data for the truss are

$$\frac{u}{L} = \tan\Theta$$

$$L' = L[1 + (u/L)^2]^{1/2} = L/\cos\theta$$

$$\lambda_{1-2} = \frac{u}{L'} = \frac{u}{L}\cos\theta = \sin\theta \qquad (\text{negative for } u > 0)$$

$$P^0_{1-2} = \frac{AE}{L}(L' - L) = AE\,\frac{1-\cos\theta}{\cos\theta}$$

With this information the total stiffness matrix may be expressed as

$$K = k + \frac{AE}{L}\lambda^2_{1-2} + \frac{P_0}{L} \times \frac{L}{L'}(1-\lambda^2_{1-2})$$

$$= k + \frac{AE}{L}(1-\cos^3\theta)$$

This last result is precisely Eq. (17). Hence, the stiffness matrices have provided the correct instantaneous stiffness for the nonlinear truss problem. The basic forms given in Eqs. (30a, b) are therefore substantiated. Similarly the need for using L' rather than L in forming $\mathbf{K}^{(1)}$ is verified.

The actual application of $\mathbf{K}$ in determining the nonlinear force-displacement curve will be taken up in the next section of the Report. The stiffness expressions given above will be basic to such calculations.

Finally, the question of truss stability under thermal gradients may be investigated. Again, suppose member 1–2, Fig. 1, is heated to T°F above normal temperature conditions prior to application of load $\mathbf{X}$. Then,

$$\varepsilon^0_T = \alpha T \qquad \sigma^0 = E_T\,\varepsilon_T = E_T\,\alpha T$$

$$P^0 = A\sigma^0 = AE_T\alpha T$$

where

ε^0_T = initial thermal strain

α = coefficient of thermal expansion

E_T = modulus of elasticity at elevated temperature T°F

A = cross-sectional area of member 1–2

Now assume an arbitrary displacement Δu to take place. Initial force P^0 will then have a horizontal component, $P^0 \cdot \Delta u/L'$, tending to augment this displacement. The resisting force will be $k \cdot \Delta u$. It is assumed that Δu is small enough that elastic straining of 1-2 may be neglected. It is then also permissible to use $P^0 \cdot \Delta u/L$ for the horizontal component of the initial thermal force. The equilibrium equation is then

$$P^0 \cdot \frac{\Delta u}{L} = k \cdot \Delta u$$

$$\left(\frac{P^0}{L} - k\right) \Delta u = 0$$

At critical temperatures, an imposed arbitrary displacement, Δu, will be maintained in the absence of external loading **X**. When this is so, $P^0 = P^0_{crit}$ and the above equation is satisfied by

$$\frac{P^0_{crit}}{L} - k = 0 \qquad \text{or} \qquad P^0_{crit} = kL$$

Since $P^0_{crit} = AE_T \alpha T_{crit}$, the critical temperature is found to be given by

$$T_{crit} = \frac{kL}{AE_T} \alpha$$

This same problem may also be investigated by using the stiffness method. However, care must be taken to keep the stiffness analysis consistent with the elementary calculations given above. For example, $\mathbf{K}^{(0)}_{1-2}$ must be dropped, since elastic behavior of 1-2 has been omitted. Also, $P^0 = -AE_T\alpha T$ should be used, rather than the form given previously which reflects elastic straining. The negative sign is introduced because heating causes an initial compressive force in the member. With these points in mind, Eq. (31) takes the form

$$\mathbf{X} = \left[k + \frac{-AE_T \alpha T}{L} (1 - \lambda^2_{1-2}) \right] \mathbf{u}$$

Note that L' has been replaced by L in writing $\mathbf{K}^{(1)}_{1-2}$. This is also consistent with the previous solution.

In the above equation $\mathbf{X} = 0$, $P^0_{1-3} = 0$, and, in the initial position ($\theta = 0$), $\lambda_{1-2} = 0$. Hence, the equation reduces to

$$\mathbf{X} = \left(k - \frac{AE_T \alpha T}{L} \right) \mathbf{u}$$

Again at T_{crit} an arbitrary displacement u will be maintained in the absence of load **X**. Hence,

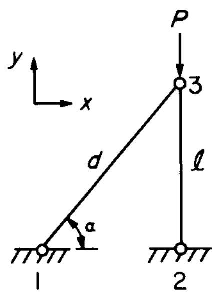
$$k - \frac{AE_T \alpha T_{crit}}{L} = 0$$

The result for T_{crit} is the same as that found previously.

The essential role of $\mathbf{K}^{(1)}$, when dealing with stability problems, can be readily appreciated from the above example.

3. Truss Stability

A different stability problem is presented in Fig. 8. It is desired to find the load P which will cause the truss to become unstable in its own plane. This problem is solved in Ref. 4, p. 147, by a procedure which may be termed the conventional approach. Even for this simple problem, it becomes surprisingly awkward to find the solution by classical techniques. As a result, it is of considerable interest to apply the stiffness method to this simple problem. Notation used is that of Ref. 4.



d = LENGTH OF MEMBER 1-3

A_d = AREA OF MEMBER 1-3

ℓ = LENGTH OF MEMBER 2-3

A_v = AREA OF MEMBER 2-3

$\lambda_{2-3} = 0, \ \mu_{2-3} = 1$

$\lambda_{1-3} = \cos\alpha, \ \mu_{1-3} = \sin\alpha$

Fig. 8. Truss stability problem

The initial member loads are $P^0_{2-3} = -P, P^0_{1-3} = 0$. (In a complex problem the stiffness method would be applied to find these initial forces.)

Normally, Eq. (31) would be set up in complete form when investigating a nonlinear problem. However, that is not necessary when studying stability problems. The reason is that the basic definition of stability makes it necessary to only set up $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$.

The critical loading will be defined as being that initial loading which will hold the structure in a displaced configuration in the absence of subsequent external forces. This can be understood by considering the simple column. The axial loading is the initial loading which can assume a critical value. However, instability occurs as a bending displacement, not as an axial compression. Therefore, when applied to the column, the definition states that at critical axial loading, an arbitrary bending displacement can be maintained in the absence of transverse forces or bending moments.

Under this definition, Eq. (31) assumes the following general form:

$$\mathbf{O} = (\mathbf{K}^{(0)} + \mathbf{K}^{(1)})\mathbf{u}$$

where $\mathbf{u}$ is the column of displacement components. Mathematically, this equation is satisfied for arbitrary displacements $\mathbf{u}$ if

$$|\mathbf{K}^{(0)} + \mathbf{K}^{(1)}| = 0 \tag{33}$$

For the truss of Fig. 8, only u_3, v_3 are nonvanishing. Hence, the stiffness matrices assume the following forms after imposing the boundary conditions:

$$\mathbf{K}^{(0)}_{1-3} = \frac{A_d E}{d} \begin{array}{cc} u_3 & v_3 \\ \end{array} \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$$

$$\mathbf{K}^{(1)}_{1-3} = 0$$

$$\mathbf{K}^{(0)}_{2-3} = \frac{A_v E}{l} \begin{array}{cc} u_3 & v_3 \\ \end{array} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{K}^{(1)}_{2-3} = \frac{-P}{l} \begin{array}{cc} u_3 & v_3 \\ \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The total $\mathbf{K}$ matrix is then found by summing the above. This gives

$$\mathbf{K} = \begin{bmatrix} \frac{A_d E}{d}\cos^2\alpha - \frac{P}{l} & \frac{A_d E}{d}\sin\alpha\cos\alpha \\ \frac{A_d E}{d}\sin\alpha\cos\alpha & \frac{A_d E}{d}\sin^2\alpha + \frac{A_v E}{l} \end{bmatrix}$$

The determinant of this matrix is now set equal to zero. This is the condition for P to equal P_{crit}. Expanding the determinant and solving for P_{crit} gives

$$P_{\mathrm{crit}} = \frac{A_d E \sin\alpha\cos^2\alpha}{1 + \frac{A_d}{A_v}\sin^3\alpha}$$

This result agrees with that found in Ref. 4. The straightforward procedure offered by the stiffness method is evident from the above calculation.

D. *Concluding Comments*

As yet, the piecewise linear technique of using $\mathbf{K}^{(0)} + \mathbf{K}^{(1)}$ to approximate the nonlinear force–deflection relationship has not been demonstrated. This will follow in the next section of the Report.

IV. THE PIECEWISE LINEAR CALCULATION PROCEDURE

A. *Incremental Step Procedure—Discussion*

The incremental step or piecewise linear procedure treats the nonlinear problem in a sequence of linear steps. Generally the loading is applied in increments; however, in certain stability calculations it may be necessary to apply incremental displacements. This Report will only treat the case of load increments.

For any arbitrary step, $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ are calculated in the ususal manner; that is, use is made of the geometry and initial forces existing at the start of the step. Incremental loading then produces incremental displacements. Internal forces developed during the step are calculated from the incremental displacements in the conventional linear manner. Total values for displacements and internal forces or stresses are obtained by summing the incremental values. This concept for treating the geometrically nonlinear problem by the stiffness method was first given in Ref. 3.

The incremental step procedure therefore corrects for deformation changes as loading takes place. At the same time the initial stress matrix is brought into the analysis. As the number of steps used to represent a given problem increases, the accuracy of results will likewise increase.

This general procedure will now be applied to the simple truss of Fig. 1 for which an exact solution can be found. This will make clear the calculation details and, at the same time, give some indication of the accuracy which can be obtained by using a small number of linear steps.

B. Simple Truss—Exact Solution

The exact force–deflection curve for the simple, indeterminate truss of Fig. 1 can be calculated from Eq. (18). It is, however, convenient to first replace θ by displacement u. Doing this permits the equation to be expressed as

$$\mathbf{X} = \left[kL + (AE)_{1-2} \frac{[1 + (u/L)^2]^{1/2} - 1}{[1 + (u/L)^2]^{1/2}} \right] \frac{u}{L}$$

Numerical data are selected as follows: $kL = 1$, $(AE)_{1-2} = 4$. It will be simplest to choose values of u/L and calculate the corresponding force $\mathbf{X}$ from the above equation. Elastic behavior is assumed for all u/L values. The results obtained from the equation above are shown in Table 1. Plotted data are given in Fig. 9.

Table 1. Exact solution data—truss of Fig. 1

u/L	0.1	0.2	0.3	0.4	0.5	0.6
X	0.10	0.22	0.35	0.51	0.71	0.94

Although no initial internal member forces exist at $u = 0$, the presence of the internal force in 1-2 must be considered as soon as displacements occur. Therefore, tracing the nonlinear curve of Fig. 9 is equivalent to tracing curve OA' of Fig. 2. Had an initial internal force been present in member 1-2 at $u = 0$, the resulting problem would have been equivalent to tracing curve OB' in Fig. 2. The nonlinear nature of the problem is just as effectively represented by curve OA' as by OB'. As a matter of fact, it has already been shown that the correct initial slope of OB' can be determined by using $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$.

C. Simple Truss—Incremental Step Solution

We again treat the truss of Fig. 1. The goal will be to approximate the exact solution shown on Fig. 9 with a sequence of linear segments as given by the stiffness method. These will be obtained by applying Eq. (31).

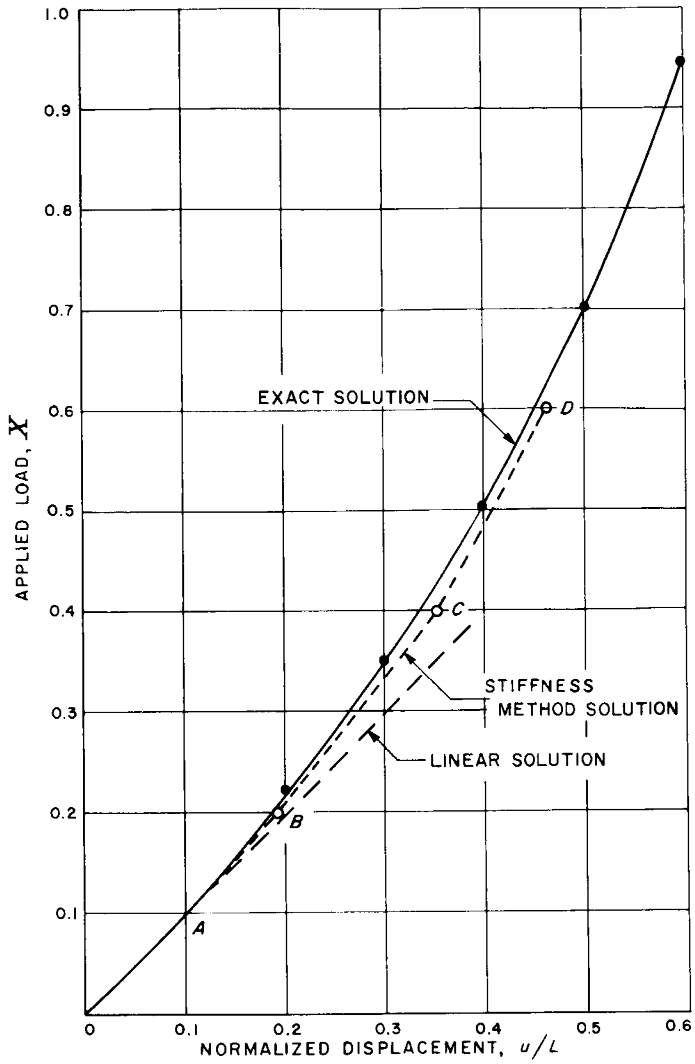


Fig. 9. Solutions for truss of Fig. 1

It has already been shown that for this problem the stiffness matrix reduces to the following scalar form:

$$\mathbf{K} = k + \left(\frac{AE}{L}\lambda^2\right)_{1-2} + \frac{P^0_{1-2}}{L} \times \frac{L}{L'}(1 - \lambda^2_{1-2})$$

Numerical values for structural parameters are chosen as follows:

$$kL = 1 \qquad (AE)_{1-2} = 4$$

The incremental stiffness equation may then be written as

$$\Delta X = \left[1 + 4\lambda^2_{1-2} + P^0_{1-2}\frac{L}{L'}(1 - \lambda^2_{1-2})\right]\frac{\Delta u}{L} \qquad (34)$$

This is the basic form of the incremental stiffness equation for the nonlinear truss of Fig. 1. It will now be applied by using four incremental steps. Load increments ΔX will be taken as 0.1, 0.1, 0.2, and 0.2 lb, respectively.

Step 1

Prior to application of any loading, the following values exist: $\lambda_{1-2} = 0$, $\lambda_{1-3} = -1$, $P^0_{1-2} = P^0_{1-3} = 0$. Equation (34) then yields the first incremental displacement as

$$\frac{\Delta u_{(1)}}{L} = 0.10$$

This is the ordinary linear solution. Plotted, it gives linear segment OA on Fig. 9.

Using the initial geometry we also find internal forces to be

$$(\Delta P^0_{1-2})_{(1)} = 0 \qquad (\Delta P^0_{1-3})_{(1)} = 0.10\,\text{lb}$$

Step 2

Because of deformation of Step 1, we now have an initial geometry defined by

$$L' = L\left[1 + \left(\frac{\Delta u_{(1)}}{L}\right)^2\right]^{1/2} = 1.005\,L$$

$$\lambda_{1-2} = -\frac{\Delta u_{(1)}}{L} = -0.10\frac{1}{1.005} = -0.0995$$

$$\lambda^2_{1-2} = 0.0099 \qquad 1 - \lambda^2_{1-2} = 0.9901$$

The initial loads for this step are those found in Step 1. With this data Eq. (34) may now be written as

$$0.10 = [1 + 4(0.0099) + 0]\,\frac{\Delta u_{(2)}}{L}$$

from which

$$\frac{\Delta u_{(2)}}{L} = 0.0962$$

At the end of Step 2 the total displacement $u_{(2)}/L$ is given by

$$\frac{u_{(2)}}{L} = 0.10 + 0.0962 = 0.1962$$

Based on geometry existing at the start of Step 2, internal forces developed during this step are given by*

$$(\Delta P^0_{1-2})_{(2)} = AE\,[\lambda \;\; \mu]_{1-2}\begin{Bmatrix}\Delta(u_2 - u_1)_{(2)}/L\\ \Delta(v_2 - v_1)_{(2)}/L\end{Bmatrix}$$

$$= 4\,[-0.0995 \;\cdots\;]\begin{Bmatrix}-0.0962\\ 0\end{Bmatrix}$$

$$= 0.0383\,\text{lb}$$

$$(\Delta P^0_{1-3})_{(2)} = kL\,[\lambda \;\; \mu]_{1-3}\begin{Bmatrix}\Delta(u_3 - u_1)_{(2)}/L\\ \Delta(v_3 - v_1)_{(2)}/L\end{Bmatrix}$$

$$= 1\,[-1 \;\; 0]\begin{Bmatrix}-0.0962\\ 0\end{Bmatrix}$$

$$= 0.0962\,\text{lb}$$

Total internal loads are now the sum of results found in Steps 1 and 2, or

$$(P^0_{1-2})_{(2)} = 0.383\,\text{lb}, \qquad (P^0_{1-3})_{(2)} = 0.1962\,\text{lb}$$

Step 3

Due to deformations accumulated during Steps 1 and 2, we now have an initial geometry defined by

$$L' = L\,[1 + (0.1962)^2]^{1/2} = 1.0193L$$

$$\lambda_{1-2} = -0.1962\frac{1}{1.0193} = -0.1927$$

$$\lambda^2_{1-2} = 0.0371, \qquad 1 - \lambda^2_{1-2} = 0.9629$$

Equation (34) then becomes under the 0.2 lb load increment

$$0.20 = \left[1 + 4(0.0371) + 0.0383\frac{1}{1.0193}(0.9629)\right]\frac{\Delta u_{(3)}}{L}$$

This gives

$$\frac{\Delta u_{(3)}}{L} = \frac{0.20}{1.1846} = 0.1687$$

The total displacement is now

$$\frac{u_{(3)}}{L} = 0.1962 + 0.1687 = 0.3649$$

*The reader who may be unfamiliar with the ideas and procedure for calculating internal forces should consult Ref. 12, pp. 33–34.

Internal force increments developed during this step are calculated as before. We obtain

$$(\Delta P^0_{1-2})_{(3)} = 4\,[-0.1927 \quad -] \begin{Bmatrix} -0.1687 \\ 0 \end{Bmatrix} = 0.1298 \text{ lb}$$

$$(\Delta P^0_{1-3})_{(3)} = 1\,[-1 \quad 0] \begin{Bmatrix} -0.1687 \\ 0 \end{Bmatrix} = 0.1687 \text{ lb}$$

Total internal member loads at the end of Step 3 are therefore

$$(P^0_{1-2})_{(3)} = 0.1681 \text{ lb} \qquad (P^0_{1-3})_{(3)} = 0.3649 \text{ lb}$$

Step 4

The load increment is again 0.20 lb. Other pertinent data are as follows:

$$L' = L\,[1 + (0.3649)^2]^{1/2} = 1.063L$$

$$\lambda_{1-2} = -0.3649 \frac{1}{1.063} = -0.3430$$

$$\lambda^2_{1-2} = 0.1177\,, \qquad 1 - \lambda^2_{1-2} = 0.8823$$

Equation (34) becomes

$$0.20 = \left[1 + 4\,(0.1177) + 0.1681 \frac{1}{1.063}\,(0.8823)\right] \frac{\Delta u_{(4)}}{L}$$

from which

$$\frac{\Delta u_{(4)}}{L} = 0.1240$$

Total displacement is now 0.3649 + 0.1240 = 0.4889 for a total loading of 0.60 lb. Since no additional steps are to be taken, the increments in internal loading for this step will not be calculated.

The results of the incremental step calculations are shown in Fig. 9 by the linear segments *OA*, *AB*, etc. Even with the gross steps assumed for the hand calculations the results are seen to be quite good. In particular the stiffness is obviously approximated to a satisfactory extent by the linear step procedure. This can be seen for example by assuming that the vibration characteristics of the system are to be calculated at $X = 0.50$ lb. The stiffness obtained at the start of Step 4 (slope of *CD*, Fig. 9) represents the true stiffness quite well (in error by 9.5%), whereas the linear solution (*OA*, Fig. 9) is in error by an appreciable amount (44%).

Some points to be kept in mind are the following:

a. Accuracy increases as the number of linear steps are increased.

b. Relatively large linear steps may give a good approximation to the true behavior.

c. The linear steps require the use of stiffness $\mathbf{K}^{(1)}$, in addition to $\mathbf{K}^{(0)}$.

d. Each linear step represents a complete problem in itself. For a truly complex problem, therefore, the numerical effort required in carrying out the incremental step procedure can be very large.

e. Careful programming becomes a matter of utmost importance when nonlinear problems are to be investigated. Otherwise computing time can increase drastically.

V. THE BEAM-COLUMN

A. Present State of Development

The slender, elongated member carrying axial loading in addition to bending will be referred to as the beam-column. No restriction is made as to the sense of the axial loading; furthermore, the causes (thermal or otherwise) of the axial loading need not be specified in detail.

For example, a beam with fixed ends and subjected to heating will develop an internal compressive force. Due to this initial load, the stiffness of the member in resisting subsequently applied transverse (bending) loads will be decreased. The behavior is analogous to that already described for the truss of Fig. 1. Again, the need for a $\mathbf{K}^{(1)}$ stiffness matrix, reflecting the presence of the initial axial loading, becomes evident.

It is an interesting fact that, as yet (May 1963), no publication has appeared which gives the derivation of $\mathbf{K}^{(1)}$ for the beam-column. The correct result was first given in an unpublished document, Ref. 5. At that time there were some questions connected with the correct form for $\mathbf{K}^{(1)}$ for the beam-column. The derivation which follows will yield what can be considered to be the correct form for this matrix. Later an interesting simplification in point of view will be discussed and also applied to a problem.

A very important reason for carefully examining the beam-column is that it provides a firm basis for proceeding on to the much more difficult problem of the plate element. This is, of course, in addition to the fact that beam problems are sufficiently important in engineering design to warrant special attention.

B. Stiffness Matrix—Beam-Column

The typical element is shown in Fig. 10. Positive directions for nodal forces and moments are shown in this figure. The beam will be assumed to lie in the x, y-plane both before and after deformation takes place. Moments, shown as double-headed vectors, follow the conventional right hand rule. It is further assumed that EI is uniform and y and z are principal axes of the cross section of the beam. The general scheme of the derivation is the same as for the truss member.

Again, the strain is split into two parts, namely $\varepsilon = \varepsilon^0 + \varepsilon^a$. However, ε^a must now be generalized from

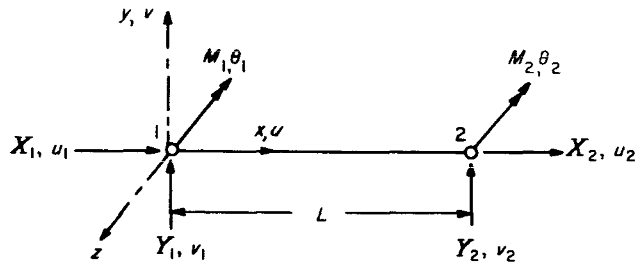


Fig. 10. Beam-column nodal forces and displacements

the value used in Eq. (22), to include bending strain. The necessary form is

$$\varepsilon^a = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - y\frac{d^2v}{dx^2} \tag{35}$$

The structural engineer can establish the validity of the bending strain term $(-y\,d^2v/dx^2)$ from basic beam theory. It is developed in more rigorous fashion in Ref. 1 (pp. 177–185).

The next step is the very important one of choosing displacement functions $u(x)$ and $v(x)$. As for the case of the truss member, we can again choose $u(x)$ as a linear function. This is consistent with the assumption of a uniform member having a constant average axial strain along its length. In the case of $v(x)$ we require a function which will be consistent with the load states permitted the element. See Fig. 10. The member is restricted to carrying a constant shear load and a linearly varying moment. Choosing $v(x)$ as a cubic in x will satisfy these conditions. Hence, displacement functions for the beam-column are taken as

$$\begin{aligned} u(x) &= a_0 + a_1 x \\ v(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 \end{aligned} \tag{36}$$

From Eqs. (36)

$$\frac{du}{dx} = a_1$$

$$\frac{dv}{dx} = b_1 + 2b_2 x + 3b_3 x^2$$

$$\frac{d^2v}{dx^2} = 2b_2 + 6b_3 x$$

These are the terms which define the additional strain, Eq. (35).

Also from Eqs. (36) nodal displacements can be written as follows:

$$u_1 = u(x=0) = a_0$$
$$u_2 = u(x=L) = a_0 + a_1 L$$
$$v_1 = v(x=0) = b_0$$
$$v_2 = v(x=L) = b_0 + b_1 L + b_2 L^2 + b_3 L^3$$
$$\theta_1 = \left.\frac{dv}{dx}\right|_{x=0} = b_1$$
$$\theta_2 = \left.\frac{dv}{dx}\right|_{x=L} = b_1 + 2b_2 L + 3b_3 L^2$$

The six equations can now be solved for the constants $a_0, a_1, \cdots, b_3$ in terms of the nodal displacements. Doing this yields

$$a_0 = u_1 \qquad a_1 = \frac{u_2 - u_1}{L}$$
$$b_0 = v_1 \qquad b_1 = \theta_1$$
$$b_2 = \frac{3}{L^2}(v_2 - v_1) - \frac{1}{L}(2\theta_1 + \theta_2)$$
$$b_3 = \frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2)$$

With these results additional strain ε^a may be written as

$$\varepsilon^a = \frac{u_2 - u_1}{L} + \frac{1}{2}\left[\theta_1 + \left\{\frac{6}{L^2}(v_2 - v_1) - \frac{2}{L}(2\theta_1 + \theta_2)\right\} x + \left\{\frac{6}{L^3}(v_1 - v_2) + \frac{3}{L^2}(\theta_1 + \theta_2)\right\} x^2\right]$$
$$- y\left[\left\{\frac{6}{L^2}(v_2 - v_1) - \frac{2}{L}(2\theta_1 + \theta_2)\right\} + \left\{\frac{12}{L^3}(v_1 - v_2) + \frac{6}{L^2}(\theta_1 + \theta_2)\right\} x\right] \qquad (37)$$

Strain energy U is again given by Eq. (24). The first term, Eq. (24), depends on ε^0; hence it must yield the initial stress stiffness matrix $\mathbf{K}^{(1)}$. The second term depends only on ε^a. Therefore, it is the conventional energy due to elastic deformation of the member. As a result it must lead to the conventional stiffness matrix $\mathbf{K}^{(0)}$. It is convenient to let the integrals in Eq. (24) be represented by $U^{(1)}$ and $U^{(2)}$ respectively. They can then be treated separately in determining $\mathbf{K}$.

The remaining part of the derivation parallels that already employed for the stringer. Stiffness matrix elements are obtained by once again applying Eq. (20). To indicate some of the details, two elements of the overall stiffness matrix will now be calculated.

1. $k_{11}^{u_1 u_1}$

$$k_{11}^{u_1 u_1} = \frac{\partial^2}{\partial u_1^2}(U^{(1)} + U^{(2)})$$

where

$$\frac{\partial^2 U^{(1)}}{\partial u_1^2} = E\varepsilon^0 \int\int\int \frac{\partial^2 \varepsilon^a}{\partial u_1^2}\, dx\, dy\, dz$$

$$\frac{\partial^2 U^{(2)}}{\partial u_1^2} = E \int\int\int \left[\varepsilon^a \frac{\partial^2 \varepsilon^a}{\partial u_1^2} + \left(\frac{\partial \varepsilon^a}{\partial u_1}\right)^2\right] dx\, dy\, dz$$

in which

$$\frac{\partial \varepsilon^a}{\partial u_1} = -\frac{1}{L} \qquad \frac{\partial^2 \varepsilon^a}{\partial u_1^2} = 0$$

Therefore

$$\frac{\partial^2 U^{(1)}}{\partial u_1^2} = 0$$

$$\frac{\partial^2 U^{(2)}}{\partial u_1^2} = E \int\int\int \frac{1}{L^2}\, dx\, dy\, dz = \frac{AE}{L}$$

We now have

$$k_{11}^{u_1 u_1} = \frac{AE}{L}$$

2. $k_{11}^{v_1 v_1}$

The expressions for the derivatives of $U^{(1)}$ and $U^{(2)}$ are similar to those in (1) except v_1 now replaces u_1. Consequently

$$\frac{\partial \varepsilon^a}{\partial v_1} = \left[\theta_1 + \left\{\frac{6}{L^2}(v_2 - v_1) - \frac{2}{L}(2\theta_1 + \theta_2)\right\} x + \left\{\frac{6}{L^3}(v_1 - v_2) + \frac{3}{L^2}(\theta_1 + \theta_2)\right\} x^2\right] \cdot \left[-\frac{6}{L^2}x + \frac{6}{L^3}x^2\right] - y\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)$$

and hence

$$\frac{\partial^2 \varepsilon^a}{\partial v_1^2} = \left(-\frac{6}{L^2}x + \frac{6}{L^3}x^2\right)^2$$

We now have

$$\frac{\partial^2 U^{(1)}}{\partial v_1^2} = E\varepsilon^0 \iiint \left(-\frac{6}{L^2}x + \frac{6}{L^3}x^2\right)^2 dx\,dy\,dz$$

$$= \frac{6}{5L} AE\,\varepsilon^0 = \frac{6P^0}{5L}$$

$$\frac{\partial^2 U^{(2)}}{\partial v_1^2} = E \iiint \left[\varepsilon^a\left(-\frac{6}{L^2}x + \frac{6}{L^3}x^2\right)^2 + \left(\frac{\partial \varepsilon^a}{\partial v_1}\right)^2\right] dx\,dy\,dz$$

The last integral need not be treated in full since most of the terms in the integrand include displacements. As pointed out for the case of the axial force member, such terms are dropped. In view of this we need only consider that part of $(\partial\varepsilon^a/\partial v_1)^2$ which is represented by $\{-y(-6/L^2 + 12x/L^3)\}^2$. Integrating this term over the volume of the member gives $12EI/L^3$. Here I is the moment of inertia of the cross-sectional area about the z-axis. For simplicity we assume this to be a principal axis of the cross-section.

Collecting results therefore gives

$$k_{11}^{v_1 v_1} = \frac{12EI}{L^3} + \frac{6P^0}{5L}$$

Remaining terms can be calculated in a similar manner. Doing this leads to the following forms for $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$:

$$\mathbf{K}^{(0)} = \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{AE}{L} & & & & \text{SYM.} & \\ 0 & \frac{12EI}{L^3} & & & & \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \qquad (38)$$

$$
\mathbf{K}^{(1)} = P^0
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\
\begin{bmatrix}
0 & & & & \text{SYM.} & \\
0 & \frac{6}{5L} & & & & \\
0 & \frac{1}{10} & \frac{2L}{15} & & & \\
0 & 0 & 0 & 0 & & \\
0 & -\frac{6}{5L} & -\frac{1}{10} & 0 & \frac{6}{5L} & \\
0 & \frac{1}{10} & -\frac{L}{30} & 0 & -\frac{1}{10} & \frac{2L}{15}
\end{bmatrix}
\end{array}
\qquad (39)
$$

It should be noted that $\mathbf{K}^{(0)}$, Eq. (38), includes both axial and bending stiffness. On the other hand $\mathbf{K}^{(1)}$ reflects only the presence of the initial loading P^0.

Equation (39) makes it clear that $\mathbf{K}^{(1)}$ possesses non-vanishing terms which represent deformations out of the line of action of initial P^0. This is precisely the nature of the deformation mode which occurs when instability takes place. It is also similar to the pattern previously discovered for the stringer, Eq. (28).

The stiffness matrices of Eqs. (38) and (39) can be easily written in terms of arbitrary orientation of the member. For example, if we restrict deformations so that the member remains in the xy plane we can use $\mathbf{T}'\mathbf{K}\mathbf{T}$ as for the stringer. However, in this present case we must take transformation matrix $\mathbf{T}$ as

$$
\mathbf{T} =
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\
\begin{bmatrix}
\lambda & \mu & 0 & & & \\
-\mu & \lambda & 0 & & & \\
0 & 0 & 1 & & & \\
 & & & \lambda & \mu & 0 \\
 & & & -\mu & \lambda & 0 \\
 & & & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

All elements not shown in $\mathbf{T}$ are understood to be zero. Also, λ and μ are defined as for the stringer. Generalization to three-dimensional space may be carried out in a similar manner. See Refs. 2 and 12 for details.

C. Column Stability—Example

The beam column of Fig. 11 will now be analyzed by the stiffness method. Only two elements will be used to represent the actual member. This is the coarsest idealization which will still permit a solution to be found. Hence, the answers obtained will be of minimum accuracy.

The theoretical solution for the critical axial loading is given by

$$
P^0_{\text{crit}} = \frac{\pi^2 EI}{l^2} = 2.465 \frac{EI}{L^2}
$$

In proceeding with the stiffness solution, we first note that boundary conditions require $v_1 = v_3 = 0$. Symmetry furthermore requires that $u_1 = -u_3$, $\theta_1 = -\theta_3$, and $u_2 = 0$. Unknown displacements may therefore be taken as u_1, θ_1, and v_2. Using Eqs. (38) and (39), we therefore establish the stiffness matrix for this problem as

$$
\mathbf{K} = \mathbf{K}^{(0)} + \mathbf{K}^{(1)} =
\begin{array}{c}
\begin{array}{ccc} u_1 & \theta_1 & v_2 \end{array} \\
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 \\
0 & \frac{4EI}{L} + \frac{2P^0L}{15} & -\left(\frac{6EI}{L^2} + \frac{P^0}{10}\right) \\
0 & -2\left(\frac{6EI}{L^2} + \frac{P^0}{10}\right) & 2\left(\frac{12EI}{L^3} + \frac{6P^0}{5L}\right)
\end{bmatrix}
\end{array}
\qquad (40)
$$

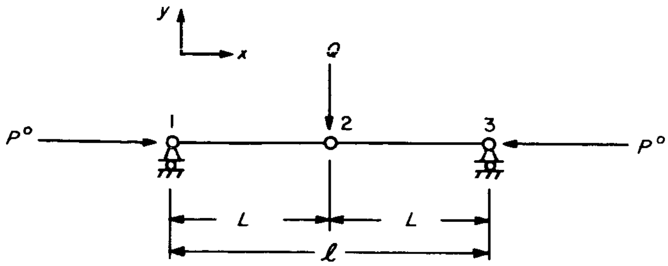


Fig. 11. Loading for simple beam-column

For the eigenvalue problem we notice that the bending degrees of freedom (θ, v) are uncoupled from the axial force problem. This corresponds to the inextensional condition usually introduced when deriving the differential equation for column stability.

Applying Eq. (33) to the stiffness matrix of Eq. (40) now gives

$$(P^0_{\text{crit}})^2 + \frac{104}{3}\frac{EI}{L^2}P^0_{\text{crit}} + 80\left(\frac{EI}{L^2}\right)^2 = 0$$

Solving for minimum P^0_{crit},

$$P^0_{\text{crit}} = -2.48\,EI/L^2$$

The negative sign indicates that the critical condition as imposed by Eq. (33) can only be attained when P^0 is compressive. The agreement between the stiffness and exact answers is striking.

A more severe test of the stiffness procedure will be given in Sec. VI-G. There a tapered column will be analyzed for several choices of idealization.

The foregoing establishes the basic theory for analyzing the most general cases of trusses and frames. However, in order to use the theory, suitable numerical methods must be available for handling large order problems. This fortunately does not require the development of new schemes. As will be shown in Sec. VI, well known existing techniques may be employed for this purpose.

Finally it is pointed out that a simpler point of view might have been adopted in the foregoing derivation of $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ for the beam-column. Briefly this proceeds as follows: Eqs. (23) are used in forming du/dx and $(dv/dx)^2/2$ in ε^a, Eq. (35), while the cubic in $v(x)$, Eq. (36), is used for the curvature term in ε^a. The analysis proceeds in the same manner as above. We again obtain $\mathbf{K}^{(0)}$ as given by Eq. (38). However, in place of Eq. (39) we now obtain the same $K^{(1)}$ as for the stringer, Eq. (28). These results can be explained in physical terms. The cubic only enters into the bending term in ε^a, whereas the rotational term $(dv/dx)^2/2$ is defined in terms of the linear $v(x)$ of Eq. (23). Consequently we have in effect the superposition of a string and a beam without any interaction between the two. This combination then leads to the expected result of $\mathbf{K}^{(0)}$ for the beam, plus $\mathbf{K}^{(1)}$ for the string. In fact this physical picture led to the earliest calculation of beam stability problems and these investigations showed that satisfactory results could be obtained in this manner. Nevertheless $\mathbf{K}^{(1)}$ as given by Eq. (39) is rightly considered as being the correct initial stress stiffness matrix for the beam-column.

If the stability problem of Fig. 11 is solved by using the stringer $\mathbf{K}^{(1)}$ matrix, we find in place of Eq. (40)

$$\mathbf{K} = \begin{array}{c} \begin{array}{ccc} u_1 & \theta_1 & v_2 \end{array} \\ \begin{bmatrix} \dfrac{AE}{L} & 0 & 0 \\ 0 & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} \\ 0 & -\dfrac{12EI}{L^2} & 2\left(\dfrac{12EI}{L^3} + \dfrac{P^0}{L}\right) \end{bmatrix} \end{array} \tag{41}$$

Setting the determinant of $\mathbf{K}$ equal to zero yields

$$P^0_{\text{crit}} = -3EI/L^2$$

The error is now 21.7% compared to 1.2% for the previous solution.

If a four-element idealization is used, the stringer $\mathbf{K}^{(1)}$ matrix leads to

$$P^0_{\text{crit}} = -2.61\,EI/L^2$$

The error is now reduced to 5.9%. As far as is known, the stiffness solution based on the stringer $\mathbf{K}^{(1)}$ matrix will converge to the correct solution as the number of elements increases. From a practical point of view it is obviously advantageous to use the $\mathbf{K}^{(1)}$ matrix of Eq. (39) when bending stability is being investigated.

D. Tie Rod Deflection—Example

The problem of determining the deflection of the "so-called" tie rod is now taken up. Figure 11 again applies; however, the direction of P^0 is now reversed. The problem is to find v_2. The exact solution as found from the appropriate differential equation is (Ref. 6, p. 43),

$$v_2 = \frac{Ql^3}{48\,EI} \times \frac{\alpha - \tan h\alpha}{\alpha^3/3}$$

where

$$\alpha = \frac{1}{2}\left(\frac{P^0 l^2}{EI}\right)^{1/2}$$

In order to compare results given by conventional theory with the stiffness method solution, it is convenient to establish specific numerical data. The following are chosen: the section is ¼ in. × ⅛ in.

$$A = 1/32 \text{ in.}^2$$

$$I = \frac{(1/8)(1/4)^3}{12} = \frac{1}{6144}\text{ in.}^4$$

Length, $l = 10$ in.

Modulus, $E = 10^7$ psi

Axial tension, $P^0 = 1000$ lb

Transverse load, $Q = 100$ lb

For this data the linear problem ($P^0 = 0$) has the solution

$$v_2 = \frac{Ql^3}{48\,EI} = 1.28\text{ in.}$$

In the alternative case of the nonlinear problem,

$$v_2 = 1.28\frac{\alpha - \tan h\alpha}{\alpha^3/3} = 0.186\text{ in.}$$

The decrease in deflection due to the axial tension is seen to be significant.

The stiffness matrix for this problem will first be taken as that given by Eq. (40). Since P^0 is assumed to remain constant, the incremental step procedure is again unnecessary. As a result the stiffness equation takes the form

$$\begin{Bmatrix} X_1 \\ M_1 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \left(\frac{4EI}{L} + \frac{2P^0L}{15}\right) & -\left(\frac{6EI}{L^2} + \frac{P^0}{10}\right) \\ 0 & -2\left(\frac{6EI}{L^2} + \frac{P^0}{10}\right) & 2\left(\frac{12EI}{L^3} + \frac{6P^0}{5L}\right) \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ v_2 \end{Bmatrix} \tag{42}$$

It is desired to find v_2. The lack of coupling between u_1 and the other displacements simplifies the calculations. Since $M_1 = 0$ and $Y_2 = -Q$ we find that

$$v_2 = -0.182\text{ in.}$$

The agreement between the stiffness answer and that found from basic theory is again excellent. Also if $\mathbf{K}^{(1)}$ is omitted the result of the stiffness calculation will be that given above for the strictly linear problem.

If this solution is now repeated using the stringer $\mathbf{K}^{(1)}$ matrix the result will be $V_2 = -0.209$ in. The error in this instance is 16.2%.

E. Comments

The implication is raised in this part of the Report that the stiffness matrix for a given element need not be unique. This seems to be particularly true with respect to the $\mathbf{K}^{(1)}$ stiffness matrices.

Actually it is true that uniqueness need not exist, except in the simplest cases (e.g., $\mathbf{K}^{(0)}$ for a uniform truss member). For the more complex structural elements such as a triangular or rectangular thin plate in bending, quite different $\mathbf{K}^{(0)}$ stiffnesses have been found and used. For certain applications, as far as accuracy of results are concerned superiority may lie with one matrix or the other. However, by using more elements to represent the overall structural unit, accuracy should improve no matter which matrix is used. This objective may not of course always be realized, especially as the element density in the idealization becomes extremely large. The whole question of convergence has yet to be studied with care. Quite obviously, important theoretical and numerical questions will enter into such studies.

Up to the present time (1962–1963) the tendency has been to favor the use of the simplest stiffness matrices for any given element. Departures from the simplest form generally introduce additional complications—as adding extra nodes to the element. Further experience with this relatively unexplored field of the basic stiffness method procedure will have to be accumulated before meaningful recommendations can be offered.

VI. STABILITY CALCULATIONS

A. Preliminary Comments

Several facts concerning stability calculations have already appeared in this Report. First, the presence of $\mathbf{K}^{(1)}$ in addition to $\mathbf{K}^{(0)}$ is necessary if stability problems are to be investigated. Second, by forming the total stiffness matrix and setting its determinant equal to zero, the critical load can be determined. Third, the stiffness process gives a simple solution for problems which otherwise might be quite difficult to analyze. Fourth, in some instances (e.g., the beam-column) more accurate results are obtained if a greater number of elements are used to represent the member. Fifth, the method would seem to be potentially useful for truly complex problems; however, a suitable numerical procedure would then have to be devised.

To a considerable extent, the application of the stiffness method to stability problems is still in its infancy. A great deal of work remains to be done in this field. This most generally involves the accumulation of experience in actually solving problems. Very little is known about the structural idealizations which should be used in stability analyses; furthermore, the best numerical procedures have not yet been established.

This Report will indicate some of the numerical procedures which may be used. Whenever possible, the results of actual calculations will be given to indicate the order of accuracy which has been achieved to date.

B. Initial Loading Intensity Factor

In a complex problem, it is necessary to write the initial member forces in terms of an initial loading intensity factor. For example, if $\mathbf{X}^0$ is the column of initial forces, this may be replaced by

$$\mathbf{X}^0 = F\mathbf{X}^{0*} \tag{43}$$

where

F = Initial load intensity factor

$\mathbf{X}^{0*}$ = Column matrix of initial load ratios (which are constants)

It may be helpful to consider Eq. (43) with a truss in mind. Let initial external forces, capable of contributing to instability, be designated by $P_1^0, P_2^0, \cdots, P_n^0$. These may be expressed as $P_1^0(r_1, r_2, \cdots, r_n)$, where $r_1 = 1$, $r_2 = P_2^0/P_1^0, \cdots, r_n = P_n^0/P_1^0$. In turn all the initial member forces may be expressed in terms of $P_1^0, P_2^0, \cdots, P_n^0$ or alternatively in terms of $P_1^0(r_1, r_2, \cdots, r_n)$. The load P_1^0 can then be taken to be the initial load intensity factor, F.

A similar line of reasoning may be applied to a frame, plate, shell, or any other structure. Instability occurs when F assumes a critical value. This critical state is reached when F makes the determinant of the stiffness matrix vanish. At the critical value of F, an arbitrary displacement state may be sustained in the sense previously discussed in the Report.

C. Plotting the Determinant of the Stiffness Matrix

In low order problems, it may be convenient to expand the determinant of the stiffness matrix (set equal to zero) and solve for F_{crit}. In more extensive problems, this is not feasible. However, it may be quite reasonable to calculate the numerical value of $|\mathbf{K}|$ for such cases. Assuming $F < F_{crit}$, the value of this determinant will be positive and will tend toward zero as $F \rightarrow F_{crit}$. Consequently, a rather obvious scheme would be to plot $|\mathbf{K}|$ vs F (see Fig. 12). Three points, determined for reasonable values of the load intensity factor F, will then enable F_{crit} to be approximated by extrapolation, as shown in Fig. 12.

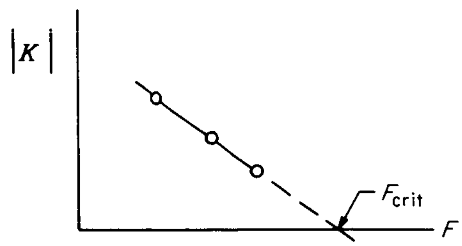


Fig. 12. Extrapolation for critical load

For a complex problem, the order of **K** may be high, and the magnitude of F_{crit} may be entirely unknown; i.e., not subject to a reasonable estimate. Forming Fig. 12 in such a case may be neither simple nor straightforward. In addition, the mode shape associated with F_{crit} remains undetermined by this procedure. Nevertheless, in certain problems, plotting $|\mathbf{K}|$ may provide a very useful means for establishing the critical loading.

D. Adaptation of Southwell's Method

Instead of calculating $|\mathbf{K}|$ as just discussed, it may be simpler to calculate a significant deflection component due to the applied loading. This opens the way for applying Southwell's procedure for determining the critical loading.

Southwell's method can be found in many elementary texts on structural analysis. It will be defined here with the beam-column of Fig. 11 in mind.

The deflection of the beam-column can be demonstrated to be given with very good accuracy by the simple equation (Ref. 6, p. 49)

$$v_2 = (v_2)_0 \frac{1}{1 - \dfrac{P^0}{P^0_{crit}}} \tag{44}$$

where $(v_2)_0$ is the deflection at node 2 due to load Q only (Fig. 11). Also v_2 is the actual deflection due to both Q and P^0. Multiplying each side of Eq. (44) by P^0_{crit}/P^0 and rewriting gives

$$\frac{v_2}{P^0} P^0_{crit} - v_2 = \frac{(v_2)_0}{P^0} P^0_{crit}$$

Dropping subscript 2 and superscript 0, and defining $(v_2)_0$ as v_0 this becomes

$$\frac{v}{P} P_{crit} - v = \frac{v_0}{P} P_{crit} \tag{45}$$

Equation (45) may now be plotted with v/P as ordinate and v as abscissa (see Fig. 13). The curve will be a straight line. From Eq. (45) it is found that the reciprocal of the slope of this curve is P_{crit}.

For the complex problem, Eq. (45) still applies. However, intensity factor F replaces P, and v is defined as some significant displacement component defining the unstable mode. An example will clarify the detailed procedure to be followed.

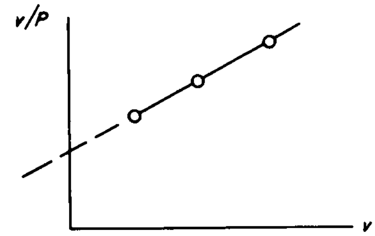


Fig. 13. Application of Southwell's method

E. Stability of Rectangular Plate—Example

The first application of the Southwell technique to a difficult stability problem was given in Ref. 7. There the stability of a square plate under uniform compression was investigated. A more complete discussion of a rectangular plate will be given in this Report. The author is indebted to The Boeing Company for the numerical data from which the Southwell type plot was made.

Figure 14 shows a rectangular plate under uniform edge compression. A 2.5-lb concentrated load also acts at

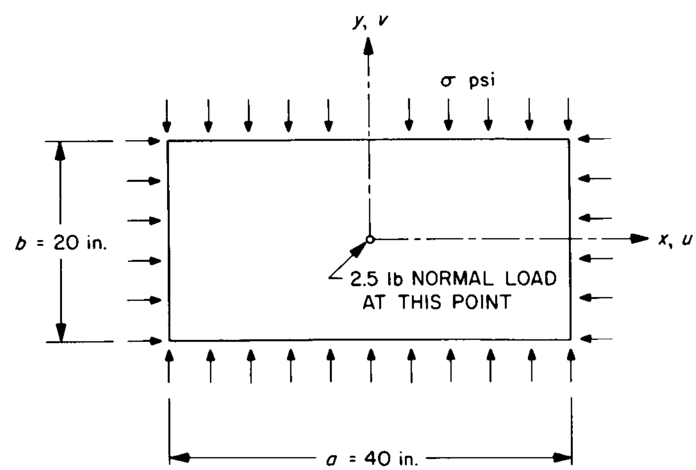


Fig. 14. Rectangular plate stability problem

the center of the plate. The plate is simply supported on all edges and is further defined by

$$E = 30 \times 10^6 \text{ psi}$$
$$\nu = 0.30$$
$$h = 0.10 \text{ (thickness)}$$

Due to symmetry only one quadrant of the plate need be considered. The idealization used in the formation of the stiffness matrix is shown in Fig. 15.

Prior to imposing boundary conditions, three rectilinear displacements ($u, v,$ and w) and two rotations ($\theta x, \theta y$) exist at each node, Fig. 15. Consequently the gross stiffness matrix **K** will initially be of order 125 × 125. This is reduced by applying boundary and symmetry conditions on the displacements. For example, boundary conditions for all simply supported edges require that,

$w = 0$ at nodes 5, 10, 15, 20, 21, 22, 23, 24, 25

$\theta x = 0$ at nodes 5, 10, 15, 20, 25

$\theta y = 0$ at nodes 21, 22, 23, 24, 25

These 19 conditions will reduce **K** to order 106 × 106.

Symmetry of displacements enable certain conditions to be written for the nodes lying on the two centerlines (x and y axes, Fig. 15). These are

$u = 0$ at nodes 1, 6, 11, 16, 21

$\theta y = 0$ at nodes 1, 6, 11, 16

$v = 0$ at nodes 1, 2, 3, 4, 5

$\theta x = 0$ at nodes 1, 2, 3, 4

These 18 conditions may now be used to further reduce the order of **K**—this time to an 88 × 88 matrix.

In forming the stiffness matrix, $\mathbf{K}^{(1)}$ for the individual triangular element was taken as the form given in Ref. 3. This matrix is for the triangle in plane stress; hence, it may be considered as a two-dimensional analog of the string, or stringer, $\mathbf{K}^{(1)}$ matrix. For the triangle, $\mathbf{K}^{(0)}$ was represented by a stiffness matrix which has not yet appeared in published form. It is based on a sandwich type element and has been in use at The Boeing Company since 1959. It is known to give good results for deflections and stresses in thin plates. The point of view adopted in this analysis is therefore similar to that used for the beam-column when $\mathbf{K}^{(1)}$ for the stringer is taken as the initial stress matrix.

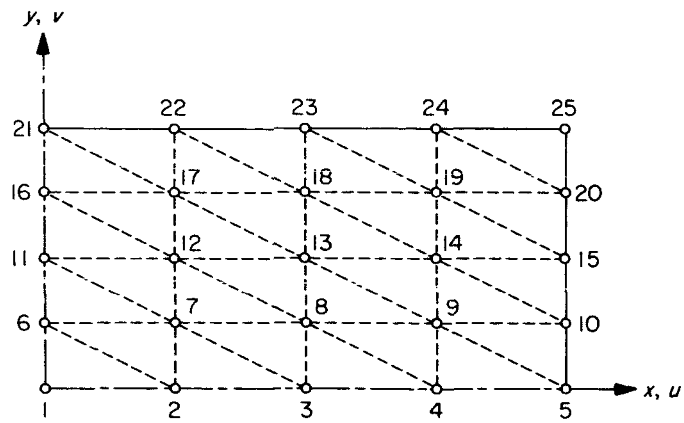


Fig. 15. Idealized plate quadrant containing 32 triangular elements

The incremental step procedure was carried out as follows:

1. Step 1 consisted of the strictly linear problem in which the 2.5-lb concentrated load was applied at the center of the plate. Nodal displacements, mid-plane stresses, and bending stresses were then determined for this step.

2. Step 2 started with the initial geometry and internal forces developed in Step 1. The load increment was taken as 100 psi compression applied on all four edges. This compressive stress was actually replaced by statically equivalent nodal forces in forming the stiffness equations. Nodal displacements and internal stresses were again calculated.

3. Steps, 3, 4, · · · consisted of similar calculations for edge stress increments of 100 psi. Results of these calculations are given in Table 2.

Table 2. Computer results—rectangular plate stability problem

Step	$\Delta\sigma$, psi	$\sigma = \Sigma\Delta\sigma$	Δw at $x = y = 0$	$\Sigma\Delta w$	$w^* = \Sigma\Delta w - 0.00575$	$\frac{w^*}{\sigma} \times 10^6$
1	0	0	0.00575	0.00575	0	–
2	100	100	0.000587	0.006337	0.00059	5.9
3	100	200	0.000736	0.007073	0.00132	6.6
4	100	300	0.000951	0.008024	0.00227	7.6
5	100	400	0.001287	0.009311	0.00356	8.9
6	100	500	0.001836	0.011147	0.00540	10.8
7	100	600	0.002823	0.013970	0.00822	13.7
8	100	700	0.004922	0.018892	0.01314	18.7
9	100	800	0.010558	0.029450	0.02370	29.6

From Table 2 we can now make the standard Southwell type plot as shown in Fig. 16. The decidedly straight line given by the data indicates the feasibility of this technique. From the theory σ_{crit} is given by the reciprocal of the slope of the straight line in Fig. 16 or

$$\sigma_{crit} = \frac{0.0237 - 0.00059}{(29.6 - 5.9)\,10^{-6}} = 975\,\text{psi}$$

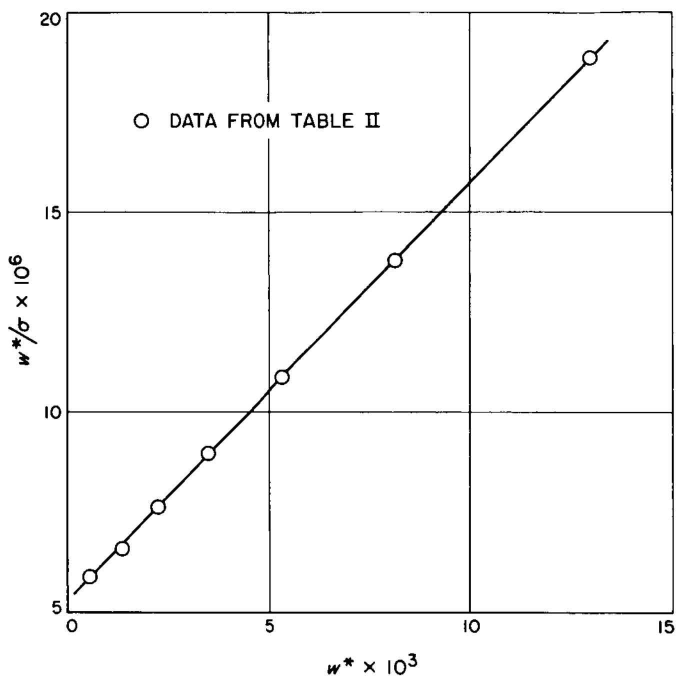


Fig. 16. Southwell type plot for plate stability problem

The corresponding theoretical solution can be computed (Ref. 4).

$$(\sigma_{crit})_{theor} \equiv 850\,\text{psi}$$

The error in the stiffness calculation is therefore 14.7%.

These initial calculations on plate stability are exploratory in the sense that they are too limited to give reliable indications of the eventual accuracy which may be achieved. Much serious work remains to be carried out before such questions can be realistically discussed. The purpose here is merely to demonstrate that the Southwell technique offers a useful approach for determining critical loadings.

F. Matrix Iterative Procedure

The practical value of the matrix iterative procedure for determining natural frequencies and principal modes of vibration of complex structural systems is well known to specialists in the field of structural dynamics. This numerical technique may be borrowed directly to compute critical loadings and the corresponding modes of buckling when using the stiffness method.

The stiffness matrix **K** for the structure (after imposing boundary conditions) is composed of $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ or, $\mathbf{K} = \mathbf{K}^{(0)} + \mathbf{K}^{(1)}$. If initial loading intensity factor F is separated out of $\mathbf{K}^{(1)}$, the expression may be written as

$$\mathbf{K} = \mathbf{K}^{(0)} + F\mathbf{K}^{(1)*}$$

At the critical load condition ($F = F_{crit}$) the force–deflection equation expresses the possibility of an arbitrary displacement state, **u**, existing in the absence of a column of external loads **X**. Or

$$(\mathbf{K}^{(0)} + F_{crit}\mathbf{K}^{(1)*})\,\mathbf{u} = \{0\}$$

This equation may be rewritten as

$$\frac{1}{F_{crit}}\mathbf{u} = -\,(\mathbf{K}^{(0)})^{-1}\mathbf{K}^{(1)*}\mathbf{u} \qquad (46)$$

which is the standard form for applying the matrix iterative method.

G. Stability of Tapered Column

This procedure and its application to stability calculations were first introduced in Ref. 5 and later reported in Ref. 7. In that reference, the critical loading for the tapered

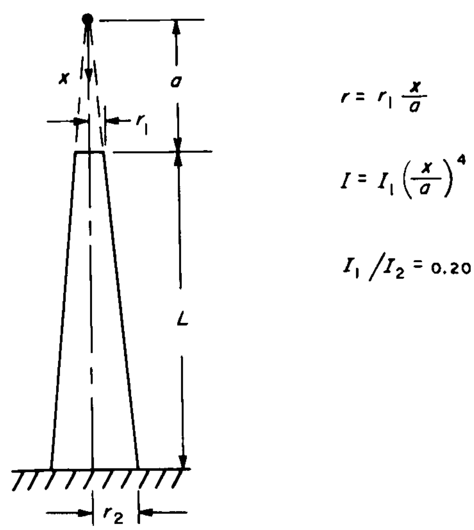


Fig. 17. Tapered column

column of Fig. 17 is investigated. The theoretical solution may be found in Ref. 4, and is given as

$$P_{\mathrm{crit}} = 1.505 \frac{EI_2}{L^2}$$

In Refs. 5 and 7 the iterative procedure is applied to stiffness matrices developed for two, four, and eight segment representations for the actual column. The numerical factors in the above equation were found to be 1.491, 1.502, and 1.5035 respectively.

Again, the calculations have only been carried out in a preliminary manner. For example, modes higher than that corresponding to the least value of P_{crit} have not been calculated. These can be readily obtained; on the other hand, theoretical results for the higher modes are not usually available for checking data provided by the stiffness method.

Application of the iterative procedure to plate and shell problems should, of course, be carried out at the earliest opportunity. A great deal of valuable practical knowledge will undoubtedly stem from such efforts. This will provide useful background information for properly exploiting the full potential of the stiffness method in treating stability problems of a truly complex character.

H. Additional Cases

Other interesting and important stability problems can be studied by applying the direct stiffness method. "Snap-through" problems represent one example. Such problems are characterized by a force–deflection diagram having a region of negative slope. In other words, several equilibrium states may exist for a fixed value of loading.

The study of such problems has only been undertaken in a very preliminary manner. They will not be discussed in the present Report.

VII. THE THIN PLATE ELEMENT

A. Introductory Remarks

A thin plate element is usually taken as a triangle or rectangle. Generalized nodal forces are established which are equivalent to a prescribed displacement state for the element. The stiffness matrix relates these generalized nodal forces to the corresponding nodal displacements. Such an element possesses in-plane or membrane stiffness, plus bending stiffness. This is analogous to axial force stiffness, plus bending stiffness, for the beam-column.

B. Membrane Stiffness Matrices

The membrane stiffness matrices for the triangle and the rectangle were first given in Ref. 2. Their derivation may be found in that reference. An alternative (and in some ways more satisfying) derivation of the membrane stiffness may be found in Refs. 8 and 11.

These membrane stiffness matrices have been used to determine displacements and stresses in a number of problems. It is now known that they provide a convenient and reliable tool for solving two-dimensional elasticity problems.

C. Bending Stiffness Matrices

Bending of thin plate elements has also been investigated. Published results for stiffness matrices have, however, been confined to the rectangular element. Several unpublished solutions for the determination of $\mathbf{K}^{(0)}$ of the triangular element in bending have been derived and used. Several have been in use at The Boeing Company in Seattle, Wash. One has appeared as a master's thesis at the University of Washington, Ref. 9.

In all, perhaps six derivations exist for bending of thin plate elements. It is known that at least some of these give good results for deflections, moments, shears, and vibration characteristics. The others have not been adequately checked due to lack of programming.

D. *Initial Force Stiffness Matrices*

Reference 3 presents a derivation for the $\mathbf{K}^{(1)}$ matrix of a triangular element carrying initial membrane stresses. This is a complex problem and the final matrix form is, in itself, quite formidable. It is suitable only for application on the high speed digital computer. It presents a large order programming problem and its application presents a large order computing problem.

The matrix given in Ref. 3 has been carefully checked, and indications are that the form as given is correct. For example, the solution of the plate problem in Section VI of this Report was obtained by using this $\mathbf{K}^{(1)}$ matrix. From this point of view, it can be programmed with confidence.

It is also believed that the form of $\mathbf{K}^{(1)}$ as given in Ref. 3 is unnecessarily complex. Therefore, a saving in programming and computing time could be gained by finding the *simplest form for this matrix.*

Some preliminary results have been obtained for $\mathbf{K}^{(1)}$ for the rectangular element. Since the bending of thin plates can often be conveniently and accurately described by the matrix for the rectangular element, it is undoubtedly of value to have the corresponding $\mathbf{K}^{(1)}$ matrix available. At present, however, it is more important to settle all questions of the form of $\mathbf{K}^{(1)}$ for the triangular element and then gain experience in its application to problems. Some interesting possibilities exist in this connection. For example, the matrix for the triangle may be used (by superposition) to determine $\mathbf{K}^{(1)}$ for a rectangular element. This will not be the same matrix as would be obtained from a derivation for $\mathbf{K}^{(1)}$ centered on the rectangle itself. Nevertheless, used in conjunction with $\mathbf{K}^{(0)}$ for the rectangle, it may still lead to satisfactory numerical results when applied to actual problems.

Reference 7 presents a different approach to the derivation of $\mathbf{K}^{(1)}$ for the triangular element. The derivation is based on some rather involved geometrical arguments and is not readily checked.

The derivation of $\mathbf{K}^{(1)}$ for a triangular thin plate element based on the equations of the nonlinear theory of elasticity and Castigliano's Theorem has been investigated in a thesis at the University of Washington (Ref. 10). The basis for selecting the necessary displacement functions u, v, and w is not carefully discussed, however. A detailed treatment of this question is a separate study in itself and beyond the scope of this present Report. The derivation in Ref. 10 is carried out in detail only for the special case of the isosceles right triangle.

In a recent paper (Ref. 11) the present author has applied the methods of this Report for deriving $\mathbf{K}^{(1)}$ matrices to the case of the arbitrary triangle in plane stress. The resulting matrix is very much simplified compared to the forms previously published in Refs. 3 and 7. When applied to the rectangular plate problem discussed in Sec. VI-E of this Report, the simpler form for $\mathbf{K}^{(1)}$ for the triangle, used in conjunction with the matrix iterative procedure of Sec. VI-F, leads to a first critical stress which agrees with the theoretical result within 2%.

NOMENCLATURE

A Cross-sectional area

e_{ij} Strains as defined in linear elasticity

E Modulus of elasticity

E_{ij} True physical strains

E_T Modulus of elasticity at elevated temperatures

I Moment of inertia in bending

$k_{ij}^{\alpha\beta}$ Stiffness coefficient

$\mathbf{K}$ Total stiffness matrix

$\mathbf{K}^{(0)}$ Conventional stiffness matrix, linear theory

$\mathbf{K}^{(1)}$ Initial stress stiffness matrix, nonlinear (elastic) theory

L Length

P^0 Initial loading present prior to incremental step

T Temperature increase, °F

$\mathbf{T}$ Transformation matrix

u, v, w Displacements of a point in the undeformed body

$\mathbf{u}$ Generalized displacements

U Strain energy

x, y, z Coordinates of a point in the undeformed body

$\mathbf{X}$ Generalized loads

$\Delta\mathbf{u}$ Incremental displacements

$\Delta\mathbf{X}$ Incremental loads

ε^0 Initial strain present prior to incremental step

ε^a Additional strain developed during incremental step

ε_{ij} Mathematically defined strains used in stiffness derivations

θ Slope of bent beam

λ, μ Direction cosines

ν Poisson's ratio

ξ, η, ρ Coordinates of the point after deformation

σ, τ Normal and shear stresses

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