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DETERMINATION OF ORBIT OF A SPACECRAFT WITH RESPECT TO AN OBJECT IN A KNOWN CIRCULAR ORBIT

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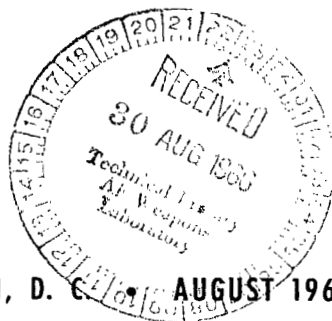
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SUMMARY

A set of second-order differential equations of motion for a body in a planetocentric orbit has been derived and solved with the use of a cylindrical "shell" coordinate system which describes the motion of an orbital vehicle as observed from another body in a known circular orbit about the planet. The solutions have been investigated for the specific case of a body in orbit about the moon. The results of the study indicate improvement in accuracy over first-order theory with no serious adverse sensitivity to initial conditions and with applicability over a wide range of flight conditions. Orbit prediction is accurate over relatively long trajectories with large spatial separation of the vehicles. Some parts of the study are applicable to a manual apogee and perigee prediction scheme.

INTRODUCTION

In this paper, solutions are presented to a set of equations which describe the motion of a space vehicle in orbit about a planetary body as this motion would appear if seen from another space vehicle in orbit about the same planetary body. Thus, this report is concerned primarily with the analysis of the motion of a body in a moving relative coordinate system.

The method differs from the second-order method, which has been used successfully in references 1 and 2, in that the coordinate system is chosen in such a manner as to eliminate or render negligible coupling between the various directions of translational motion in the mathematical treatment. The coordinate system used is similar to the shell system of reference 3 where, however, only first-order terms were considered. In this study, a cylindrical shell system has been adopted in preference to the spherical shell system of the latter reference. The method of attack is to examine the equations of motion of one vehicle as viewed from the frame of reference of the other vehicle by means of a set of second-order differential equations of motion. By making use of an exact first integral of one of the equations, solutions can be obtained that are accurate to second order. A simple perigee prediction method is also suggested which may be suitable for manual solution.

A specific application of this analysis is the problem of trajectory prediction for a vehicle which is launched from a lunar orbit in an attempt to land on the lunar surface. The numerical computations are based on a circular orbit 200 kilometers above the lunar surface. The mathematical development is, however, thought to be quite general and, hence, is thought to be applicable to orbits about any spherical, gravitational body.

SYMBOLS

A   integration constant for equation (17) defined in equation (20)

a   constant of integration which describes amplitude of departure from reference circular orbit

c   constant of integration determined by initial conditions, defined in equation (20)

B, C, D, E, F   constants determined by initial conditions defined in equation (20)

H   total energy of landing vehicle

i, j, k   orthogonal coordinates

K   constant, $\dfrac{\dot{x}_o}{\omega r_s} - 1$

L   Lagrangian, potential energy subtracted from kinetic energy

M   translation substitution used to solve equation (A30) and defined in equation (A36)

m   mass of landing vehicle

r   distance from center of planet to orbital vehicle

r_m radius of attracting body

r_s distance from center of attracting body to reference orbit

t   time

u	dummy variable, $a \cos \epsilon$
V	substitution variable used to solve equation (A30)
X,Y,Z	nondimensional coordinates of shell coordinate system centered on a body moving in a circular orbit about an attracting planet or satellite
x,y,z	dimensional coordinates of shell coordinate system centered on a body moving in a circular orbit about an attracting planet or satellite (see fig. 1)
α, β, λ	functions of initial velocity component in x-direction defined in equation (20)
Δ	function of a power expansion in ρ defined in equation (20)
ϵ	arbitrary constant of integration which describes epoch angle with respect to reference circular orbit
θ	angular coordinate in a cylindrical system with origin at center of attracting body and lying in plane of reference circular orbit, measured from positive x-direction clockwise
γ, η, κ	constant functions of initial out-of-plane velocity defined in equation (20)
μ	gravitational constant
ρ	function of α, β, and λ defined in equation (20), hence a function of initial velocity in x-direction only
τ	nondimensionalized or scaled time, ωt
ϕ	ejection angle (see fig. 1)
ψ	function of a power expansion in ρ defined in equation (20)
ω	angular velocity
$\overline{\Omega}$	directional angular velocity

Subscripts:

max   maximum

o   initial conditions

p   perigee

Dots over symbols denote differentiation with respect to time or scaled time depending upon the section of the report in which they appear. Primes refer to first-order solution quantities. For constants, any consistent set of units may be used. In this paper the following values, which apply to the moon, were chosen:

$$r_m = 1736.5 \text{ kilometers}$$

$$r_s = r_m + 200 = 1936.5 \text{ kilometers}$$

$$\mu = 4.8936 \times 10^{12} \text{ meter}^3\big/\text{second}^2$$

$$\omega = 0.00082086 \text{ radians/second}$$

COORDINATES AND EQUATIONS OF MOTION

Two assumptions are made about the physical nature of the problem:

(1) The attracting planetary mass is a gravitational sphere.

(2) The body upon which the coordinate system is centered is in a circular orbit.

The coordinate system employed in this development is shown in figure 1 where r is the projection of a line connecting the center of the planet to the orbital vehicle upon the plane of the reference vehicle, z is a normal to this plane passing through the orbital vehicle, y is measured along r from the reference vehicle altitude to the projection of the maneuvering vehicle with the positive direction upward, and x is measured in a curved arc backward along the flight path of the reference vehicle in the plane of the reference vehicle orbit to r. The coordinate system rotates about the origin with angular velocity ω.
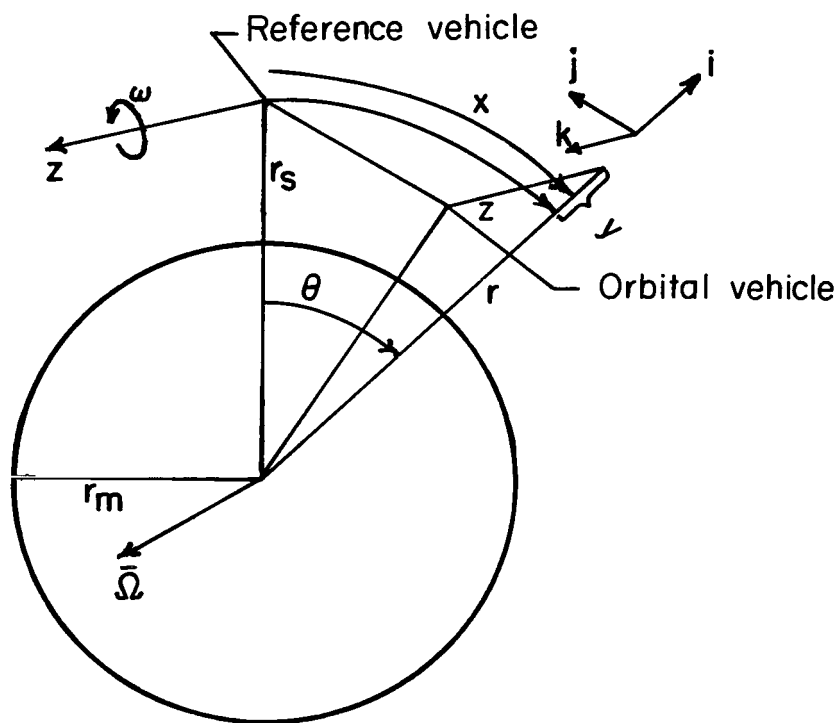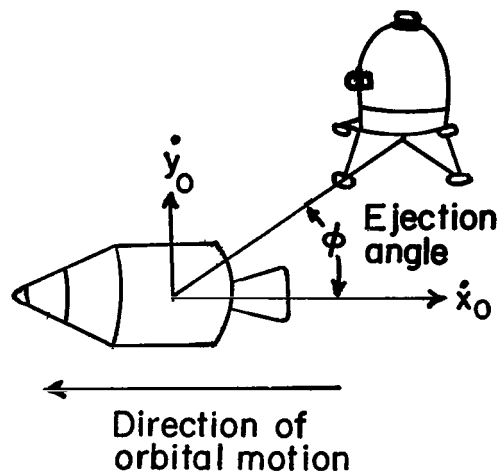
Figure 1.- Coordinates employed in describing the motions of the vehicles.

This motion may be determined from the equation for the total energy and this method leads to an exact solution. However, this approach yields time as a function of y transcendentally and thus is not thought to be desirable since time normally is the available independent variable.

The derivation of the equations of relative motion is given in detail in appendix A; therefore, only the more pertinent results are reviewed in this section.

The equations of motion can be derived through the use of the Lagrangian which, for cylindrical coordinates, is

$$L = \frac{1}{2} m \left[\dot{r}^2 + r^2(\dot{\theta} - \omega)^2 + \dot{z}^2 \right] + m \frac{\mu}{\sqrt{r^2 + z^2}} \tag{1}$$

Equation (1) is converted to shell coordinates (see appendix B) by means of the following substitutions:

$$y + r_s = r$$

$$x = r_s \theta$$

$$z = z$$

and use is made of the exact orbital expression for the angular velocity

$$\omega^2 = \frac{\mu}{r_s^3}$$

to express the equations in the cylindrical shell system. Since the x-coordinate is cyclic, a first integral to its differential equation of motion is found immediately. The resulting differential equations of motion are

$$\left(\frac{\dot{x}}{\omega r_s} - 1 \right) \left(1 + \frac{y}{r_s} \right)^2 = K \tag{2}$$

$$\frac{\ddot{y}}{\omega^2 r_s} - \left(1 + \frac{y}{r_s} \right) \left\{ \left(1 - \frac{\dot{x}}{\omega r_s} \right)^2 - \left[\left(1 + \frac{y}{r_s} \right)^2 + \left(\frac{z}{r_s} \right)^2 \right]^{-3/2} \right\} = 0 \tag{3}$$

$$\frac{\ddot{z}}{\omega^2 r_s} + \frac{z}{r_s} \left[1 + \left(\frac{y}{r_s} \right)^2 + \left(\frac{z}{r_s} \right)^2 \right]^{-3/2} = 0 \tag{4}$$

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By solving equation (2) for $1 - \dfrac{\dot{x}}{\omega r_s}$, it becomes possible to substitute the solution into equation (3) and hence remove all terms in x from the latter. Then, equation (3) is coupled to equation (4) only through the z-term. It is, however, found from experience that in orbits which are economically feasible, z is much smaller than y. Hence, there will be only a small amount of error in equation (3) if the term $\left(\dfrac{z}{r_s}\right)^2$ is assumed to be negligible even though this term is, strictly speaking, a second-order term in the mathematical sense. A specific example of this relationship can be seen in figure 2 which shows the ratio of maximum z amplitude to maximum y amplitude for a typical lunar synchronous orbit over a range of economically feasible initial z velocity values. It is also found expedient to expand terms of the form $\dfrac{1}{\left(1 + \dfrac{y}{r_s}\right)^n}$ when they occur.

Figure 2.- Effect of maximum z excursions on perigee altitude for synchronous orbits using exact equations.

If the assumptions indicated are made, the equations which are to be solved are

$$\frac{\dot{x}}{\omega r_s} + \frac{2Ky}{r_s} - (K - 1) = \frac{3Ky^2}{r_s^2} \tag{5}$$

$$\frac{\ddot{y}}{\omega^2 r_s} + \frac{\alpha^2 y}{r_s} - \beta^2 = -\frac{\lambda y^2}{r_s^2} \tag{6}$$

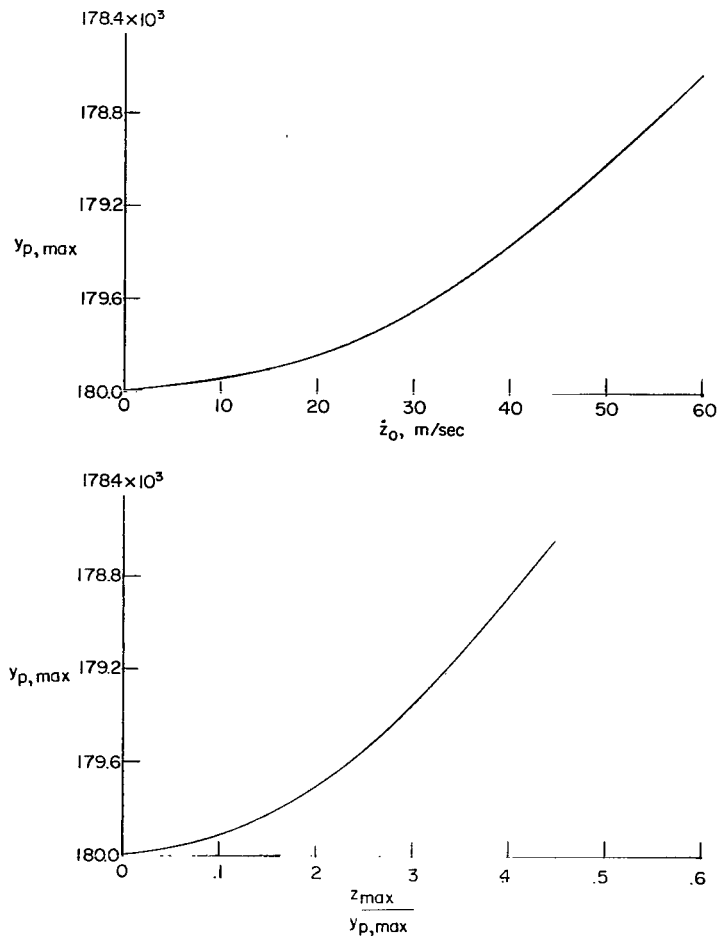$$\frac{\ddot{z}}{\omega^2 r_s} + \frac{z}{r_s} = \frac{3yz}{r_s^2} \tag{7}$$

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where $\dot{x}_o$, $\dot{y}_o$, and $\dot{z}_o$ are arbitrary initial velocities and where

$$K = \frac{\dot{x}_o}{\omega r_s} - 1$$

and

$$\alpha^2 = 3K^2 - 2$$

$$\beta^2 = K^2 - 1$$

$$\lambda = -6K^2 + 3$$

In order to get an intuitive feel for the values of these constants, it is pointed out that ωr_s is the speed of the orbiting vehicle and hence that $\frac{\dot{x}_o}{\omega r_s}$ is never likely in a practical case to approach unity. In fact, for purposes of this report, it is assumed that $\dot{x}_o$ is around 150 m/sec or less for the moon. Under these circumstances $\frac{\dot{x}_o}{\omega r_s}$ will be less than 0.1. Thus, for a rough approximation as to order of magnitude, K is on the order of -1, α^2 is approximately unity, β^2 is very small but can be either positive or negative, and λ will be on the order of -3.

The initial conditions assigned are those at the time of ejection when $t = 0$, $x = y = z = 0$, and $\dot{x} = \dot{x}_o$, $\dot{y} = \dot{y}_o$, $\dot{z} = \dot{z}_o$ for all the cases studied in this report. Equation (6) is cast in the form of a one-dimensional anharmonic oscillator and hence can be solved with y as a time-dependent variable (ref. 4 and eq. (A31)). The solution of the equation of a one-dimensional anharmonic oscillator however is an elliptic integral and since this solution would prove difficult to incorporate in the solution of equations (5) and (7), a perturbation solution is accepted. This solution is then used in a similar manner to obtain an integral of equation (5) and, along with a first-order solution for equation (7), can also be used to provide a second-order perturbation solution for equation (7). Hence, equations (5), (6), and (7) can be solved. A complete treatment of the solution of these equations is given in appendix A.

The following solutions are to first order, where primes have been used to distinguish the integration constants from those which will occur in second-order theory

$$\frac{x}{r_s} = \left(\frac{\dot{x}_o}{\omega r_s} - \frac{2K\beta^2}{\alpha^2}\right)\omega t - \frac{2Ka'}{\alpha}\sin(\alpha\omega t + \epsilon') + \frac{2Ka'}{\alpha}\sin\epsilon' \qquad (8)$$

$$\frac{y}{r_s} = a' \cos(\alpha\omega t + \epsilon') + \frac{\beta^2}{\alpha^2} \tag{9}$$

$$\frac{z}{r_s} = \frac{\dot{z}_0}{\omega r_s} \sin \omega t \tag{10}$$

and where the integration constants a' and ϵ' are given by

$$a' = \frac{1}{\alpha}\sqrt{\frac{\beta^4}{\alpha^2} + \frac{\dot{y}_0^2}{\omega^2 r_s^2}} \tag{11}$$

$$\epsilon' = \cos^{-1} \frac{-\beta^2}{\sqrt{\beta^4 + \frac{\alpha^2 \dot{y}_0^2}{\omega^2 r_s^2}}} \tag{12}$$

Equations (11) and (12) must be used with a sign selection scheme. In order to do so, equation (9) can be expressed in a deterministic form without the epoch angle as

$$\frac{y}{r_s} = -\frac{\beta^2}{\alpha^2} \cos \alpha\omega t + \frac{\dot{y}_0}{\alpha\omega r_s} \sin \alpha\omega t + \frac{\beta^2}{\alpha^2} \tag{13}$$

Comparison of equations (13) and (9) gives

$$a' \cos \epsilon' = -\frac{\beta^2}{\alpha^2} \tag{14}$$

$$a' \sin \epsilon' = -\frac{\dot{y}_0}{\alpha\omega r_s} \tag{15}$$

Hence,

$$\tan \epsilon' = \frac{-\dfrac{\dot{y}_0}{\alpha\omega r_s}}{-\dfrac{\beta^2}{\alpha^2}} \tag{16}$$

Since α is a frequency, which is a physical quantity, a negative α has no meaning. Hence, with α positive, the quadrant for ϵ' is determined by the respective signs in the numerator and denominator. Also β^2 is strictly a function of $\dot{x}_0$ and always has the opposite sign, $\beta^2 = \dfrac{\dot{x}_0}{\omega r_s}\left(\dfrac{\dot{x}_0}{\omega r_s} - 2\right)$, since $\dfrac{\dot{x}_0}{\omega r_s} \ll 2$. It then follows, once the quadrant for ϵ' is known, that equations (14) and (15) give the sign for a' to be selected. It is found that a' is always positive. These characteristics for all possible ejection quadrants are given in table I.

Solutions to the same differential equations but carried out to second order are

$$x = r_s\left[A + B\omega t + C\,\sin(\rho\omega t + \epsilon) + D\,\sin 2(\rho\omega t + \epsilon) + E\,\sin 3(\rho\omega t + \epsilon) + F\,\sin 4(\rho\omega t + \epsilon)\right]$$

(17)

$$y = r_s\left\{a\,\cos(\rho\omega t + \epsilon) - \frac{\lambda a^2}{2\rho^2}\left[1 - \frac{1}{3}\cos 2(\rho\omega t + \epsilon)\right] - \frac{\left(\alpha^2 - \rho^2\right)}{2\lambda}\right\}$$

(18)

$$z = r_s\left\{\frac{\dot{z}_0}{\omega r_s}\sin\omega t + \sin\omega t\left[\frac{\gamma\psi}{4}\cos(\rho\omega t + \epsilon) + \frac{\eta\Delta}{4}\cos 2(\rho\omega t + \epsilon) + \frac{\kappa}{4}\right]\right.$$

$$\left. - \cos\omega t\left[\frac{\gamma\psi}{2\rho}\sin(\rho\omega t + \epsilon) + \frac{\eta\Delta}{4\rho}\sin 2(\rho\omega t + \epsilon) + \frac{\kappa\omega t}{2} + \frac{c}{2}\right]\right\}$$

(19)

where the constant terms are given by

$$K = \frac{\dot{x}_0}{\omega r_s} - 1$$

$$\alpha^2 = 3K^2 - 2$$

$$\beta^2 = K^2 - 1$$

$$\lambda = -6K^2 + 3$$

$$\rho = \left(4\lambda\beta^2 + \alpha^4\right)^{1/4} = \left(-15K^4 + 24K^2 - 8\right)^{1/4}$$

$$A = -(C \sin \epsilon + D \sin 2\epsilon + E \sin 3\epsilon + F \sin 4\epsilon)$$

$$B = K\left[\frac{\dot{x}_0}{\omega r_s K} + \frac{\left(\alpha^2 - \rho^2\right)}{\lambda} + \frac{3\left(\alpha^2 - \rho^2\right)^2}{4\lambda^2} + \frac{a^2\lambda}{\rho^2} + \frac{3\alpha^2 a^2}{2\rho^2} + \frac{19}{24}\frac{\lambda^2 a^4}{\rho^4}\right]$$

$$C = -K\left[\frac{2a}{\rho} + \frac{3a\left(\alpha^2 - \rho^2\right)}{\lambda\rho} + \frac{5\lambda a^3}{2\rho^3}\right]$$

$$D = K\left(\frac{a^2}{\rho} - \frac{a^2\alpha^2}{4\rho^3} - \frac{\lambda\alpha^2}{6\rho^3} - \frac{\lambda^2 a^4}{4\rho^5}\right)$$

$$E = \frac{K\lambda a^3}{6\rho^3}$$

$$F = \frac{K\lambda^2 a^4}{96\rho^5}$$

$$\kappa = \frac{-3\dot{z}_0}{\omega r_s}\left[\frac{\lambda a^2}{2\rho^2} + \frac{\left(\alpha^2 - \rho^2\right)}{2\lambda}\right]$$

$$\eta = \frac{\lambda a^2 \dot{z}_0}{2\omega r_s \rho^2}$$

$$\gamma = \frac{3a\dot{z}_0}{\omega r_s}$$

$$\Delta = 1 + \rho^2 + \rho^4$$

$$\psi = 1 + \frac{\rho^2}{4} + \frac{\rho^4}{16}$$

$$c = -\frac{\gamma\psi}{\rho} \sin \epsilon - \frac{\eta\Delta}{2\rho} \sin 2\epsilon$$

$$\left.\right\} \quad (20)$$

The determination of the integration constants, a and ϵ, is somewhat more involved under second-order theory because of the necessity of solving a set of simultaneous transcendental equations. Taking the y-equation and its first time derivative with the same end conditions as previously applied (namely, at $t = 0$, $x = 0$, $y = 0$, $\dot{x} = \dot{x}_0$, and $\dot{y} = \dot{y}_c$) and making the substitution $u = a \cos \epsilon$ yields the quartic equation

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$$-\frac{2\lambda^2}{9\rho^4} u^4 + \left(\frac{3}{2} + \frac{\rho^2 - \alpha^2}{3\rho^2}\right)u^2 + \left(\frac{5\rho^2 - 2\alpha^2}{2\lambda}\right)u + \left[\frac{3}{4}\frac{\rho^2(\rho^2 - \alpha^2)}{\lambda^2}\right] - \left(\frac{\dot{y}_0}{\omega r_s}\right)^2 \frac{1}{\rho^2} = 0 \qquad (21)$$

It is necessary to solve this equation numerically. Of the four possible values of u which satisfy this equation, only one will apply; this value is the value which approximates the expected amplitude of the y-expression. The computations made for this report give two imaginary roots and two real roots. The extraneous real root was found to be an order of magnitude larger than the correct root. Hence, comparison with first-order theory $\left(\text{where it is found that } u' = -\frac{\beta^2}{\alpha^2}\right)$ affords a comparatively simple method of separating the roots.

From equation (A58) and the definition of u,

$$a = \sqrt{\left[\frac{3\rho\dot{y}_0}{\omega r_s\left(3\rho^2 + 2\lambda u\right)}\right]^2 + u^2} \qquad (22)$$

and

$$\epsilon = \cos^{-1}\frac{u}{a} \qquad (23)$$

It is assumed that approximately the same sign and quadrant selection scheme holds for the second-order integration constants as for the first-order integration constants, that is, a is always positive with the quadrant for ϵ determined by ϕ. This relationship, however, is only approximately true. Examination of equation (21) for the special case where $u = 0$ gives for $\dfrac{\dot{y}_0}{\omega r_s}$

$$\frac{\dot{y}_0}{\omega r_s} = \pm\sqrt{-\frac{3}{4}\frac{\rho^4\left(\alpha^2 - \rho^2\right)}{\lambda^2}} \qquad (24)$$

The terms on the right in equation (24) are functions of $\dot{x}_0$ only. Sketch (1) (see appendix A) shows how $\dot{X}_0$ is related to $\dot{Y}_0$ when $u = 0$. This difference is the primary difference between first- and second-order theory. The sign on u along with the ejection angle specifies the quadrant for ϵ. This quadrant selection is summarized in

table II. Since the terms under the radical are all functions of $\dot{x}_O$ only, the ejection angle is related to $\dot{x}_O$ by

$$\tan \phi = \frac{\dot{y}_O}{\dot{x}_O} = \pm \frac{1}{\dot{x}_O} \sqrt{\frac{-3\rho^4\left(\alpha^2 - \rho^2\right)}{4\lambda^2}}$$

The parameters ρ^2, α^2, and λ^2 are functions of $\dot{x}_O$ only; thus the quadrant of ϵ changes depending upon ϕ, but no longer simply at $\pi/2$, π, $3\pi/2$, and 2π as was true for the first-order solutions. Figure 3 is a plot of $\dot{x}_O$ as a function of ϕ for $u = 0$ in dimensional terms which apply to the moon. The figure can be used therefore to determine for the known value of $\dot{x}_O$ and ϕ, the proper quadrant for ϵ. As in first-order theory, a is always selected positive.
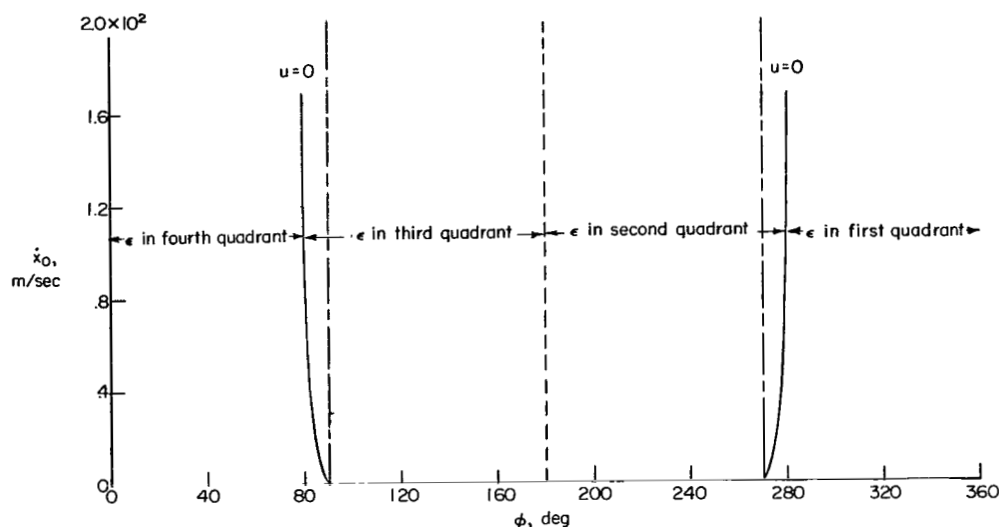

Figure 3.- $u = 0$ as a function of $\dot{x}_O$ and ejection angle ϕ.

PHYSICAL INTERPRETATION OF TERMS

The physical meaning of several of the terms in these solutions can be seen if the first-order solutions and second-order solutions are compared.

y-Equation

The most important from the standpoint of applications is the y-equation. It will be best to begin by comparing equation (9) (first-order solution) with equation (18) (second-order solution). In first-order theory, a sinusoidal term is added to a constant which is a function of initial velocity in the x-direction. Hence, the constant merely reflects a change in energy produced by ejection from the reference orbit and is a function of the

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speeding up or slowing down of the orbital vehicle's angular rate. Superimposed upon this motion is an oscillatory motion above and below the mean altitude, the maximum amplitudes of which are apogee and perigee. In second-order theory, the same fundamental characteristics appear. In this case, however, they are modified slightly to take better account of the inverse square nature of the planetary gravity field. The form is essentially the same as first-order theory presents, but the constant term is slightly modified to become the coefficient of the second term combined with the last term on the right in equation (18). In like manner, the oscillatory term is present although the coefficient is changed slightly. Hence, these two terms represent the same type of motion as was present in the first-order theory. Superimposed upon this term is an oscillatory term of twice the frequency of the primary term acting very much as a second harmonic. The primary effect of this term is to increase the apogee and to decrease the perigee. The function of this term is then to take account of the weakening of the restoring force as the gravity field decreases at longer distances from the center of the planet.

Another way of viewing this phenomenon can be seen in figure 4 which shows the acceleration of the lander relative to the reference orbit in terms of exact first-order and second-order theory for a typical case. It can be seen that acceleration is a linear function of displacement over a reasonable altitude range under first-order theory.
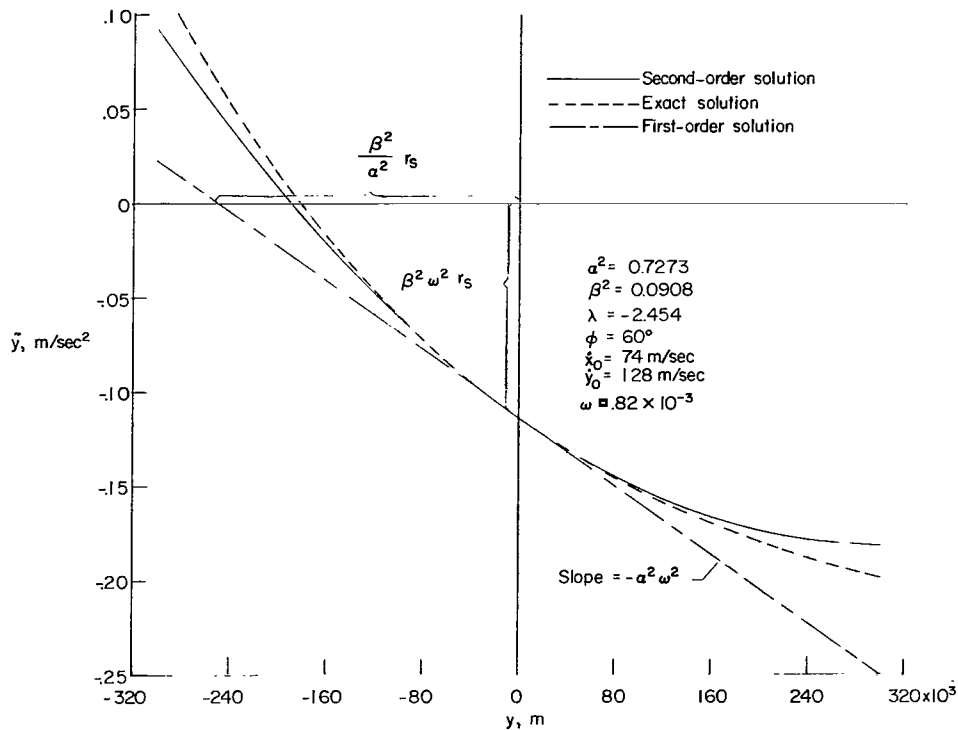

Figure 4.- Acceleration of the orbital vehicle lander as a function of departure distance from the reference orbit for a typical case.

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Under second-order theory, the acceleration is quadratic and forms a much better approximation to the exact conditions. In figure 4, β^2/α^2 constitutes a change in mean equilibrium altitude and shows that, in general, the point of equilibrium is displaced, in this case downward. It is pointed out that this change will occur even for orbits of the same total energy as the reference orbit since orbits of the same period but different eccentricity will have a different mean altitude. Under second-order theory this mean displacement is defined a little more precisely by

$$-r_s \left(\frac{\alpha^2 - \rho^2}{2\lambda} + \frac{\lambda a^2}{2\rho^2} \right)$$

(not shown in fig. 4). It can be seen that under the special condition that $\lambda = 0$, this relation reduces to β^2/α^2 as in first-order theory.

x-Equation

Comparison of the x-equation (eq. (17)) with first-order theory (eq. (8)) shows that the first-order equation contains three main terms: a constant, a secular term, and a sinusoidal term. From the physics of the situation, it is clear that the phenomenon which was manifested as a constant mean separation in the y-equation (eq. (18)) is a time-dependent linear mean drift rate in the x-equation (eq. (17)). In general, the ejected vehicle would be expected to drift away over an interval of several orbital periods due to its difference in energy and angular momentum. For a synchronous orbit, of course, the coefficient of the secular term would be zero; thus, this term controls the rate of drift.

In other respects the form of this equation is about the same as is obtained for the y-equation, that is, a series of higher order trigonometric terms.

z-Equation

In the z-equation, first-order theory predicts simple harmonic motion; so again there is a linear restoring force. This theory then assumes that altitude y does not have any appreciable effect upon the out-of-plane mode z. Although in trajectories of practical interest it is quite true that z does not have any marked effect upon y because the orbits are nearly coplanar, the converse of this statement is not true, and it is found that for orbits of moderately large eccentricity (for instance, on the order of 0.1), y-coupling into the z-equation can be very significant relative to the total, but admittedly small, amplitude in z. In fact, for the typical snychronous lunar orbit used for the numerical calculations in this paper, the coupled term in y and z amounts to one-third of the value of the pure z-term. Consequently, the second-order terms in equation (19) represent coupling terms with y.

APOGEE AND PERIGEE PREDICTION

The total orbital energy can be used in the following manner to predict apogee and perigee:

Equation (5) is substituted into the orbital energy expression

$$H = \frac{1}{2} m \left[\dot{y}^2 + \dot{z}^2 + \left(y + r_s\right)^2 \left(\frac{\dot{x}}{r_s} - \omega\right)^2 - \frac{2\mu}{\sqrt{\left(y + r_s\right)^2 + z^2}} \right] \tag{25}$$

The out-of-plane terms, z and $\dot{z}$, are neglected since they are small.

Use of equation (5) yields

$$\overline{C} = \frac{H}{r_s^2 \omega^2 m} = \frac{1}{2} \left[\frac{\dot{y}^2}{\omega^2 r_s^2} + \frac{\left(1 + \frac{y_0}{r_s}\right)^2}{\left(1 + \frac{y}{r_s}\right)^2} \left(\frac{\dot{x}_0}{\omega r_s} - 1\right)^2 - \frac{2}{\left(1 + \frac{y}{r_s}\right)} \right] \tag{26}$$

where $\overline{C}$ is constant, and under the initial conditions assumed, for example, at $t = 0$, $y = 0$, $\overline{C}$ becomes

$$\overline{C} = \frac{1}{2} \left[\frac{\dot{y}_0^2}{\omega^2 r_s^2} + \left(\frac{\dot{x}_0}{\omega r_s} - 1\right)^2 - 2 \right] \tag{26a}$$

The extremals are then found to be

$$y_{extremal} = \left[\frac{-1 \pm \sqrt{1 + 2 \left(\frac{\dot{x}_0}{\omega r_s} - 1\right)^2 \overline{C}}}{2\overline{C}} - 1 \right] r_s \tag{27}$$

The positive value is apogee and the negative value is perigee as measured from orbital altitude. It is felt that this relation is simple enough for a pilot to use in a manual computation.

TEST CASES

In order to test the properties of the solutions, the exact differential equations, first-order solution equations, and second-order solution equations were programed on a

digital computer. The test orbital conditions were taken as those for ejection from a 200-kilometer circular orbit about the moon. The period of the orbit is 7600 seconds. Two particular trajectories were studied in detail. These trajectories were a synchronous orbit and a Hohmann transfer to the vicinity of the surface. The latter trajectory will henceforth be referred to as the Hohmann case. Both trajectories had a perigee point 20 kilometers above the surface of the moon. Such orbits are suitable either for reconnaissance or for landing, and both orbits are of interest for the Apollo mission.
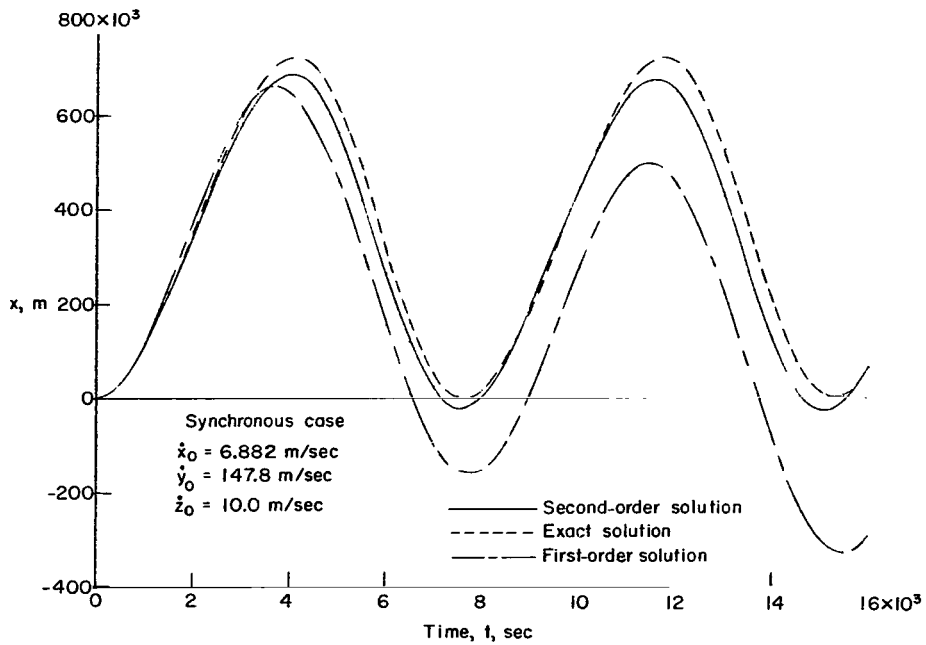
Synchronous Case

The chief advantage of selecting the synchronous orbit is that since it returns to its initial relative position after one revolution, the investigator is in a position to interpret errors in the trajectories in terms of the particular terms in the solutions to which they are due. Trajectories were run for two separate synchronous cases: one in which there was no out-of-plane velocity component and one in which the initial out-of-plane velocity component was 10 meters per second. The latter corresponds to an orbit plane change of 0.36°. This out-of-plane velocity was used to show that coupling in the z-direction is very slight. In fact, it can be seen in figures 5 and 6 that there is no observable difference between the two figures to the accuracy of the plots. It can also be seen that the second-order solutions predict the time of occurrence of apogee and perigee much better than do the first-order solutions.

Ejection was in an upward direction so that apogee is reached at approximately one-quarter orbit and perigee at three-quarters orbit. Time histories showing the main results of this study are shown in figures 5 and 6. Particular points of interest are:

(1) The error at perigee in these cases for the second-order solution is 5 kilometers. The second-order solution is therefore an improvement over the first-order solution where the error is 20 kilometers and seems to indicate that the equations will indeed be useful as a prediction method for lunar landing or reconnaissance vehicles.

(2) The second-order y-equation is in error timewise at the end of one orbit by 200 seconds. Since the period for this orbit is about 6800 seconds, this error is considered to be relatively large. The x-equation is in error timewise by the same amount. The reason for this time error is found in the nature of the anharmonic oscillator equation employed in solving for y. It can be shown that when the equation is restricted to second-order terms, the period is not a function of amplitude. If terms of higher order had been carried, the period would be a function of amplitude and the time error would be considerably smaller (ref. 4).
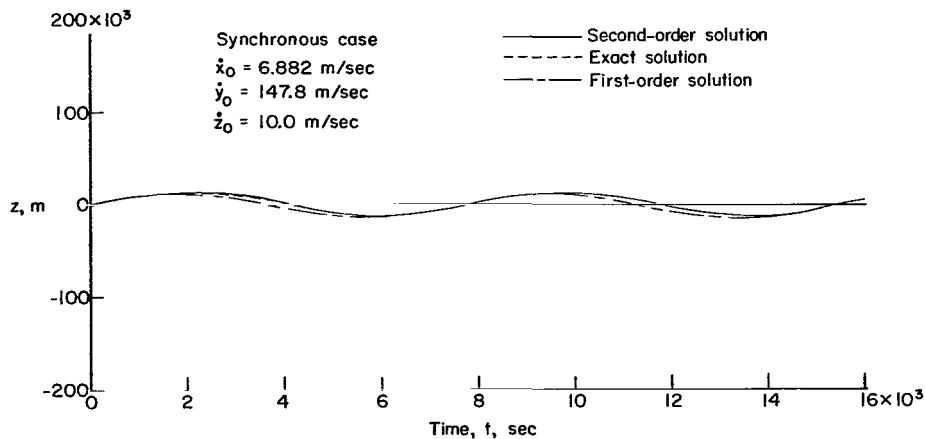
(a) x-direction.

Figure 5.- Time history for a 200-kilometer synchronous lunar orbit showing exact, first-order, and second-order solutions.
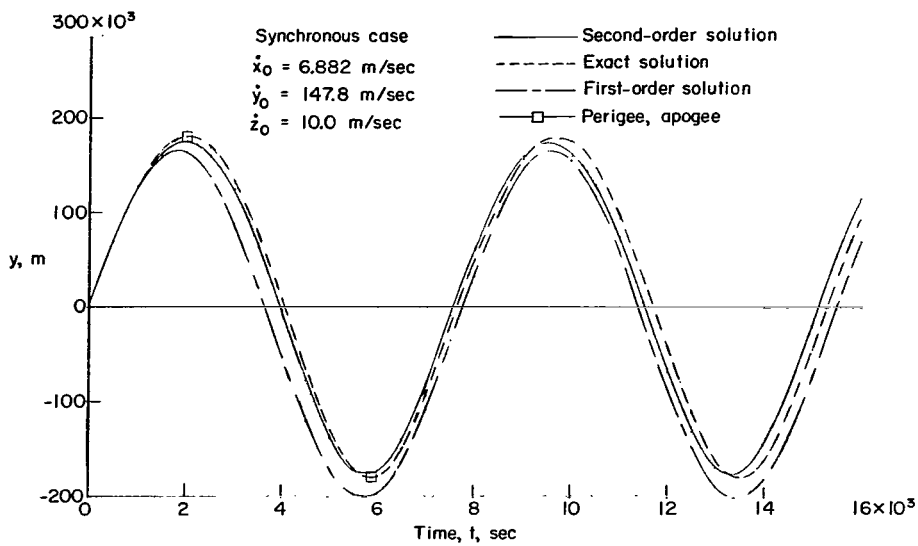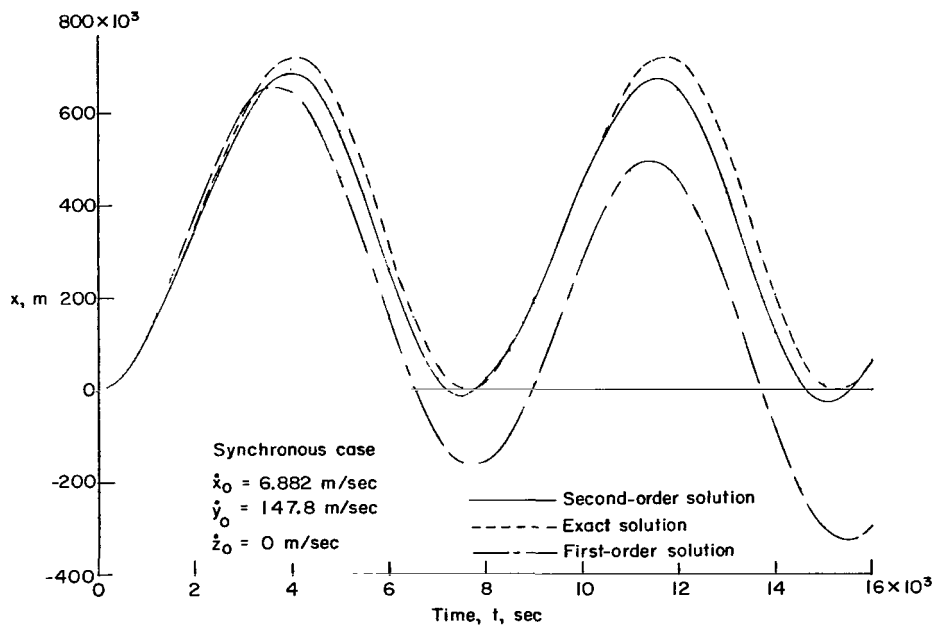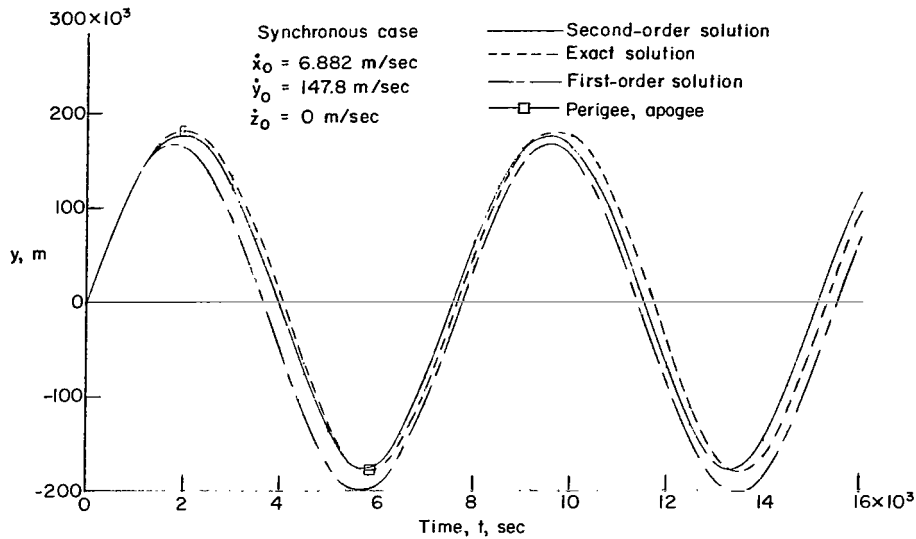

(b) y-direction.

Figure 5.- Continued.

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(c) z-direction.

Figure 5.- Concluded.

(a) x-direction.

Figure 6.- Time history for a 200-kilometer synchronous lunar orbit showing exact, first-order, and second-order solutions with no initial out-of-plane velocity component.

Synchronous case

$\dot{x}_0$ = 6.882 m/sec
$\dot{y}_0$ = 147.8 m/sec
$\dot{z}_0$ = 0 m/sec

—————— Second-order solution
— — — — Exact solution
—·—·— First-order solution
—□— Perigee, apogee

(b) y-direction.

Figure 6.- Concluded.

Figure 7 is a plot of the variation of x with y for the synchronous orbit of figure 5. Hence, this shows the position of the ejected vehicle as it would be seen from the ejecting vehicle if the out-of-plane motion is disregarded. It is to be observed that in spite of the time error in both x and y, the spatial agreement between the exact and approximate second-order equations is very good.

—————— Second-order solution
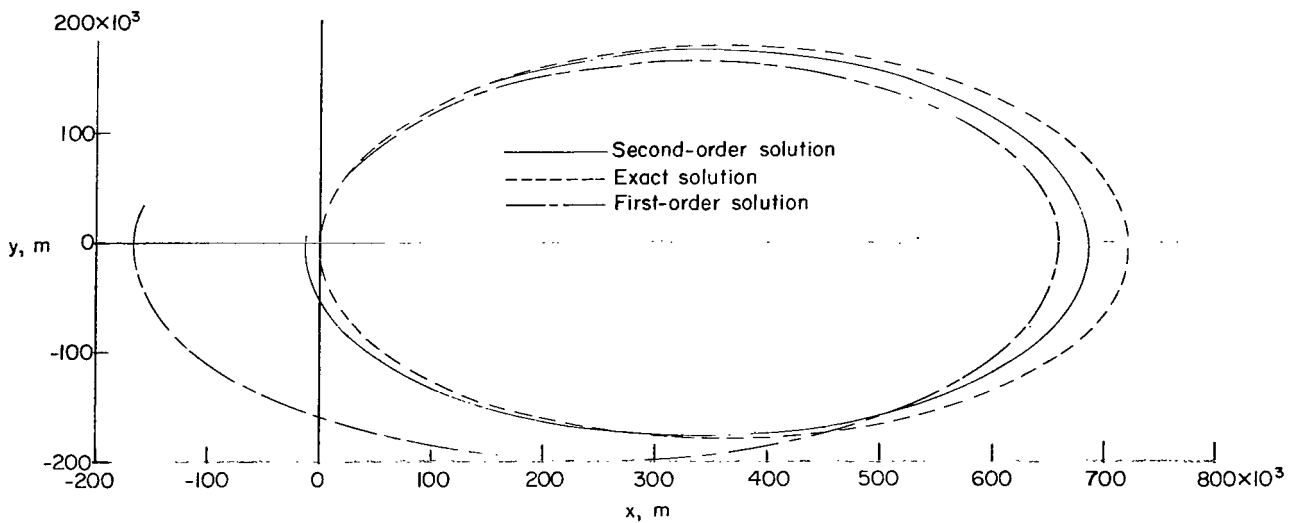— — — — Exact solution
—·—·— First-order solution

Figure 7.- Variation of x with y for a 200-kilometer synchronous lunar orbit showing the respective positions of the landing vehicle as seen from a vehicle in reference orbit.

Hohmann Case

Figures 8, 9, and 10 show essentially the same information as figures 5, 6, and 7, but for the Hohmann case. First-order data have not been included in these cases. The agreement between exact and second-order solutions is better for the y-motion than it is for the synchronous orbit since the ejection velocity here is much smaller. The data were carried for two complete orbits in order to show more clearly the nature of the error buildup which occurs at the end of the first orbit in the x-equation. It is observed that the approximate solution is better over certain portions of the trajectory than over others and that the error is most significant toward the end of each orbital period. This result has also been obtained by the authors of reference 3 for a rectangular coordinate system and suggests that the cause of the error is something which is retained by both approaches. Figure 10 demonstrates a gradual departure from the exact solutions for each successive orbit because of the secular term of equation (8).

Other Launch Angles

In figure 11, a series of trajectories is shown for different ejection angles ϕ spaced at 20° intervals. The ejection speed is the same as for a synchronous orbit. None of these are synchronous orbits, however, because of the direction in which ejection takes place. For the sake of clarity, only the exact and second-order solutions are shown.

Figure 12 contains the same information as figure 11, but the speed is the same as that for a Hohmann transfer. Because of the lower ejection velocity, these trajectories are in much better agreement than those of figure 11.
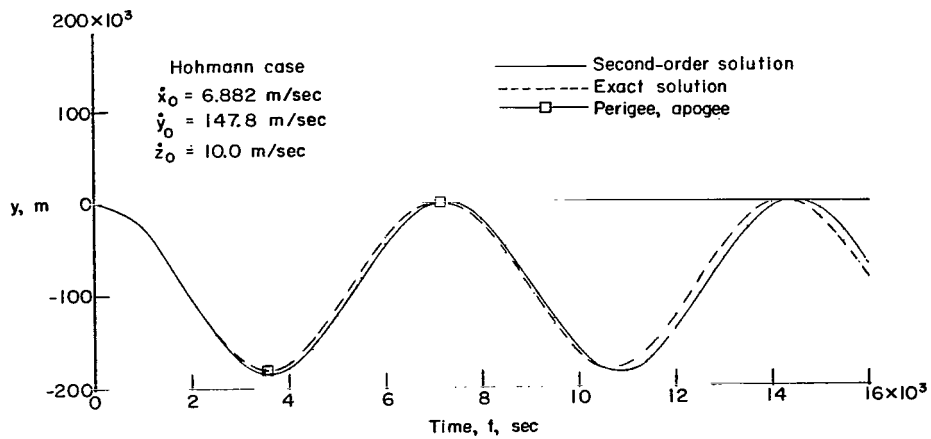
LIMITATIONS ON APPROXIMATE SOLUTIONS

In figure 11 no approximate solutions are presented over a range of ejection angles from -120° to $+120^O$. In addition, the solutions which are presented for $\pm110^O$ are considerably in error. A fundamental assumption in solving the anharmonic oscillator equation is that the first-order term is large in comparison with the second-order term. For the relatively high ejection speed considered for these cases, the assumption is violated over this range of ejection angles. A similar situation is not encountered for the lower launch speed of figure 12. Mathematically, the reason for this difficulty is that for these trajectories, ρ^2 becomes progressively small in relation to the second-order term and then imaginary. When ρ^2 is nearly as small as the second-order term, the solutions are in error, and when ρ^2 becomes imaginary, ρ is undefined and no solutions exist at all.
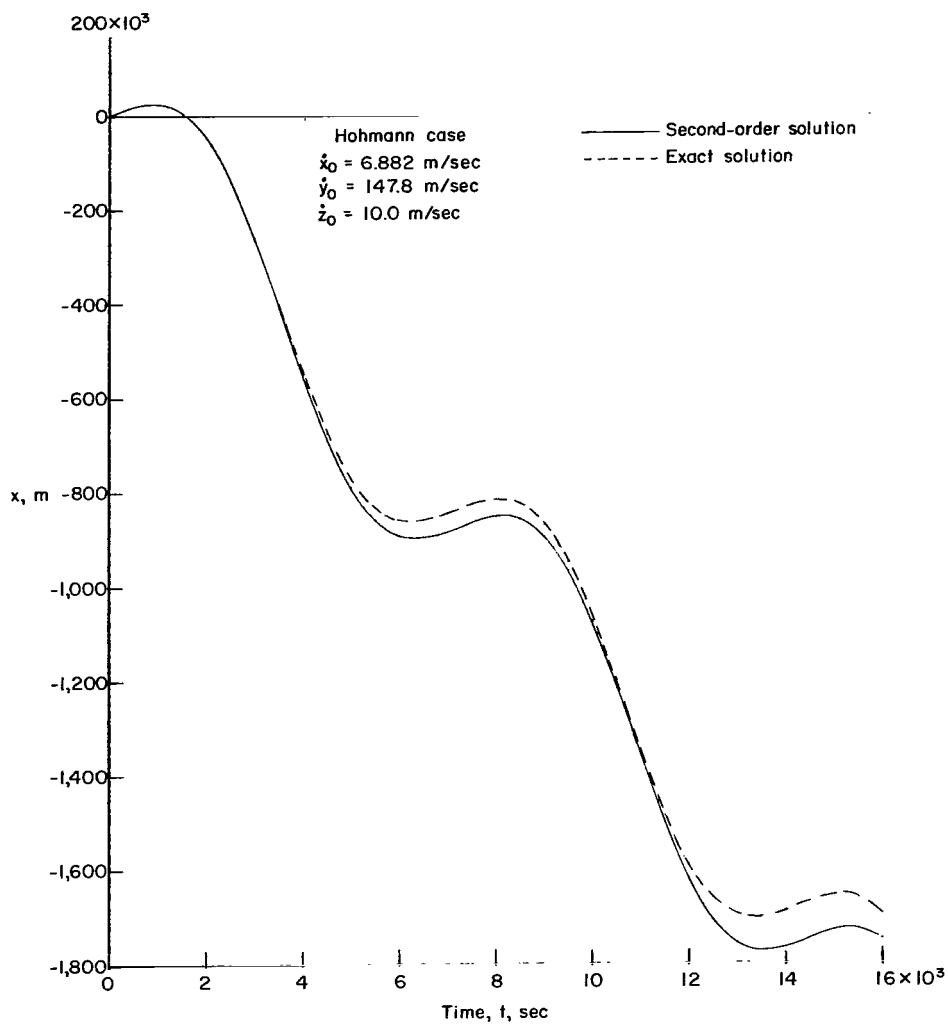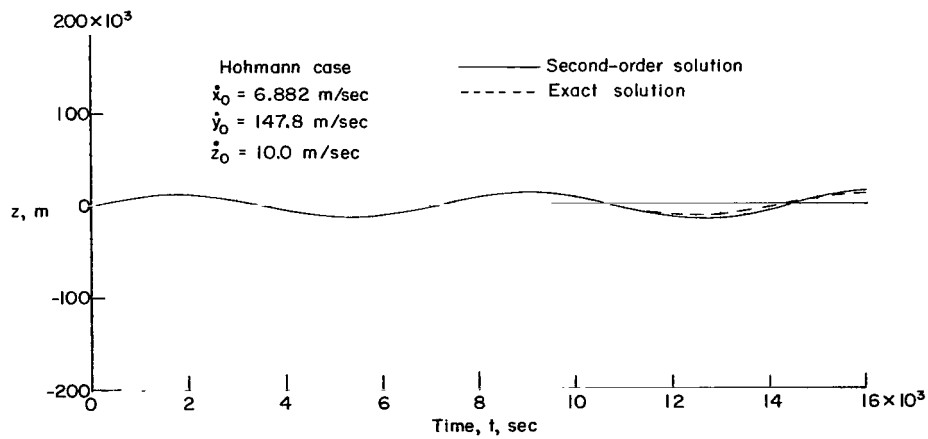
(a) x-direction.

Figure 8.- Time history for a 200-kilometer Hohmann transfer to near the lunar surface showing exact and second-order solutions.

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Hohmann case

$\dot{x}_0 = 6.882$ m/sec

$\dot{y}_0 = 147.8$ m/sec

$\dot{z}_0 = 10.0$ m/sec

——— Second-order solution

- - - - - Exact solution

—□— Perigee, apogee

(b) y-direction.

Figure 8.- Continued.

Hohmann case

$\dot{x}_0 = 6.882$ m/sec

$\dot{y}_0 = 147.8$ m/sec

$\dot{z}_0 = 10.0$ m/sec

——— Second-order solution

- - - - - Exact solution

(c) z-direction.

Figure 8.- Concluded.

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(a) x-direction.

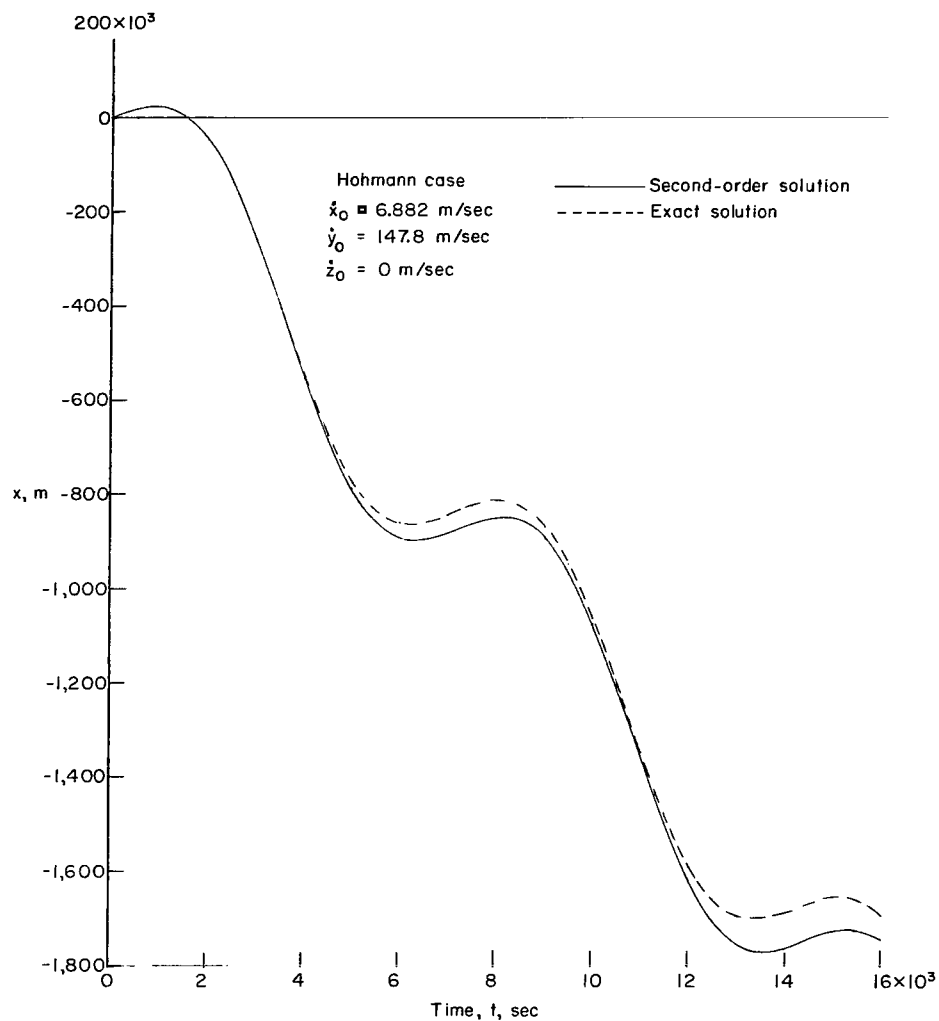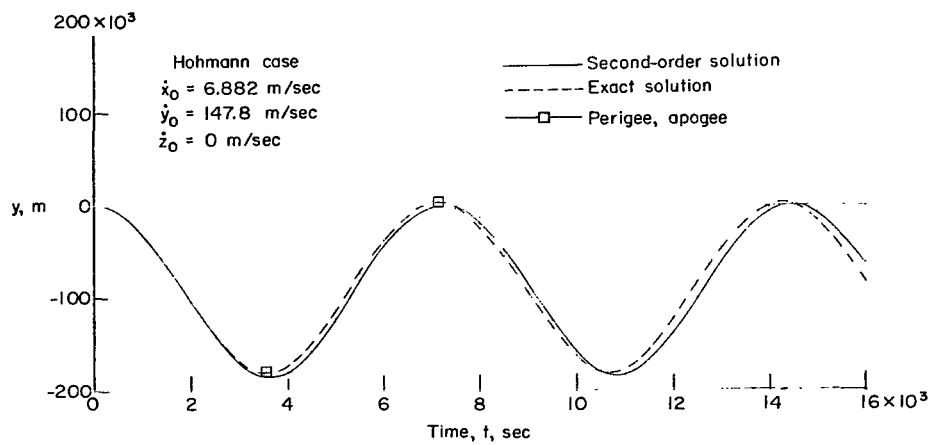Figure 9.- Time history for a 200-kilometer Hohmann transfer to near the lunar surface showing exact and second-order solutions with no initial out-of-plane velocity component.
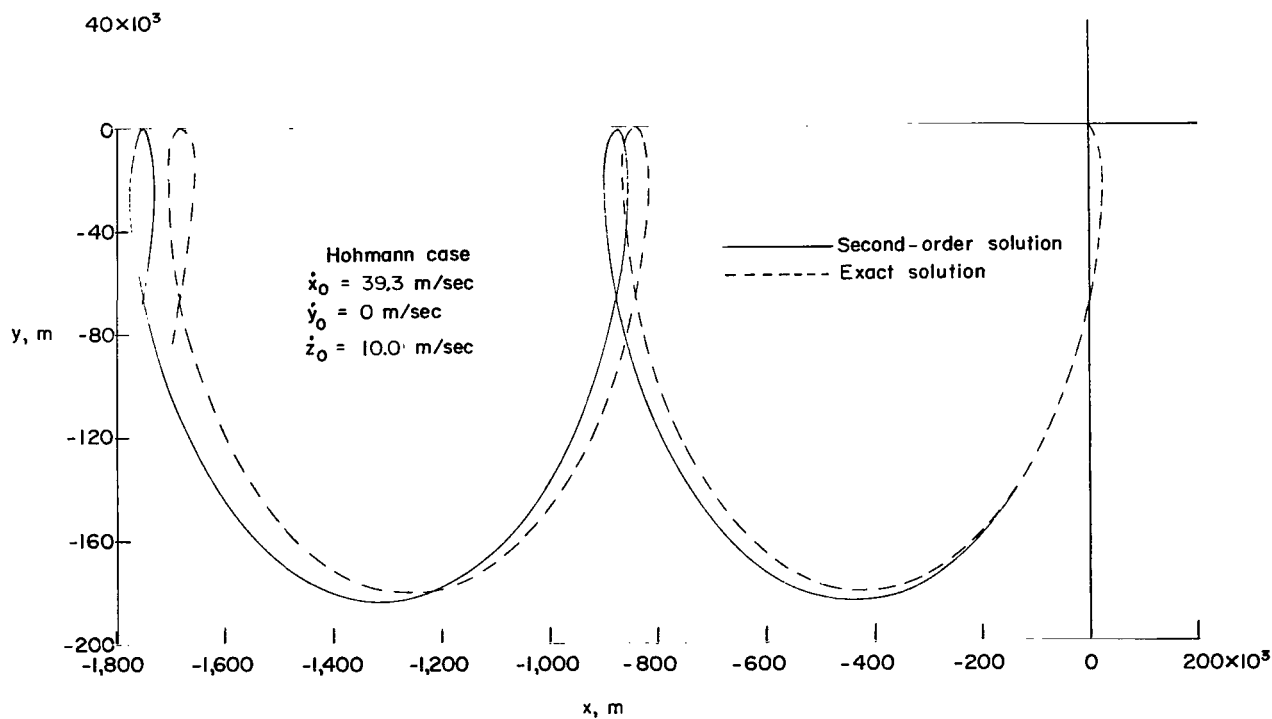
(b) y-direction.

Figure 9.- Concluded.

Figure 10.- Variation of x with y for a Hohmann transfer from 200-kilometer trajectory showing successive positions of the vehicle as seen by an observer in the orbiting vehicle in the reference orbit.
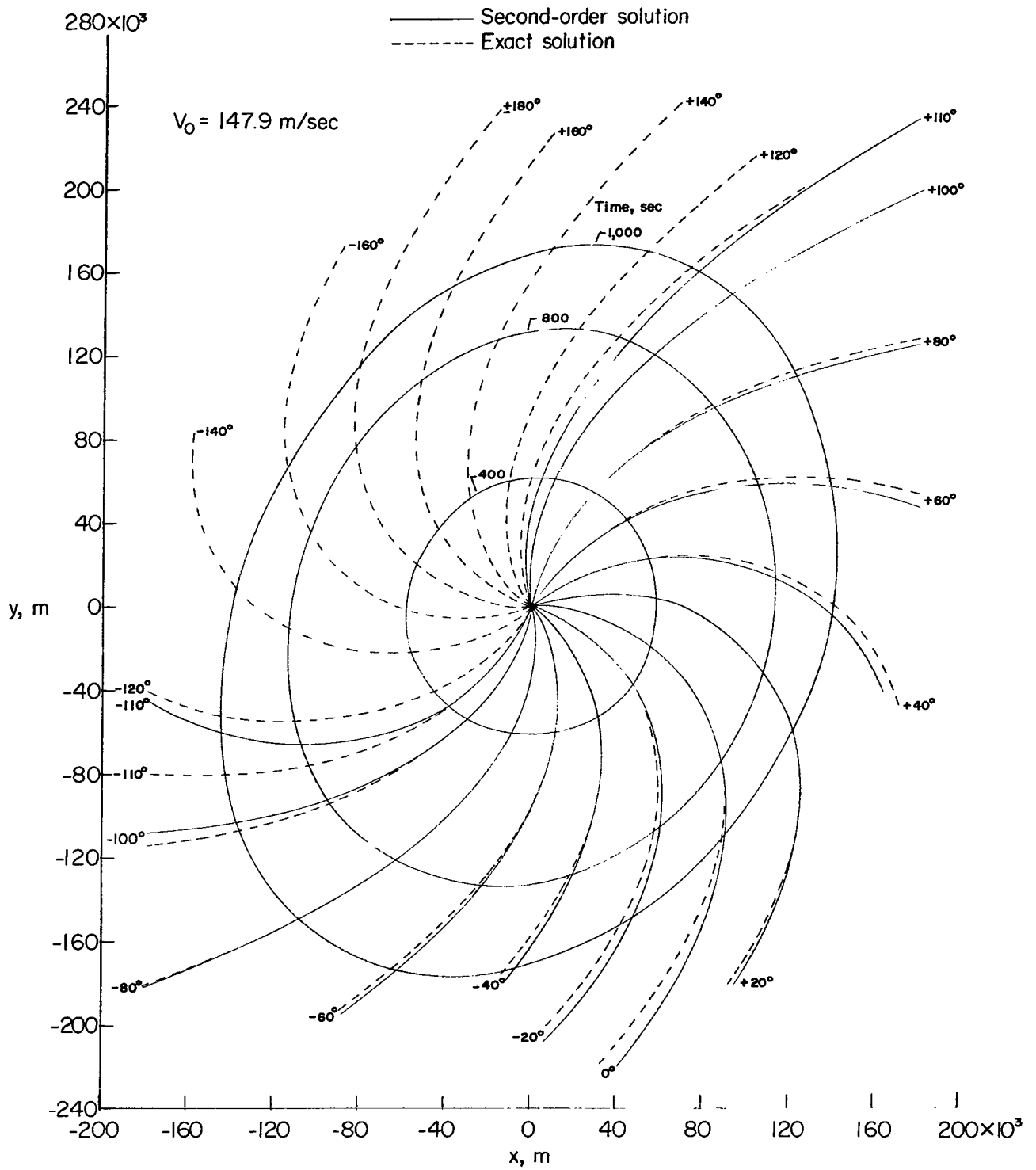
Figure 11.- A series of trajectories for different ejection angles. Ejection speed is the same as for a synchronous orbit.
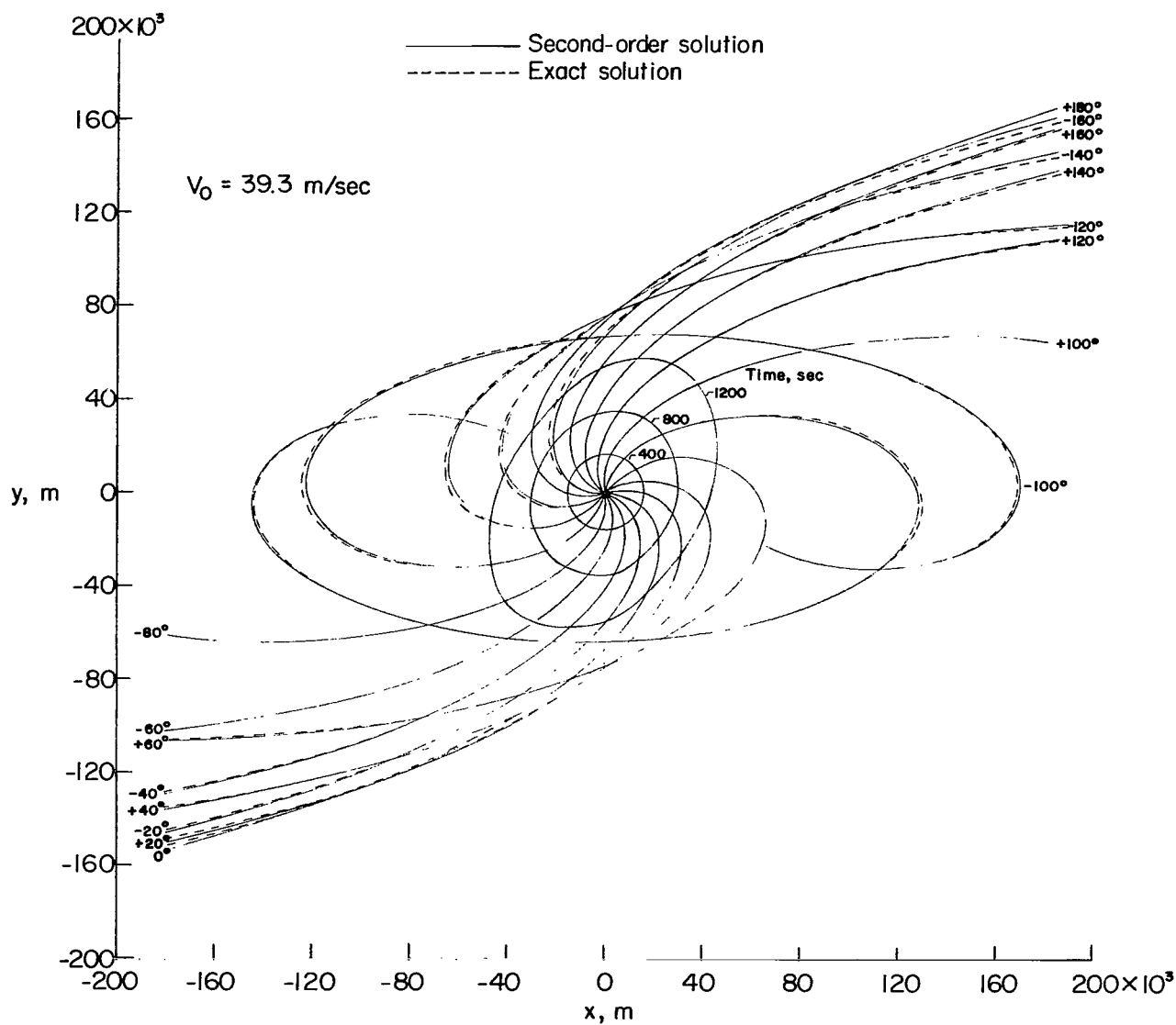
Figure 12.- A series of trajectories for different ejection angles. Ejection speed is the same as for a Hohmann transfer.

CONCLUDING REMARKS

Second-order solutions to the equations of relative motion for two bodies in orbit about a planet have been presented in this paper. Also, since both a Hohmann transfer and a synchronous transfer from orbit to the letdown phase of a lunar landing maneuver are considered important with regard to current projects to explore the moon, these two trajectories have been studied in detail. Various other trajectories having the same initial speeds as these two cases but with different initial directions of motion have also been investigated. As a result of these studies, the second-order solutions have been found to be more accurate than the corresponding first-order solutions for the same types of motion. It has been established that errors are small for the Hohmann transfer, or indeed for any kind of trajectory at Hohmann speed. However, for a synchronous transfer the errors are somewhat larger. A certain range of trajectories at synchronous speed, but in nonsynchronous directions, are found to give large errors which are due to violation of the conditions under which the second-order solutions are derived, and within this range is a second range in which no solutions are possible at all. These conditions are not thought to arise in any practical situation, however. It is therefore concluded that the second-order solutions may be used as a part of a guidance system where prediction of the future position of the space vehicle is required.

In addition, a simple technique for computing apogee and perigee altitudes is illustrated for the landing vehicle transfer trajectory when the initial separation conditions are known. Because of its simplicity, this technique may be used in manual guidance as well as automatic guidance.

Langley Research Center,
    National Aeronautics and Space Administration,
        Langley Station, Hampton, Va., March 29, 1966.

APPENDIX A

DERIVATION OF THE EQUATIONS OF MOTION

Differential Equations

The coordinate system in this development is shown in figure 1. The Lagrangian is set up formally in a cylindrical coordinate system centered on the planet and then converted to shell coordinates before taking the appropriate derivatives. The Lagrangian in cylindrical coordinates is given by

$$L = \frac{1}{2} m \left[\dot{r}^2 + r^2 (\dot{\theta} - \omega)^2 + \dot{z}^2 \right] + \frac{m\mu}{\sqrt{r^2 + z^2}} \tag{A1}$$

This equation is converted to shell coordinates by means of the following substitutions:

$$\left. \begin{aligned} y + r_s &= r \\ x &= r_s \theta \\ z &= z \end{aligned} \right\} \tag{A2}$$

Hence,

$$\left. \begin{aligned} \dot{y} &= \dot{r} \\ \dot{x} &= r_s \dot{\theta} \\ \dot{z} &= \dot{z} \end{aligned} \right\} \tag{A3}$$

Then the Lagrangian becomes

$$L = \frac{1}{2} m \left[\dot{y}^2 + \dot{z}^2 + \left(y + r_s \right)^2 \left(\frac{\dot{x}}{r_s} - \omega \right)^2 + \frac{2\mu}{\sqrt{\left(y + r_s \right)^2 + z^2}} \right] \tag{A4}$$

It can be seen that equation (A4) is cyclic in the x-coordinate; thus a first integral of the equation of motion in the x-direction is found immediately. If the appropriate derivatives of the Lagrangian are taken and use is made of the exact orbital expression

$$\omega^2 = \frac{\mu}{r_s{}^3} \tag{A5}$$

the equations of motion become

$$\left(\frac{\dot{x}}{\omega r_s} - 1\right)\left(1 + \frac{y}{r_s}\right)^2 = K \tag{A6}$$

$$\ddot{y} - (y + r_s)\left\{\left(\frac{\dot{x}}{r_s} - \omega\right)^2 - \omega^2\left[\left(1 + \frac{y}{r_s}\right)^2 + \left(\frac{z}{r_s}\right)^2\right]^{-3/2}\right\} = 0 \tag{A7}$$

$$\ddot{z} + \omega^2 z\left[\left(1 + \frac{y}{r_s}\right)^2 + \left(\frac{z}{r_s}\right)^2\right]^{-3/2} = 0 \tag{A8}$$

For convenience, let

$$\left.\begin{array}{l} X = \dfrac{x}{r_s} \\[2ex] Y = \dfrac{y}{r_s} \\[2ex] Z = \dfrac{z}{r_s} \\[2ex] \tau = \omega t \end{array}\right\} \tag{A9}$$

The resulting equations are

$$(\dot{X} - 1) = \frac{K}{(1 + Y)^2} \tag{A10}$$

$$\ddot{Y} - (Y + 1)\left\{(\dot{X} - 1)^2 - \left[(1 + Y)^2 + Z^2\right]^{-3/2}\right\} = 0 \tag{A11}$$

$$\ddot{Z} + Z\left[(1 + Y)^2 + Z^2\right]^{-3/2} = 0 \tag{A12}$$

where the dot over the symbols now refers to derivatives with respect to τ. The approach taken in obtaining the solutions is the following: Since equation (A11) contains

terms in $\dot{X}$ but not in X, if Z^2 is assumed to be small in relation to $(1 + Y)^2$ and can hence be neglected, it is possible to cast equation (A11) as an equation in Y and $\dot{Y}$ by direct substitution of equation (A10). This equation can then be solved approximately for Y as an explicit function of time. This solution may then be used in the solution of equations (A10) and (A12).

In order to establish a value for K, assume that at

$$\tau = 0 \ (t = 0)$$

$$\left. \begin{aligned} \dot{X} &= \dot{X}_o \\ \\ Y &= 0 \end{aligned} \right\} \tag{A13}$$

Then $K = \left(\dot{X}_o - 1\right)$. Substituting equation (A10) into equation (A11) and neglecting the out-of-plane term Z results in

$$\ddot{Y} - \frac{K^2}{(1 + Y)^3} + \frac{1}{(1 + Y)^2} = 0 \tag{A14}$$

If the forms of

$$\frac{1}{(1 + Y)^n}$$

where $n = 2$ or 3 are expanded, and the terms of the second order and lower are retained, equations (A10), (A11), and (A12) become

$$\dot{X} + 2KY - (K + 1) = 3KY^2 \tag{A15}$$

$$\ddot{Y} + \left(\alpha^2\right)Y - \left(\beta^2\right) = (-\lambda)Y^2 \tag{A16}$$

$$\ddot{Z} + Z = 3YZ \tag{A17}$$

where

$$\left. \begin{aligned} K &= \left(\dot{X}_o - 1\right) \\ \alpha^2 &= 3K^2 - 2 \\ \beta^2 &= K^2 - 1 \\ \lambda &= -6K^2 + 3 \end{aligned} \right\} \tag{A18}$$

APPENDIX A

First-Order Solutions

The first-order solutions to equations (A15), (A16), and (A17) are obtained by dropping the second-order terms (setting the terms on the right-hand side equal to zero). The Y-equation is solved by inspection. Its solution is

$$Y = a' \cos(\alpha\tau + \epsilon') + \frac{\beta^2}{\alpha^2} \tag{A19}$$

where primes are used to distinguish first-order equation integration constants from those of the second order.

The X-equation becomes, upon replacing $K + 1$ with its equivalent by definition $\dot{X}_0$

$$\dot{X} = \dot{X}_0 - 2KY \tag{A20}$$

Then substituting equation (A19) into equation (A20) and integrating, the X-equation is

$$X = \left(\dot{X}_0 - \frac{2K\beta^2}{\alpha^2}\right)\tau - \frac{2Ka'}{\alpha} \sin(\alpha\tau + \epsilon') + \frac{2Ka'}{\alpha} \sin \epsilon' \tag{A21}$$

A solution to the Z-equation is

$$Z = \dot{Z}_0 \sin \tau \tag{A22}$$

where $Z = 0$ at $\tau = 0$ and $\dot{Z} = \dot{Z}_0$.

To compute the integration constants from the initial conditions, set $Y = 0$ at $\tau = 0$ and $\dot{Y} = \dot{Y}_0$. Then solve for a' and ϵ'. This solution can be found in a straightforward manner; but, for purposes of making the second-order equivalent development more understandable, it is advisable to make the substitution

$$u' = a' \cos \epsilon'$$

and hence

$$u' + \frac{\beta^2}{\alpha^2} = 0 \tag{A23}$$

Then

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$$a' = \sqrt{u'^2 + \left(\frac{\dot{Y}_0}{\alpha}\right)^2} \qquad (A24)$$

and

$$\epsilon' = \cos^{-1}\frac{u'}{a'} \qquad (A25)$$

The sign selection on equations (A24) and (A25) can be done empirically. For instance, a' can be chosen always positive and then the quadrant for ϵ' depends upon the launch angle ϕ. Equation (A19) can be expressed in a deterministic form without the epoch angle as

$$Y = -\frac{\beta^2}{\alpha^2}\cos\alpha\tau + \frac{\dot{Y}_0}{\alpha}\sin\alpha\tau + \frac{\beta^2}{\alpha^2} \qquad (A26)$$

Expanding $\cos(\alpha\tau + \epsilon')$ in equation (A19) yields

$$Y = (a'\cos\epsilon')\cos\alpha\tau - (a'\sin\epsilon')\sin\alpha\tau + \frac{\beta^2}{\alpha^2} \qquad (A27)$$

Comparison of equations (A26) and (A27) gives

$$\left.\begin{aligned} a'\cos\epsilon' &= -\frac{\beta^2}{\alpha^2} \\[2mm] a'\sin\epsilon' &= -\frac{\dot{Y}_0}{\alpha} \end{aligned}\right\} \qquad (A28)$$

Hence,

$$\tan\epsilon' = \frac{-\dot{Y}_0/\alpha}{-\beta^2/\alpha^2} \qquad (A29)$$

Since α is a frequency which is a physical quantity, a negative α has no meaning. Hence, with α positive, the quadrant for ϵ' is determined by the respective signs in the numerator and denominator. Also β^2 is strictly a function of $\dot{X}_0$ and always has the opposite sign $\left(\beta^2 = \dot{X}_0\left(\dot{X}_0 - 2\right),\text{ since } \dot{X}_0 \ll 2\right)$. It then follows, once the quadrant of ϵ' is known, that equations (A28) give the sign to be selected for a' in equation (A23). It is found that a' is always positive. These characteristics for all possible ejection quadrants are given in table I. Table I relates the ϵ'-quadrant and the ϕ-quadrant for the first-order solutions.

APPENDIX A

Second-Order Solutions

<u>Solution for Y.</u>- Equation (A16) can be expressed as

$$\ddot{Y} + \alpha^2 Y + \lambda Y^2 = \beta^2 \tag{A30}$$

It is advantageous to transform equation (A30) into the form (to eliminate the constant term)

$$\ddot{V} + \rho^2 V + \lambda V^2 = 0 \tag{A31}$$

This transformation can be accomplished by a change in variable $Y = V + M$ which results in

$$\ddot{V} + \left(\alpha^2 + 2\lambda M\right)V + \lambda V^2 = \beta^2 - \lambda M^2 - \alpha^2 M \tag{A32}$$

Comparing equations (A30) and (A31) shows that

$$\beta^2 - \lambda M^2 - \alpha^2 M = 0 \tag{A33}$$

and

$$\alpha^2 + 2\lambda M = \rho^2 \tag{A34}$$

which can also be written as

$$\rho^2 = \sqrt{-15K^4 - 24K^2 - 8} \tag{A35}$$

Solving equation (A33) for M yields

$$M = -\frac{\alpha^2}{2\lambda} \pm \left(\frac{\beta^2}{\lambda} + \frac{\alpha^4}{4\lambda^2}\right)^{1/2} \tag{A36}$$

which can also be expressed in terms of ρ^2 as

$$M = -\frac{\alpha^2 - \rho^2}{2\lambda} \tag{A37}$$

Equation (A31) may be recognized as the equation of a one-dimensional anharmonic oscillator. An exact second integral of this equation may be obtained if desired; however,

as this solution is an elliptic integral, the results are not very useful for solving the X- and Z-equations. For this reason an approximate perturbation solution is sought. A more complete treatment than that given here using higher order terms can be found in reference 4. The solution V is obtained as the sum of a first-order solution V_1 and a second-order correction term V_2. Let

$$V = V_1 + V_2 \tag{A38}$$

The first-order solution to this equation (by assuming $\rho^2 \gg \lambda V$) is

$$V_1 = a \cos(\rho\tau + \epsilon) \tag{A39}$$

from which equation (A38) becomes

$$V = a \cos(\rho\tau + \epsilon) + V_2 \tag{A40}$$

Applying these relations to equation (A31) gives

$$-\rho^2 a \cos(\rho\tau + \epsilon) + \ddot{V}_2 + \rho^2 a \cos(\rho\tau + \epsilon) + \rho^2 V_2 + \lambda a^2 \cos^2(\rho\tau + \epsilon)$$

$$+ 2aV_2 \cos(\rho\tau + \epsilon) + V_2^{\,2} = 0 \tag{A41}$$

After some cancellation and rearrangement of terms,

$$\ddot{V}_2 + \rho^2 V_2 = -\lambda a^2 \cos^2(\rho\tau + \epsilon) - 2\lambda aV_2 \cos(\rho\tau + \epsilon) - \lambda V_2^{\,2} \tag{A42}$$

Omitting terms of higher order than the second (last two terms on right), results in,

$$\ddot{V}_2 + \rho^2 V_2 = -\lambda a^2 \cos^2(\rho\tau + \epsilon)$$

$$= -\frac{1}{2} \lambda a^2 - \frac{1}{2} \lambda a^2 \cos 2(\rho\tau + \epsilon) \tag{A43}$$

Solving the inhomogeneous linear equation in the usual way yields

$$V_2 = -\frac{\lambda a^2}{2\rho^2} + \frac{\lambda a^2}{6\rho^2} \cos 2(\rho\tau + \epsilon) \tag{A44}$$

Hence, the second-order solution for V is

$$V = a \cos(\rho\tau + \epsilon) - \frac{\lambda a^2}{2\rho^2} + \frac{\lambda a^2}{6\rho^2} \cos 2(\rho\tau + \epsilon) \tag{A45}$$

The constants a and ϵ are integration constants which depend on initial conditions. This solution is, of course, limited to cases where the first-order assumption is approximately valid; that is,

$$\lambda V \ll \rho^2$$

or

$$\lambda(Y - M) \ll \rho^2$$

It can be seen from numerical solution that this relationship is usually the case since Y seldom exceeds 0.1 for cases which are physically practical and M will be small provided the departure velocity is small enough.

Converting equation (A45) to Y notation yields,

$$Y = a \cos(\rho\tau + \epsilon) - \frac{\lambda a^2}{2\rho^2}\left[1 - \frac{1}{3}\cos 2(\rho\tau + \epsilon)\right] - \frac{\alpha^2 - \rho^2}{2\lambda} \tag{A46}$$

The last term on the right is part of a small correction term which takes account of the change in energy and hence the change in average altitude between the original circular orbit and the new orbit into which the vehicle is launched.

Solution for X.- The second-order solution for X is obtained by substituting equation (A46) for Y into equation (A15) for $\dot{X}$ and then integrating. The differential equation is

$$\dot{X} = \dot{X}_0 - 2K\left[a\cos(\rho\tau + \epsilon) + \frac{\lambda a^2}{6\rho^2}\cos 2(\rho\tau + \epsilon) - \frac{\lambda a^2}{2\rho} - \frac{\alpha^2}{2\lambda} + \frac{\rho^2}{2\lambda}\right] + 3K\left[a^2\cos^2(\rho\tau + \epsilon)\right.$$

$$+ \frac{\lambda^2\alpha^4}{36\rho^4}\cos^2 2(\rho\tau + \epsilon) + \frac{\lambda^2\alpha^4}{4\rho^4} + \frac{a^2\alpha^2}{2\rho^2} + \frac{\alpha^4}{2\lambda^2} + \frac{\beta^2}{\lambda} - \left(\frac{\lambda a^2}{\rho^2} + \frac{\alpha^2}{\lambda}\right)\frac{\rho^2}{2\lambda} + \frac{2\lambda a^3}{3\rho^2}\cos^3(\rho\tau + \epsilon)$$

$$- \frac{4\lambda a^3}{3\rho^2}\cos(\rho\tau + \epsilon) - \frac{\alpha^2 a}{\lambda}\cos(\rho\tau + \epsilon) + \frac{\rho^2 a}{\lambda}\cos(\rho\tau + \epsilon) - \frac{\lambda^2 a^4}{6\rho^4}\cos 2(\rho\tau + \epsilon)$$

$$\left. - \frac{a^2\alpha^2}{6\rho^2}\cos 2(\rho\tau + \epsilon) + \frac{a^2}{6}\cos 2(\rho\tau + \epsilon)\right] \tag{A47}$$

Integrating equation (A47) with respect to τ with A as an arbitrary constant of integration yields

$$X = A + B\tau + C \sin(\rho\tau + \epsilon) + D \sin 2(\rho\tau + \epsilon) + E \sin 3(\rho\tau + \epsilon) + F \sin 4(\rho\tau + \epsilon) \tag{A48}$$

where

$$
\left.
\begin{aligned}
A &= -(C \sin \epsilon + D \sin 2\epsilon + E \sin 3\epsilon + F \sin 4\epsilon) \\[2mm]
B &= K\left[\frac{\dot{X}_0}{K} + \frac{\left(\alpha^2 - \rho^2\right)}{\lambda} + \frac{3\left(\alpha^2 - \rho^2\right)^2}{4\lambda^2} + \frac{\alpha^2\lambda}{\rho^2} + \frac{3\alpha^2 a^2}{2\rho^2} + \frac{19}{24}\frac{\lambda^2 a^4}{\rho^4}\right] \\[2mm]
C &= -K\left[\frac{2a}{\rho} + \frac{3a\left(\alpha^2 - \rho^2\right)}{\lambda\rho} + \frac{5\lambda a^3}{2\rho^3}\right] \\[2mm]
D &= K\left(\frac{a^2}{\rho} - \frac{a^2\alpha^2}{4\rho^3} - \frac{\lambda a^2}{6\rho^3} - \frac{\lambda^2 a^4}{4\rho^5}\right) \\[2mm]
E &= \frac{K\lambda a^3}{6\rho^3} \\[2mm]
F &= \frac{K\lambda^2 a^4}{96\rho^5}
\end{aligned}
\right\} \tag{A49}
$$

Terms E and F have been found to be negligible in all cases tested, but are included here for the sake of completeness. The constant A is evaluated on the usual assumption that at $\tau = 0$, $X = 0$.

Solution for Z.- The solution for Z is obtained by replacing the right-hand side of equation (A17) by the first-order term for Z and the appropriate second-order term for Y to obtain a time-dependent right-hand side. In doing so, of course, terms are carried which are higher than the second order of smallness. However, if only second-order terms are retained, new end conditions corresponding to a and ϵ would have to be computed as a peripheral calculation; this calculation is thought to be unnecessary. Therefore, equation (A17) becomes

$$\ddot{Z} + Z = 3\left(\dot{Z}_0 \sin \tau\right)\left\{a \cos(\rho\tau + \epsilon) - \frac{\lambda a^2}{2\rho^2}\left[1 - \frac{1}{3}\cos 2(\rho\tau + \epsilon)\right] - \frac{\alpha^2}{2\lambda} + \frac{\rho^2}{2\lambda}\right\} \tag{A50}$$

Let

$$-3\dot{Z}_0 \left[\frac{\lambda a^2}{2\rho^2} + \frac{(\alpha^2 - \rho^2)}{2\lambda} \right] = \kappa$$

$$\left. \frac{\lambda a^2 \dot{Z}_0}{2\rho^2} = \eta \right\}$$ (A51)

$$3a\dot{Z}_0 = \gamma$$

Then

$$\ddot{Z} + Z = \gamma \cos(\rho\tau + \epsilon)\sin \tau + \eta \cos 2(\rho\tau + \epsilon)\sin \tau + \kappa \sin \tau$$ (A52)

The solution of this equation is

$$Z = \dot{Z}_0 \sin \tau + \frac{1}{4}\left[\gamma\psi \cos(\rho\tau + \epsilon) + \eta\Delta \cos 2(\rho\tau + \epsilon) + \kappa\right]\sin \tau$$

$$- \frac{1}{2\rho}\left[\gamma\psi \sin(\rho\tau + \epsilon) + \frac{\eta\Delta}{2} \sin 2(\rho\tau + \epsilon) + \rho c\right]\cos \tau$$ (A53)

where

$$\Delta = 1 + \rho^2 + \rho^4$$

$$\left. \psi = 1 + \frac{\rho^2}{4} + \frac{\rho^4}{16} \right\}$$ (A54)

$$c = - \frac{\gamma\psi}{\rho} \sin \epsilon - \frac{\eta\Delta}{2\rho} \sin 2\epsilon$$

Evaluation of the Integration Constants a and ϵ

In order to make effective use of equation (A46), it is necessary to compute values for the two integration constants a and ϵ. It is found from experience that when synchronous or very nearly snychronous orbits are under consideration, one can obtain excellent results by using the values which are obtained from first-order theory. For even moderate departures from the synchronous condition, for instance, on the order of 5° in launch direction, however, this is not the case, especially for ϵ. Therefore, it will be necessary in most cases to evaluate the integration constants. A method for evaluating these constants is outlined in the following.

APPENDIX A

Differentiating equation (A46) and applying the initial conditions yields the simultaneous set of transcendental equations

$$0 = a \cos \epsilon - \frac{\lambda a^2}{2\rho^2}\left(1 - \frac{1}{3}\cos 2\epsilon\right) + \frac{\rho^2 - \alpha^2}{2\lambda} \tag{A55}$$

$$\dot{Y}_o = -\rho a \sin \epsilon - \frac{\lambda a^2}{3\rho} \sin 2\epsilon \tag{A56}$$

where a and ϵ are the two unknowns. These two equations can be solved by eliminating ϵ in favor of the two variables u and a where

$$u = a \cos \epsilon$$

After substitution and some manipulation,

$$\frac{\lambda u^2}{3\rho^2} + u - \frac{2}{3}\frac{\lambda a^2}{\rho^2} + \frac{\rho^2 - \alpha^2}{2\lambda} = 0 \tag{A57}$$

and

$$\frac{\dot{Y}_o}{\rho} + \left(1 + \frac{2\lambda u}{3\rho^2}\right)\sqrt{a^2 - u^2} = 0 \tag{A58}$$

Solving this set by eliminating a^2 gives a quartic equation in u

$$\left(\frac{2\lambda^2}{9\rho^4}\right)u^4 - \left(\frac{3}{2} + \frac{\rho^2 - \alpha^2}{3\rho^2}\right)u^2 - \left(\frac{5\rho^2 - 2\alpha^2}{2\lambda}\right)u + \left[\frac{3}{4}\frac{\rho^2(\alpha^2 - \rho^2)}{\lambda^2} + \frac{\dot{Y}_o^2}{\rho^2}\right] = 0 \tag{A59}$$

In equation (A59) it can be seen that u is a function of both $\dot{Y}_o$ and $\dot{X}_o$ (through α, ρ, and λ) instead of just $\dot{X}_o$ as was the case under first-order theory (eq. (A23)). As a result, the special case where $u = 0$ will no longer occur independent of $\dot{Y}_o$. It is seen that when $u = 0$, the following relation is obtained:

$$\frac{3}{4}\frac{\rho^2(\alpha^2 - \rho^2)}{\lambda^2} + \frac{\dot{Y}_o^2}{\rho^2} = 0 \tag{A60}$$

or

$$\dot{Y}_o = \pm \sqrt{-\frac{3}{4}\frac{\rho^4(\alpha^2 - \rho^2)}{\lambda^2}} \tag{A61}$$

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This relationship is shown in sketch 1. It can be seen that under first-order theory, this curve degenerates into a straight vertical line.

This curve forms the boundary for the selection of the quadrant for ϵ since the sign on u along with the positive selection for a determines the sign of $\cos \epsilon$. Since the second-order term in equation (A46) is a function of $\cos^2(\rho\tau + \epsilon)$, the quadrant is not ambiguous and all four quadrants must be used. Table II summarizes the second-order sign selection. In terms of ejection angle ϕ, equation (A61) becomes

Sketch 1.

$$\tan \phi = \frac{\dot{Y}_O}{\dot{X}_O} = \pm \sqrt{-\frac{3}{4} \frac{\rho^4}{\lambda^2} \frac{\left(\alpha^2 - \rho^2\right)}{\dot{X}_O^2}} \tag{A62}$$
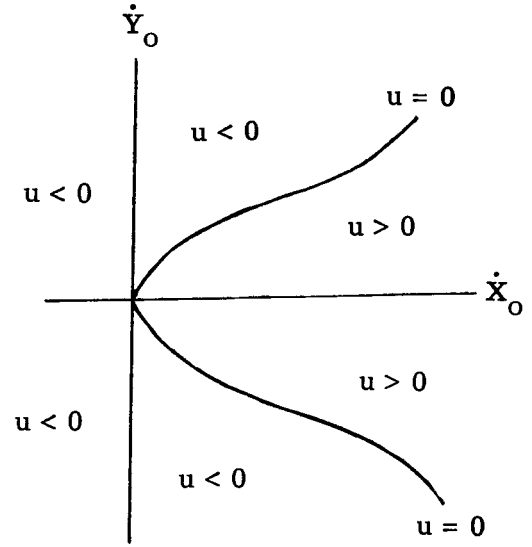
Equation (A62) relates ϕ and $\dot{X}_O$ for the condition that $u = 0$ or in other words, it specifies the combinations of ϕ and $\dot{X}_O$ at which ϵ changes quadrants.

Figure 3 is a plot of $\dot{x}_O$ as a function of ϕ for $u = 0$. The figure can be used therefore to determine for the known value of $\dot{x}_O$ and ϕ, the proper quadrant for ϵ.

LIMITATIONS ON THE APPROXIMATE SOLUTIONS

It will be recalled that one of the conditions implicit in solving the equation of an anharmonic oscillator was that $\rho^2 \gg \lambda V$ where V was defined as $V = Y - M$. Hence, in order for the solution to be valid,

$$\rho^2 \gg -\lambda(Y - M) \tag{A63}$$

where Y is inherently nondimensionalized in such a way that for all practical purposes the term Y in the expression can be neglected. Thus, the criterion which sets a limit on the validity of the solutions in actual practice is
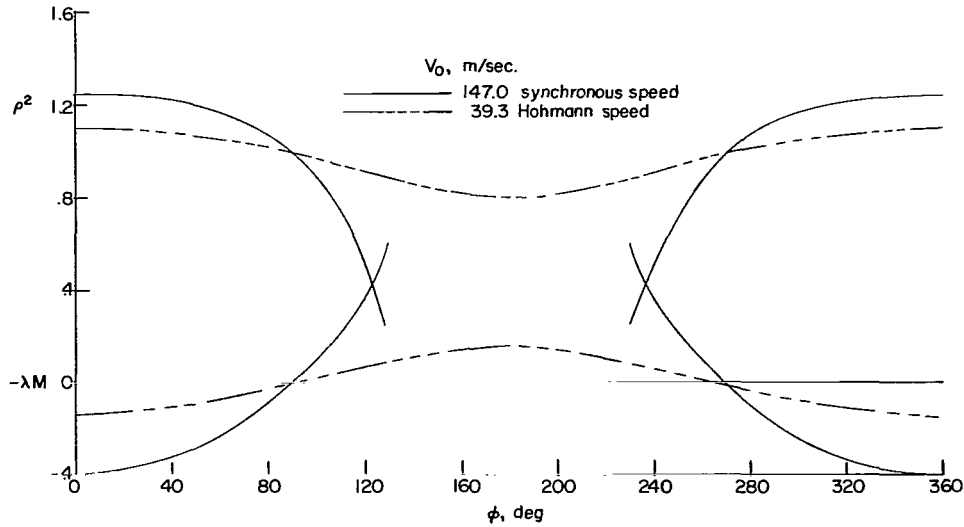
$$\rho^2 \gg -\lambda M \tag{A64}$$

APPENDIX A

Figure 13.- Comparison of the magnitudes of ρ^2 and $-\lambda M$ for Hohmann and synchronous speeds.

For synchronous speed at ejection angles between 110° and 250°, equation (A64) is not satisfied as can be seen in figure 13. Hence, one would not reasonably expect equation (A46) to describe the physical situation. That this is actually the case can be seen by looking at figure 11 for ejection angles of 110° and -110° (250°). It is observed that for these cases the error is considerable. At ejection angles of slightly larger magnitude than this value, ρ^2 becomes imaginary $\left(\rho^2 = \sqrt{4\lambda\beta^2 + \alpha^4}\right)$. Hence, a value for ρ is undefined. Inspection of equation (A46) shows that ρ is the frequency term. Therefore, no solutions would be expected to exist at all with ρ^2 imaginary. This situation occurs for those cases in figure 11 for which no solutions are shown. For the Hohmann case, equation (A46) is satisfied for all ejection angles and ρ^2 remains real; hence, breakdown does not occur.

APPENDIX B

CONVERSION OF DATA FROM RECTANGULAR TO SHELL COORDINATES

If initial measurements are taken in an x',y',z' rectangular coordinate system, the conversion to an x,y,z shell coordinate system can be derived as follows where all terms are dimensional.
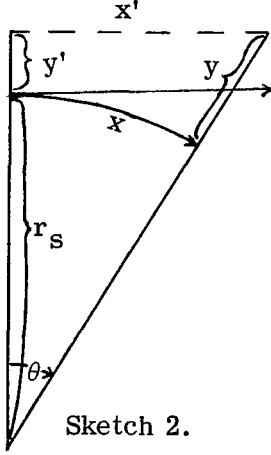
Sketch 2.

From sketch 2, where it is assumed, as before, that r_s is measured toward the center of the planet,

$$\tan \theta = \frac{x'}{r_s + y'}$$

but $\theta = \frac{x}{r_s}$. Therefore,

$$x = r_s \tan^{-1} \frac{x'}{r_s + y'} \tag{B1}$$

and from the large right triangle in the sketch

$$y = + \sqrt{x'^2 + \left(y' + r_s\right)^2} - r_s \tag{B2}$$

where the positive sign is selected for the radical because r_s and the radical are of the same order of magnitude.

Equations (B1) and (B2) serve to express spatial positions x and y in shell coordinates in terms of x' and y' in rectangular coordinates. Of course, the out-of-plane measurement is trivial since

$$z = z' \tag{B3}$$

The velocity terms are obtained by differentiation of equations (B1), (B2), and (B3) with respect to time to get

$$\dot{x} = r_s \left[\frac{\left(r_s + y'\right)\dot{x}' - x'\dot{y}'}{x'^2 + \left(r_s + y'\right)^2} \right] \tag{B4}$$

$$\dot{y} = \frac{x'\dot{x}' + \left(y + r_s\right)\dot{y}'}{\sqrt{x'^2 + \left(y' + r_s\right)^2}} \tag{B5}$$

$$\dot{z} = \dot{z} \tag{B6}$$

REFERENCES

1. London, Howard S.: Second Approximation to the Solution of the Rendezvous Equations. AIAA J. (Tech. Notes), vol. 1, no. 7, July 1963, pp. 1691-1693.

2. Anthony, Maurice L.; and Sasaki, Frank T.: The Rendezvous Problem for Nearly Circular Orbits. Paper No. 65-32, Am. Inst. Aeron. Astronaut., Jan. 1965.

3. Swanson, Robert S.; and Soule, Peter W.: Rendezvous Guidance Technology. Proceedings of the National Meeting on Manned Space Flight, Inst. Aerospace Sci., 1962, pp. 106-129.

4. Landau, L. D.; and Lifshitz, E. M. (J. B. Dykes and J. S. Bell, trans.): Mechanics. Pergamon Press, c.1960.

TABLE I.- FIRST-ORDER QUADRANT AND SIGN SELECTION

$\dot{X}_O$	$\dot{Y}_O$	β^2	ϵ' quadrant	a'	ϕ quadrant
+	+	−	4	+	1
−	+	+	3	+	2
−	−	+	2	+	3
+	−	−	1	+	4

TABLE II.- SECOND-ORDER QUADRANT AND SIGN SELECTION

Ejection quadrant	$\dot{Y}_O$	u	Sign on a	Quadrant for ϵ
$0 \leqq \phi \leqq \pi$	$\dot{Y}_O > 0$	$u > 0$	+	ϵ in fourth quadrant
$0 \leqq \phi \leqq \pi$	$\dot{Y}_O > 0$	$u < 0$	+	ϵ in third quadrant
$\pi \leqq \phi \leqq 2\pi$	$\dot{Y}_O < 0$	$u < 0$	+	ϵ in second quadrant
$\pi \leqq \phi \leqq 2\pi$	$Y < 0$	$u > 0$	+	ϵ in first quadrant

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