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THE EIGENVALUE PROBLEM FOR BEAMS AND RECTANGULAR PLATES WITH LINEARLY VARYING IN-PLANE AND AXIAL LOAD

by Guy Fauconneau and William M. Laird

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ABSTRACT

Part I - The Uniform Beam

The equation of motion and boundary conditions for uniform beams carrying constant end thrust and linearly varying axial load are derived from Hamilton's principle. Previous studies on the determination of natural frequencies of beams with varying axial loads are reviewed. The eigenvalue problem is formulated in a variational form to facilitate approximate solutions.

Part II - The Rectangular Plate

The equation of motion of uniform plates carrying in-plane loads is derived from Hamilton's principle. The eigenvalue problem for a rectangular plate carrying a linearly varying in-plane load parallel to one side is transformed into that of a uniform beam subjected to a uniformly distributed axial load whenever the sides of the plate parallel to the loading are simply supported. The eigenvalues for the beam are presently being computed and will be tabulated in a later report.

CONTENTS

Part I - The Uniform Beam

Part II - The Rectangular Plate

PART I - THE UNIFORM BEAM

LIST OF SYMBOLS

A Differential Operator of Loaded Beams (IV - 19)*

EI Flexural Rigidity

$F(t)$ Function of Time (IV - 1)

g Gravitational Acceleration

K, K_b, K_v Classes of Functions Admissable for the Variational Principles (IV - 16), (IV - 23)

L Length of the Beam

P_1 Constant End Thrust

T Kinetic Energy (III - 2)

U Strain Energy of Bending (III - 3)

u Function

V Total Potential Energy

v Function

W Potential Energy of the External Forces (III - 4)

x Axial Coordinate

y Lateral Deflection

$\alpha = \dfrac{\omega L^3}{EI}$ Distributed Axial Load Parameter (IV - 4)

α_c Critical Value of Axial Load Parameter

$\beta = \dfrac{P_1}{\omega L}$ Ratio of End Load to Total Distributed Load (IV - 4)

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*Numbers in parentheses refer to the equations in the text

δ	Symbol for Variation
λ_j	j-th Eigenvalue
ρ	Density
$\xi = \dfrac{x}{L}$	Non-dimensional Axial Variable
$\Psi(x)$	Function of x (IV -1)
Ψ_i	Eigenfunction
i, j	Subscripts (Integers)
γ^4	Separation constant (IV -2)

1. INTRODUCTION

The trend in the development of rocket vehicles appears to be in the direction of larger thrusts and more flexible structures to achieve higher payloads. Corresponding to this trend the significance of the longitudinal inertia forces due to the thrust on the vibration and stability characteristics of the vehicle and its components becomes more important.

Recent studies[1,2,3]* indicates that a modern rocket is so stiff that the acceleration required to make it elastically unstable is at least 5 g's less than the maximum acceleration experienced. Thus, there is no immediate danger of present rockets becoming elastically unstable, although it is possible that future rocket vehicles, particularly solid-fueled boosters, will reach higher accelerations, or that more flexible structures will be designed for which the problem will exist. The danger, however, exists that components of the vehicle or of the payload may be flexible enough to have critical accelerations in the neighborhood of those presently attained. Furthermore, the thrust acting on a structure tends to reduce its natural frequencies[4] which may then fall within the spectrum of excitations experienced. Hence, as further increases in payload are required, closer determination of system characteristics will be needed.

The purpose of this report is to review the studies that have been made to date on the effect of the linearly varying axial load created by the thrust on the natural frequencies and stability characteristics of uniform beams, and to present a derivation of the corresponding eigenvalue problem.

II. PREVIOUS STUDIES

The literature abounds in determinations of natural frequencies and buckling loads for uniform beams subjected to constant end loads. See, for instance, references 5 and 6. However, the case where the structure is subjected to a linearly varying axial load has received little consideration, probably because of the difficulties involved in obtaining solutions to differential equations with variable coefficients.

McKinney[7] considered single span and multiple span beams with distributed axial load and constant end load. He obtained approximations to the fundamental frequency by the Rayleigh-Ritz method, and by the perturbation method for small values of the axial load. He gave some numerical results for a simply supported single-span beam column, for

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*Parenthetical references placed superior to the line of the text refer to the bibliography.

a continuous double-span beam column pinned at either two or three supports, and for a cantilever beam column. In every case, the axial load was taken as the weight of the beam. The two-terms perturbation solution and the Rayleigh-Ritz solution for trial functions consisting of two and three modes of the unloaded beams are compared for the simply supported beam and for the cantilever beam. In every case, the perturbation solution gave a frequency larger than that obtained by the Rayleigh-Ritz method.

Seide[2] considered the effect of a constant longitudinal acceleration on the transverse vibrations of uniform free-free beams. He obtained natural frequency approximations by using linear combinations of the mode shapes of the free-free beam without axial loading in the Rayleigh-Ritz method. He found that a reasonable approximation to the variation of the frequencies with the axial load could be expressed by

$$\frac{f}{f_{on}} = \left[1 - \frac{P}{P_{crn}} \right]^{2/3}$$

where f is the frequency of vibration

f_{on} is the nth mode frequency of the beam without axial load

P is the end load producing the acceleration

P_{crn} is the nth mode critical end load

This relation is said to give a good approximation for the first three modes and possibly the fourth mode. Some results are also given as the effect of tension on the first two frequencies. As expected, the frequencies increase with the tension.

Tu and Handelman[8] considered the effect of a distributed axial load on the fundamental frequency of a uniform cantilever beam carrying a constant axial load at its free end. They obtained approximate solutions by expanding the eigenfunctions and eigenvalues in power series of a loading parameter γ^2:

$$\gamma^2 = \frac{\rho g A L}{E I}$$

where :
ρ is the density
A is the cross sectional area
L is the length of the beam
EI is the flexual rigidity

For small values γ^2 they used a standard perturbation technique, and for large values of γ^2 a singular perturbation method while retaining three terms in the expansions. They also obtained upper bounds by using "simple polynominals" in the Rayleigh quotient and Schwarz iterations, and lower

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bounds by Southwell's method and Schwarz iterations. Their numerical results are for ratios of end load to total distributed load of -2 (beams in tensions), -.25 (beam partially in tension and partially in compression), 0 (beams in compression) and 1 (beams in compression) for various values of the parameter γ^2. The perturbation solutions are fairly good for small γ^2 when the beam is in compression although they are not consistently between the upper and lower bounds. For large values of γ^2 the singular perturbation approximates the eigenvalue from below in the given numerical results. The Schwarz iterations give very narrow bounds for small γ^2 but the gaps become quite large as γ^2 increases.

Beal[3] gave some graphical results on the effect of axial thrusts on the natural frequencies of a uniform free-free beam. The case he considered is, however, different from the one considered by Seide[2] in that the thrust in his problem remains tangent to the beam whereas in Seide's problem it has a fixed line of action. Thus Beal's problem is not self-adjoint whereas Seide's is. Beal's method of attack consists in using linear combinations of the mode shapes of the free-free beam without axial load in Galerkin's method.

Glaser[1] considered also a free-free beam with end thrust remaining tangent to the beam. His method of attack is to represent the beam as a lumped-mass system. He gave graphical results on the effect of the axial thrust on the first three frequencies of Bernoulli-Euler and Timoshenko beams for various distributions of the point masses.

Several authors have considered the problem of the elastic stability of uniform beams with uniformly distributed axial load, and obtained exact solutions because they were able to transform the governing differential equation into known equations. Timoshenko and Gere[6, p. 101] present the solution of the buckling problem for a cantilever beam-column loaded by its own weight. The lowest buckling load is obtained in terms of Bessel functions of the first kind of orders 1/3 and -1/3. A list of references on the problem, going back to Euler, is also included The same problem is solved in the same fashion by McLachlan[9, p. 37] and Bowman [10, p. 125]. Tyler and Rouleau[11] have considered the problem of buckling for a simply supported beam which may be partly in tension and partly in compression. Their solution is in terms of Airy functions which may be related to Bessel functions of orders 1/3 and -1/3.

Przemieniecki[12] considered the problem for various end conditions and gave stability criteria in a series of curves relating the maximum compressive axial load with the maximum distributive load.

Beal[3] and Glaser[1] have considered the problem of buckling for the free-free beam by the methods described above. Their analyses present the interesting point that, since their systems are non-conservative, the dynamic stability criterion must be applied, by which buckling may occur by either the coalescence of adjacent frequencies or by the reduction of a frequency to zero.

In summary, one finds several solutions to the buckling problem, but only a few for the determination of the natural frequencies of beams with uniformly distributed axial load. Most of the approximations appearing in the literature are upper bounds to the true solutions. The closeness of these approximations to the true solutions cannot be established without the knowledge of lower bounds, unless they can be compared with results from experiments.

In the following section, the equation of motion is derived from Hamilton's principle. This method has the advantage over the strength of materials approach that it yields at the same time the natural boundary conditions of importance in the variational characterization of the eigenvalues. This characterization is presented in section IV. It is the basis for upper and lower bounds computations of eigenvalues.

III. EQUATION OF MOTION OF BEAMS WITH UNIFORMLY DISTRIBUTED AXIAL LOAD

We consider a Bernoulli-Euler beam of uniform flexural rigidity EI, subjected to a constant end load P_1 and a uniformly distributed axial load given per unit of length. P_1 and ω are taken to be positive when acting in compression. The loading system is illustrated in figure 1.
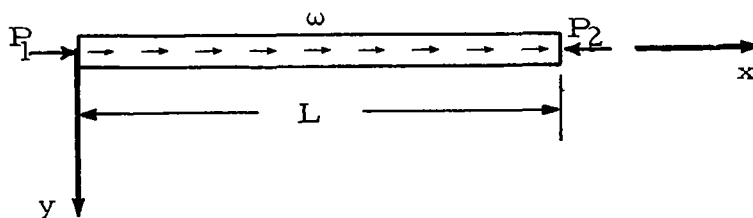

Figure 1. Loading System

The motion of the beam is governed by Hamilton's principle which is formally expressed by

$$\delta \int_{t_0}^{t_1} \left[T - V \right] dt = 0 \qquad \text{(III-1)}$$

where T represents the kinetic energy, and V the potential energy of the system. T is given by

$$T = \frac{1}{2} \int_0^L \rho \left(\frac{\partial y}{\partial t} \right)^2 dt \qquad \text{(III-2)}$$

where p denotes the material's density and y is the beam's lateral deflection. The potential energy of the system may be decomposed into U, the potential energy of the internal forces, given by

$$U = \frac{1}{2} EI \int_0^L \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \qquad \text{(III-3)}$$

and W, the potential energy of the external forces, given by

$$W = -\frac{1}{2} \int_0^L \left(P_1 + \omega x \right) \left(\frac{\partial y}{\partial x} \right)^2 dx \qquad \text{(III-4)}$$

U is the strain energy of bending, and W the potential energy of the axial loads. Substitution of T and V into (III-1) yields, after carrying the variation under the integral sign and integration by parts, that the motion must be such that

$$\int_{t_0}^{t_1} \int_0^L \left\{ \rho \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x} \left[(P_1 + \omega x) \frac{\partial y}{\partial x} \right] \right\} \delta y \, dx \, dt$$

$$\qquad \text{(III-5)}$$

$$- \int_{t_0}^{t_1} \left[EI \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} (\delta y) \Big|_0^L - EI \frac{\partial^3 y}{\partial x^3} \delta y \Big|_0^L - (P_1 + \omega x) \frac{\partial y}{\partial x} \delta y \Big|_0^L \right] dt = 0$$

This equation holds for arbitrary variations δy. Choosing δy and its derivative to vanish at the end points for any time t, but to be arbitrary within the domain, the following Euler equation results from the variational principle:

$$\rho \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x} \left[\left(P_1 + \omega x \right) \frac{\partial y}{\partial x} \right] = 0 \qquad \text{(III-6)}$$

Removal of the restriction on δy and its derivative yeilds, as necessary conditions for the vanishing of the variation of (III-1), the following natural boundary conditions:

$$\left. \begin{array}{l} \dfrac{\partial^2 y}{\partial x^2} = 0 \\[2em] \dfrac{\partial^3 y}{\partial x^3} + \dfrac{1}{EI} \left(P_1 + \omega x \right) \dfrac{\partial y}{\partial x} = 0 \end{array} \right\} \qquad \text{(III-7)}$$

The motion of the beam is therefore governed by equation (III-6) and boundary conditions appropriate to the type of end support. At a free end, the deflection and its derivative with respect to x are arbitrary which from (III-5), indicates that all the boundary conditions are natural, ie.,

$$\left. \begin{array}{l} \dfrac{\partial^2 y}{\partial x^2} = 0 \\[2em] \dfrac{\partial^3 y}{\partial x^3} + \dfrac{1}{EI} \left(P_1 + \omega x \right) \dfrac{\partial y}{\partial x} = 0 \end{array} \right\} \qquad \text{(III-8)}$$

At a simply supported end, the deflection must vanish and its derivative is arbitrary. Hence, the following boundary conditions apply:

$y = 0$ (prescribed boundary condition)*

at a simply supported end (III-9)

$\dfrac{\partial^2 y}{\partial x^2} = 0$ (natural boundary condition)

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*The prescribed boundary conditions are also called essential, or principal, or stable boundary conditions, while the natural boundary conditions are sometimes called unstable.

At the clamped end, both the deflection and its derivative must vanish. Hence, the following are the appropriate boundary conditions:

$y = 0$ (prescribed boundary condition)

at a clamped end.   (III-10)

$\dfrac{\partial y}{\partial x} = 0$ (prescribed boundary condition)

Before leaving this section, it should be noted that the equation derived above is based on the small deflection theory, by which it is admissible to consider that the axial loads remain constant in magnitude during the beam's motion, and that the supports are free to slide in the axial direction. If these assumptions are not made, the equation involved is no longer linear*.

IV.   THE EIGENVALUE PROBLEM

The eigenvalue problem is obtained by separating the variables x and t, ie by looking for solutions to (III-6) of the form

$$y(x,t) = \Psi(x) \cdot F(t) \qquad \text{(IV-1)}$$

Substitution of this function in (III-6) yields the pair of equations

$$\frac{d^2 F}{dt^2} + \gamma^4 F = 0 \qquad \text{(IV-2)}$$

$$\frac{d^4 \Psi}{dx^4} + \frac{1}{EI} \frac{d}{dx}\left[\left(P_1 + \omega x \right) \frac{d\Psi}{dx} \right] - \frac{\gamma^4 g \Psi}{EI} = 0 \qquad \text{(IV-3)}$$

where γ^4 is the separation constant.

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*For some examples of systems with fixed supports see references 13 and 14.

Introduction of the non-dimensional variable $\xi = \dfrac{x}{L}$, and of the parameters

$$\alpha = \frac{\omega L^3}{EI} \quad , \quad \beta = \frac{P_1}{\omega L} \quad , \quad \lambda = \frac{\gamma^4 g L^4}{EI} \tag{IV-4}$$

transforms (IV-3) into

$$\frac{d^4\psi}{d\xi^4} + \alpha \frac{d}{d\xi}\left[(\beta+\xi)\frac{d\psi}{d\xi} \right] - \lambda \psi = 0 \tag{IV-5}$$

and the boundary conditions into

i) $\dfrac{d^2\psi}{d\xi^2} = 0 \quad , \quad \dfrac{d^3\psi}{d\xi^3} + \alpha\left(\beta+\xi\right)\dfrac{d\psi}{d\xi} = 0$ at a free end $\tag{IV-6}$

ii) $\psi = 0 \quad , \quad \dfrac{d^2\psi}{d\xi^2} = 0$ at a simply supported end $\tag{IV-7}$

iii) $\psi = 0 \quad , \quad \dfrac{d\psi}{d\xi} = 0$ at a clamped end $\tag{IV-8}$

In view of the definition of the parameter β , it is clear that for a given compressive distributed load ω , the following cases may occur:

1) $\beta > 0$, the beam is entirely in compression

2) $-1 < \beta < 0$, the beam is partly in tension and partly in compression

3) $\beta \leq -1$, the beam is entirely in tension since the tensile and load P_1 is larger than the total distributed load

In the last case, the problem of elastic stability does not exist.

The determination of the mode shapes and natural frequencies involves the solution of the differential engenvalue problem specified by (IV-5) subject to the boundary conditions appropriate for the type of end condition. We note that (IV-5) is a linear differential equation with a variable coefficient for which exact solutions are very difficult to obtain.* It is then in order to consider approximation techniques. If the problem is considered in its differential form, one can use pertubation techniques, but it is then difficult to state in which direction

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*In fact, as early as 1932, Meyer zur Capellen[15] considered a similar equation and stated that it is impossible to find solutions in terms of "ordinary" functions.

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the error lies. A more fruitful approach is to consider the problem in a variational form, whenever possible, and to obtain approximate solutions to the variational problem.* As is well known, a variational principle can always be constructed from a self-adjoint operator in such a way that the corresponding Euler equation is the given differential equation. To establish the character of our operator, we introduce the following notation: let A denote the differential operator in (IV-5), ie

$$A = \frac{d^4}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{(\beta+\xi)\frac{d}{d\xi}\right\} \tag{IV-9}$$

and let $\langle u,v\rangle$ denote the inner product between two functions u and v, where

$$\langle u,v\rangle = \int_0^1 u\, v\, d\xi \tag{IV-10}$$

We consider the inner product

$$\langle Au,v\rangle = \int_0^1 v\left[\frac{d^4 u}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{(\beta+\xi)\frac{du}{d\xi}\right\}\right] d\xi \tag{IV-11}$$

defined for any two functions u and v of class C^4 . By two integrations by parts, it is transformed into

$$\langle Au,v\rangle = \int_0^1 \left[\frac{d^2 u}{d\xi^2}\frac{d^2 v}{d\xi^2} - \alpha\left(\beta+\xi\right)\frac{du}{d\xi}\frac{dv}{d\xi}\right] d\xi$$

$$+ \left[v\frac{d^3 u}{d\xi^3} - \frac{dv}{d\xi}\frac{d^2 u}{d\xi^2} + \alpha\left(\beta+\xi\right)v\frac{du}{d\xi}\right]_0^1 \tag{IV-12}$$

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*For details, see for instance references 16, 17, 18 and 20.

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Further integration by parts yields

$$\langle A u, v \rangle = \int_0^1 u \left\{ \frac{d^4 v}{d\xi^4} + \alpha \frac{d}{d\xi}\left[(\beta+\xi)\frac{dv}{d\xi}\right] \right\} d\xi \tag{IV-13}$$

$$+ \left[v \frac{d^3 u}{d\xi^3} - \frac{dv}{d\xi}\frac{d^2 u}{d\xi^2} + \alpha(\beta+\xi)v\frac{du}{d\xi} \right]_0^1$$

$$- \left[u \frac{d^3 v}{d\xi^3} - \frac{du}{d\xi}\frac{d^2 v}{d\xi^2} + \alpha(\beta+\xi)u\frac{dv}{d\xi} \right]_0^1$$

When A operates over the classes of functions satisfying the conditions given by (IV-6), or (IV-7), or (IV-8), the boundary term in (IV-13) vanishes, and leaves

$$\langle A u, v \rangle = \langle u, A v \rangle \tag{IV-14}$$

which indicates that over those classes A is self-adjoint.

The problem is therefore self-adjoint for free, clamped, or simply supported end conditions, as well as combinations of these, and in each case the eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal.*

Before going into the variational formulation of the eigenvalues of A, we consider the related problem of the elastic stability of the beam as it yields some insight in the nature of the operator A. By setting λ equal to zero in (IV-5) we obtain

$$\frac{d^4 \psi}{d\xi^4} + \alpha \frac{d}{d\xi}\left[(\beta+\xi)\frac{d\psi}{d\xi}\right] = 0 \tag{IV-15}$$

along with the same boundary conditions as before. The last approach is

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*See reference 19 for a discussion on the methods of solution of stability problems.

satisfactory because the system is conservative which implies that buckling occurs whenever the loads are such that the natural frequencies become zero.

For $\beta \leq -1$, the beam is entirely in tension, and no value of the parameter α exist that will make it elastically unstable. For $\beta > -1$, the beam is either partly or entirely in compression, and there may exist discrete values of α for which (IV-15) has solutions. The smallest of these, α_c is the critical axial load. It may be characterized by the following variational principle:*

$$\alpha_c = \min_{u \in K_b} \frac{\int_0^1 \left(\frac{d^2 u}{d\xi^2}\right)^2 d\xi}{\int_0^1 (\beta + \xi)\left(\frac{du}{d\xi}\right)^2 d\xi} \qquad (1V\text{-}16)$$

where K_b is the class of admissible functions μ satisfying the following conditions:

1) μ Satisfies the prescribed boundary conditions

2) $\frac{d^2 u}{d}$ is square integrable.

3) $\qquad \int_0^1 (\beta + \xi)\left(\frac{du}{d\xi}\right)^2 d\xi \ > 0 \qquad (IV\text{-}17)$

To the smallest eigenvalue α there corresponds in K_b a function for which

$$\alpha_c = \frac{\int_0^1 \left(\frac{d^2 u}{d\xi^2}\right)^2 d\xi}{\int_0^1 (\beta + \xi)\left(\frac{du}{d\xi}\right)^2 d\xi} \qquad (1V\text{-}18)$$

or

$$\int_0^1 \left[\left(\frac{d^2 u}{d\xi^2}\right)^2 - \alpha_c (\beta + \xi)\left(\frac{du}{d\xi}\right)^2 \right] d\xi = 0 \qquad (1V\text{-}19)$$

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*See for instance references 17, 18, and 20.

For any other function $v \in K_b$ different from zero,

$$\int_0^1 \left[\left(\frac{d^2 v}{d\xi^2}\right)^2 - \alpha_c \left(\beta+\xi\right)\left(\frac{dv}{d\xi}\right)^2 \right] d\xi > 0 \qquad \text{(IV-20)}$$

and for any $0 \le \alpha \le \alpha_c$,

$$\int_0^1 \left[\left(\frac{d^2 v}{d\xi^2}\right)^2 - \alpha \left(\beta+\xi\right)\left(\frac{dv}{d\xi}\right)^2 \right] d\xi > 0 \qquad \text{(IV-21)}$$

Now, the lowest eigenvalue of A can be characterized by the minimum principle.*

$$\lambda_1 = \min_{u \in K} \frac{\langle A u, u \rangle}{\langle u, u \rangle} = \min_{u \in K} \frac{\int_0^1 u \left[\frac{d^4 u}{d\xi^4} + \alpha \frac{d}{d\xi}\left[(\beta+\xi)\frac{du}{d\xi}\right] \right] d\xi}{\int_0^1 u^2 \, d\xi} \qquad \text{(IV-22)}$$

where K is the class of functions constituting the field of definition of the operator A and, hence, satisfying both the prescribed and the natural boundary conditions. The minimum principle can also be written as

$$\lambda_1 = \min_{u \in K_v} \frac{\int_0^1 \left[\left(\frac{d^2 u}{d\xi^2}\right)^2 - \alpha (\beta+\xi)\left(\frac{du}{d\xi}\right)^2 \right] d\xi}{\int u^2 \, d\xi} \qquad \text{(1V-23)}$$

where Kv is the class of admissable functions required to satisfy only the prescribed boundary conditions.*

We note that since K_b is a subclass of Kv, for all $0 \le \alpha \le \alpha_c$ the following inequality is satisfied for $u \in Kv$:

$$\int_0^1 \left[\left(\frac{d^2 u}{d\xi^2}\right)^2 - \alpha (\beta+\xi) \left(\frac{du}{d\xi}\right)^2 \right] d\xi \ge 0 \qquad \text{(IV-24)}$$

where the equality sign holds for u = o only.

The operator A is therefore positive definite.

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*This function is known as Rayleigh's quotient.

**In the terminology of Mikhlin, ref. 20, they are the functions with finite energy since the numerator of the Rayleigh quotient corresponds to the potential energy of the system.

The other eigenvalues of A may be characterized by either the recursive characterization* or by Courant's maximum-minimum characterization.*

In the latter form, the j-th eigenvalue of A is given by

$$\lambda_j = \max_{\{u_i\}} \left\{ \min_{<\phi, u_i>=0} \frac{\int_0^1 \left[\left(\frac{d^2\phi}{d\xi^2}\right)^2 - \alpha(\beta+\xi)\left(\frac{d\phi}{d\xi}\right)^2 \right] d\xi}{\int_0^1 \phi^2 d\xi} \right\} \quad \text{(IV-25)}$$

where ϕ and u_i belong to K_v.

In resumé if β is such that buckling may occur, there exists a critical value of the distributed axial load parameters, α_c, for which the beam is unstable and potential energy is equal to zero. For any value of α less than c, the potential energy is positive, and the beam has discrete natural frequencies whose square are proportional to the eigenvalues of the operator A. These eigenvalues are assumed to be ordered in the non-decreasing sequence:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \qquad \text{(IV-26)}$$

The eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal, and correspond to the mode shapes of the beam. For a given value of β, as α increases the denominator of the Rayleigh quotient decreases and eigenvalues decrease. Buckling occurs when α becomes equal to α_c, for which the first eigenvalue goes to zero.

The problem is now transformed into obtaining solutions to the variational principle (IV-25).

V. CONCLUDING REMARKS

The study of the literature indicates that an exact solution to the determination of the eigenvalues of uniform beams with linearly varying axial load is extremely difficult. Most of the studies conducted furnish upper bounds to the true solutions. In order to facilitate obtaining upper and lower bounds to the eigenvalues, the equation of motion and the boundary conditions have been derived, and the eigenvalue problem has been cast in a variational form.

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*See for instance reference 16 Chapter III.

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PART II - THE RECTANGULAR PLATE

LIST OF SYMBOLS

a, b	Dimensions of a rectangular plate
$D = \dfrac{Eh^3}{12(1-\nu^2)}$	Flexural rigidity of a plate
E	Modulus of elasticity
$G(t)$	Function of time (25)*
g	Acceleration of gravity
M_n	Bending moment
M_{nt}	Twisting moment
m	Density per unit area
N_x, N_y, N_{xy}	Stress resultants (3)
$\underline{n}$	Unit normal vector
P_1	Uniform pressure
Q	Shearing force
q	Lateral load
s	Coordinate along plate boundary
T	Kinetic energy (6)
$\underline{t}$	Tangent unit vector
V	Potential energy (5)
w	Lateral deflection
X, Y	Body face components (4)
x, y	Coordinates
α	Loading parameter (31)
β	Loading parameter (31)
δ	Variation symbol
ϵ	Eigenvalue, square of natural frequency (26)
λ	Eigenvalue (31)
$\phi(x,y)$	Mode shape (25)
$\Psi(x)$	Function of x (28)
$\Omega(y)$	Function of y (28)
ξ	Nondimensional coordinate (31)
$\sigma_{yy}, \sigma_{xx}, \sigma_{xy}$	Stress components
θ	First stress invariant
Θ	Angle between $\underline{n}$ and x axis

*Numbers in parentheses in this section refer to the equations.

18

I. INTRODUCTION

Structures subjected to accelerations develop inertia forces which tend to reduce their natural frequencies. This phenomenon may be of concern in the design of rocket components as higher payloads are required and higher accelerations attained. For instance, when its acceleration has a component in the plane of a plate, the plate develops in-plane loads which must be carried in addition to any external loads. This produces an alteration in the natural frequencies which may then fall within the spectrum of excitations. The problem of the determination of the natural frequencies and buckling loads of plates with in-plane loads is a difficult one. For constant in-plane loads, some solutions appear in the literature. See for instance references*[1] through [4] for rectangular plates, [5] for a triangular plate, [6] for a circular plate, and in particular reference [9] which contains an extensive bibliography. For varying in-plane loads, however, few solutions are available. References [7] and [8] present solutions for the buckling problem of rectangular plates.

The object of this report is to derive the equation of motion of plates with in-plane loads and to show that in special cases the eigenvalue problem reduces to that of a beam with linearly varying axial load which was considered in a previous report [11].

*Parenthetical references superior to the line of the text refer to the bibliography.

II. EQUATION OF MOTION OF A UNIFORM PLATE WITH IN-PLANE LOADS

Consider a rectangular plate having uniform thickness, h, small in comparison to its other dimensions, a and b. Under the small deflection assumptions, the middle plane of the plate does not change during the motion. This stress distribution is obtained by solving the plane stress problem governed by the equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho F_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho F_y = 0 \tag{1}$$

and the compatibility equations in terms of stresses

$$\nabla^2 \sigma_{xx} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x^2} = \frac{-\rho\nu}{1-\nu}\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right] - 2\rho\frac{\partial F_x}{\partial x}$$

$$\nabla^2 \sigma_{yy} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y^2} = \frac{-\rho\nu}{1-\nu}\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right] - 2\rho\frac{\partial F_y}{\partial y}$$

$$\frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial z^2} = \frac{-\rho\nu}{1-\nu}\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right] \tag{2}$$

$$\nabla^2 \sigma_{xy} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x \partial y} = -\rho\left[\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}\right]$$

$$\frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y \partial z} = 0$$

$$\frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial z \partial x} = 0$$

where

$$\Theta = \sigma_{xx} + \sigma_{yy}$$

The in-plane loads per unit length are taken as the stress resultants defined by

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}\, dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_{yy}\, dz \tag{3}$$

$$N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz$$

and the body force components per unit area are defined by

$$X = \rho h F_x$$

$$Y = \rho h F_y \tag{4}$$

The potential energy of the system consists of the strain energy due to bending, V_1, the work done by the in-plane loads, V_2, and the energy, V_3, due to the external normal forces, boundary forces, and bending moment on the boundary. V is given by

$$V = \frac{1}{2} D \int_0^b \int_0^a \left\{ \left[\nabla^2 w \right]^2 - 2\left[1-\nu\right]\left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx\, dy$$

$$+ \frac{1}{2} \int_0^b \int_0^a \left\{ N_x \left[\frac{\partial w}{\partial x} \right]^2 + N_y \left[\frac{\partial w}{\partial y} \right]^2 + 2 N_{xy} \left[\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \right\} dx\, dy \tag{5}$$

$$- \int_0^b \int_0^a q w\, dx\, dy + \int_\Gamma M_n \frac{\partial w}{\partial n}\, ds - \int_\Gamma \left\{ Q_n - \frac{\partial M_{nt}}{\partial s} \right\} w\, ds$$

where w denotes the lateral deflection, D the flexural rigidity, q the lateral load, M_n the distributed bending moment along the boundary Γ , and $Q_n - \frac{\partial M_{nt}}{\partial s}$ the transverse force along the boundary.

The notation and sign convention are the same as those used by Timoshenko and Woinowsky-Krieger[10]. Positive moments and shears are as shown in Figure 1.
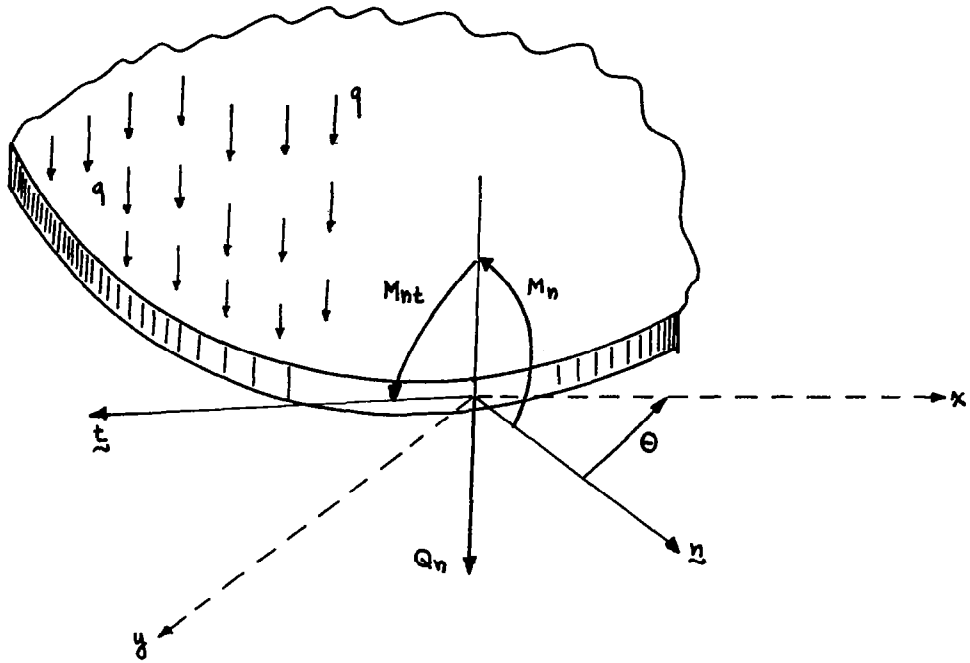

Figure 1

NOTATION AND POSITIVE ORIENTATION OF MOMENTS AND SHEARS

The kinetic energy of the plate is given by

$$T = \frac{1}{2} \int_0^b \int_0^a m \left[\frac{\partial w}{\partial t} \right] dx\, dy \tag{6}$$

where m denotes the density per unit area.

Now, Hamilton's principle reads

$$\delta \int_{t_0}^{t_1} \left[T - V \right] dt = 0 \tag{7}$$

Taking each term in turn,

$$\delta T = \tfrac{1}{2}\, \delta \int_0^b \int_0^a m \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy \;=\; \int_0^b \int_0^a m\, \frac{\partial w}{\partial t}\, \frac{\partial(\delta w)}{\partial t}\, dx\, dy$$

$$= \int_0^b \int_0^a m\, \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\, \delta w\right) dx\, dy \;-\; \int_0^b \int_0^a m\, \frac{\partial^2 w}{\partial t^2}\, \delta w\, dx\, dy \tag{8}$$

The variation of the first integral of V is given by Timoshenko and Woinowsky-Krieger[10], p. 23.

$$\delta V_1 = D \int_0^b \int_0^a \nabla^4 w\, \delta w\, dx\, dy \;+\; D \int_\Gamma \left\{ \left[1-\nu\right]\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta\right]\right\}\frac{\partial \delta w}{\partial n}\, ds$$

$$+\; D \int_\Gamma \nu \nabla^2 w\, \frac{\partial \delta w}{\partial n}\, ds \;+\; D \int_\Gamma \left[1-\nu\right]\frac{\partial}{\partial s}\left[\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\sin\theta\cos\theta - \frac{\partial^2 w}{\partial x\partial y}\left(\cos^2\theta-\sin^2\theta\right)\right]\delta w\, ds$$

$$-\; D \int_\Gamma \left\{\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)\cos\theta + \left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)\sin\theta\right\}\delta w\, ds. \tag{9}$$

where θ is defined in Figure 1 as the angle between the normal $\underset{\sim}{n}$ and the x-axis. Similarly for the second integral of V

$$\delta V_2 = \int_0^b \int_0^a \left\{ N_x\, \frac{\partial w}{\partial x}\, \frac{\partial \delta w}{\partial x} + N_y\, \frac{\partial w}{\partial y}\, \frac{\partial \delta w}{\partial y} + N_{xy}\, \frac{\partial \delta w}{\partial x}\, \frac{\partial w}{\partial y} + N_{xy}\, \frac{\partial w}{\partial x}\, \frac{\partial \delta w}{\partial y} \right\} dx\, dy \tag{10}$$

But

$$N_x\, \frac{\partial w}{\partial x}\, \frac{\partial \delta w}{\partial x} = \frac{\partial}{\partial x}\left\{ N_x\, \frac{\partial w}{\partial x}\, \delta w\right\} - \frac{\partial N_x}{\partial x}\, \frac{\partial w}{\partial x}\, \delta w - N_x\, \frac{\partial^2 w}{\partial x^2}\, \delta w$$

$$N_y\, \frac{\partial w}{\partial y}\, \frac{\partial \delta w}{\partial y} = \frac{\partial}{\partial y}\left\{ N_y\, \frac{\partial w}{\partial y}\, \delta w\right\} - \frac{\partial N_y}{\partial y}\, \frac{\partial w}{\partial y}\, \delta w - N_y\, \frac{\partial^2 w}{\partial y^2}\, \delta w \tag{11}$$

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and

$$N_{xy} \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} = \frac{\partial}{\partial y} \left| N_{xy} \frac{\partial w}{\partial x} \delta w \right| - \frac{\partial N_{xy}}{\partial y} \frac{\partial w}{\partial x} \delta w - N_{xy} \frac{\partial^2 w}{\partial x \partial y} \delta w$$

$$N_{xy} \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} = \frac{\partial}{\partial x} \left| N_{xy} \frac{\partial w}{\partial y} \delta w \right| - \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} \delta w - N_{xy} \frac{\partial^2 w}{\partial x \partial y} \delta w$$

so that

$$\delta V_2 = - \int_0^b \int_0^a \left| N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} - X \frac{\partial w}{\partial x} - Y \frac{\partial w}{\partial y} \right| \delta w \, dx \, dy$$

$$+ \int_0^b \int_0^a \left| \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} \delta w + N_{yx} \frac{\partial w}{\partial y} \delta w \right) + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} \delta w + N_{xy} \frac{\partial w}{\partial x} \delta w \right) \right| dx \, dy$$

(12)

where use has been made of the equilibrium equations for the in-plane loads. The second integral in (12) can be converted into a line integral using

$$\int_A \frac{\partial F}{\partial x} dA = \int_\Gamma F \cos \theta \, ds$$

(13)

$$\int_A \frac{\partial F}{\partial y} dA = \int_\Gamma F \sin \theta \, ds$$

to give

$$\delta V_2 = - \int_0^b \int_0^a \left| N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} - X \frac{\partial w}{\partial x} - Y \frac{\partial w}{\partial y} \right| \delta w \, dx \, dy$$

$$+ \int_\Gamma \left| \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \cos \theta + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \sin \theta \right| \delta w \, ds$$

(14)

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Finally,

$$\delta V_3 = -\int_0^b \int_0^a q \, \delta w \, dx \, dy \; + \; \int_\Gamma M_n \frac{\partial \delta w}{\partial n} \, ds \; - \; \int_\Gamma \left(Q_n - \frac{\partial M_{nt}}{\partial s} \right) \delta w \, ds \tag{15}$$

Substitution of (8), (9), (14), and (15) into (7) yields the differential equation

$$m \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + X \frac{\partial w}{\partial x} + Y \frac{\partial w}{\partial y} - q = 0 \tag{16}$$

and the natural boundary conditions

$$D(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \cos^2\theta + 2 \frac{\partial^2 w}{\partial x \partial y} \sin\theta \cos\theta + \frac{\partial^2 w}{\partial y^2} \sin^2\theta \right] + D\nu \nabla^2 w + M_n = 0 \tag{17}$$

and

$$D(1-\nu) \frac{\partial}{\partial s} \left[\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) \sin\theta \cos\theta \; - \; \frac{\partial^2 w}{\partial x \partial y} \left(\cos^2\theta - \sin^2\theta \right) \right]$$

$$- D \left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right] \cos\theta \; - \; D \left[\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \right] \sin\theta \tag{18}$$

$$+ \left[N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right] \cos\theta + \left[N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right] \sin\theta \; - \left[Q_n - \frac{\partial M_{nt}}{\partial s} \right] = 0$$

Hence the following boundary conditions apply:

1)   For built-in edges: $\left(\delta w = 0 \quad , \quad \dfrac{\partial \delta w}{\partial n} = 0 \right)$

$$w = 0 \qquad , \qquad \frac{\partial w}{\partial n} = 0 \tag{19}$$

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2) For simply-supported edges: $\left(\delta w=0 \ , \ M_n=0\right)$

$$w=0 \ , \quad (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta\right] + \nu \nabla^2 w = 0 \qquad (20)$$

3) For free edges: $\left(\delta w \text{ and } \frac{\partial \delta w}{\partial n} \text{ arbitrary}, \ M_n=0 \ , Q_n - \frac{\partial M_{nt}}{\partial s} = 0\right)$

$$(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta\right] + \nu \nabla^2 w = 0 \qquad (21)$$

and

$$D(1-\nu)\frac{\partial}{\partial s}\left[\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right)\sin\theta\cos\theta - \frac{\partial^2 w}{\partial x \partial y}\left(\cos^2\theta - \sin^2\theta\right)\right] - D\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right]\cos\theta$$

$$- D\left[\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right]\sin\theta \quad + \left[N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right]\cos\theta + \left[N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right]\sin\theta = 0 \qquad (22)$$

For rectilinear edges parallel to the coordinate axes, these equations can be simplified using the following conditions on θ :

i) Edge parallel to the x-axis: $\theta = \frac{\pi}{2}$, $\cos\theta = 0$, $\sin\theta = 1$

2) Edge parallel to the y-axis: $\theta = 0$, $\cos\theta = 1$, $\sin\theta = 0$

In the question of motion derived above, equation (16), the in-plane loads are in general form. They must be determined through the solution of equations (1) through (3) before any attempt at a solution to (16) can be made. In this report we will consider the following type of loading:

$$N_x = -\left[P_1 + mgx\right] \ , \quad N_y = N_{xy} = 0 \ , \quad X = mg \ , \ Y = 0 \qquad (23)$$

That such a state of stress is possible can be seen by substitution in the equilibrium equations and the compatibility equations for stress. Such a state of stress can be thought as arising in a vertical plate loaded by a uniform pressure P_1 and gravity with gravitational constate g, as shown in Figure 2.

This interpretation, however, involves the assumption that the plate is free to deform in its plane.
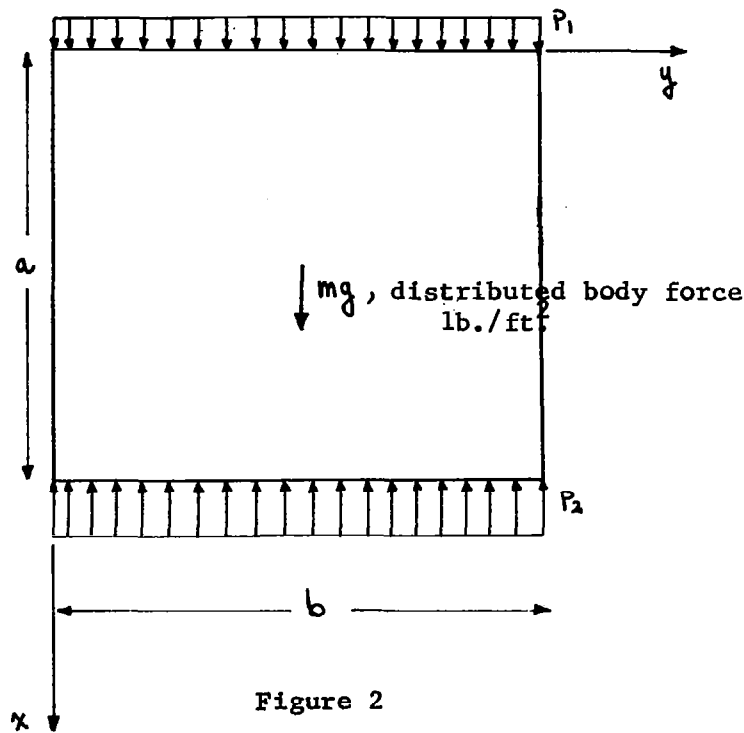

Figure 2

GEOMETRY FOR RECTANGULAR PLATES

With these in-plane loads, equation (16) in the absence of lateral load, q , becomes

$$m \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w + \left(P_1 + mg x \right) \frac{\partial^2 w}{\partial x^2} + mg \frac{\partial w}{\partial x} = 0 \qquad (24)$$

In the following section, the corresponding eigenvalue problem is considered.

III. EIGENVALUE PROBLEM

The mode shapes and natural frequencies of the plate are obtained by searching for eigenvibrations of the form

$$w(x,t) = \phi(x,y) \cdot G(t) \qquad (25)$$

Equation (24) yields

$$\frac{d^2 G}{dt^2} + \epsilon^4 G = 0 \qquad (26)$$

and

$$\nabla^4 \phi + \frac{1}{D}\left(P_1 + mgx\right)\frac{d^2 \phi}{dx^2} + mg\,\frac{d\phi}{dx} - \frac{m\epsilon^4}{D}\phi = 0 \qquad (27)$$

The last equation involves both of the independent variables x and y. Following Nowacki(2,) p. 206, it can be separated to obtain ϕ in the form

$$\phi = \Psi(x)\cdot\Omega(y) \qquad (28)$$

provides Ω(y) satisfies

$$\frac{d^2 \Omega}{dy^2} = -\mu^2 \Omega \quad , \quad \frac{d^4 \Omega}{dy^4} = \mu^4 \Omega \qquad (29)$$

where μ is constant.

These conditions are satisfied only by trigonometric functions. Hence, for equation (27) to separate, the plate must be simply supported at the edges y = 0 and y = b. This requirement being satisfied, the function ψ(x) must satisfy the equation

$$\frac{d^4 \Psi}{dx^4} + \left(\frac{P_1}{D} + \frac{mgx}{D} - 2\mu^2\right)\frac{d^2 \Psi}{dx^2} + \frac{mg}{D}\frac{d\Psi}{dx} - \left(\frac{\epsilon^4 m}{D} - \mu^4\right)\Psi = 0 \qquad (30)$$

which, by changing to the following parameters

$$\xi = \frac{x}{a} \quad , \quad \alpha = \frac{mga^3}{D} \quad , \quad \beta = \left(\frac{P_1}{D} - 2\mu^2\right)\frac{D}{mga} \quad , \quad \lambda = a^4\left(\frac{\epsilon^4 m}{D} - \mu^4\right) \qquad (31)$$

takes the form

$$\frac{d^4 \Psi}{d\xi^4} + \alpha\frac{d}{d\xi}\left[(\beta+\xi)\frac{d\Psi}{d\xi}\right] - \lambda\Psi = 0 \qquad (32)$$

i.e., the same form as the eigenvalue problem for a beam with linearly distributed axial load.

This equation has been considered in Part I. Its eigenvalues may be computed by numerical methods. The eigenvalues of the plate can then be obtained from the knowledge of λ_i and μ_i^4.

IV. CONCLUDING REMARKS

a) Extension to More General Distributed Load.

The foregoing analysis has assumed that the uniformly distributed in-plane loading arises solely from the distributed weight of a vertical plate. Such a loading configuration could result from any in-plane tractive load (friction, for example) imposed on the surface of the plate. For such an application the term mg in eq. (24) should be replaced by a suitable expression representing the total in-plane uniformly distributed tractive load.

b) Summary.

The eigenvalue problem for a uniform rectangular plate free to deform in its plane, and subjected to a linearly varying in-plane load has been reduced to that of a uniform beam carrying a linearly varying axial load when the plate is simply supported along the edges parallel to the load. The type of support along the other edges is arbitrary. Solutions will be presented in a future report for the cases where the edges perpendicular to the load are either simply supported or clamped.

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