TECHNICAL NOTE R-182

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EFFECTIVE MOMENT OF INERTIA AND VELOCITY RATIO FOR LIQUID-FILLED CYLINDRICAL TANKS OSCILLATING ABOUT THE LONGITUDINAL AXIS

by J. L. Diven

March 1966

RESEARCH LABORATORIES

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ABSTRACT

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The Navier-Stokes equations are solved exactly for the motion of the liquid in a liquid-filled cylindrical tank with oscillation about the longitudinal axis. The moment due to the viscous shear at the tank wall is calculated. Equations for the effective moment of inertia of the liquid are presented for both a very large coefficient of viscosity and a normal coefficient of viscosity. An equation for the velocity ratio is derived and velocity ratio profiles are presented. A comparison of the analytical and experimental effective moment of inertia ratios shows good agreement.

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LIST OF SYMBOLS

A	Constant from a table of the ber_0 and bei_0 functions (Equation 47)
A_1	Constant in Equation 17
a	Tank radius
B	Constant from a table of the ber_1 and bei_1 functions (Equation 47)
B_1	Constant in Equation 17
F	Exterior force exerted on the liquid
h	Height of liquid in the tank
I_{eff}	Effective moment of inertia of the liquid in a liquid-filled cylindrical tank
$I_0(x)$	Modified Bessel function of the first kind and zeroth order
$I_1(x)$	Modified Bessel function of the first kind and first order
i	$\sqrt{-1}$
$J_0(x)$	Bessel function of the first kind and zeroth order
$J_1(x)$	Bessel function of the first kind and first order
$K_1(x)$	Modified Bessel function of the second kind and first order
$M_0(x)$	Constant from a table of the ber_0 and bei_0 functions (Equation 42)
$M_1(x)$	Constant from a table of the ber_1 and bei_1 functions (Equation 42)
$M_{(r,t)_{r=a}}$	Moment resulting from the shear stress at the tank wall
p	Static pressure in the liquid

R	Symbol replacing the argument of the ber and bei functions (Equation 43)
$R(r)$	Velocity in the ϕ direction as a function of r only
r, ϕ, z	Cylindrical coordinates (Figure 1)
$T(t)$	Velocity in the ϕ direction as a function of t only
t	Time
V	Velocity with a subscript denoting direction
$\theta_0(x)$	Phase angle of the ber_0 and bei_0 functions (Equation 43)
$\theta_1(x)$	Phase angle of the ber_1 and bei_1 functions (Equation 43)
λ	Constant used in the solution of Equation 16
μ	Coefficient of viscosity of the liquid
ν	Kinetic coefficient of viscosity of the liquid
ρ	Liquid density
τ	Shear stress in the liquid
ϕ	Amplitude of the angular displacement of the tank
ω	Tank oscillation frequency

Subscripts

r, ϕ, z	Indicate appropriate direction of the liquid velocity
$r\phi$, rz, ϕz	Indicate appropriate plane in which the shear stress occurs

INTRODUCTION

The problem of the oscillating liquid-filled cylindrical propellant tank is important in the state-of-the-art missile design. This problem must be considered for both structural evaluation and flight control requirements of a large missile.

Additional structural loads are imposed on the propellant tank and associated tank structure by the relative motion of the liquid propellant. The viscous shear stress of the liquid propellant at the tank wall creates a moment. This moment is in the direction opposite to the tank motion and resists additional tank displacement. The effective moment of inertia of the liquid is defined by the ratio

$$I_{eff} = \frac{M_{r=a}}{\ddot{\theta}}$$

where $M_{r=a}$ is the moment due to the viscous shear at the tank wall and $\ddot{\theta}$ is the angular acceleration of the tank.

A theoretical analysis of the effective moment of inertia of a liquid-filled cylindrical tank with oscillation about the longitudinal axis is presented in this report. Equations are given for the special cases of a very large liquid propellant coefficient of viscosity and a small coefficient of viscosity. An analysis of the velocity profiles is also included. This information is presented in both analytical and graphical form.

THEORETICAL INVESTIGATION

FORMULATION OF THE EQUATIONS

The coordinate system for the cylindrical tank is shown in Figure 1. A cylindrical tank with smooth walls, i.e. no internal obstructions, is assumed for simplicity. It is further assumed that the tank is long in order to neglect the end effects.

The liquid dynamics of this system are represented by the Navier - Stokes and Continuity equations. These equations are taken from Reference 1.

Navier-Stokes Equations (Cylindrical Coordinates)

$$\rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_r}{\partial \phi} - \frac{V_\phi^2}{r} + V_z\frac{\partial V_r}{\partial z}\right) = F_r - \frac{\partial p}{\partial r}$$

$$+ \mu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{1}{r}\frac{\partial^2 V_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial^2 V_r}{\partial z^2}\right) \quad (1)$$

$$\rho\left(\frac{\partial V_\phi}{\partial t} + V_r\frac{\partial V_\phi}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} + V_z\frac{\partial V_\phi}{\partial z}\right) = F_\phi - \frac{1}{r}\frac{\partial p}{\partial \phi}$$

$$+ \mu\left(\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \phi} + \frac{\partial^2 V_\phi}{\partial z^2}\right) \quad (2)$$

$$\rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_z}{\partial \phi} + V_z\frac{\partial V_z}{\partial z}\right) = F_z - \frac{\partial p}{\partial z}$$

$$+ \mu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \phi^2} + \frac{\partial^2 V_z}{\partial z^2}\right) \quad . \quad (3)$$

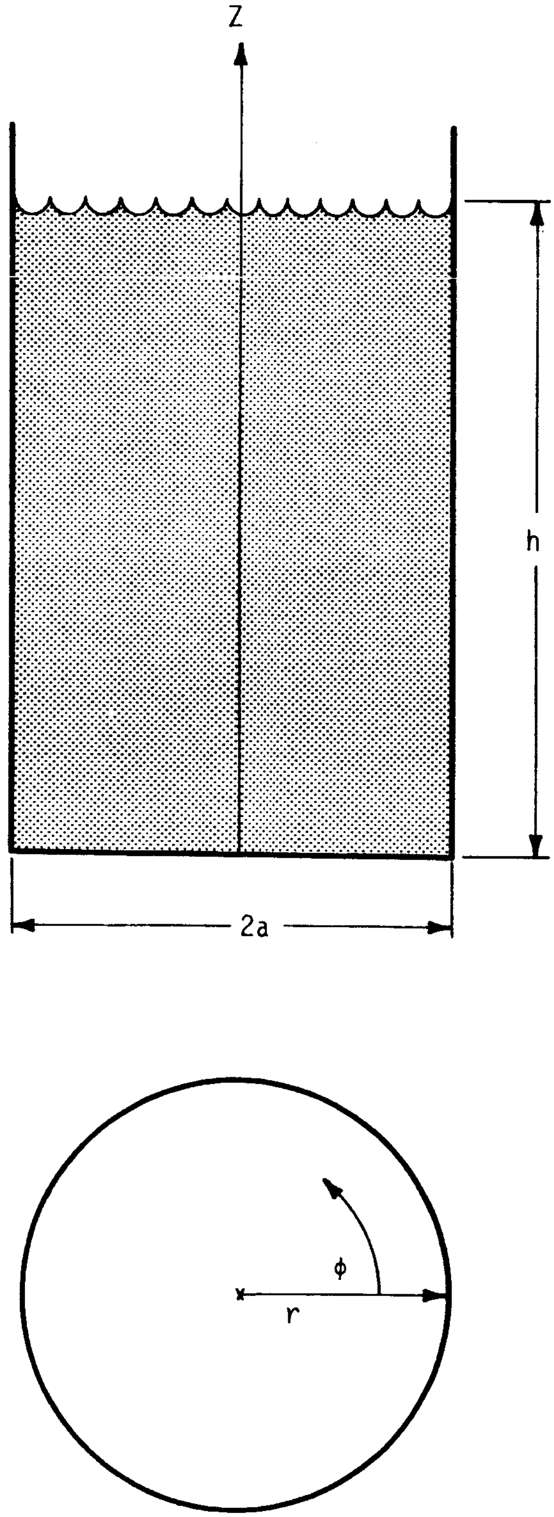

Figure 1. Cylindrical Tank Coordinate System

Continuity Equation (Cylindrical Coordinates)

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_t}{\partial z} = 0 \quad . \tag{4}$$

The equations for the shear stress are also taken from Reference 1.

Stress Components

$$\tau_{r\phi} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \phi} \right] \tag{5}$$

$$\tau_{\phi z} = \mu \left(\frac{\partial V_\phi}{\partial z} + \frac{1}{r}\frac{\partial V_z}{\partial r} \right) \tag{6}$$

$$\tau_{rz} = \mu \left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \right) \tag{7}$$

Equations 1 through 7 must be solved for the desired boundary conditions imposed upon the tank motion.

ASSUMED TANK MOTION AND RESULTING LIQUID DYNAMICS

A smooth-walled cylindrical tank with motion about the longitudinal axis is assumed. The motion of the tank is assumed to be an undamped, forced oscillation that may be described by

$$\phi = \phi_0 \, e^{i\omega t} \tag{8}$$

where ϕ_0 is the amplitude of the angular displacement and ω is the frequency of the oscillations. It is further assumed that the tank is long and the end effects may be neglected.

These assumptions are satisfied if the fluid velocity components are

$$V_r = 0 \tag{9}$$

$$V_z = 0 \tag{10}$$

$$V_{\phi_{r=a}} = a\dot{\phi} = i a \omega \phi_0 e^{i\omega t} \quad . \tag{11}$$

The Navier-Stokes, continuity, and stress-component equations may be simplified by the substitution of these boundary conditions in Equations 1 through 7.

Navier-Stokes Equations

$$\frac{\partial p}{\partial r} = \rho \frac{V_\phi^2}{r} \tag{12}$$

$$\frac{\partial V_\phi}{\partial t} = \frac{\mu}{\rho} \left(\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r^2} \right) \tag{13}$$

Stress Component Equation

$$\tau_{r\phi} = r\mu \frac{\partial}{\partial r} \left(\frac{V_\phi}{r} \right) \tag{14}$$

Equations 13 and 14 must be solved for the fluid velocity distribution and shear stress respectively. Equation 12 may be solved for the pressure as a function of the tank radius if desired.

SOLUTION OF THE SIMPLIFIED NAVIER-STOKES EQUATION

The simplified Navier-Stokes equation is an equation of the form

$$V_\phi = R(r) \cdot T(t) \quad . \tag{15}$$

This equation may be solved by the standard separation of variable technique since

$$\frac{T'}{T} = \nu \left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} \right) = \pm \lambda \tag{16}$$

where λ is a constant that must satisfy the boundary conditions. The complete solution to Equation 16 is

$$V_{\phi(r,t)} = \left\{ A_1 I_1 \left[r \left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} \right] + B_1 K_1 \left[r \left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} \right] \right\} e^{\lambda t} . \tag{17}$$

A_1 and B_1 are constants to be determined from the boundary conditions and $I_1 [r(i\omega/\nu)^{\frac{1}{2}}]$ and $K_1 [r(i\omega/\nu)^{\frac{1}{2}}]$ are modified Bessel functions of the first and second kind and first order respectively.

The boundary condition of a zero velocity at the tank axis requires the constant $B_1 = 0$ since $I_1(o) = 0$ and $K_1(o) \neq 0$. The boundary condition

$$V_{\phi_{r=a}} = ia\omega\phi_0 e^{i\omega t} \tag{11}$$

may be used to evaluate the constant A_1 as

$$A_1 = \frac{ia\omega\phi_0}{\left[a \left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} \right]} \tag{18}$$

The equation for the fluid velocity as a function of the radius, fluid properties, and tank oscillation frequency is

$$V_\phi = ia\omega\phi_0 e^{i\omega t} \cdot \frac{I_1 \left[r \left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} \right]}{I_1 \left[a \left(\frac{i\omega}{\nu} \right)^{\frac{1}{2}} \right]} . \tag{19}$$

Equation 19 may be interpreted as a vector at a given phase angle with respect to the tank displacement vector. Using this interpretation, Equation 19 may be written

$$V_\phi = a\omega\phi_0 \, e^{i[\omega t + (\pi/2)]} \cdot \frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \tag{20}$$

The imaginary arguments of the modified Bessel function may be simplified. A complete analysis including velocity profiles is presented in Appendix D.

SOLUTION FOR THE SHEAR STRESS

The equation for the shear stress is rewritten for convenience.

$$\tau_{r\phi} = \mu r \frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right) \tag{14}$$

After substitution of the equation for the fluid velocity, Equation 14 becomes

$$\tau_{r\phi} = \frac{\mu r a \omega \phi_0 \, e^{i[\omega t + (\pi/2)]}}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \cdot \frac{\partial}{\partial r}\left\{\frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r}\right\}. \tag{21}$$

The procedure for the evaluation of the derivative in Equation 21 is presented in Appendix A. After substitution of the value for the derivative, Equation 21 may be written

$$\tau_{r\phi} = \frac{\mu r a \omega \phi_0 \, e^{i[\omega t + (\pi/2)]}}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \cdot \left\{\frac{r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} I_0\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right] - 2\, I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r^2}\right\}. \tag{22}$$

Equation 22 may be simplified

$$\tau_{r\phi} = \frac{\mu a \omega \phi_0 \, e^{i[\omega t + (\pi/2)]}}{r} \cdot \left\{ \frac{r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} I_0\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right] - 2 I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \right\}. \quad (23)$$

Equation 23 is an equation for the shear stress profile as a function of the tank radius, fluid properties, and tank oscillation frequency.

The shear stress at the tank wall is determined from Equation 23 evaluated at r=a.

$$\tau_{r\phi_{r=a}} = \mu \omega \phi_0 \, e^{i[\omega t + (\pi/2)]} \cdot \left\{ a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} \frac{I_0\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} - 2 \right\} \quad (24)$$

Equation 24 in this form is difficult to use due to the imaginary arguments of the modified Bessel functions. Another function will be introduced in a later section of this report.

EFFECTIVE MOMENT OF INERTIA

The effective moment of inertia of the liquid-filled oscillating cylindrical tank is defined as

$$I_{eff} = \frac{M_{r=a}}{\ddot{\phi}} \quad (25)$$

where

$M_{r=a}$ - moment due to the shear stress at the tank wall

$\ddot{\phi}$ - angular acceleration of the oscillating tank

The angular acceleration of the oscillating tank is obtained by differentiating Equation 8 twice with respect to time.

The moment due to the shear stress at the tank wall is

$$M_{r=a} = 2\pi a^2 h \, \tau_{r\phi_{r=a}} \quad . \tag{26}$$

The shear stress at the tank wall is given by Equation 24. After substitution of Equation 24 into Equation 26, the equation for the moment at the tank wall is

$$M_{r=a} = 2\pi a^2 h\mu\omega\phi_0 \left\{ a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} \cdot \frac{I_0\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} - 2 \right\} e^{i[\omega t + (\pi/2)]} \tag{27}$$

The effective moment of inertia of the liquid-filled oscillating tank is evaluated by substitution of the equations for the moment and angular acceleration in Equation 25. Making these substitutions, the equation for the effective moment of inertia is

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \left\{ 2 - a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} \frac{I_0\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \right\} \tag{28}$$

The modified Bessel functions with imaginary arguments may be written in another manner to improve the usefulness of Equation 28.

In Appendix B it is shown that the modified Bessel functions with imaginary arguments may be written in terms of Bessel functions of the first kind. Writing Equation 28 in this manner

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \left\{ 2 - a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} i^{\frac{3}{2}} \frac{J_0\left[i^{\frac{3}{2}} a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]}{J_1\left[i^{\frac{3}{2}} a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]} \right\} e^{i(\pi/2)} \quad . \tag{29}$$

The validity of Equation 29 may be proven by showing that I_{eff} is equal to the moment of inertia of a solid with the same dimensions as the coefficient of viscosity approaches infinity, i.e.,

$$\lim_{\mu \to \infty} I_{eff} = I_{solid} = \frac{\pi a^4 h \rho}{2} \quad . \tag{30}$$

Before Equation 29 can be evaluated for a fluid with an infinite coefficient of viscosity, Equation 29 must be written in an alternate form

$$I_{eff} = \frac{2\pi a^2 h \mu}{\omega} \left\{ 2 - i^{\frac{3}{2}} a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \quad \frac{J_0\left[i^{\frac{3}{2}} a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]}{J_1\left[i^{\frac{3}{2}} a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]} \right\} \cdot \ i \tag{31}$$

A direct substitution of an infinite coefficient of viscosity into Equation 31 results in an indeterminate form. An application of L'Hopital's rule still yields an indeterminate form. In Appendix C the series definition of the Bessel functions is utilized to evaluate the effective moment of inertia of an oscillating cylindrical tank filled with a fluid having an infinite coefficient of viscosity.

A more useful form of Equation 31 would be an equation that did not contain imaginary quantities and Bessel functions with imaginary arguments. The following definition of $J_n(i^{\frac{3}{2}}x)$ was obtained from Reference 2

$$J_n(i^{\frac{3}{2}} x) = i^{3n/2} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \, x^{n+4m}}{2^{n+4m} \, (2m)! \, (n+2m)!} \right.$$

$$\left. + \, i \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{n+2+4m}}{2^{n+2+4m} \, (2m+1)! \, (n+2m+1)!} \right] \quad . \tag{32}$$

For conservation of space, Equation 32 will be written

$$J_n(i^{\frac{3}{2}} x) = i^{3n/2} \left(\sum\nolimits_1 + \sum\nolimits_2 \right) \tag{33}$$

where $\sum_1$ and $\sum_2$ represent the first and second summations in Equation 32 respectively. Using deMoivre's theorem, $i^{3n/2}$ may be written

$$i^{3n/2} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{3n/2} = \cos \frac{3n\pi}{4} + i \sin \frac{3n\pi}{4} \quad . \tag{34}$$

Equation 33 may be simplified using the results of Equation 34.

$$J_n(i^{\frac{3}{2}} x) = \left(\cos \frac{3n\pi}{4} \sum\nolimits_1 - \sin \frac{3n\pi}{4} \sum\nolimits_2 \right)$$

$$+ i \left(\cos \frac{3n\pi}{4} \sum\nolimits_2 + \sin \frac{3n\pi}{4} \sum\nolimits_1 \right) \quad . \tag{35}$$

The real and imaginary parts of Equation 35 are defined as the functions $ber_n(x)$ and $bei_n(x)$ such that

$$J_n(i^{\frac{3}{2}} x) = ber_n(x) + bei_n(x) \quad . \tag{36}$$

Equation 31 may be rewritten in terms of these functions with the aid of Equation 36.

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \left\{ 2 - i^{\frac{3}{2}} a \left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \frac{ber_0 \left[a \left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \right] + i\, bei_0 \left[a \left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \right]}{ber_1 \left[a \left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \right] + i\, bei_1 \left[a \left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \right]} \right\} i \tag{37}$$

The real and imaginary parts of Equation 37 may be separated using the following relationships:

$$i^{\frac{3}{2}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \tag{38}$$

and

$$\frac{\mathrm{ber}_0(x) + i\,\mathrm{bei}_0(x)}{\mathrm{ber}_1(x) + i\,\mathrm{bei}_1(x)} = [\mathrm{ber}_0(x)\,\mathrm{ber}_1(x) + \mathrm{bei}_0(x)\,\mathrm{bei}_1(x)]\,\{[\mathrm{ber}_1(x)]^2 + [\mathrm{bei}_1(x)]^2\}^{-1}$$

$$+ i\,[\mathrm{bei}_0(x)\,\mathrm{ber}_1(x) - \mathrm{ber}_0(x)\,\mathrm{bei}_1(x)]\,\{[\mathrm{ber}_1(x)]^2 + [\mathrm{bei}_1(x)]^2\}^{-1} \tag{39}$$

where

$$x = a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} .$$

When Equations 38 and 39 are substituted in Equation 37 and the indicated multiplications are completed, the following equation for the effective moment of inertia is obtained.

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega}\left[\left(a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\left\{\frac{\cos\frac{3\pi}{4}\,[\mathrm{bei}_0(x)\,\mathrm{ber}_1(x) - \mathrm{ber}_0(x)\,\mathrm{bei}_1(x)]}{\mathrm{ber}_1{}^2(x) + \mathrm{bei}_1{}^2(x)}\right.\right.\right.$$

$$\left.\left. + \frac{\sin\frac{3\pi}{4}\,[\mathrm{ber}_0(x)\,\mathrm{ber}_1(x) + \mathrm{bei}_0(x)\,\mathrm{bei}_1(x)]}{\mathrm{ber}_1{}^2(x) + \mathrm{bei}_1{}^2(x)}\right\}\right)$$

$$+ i\left(2 + a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\left\{\frac{\sin\frac{3\pi}{4}\,[\mathrm{bei}_0(x)\,\mathrm{ber}_1(x) - \mathrm{ber}_0(x)\,\mathrm{bei}_1(x)]}{\mathrm{ber}_1{}^2(x) + \mathrm{bei}_1{}^2(x)}\right.\right.$$

$$\left.\left.\left. - \frac{\cos\frac{3\pi}{4}\,[\mathrm{ber}_0(x)\,\mathrm{ber}_1(x) + \mathrm{bei}_0(x)\,\mathrm{bei}_1(x)]}{\mathrm{ber}_1{}^2(x) + \mathrm{bei}_1{}^2(x)}\right\}\right)\right] \tag{40}$$

One final simplification in this equation may be made since

$$\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}} \quad .$$

Making this simplification:

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega}\left(\left\{ a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \left[\frac{-bei_0(x)\,ber_1(x) + ber_0(x)\,bei_1(x)}{ber_1{}^2(x) + bei_1{}^2(x)}\right.\right.\right.$$

$$\left.\left. + \frac{ber_0(x)\,ber_1(x) + bei_0(x)\,bei_1(x)}{ber_1{}^2(x) + bei_1{}^2(x)}\right]\right\}$$

$$+ i\left\{ 2 + a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \left[\frac{bei_0(x)\,ber_1(x) - ber_0(x)\,bei_1(x)}{ber_1{}^2(x) + bei_1{}^2(x)}\right.\right.$$

$$\left.\left.\left. + \frac{ber_0(x)\,ber_1(x) + bei_0(x)\,bei_1(x)}{ber_1{}^2(x) + bei_1{}^2(x)}\right]\right\}\right) \tag{41}$$

where

$$x = a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \quad .$$

This equation for the effective moment of inertia is valid for both large and small arguments of the ber and bei functions.

For most applications, the argument of the ber and bei functions will be large since the viscosity of most liquids of interest will be small. An investigation of any table of ber and bei functions (see Table 1) indicates that the following expansions may be made:

TABLE 1

TABLE OF ber AND bei FUNCTIONS FOR LARGE ARGUMENTS

x^{-1}	$x^{\frac{1}{2}} e^{-x/\sqrt{2}} M_0(x)$	$\theta_0(x) - (x/\sqrt{2})$	$x^{\frac{1}{2}} e^{-x/\sqrt{2}} M_1(x)$	$\theta_1(x) - (x/\sqrt{2})$	$<x>^*$
0.15	0.40418	-0.40758	0.38359	1.22254	7
0.14	0.40383	-0.40644	0.38457	1.21922	7
0.13	0.40349	-0.40534	0.38556	1.21598	8
0.12	0.40315	-0.40427	0.38655	1.21280	8
0.11	0.40281	-0.40323	0.38755	1.20968	9
0.10	0.40246	-0.40221	0.38856	1.20660	10
0.09	0.40211	-0.40119	0.38957	1.20356	11
0.08	0.40176	-0.40019	0.39060	1.20057	13
0.07	0.40141	-0.39921	0.39162	1.19762	14
0.06	0.40106	-0.39824	0.39266	1.19471	17
0.05	0.40071	-0.39728	0.39369	1.19184	20
0.04	0.40035	-0.39634	0.39474	1.18901	25
0.03	0.40000	-0.39541	0.39578	1.18622	33
0.02	0.39965	-0.39449	0.39683	1.18348	50
0.01	0.39930	-0.39359	0.39789	1.18077	100
0.00	0.39894 $\begin{bmatrix}(-5)1\\2\end{bmatrix}$	-0.39270 $\begin{bmatrix}(-5)1\\2\end{bmatrix}$	0.39894 $\begin{bmatrix}(-6)3\\2\end{bmatrix}$	1.17810 $\begin{bmatrix}(-5)1\\2\end{bmatrix}$	∞

* <x> - nearest integer to x.

$$
\left.\begin{aligned}
ber_0(x) &= M_0(x)\cos\theta_0(x)\\
bei_0(x) &= M_0(x)\sin\theta_0(x)\\
ber_1(x) &= M_1(x)\cos\theta_1(x)\\
bei_1(x) &= M_1(x)\sin\theta_1(x)
\end{aligned}\right\} \quad (42)
$$

where

$$
\left.\begin{aligned}
\theta_0(x) &= (x/\sqrt{2}) - A = R - A\\
\theta_1(x) &= (x/\sqrt{2}) + B = R + B\\
M_0(x) &= (C/\sqrt{x})\, e^{x/\sqrt{2}}\\
M_1(x) &= (D/\sqrt{x})\, e^{x/\sqrt{2}} \quad .
\end{aligned}\right\} \quad (43)
$$

In the expressions above, A, B, C, and D are constants listed in the tables of these functions.

The products of the terms in Equation 42 necessary for substitution in Equation 41 are:

$$
\left.\begin{aligned}
ber_0(x)\, ber_1(x) &= M_0(x)\, M_1(x)\cos(R-A)\cos(R+B)\\
bei_0(x)\, bei_1(x) &= M_0(x)\, M_1(x)\sin(R-A)\sin(R+B)\\
bei_0(x)\, ber_1(x) &= M_0(x)\, M_1(x)\sin(R-A)\cos(R+B)\\
ber_0(x)\, bei_1(x) &= M_0(x)\, M_1(x)\cos(R-A)\sin(R+B) \quad .
\end{aligned}\right\} \quad (44)
$$

The substitution of trigonometric identities and simplification of the above results in the following equations:

$$\left.\begin{aligned}
\mathrm{ber}_0(x)\,\mathrm{ber}_1(x) &= M_0(x)\,M_1(x)\big[-\tfrac{1}{2}\sin 2R\sin(B-A) \\
&\quad + \cos^2 R\cos A\cos B - \sin^2 R\sin A\sin B\big] \\
\mathrm{bei}_0(x)\,\mathrm{bei}_1(x) &= M_0(x)\,M_1(x)\big[\tfrac{1}{2}\sin 2R\sin(B-A) \\
&\quad + \sin^2 R\cos A\cos B - \cos^2 R\sin A\sin B\big] \\
\mathrm{bei}_0(x)\,\mathrm{ber}_1(x) &= M_0(x)\,M_1(x)\big[\tfrac{1}{2}\sin 2R\cos(B-A) \\
&\quad - \sin^2 R\cos A\sin B - \cos^2 R\sin A\cos B\big] \\
\mathrm{ber}_0(x)\,\mathrm{bei}_1(x) &= M_0(x)\,M_1(x)\big[\tfrac{1}{2}\sin 2R\cos(B-A) \\
&\quad + \cos^2 R\cos A\sin B + \sin^2 R\sin A\cos B\big]
\end{aligned}\right\} \tag{45}$$

The final equation for the effective moment of inertia is obtained by substitution of Equation 45 into Equation 41.

$$\begin{aligned}
I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \Bigg(& a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \frac{M_0(x)}{M_1(x)} \left[\sin(B+A) + \cos(B+A)\right] \\
& + i\left\{2 + a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \frac{M_0(x)}{M_1(x)} \left[\cos(B+A) - \sin(B+A)\right]\right\}\Bigg)
\end{aligned} \tag{46}$$

where

$$x = a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} .$$

Equation 46 is valid for any argument x if the appropriate values of the constants are taken from a table of the ber and bei functions.

A trigonometric expansion of the ber and bei functions may be made for large arguments to simplify Equation 46. Since the arguments of the ber and bei functions will be large for typical values of tank oscillation frequency and liquid parameters, an expansion of the ber and bei functions is presented in the following section.

EFFECTIVE MOMENT OF INERTIA FOR LARGE ARGUMENTS

An examination of a table of values for ber and bei functions with arguments greater than 1000 indicates that the constants in Equation 43 do not change with increasing arguments. For arguments having at least this order of magnitude, the following values of the necessary constants are taken from Table 1:

$$\left.\begin{aligned} A &= 0.39270 \\ B &= 1.17810 \\ M_0(x) &= M_1(x) \end{aligned}\right\} \quad . \tag{47}$$

The value of the trigonometric functions necessary for substitution in Equation 46 are

$$\left.\begin{aligned} \sin(B+A) &= \sin(1.57080) = \sin \pi/2 \doteq 1 \\ \cos(B+A) &= \cos(1.57080) = \cos \pi/2 = 0 \end{aligned}\right\} \tag{48}$$

The equation for the effective moment of inertia for oscillating cylindrical tanks valid for large arguments of the ber and bei functions may be obtained by substitution of Equation 48 into Equation 46.

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \left\{ a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} + i \left[2 - a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \right] \right\} \quad . \tag{49}$$

Since it was assumed that

$$a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \approx 1000$$

the following approximation for Equation 49 may be made without significant error.

$$I_{eff} = \pi a^3 h \left(\frac{2\rho\mu}{\omega}\right)^{\frac{1}{2}} \cdot [1 - i] \quad . \tag{50}$$

Equation 50 consists of both a real and imaginary part. Another manner of presentation would be a vector at a phase angle with respect to the tank displacement. Using this notation, the effective moment of inertia may be written

$$I_{eff} = 2\pi a^3 h \left(\frac{\rho\mu}{\omega}\right)^{\frac{1}{2}} e^{-i^{\pi/4}} \tag{51}$$

Equation 51 indicates that the vector for the effective moment of inertia lags the tank displacement vector by $\pi/4$.

MOMENT OF INERTIA RATIO

The available state-of-the-art literature indicates that the ratio of the effective moment of inertia to the moment of inertia of a solid with the same dimensions is a useful parameter. The moment of inertia of a solid cylinder about the longitudinal axis is

$$I_{solid} = \frac{1}{2} m a^2 \tag{52}$$

where m is the mass and a is the radius of the cylinder. Substitution of the mass in terms of density and radius into Equation 52 will result in the following equation for the effective moment of inertia of a solid cylinder.

$$I_{solid} = \frac{\pi a^4 \rho h}{2} \tag{53}$$

The effective moment of inertia ratio may be calculated by the ratio of Equations 46 and 53 or by Equation 51 and 53. These ratios are

$$\frac{I_{eff}}{I_{solid}} = \frac{4\mu}{\omega a^2\rho}\left(\left\{a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \frac{M_0(x)}{M_1(x)}\left[\sin(B+A) + \cos(B+A)\right]\right\}\right. \tag{54}$$

$$\left. + i\left\{2 + a\left(\frac{\omega\rho}{2\mu}\right)^{\frac{1}{2}} \frac{M_0(x)}{M_1(x)}\left[\cos(B+A) - \sin(B+A)\right]\right\}\right)$$

and

$$\frac{I_{eff}}{I_{solid}} = 4\left(\frac{\mu}{a^2\rho\omega}\right)^{\frac{1}{2}} e^{-i^{\pi/4}} \quad . \tag{55}$$

Equation 54 is valid for both large and small values of $(\omega/\nu)^{\frac{1}{2}}$ but Equation 55 is valid only for values

$$a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \geq 1000 \quad .$$

A plot of Equation 55 is presented in Figure 2 for

$$10^2 \leq a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \leq 10^6 \quad .$$

VELOCITY RATIO AS A FUNCTION OF RADIUS RATIO

The fluid velocity ratio in the tank is also of interest. Large velocity ratios will indicate the location where baffles are required to decrease the fluid motion. Small velocity ratios will indicate where propellant lines or other obstructions could be located to influence the fluid motion the least.

The ratio of the velocity at any radius to that at the tank wall is found by evaluating Equation 19 at these boundary conditions and taking the ratio. This ratio is

$$\frac{V_\phi}{V_{\phi_{r=a}}} = \frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{I_1\left[a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]} \tag{56}$$

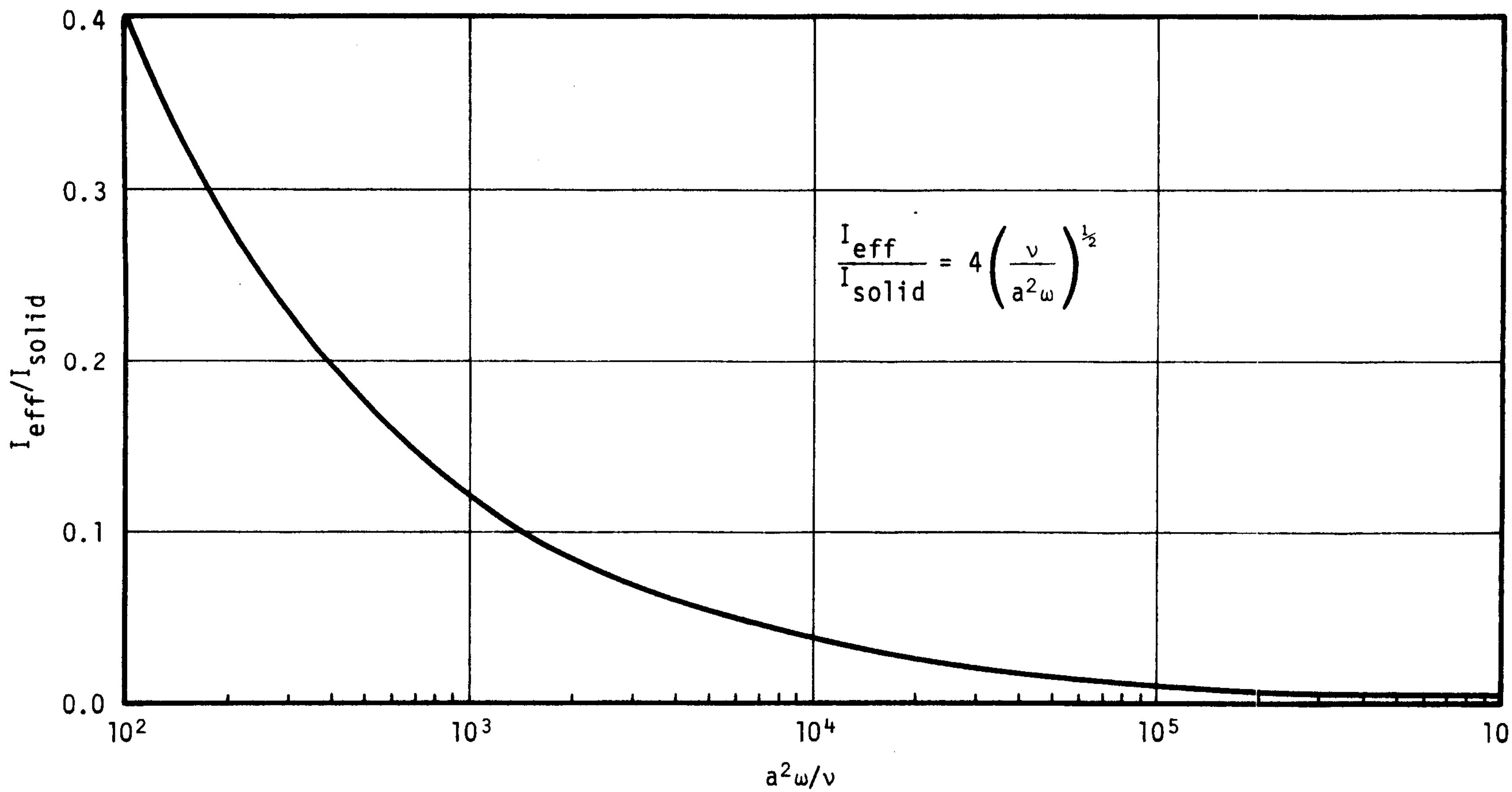

Figure 2. Effective Moment of Inertia Ratio for Large Arguments

Equation 56 may be simplified by a similar approach to the one taken for the effective moment of inertia. The detailed derivation is presented in Appendix D. The final equation for the velocity ratio is

$$\frac{V_{\phi_r}}{V_{\phi_a}} = \frac{M_1\left[\frac{r}{a}\cdot a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]}{M_1\left[a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]} \cdot \exp i\left\{\theta_1\left[\frac{r}{a}\cdot a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right] - \theta_1\left[a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]\right\} \tag{57}$$

where

$M_1(x)$ - A constant taken from Table 1 at the proper argument and used in the evaluation of the magnitude of the ber_1 and bei_1 functions

$\theta_1(x)$ - A constant taken from Table 1 at the proper argument and used in the evaluation of the phase angle of the ber_1 and bei_1 functions.

The absolute value of the velocity ratio is presented in Figure 3 as a function of the parameter $a(\omega\rho/\mu)^{\frac{1}{2}}$. Figure 4 shows the velocity ratio near the tank wall in more detail. As would be expected, the inertia forces dominate at the higher values of $a(\omega\rho/\mu)^{\frac{1}{2}}$, and the viscous forces influence the fluid motion at the lower values.

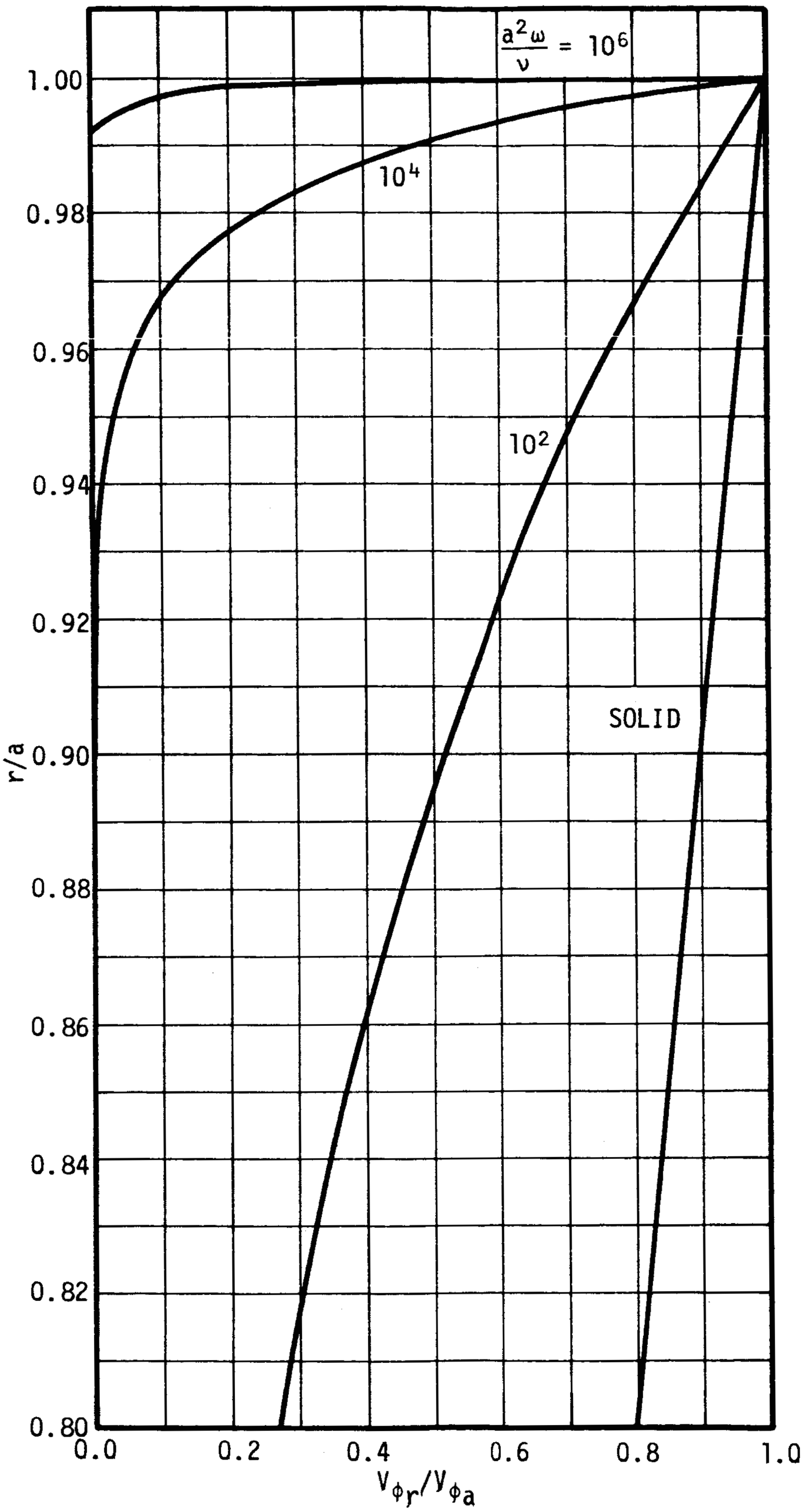

Figure 3. Velocity Ratio

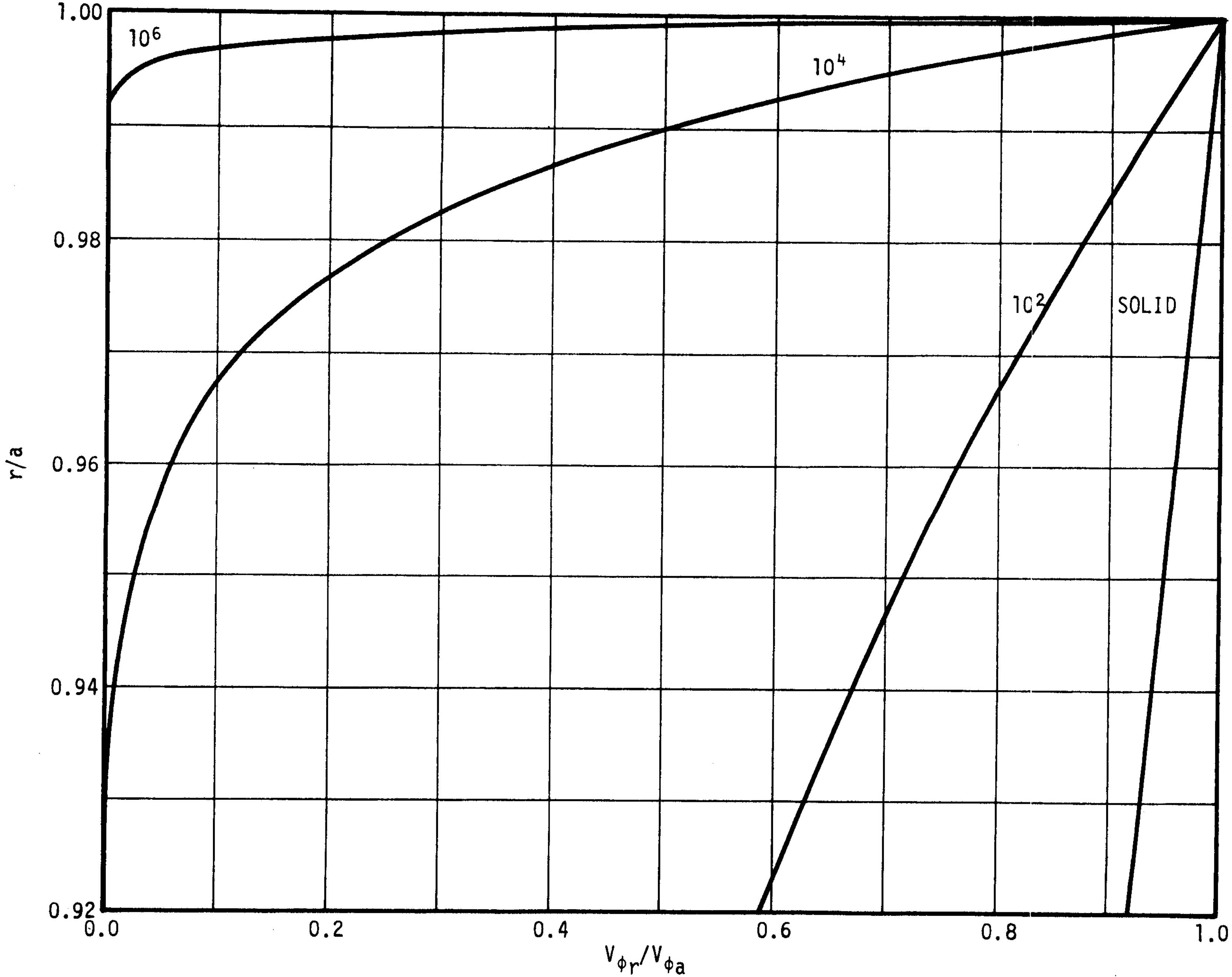

Figure 4. Fluid Velocity Ratio Near the Tank Wall

COMPARISON OF THEORY WITH EXPERIMENTAL VALUES

The theoretical analysis of the effective moment of inertia ratios discussed in this report was compared with experimental values. Only a minimum of applicable experimental information was discovered in the literature (Reference 4). A sketch of the experimental equipment that was used in Reference 4 is shown in Figure 5. The experimental procedure was to rotate the air bearing table from its equilibrium position. The torsional stress in the torsion bar exerted a restoring force to the air table. Assuming that the losses in the air bearing table and the torsion bar are small, the forced oscillations produced by this experimental system are essentially the same as the boundary condition given by Equation 8.

Experimental information was obtained for two tank sizes. The dimensions of these tanks and the experimental values are listed in Table 2. A comparison of the analytical and experimental values of the effective moment of inertia ratios is presented in Figure 6. The approximate equation for the effective moment of inertia ratio is used because

$$\left(\frac{a^2\omega}{\nu}\right) \geqq 2.0 \times 10^4 \quad .$$

The agreement at large values of $(a^2\omega/\nu)$ is good and indicates that the theory derived in this report is valid. Additional comparison with experimental values at other values of $(a^2\omega/\nu)$ outside the range

$$2.0 \times 10^4 \leqq \left(\frac{a^2\omega}{\nu}\right) \leqq 10^6 \quad .$$

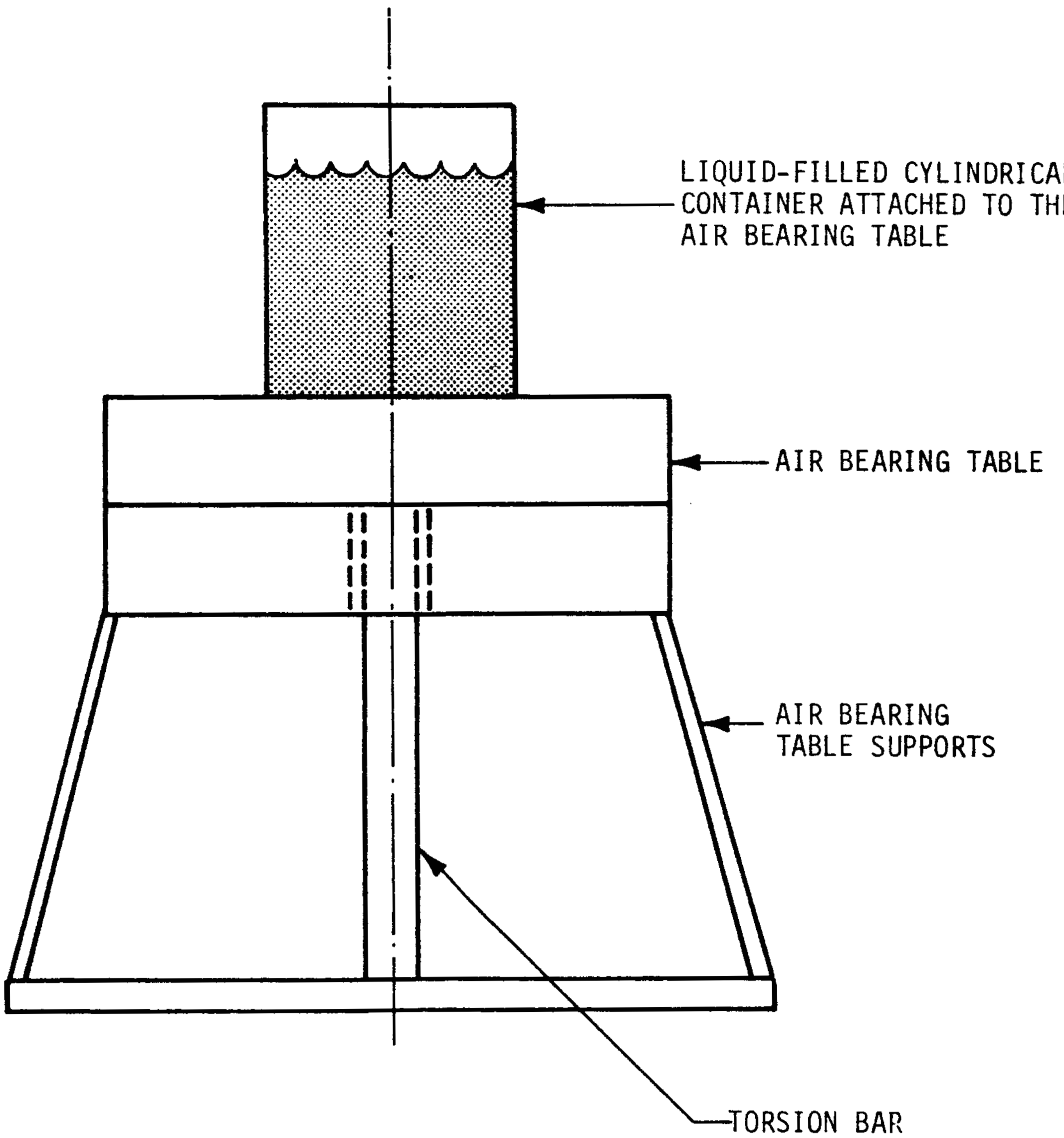

Figure 5. Sketch of Experimental Equipment

TABLE 2

PHYSICAL CHARACTERISTICS OF THE CYLINDRICAL TANKS

Tank	h (in.)	a (in.)	h/a	I_{eff} (in.-lb-sec^2)	$1/\omega$ (sec)
A	6.0	11.875	0.506	0.10	0.51098
B	12.0	11.875	1.010	0.17	0.53993
C	19.25	11.875	1.622	0.22	0.68313
D	36.0	11.875	3.03	0.52	0.81227
A_1	1.93	3.75	0.516	0.0014	0.3855
B_1	3.80	3.75	1.014	0.0020	0.3859
C_1	5.65	3.75	1.506	0.0027	0.3864
D_1	7.53	3.75	2.00	0.0034	0.3868
E_1	11.35	3.75	3.03	0.0047	0.3878

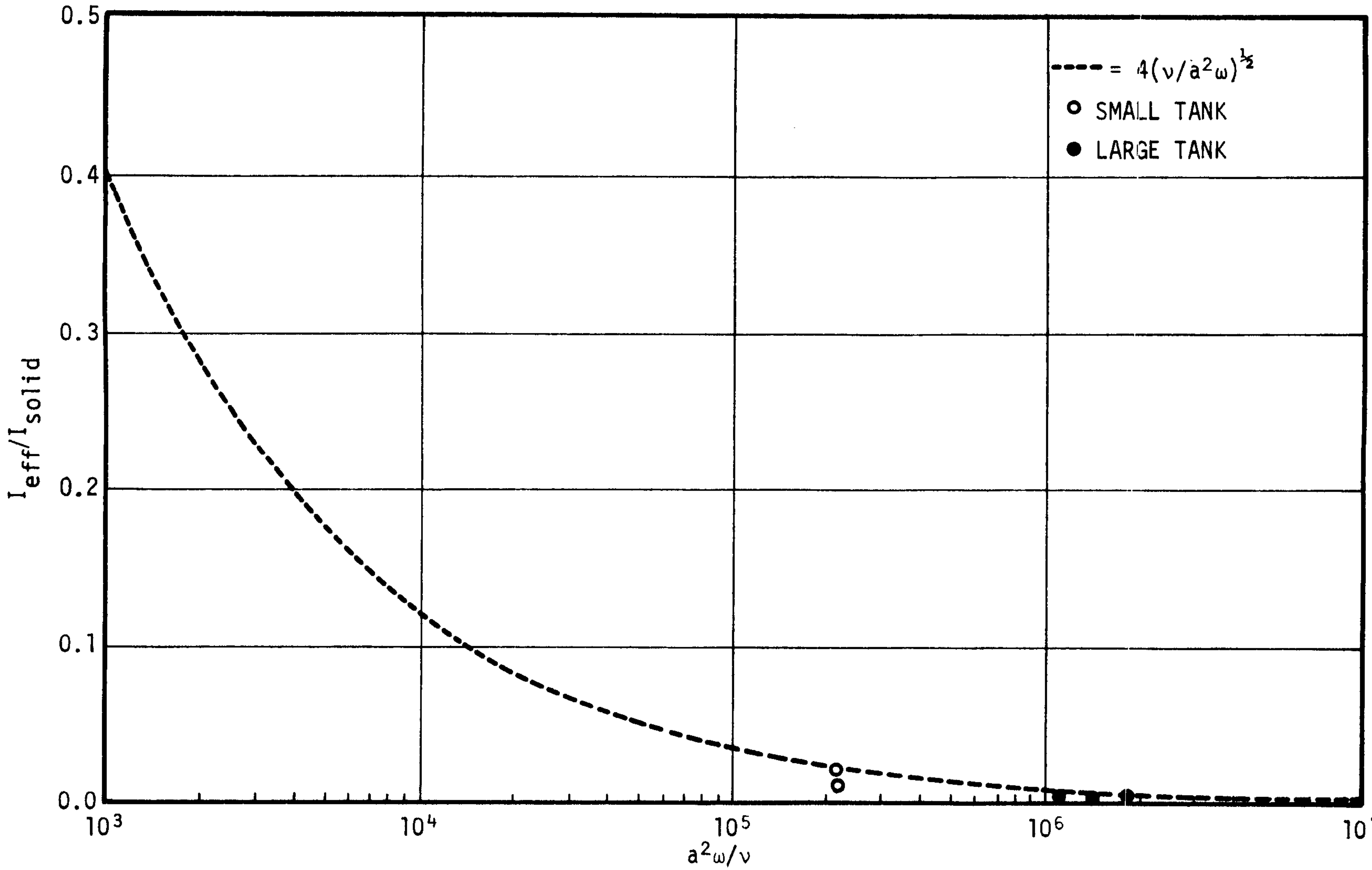

Figure 6. Comparison of Experimental and Analytical Moment of Inertia Ratios

REFERENCES

1. Schlichting, Herman, Boundary Layer Theory, McGraw-Hill Book Company, Inc., New York, 1960, 4th ed.

2. Wylie, C. R. Jr., Advanced Engineering Mathematics, McGraw-Hill Book Company, Inc., 1951

3. Abramowitz, M. and Stegun, I. A., "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55, June 1964

4. Unpublished experimental data on effective moments of inertia of liquid-filled cylindrical tanks, SPACO, Inc., Huntsville, Alabama

APPENDIX A

EVALUATION OF $\frac{\partial}{\partial r}\left\{\frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r}\right\}$

$$\frac{d}{dr}\quad\frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r} = \frac{r\,\frac{d}{dr}\left\{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]- I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right\}}{r^2} \tag{A. 1}$$

Reference 2 may be used to obtain an equation for the evaluation of the derivative of the modified Bessel function as

$$\frac{d}{dr}\left[I_n(u)\right] = \left[I_{n-1}(u) - \frac{n}{u}\,I_n(u)\right]\frac{du}{dr}\quad . \tag{A. 2}$$

After application of the results of Equation A. 2 in Equation A. 1.

$$\frac{d}{dr}\left\{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right\} = \left\{I_0\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right.$$

$$\left. - \frac{1}{r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}}\cdot I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right\}\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} \tag{A. 3}$$

Equation A. 3 may be simplified by performing the indicated multiplication.

$$\frac{d}{dr}\left\{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right\}=\left\{\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} I_0\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right] - \frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r}\right\} \tag{A. 4}$$

The final equation for the evaluation of the derivative is

$$\frac{d}{dr}\left\{\frac{I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r}\right\} = \left\{\frac{r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} I_0\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right] - 2\,I_1\left[r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]}{r^2}\right\}. \quad \text{(A. 5)}$$

APPENDIX B

SIMPLIFICATION OF $I_0\left(i^{\frac{1}{2}}w\right)$ AND $I_1\left(i^{\frac{1}{2}}w\right)$

In order to simplify Equation 28, the series definition of the modified Bessel function was written from Reference 2.

$$I_n(\beta) = \sum_{m=0}^{\infty}\left(\frac{\beta}{n}\right)^{2m+n} \cdot \frac{1}{m!\ (m+n)!} \quad . \qquad \text{(B. 1)}$$

This equation may be written in an alternate form as

$$I_n(\beta) = \left(\frac{\beta}{n}\right)^{n} \sum_{m=0}^{\infty}\left(\frac{\beta}{n}\right)^{2m} \cdot \frac{1}{m!\ (m+n)!} \quad . \qquad \text{(B. 2)}$$

A similar definition of the Bessel function of the first kind may be written as

$$J_n(\beta) = \sum_{m=0}^{\infty}(-1)^{m}\left(\frac{\beta}{2}\right)^{2m+n} \cdot \frac{1}{m!\ (m+n)!} \quad . \qquad \text{(B. 3)}$$

After substitution of the identity $i^2 = -1$, the above equation may be written as

$$J_n(\beta) = \left(\frac{\beta}{2}\right)^{n} \sum_{m=0}^{\infty}\left(\frac{\beta i}{2}\right)^{2m} \frac{1}{m!\ (m+1)!} \quad . \qquad \text{(B. 4)}$$

Writing the series for $I_0\ (i^{\frac{1}{2}}\ w)$ from Equation B. 2:

$$I_0\ (i^{\frac{1}{2}}\ w) = \sum_{m=0}^{\infty}\left(\frac{i^{\frac{1}{2}}\ w}{2}\right)^{2m} \frac{1}{m!\ m!} \qquad \text{(B. 5)}$$

$$I_0\ (i^{\frac{1}{2}}\ w) = \sum_{m=0}^{\infty}\left(\frac{w\ i}{2\ i^{\frac{1}{2}}}\right)^{2m} \frac{1}{m!\ m!} \qquad \text{(B. 6)}$$

By inspection, Equation B. 6 is equal to $J_0\ (i^{-\frac{1}{2}} w)$, i. e.

$$J_0\ (i^{-\frac{1}{2}} w) = \sum_{m=0}^{\infty} \left(\frac{w}{2\, i^{\frac{1}{2}}}\ i \right)^{2m} \frac{1}{m!\ m!} \quad . \tag{B. 7}$$

It is therefore shown that

$$I_0\ (i^{\frac{1}{2}} w) = J_0\ (i^{-\frac{1}{2}} w) \tag{B. 8}$$

Writing the series $I_1\ (i^{\frac{1}{2}} w)$ from Equation B. 2:

$$I_1\ (i^{\frac{1}{2}} w) = \left(\frac{i^{\frac{1}{2}} w}{2} \right) \sum_{m=0}^{\infty} \left(\frac{i^{\frac{1}{2}} w}{2} \right)^{2m} \frac{1}{m!\ (m+1)!} \quad . \tag{B. 9}$$

$$I_1\ (i^{\frac{1}{2}} w) = \left(\frac{i^{\frac{1}{2}} w}{2} \right) \sum_{m=0}^{\infty} \left(\frac{w}{2\, i^{\frac{1}{2}}} \cdot i \right)^{2m} \frac{1}{m!\ (m+1)!} \quad . \tag{B. 10}$$

But

$$J_1\ (i^{-\frac{1}{2}} w) = \left(\frac{w}{2\, i^{\frac{1}{2}}} \right) \sum_{m=0}^{\infty} \left(\frac{w}{2\, i^{\frac{1}{2}}} \cdot i \right)^{2m} \frac{1}{m!\ (m+1)!} \tag{B. 11}$$

By comparison of Equations B. 10 and B. 11 it is seen that

$$I_1\ (i^{\frac{1}{2}} w) = J_1 \left(\frac{w}{i^{\frac{1}{2}}} \right) \tag{B. 12}$$

The functions $J_0\ (w/i^{\frac{1}{2}})$ and $J_1\ (w/i^{\frac{1}{2}})$ are difficult to use due to the imaginary argument. This may be simplified as follows:

$$\frac{w}{i^{\frac{1}{2}}} \cdot \frac{i^{\frac{1}{2}}}{i^{\frac{1}{2}}} \cdot \frac{i}{i} = -i^{\frac{3}{2}} w \quad . \tag{B. 13}$$

The functions $J_0(-i^{\frac{3}{2}} w)$ and $J_1(-i^{\frac{3}{2}} w)$ may be expanded in Equation B.3 to show that

$$J_0(-i^{\frac{3}{2}} w) = J_0(i^{\frac{3}{2}} w) \tag{B.14}$$

$$J_1(-i^{\frac{3}{2}} w) = -J_1(i^{\frac{3}{2}} w) \quad . \tag{B.15}$$

The final expressions obtained for $I_0(i^{\frac{1}{2}} w)$ and $I_1(i^{\frac{1}{2}} w)$ are

$$I_0(i^{\frac{1}{2}} w) = J_0(i^{\frac{3}{2}} w) \tag{B.16}$$

$$I_1(i^{\frac{1}{2}} w) = -J_1(i^{\frac{3}{2}} w) \quad . \tag{B.17}$$

These relations have been substituted in Equation 29.

APPENDIX C

EVALUATION OF $\lim_{\mu \to \infty} I_{eff}$

The validity of Equation 29

$$I_{eff} = \frac{2\pi a^2 h\mu}{\omega} \left\{ 2 - i^{\frac{3}{2}} a \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \frac{J_0\left[i^{\frac{3}{2}} a \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]}{J_1\left[i^{\frac{3}{2}} a \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]} \right\} \cdot e^{i(\pi/2)} \qquad (C.1)$$

may be shown by expanding $J_0 [i^{\frac{3}{2}} a (\omega/\nu)^{\frac{1}{2}}]$ and $J_1 [i^{\frac{3}{2}} a (\omega/\nu)^{\frac{1}{2}}]$ in the series definition of the Bessel function and taking the limit of the resulting expression. For convenience, let

$$x = i^{\frac{3}{2}} a \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} = i^{\frac{3}{2}} a \left(\frac{\omega \rho}{\mu}\right)^{\frac{1}{2}} . \qquad (C.2)$$

The series definition of $J_0 (x)$ and $J_1 (x)$ are

$$J_0 = \sum_{m=0}^{\infty} \left(\frac{x}{2} \cdot i\right)^{2m} \frac{1}{m!\ m!}$$

$$= \left[1 + \left(\frac{x}{2}\right)^2 \frac{(-1)}{1!\ 1!} + \left(\frac{x}{2}\right)^4 \frac{1}{2!\ 2!} + \ldots\right] \qquad (C.3)$$

and

$$J_1 (x) = \left(\frac{x}{2}\right) \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{2m} \cdot \frac{i^{2m}}{m!\ (m+1)!}$$

$$= \left(\frac{x}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2 \frac{(-1)}{2!} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2!\ 3!} + \ldots\right] \qquad (C.4)$$

For convenience, let

$$i^{\frac{3}{2}} a \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \quad \frac{J_0\left[i^{\frac{3}{2}}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]}{J_1\left[i^{\frac{3}{2}}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right]} = \frac{x\,J_0\,(x)}{J\,(x)}$$

$$= \frac{x\left[1+\left(\frac{x}{2}\right)^2\frac{(-1)}{1!\;1!}+\left(\frac{x}{2}\right)^4\frac{1}{2!\;2!}+\ldots\right]}{\left(\frac{x}{2}\right)\left[1+\left(\frac{x}{2}\right)^2\frac{(-1)}{2!}+\left(\frac{x}{2}\right)^4\frac{1}{2!\;3!}+\ldots\right]} \tag{C. 5}$$

Taking the limit of a large coefficient of viscosity, the value x approaches zero. The series given by Equation C. 3 and C. 4 may be approximated by the first few terms because x is small. Making this approximation,

$$\frac{x\,J_0\,(x)}{J_1\,(x)} \approx \frac{2\left[1+\left(\frac{x}{2}\right)^2\right]}{\left[1+\left(\frac{x}{2}\right)^2\frac{(-1)}{1!\;2!}\right]} \tag{C. 6}$$

Performing the indicated division and retaining terms of the second order

$$\frac{x\,J_0\,(x)}{J_1\,(x)} = 2\left[1-\frac{1}{2}\left(\frac{x}{2}\right)^2\right]$$

$$= 2-\left(\frac{x}{2}\right)^2 \tag{C. 7}$$

Substituting the results of Equation C. 7 into Equation C. 1:

$$\lim_{\mu\to\infty} I_{eff} = \frac{2\pi a^2 h\mu}{\omega}\left\{2-2+\left[\frac{i^{\frac{3}{2}}a}{2}\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]^2\right\}\cdot i$$

$$= \frac{2\pi a^2 h\mu}{\omega}\left[\frac{i^3 a^2}{4} \cdot \frac{\omega\rho}{\mu}\right] \cdot i$$

$$= \frac{\pi a^4 h \rho}{2} \qquad (C.8)$$

The moment of inertia of a cylinder about its axis is

$$I_{\substack{solid \\ cyl.}} = \frac{1}{2} m a^2$$

$$= \frac{1}{2}(\pi a^2 h \rho) a^2 = \frac{1}{2}\pi a^4 h \rho \qquad (C.9)$$

A comparison of Equations C. i and C. 9 indicate that

$$\lim_{\mu\to\infty} I_{eff} = I_{\substack{solid \\ cyl.}} = \frac{1}{2}\pi a^4 h \rho$$

The solution obtained in this report for the effective moment of inertia of a liquid-filled cylinder about its longitudinal axis is valid in the limit of a very viscous liquid. The value of the effective moment of inertia for a cylinder filled with a liquid having a small coefficient of viscosity is presented in the main text.

$$a' = a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}} \quad .$$

The validity of this equation is shown by the fact that it satisfies the boundary conditions at the tank wall and center of the tank. These boundary conditions are

$$\text{at } r = 0, \quad \frac{V_{\phi_r}}{V_{\phi_a}} = 0$$

and

$$\text{at } r = a, \quad \frac{V_{\phi_r}}{V_{\phi_r}} = 1 \quad .$$

The ber_1 and bei_1 functions expanded in terms of a constant and a trigonometric function are

$$\left.\begin{aligned} ber_1(a') &= M_1(a')\cos\theta_1(a') \\ bei_1(a') &= M_1(a')\sin\theta_1(a') \\ ber_1(r') &= M_1(r')\cos\theta_1(r') \\ bei_1(r') &= M_1(r')\sin\theta_1(r') \quad . \end{aligned}\right\} \quad (D.4)$$

The relations above when substituted into Equation D. 3 yield

$$\frac{V_{\phi_r}}{V_{\phi_a}} = \frac{M_1(r')}{M_1(a')}\Big\{[\cos\theta_1(r')\cos\theta_1(a') + \sin\theta_1(r')\sin\theta_1(a')]$$

$$+ i[\sin\theta_1(r')\cos\theta_1(a') - \cos\theta_1(r')\sin\theta_1(a')]\Big\} \quad (D.5)$$

The properties of the trigonometric identities may be used to reduce the above equation to a vector notation.

$$\frac{V\phi_r}{V\phi_a} = \frac{M_1(r')}{M_1(a')} \exp i\,[\theta_1(r') - \theta_1(a')] \tag{D. 6}$$

The final equation for the velocity ratio used in the text is obtained by substituting the values of r' and a' in Equation D. 6:

$$\frac{V\phi_r}{V\phi_a} = \frac{M_1\left[\frac{r}{a} \cdot a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]}{M_1\left[a\left(\frac{\omega\rho}{\mu}\right)^{\frac{1}{2}}\right]} \cdot \exp i\{\theta_1[r/a \cdot a\,(\omega\,\rho/\mu)^{\frac{1}{2}}] - \theta_1[a\,(\omega\rho/\mu)^{\frac{1}{2}}]\} \tag{D. 7}$$

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13. ABSTRACT

The Navier-Stokes equations are solved exactly for the motion of the liquid in a liquid-filled cylindrical tank with oscillation about the longitudinal axis. The moment due to the viscous shear at the tank wall is calculated. Equations for the effective moment of inertia of the liquid are presented for both a very large coefficient of viscosity and a normal coefficient of viscosity. An equation for the velocity ratio is derived and velocity ratio profies are presented. A comparison of the analytical and experimental effective moment of inertia ratios shows good agreement.

Author

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