A Discussion of the paper "Asymptotic Stability of Sampled Nonlinear Systems with Variable Sampling Periods -- A Bionic Application" by P. Vidal and F. Laurent

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NGR-40-002-015

Messrs. Vidal and Laurent develop conditions for the stability of a certain class of nonlinear sampled systems and apply them to several examples. These same conditions can be easily developed with somewhat more generality and rigor through the use of a more recent form of Liapunov's second method. The purpose of this discussion is to present the relevant theorem and its corollary and to illustrate how Vidal's and Laurent's criteria can be developed using Liapunov functions.

Given a continuous map $f: E^m \to E^m$ with $f(0) = 0$, the problem is to determine criteria for, and a domain of, stability for the origin of the difference equation (1).

$$x(n+1) = f(x(n)) , \quad x(0) = x_o \tag{1}$$

A set $M \subset E^m$ is called an invariant set for (1) if for each x_o in M there is a solution $x(n)$ of (1) such that $x(n)$ is in M for all $n = 0, \pm 1, \pm 2, \ldots$.

<u>THEOREM:</u> Let S be a closed, bounded set in E^m. If there exists a continuous function $V(x)$ defined on S such that $\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all x in S, then all solutions $x(n)$ of (1) which remain in S for all $n \geq 0$ approach the largest invariant set M contained in the set $S_o = \{x \in S: \Delta V(x) = 0\}$ as $n \to \infty$.

This theorem is the difference analog to LaSalle's theorem for differential equations [1]. The advantage of this theorem over the usual Liapunov theorems is that it can be used to define a region of attraction for M. Usually, the set S is constructed so that all the solutions which start in S remain in S, making S a domain of stability.

COROLLARY: In addition to the conditions of the theorem, let the set S have the form $S = \{x: V(x) \leq c\}$ for some $c > 0$. Then all solutions which start in S approach M as $n \to \infty$.

For some m-vector x, consider a norm $\|x\|$ and define the norm of a matrix L, $\|L\|$, by

$$\|L\| = \max_{\|x\|=1} \|Lx\| .$$

Notice that $\|Lx\| \leq \|L\| \, \|x\|$ for all m-vectors x. See [2] for a discussion of various vector and matrix norms.

Messrs. Vidal and Laurent consider difference equations of the type given in equation (2)

$$x(n+1) = L(x(n))x(n) \tag{2}$$

where $L(x)$ is a matrix valued function defined on E^m. Consider the functional $V(x) = \|x\|$, which is positive definite on all of E^m. Then $\Delta V(x) = \|L(x)x\| - \|x\| \leq (\|L(x)\|-1)\|x\| = (\|L(x)\|-1)V(x)$. Define the sets $S_1(\delta) = \{x: \|L(x)\| \leq 1-\delta\}$ and $S_2(c) = \{x: V(x) \leq c\}$. Define the set $S(\delta)$

as $S_2(c)$ where $c > 0$ is chosen such that $S_2(c) \subset S_1(\delta)$. Then all the conditions of the corollary are satisfied, implying stability. Furthermore, for $\delta > 0$, the only possible set M is the origin since $V(x)n)) = \|x(n)\|$ approaches zero exponentially as $n \to \infty$.

$$\begin{aligned} V(x(n)) &= V(x(n-1)) + \Delta V(x(n-1)) \\ &\leq (1-\delta)V(x(n-1)) \leq (1-\delta)^n V(x(0)) . \end{aligned}$$

In this case the origin is exponentially asymptotically stable with a domain of stability $S(\delta)$. Notice that if $L(x)$ is continuous in x and $\|L(x)\| < 1-\delta$, then $S(\delta)$ is non-empty. There is also the possibility of finding a larger domain of stability by using $V^k(x)$ for some $k > 1$ in place of $V(x)$ in determining $S(\delta)$.

Let the vector x have elements x_i and the matrix L have elements L_{ij}. Then, if $\|x\| = \max\limits_{1\leq i\leq m} |x_i|$, the norm of L is given by $\|L\| = \max\limits_{1\leq i\leq m} \sum_{j=1}^{m} |L_{ij}|$. The condition that $\|L(x)\| \leq 1-\delta$ is the first criterion arrived at by Messrs. Vidal and Laurent. Another choice for the norm, namely $\|x\| = \sum_{j=1}^{m} |x_j|$, results in the second criterion. Each choice of a vector norm would result in a criterion for, and a domain of, stability. Any such choice gives sufficient conditions for stability.

For the nonautonomous case, where $L = L(x,n)$, the above results hold provided that $\delta > 0$. Thus, inequality (6) of the paper gives a domain of stability for the problem considered. This is also true for the inequalities developed in each of the examples.

REFERENCES

[1] LaSalle, J. P., An invariance principle in the theory of stability, Center for Dynamical Systems, Brown University, Technical Report 66-1, April, 1966.

[2] Faddeeva, V. N., Computational Methods of Linear Algebra, Dover, New York, 1959, 54-60.