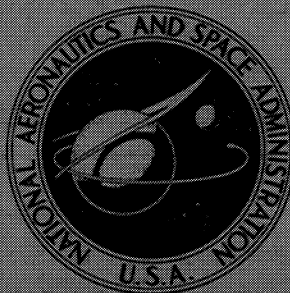


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THE INFLUENCE OF THE  
BOUNDARY CONDITIONS ON THE  
BUCKLING LOAD OF CYLINDRICAL  
SHELLS UNDER AXIAL COMPRESSION

by Shigeo Kobayashi

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# DEFINITION OF SYMBOLS

$A_j (j = 1, 2)$	$q \pm i \sqrt{1 - q^2}$
$C_j (j = 1 \dots 8)$	arbitrary real constants
$c_1, c_2, c_3$	spring constants
$D$	$Et^3/12(1 - \nu^2)$
$D_{S1}, \text{ etc.}$	determinants for buckling load equations, suffix shows the boundary conditions
$E$	Young's modulus
$F$	stress function
$f_n$	dimensionless stress function
$k_1, k_2, k_3$	dimensionless spring constants
$M_x$	bending moment
$N_o$	axial compression load per unit length
$N_x, N_y, N_{xy}$	additional in-plane stress resultants induced by buckling
$n$	number of waves in circumferential direction
$q$	$N_o R \sqrt{3(1 - \nu^2)} / Et^2 = N_o / t \sigma_{cl}$
$R$	radius of cylindrical shell
$t$	shell thickness
$u, v, w$	displacements in axial, circumferential and radial directions
$u_n, v_n, w_n$	dimensionless displacements
$x, y$	axial and circumferential coordinates



$\alpha_1, \alpha_2$	constants defined by equation (14)
$\beta$	$\sqrt{\frac{R}{t}} \sqrt[2]{\frac{n}{12(1 - \nu^2)}}$
$\xi$	$x \sqrt{\sqrt{12(1 - \nu^2)}/Rt}$
$\xi_j (j = 1, 2)$	$w_n \pm A_j f_n$
$\kappa$	$\sqrt{\frac{R}{t}} \sqrt{12(1 - \nu^2)}$
$\lambda_{jk}$	roots of characteristic equation
$\mu_1, \mu_2$	constants defined by equation (14)
$\nu$	Poisson's ratio
$\phi$	$\cos^{-1} q$
$\sigma_{cl}$	$Et/R \sqrt{3(1 - \nu^2)}$
$\rho$	constant defined by equation (12)

THE INFLUENCE OF THE BOUNDARY CONDITIONS  
ON THE BUCKLING LOAD OF CYLINDRICAL  
SHELLS UNDER AXIAL COMPRESSION

By Shigeo Kobayashi<sup>\*</sup>

SUMMARY

A new approach to the study of the effect of boundary conditions on thin walled axial cylinders loaded with uniform axial compression leads to a graphical representation that establishes limits on the buckling stress without detailed calculation. The effect of elastic boundary conditions is also studied.

*Author*

INTRODUCTION

The problem of the buckling of a cylindrical shell under uniform axial compression still contains elements which are not completely understood. Several recent papers have discussed the effect of the prebuckling deformation and the details of the boundary conditions on the buckling load. Ohira (Refs. 1 and 2) has shown that the shell will buckle at values of one-half the classical buckling load ( $P_{cl}$ ) for certain boundary conditions. This result was obtained by studying the local buckling of a semi-infinite cylinder using the linear theory. Nachbar and Hoff (Ref. 3) found that, in the case of a free boundary, the buckling load is  $0.38 P_{cl}$ . Stein (Ref. 4) determined the buckling load taking into account the prebuckling deformation. Stein's analysis used a finite difference approach

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and showed that for the boundary condition S-3 the buckling load was approximately  $0.42 P_{cl}$ . Ohira's result for the boundary conditions S-3 suggests that this  $0.58 P_{cl}$  reduction is due to the  $0.50 P_{cl}$  reduction for the S-3 boundary condition and an additional  $0.08 P_{cl}$  reduction due to the prebuckling deformation. Fischer (Ref. 5) and Almroth (Ref. 6) have also obtained buckling loads for various boundary conditions taking into account the effect of the prebuckling deformations. When these results are compared with Ohira's, the effect of prebuckling deformation can be determined for other boundary conditions.

The present work deals with the effect of elastically supported boundary conditions on the buckling load. The analysis uses the linear theory and neglects the effect of the prebuckling deformation.

## DIFFERENTIAL EQUATIONS AND GENERAL SOLUTION

If the effect of the prebuckling deformation is neglected, the differential equations which govern the behavior during buckling of an axially loaded cylindrical shell are expressed as:

$$D \nabla^4 w + \frac{1}{R} \frac{d^2 F}{dx^2} + N_0 \frac{d^2 w}{dx^2} = 0$$

$$\nabla^4 F = \frac{Et}{R} \frac{d^2 w}{dx^2} \quad (1)$$

where  $N_0$  is the axial load per unit length and

$$\nabla^4 = \frac{d^4}{dx^4} + 2 \frac{d^4}{dx^2 dy^2} + \frac{d^4}{dy^4}$$

The shell geometry and coordinate system are shown in Figure 1.

These equations are the ones given by Donnell (Ref. 7) and use the following definition of the stress function  $F$ :

$$N_x = \frac{\partial^2 F}{\partial y^2} \quad N_y = \frac{\partial^2 F}{\partial x^2} \quad N_{xy} = - \frac{\partial^2 F}{\partial x \partial y} \quad (2)$$

In order to make the equations complete the following relations, also given by Donnell, are necessary

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{Et} \left( \frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) & \frac{\partial v}{\partial y} + \frac{w}{R} &= \frac{1}{Et} \left( \frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \right) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= - \frac{2(1+\nu)}{Et} \frac{\partial^2 F}{\partial x \partial y} \end{aligned} \quad (3)$$

The buckling deformations  $u$ ,  $v$ ,  $w$ , and the stress function  $F$  are expressed in a Fourier series as follows:

$$w = \frac{t}{\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} w_n(x) \sin \frac{ny}{R} \quad (4.1)$$

$$u = \frac{t}{\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} u_n(x) \sin \frac{ny}{R} \quad (4.2)$$

$$v = \frac{t}{\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} v_n(x) \cos \frac{ny}{R} \quad (4.3)$$

$$F = \frac{Et^3}{12(1-\nu^2)} \sum_{n=2}^{\infty} f_n(x) \sin \frac{ny}{R} \quad (4.4)$$

As is well known, separate eigenvalue problems are derived for each  $n$ . Substituting equation (4) into equation (1), and changing the variable from  $x$  to  $\xi$  where

$$\xi = \sqrt{\frac{\sqrt{12(1-\nu^2)}}{Rt}} x$$

the following equations result

$$\left(\frac{d^2}{d\xi^2} - \beta^2\right)\left(\frac{d^2}{d\xi^2} - \beta^2\right)w_n + \frac{d^2 f_n}{d\xi^2} + 2g \frac{d^2 w_n}{d\xi^2} = 0 \quad (5.1)$$

$$\left(\frac{d^2}{d\xi^2} - \beta^2\right)\left(\frac{d^2}{d\xi^2} - \beta^2\right)f_n = \frac{d^2 w_n}{d\xi^2} \quad (5.2)$$

where

$$\beta = \frac{n}{k} \quad \kappa = \sqrt{\frac{R}{t} \sqrt{12(1-\nu^2)}} \quad g = \frac{N_0/t}{\sigma_{cl}} \quad (5.3)$$

and  $\sigma_{cl} = \frac{Et}{R \sqrt{3(1-\nu^2)}}$  (5.4)

is the classical buckling stress.

Equations (5.1) and (5.2) are solved as follows. Adding (5.1) to  $A \times (5.2)$  (where  $A$  is an undetermined constant) leads to:

$$\left(\frac{d^2}{d\xi^2} - \beta^2\right)\left(\frac{d^2}{d\xi^2} - \beta^2\right)(w_n + A f_n) + \frac{d^2}{d\xi^2} [(2g - A)w_n + f_n] = 0 \quad (6)$$

Then  $A$  is selected as  $2g - A = \frac{1}{A}$

In the case  $q < 1$ , (the region of interest in this report) the roots of the above equation are

$$A_{1,2} = g \pm i\sqrt{1-g^2} \equiv e^{\pm i\phi} \quad (7)$$

Using the definition

$$w_n + A_j f_n = \xi_j \quad j = 1, 2 \quad (8)$$

equation (6) can be written as follows:

$$\left(\frac{d^2}{d\xi^2} - \beta^2\right)\left(\frac{d^2}{d\xi^2} - \beta^2\right)\xi_j + \frac{1}{A_j} \frac{d^2 \xi_j}{d\xi^2} = 0 \quad j = 1, 2 \quad (9)$$

Expressing  $\xi_j$  as  $\xi_j = e^{\lambda_j \xi}$  the characteristic equations of (9) are

$$\begin{aligned} \lambda_1^4 + (-2\beta^2 + e^{-i\phi})\lambda_1^2 + \beta^4 &= 0 \\ \lambda_2^4 + (-2\beta^2 + e^{i\phi})\lambda_2^2 + \beta^4 &= 0 \end{aligned} \quad (10)$$

The roots of these equations are as follows:

$$\begin{aligned} \lambda_{1m} &= \frac{i}{2} \left[ \pm e^{-\frac{i\phi}{2}} \pm \sqrt{e^{-i\phi} - 4\beta^2} \right] \\ \lambda_{2m} &= \frac{i}{2} \left[ \pm e^{\frac{i\phi}{2}} \pm \sqrt{e^{i\phi} - 4\beta^2} \right] \end{aligned} \quad m = 1, 2, 3, 4 \quad (11)$$

If the following relation is introduced

$$e^{\mp i\phi} - 4\beta^2 = \rho e^{\mp i\psi} \quad (12.1)$$

or written in another manner

$$\rho \sin \psi = \sin \phi \quad ; \quad \rho \cos \psi = \cos \phi - 4\beta^2 \quad (12.2)$$

the eight roots can be easily written as

$$\begin{aligned}
 \lambda_{11} &= \mu_1 + i\alpha_1 & \lambda_{21} &= -\mu_1 + i\alpha_1 \\
 \lambda_{12} &= -\mu_2 + i\alpha_2 & \lambda_{22} &= \mu_2 + i\alpha_2 \\
 \lambda_{13} &= -\lambda_{12} & \lambda_{23} &= -\lambda_{22} \\
 \lambda_{14} &= -\lambda_{11} & \lambda_{24} &= -\lambda_{21}
 \end{aligned} \tag{13}$$

where

$$\begin{aligned}
 \mu_1 &= \frac{\sqrt{\rho}}{2} \sin \frac{\psi}{2} + \frac{1}{2} \sin \frac{\phi}{2} \\
 \mu_2 &= \frac{\sqrt{\rho}}{2} \sin \frac{\psi}{2} - \frac{1}{2} \sin \frac{\phi}{2} \\
 \alpha_1 &= \frac{1}{2} \cos \frac{\phi}{2} + \frac{\sqrt{\rho}}{2} \cos \frac{\psi}{2} \\
 \alpha_2 &= \frac{1}{2} \cos \frac{\phi}{2} - \frac{\sqrt{\rho}}{2} \cos \frac{\psi}{2}
 \end{aligned} \tag{14}$$

The relation between  $\rho$ ,  $\psi$ , and  $\phi$  and the parameters of the problem  $q$  and  $4\beta^2$  is illustrated graphically in Figure 2. As will be shown, since the values of  $\rho$ ,  $\phi$ ,  $\psi$ , and  $\beta$  depend on the boundary conditions, this figure represents all possible values of  $q$ , or  $N_0 t / \sigma_{cl}$ .

For the range of  $q$  given by  $0 < q < 1$ , the ranges of  $\phi$  and  $\psi$  are

$$0 < \phi < \frac{\pi}{2} \qquad \phi < \psi < \pi \tag{15.1}$$

Therefore

$$\begin{aligned}
 \sin \frac{\phi}{2} &> 0 & \cos \frac{\phi}{2} &> 0 \\
 \sin \frac{\psi}{2} &> 0 & \cos \frac{\psi}{2} &> 0
 \end{aligned} \tag{15.2}$$

It therefore follows that

$$\mu_1 > 0 \quad \alpha_1 > 0 \quad (16.1)$$

In order to show that  $\mu_2 > 0$ ,  $\alpha_2 > 0$  the following calculation is performed

$$a = \left( \sqrt{\rho} \sin \frac{\psi}{2} \right)^2 - \left( \sin \frac{\phi}{2} \right)^2 = \frac{1}{2} [\rho - \rho \cos \psi - 1 + \cos \phi] = \frac{1}{2} (\rho + 4\beta^2 - 1)$$

$$b = \left( \cos \frac{\phi}{2} \right)^2 - \left( \sqrt{\rho} \cos \frac{\psi}{2} \right)^2 = \frac{1}{2} [1 + \cos \phi - \rho - \rho \cos \psi] = \frac{1}{2} (1 + 4\beta^2 - \rho)$$

Using equation (15) and noting from Figure 2 that

$$\rho + 4\beta^2 > 1 \quad 1 + 4\beta^2 > \rho$$

it is concluded that  $a > 0$ ,  $b > 0$ , therefore

$$\mu_2 > 0 \quad \alpha_2 > 0 \quad (16.2)$$

Substituting the general solutions of  $\xi_1$  and  $\xi_2$  into the expressions for  $w_n$  and  $f_n$ , which can be derived from equation (8),

$$w_n = \frac{\xi_2 A_1 - \xi_1 A_2}{A_1 - A_2} \quad f_n = \frac{\xi_1 - \xi_2}{A_1 - A_2}$$

The following equations are obtained:

$$\begin{aligned} w_n = & C_1 e^{-\mu_1 \xi} \cos(\alpha_1 \xi + \phi) + C_2 e^{-\mu_1 \xi} \sin(\alpha_1 \xi + \phi) + C_3 e^{-\mu_2 \xi} \cos(\alpha_2 \xi - \phi) \\ & + C_4 e^{-\mu_2 \xi} \sin(\alpha_2 \xi - \phi) + C_5 e^{\mu_1 \xi} \cos(\alpha_1 \xi - \phi) + C_6 e^{\mu_1 \xi} \sin(\alpha_1 \xi - \phi) \\ & + C_7 e^{\mu_2 \xi} \cos(\alpha_2 \xi + \phi) + C_8 e^{\mu_2 \xi} \sin(\alpha_2 \xi + \phi) \end{aligned} \quad (17)$$

$$\begin{aligned} f_n = & - \left[ C_1 e^{-\mu_1 \xi} \cos \alpha_1 \xi + C_2 e^{-\mu_1 \xi} \sin \alpha_1 \xi + C_3 e^{-\mu_2 \xi} \cos \alpha_2 \xi + C_4 e^{-\mu_2 \xi} \sin \alpha_2 \xi \right. \\ & \left. + C_5 e^{\mu_1 \xi} \cos \alpha_1 \xi + C_6 e^{\mu_1 \xi} \sin \alpha_1 \xi + C_7 e^{\mu_2 \xi} \cos \alpha_2 \xi + C_8 e^{\mu_2 \xi} \sin \alpha_2 \xi \right] \end{aligned}$$

where  $C_i$  ( $i = 1 \dots 8$ ) are arbitrary real constants.



## BUCKLING LOAD EQUATION

The equation for the buckling load, or eigenvalue equation, is now obtained by applying the boundary conditions to the general solution given by equation (17). Taking the coordinate system at one end of the shell as shown in Figure 1, and considering only local buckling in the region close to the boundary, the constants  $C_5, C_6, C_7, C_8$  can be taken equal to zero under the condition that  $w_n$  and  $f_n$  are finite as  $\xi \rightarrow \infty$ . Therefore, only four boundary conditions are needed at the end  $\xi = 0$ .

The expressions for  $u_n$  and  $v_n$  can be obtained from equation (3) and  $N_x$  and  $N_{xy}$  from equation (2).

$$\begin{aligned}
 n u_n &= \frac{1}{\beta} \frac{d^3 f_n}{d\xi^3} - (2+\nu) \beta \frac{df_n}{d\xi} - \frac{1}{\beta} \frac{dw_n}{d\xi} \\
 n v_n &= -\frac{d^2 f_n}{d\xi^2} - \nu \beta^2 f_n + w_n \\
 N_x &= -\frac{Et^2}{R\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} \beta^2 f_n \sin \frac{n\eta}{R} \\
 N_{xy} &= -\frac{Et^2}{R\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} \beta \frac{df_n}{d\xi} \cos \frac{n\eta}{R}
 \end{aligned} \tag{18}$$

Using the relation between the moment  $M_x$  and the deflection  $w$

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = -\frac{Et^3}{12(1-\nu^2)R} \sum_{n=2}^{\infty} \left( \frac{d^2 w_n}{d\xi^2} - \nu \beta^2 w_n \right) \sin \frac{n\eta}{R} \quad (19)$$

all of the boundary conditions can now be expressed as functions of  $w_n$  and  $f_n$ .

The eigenvalue equation has been obtained for eight sets of boundary conditions. The procedure will be illustrated for the following set of boundary conditions:

$$\begin{aligned} w = 0 \quad \frac{\partial w}{\partial x} = 0 \quad N_x = 0 \quad N_{xy} = 0 \\ \text{at } x = 0 \end{aligned} \quad (20)$$

Using equations (17), (18) and (20) the following set of linear homogeneous equations is obtained for the four constants  $C_1, C_2, C_3$ , and  $C_4$ :

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos \phi & \sin \phi & \cos \phi & -\sin \phi \\ \mu_1 & -\alpha_1 & \mu_2 & -\alpha_2 \\ (-\mu_1 \cos \phi - \alpha_1 \sin \phi) & (-\mu_1 \sin \phi + \alpha_1 \cos \phi) & (-\mu_2 \cos \phi + \alpha_2 \sin \phi) & (-\mu_2 \sin \phi + \alpha_2 \cos \phi) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0$$

In order that a nontrivial solution of these equations exists the determinant of the coefficients must vanish. This condition gives the following eigenvalue equation:

$$\sin^2 \phi = 0$$

In a similar manner the other seven sets of boundary conditions considered lead to an eigenvalue equation. The boundary conditions and the eigenvalue equations are given below.

S-1\*

$$W=0 \quad M_x=0 \quad u=0 \quad v=0$$

$$-\frac{d^3 f_n}{d\xi^3} + (2+\nu)\beta^2 \frac{df_n}{d\xi} + \frac{dW_n}{d\xi} = 0$$

$$W_n = 0$$

$$\frac{d^2 f_n}{d\xi^2} + \nu\beta^2 f_n = 0$$

$$\frac{d^2 W_n}{d\xi^2} = 0$$

(21)

$$D_{S1} = \frac{\sqrt{p}\beta^2}{4} \sin\phi \cos\frac{\psi}{2} \left[ (\rho + 4\beta^2) \left\{ (1+\rho^2) + \frac{\nu}{2}(1+\rho)^2 + (1-\nu^2)\rho \right\} + (1-\nu)^2 \right]$$

S-2

$$W=0 \quad M_x=0 \quad N_x=0 \quad v=0$$

$$-f_n = 0$$

$$W_n = 0$$

$$\frac{d^2 f_n}{d\xi^2} = 0$$

$$\frac{d^2 W_n}{d\xi^2} = 0$$

(22)

$$D_{S2} = \rho (\sin\phi)^2$$

S-3

$$W=0 \quad M_x=0 \quad u=0 \quad N_{xy}=0$$

$$-\frac{d^3 f_n}{d\xi^3} + \frac{dW_n}{d\xi} = 0$$

$$W_n = 0$$

$$-\frac{df_n}{d\xi} = 0$$

$$\frac{d^2 W_n}{d\xi^2} = 0$$

(23)

$$D_{S3} = -\rho\beta^2 \cos\left(\phi - \frac{\psi}{2}\right) \cos\left(\phi + \frac{\psi}{2}\right)$$

\*

This designation of the boundary conditions is the same as that of Ohira (Ref. 2) and Almroth (Ref. 6), but is different from that of Sobel (Ref. 8).

S-4

$$W=0 \quad M_x=0 \quad N_x=0 \quad N_{xy}=0$$

$$f_n=0$$

$$w_n=0$$

$$\frac{df_n}{d\xi}=0$$

$$\frac{d^2 w_n}{d\xi^2}=0$$

$$D_{S4} = -\sqrt{\rho} \sin \phi \cos \left( \phi + \frac{\psi}{2} \right)$$

(24)

C-1

$$W=0 \quad \frac{\partial W}{\partial x}=0 \quad u=0 \quad v=0$$

$$-\frac{d^3 f_n}{d\xi^3} + (2+\nu)\beta^2 \frac{df_n}{d\xi} = 0$$

$$w_n=0$$

$$\frac{d^2 f_n}{d\xi^2} + \nu\beta^2 f_n = 0$$

$$\frac{dw_n}{d\xi}=0$$

(25)

$$D_{C1} = 2\beta^2 \left( \cos \frac{\psi}{2} \right)^2 + \frac{2\beta^4}{\rho} (\sin \phi)^2 (3+2\nu-\nu^2)$$

C-2

$$W=0 \quad \frac{\partial W}{\partial x}=0 \quad N_x=0 \quad v=0$$

$$f_n=0$$

$$w_n=0$$

$$\frac{d^2 f_n}{d\xi^2}=0$$

$$\frac{dw_n}{d\xi}=0$$

(26)

$$D_{C2} = \sqrt{\rho} \sin \phi \cos \left( \phi - \frac{\psi}{2} \right)$$

$$\begin{aligned}
\text{C-3} \quad w=0 \quad \frac{dw}{dx}=0 \quad u=0 \quad N_{xy}=0 \\
\frac{d^3 f_n}{d\xi^3}=0 \\
w_n=0 \\
\frac{df_n}{d\xi}=0 \\
\frac{dw_n}{d\xi}=0
\end{aligned} \tag{27}$$

$$D_{C3} = \sqrt{\rho} \beta^2 \cos(\phi - \frac{\psi}{2})$$

$$\begin{aligned}
\text{C-4} \quad w=0 \quad \frac{dw}{dx}=0 \quad N_x=0 \quad N_{xy}=0 \\
f_n=0 \\
w_n=0 \\
\frac{df_n}{d\xi}=0 \\
\frac{dw_n}{d\xi}=0
\end{aligned} \tag{28}$$

$$D_{C4} = (\sin \phi)^2$$

Since  $\phi$  and  $\psi/2$  have the following ranges

$$0 < \phi < \frac{\pi}{2} \quad \phi/2 < \psi/2 < \frac{\pi}{2} \tag{29}$$

then

$$\sin \phi > 0 \quad \cos \frac{\psi}{2} > 0 \tag{30}$$

In addition

$$\cos(\phi - \frac{\psi}{2}) > 0 \tag{31}$$

since

$$\frac{\pi}{4} > (\phi - \frac{\psi}{2}) > -\frac{\pi}{2} \tag{32}$$

With this information, it can easily be seen that the eigenvalue equations for the boundary conditions S-1, S-2, C-1, C-2, C-3, C-4 do not have any solution with an eigenvalue lower than the classical buckling load  $q=1$ .

In order to examine the boundary conditions S-3, S-4, the value of  $\cos(\phi + \frac{\psi}{2})$  must be determined. Since  $\pi > (\phi + \frac{\psi}{2}) > \frac{3}{2}\phi$ , for the region of  $\phi > \frac{\pi}{3}$  (i. e.  $q < \frac{1}{2}$ ),  $\pi > (\phi + \frac{\psi}{2}) > \frac{\pi}{2}$ , which implies that  $\cos(\phi + \frac{\psi}{2}) < 0$ . However, in the region  $\phi < \frac{\pi}{3}$ , it is possible for the value of  $\cos(\phi + \frac{\psi}{2})$  to be greater than zero, depending on the value of  $4\beta^2$ . The smaller the value of  $4\beta^2$ , the nearer to  $q = 1/2$  this transition occurs. The value of  $4\beta^2$  is not equal to zero, but becomes quite small for the smallest value of  $n$  which is equal to 2. This is true since  $4\beta^2 = \frac{2n^2}{\sqrt{3(1-\nu^2)}}(\frac{t}{R})$  and  $\frac{t}{R}$  is small for thin shells. Therefore, it is easily concluded that the cases S-3 and S-4 have a buckling load lower than  $q = 1$ . The value of  $q_{cr}$  is very close to  $1/2$ , corresponding to  $n = 2$ . These conclusions are the same as those of Ohira (Ref. 2). However, the present expressions of the eigenvalue determinant directly leads to this conclusion without any numerical calculation.

### ELASTICALLY SUPPORTED BOUNDARY

The following uncoupled spring type linear elastic support at the boundary  $x = 0$  is now considered:

$$M_x = -c_1 \frac{\partial w}{\partial x} \quad (33.1)$$

$$N_x = c_2 u \quad (33.2)$$

$$N_{xy} = c_3 v \quad (33.3)$$

where  $c_1$ ,  $c_2$  and  $c_3$  are the spring constants for the rotation, axial and circumferential constraints. It is assumed that the radial restraint is rigid, giving the condition

$$W_n = 0 \quad (34.1)$$

Substituting equations (4), (18), (19) into equations (33) the following expressions for the elastically supported edges are obtained.

$$\frac{d^2 W_n}{d\xi^2} - k_1 \frac{dW_n}{d\xi} = 0 \quad (34.2)$$

$$-f_n - k_2 \left[ \frac{d^3 f_n}{d\xi^3} - (2+\nu)\beta^2 \frac{df_n}{d\xi} - \frac{dW_n}{d\xi} \right] = 0 \quad (34.3)$$

$$-\frac{df_n}{d\xi} + k_3 \left[ \frac{d^2 f_n}{d\xi^2} + \nu\beta^2 f_n \right] = 0 \quad (34.4)$$

where

$$\begin{aligned} k_1 &= \frac{c_1 \kappa \sqrt{12(1-\nu^2)}}{Et^2} \\ k_2 &= \frac{c_2 R}{Et\beta^4 \kappa} \\ k_3 &= \frac{c_3 R}{Et\beta^2 \kappa} \end{aligned} \quad (35)$$

are the dimensionless spring constants. Substituting equations (17) into equations (34) a set of linear homogeneous equations for the four unknown constants  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  is obtained. The determinant of the coefficients of these equations is as follows:

$$D_E = \begin{vmatrix} 1 + k_2[\mu_1(3\alpha_1^2 - \mu_1^2) + (-\mu_1 \cos \phi - \alpha_1 \sin \phi) + (2+\nu)\beta^2 \mu_1] & 0 + k_2[\alpha_1(3\mu_1^2 - \alpha_1^2) + (-\mu_1 \sin \phi + \alpha_1 \cos \phi) - (2+\nu)\beta^2 \alpha_1] & 1 + k_2[\mu_2(3\alpha_2^2 - \mu_2^2) + (-\mu_2 \cos \phi + \alpha_2 \sin \phi) + (2+\nu)\beta^2 \mu_2] & 0 + k_2[\alpha_2(3\mu_2^2 - \alpha_2^2) + (\mu_2 \sin \phi + \alpha_2 \cos \phi) - (2+\nu)\beta^2 \alpha_2] \\ \cos \phi & \sin \phi & \cos \phi & -\sin \phi \\ -\mu_1 + k_3[-(\mu_1^2 - \alpha_1^2) - \nu\beta^2] & \alpha_1 + k_3(2\mu_1\alpha_1) & -\mu_2 + k_3[-(\mu_2^2 - \alpha_2^2) - \nu\beta^2] & \alpha_2 + k_3(2\mu_2\alpha_2) \\ (\mu_1^2 - \alpha_1^2)\cos \phi + 2\mu_1\alpha_1\sin \phi + k_1[\mu_1\cos \phi + \alpha_1\sin \phi] & (\mu_1^2 - \alpha_1^2)\sin \phi - 2\mu_1\alpha_1\cos \phi + k_1[\mu_1\sin \phi - \alpha_1\cos \phi] & (\mu_2^2 - \alpha_2^2)\cos \phi - 2\mu_2\alpha_2\sin \phi + k_1[\mu_2\cos \phi - \alpha_2\sin \phi] & -(\mu_2^2 - \alpha_2^2)\sin \phi - 2\mu_2\alpha_2\cos \phi + k_1[-\mu_2\sin \phi - \alpha_2\cos \phi] \end{vmatrix}$$



Evaluation of this determinant gives the following result.

$$D_E = [D_{S4} + k_1 D_{C4} + k_3 D_{S2} + k_1 k_3 D_{C2}] \\ + k_2 [D_{S3} + k_1 D_{C3} + k_3 D_{S1} + k_1 k_3 D_{C1}] \quad (36)$$

If the values of the spring constants  $k_1$ ,  $k_2$  and  $k_3$  are given, the eigenvalue  $q$  can be determined from setting equation (36) equal to zero.

The results of the previous section have shown that the lowest eigenvalue ( $0.5 P_{cl}$ ) occurs for the boundary conditions  $N_{xy} = 0$ , and  $M_x = 0$ . The effect of the axial boundary condition is not important since either condition  $N_x = 0$  or  $u = 0$  gives the same eigenvalue. In order to show the dependence of  $q_{cr}$  on the values of the  $k_1$ ,  $k_3$  three cases will be treated. For all three cases the condition  $N_x = 0$  is taken for simplicity. The relation of these three cases is illustrated in Figure 3.

Case A,  $k_2 = k_1 = 0$

For this case the determinant, equation (36), simplifies to

$$D_{S4} + k_3 D_{S2} = 0 \quad (37)$$

Using equations (22) and (24) this becomes

$$k_3 = \frac{\cos(\phi + \frac{\psi}{2})\sqrt{\sin \psi}}{\sin \phi \sqrt{\sin \phi}} \quad (38)$$

For a given  $k_3$  the smallest value of the eigenvalue  $q$  is found by varying the value of the circumferential wave number  $4\beta^2$ . This calculation, therefore, determines the largest value of  $\phi$  for a given  $k_3$ . The calculation can also be performed by determining the largest value of  $k_3$  for a given value of  $\phi$ , or  $q$ , by changing the value of  $4\beta^2$ . Therefore, the maximum value of  $k_3$  with respect to  $\psi$  is found by the following condition,

$$\frac{dk_3}{d\psi} = 0$$

which gives the result

$$\cos\left(\phi + \frac{3\psi}{2}\right) = 0 \quad (39)$$

The conditions on the ranges of  $\phi$  and  $\psi$ , equation (15.1) limits the range of  $\phi + \frac{3\psi}{2}$  to the following.

$$0 < \left(\phi + \frac{3\psi}{2}\right) < \frac{3\pi}{2}$$

Therefore, the maximum  $k_3$  is given by  $\phi + \frac{3\psi}{2} = \frac{\pi}{2}$ . In addition, there is the condition that  $\psi > \phi$ . This shows that the maximum value of  $k_3$  is given by the following values of  $\psi$ .

$$\psi = \phi \quad 60^\circ \geq \phi \geq 36^\circ \quad (40.1)$$

$$\psi = \frac{\pi}{3} - \frac{2\phi}{3} \quad 36^\circ \geq \phi > 0 \quad (40.2)$$

Using equations (38) and (40) the relation between  $k_3$  and  $\phi$ , or  $q$ , is found to be

$$60^\circ \geq \phi \geq 36^\circ \quad k_3 = \frac{\cos \frac{3}{2}\phi}{\sin \phi} \quad 4\beta^2 \approx 0 \quad (41.1)$$

$$36^\circ \geq \phi > 0 \quad k_3 = \left[ \frac{\cos(\frac{2}{3}\phi + \frac{\pi}{6})}{\sin \phi} \right]^{\frac{3}{2}} \quad (41.2)$$

$$4\beta^2 = \frac{\sin(\frac{\pi}{3} - \frac{5}{3}\phi)}{\sin(\frac{\pi}{3} - \frac{2}{3}\phi)}$$

These results are shown in Figure 4. The circumferential wave number  $n$  has its lowest value ( $n = 2$ ) in the range  $k_3 \leq 1$ , but increases for  $k_3 > 1$ .

Case B,  $k_2 = k_3 = 0$

In this case the buckling load equation, equation (36), becomes

$$D_{s4} + k_1 D_{c4} = 0 \quad (42)$$

Using equations (24) and (28) the following relation is obtained for  $k_1$ .

$$k_1 = \frac{\cos(\phi + \frac{\psi}{2})}{\sqrt{\sin \phi} \sqrt{\sin \psi}} \quad (43)$$

Again looking for the largest value of  $k_1$  for a fixed  $\phi$

$$\frac{dk_1}{d\psi} = 0$$

the extreme conditions is given by

$$\cos\left(\phi - \frac{\psi}{2}\right) = 0 \quad (44)$$

However, this condition can not be satisfied due to the restrictions on  $\phi - \frac{\psi}{2}$  given by equation (32). Examination of equation (43) shows that for the range of  $\psi$  that is permitted, the maximum value of  $k_1$  occurs for  $\psi = \phi$ . This gives us the final equation for this case.

$$k_1 = \frac{\cos \frac{3}{2}\phi}{\sin \phi} \quad 4\beta^2 \approx 0 \quad (45)$$

The result of this calculation is shown in Figure 4. The part corresponding to  $k_1 < 1$ , is the same as Case A for  $k_3 < 1$ .

Case C,  $k_3 = 0$

In this case only the axial constraint is zero and the buckling load equation, equation (36), becomes,

$$D_E = D_{S4} + k_1 D_{C4} + k_3 D_{S2} + k_1 k_3 D_{C2} \quad (46)$$

Using equations (22), (24), (26), and (28) this buckling load equation can be written as follows:

$$D_E = \sin \phi \left[ \sin \phi \left\{ 2\sqrt{\rho} \sin \frac{\psi}{2} + k_1 + \rho k_3 \right\} + \left\{ k_1 k_3 - 1 \right\} \sqrt{\rho} \cos \left( \phi - \frac{\psi}{2} \right) \right] \quad (47)$$

This equation leads to the simple conclusion that if

$$k_1 k_3 - 1 > 0 \quad (48)$$

there exist no eigenvalues lower than the value  $q = 1$ , which is the classical buckling load. Therefore, if an experimental set up has end fixtures that are rigid enough such that  $k_1 \cdot k_3 > 1$ , this analysis shows that there is no decrease in the buckling load due to the elastically supported boundaries. The result of equation (48) is illustrated in Figure 5.

## CONCLUSION

The buckling load equations for a semi-infinite cylindrical shell have been obtained for eight sets of boundary conditions using the linear Donnell equations. These expressions directly lead to the same conclusions as that of Ohira (Ref. 2) without any numerical computation. It is found that when the constraint of the boundary in the circumferential direction is released for the simply supported cases (S-3 and S-4) the buckling load drops to approximately  $1/2$  the classical buckling load. The minimum buckling load occurs for the smallest number of circumferential waves ( $n = 2$ ). However, the accuracy of the Donnell equations is somewhat in doubt for small number of circumferential waves. Nevertheless, it can easily be shown from the buckling load equations for these two cases that even for  $n = 5$  and  $R/t \geq 100$  the buckling load is always less than  $0.60 P_{cl}$ . Therefore, the drop in the buckling load due to the lack of circumferential constraint as predicted by this analysis is real.

In addition, the effect of elastically supported boundaries has been calculated. An uncoupled spring type support has been assumed for the circumferential, axial and rotation constraints. The radial constraint was assumed to be rigid. The buckling load has been calculated for three different cases. It was found that the effect of the rotation constraint and the circumferential constraints is about the same if the other constraints are zero. In addition, if both moment and circumferential constraint are acting with the axial constraint zero, there are no buckling loads lower than the classical buckling load if the product of the two non-dimensional spring constants is greater than 1.

## REFERENCES

1. Ohira, H.: Local Buckling Theory of Axially Compressed Cylinders. Proc. 11th Japan National Congress for Applied Mechanics, 1961, pp. 37-41.
2. Ohira, H.: Linear Local Buckling Theory of Axially Compressed Cylinders and Various Eigenvalues. Proc. 5th International Symposium on Space Technology and Science, Tokyo, 1963, pp. 511-526.
3. Nachbar, W. and Hoff, N. J.: On Edge Buckling of Axially Compressed Circular Cylindrical Shells, Quart. Appl. Math., Vol. 20, No. 3, 1962, pp. 269-298.
4. Stein, M.: The Influence of Prebuckling Deformations and Stresses on the Buckling of Perfect Cylinders. NASA TR R-190, February 1964.
5. Fischer, G.: Influence of Boundary Conditions on Stability of Thin-Walled Cylindrical Shells under Axial Load and Internal Pressure. AIAA Journal, Vol. 3, No. 4, April 1965.
6. Almroth, B. O.: Influence of Edge Conditions on the Stability of Axially Compressed Cylindrical Shells. NASA CR-161, 1965.
7. Donnell, L. H.: Stability of Thin-Walled Tubes under Torsion. NACA Report 479, 1933.
8. Sobel, L. H.: Effect of Boundary Conditions on the Stability of Cylinders Subjected to Lateral and Axial Pressures. AIAA Journal, Vol. 2, No. 8, 1964, pp. 1437-1440.

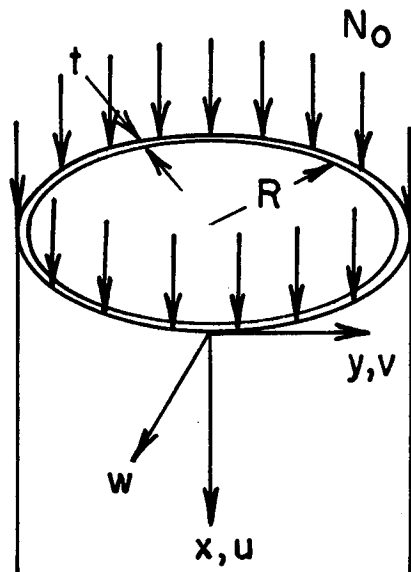


Fig. 1 Shell Geometry and Coordinate System

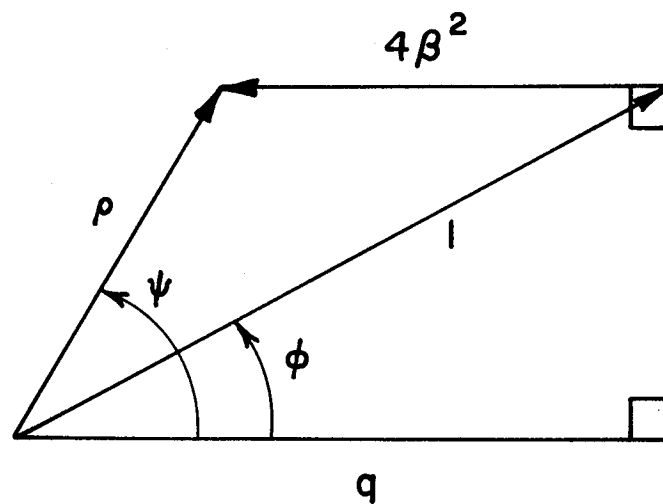


Fig. 2 Geometric Relation between  $q, \phi, 4\beta^2, \rho$  and  $\psi$

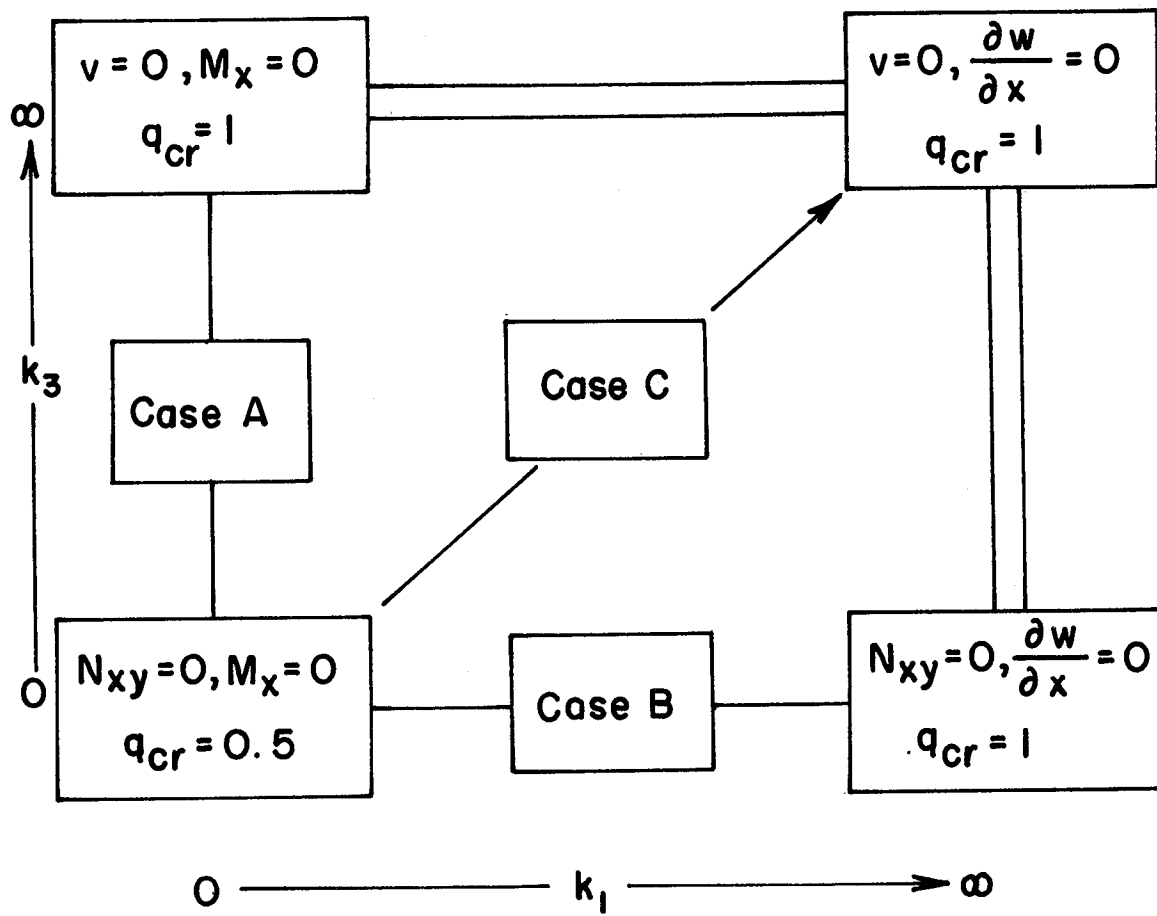


Fig. 3 Relationship of Cases A, B and C



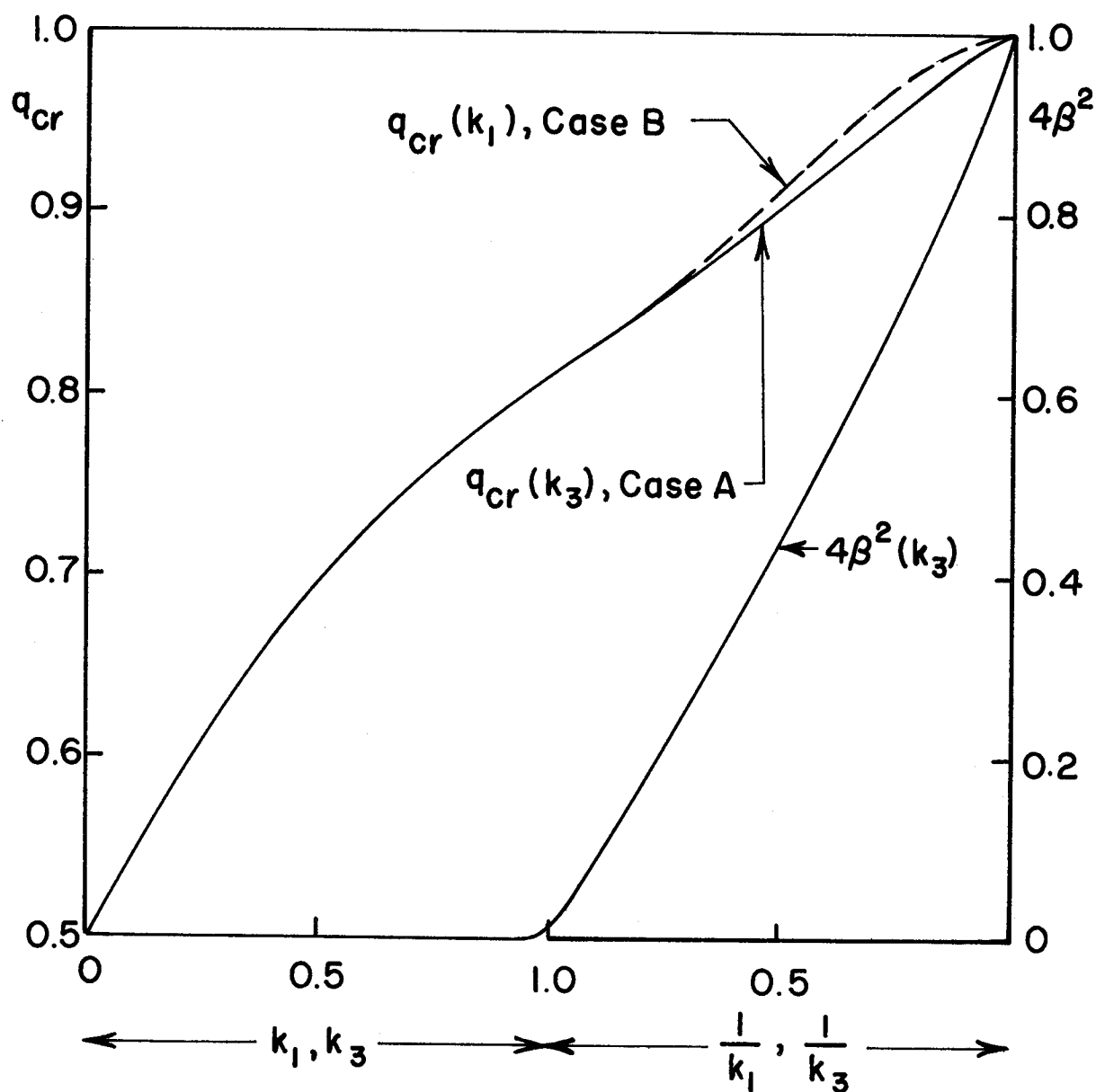


Fig. 4  $q_{cr}$  and  $4\beta^2$  For Cases A and B

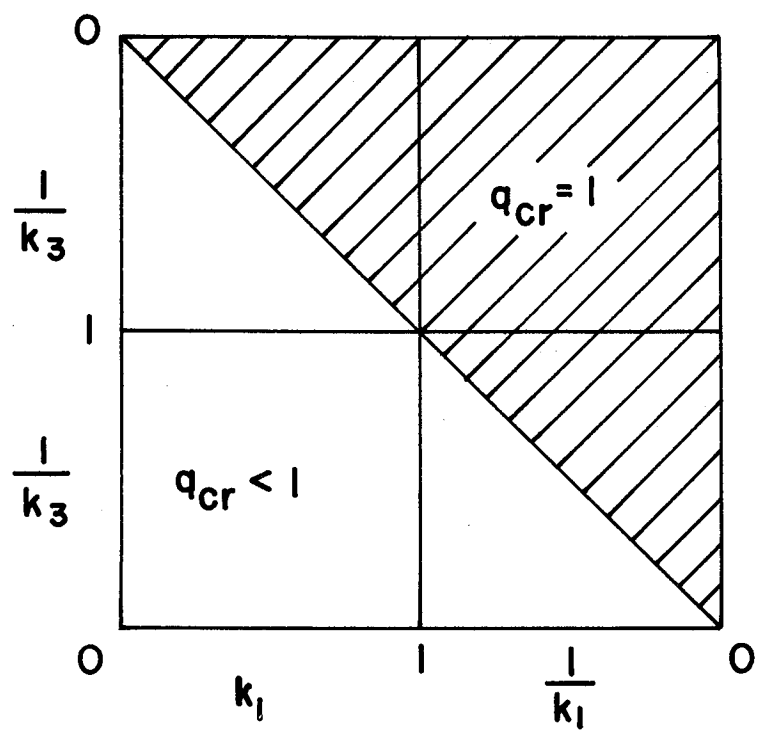


Fig. 5  $q_{cr}$  For Case C

