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EXPLICIT RUNGE-KUTTA INTEGRATION

By LYLE R. DICKEY
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

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Lyle R. Dickey

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

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A fourth order Runge-Kutta method is applied to a system of linear differential equations. The solution over one time step is obtained explicitly as a function of initial conditions and the forcing function evaluated at the initial time, the final time, and the midpoint. Successive applications of this result can be employed to obtain an explicit solution for a number of time points which can later be used to rapidly determine solutions for a large number of different sets of initial conditions and forcing functions.

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TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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EXPLICIT RUNGE-KUTTA INTEGRATION

SUMMARY

A fourth order Runge-Kutta method is applied to a system of linear differential equations. The solution over one time step is obtained explicitly as a function of initial conditions and the forcing function evaluated at the initial time, the final time, and the midpoint. Successive applications of this result can be employed to obtain an explicit solution for a number of time points which can later be used to rapidly determine solutions for a large number of different sets of initial conditions and forcing functions.

EXPLICIT RUNGE-KUTTA

Quite often it is necessary to solve the same system of differential equations for a large number of different initial conditions and forcing functions. In the event that the differential equations are linear, considerable computer time can be saved by taking advantage of the fact that the Runge-Kutta method of integration is a linear operation. With a little additional effort, the operation can be carried out explicitly resulting in a solution which is a linear combination of values of the forcing function evaluated at discrete points and the initial conditions. Having obtained the solution in this form, it is a simple matter for a computer to rapidly evaluate the solution for an extremely large number of different conditions. If this can be accomplished over one time step, it is only a matter of bookkeeping to repeat the operation over a number of time steps. With this in mind, the following system of linear differential equations will be solved explicitly over one such time step by fourth order Runge-Kutta:

$$\dot{X} = F(X, t) = A(t)\,X + H(t). \qquad (1)$$

The solution at a point t_1 can be obtained from the initial conditions at t_0 and the forcing function, $H(t)$, evaluated at t_0, t_1 and the midpoint $\bar{t}$ by the following method:

$$X_1 = X_o + \frac{1}{6}(K_o + 2K_1 + 2K_2 + K_3),$$

where

$$
\begin{aligned}
K_o &= hF(X_o,\ t_o) \\
K_1 &= hF\left(X_o + \frac{K_o}{2},\ \bar{t}\right) \\
K_2 &= hF\left(X_o + \frac{K_1}{2},\ \bar{t}\right) \\
K_3 &= hF(X_o + K_2,\ t_1) \\
X_o &= X(t_o) \\
X_1 &= X(t_1) \\
h &= t_1 - t_o\ .
\end{aligned}
\qquad (2)
$$

Since the vector equation (1) is linear and the operations described in equations (2) are also linear, the solution is of the following form:

$$X_1 = UX_o + C_oH_o + \bar{C}\bar{H} + C_1H_1, \qquad (3)$$

where

$$
\begin{aligned}
H_o &= H(t_o) \\
\bar{H} &= H(\bar{t}) \\
H_1 &= H(t_1).
\end{aligned}
$$

Successive substitution of equation (1) into the expressions described in equations (2) yield the following equations for U, C_o, $\bar{C}$, and C_1.

$$U = I + \frac{h}{6}(A_o + 4\bar{A} + A_1) + \frac{h^2}{6}(\bar{A}\,A_o + \bar{A}^2 + A_1\,\bar{A})$$

$$+ \frac{h^3}{12}(\bar{A}^2\,A_o + A_1\,\bar{A}^2) + \frac{h^4}{24}(A_1\,\bar{A}^2\,A_o),$$

$$C_o = \frac{h}{6} I + \frac{h^2}{6}\bar{A} + \frac{h^3}{12}\bar{A}^2 + \frac{h^4}{24} A_1\,\bar{A}^2$$

$$\bar{C} = \frac{2h}{3} I + \frac{h^2}{6}(\bar{A} + A_1) + \frac{h^3}{12} A_1\,\bar{A}$$

$$C_1 = \frac{h}{6} I,$$

where

$$A_o = A(t_o)$$

$$\bar{A} = A(\bar{t})$$

$$A_1 = A(t_1).$$

If the function H(t) is in tabular form and available at time increments which differ from h, some sort of interpolation method is required. However, since most of the useful interpolation formulas are linear with respect to the dependent variable, H_o, $\bar{H}$ and H_1 can be expressed as a linear combination of the values of H(t) at the points for which they are tabulated. Although the bookkeeping may become somewhat tedious when the solution is desired at a large number of time points, the use of matrix notation and digital computers, both of which are well designed for such operations, makes this an extremely small point for consideration.

The explicit method outlined above is extremely useful when a large number of different initial conditions and forcing functions are to be used. In particular, if the statistics of these inputs are available, the mean and standard deviation of the output can be evaluated in a straightforward manner and achieve results that would require an extremely large amount of computer time if the differential equations were to be solved over and over again for each individual case.

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