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THE GRAVITY POTENTIAL AND FORCE FIELD OF THE EARTH THROUGH FOURTH ORDER

by C. A. Wagner

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Greenbelt, Md.

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ABSTRACT

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In most references, for the great convenience in calculations, the exterior gravity potential of the spheroidal earth is presented as a series of spherical harmonic terms. In an early part of this paper that series is derived in full detail as a classical solution of Laplace's equation in spherical coordinates. In a later part of the report the series is truncated at the $1/r^3$ terms as is often done in the literature. It is shown that this truncation gives the potential for a rotating triaxial earth, having small oblateness and equatorial eccentricity, with an equipotential surface. The coefficients of the truncated series are interpreted as functions of the geometric parameters of the matching triaxial earth model and its rotation. The harmonic coefficients, developed according to the theory presented in this paper and measured to date from gravity surveys and satellite data, show that the earth has:

1. an oblateness of 1/298.2
2. an equatorial eccentricity of 1.14×10^{-5}, which corresponds to a difference of major and minor equatorial diameters for the earth of about 475 feet.

In addition to the harmonics contributing principally to the triaxial earth, the higher order harmonics are interpreted in the latter part of the paper as reflecting additional small longitude and latitude-dependent surface deviations from this triaxial earth. The effect on the earth surface model (the geoid) of terms through J_{44} is calculated, but the longitude-dependent deviations themselves are left in general terms as functions of the harmonic coefficients. These latter coefficients must be regarded as largely unknown except for the second tesseral harmonic J_{22} which (together with J_{20} and the centrifugal potential) fixes the surface shape of the triaxial geoid. Longitude independent (zonal) geoid deviations through J_{40} are calculated from a zonal potential due to Kozai (1962), which is probably the most accurate to date.

While the details of the topography of the geoid remain to be discovered by geodetic research, certain large-scale features appear to be discernable already. These are discussed in a later section of this report. Except perhaps for J_{22} determined from 24 hour satellite data, it appears necessary at this date to consider all longitude gravity effects as a set through at least the fourth order in the long term applications.

Only the second order longitude gravity effect probably needs to be considered in the applications concerning the 24-hour satellite, though the effects of third- and fourth-order earth gravity may be discernable over long periods of time.

In the last section of this report the gravity and geometric effects of the small nutations of the north pole are discussed. The gravity effects are shown to be entirely negligible for most applications. The geometric effects might well be discernable by close long-term study of the geoid coordinate errors of stations used as references in the observation of earth satellites.

CONTENTS

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I. INTRODUCTION

The basic purpose of this report is to gather in one document information on the earth's external gravity field necessary for the accurate determination of spacecraft trajectories in its vicinity. Much of the material on the mathematical representation of the field is not new. In most references, for convenience of calculation and because the earth itself is so nearly spherically symmetric, the general gravity field has been developed into an infinite series of spherical harmonic components. The first two terms of this series have been known to some accuracy since Isaac Newton's time. In recent years, with the results of close tracking of artificial satellites available, many more terms of this series have become known. It seemed strange, with so much history behind it, that in 1962 when the author began his investigations of the gravity effects on a 24-hour satellite, no single reference could answer adequately the basic question: Where does the familiar harmonic series representation of the earth's gravity field come from? That one question, and the attempt to answer it, gave rise to this document. But in the course of answering that basic question other questions arose on the physical and geometric interpretation of the individual gravity components for which, likewise, no adequate answers could be found in the literature. The interpretation of satellite observations is always with respect to specific stations fixed on the earth's surface. The form of that surface, called the geoid or mean sea level surface, is a reflection of the mass distribution of the earth beneath it just as the gravity series is. Thus the problems of accurate orbit determination and accurate ground station locations are interrelated. Their common point of departure is the geometric (geoid) interpretation of the terms in the gravity field. These are just some of the additional questions about the field which are important in the applications to geodesy and satellite orbit determination and for which answers are given in this report.

For example, in the last section of this report the gravity and geometric effects of the small nutations of the north pole are discussed and calculated, so far as is known, for the first time in the literature.

II. THE GRAVITY POTENTIAL AND THE GRAVITY FIELD OF THE EARTH

In this study we are interested fundamentally in determining the exterior gravity force field of the earth in a form most efficient for the calculation of the trajectories of small bodies in the vicinity of the earth. In Figure 1 the earth is represented by an extended domain, D_e, in the space X_1, X_2, X_3. The gravitational force exerted by the small mass dm in the extended earth domain D_e on a particle of unit mass at P is, from Newton's law of gravitation,

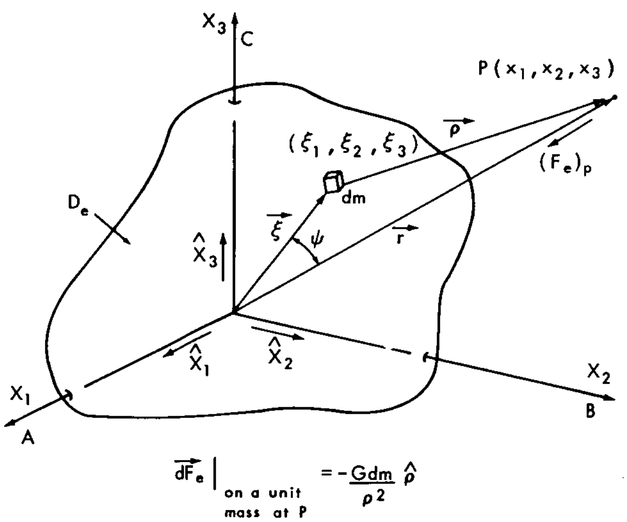


Figure 1—The gravity force field of the earth.

$$(\vec{dF}_e)_P = \frac{-G dm}{\rho^2}\hat{\rho}\ , \tag{1}$$

where G is the universal gravity constant ($= 6.67 \times 10^{-8}$ dynes-cm^2/gm^2). The total gravity force of D_e on the unit mass at P is the integral of Equation 1 over D_e,

$$(\vec{F}_e)_P = \int_{D_e} \frac{-G dm}{\rho^2}\hat{\rho}\ . \tag{2}$$

The unit vector $\hat{\rho}$ can be decomposed as follows:

$$\hat{\rho} = \frac{\vec{\rho}}{\rho} = \frac{\hat{x}_1(x_1-\xi_1)+\hat{x}_2(x_2-\xi_2)+\hat{x}_3(x_3-\xi_3)}{\left\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right\}^{1/2}}\ .$$

Thus Equation 2 can be rewritten as the vector equation

$$\vec{F}_e = -G\int_{D_e} \frac{\left\{\hat{x}_1(x_1-\xi_1)+\hat{x}_2(x_2-\xi_2)+\hat{x}_3(x_3-\xi_3)\right\} dm}{\left\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right\}^{3/2}}\ . \tag{3}$$

Equation 3 implies that the three scalar equations for the $(F_e)_{x_1}$, $(F_e)_{x_2}$ and $(F_e)_{x_3}$ coordinate components of total attraction, $(\vec{F}_e)_P = (F_e)_{x_1}\hat{x}_1 + (F_e)_{x_2}\hat{x}_2 + (F_e)_{x_3}\hat{x}_3$, are

$$(F_e)_{x_i} = -G\int_{D_e} \frac{(x_i-\xi_i)\, dm}{\left\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right\}^{3/2}}$$

$$= -G\int_{D_e} \frac{(x_i-\xi_i)\, dm}{\rho^3}\ , \qquad i = 1, 2, 3\ . \tag{4}$$

For an arbitrary mass distribution in D_e it would be extremely cumbersome to compute the gravity field through the three triple integrals of Equation 4. For most applications it is far more desirable and instructive to compute the field with a single intermediate scalar function V, called the potential, whose easily computed gradient $(\nabla V = \hat{x}_1\,\partial V/\partial x_1 + \hat{x}_2\,\partial V/\partial x_2 + \hat{x}_3\,\partial V/\partial x_3)$ gives the force field represented by Equation 4. The gravity potential of D_e at P, $(V_e)_P$ is defined by the scalar triple integral

$$(V_e)_P = \int_{D_e} \frac{G\,dm}{\rho} = G\int_{D_e} \frac{dm}{\left\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right\}^{1/2}} \,. \tag{5}$$

Note this implies that each small mass contributes $d(V_e)_P = G\,dm/\rho$ to the total potential. For example, the scalar potential of a point mass, m, on a unit mass at ρ from m in space is

$$V_m = \frac{Gm}{\rho} \,. \tag{6}$$

The gradient is a variation of the field with respect to the coordinates of the field point only and is therefore independent of the limits of integration for the domain D_e. Thus the ∇ operation can be taken on the integrand and the gradient of Equation 5 with respect to the space coordinates of the arbitrary field point P is

$$\nabla(V_e)_P = \nabla\int_{D_e}\frac{G\,dm}{\rho} = G\int_{D_e} dm\,\nabla\left(\frac{1}{\rho}\right) \,. \tag{7}$$

But

$$\rho = \left\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right\}^{1/2}$$

and

$$\frac{\partial(1/\rho)}{\partial x_i} = -\frac{1}{\rho^2}\left\{\frac{1}{2}\left[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2\right]^{-1/2}\left[2(x_i-\xi_i)\right]\right\}$$

$$= \frac{-(x_i-\xi_i)}{\rho^3}\,, \quad i = 1, 2, 3 \,. \tag{8}$$

Therefore, from Equation 8 and the definition of the gradient Equation 7 becomes:

$$\nabla(V_e)_P = -G\int_{D_e}\frac{dm}{\rho^3}\left\{\hat{x}_1(x_1-\xi_1)+\hat{x}_2(x_2-\xi_2)+\hat{x}_3(x_3-\xi_3)\right\} \,.$$

or

$$\hat{x}_1 \frac{\partial (V_e)_P}{\partial x_1} + \hat{x}_2 \frac{\partial (V_e)_P}{\partial x_2} + \hat{x}_3 \frac{\partial (V_e)_P}{\partial x_3} = \hat{x}_1 \left[-G \int_{D_e} \frac{(x_1 - \xi_1)\, dm}{\rho^3} \right]$$

$$+ \hat{x}_2 \left[-G \int_{D_e} \frac{(x_2 - \xi_2)\, dm}{\rho^3} \right] + \hat{x}_3 \left[-G \int_{D_e} \frac{(x_3 - \xi_3)\, dm}{\rho^3} \right] . \tag{9}$$

Equation 9 is equivalent to three scalar equations:

$$\frac{\partial (V_e)_P}{\partial x_i} = -G \int_{D_e} \frac{(x_i - \xi_i)\, dm}{\rho^3} , \qquad i = 1, 2, 3 . \tag{10}$$

From Equations 10 and 4 the components of the gravity force and gravity potential gradient vectors are identical:

$$\frac{\partial (V_e)_P}{\partial x_i} = (F_e)_{x_i} , \qquad i = 1, 2, 3 . \tag{11}$$

Thus the problem of finding the vector gravity field of the earth has been broken down to the simpler problem of finding the single scalar gravity potential of the earth (Equation 5). The three gradient derivatives of this scalar function will be the coordinate components of the vector gravity force field (Equation 11).

III. THE EXTERNAL POTENTIAL OF THE EARTH AS AN INFINITE SERIES OF MASS INTEGRALS

In determining the gravity potential of the earth at P from Equation 5, there are three methods of proceeding:

1. If the precise nature or form of the mass domain of the earth is known or can be assumed, the elemental potential $G\,dm/\rho$ can be expressed in accordance with this form to allow an exact integration of Equation 5. In such a manner Newton first proved that the external gravity potential of a spherical mass whose density is only a function of distance from the center of mass (c.m.) is given simply as

$$(V)_P = GM/r , \tag{12}$$

where M is the total mass of the extended sphere and r is the distance from the c.m. of the sphere to the field point P. In other words, Equation 12 states that an extended earth of radially symmetric mass distribution can be treated as a mass point with a simple central force field given by

$$\left(\vec{F}_e\right)_P = -\frac{GM_e}{r^2}\hat{r} \quad . \tag{13}$$

Since the real mass distribution of the earth can be closely approximated by such a spherically symmetric mass domain, the exact form of the earth's potential is Equation 12 with small, but important departures which depend, in general, on all three coordinates of the field point independently and not just the distance from the c.m. of the earth.

2. In the general case of a nearly spherical mass domain the integrand of Equation 5 can be expanded in an infinite harmonic series of terms in $1/r^n$ $(n = 1, 2, 3, \cdots)$. Integration of the leading term $(1/r)$ will give the dominant spherically symmetric part of the potential. The remaining series, integrated term-by-term with respect to the domain D_e, will give mass integrals of small order related to the nonspherical part of the mass distribution.

3. In the general case of an arbitrary domain the indicated mass integration over the domain in Equation 5 for V_e can be replaced by a differential equation which V_e must satisfy independently of the nature of the domain. The solution of the differential equation in spherical coordinates as an infinite series of spherical harmonics will yield the same series solution as in method 2. But in this case, the coefficients of the series will be arbitrary. Spherical coordinates are chosen to give a series whose first few terms approximate well the nearly spherically symmetric earth. The form of this series is also convenient for calculating the perturbations of nearly circular satellite orbits, as well as displaying the gravity field on the earth's nearly spherical surface. The coefficients themselves can be related to the integrated mass properties of the earth by matching term-by-term with the solution from method 2 (see part VI). A large number of coefficients of this series have been determined to date by many kinds of geodetic measurements which essentially map and sample the gravity field over the earth's surface (gravimeter surveys and astro-geodetic survey mappings) and in space (satellite perturbation studies).

Proceeding with the direct integration of Equation 5 (see Figure 1), we note from the law of cosines that

$$\rho^2 = \xi^2 + r^2 - 2\xi r \cos\psi \ ,$$

or

$$\rho = r\left\{1 + (\xi/r)^2 - 2(\xi/r)\cos\psi\right\}^{1/2} \quad . \tag{14}$$

Equation 14 in Equation 5 gives (noting r is independent of D_e)

$$\left(V_e\right)_P = \frac{G}{r}\int_{D_e} dm\left[1 + (\xi/r)^2 - 2(\xi/r)\cos\psi\right]^{-1/2}$$

$$= \frac{G}{r}\int dm\left[1 + \left(\frac{\xi}{r}\right)^2\right]^{-1/2}\left[1 - \frac{2\left(\frac{\xi}{r}\right)\cos\psi}{1 + \left(\frac{\xi}{r}\right)^2}\right]^{-1/2} \quad . \tag{15}$$

It is evident from Equation 15 that for field points far from the earth (assuming the center of coordinates is somewhere inside the earth), $(V_e)_{P,\ \text{far from the earth}} \doteq GM_e/r$. This implies the earth acts as an effective point mass (independent of the exact nature of its mass distribution) at a point sufficiently far from its vicinity.

For exterior points P for which $(\xi/r) < 1$, since $2(\xi/r)\cos\psi/1 + (\xi/r)^2 < 1$, the integrand of Equation 15 can be expanded into two convergent binomial series, yielding

$$(V_e)_P = \frac{G}{r}\int_{D_e} dm \left\{1 - \frac{\left(\frac{\xi}{r}\right)^2}{2} + \cdots\right\}\left\{1 + \left(\frac{\xi}{r}\right)\cos\psi\left[1 - \left(\frac{\xi}{r}\right)^2 + \cdots\right] + \frac{3\left(\frac{\xi}{r}\right)^2}{2}\cos^2\psi\left[1 - 2\left(\frac{\xi}{r}\right)^2 + \cdots\right]\right.$$
$$\left. + \frac{5\left(\frac{\xi}{r}\right)^3}{2}\cos^3\psi\left[1 - 3\left(\frac{\xi}{r}\right)^2 + \cdots\right] + \cdots\right\} . \tag{15a}$$

Collecting terms of Equation 15a in powers of (ξ/r) gives

$$(V_e)_P = \frac{G}{r}\int_{D_e} dm \left\{1 + \left(\frac{\xi}{r}\right)\cos\psi + \frac{\left(\frac{\xi}{r}\right)^2}{2}\left[2 - 3\sin^2\psi\right] + 0\left(\frac{\xi}{r}\right)^3 + \cdots\right\} . \tag{16}$$

If the origin of coordinates is the c.m. of the earth,

$$\int_{D_e} dm[\xi\cos\psi] = 0 , \tag{17}$$

from the definition of the c.m. of a body. This property is independent of the kind of body-fixed coordinate system chosen. Thus, for orthogonal axes x_1^1, x_2^1, x_3^1 with origin at the c.m. and x_1^1 oriented along $\vec{r}$ to the field point P, $\xi_1^1 = \xi\cos\psi$, and $\int_{D_e} dm(\xi\cos\psi) = \int_{D_e}\xi_1^1\,dm = 0$. Also note from Figure 1 that

$$\int_{D_e} \xi^2\,dm = I_0 , \tag{18}$$

the total moment of inertia of the mass of the earth about its center of mass (c.m.). This quantity is independent of the field point P. In addition note that

$$\int_{D_e} dm(\xi\sin\psi)^2 = I_r , \tag{19}$$

the moment of inertia of the earth about $\vec{r}$. This quantity is a function of the *direction* of P from the c.m. of the earth, but not its *distance*. Integration of Equation 16 term-by-term with the results of Equations 17, 18 and 19 gives the first two mass integrals of the external gravity potential of the earth as

$$\left(V_e\right)_P = G\left\{\frac{M_e}{r} + \frac{2I_0 - 3I_r}{2r^3} + 0\left(\frac{\xi}{r}\right)^4 + \cdots\right\} . \tag{20}$$

I_r can be shown to be a simple function of the direction of P from the c.m. and the three principal moments of inertia of the earth.

For example, let x_1, x_2, x_3 be principal c.m. fixed axes for the earth and A, B, C be the corresponding principal moments of inertia for the earth defined as:

$$A = \int_{D_e} dm\left(\xi_2^2 + \xi_3^2\right)$$

$$B = \int_{D_e} dm\left(\xi_1^2 + \xi_3^2\right)$$

$$C = \int_{D_e} dm\left(\xi_1^2 + \xi_2^2\right) .$$

But from the law of cosines

$$I_r = \int_{D_e} \xi^2 \sin^2\psi\, dm = \int_{D_e} \xi^2\left(1 - \cos^2\psi\right) dm = \int_{D_e} dm\, \xi^2\left[1 - \frac{\left(\xi^2 + r^2 - \rho^2\right)^2}{4\xi^2 r^2}\right] .$$

Thus

$$I_r = \int_{D_e} dm\left\{\xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{\left[\xi_1^2 + \xi_2^2 + \xi_3^2 + x_1^2 + x_2^2 + x_3^2 - \left(x_1 - \xi_1\right)^2 - \left(x_2 - \xi_2\right)^2 - \left(x_3 - \xi_3\right)^2\right]^2}{4\left(x_1^2 + x_2^2 + x_3^2\right)}\right\}$$

$$= \int_{D_e} dm\left[\xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{\left(x_1\xi_1 + x_2\xi_2 + x_3\xi_3\right)^2}{x_1^2 + x_2^2 + x_3^2}\right]$$

$$= \int_{D_e} dm\left[\frac{x_1^2\left(\xi_2^2 + \xi_3^2\right) + x_2^2\left(\xi_1^2 + \xi_3^2\right) + x_3^2\left(\xi_1^2 + \xi_2^2\right) - 2\left(x_1 x_2 \xi_1 \xi_2 + x_1 x_3 \xi_1 \xi_3 + x_2 x_3 \xi_2 \xi_3\right)}{r^2}\right] . \tag{20a}$$

But, for principal axes of inertia x_1, x_2, x_3,

$$\int_{D_e} \xi_1 \xi_2\, dm = \int_{D_e} \xi_1 \xi_3\, dm = \int_{D_e} \xi_2 \xi_3\, dm = 0 ,$$

and therefore

$$I_r = \left[A\left(x_1\right)^2 + B\left(x_2\right)^2 + C\left(x_3\right)^2\right]/r^2$$

$$= A\ell^2 + Bm^2 + Cn^2 , \tag{20b}$$

where ℓ, m and n are the direction cosines of P from the c.m. of the earth with respect to earth-fixed principal axes of inertia. From Equation 18 we have:

$$2I_0 = 2\int_{D_e} (\xi_1^2 + \xi_2^2 + \xi_3^2)\, dm = 2\int_{D_e} \left[\frac{1}{2}(\xi_1^2 + \xi_2^2) + \frac{1}{2}(\xi_1^2 + \xi_3^2) + \frac{1}{2}(\xi_2^2 + \xi_3^2)\right] dm$$

$$= A + B + C \quad . \tag{20c}$$

For a spherically symmetric earth mass distribution

$$A = B = C \ ,$$

and from Equations 20c and 20b, $2I_0 = 3A$ and

$$3I_r = 3\left[A(x_1)^2 + B(x_2)^2 + C(x_3)^2\right]\Big/ r^2 = 3A\left[(x_1)^2 + (x_2)^2 + (x_3)^2\right]\Big/ r^2 = 3A \tag{20d}$$

(for a spherically symmetric earth). Thus from the above results in Equation 20 it is seen that the $1/r^3$ term of the gravity potential is zero for a spherically symmetric mass distribution. Thus, for a triaxial spheroidal earth, not differing greatly from a spherically symmetric mass domain, the second term of the potential function, Equation 20, will be close to zero even for field points in the near vicinity of the earth. The same argument can be made for the higher order terms of Equation 20. These can be shown to measure earth mass deviations with planes of symmetry not necessarily orthogonal.

A summary of the development of the gravity potential of the earth thus far shows:

1. The first term (the $1/r$ term) of the infinite harmonic series of mass integrals of the gravity potential of the earth gives the effect of the earth's mass distribution concentrated at a point.

2. The second term (the $1/r^3$ term) gives the gravity effect due to the deviation of the earth's ellipsoid of inertia from a sphere. The $1/r^2$ term in the potential is absent if the c.m. is the center of the coordinate reference.

IV. THE EXTERNAL GRAVITY POTENTIAL OF THE EARTH IN TERMS OF LAPLACE'S EQUATION

The Gravity Potential as a Solution to Laplace's Equation

The operator $\nabla^2(\ )$ of a scalar function $(\ )$ is defined as

$$\nabla^2(\ ) = \frac{\partial^2(\ )}{\partial x_1^2} + \frac{\partial^2(\ )}{\partial x_2^2} + \frac{\partial^2(\ )}{\partial x_3^2} \quad .$$

Taking $\nabla^2 (V_e)_P$ in Equation 5 and noting that this operation is independent of the manner in which the mass domain D_e is integrated gives

$$\nabla^2 (V_e)_P = \nabla^2 \int_{D_e} \frac{G dm}{\rho} = G \int_{D_e} dm \, \nabla^2 (1/\rho) \quad . \tag{21}$$

But

$$\nabla^2 (1/\rho) = \frac{\partial^2 (1/\rho)}{\partial x_1^2} + \frac{\partial^2 (1/\rho)}{\partial x_2^2} + \frac{\partial^2 (1/\rho)}{\partial x_3^2} \quad ,$$

and

$$\frac{\partial^2 (1/\rho)}{\partial x_i^2} = \frac{\partial \left(\frac{\partial (1/\rho)}{\partial x_i} \right)}{\partial x_i} = \frac{\partial \left[\frac{\partial (1/\rho)}{\partial \rho} \frac{\partial \rho}{\partial x_i} \right]}{\partial x_i}$$

$$= \partial \frac{\left\{ -\frac{1}{\rho^2} \left[\frac{1}{2} \left(\rho^{-1} \right) 2 \left(x_i - \xi_i \right) \right] \right\}}{\partial x_i}$$

$$= \frac{-\partial \left[\rho^{-3} \left(x_i - \xi_i \right) \right]}{\partial x_i}$$

$$= -\rho^{-3} + 3 \left(x_i - \xi_i \right)^2 \rho^{-5} \quad , \qquad i = 1, 2, 3 \; .$$

Thus

$$\nabla^2 (1/\rho) = -3\rho^{-3} + 3\rho^{-5} \left\{ \left(x_1 - \xi_1 \right)^2 + \left(x_2 - \xi_2 \right)^2 + \left(x_3 - \xi_3 \right)^2 \right\}$$

$$= -3\rho^{-3} + 3\rho^{-5} \rho^2 = 0 \; .$$

Thus since the integrand in Equation 21 is zero at every mass point (ξ_1, ξ_2, ξ_3) in the domain D_e, the integral is also zero. Therefore, the exterior gravity potential (where $\rho > 0$ everywhere, and assuming no singularities in the potential at P) must satisfy Laplace's Equation,

$$\nabla^2 (V_e)_P = 0 \; . \tag{22}$$

(See Reference 1, Section II for a more complete discussion of the derivation of Equation 22 and the mass integral potential series, Equation 20.)

The External Gravity Potential as an Infinite Series of Spherical Harmonics

From the suggestion of part III, since the earth is nearly spherically symmetric in form, solutions of Laplace's equation in spherical coordinates will be developed; and the number of terms necessary to obtain an accurate gravity field for most purposes will be a minimum. (Ellipsoidal coordinates would require fewer terms for accuracy but they would be computationally more cumbersome.)

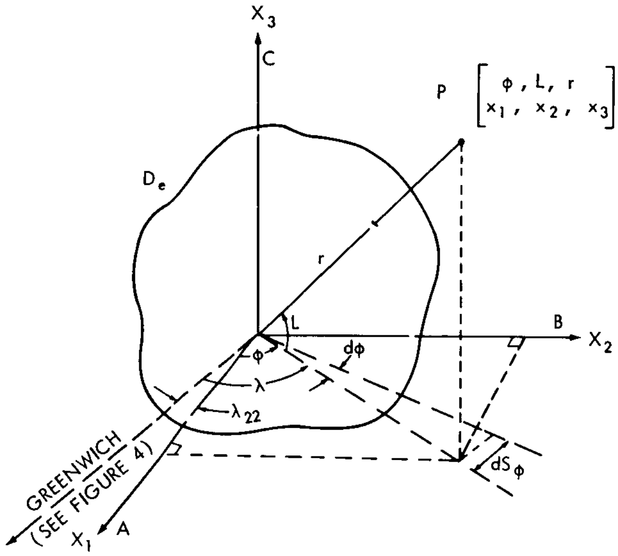


Figure 2—Domain of the earth and coordinate systems.

Consider body "D_e" with the origin of coordinates x_1, x_2, x_3 fixed inside "D_e", and the exterior field point P specified by spherical coordinates ϕ, L, r (Figure 2).

In general orthogonal coordinates q_1, q_2, q_3

$$\nabla^2 V = \frac{1}{Q_1 Q_2 Q_3}\left\{\frac{\partial}{\partial q_1}\left[\frac{Q_2 Q_3}{Q_1}\frac{\partial V}{\partial q_1}\right] + \frac{\partial}{\partial q_2}\left[\frac{Q_1 Q_3}{Q_2}\frac{\partial V}{\partial q_2}\right] + \frac{\partial}{\partial q_3}\left[\frac{Q_1 Q_2}{Q_3}\frac{\partial V}{\partial q_3}\right]\right\}, \tag{23}$$

where $ds_i = Q_i\, dq_i$, a small line element in the direction of increasing q_i coordinate at P, all other coordinates being constant (Reference 2, pp. 173 and 175).

From the spherical coordinates of Figure 2 the following terms are identified:

$$ds_\phi = r \cos L d\phi, \quad \text{implying } Q_\phi = r\cos L$$

$$ds_L = r\, dL, \quad \text{implying } Q_L = r$$

$$ds_r = 1\, dr, \quad \text{implying } Q_r = 1 .$$

With these identifications in Equation 23, Equation 22 becomes

$$\frac{1}{r^2 \cos L}\left\{\frac{\partial}{\partial \phi}\left[\frac{1}{\cos L}\frac{\partial V}{\partial \phi}\right] + \frac{\partial}{\partial L}\left[\cos L \frac{\partial V}{\partial L}\right] + \frac{\partial}{\partial r}\left[r^2 \cos L \frac{\partial V}{\partial r}\right]\right\} = 0 .$$

Expansion of this equation will give

$$\frac{1}{r^2 \cos L}\left\{\frac{1}{\cos L}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial}{\partial L}\left[\cos L \frac{\partial V}{\partial L}\right] + \cos L \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)\right\} = 0 ,$$

or,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\cos L}\frac{\partial}{\partial L}\left(\cos L\frac{\partial V}{\partial L}\right)+\frac{1}{r^2\cos^2 L}\frac{\partial^2 V}{\partial\phi^2} = 0 . \tag{24}$$

To separate variables, assume

$$V = R(r)\,H(L)\,\Phi(\phi) . \tag{25}$$

Substituting Equation 25 into Equation 24, taking the indicated partial derivatives, and rearranging terms will transform Equation 24 to

$$\frac{H\Phi}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R\Phi}{r^2\cos L}\frac{\partial}{\partial L}\left(\cos L\frac{\partial H}{\partial L}\right)+\frac{RH}{r^2\cos^2 L}\frac{\partial^2\Phi}{\partial\phi^2} = 0 . \tag{26}$$

Multiply Equation 26 by $r^2\cos^2 L$ to get

$$H\Phi\cos^2 L\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+R\Phi\cos L\frac{\partial}{\partial L}\left(\cos L\frac{\partial H}{\partial L}\right)+RH\frac{\partial^2\Phi}{\partial\phi^2} = 0 . \tag{27}$$

Next divide Equation 27 through by $R\Phi H$ to isolate $\Phi(\phi)$. Equation 27 now becomes

$$\frac{\cos^2 L}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\cos L}{H}\frac{\partial}{\partial L}\left(\cos L\frac{\partial H}{\partial L}\right) = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = +m^2 . \tag{28}$$

Small m is a constant (to be determined later) since the far left-hand side of Equation 28 is a function of r and L only, while the right-hand side, $\left(-1/\Phi\ \partial^2\Phi/\partial\phi^2\right)$, is a function of ϕ only.

From Equation 28

$$\frac{\partial^2\Phi}{\partial\phi^2}+m^2\Phi = 0 ,$$

which implies

$$\Phi(\phi) = K_0\cos m\left(\phi-\phi_0\right) , \tag{29}$$

where K_0 and ϕ_0 are constants of integration. Since V must be periodic at least every 2π radians, and $\cos m\left(\phi-\phi_0\right) = \cos\left[-m\left(\phi-\phi_0\right)\right]$, it is necessary and sufficient that m be restricted to all positive integers including zero.

Dividing Equation 28 by $\cos^2 L$ and rearranging terms allows one to isolate H and R. Equation 28 now becomes

$$\frac{1}{H\cos L}\frac{\partial}{\partial L}\left(\cos L\,\frac{\partial H}{\partial L}\right) - \frac{m^2}{\cos^2 L} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial R}{\partial r}\right) = -q\ , \tag{30}$$

where q is a constant (to be determined later), since the far left-hand side of Equation 30 is a function of L only and the near right is a function of r only.

From Equation 30

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial R}{\partial r}\right) = q\ , \tag{31}$$

and also

$$\frac{1}{H\cos L}\frac{\partial}{\partial L}\left(\cos L\,\frac{\partial H}{\partial L}\right) - \frac{m^2}{\cos^2 L} = -q\ . \tag{32}$$

The independent variable in Equation 32 is transformed from L to x by setting $x = \sin L$. Now

$$\frac{d(\ )}{dL} = \frac{d(\ )}{dx}\,\frac{dx}{dL} = \cos L\,\frac{d(\ )}{dx}\ ;$$

and

$$\cos L = \left(1 - x^2\right)^{1/2}\ ,\quad \cos^2 L = 1 - x^2\ ,\ \text{etc.} \tag{33}$$

In Equations 31 and 32 let $q = n(n+1)$, where n is also a constant which will be determined later.

With these changes Equation 32 becomes

$$\frac{1}{H}\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dH}{dx}\right] - \frac{m^2}{1 - x^2} + n(n+1) = 0\ . \tag{34}$$

Note that the partial differentiation in Equation 32 is equivalent to total differentiation since only one independent variable (L) is involved.

Expanding Equation 34 and multiplying through by H gives

$$\left(1 - x^2\right)\frac{d^2 H}{dx^2} - 2x\,\frac{dH}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]H = 0\ . \tag{35}$$

Equation 35 is recognized as the *associate Legendre equation*. It has applicable solutions

$$H_{nm} = D_{nm}\left(1 - x^2\right)^{m/2} \frac{d^m P_n(x)}{dx^m} . \qquad \text{(Reference 3, pp. 311-313)} \tag{36}$$

The $P_n(x)$ are the well known Legendre polynomials (Reference 3, pp. 305, 306, 312 and 313), and m and n are non-negative integers with $m \leq n$. H_{nm} refers to the harmonic of order n and power m.

Other solutions of 35 are possible when n is or is not an integer if $|x| \neq 1$. But $|x| = |\sin L| \leq 1$ for the exterior space surrounding the earth, and the other solutions are thus not admissible. They give infinities for $|x| = 1$ (Reference 4, pp. 192-194 and 199).

The polynomials $P_n(x)$ take the form

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left\{x^n - \frac{n(n-1)\,x^{n-2}}{(2')(1!)(2n-1)} + \frac{n(n-1)(n-2)(n-3)\,x^{n-4}}{\left(2^2\right)(2!)(2n-1)(2n-3)} - \cdots\right\} . \tag{37}$$

It may be noted from Equation 37 that if n is not an integer, the polynomial series becomes infinite and complex and thus is not applicable to the solution for V. A full treatment of the derivation of the Associated Legendre Functions H_{nm} can be found in Reference 13. The H_{nm} are there shown to be a complete, orthogonal set of functions giving the only finite solutions to Equation 35 for $|x| \leqq 1$.

In particular, the first six polynomials are:

$$\begin{aligned}
P_0(x) &= P_0(\sin L) = 1 \\
P_1(x) &= P_1(\sin L) = \sin L \\
P_2(x) &= P_2(\sin L) = \frac{1}{2}\left(3\sin^2 L - 1\right) \\
P_3(x) &= P_3(\sin L) = \frac{1}{2}\left(5\sin^3 L - 3\sin L\right) \\
P_4(x) &= P_4(\sin L) = (1/8)\left(35\sin^4 L - 30\sin^2 L + 3\right) \\
P_5(x) &= P_5(\sin L) = (1/8)\left(63\sin^5 L + 70\sin^3 L + 15\sin L\right) \\
P_6(x) &= P_6(\sin L) = (1/16)\left(231\sin^6 L - 315\sin^4 L + 105\sin^2 L - 5\right) .
\end{aligned} \tag{38}$$

To determine the variation in r, in Equation 31 we set $q = n(n+1)$ and expand to get

$$r^2 \frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n(n+1) = 0 . \tag{39}$$

Equation 39 is a "Cauchy" equation (Reference 4, p. 199) whose general solution is

$$R(r) = A_1 r^n + A_2 r^{-(n+1)} . \tag{40}$$

If (as is conventional) we arbitrarily stipulate that $V_e \to 0$ as $r \to \infty$, then A_1 must be set equal to zero to avoid the infinities from the first term in Equation 40 for $n > 0$ and to avoid a finite V_e as $r \to \infty$ for $n = 0$.

When one combines Equations 29, 36 and 40 and formulates new coefficients F_{nm} and ϕ_{nm}, one obtains the series solution of Laplace's equation in sperhical coordinates (for V_e) as

$$V_e = \sum_{n=0}^{\infty} \sum_{m=0}^{n} F_{nm} \, r^{-(n+1)} \cos^m L \, \frac{d^m P_n(\sin L)}{d(\sin L)^m} \cos m\left(\phi - \phi_{nm}\right) . \tag{41}$$

The potential field of Equation 41 is usually separated into zonal ($m = 0$) and sectorial-tesseral (or longitude) terms ($m \neq 0$):

$$V_e = \sum_{n=0}^{\infty} F_{n0} \, r^{-(n+1)} P_n(\sin L) + \cdots + \sum_{n=1}^{\infty} \sum_{m=1}^{n} F_{nm} \, r^{-(n+1)} \cos^m L \, \frac{d^m P_n(\sin L)}{d(\sin L)^m} \cos m\left(\phi - \phi_{nm}\right) . \tag{42}$$

The sectorial terms of Equation 42 are those of $m \neq 0$ for which $n = m$. The tesseral terms (distinguished from the sectorial terms) are those of $m \neq 0$ for which $n \neq m$. The ϕ_{nm} represent the location in the space (ϕ, L, r) of the principal plane of longitudinal mass symmetry for the nm harmonic ($m \neq 0$).

V. MATCHING THE FIRST TWO MASS INTEGRALS OF THE POTENTIAL WITH THEIR EQUIVALENT SPHERICAL HARMONIC REPRESENTATIONS

The series in Equation 42 terminated at $n = 2$ will give

$$V_e = \frac{1}{r} F_{00} + \frac{1}{r^2}\left[F_{10} \sin L + F_{11} \cos L \cos\left(\phi - \phi_{11}\right)\right] + \frac{1}{r^3}\left[3F_{22} \cos^2 L \cos 2\left(\phi - \phi_{22}\right)\right.$$

$$\left. + \frac{F_{20}}{2}\left(3\sin^2 L - 1\right) + \frac{3F_{21}}{2} \sin 2L \cos\left(\phi - \phi_{21}\right)\right] + \text{order}\left(\frac{\xi}{r}\right)^4 + \cdots . \tag{43}$$

Consider the potential of a finite continuous distribution of matter at a point far enough from the distribution that the radius from the origin of coordinates to the field point is greater than the radius to any part of the distribution. If the origin of coordinates is taken at the center of mass of the distribution, it was shown (in Section III) that the expansion $(1/\rho)$ in Equation 15a is absolutely and uniformly convergent, which justifies the term-by-term integration in Equation 20 to give the result

$$\frac{V_e}{G} = \frac{M_e}{r} + \frac{\left(2I_0 - 3I_r\right)}{2r^3} + \text{order}\left(\frac{\xi}{r}\right)^4 + \cdots .$$

The discussion of the physical significance of the coefficients of Equation 43 is simplified by considering axes X_1, X_2, X_3 corresponding to the principal moments of inertia A, B and C of the mass distribution (Figure 2).

For the field represented by the mass-integrals of Equation 20, the expansion in harmonics of Equation 43 is equivalent to the mass-integral harmonic series of Equation 20 if and only if

$$F_{00} = GM_e , \tag{44}$$

$$\frac{1}{G}\left[F_{10} \sin L + F_{11} \cos L \cos\left(\phi - \phi_{11}\right)\right] \equiv 0 , \tag{45}$$

$$\frac{F_{20}}{2}\left(3\sin^2 L - 1\right) + \frac{3F_{21}}{2}\left[\sin 2L \cos\left(\phi - \phi_{21}\right)\right] + 3F_{22}\cos^2 L \cos 2\left(\phi - \phi_{22}\right) = \frac{G\left(2I_0 - 3I_r\right)}{2} , \quad \text{etc.} \tag{46}$$

Equation 44 gives the familiar GM_e/r potential term which, if it were the only term of the series present, would state that the whole mass of the earth, M_e, acted effectively at its c.m. Equation 45 implies that $F_{10} \equiv F_{11} \equiv 0$ since the two terms are linearly independent. The following new constants are defined

$$\left(F_{21}\right)_0 = \frac{3F_{21}}{2G}\cos\phi_{21} , \qquad \left(F_{22}\right)_0 = \frac{F_{22}}{G}\cos 2\phi_{22} ,$$

$$\left(F_{20}\right)_0 = \frac{F_{20}}{G} , \qquad \left(F_{22}\right)_1 = \frac{F_{22}}{G}\sin 2\phi_{22} . \tag{46a}$$

$$\left(F_{21}\right)_1 = \frac{3F_{21}}{2G}\sin\phi_{21} ,$$

To facilitate the term-by-term matching of Equation 46, transform back to rectangular principal coordinates x_1, x_2, x_3, where

$$x_1 = r\cos L\cos\phi ,$$

$$x_2 = r\cos L\sin\phi ,$$

$$x_3 = r\sin L . \tag{47}$$

Expanding Equation 46 and introducing the new constants and the transformation of Equation 47, makes Equation 46 become

$$\frac{\left(F_{20}\right)_0}{2}\frac{\left[\left(3x_3\right)^2 - \left\{\left(x_1\right)^2 + \left(x_2\right)^2 + \left(x_3\right)^2\right\}\right]}{r^2} + \frac{2\left(F_{21}\right)_0 x_1 x_3}{r^2} + \frac{2\left(F_{21}\right)_1 x_2 x_3}{r^2} + \frac{3\left(F_{22}\right)_0\left[\left(x_1\right)^2 - \left(x_2\right)^2\right]}{r^2} + \frac{6\left(F_{22}\right)_1 x_1 x_2}{r^2} = \frac{2I_0 - 3I_r}{2} . \tag{47a}$$

But, from their definitions (Equations 20b and 20c),

$$2I_0 = (A+B+C)\,\frac{\left[(x_1)^2+(x_2)^2+(x_3)^2\right]}{r^2}\,, \tag{47b}$$

and

$$I_r = \frac{\left[A(x_1)^2+B(x_2)^2+C(x_3)^2\right]}{r^2}\,. \tag{47c}$$

Equation 46 thus becomes (with Equations 47b and 47c in 47a)

$$(x_1)^2\left[-\frac{(F_{20})_0}{2}+3(F_{22})_0\right]+(x_2)^2\left[-\frac{(F_{20})_0}{2}-3(F_{22})_0\right]+(x_3)^2(F_{20})_0+x_1x_2\left[6(F_{22})_1\right]+x_1x_3\left[2(F_{21})_0\right]$$

$$+x_2x_3\left[2(F_{22})_1\right] \equiv (x_1)^2\left[\frac{A+B+C}{2}-\frac{3A}{2}\right]+(x_2)^2\left[\frac{A+B+C}{2}-\frac{3B}{2}\right]+(x_3)^2\left[\frac{A+B+C}{2}-\frac{3C}{2}\right]. \tag{48}$$

Since all the terms in $x_i\,x_j$ on either side of the equality in Equation 48 are linearly independent, the only way Equation 48 can be satisfied identically for all sets of x_1, x_2, x_3 with non-zero coefficients is for

$$-\frac{(F_{20})_0}{2}+3(F_{22})_0 \equiv \frac{(B+C-2A)}{2} \tag{49}$$

$$-\frac{(F_{20})_0}{2}-3(F_{22})_0 \equiv \frac{(A+C-2B)}{2} \tag{50}$$

$$(F_{20})_0 \equiv \frac{(A+B-2C)}{2} \tag{51}$$

$$6(F_{22})_1 \equiv 0, \text{ which implies } \phi_{22} = 0^\circ \text{ or } 180^\circ\,, \tag{52}$$

$$2(F_{21})_0 \equiv 0 \tag{53}$$

$$2(F_{21})_1 \equiv 0\,. \tag{54}$$

Thus, Equations 52, 53 and 54 determine $(F_{22})_1 \equiv (F_{21})_0 \equiv (F_{21})_1 \equiv 0$.

Substituting $(F_{20})_0$ from Equation 51 into Equation 49 determines $(F_{22})_0$ as

$$(F_{22})_0 = \frac{B-A}{4}\,. \tag{55}$$

It may be verified that Equation 50 is satisfied identically with $(F_{22})_0$ and $(F_{20})_0$ given by Equations 55 and 51.

Therefore, with respect to (r, L, ϕ), or with reference to the principal axes of the mass distribution, the potential (Equation 43) becomes

$$V_e = \frac{GM_e}{r} + \frac{GM_e}{r^3}\left[\frac{(F_{20})_0}{2}\left(3\sin^2 L - 1\right) - 3(F_{22})_0 \cos^2 L \cos 2\phi\right] + \text{order}\left(\frac{\xi}{r}\right)^4 + \cdots \quad (55a)$$

In terms of the coefficients of Reference 5 this may be written as

$$V_e = \frac{GM_e}{r}\left[1 - \frac{J_{20}}{2}(\bar{a}/r)^2\left(3\sin^2 L - 1\right) - 3J_{22}(\bar{a}/r)^2 \cos^2 L \cos 2\phi\right] + \text{order}\left(\frac{\xi}{r}\right)^4 + \cdots \quad (56)$$

The constant $\bar{a}$ will be defined later as a mean equatorial radius of the model earth ellipsoid.

In Equation 56 (from Equations 51 and 55 in Equation 55a)

$$J_{20} = -\frac{(F_{20})_0}{M_3(\bar{a})^2} = \frac{2C - (A+B)}{2M_e(\bar{a})^2}, \quad (57)$$

and

$$J_{22} = -\frac{(F_{22})_0}{M_e(\bar{a})^2} = \frac{A-B}{4M_e(\bar{a})^2}. \quad (58)$$

Equations 57 and 58 thus provide links between the gross mass properties of the earth (its principal moments of inertia) and its external gravity field. In fact, the oblateness of the earth (measured by $2C - [A+B]$) is known more accurately from satellite determinations of J_{20} than from direct surface measures of oblateness by astronomical observations of the change of latitude with distance north. Similar links can be established between the higher order gross mass properties of the earth and the higher order coefficients of the external gravity field. In part VII these coefficients will be linked to the surface geometry of the geoid, or mean sea level of the earth.

The gravitational equipotential of Equation 56 with the higher order terms neglected and a centrifugal term added will be shown in part VII to give essentially an ellipsoidal surface if J_{20} and J_{22} are sufficiently small. J_{20} and J_{22} may then be determined as functions of the rotation, oblateness and ellipticity of a model triaxial-ellipsoidal earth whose surface would be at mean sea level.

Two questions remain to be resolved before giving geometric (geoid) significance to J_{20} and J_{22}. The first concerns the validity of ignoring the $1/r^2$ term in the gravitational potential for those field points (i.e., on the surface of an ellipsoidal earth away from the equatorial major axis) for which some of the mass distribution is farther from the origin than the field point. The infinite series of harmonic functions in Equation 41 is the general solution of Laplace's equation with the boundary condition $V_e\ (r \to \infty) = 0$ and represents the potential at *all* points exterior to the surface of the mass distribution. (See Reference 1, Chapter II, for a rigorous proof of this statement.) But at points more strongly exterior to the surface the direct integration by series for the potential shows that the $1/r^2$ term is absent for the choice of origin at the c.m. The numerator of the $1/r^2$ term in the harmonic series of Equation 41 is a function of L and ϕ only. On a radial line, therefore, it is constant. Since it is zero in the outer region, it is zero everywhere to the surface because the inner region to the surface can be completely covered by radial lines from the enfolding outer region.

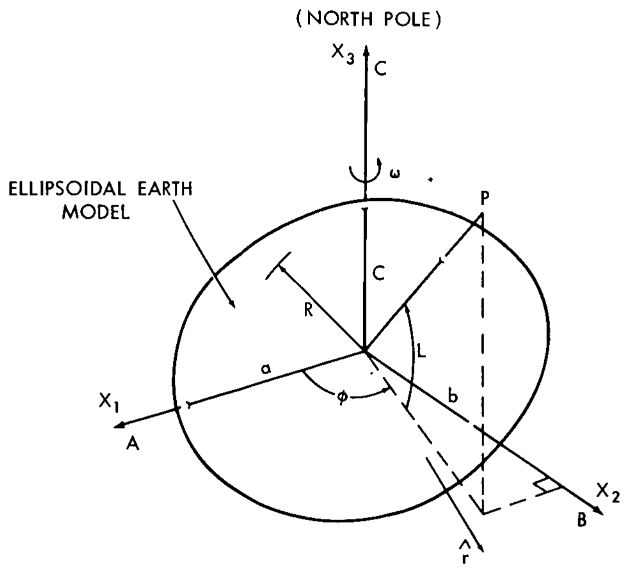


Figure 3—The triaxial geoid's geometry.

The second question concerns the orientation of the x_1, x_2, x_3 principal axes of inertia in the earth. Let us assume the earth is spun about an arbitrary axis in inertial space. From Reference 5, pp. 377-378, it is shown that for a rigid body such an initial rotation is invariable in the absence of external torques if, and only if, that initial spin axis is a principal axis of inertia. Astronomical observation shows that to a high degree of accuracy the earth's north pole spin axis is invariable in direction in inertial space after accounting for all external torques. We can thus arbitrarily (between x_1, x_2 and x_3) identify x_3 as the polar axis of rotation of the earth. In the model to which the simplified series in Equation 56 belongs, L corresponds to geocentric latitude with respect to the c.m., and ϕ is the geocentric longitude measured counterclockwise (looking down on the north pole) from the principal axis x_1 on the equator (Figures 2 and 3).

VI. MATCHING THE LEVEL SURFACE OF THE SECOND-ORDER GRAVITY AND EARTH ROTATION POTENTIAL TO THE SURFACE OF AN ELLIPSOID OF THREE AXES (THE TRIAXIAL GEOID)

In this section we will show that the mean sea level surface of the earth, the geoid, is, to second order, an ellipsoid of three axes. Arbitrarily the x_1 axis is taken along a, the semimajor equatorial axis of the triaxial ellipsoid.

In the ellipsoidal earth model of Figure 3, e is the ellipticity of the equator and f is the polar flattening; and in terms of a, b, and c, the half axes of the ellipsoid,

$$e = 1 - \frac{b}{a} = \frac{a - b}{a} \tag{59}$$

and

$$f = 1 - \frac{c}{\bar{a}} = \frac{\bar{a} - c}{\bar{a}} , \tag{60}$$

where

$$\bar{a} = \frac{a + b}{2} . \tag{61}$$

(Equations 60 and 61 are arbitrary but convenient definitions of f and $\bar{a}$.) The surface of the ellipsoid in Figure 3 is given by

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = 1 . \tag{62}$$

In terms of the coordinate transformation of Equation 47 the surface is also given by

$$\begin{aligned} x_1 &= R \cos L \cos \phi \\ x_2 &= R \cos L \sin \phi \\ x_3 &= R \sin L , \end{aligned} \tag{63}$$

where R is the radius to the surface of the ellipsoid.

Combining Equations 62 and 63 produces

$$\frac{\cos^2 L \cos^2 \phi}{a^2} + \frac{\cos^2 L \sin^2 \phi}{b^2} + \frac{\sin^2 L}{c^2} = 1/R^2 . \tag{64}$$

Multiplying Equation 64 through by b^2, solving for b/a from Equation 59 and rearranging makes Equation 64 become

$$(1 - e)^2 \left(\cos^2 L \cos^2 \phi\right) + \cos^2 L \sin^2 \phi = b^2 \left(\frac{1}{R^2} - \frac{\sin^2 L}{c^2}\right) . \tag{65}$$

Next solve for c^2 from Equation 60 and introduce this into Equation 65 which then becomes

$$(1 - e)^2 \cos^2 L \cos^2 \phi + \cos^2 L \sin^2 \phi = \frac{b^2}{\bar{a}^2} \left[\frac{\bar{a}^2}{R^2} - \frac{\sin^2 L}{(1 - f)^2}\right] . \tag{66}$$

A combination of Equations 59 and 61 leads to

$$\frac{b^2}{\bar{a}^2} = \left[\frac{2(1 - e)}{2 - e}\right]^2 . \tag{67}$$

Upon the substitution of Equation 67 into 66 this equation now becomes

$$\left(\frac{\bar{a}}{R}\right)^2 = \frac{(1-e)^2 \cos^2 L \cos 2\phi + \cos^2 L \sin^2 \phi + \dfrac{2(1-e)\sin^2 L}{(2-e)(1-f)^2}}{\left[\dfrac{2(1-e)}{2-e}\right]^2} . \tag{68}$$

When Equation 68 is somewhat simplified, it takes the form

$$(\bar{a}/R)^2 = \frac{1}{4}(2-e)^2 \cos^2 L \cos^2 \phi + \frac{\frac{1}{4}(2-e)^2 \cos^2 L \sin^2 \phi}{(1-e)^2} + \frac{\sin^2 L}{(1-f)^2} . \tag{69}$$

Two factors from terms in Equation 69 are now expanded to give

$$\left[\frac{2-e}{1-e}\right]^2 \equiv \left(4 - 4e + e^2\right)\left(1 + 2e + 3e^2 + \cdots\right) \doteq 4 + 4e + 5e^2 , \quad \text{for } e \ll 1 , \tag{70}$$

and

$$(1-f)^{-2} \doteq 1 + 2f + 3f^2 , \quad \text{for } f \ll 1 . \tag{71}$$

With the approximations of Equations 70 and 71, Equation 69 becomes

$$(\bar{a}/R)^2 = \frac{1}{4}(2-e)^2 \cos^2 L \cos^2 \phi + \frac{1}{4}\left(4 + 4e + 5e^2\right) \cos^2 L \sin^2 \phi + \left(1 + 2f + 3f^2\right) \sin^2 L . \tag{72}$$

Equation 72 describes the surface of the triaxial ellipsoid to e^2 and f^2 in the equatorial ellipticity and oblateness (polar flattening). On the surface of the rotating earth ellipsoid the centrifugal force per unit mass is

$$\left(\vec{F}_c\right)_R = \left(\frac{v^2}{r'}\right)_R \hat{r}' = \left(\omega^2 r'\right)_R \hat{r}' , \tag{73}$$

where v is the surface velocity at r' from the axis of rotation of the earth, and ω is the rotation rate of the earth. The force field of Equation 73 is derivable as the gradient ($\nabla = \hat{r}' \, \partial/\partial r'$) of a potential function of r':

$$\left(V_c\right)_R = \frac{(\omega r')^2{}_R}{2} . \tag{74}$$

The centrifugal potential at the surface of the ellipsoid is thus given as

$$\left[V_c\right]_R = \frac{\omega^2}{2}\left[(x_1)^2 + (x_2)^2\right]_R = \frac{\omega^2}{2}\left[R^2\cos^2 L\cos^2\phi + R^2\cos^2 L\sin^2\phi\right] ,$$

$$= \frac{\omega^2}{2} R^2\cos^2 L . \tag{75}$$

By combining Equations 56 and 75, the full potential at the surface of the ellipsoid can be expressed as

$$V(\text{total}) = V_e + V_c = \frac{GM}{\bar{a}}\left[\frac{\bar{a}}{R} + \frac{\alpha}{2}\left(\frac{R}{\bar{a}}\right)^2\cos^2 L - \frac{J_{20}}{2}\left(\frac{\bar{a}}{R}\right)^3\left(3\sin^2 L - 1\right) - 3J_{22}\left(\frac{\bar{a}}{R}\right)^3\cos^2 L\cos 2\phi\right], \tag{76}$$

where

$$\alpha = \frac{\omega^2\bar{a}^3}{GM} . \tag{77}$$

Next, the assumption is made that the surface of the earth ellipsoid is a level potential surface of constant amount C_0, where (from Equation 76)

$$C_0 = \frac{GM}{\bar{a}}\left[\frac{\bar{a}}{R} + \frac{\alpha}{2}\left(\frac{R}{\bar{a}}\right)^2\cos^2 L - \frac{J_{20}}{2}\left(\frac{\bar{a}}{R}\right)^3\left(3\sin^2 L - 1\right) - 3J_{22}\left(\frac{\bar{a}}{R}\right)^3\cos^2 L\cos 2\phi\right] . \tag{78}$$

Solving for R from Equation 72 will give

$$R = \bar{a}\left[1 + \left(-e + \frac{e^2}{4}\right)\cos^2 L\cos^2\phi + \left(e + \frac{5e^2}{4}\right)\cos^2 L\sin^2\phi + \left(2f + 3f^2\right)\sin^2 L\right]^{-1/2} ,$$

which can be simplified by neglecting all terms in e and f above first order. This simplification leads to

$$R = \bar{a}\left[1 - e\cos^2 L\cos 2\phi + 2f\sin^2 L\right]^{-1/2} . \tag{79}$$

The following expressions are then obtained by an expansion of Equation 79 (to first order in f and e):

$$R = \bar{a}\left(1 + \frac{e}{2}\cos^2 L\cos 2\phi - f\sin^2 L\right) , \tag{79a}$$

$$R^{-1} = (\bar{a})^{-1}\left(1 - \frac{e}{2}\cos^2 L\cos 2\phi + f\sin^2 L\right) , \tag{80}$$

$$R^2 = (\bar{a})^2\left(1 + e\cos^2 L\cos 2\phi - 2f\sin^2 L\right) , \tag{81}$$

and

$$R^3 = (\bar{a})^3 \left(1 + \frac{3e}{2} \cos^2 L \cos 2\phi - 3f \sin^2 L\right) . \tag{82}$$

When Equations 80 through 82 are substituted into Equation 78 and the second-order product terms in J_{20}, J_{22}, e, f and α are ignored, Equation 78 takes the form

$$C_0 = \frac{GM}{\bar{a}} \left[1 - \frac{e}{2} \cos^2 L \cos 2\phi + f \sin^2 L + \frac{\alpha}{2} \left(1 - \sin^2 L\right) - \frac{J_{20}}{2} \left(3 \sin^2 L - 1\right) - 3J_{22} \cos^2 L \cos 2\phi\right]. \tag{83}$$

In order for Equation 83 to be an identity for all L and ϕ on the surface of the earth ellipsoid, the following conditions must be satisfied:

$$C_0 \equiv \frac{GM}{\bar{a}} \left(1 + \frac{\alpha}{2} + \frac{J_{20}}{2}\right) , \tag{84}$$

$$-\frac{e}{2} - 3J_{22} \equiv 0 , \tag{85}$$

$$f - \frac{\alpha}{2} - \frac{3}{2} J_{20} \equiv 0 . \tag{86}$$

The existence of the geometric-potential compatibility Equations 84-86 establishes approximation of the surface equipotential of second order gravity and earth rotation as a triaxial ellipsoid. From Equation 85 it is seen that

$$J_{22} = - e/6 , \tag{87}$$

and from Equation 86 the gravity potential constant J_{20} is identified as

$$J_{20} = \frac{2f}{3} - \frac{\alpha}{3} . \tag{88}$$

Expressions 87 and 88 give approximate physical interpretation to the harmonic coefficients in the potential of Equation 56 as functions of the small parameters of the model earth triaxial ellipsoid and its rotation.

This now allows us to describe the geometry of the earth model and identify axes x_1 and x_2 from satellite and surface derived values of J_{20} and J_{22}.

According to Kozai (Reference 7) and Wagner (References 8 and 11) the best values of J_{20}, J_{22}, GM and $\bar{a}$ are

$$J_{20} = 1.08248 \times 10^{-3} \text{ (Kozai)} , \tag{89}$$

$$J_{22} = - 1.9 \times 10^{-6} \text{ (Wagner)} . \tag{90}$$

From O'Keefe, Eckels and Squires cited in Reference 5, the following constants are obtained:

$$GM = 3.98603 \times 10^{5} \text{ km}^3/\text{sec}^2 \text{ (Kozai)}$$

$$\bar{a} = 6.378165 \times 10^{3} \text{ km} \tag{91}$$

$$\omega = .7292115 \times 10^{-4} \text{ rad/sec (from Reference 9)} \tag{92}$$

Therefore, from Equations 91 and 92 in Equation 77

$$\alpha = 3.46165 \times 10^{-3} . \tag{92a}$$

Solving for the oblateness (polar flattening) of the ellipsoid model from Equation 86 with the constants of Equations 89 and 92a gives

$$f = 1.5 \times 1.08248 \times 10^{-3} + .5 \times 3.46165 \times 10^{-3} = 1/298.24 . \tag{93}$$

Similarly, a solution for the equatorial ellipticity from Equation 85 with J_{22} given in Equation 90 will give

$$e = 11.4 \times 10^{-6} . \tag{94}$$

The difference in major and minor radii of the elliptical equator of the model triaxial earth can now be found. From Equations 59 and 60

$$a(1 - e) - b = 0 , \tag{95a}$$

$$a + b = 2\bar{a} . \tag{95b}$$

Solving Equations 95a and 95b for a and b will show that

$$a = 2\bar{a}/(2 - e) ,$$

$$b = \frac{2\bar{a}(1 - e)}{2 - e} . \tag{96}$$

Therefore, from Equation 96

$$a - b = \frac{2\bar{a}\, e}{2 - e} = \frac{\bar{a}e}{1 - e/2} \doteq \bar{a}e . \tag{97}$$

With the results of Equations 91 and 94 in Equation 97 the difference in major and minor radii of the elliptical equator is calculated as 235 feet.

The order of the principal axes of inertia x_1, x_2 and x_3 is established by the signs of J_{20} and J_{22} in addition to their relative magnitudes. Since J_{22} is negative, from Equation 58 it can be seen that $B > A$. Also from Equation 57, since J_{20} is positive, $2C > (A + B)$. Therefore since $B > A$, $2C > 2A$, or $C > A$. The axis of minimum moment of inertia, x_1, is thus established. On the earth ellipsoid (Figure 3) this is the major axis of the equatorial ellipse. (Note also for $J'_{22} < 0$, $e > 0$ from Equation 85 and therefore $a > b$ from Equation 59.) To establish the probable order of principal axes x_2 and x_3, consider the earth ellipsoid as homogeneous. Then (from Reference 6, p. 292)

$$B/C = \frac{\left(c^2/a^2\right) + 1}{\left(b^2/a^2\right) + 1} ;$$

so that if we can show $b/c > 1$, then $C > B$. But from Equation 60 $c = \bar{a}(1 - f)$. This result with that from Equation 96 will give

$$\frac{b}{c} = \frac{1 - e}{\left(1 - \frac{e}{2}\right)(1 - f)} \doteq 1 + f - \frac{e}{2} > 1 ,$$

(from Equations 93 and 94). This final result establishes the absolute order of the moments of inertia (for a homogeneous ellipsoid) as

$$A < B < C , \tag{98}$$

and for the actual earth, as least $A < B, C$.

VII. THE LEVEL SURFACES OF THE GRAVITY POTENTIAL OF THE EARTH TO FOURTH ORDER

The form of the potential function in Equation 56 (and the magnitude of the second-order coefficients) suggests the interpretation that the earth attracts principally as a sphere (homogeneous in spherical shells). The higher order terms may be interpreted as arising from small mass deviations in the principal spherical model. At the surface these deviations are reflected as slight corrugations in latitude, longitude and radius above and below the surface of the average earth sphere. In particular, the previous development has shown how the principal term together with the J_{20} and J_{22} harmonics and the centrifugal potential describe an earth with an equipotential surface as a triaxial ellipsoid only very slightly deviating from a spherical shape. In the following discussion the surface deviations from an average earth sphere of radius $\bar{a}$ (chosen for convenience) due to the higher harmonics will be considered. The deviations will be such that they give an equipotential surface for the particular harmonic. The actual equipotential surface of the earth is then composed of the higher harmonic deviations superimposed on the second order triaxial elliposid described in the previous section. Higher harmonics to the fourth order will be considered.

From Equation 42, since $\phi - \phi_{nm} \approx \lambda - \lambda_{nm}$, $F_{10} = F_{11} \approx 0$ and $F_{21} \doteq 0$ as discussed in Sections V and IX, the geopotential through the J_{44} term with respect to geocentric-geographic coordinates

r, L, λ measured east from Greenwich (Figures 2 and 4) is:

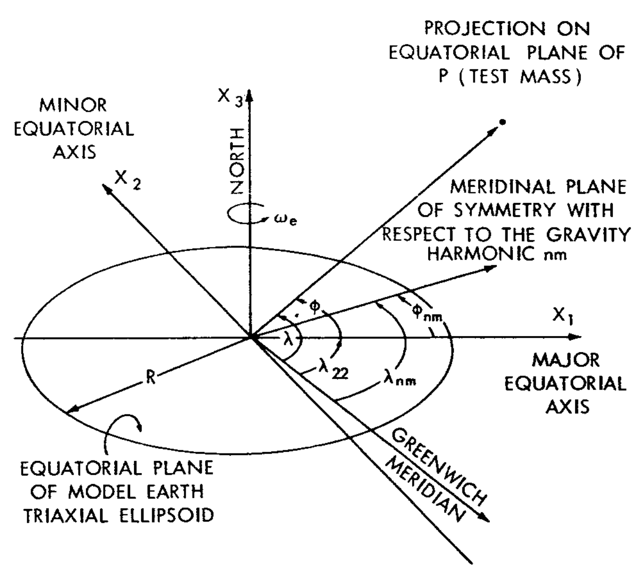
$$
\begin{aligned}
V_e \text{ (geocentric-geographical)} = \frac{GM}{r}\Big[1 &- (\bar{a}/r)^2 J_{20} P_2^0 (\sin L) - (\bar{a}/r)^2 J_{22} P_2^2 (\sin L) \cos 2(\lambda - \lambda_{22}) \\
&- (\bar{a}/r)^3 J_{30} P_3^0 (\sin L) - (\bar{a}/r)^3 J_{31} P \tfrac{1}{3} (\sin L) \cos (\lambda - \lambda_{31}) \\
&- (\bar{a}/r)^3 J_{32} P_3^2 (\sin L) \cos 2(\lambda - \lambda_{32}) - (\bar{a}/r)^3 J_{33} P_3^3 (\sin 1) \cos 3(\lambda - \lambda_{33}) \\
&- (\bar{a}/r)^4 J_{40} P_4^0 (\sin L) - (\bar{a}/r)^4 J_{41} P \tfrac{1}{4} (\sin L) \cos (\lambda - \lambda_{41}) \\
&- (\bar{a}/r)^4 J_{42} P_4^2 (\sin L) \cos 2(\lambda - \lambda_{42}) - (\bar{a}/r)^4 J_{43} P_4^3 (\sin L) \cos 3(\lambda - \lambda_{43}) \\
&- (\bar{a}/r)^4 J_{44} P_4^4 (\sin L) \cos 4(\lambda - \lambda_{44})\Big] \quad , \qquad (99)
\end{aligned}
$$

where

$$
P_n^m = \left[\frac{d^m P_n (\sin L)}{d(\sin L)^m}\right] \cos^m L \quad .
$$

The λ_{nm} above represents the geographical longitude of the principal plane of longitudinal symmetry for the nm harmonic.

Equation 99 is Equation 42 with the coefficients rewritten to conform with the J_{nm} coefficients of Equation 56. The P_n^m are the H_{nm} of Equation 36 without the coefficient D_{nm}. With the selection of the reference coordinate system as explained in Section V, the $1/r^2$ term in the potential and the J_{21} term have been ignored. Depending on the actual choice of geocentric-geographical coordinates of radius and latitude used in a given application, reported coefficients for Equation 99 may vary slightly. For example if a calculation is to be run with Equation 99 in geographic coordinates referenced to the north pole and an earth center on the equatorial plane, the reported coefficients would appear to be logically in order if the model used to derive those coefficients used the same references. However, since the north pole experiences small anomalous motions which are generally not



NOTES:
$\phi - \phi_{nm} = \lambda - \lambda_{nm}$

Figure 4—Longitude relationships in the gravity potential of the earth with respect to the elliptical equator of the earth.

corrected for in the calculations, very small discrepancies may still logically exist between otherwise correct sets of coefficients. More basic still is the consideration that the uncorrected polar spin axis is not quite an axis of principal moment of inertia for the earth. For calculations using an equatorial center and latitude with respect to the spin pole, a "best" set of coefficients should include the J_{21} term (see Section IX).

In what follows the potential of Equation 99 is referred to an origin at the c.m. of the earth and the north pole spin axis, which is assumed to be a principal axis of inertia for the earth.

Table 1 gives longitude coefficients (J_{nm}, λ_{nm}, $m \neq 0$) for the expansion Equation 99 as reported by geodesists from 1915 to 1964.

The following material in this section describes how this so-called spectral representation of the field can be translated into a spatial representation of deviations on the mean earth geoid. Summing up each harmonic spatial deviation of the geoid finally gives the actual sea level surface described by that particular set of harmonics. Figures 16 through 23 give the actual sea level surfaces for some of the more recent longitude geoids in Table 1. The zonal harmonics (J_{20}, J_{30}, J_{40}) have not been reported in this table for each investigation since they do not influence deviations of the geoid along any given latitude circle. Figures 16 through 23 show that the full surface equipotential contours are sensitive to these harmonics however. The reference geoid in each case has a flattening of 1/298.2 which includes the deviatory effect of the centrifugal potential and the oblate gravity term J_{20} (Section VI). Though not listed, all of these recent geoids (Figures 16 through 23) report J_{20}, J_{30} and J_{40} values little different from the set given by Y. Kozai in Table 1 (also Reference 7):

$$J_{20} = 1.08248 \times 10^{-3}$$

$$J_{30} = -2.6 \times 10^{-6}$$

$$J_{40} = -1.84 \times 10^{-6} .$$

A truer comparison of the geoids in Table 1 can be gained through the integrated spatial representation such as Figures 16 through 23. This is because it is the full field at various locations that is actually measured in any investigation. This representation illustrates the fact that relatively minor spatial differences can give rise to large spectral differences term-by-term. It also points up the necessity of considering the reported spectral coefficients as a set in any application, at least for the relatively ill-determined longitude ones. Considered as a set, or spatially, it will be seen that most of the recent geoids agree rather well in overall features. In theory the spectral effects are well separable (as in practice they have proved to be for the zonal terms). In the longitude reductions to date, the data has been too lacking in quantity and quality to allow meaningful separation except perhaps by order of magnitude. The one exception to this state of affairs is in regard to J_{22} (see Reference 11, for example). Nevertheless, on the basis of the spatial representations of Figures 16 through 23 derived from different kinds of gravity data, it is possible to delineate a number of large area features over the earth (sensitive to the longitude harmonics) which appear in

Table 1

Longitude Coefficients in the Gravity Potential of the Earth $\left\{ V_E = \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[1 - \left(R_0/r \right)^n P_n^m (\sin \phi) J_{nm} \cos m \left(\lambda - \lambda_{nm} \right) \right] \right\}^1$, as Reported 1915-1964[2].

Longitude Geoid[7]	J_{22}	λ_{22}	J_{31}	λ_{31}	J_{32}	λ_{32}	J_{33}	λ_{33}	J_{41}	λ_{41}	J_{42}	λ_{42}	J_{43}	λ_{43}	J_{44}	λ_{44}
(1) Wagner (1964)[3]	-1.7×10^{-6}	-19.0°														
(2) Kaula-Combined (1964)[6]	-1.51	-15.5	-1.51×10^{-6}	0.0°	$-.102 \times 10^{-6}$	0.0°	$-.149 \times 10^{-6}$	22.8°	$-.465 \times 10^{-6}$	-136.0°	$-.163 \times 10^{-6}$	37.0°	$-.061 \times 10^{-6}$	- 1.9°	$-.0053 \times 10^{-6}$	35.8°
(3) Izsak (1964)[4]	-1.00	-17.0	- .934	-15.5	-.116	19.0	- .173	38.0	- .949	-146.0	-.074	47.5	-.024	- 3.9	-.0206	25.3
(4) Kaula (1964)[4]	-1.77	-18.2	-2.12	- 5.4	-.379	10.5	- .105	23.1	- .263	-239.0	-.117	42.3	-.0473	15.0	-.0104	14.5
(5) Anderle and Oester-winter (1963)[3]	-2.09	-14.1							- .773	-140.6	-.287	34.7	-.163	- 4.3		
(6) Guier (1963)[3]	-1.80	-10.4	-1.77	6.3	-.286	- 2.6	- .204	24.1	- .73	-141.0	-.273	38.6	-.0791	- .7	-.0102	35.0
(7) Kaula (Sept. 1963)[4]	-1.51	-18.1	-1.65	5.3	-.144	46.4	- .145	15.8	- .471	-228.0	-.078	44.2	-.0265	22.6	-.0038	23.3
(8) Izsak (July 1963)[4]	-1.05	-11.2	-1.1	3.2	-.20	-21.8	- .14	20.0	- .43	-132.1	-.13	37.0	-.026	11.5	-.019	14.8
(9) Kaula (May 1963)[4]	-1.4	-21.5	-1.6	- 1.9	-.15	35.8	- .156	18.5	- .53	-233.7	-.12	44.5	-.019	10.7	-.0038	23.3
(10) Cohen (May 1963)[3]	-2.08	-14.1							- .775	-159.0	-.288	34.6	-.162	- 4.3		
(11) Kaula (Jan. 1963)[4]	-1.62	-21.4	-1.81	- 3.57	-.145	6.6	- .112	37.6	- .479	-245.5	-.072	47.7	-.0988	5.9	-.0132	28.4
(12) Uotila (1962)[5]	-1.52	-36.5	- .685	-81.0	-.409	- 5.2	- .398	19.5	- .238	-127.0	-.211	14.6	-.082	- 9.3	-.0142	- 2.6
(13) Kozai (Oct. 1962)[4]	-1.2	-26.4	-1.9	4.6	-.14	-16.8	- .10	42.6	- .52	-122.5	-.062	65.2	-.035	0.5	-.031	14.9
(14) Newton (April 1962)[3]	-2.15	-10.9							-2.53	-189.1						
(15) Newton (Jan. 1962)[3]	-4.16	-11.0														
(16) Kozai (June 1961)[4]	-2.32	-37.5	-3.21	22.0	-.41	31.0	-1.91	51.3	- .262	-196.5	-.168	54.0	-.044	-13.0	-.054	50.3
(17) Kaula (June 1961)[6]	- .55	-13.3	-1.19	20.6	-.33	- .9	- .21	22.6	- .617	-166.0	-.14	21.1	-.031	- .5	-.008	26.4
(18) Izsak (Jan. 1961)[4]	-5.35	-33.2														
(19) Kaula (1961)[5]	-1.68	-38.5							-1.15	- 13.0						
(20) Krassowski (1961?)	-5.53	15.0														
(21) Kaula (1959)[5]	- .62	-20.9	- .98	55.4	-.11	13.3	- .19	14.3	- .46	-132.3	-.081	48.6	-.01	-30.0	-.02	22.5
(22) Jeffreys (1959)[5]	-4.17	0.0														
(23) Uotila (1957)[5]	-3.5	- 6.0														
(24) Zhongolovitch (1957)[5]	-5.95	- 7.7	-2.21	-25.7	-.628	-26.4	- .54	13.0	- .78	-149.1	-.080	45.0	-.051	- 3.8	-.0224	15.9
(25) Subbotin (1949)[5]	-5.5															
(26) Niskanen (1945)[5]	-7.67	- 4.0														
(27) Jeffreys (1942)[5]	-4.1	0.0	-2.1	0.0	-.66	0.0	- .24	33.3								
(28) Heiskanen (1928)[5]	-6.34	0.0														
(29) Heiskanen (1924)[5]	-9.0	18.0														
(30) Helmert (1915)[5]	-6.0	-17.0														

[1] r is the radial distance of the field point to the center of mass of the earth, μ the earth's Gaussian gravity constant $\doteq 3.9860 \times 10^{20}$ cm^3/sec^2, R_0 the mean equatorial radius of the earth $\doteq$ 6378.2 km. ϕ is the geocentric latitude of the field point. λ is the geographic longitude of the field point. $J_{21} \doteq 0$, since the polar axis is very nearly a principal axis of inertia for the earth. $P_n^m (\sin \phi) = \cos^m \phi \sum_{t=0}^{K} T_{nmt} \sin^{n-m-2t} \phi$, where K is the integer part of $(n-m)/2$ and $T_{nmt} = \frac{(-1)^t (2n-2t)!}{2^n t! (n-t)! (n-m-2t)!}$ (See Kaula, 1964). The longitude coefficients are those for which $m \neq 0$.

[2] The J_{nm}'s and λ_{nm}'s in this table, except in one or two instances, have been converted from the original author's set of gravity coefficients. The blanks indicate the author did not consider that particular harmonic in fitting an earth potential to the observed data. In one or two instances, noted below, the author reported tesseral coefficients to higher order than the fourth.

[3] Satellite - Doppler geoid

[4] Satellite-camera geoid.

[5] Surface-gravimetric geoid.

[6] Combined astro-geodetic, gravimetric and satellite geoid.

[7] Detailed information on references for the geoids listed below given on following page.

*Zonal Coefficients in the Gravity Potential of the Earth
The best set is probably that of Y. Kozai (1962).**

$J_{20} = 1082.48 \times 10^{-6}$ $\quad J_{50} = -0.064 \times 10^{-6}$

$J_{30} = -2.562 \times 10^{-6}$ $\quad J_{60} = 0.39 \times 10^{-6}$ $\quad J_{80} = -0.02 \times 10^{-6}$

$J_{40} = -1.84 \times 10^{-6}$ $\quad J_{70} = -0.470 \times 10^{-6}$ $\quad J_{90} = 0.117 \times 10^{-6}$

**"Research in Space Science, Special Report #101," Smithsonian Astro. Obs., July 1962.

References for Geoids Listed in Table 1

Geoid	References
(1)	"Determination of the Ellipticity of the Earth's Equator from Observations on the Drift of the Syncom II Satellite" TN D-2759, May 1965.
(2)	Private communication from W. M. Kaula (July 1964).
(3),(4)	Private communication from W. M. Kaula (July 1964).
(5)	Private communication from W. M. Kaula (July 1964).
(6),(10),(14)	Guier, W. H., and Newton, R. R., "Non-Zonal Harmonic Coefficients of the Geopotential from Satellite Doppler Data," Silver Spring, Maryland: Johns Hopkins University Applied Physics Laboratory, TG-520, November 1963.
(7)	Kaula, W. M., "Improved Geodetic Results from Camera Observations of Satellites," J. Geophys. Res. 68(18):5183-5190, September 15, 1963.
(8)	Izsak, I. G., "Tesseral Harmonics in the Geopotential," Nature, 199(4889):137-139, July 13, 1963.
(9)	Private communication from W. M. Kaula (May 1963).
(11)	Kaula, W. M., "Tesseral Harmonics of the Gravitational Field and Geodetic Datum Shifts Derived from Camera Observations of Satellites," J. Geophys. Res. 68(2):473-484, January 15, 1963.
(12)	Private Communication from W. M. Kaula (July 1964).
(13)	Private communication from Y. Kozai (October 1962).
(14)	Private communication from Newton (April 1962).
(15)	Newton, R. R., "Ellipticity of the Equator Deduced from the Motion of Transit 4A," J. Geophys. Res. 67(1):415-416, January 1962.
(16)	Kozai, Y., "Tesseral Harmonics of the Gravitational Potential of the Earth as Derived from Satellite Motions," Astron. J. 66(7):355-358, September 1961.
(17)	Kaula, W. M., "A Geoid and World Geodetic System Based on a Combination of Gravimetric, Astrogeodetic and Satellite Data," J. Geophys. Res. 66(6):1799-1811, June 1961.
(18)	Izsak, I. G., "A Determination of the Ellipticity of the Earth's Equator from the Motion of Two Satellites," Astron. J. 66(5):226-229, June 1961.
(19)	Kaula, W. M., "Analysis of Satellite Observations for Longitudinal Variations of the Gravitational Field," In: Space Research II, (H. C. van de Hulst, C. de Jager, and A. F. Moore, eds.): 360-372, Amsterdam: North-Holland Publishing Co., 1961.
(20)	"Passive Dynamics in the Space Flight," Bureau of Naval Weapons Paper by J. D. Nicolaides and M. M. Macomber, 1962.
(21)	Kaula, W. M., "Statistical and Harmonic Analysis of Gravity," J. Geophys. Res. 64(12): 2401-2421, December 1959.
(22)	Jeffreys, H., "The Earth: its Origin, History, and Physical Constitution," 4th ed., London: Cambridge University Press, 1959: pp. 127-194.
(23),(26),(28),(29),(30)	Heiskanen, W. A., and Vening-Meinesz, F. A., "The Earth and its Gravity Field," New York: McGraw-Hill Book Co., 1958:79
(24)	Zhongolovich, I. D., "Potential of the Terrestrial Attraction," Akad. Nauk SSSR Inst. Theoret. Astron. 6(8):505-523, 1957 (In Russian).
(25)	Subbotin, M. F., Kurs Nebesnoi Mekhaniki, Vol. 3, 2d ed., Moscow: Gostekhizdat, 1949.
(27)	In: Monthly Notices Royal Astronom. Soc. Geophys. Suppl., 5(55):1942.

common. One or more of these common longitude-sensitive undulations of the geoid can be said to exist with the same certainty as the mass deviations responsible for the so called "pear-shaped earth."

Table 2 lists ten such common features seen in the spatial representations of Figures 16 through 23. It is noted that only two appear to be absolute in the sense of standing out strongly on all the geoids (see Table 2A). The reader is welcome to try his own eye for comparative land-scaping in these figures. Perhaps the feature called "the middle American low" ought to be combined with "the central Pacific and central Atlantic low" features to distinguish a more common strong tendency of these geoids. The same might be said of "the Afro-European" and "the Afro-Antartic" highs.

Table 2

Longitude Geoid Features (see Figures 16–23).

Geoid (Figures 16–23) (References are in Table 1 except as noted.)	Eastern Hemisphere					Western Hemisphere				
	Afro-Antartic High	Afro-European High	Indian Ocean Low	Micronesian High	Austro-Antartic Low	Alaskan High	Central Pacific Low	South American High	Middle American Low	Central Atlantic Low
Izsak (1964)*	✓	?	✓	✓	✓	?	✓	✓	?	✓
Guier (1963)	✓	✓	✓	✓	✓	?		?	✓	
Kaula (Sept. 1963)	✓	✓	✓	✓	✓		?	?	?	
Izsak (1963)	✓	✓	✓	✓	✓		?	✓	?	?
Anderle and Oester-winter (1963)	✓	✓	✓	✓	?	✓	?	✓	✓	
Kozai (1962)	✓	✓	✓	✓	✓		✓	✓	?	✓
Uotila (1962)	?	✓	✓	✓	✓	?	✓	✓	✓	?
Kaula (June 1961)	✓	✓	✓	✓	✓	?	?	✓	?	✓

*See Reference 10.

Table 2A

Longitude Geoid Features in Order of Common Strength.

1. Indian Ocean Low
2. Micronesian High
3. Afro-Antartic and Afro-European High
4. Austro-Antartic Low
5. South American High
6. Central Pacific – Central American – Central Atlantic Low

The deviations giving rise to the terms of Equation 99 are described below.

a. Deviations of the Spherical Equipotential Surface Associated with the J_{20} Term of the Gravity Field of the Earth

To consider the effect on the average earth sphere (of radius about equal to $\bar{a}$) of just the J_{20} term in Equation 99, we write the potential of that sphere modified by the J_{20} deviation as

$$V = \frac{GM}{r}\left\{1 - (\bar{a}/r)^2 \frac{J_{20}}{2}\left(3\sin^2 L - 1\right)\right\} \quad . \tag{99a}$$

According to Y. Kozai in Reference 7, $J_{20} = 1082.48 \times 10^{-6}$. The deviations from the average sphere are referenced to latitude $L = \pm 35.2°$ since the deviatory part of Equation 99a goes to zero there. Thus when $L = 35.2°$ in Equation 99a, $r = \bar{a}$. The equipotential surface for Equation 99a then has a potential of constant amount $GM/\bar{a}$. The condition equation for the equipotential surface due to Equation 99a is thus:

$$\frac{GM}{\bar{a}} = \frac{GM}{r}\left\{1 - \frac{(\bar{a}/r)^2}{2} J_{20}\left(3\sin^2 L - 1\right)\right\} \quad . \tag{99b}$$

Let

$$r = \bar{a} + \Delta r_{20} \quad , \tag{99c}$$

where

$$\Delta r_{20} << \bar{a} \quad .$$

Next introduce $\bar{a} + \Delta r_{20}$ into Equation 99b and solve for Δr_{20} to get

$$\Delta r_{20} = -\bar{a}\frac{J_{20}}{2}\left(3\sin^2 L - 1\right) \quad . \tag{99d}$$

The zeros of Equation 99d are at $L = \pm 35.2°$ (see Figure 5). Extreme radial deviations due to J_{20} are attained when

$$\frac{d\Delta r_{20}}{dL} = 0 = -3\bar{a}\frac{\sin 2L}{2} \text{ , or when } L = 0° \text{ and } \pm 90° \ .$$

The extreme radial deviations at these latitudes are

$$\left(\Delta r_{20}\right)_{min} = -\bar{a}J_{20} \text{ (at the poles)}$$

$$= -6378 \times 1082 \times 10^{-6} = \underline{-6.90 \text{ km.}} = \underline{-4.28 \text{ st. miles}};$$

and

$$\left(\Delta r_{20}\right)_{max} = \frac{\bar{a} J_{20}}{2} \text{ (at the equator)}$$

$$\doteq 2.14 \text{ st. miles (Figure 5).}$$

The deviations in Figure 5 are strictly gravitational. When the surface rotational potential of the earth is considered as well, an equipotential surface with the approximate shape of a biaxial ellipsoid is the result (ignoring much smaller higher order effects) as Section VII showed.

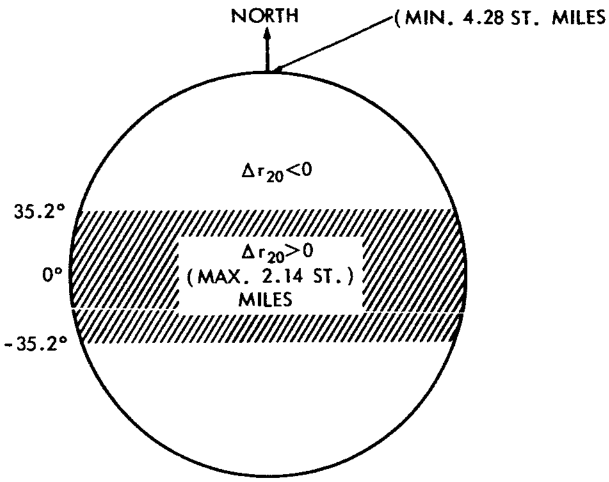


Figure 5—J_{20} deviations from an average earth sphere from a potential from Reference 7.

From Equation 60 the polar radius c of the oblate earth spheroid is given in terms of the flattening f and the mean equatorial radius $\bar{a}$ as

$$c = \bar{a}(1 - f) .$$

The difference, then, in polar and equatorial radii for the oblate earth spheroid (often used as the ellipsoid of reference for the earth or the "geoid") is

$$a - c = \bar{a}(1 + f - 1) = \bar{a}f \doteq 6378/298.2 \text{ (from Equation 93)} = 21.4 \text{ km} .$$

Thus, the maximum deviations of the oblate earth spheroid from the average earth sphere are of the order of $21.4/2 = 10.7$ km $= 6.65$ st. miles when the rotational potential is considered as well as the deviatory part of the gravitational potential. This latter result is consistent with the maximum and minimum equipotential deviations calculated above for just the gravitational part (due to J_{20}).

b. Deviations Associated with the J_{22} Term of Equation 99

(According to C. A. Wagner in Reference 11, $J_{22} = -1.9 \times 10^{-6}$, $\lambda_{22} = -20°$.) The J_{22} deviatory geopotential term in Equation 99 is

$$\left(V_e\right)_{22} = -3\left(J_{22}\right)(\bar{a}/r)^2 \cos^2 L \cos 2\left(\lambda - \lambda_{22}\right) . \tag{99e}$$

Proceeding as in the case for the deviations giving rise to the J_{20} term, we arrive at radial deviations from the surface of the average earth sphere of radius $\bar{a}$ given by

$$\Delta r_{22} = -3\bar{a} J_{22} \cos^2 L \cos 2\left(\lambda - \lambda_{22}\right) . \tag{99f}$$

The zeros of Equation 99f are at $L = \pm 90°$, and at every latitude where $\lambda = \lambda_{22} \pm 45°, \lambda_{22} \pm 135°$ (Figure 6).

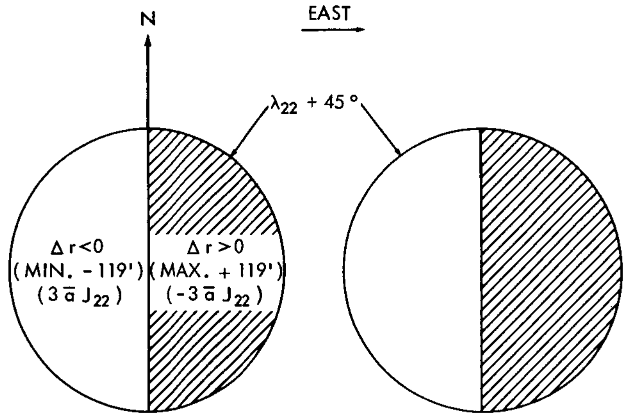


Figure 6—J_{22} deviations of the mean earth sphere for $J_{22} < 0$, from Reference 11.

Extreme deviations due to J_{22} are reached when

$$\frac{\partial \Delta r_{22}}{\partial L} = 0 = 3\bar{a} J_{22} \sin 2L \cos 2(\lambda - \lambda_{22})$$

and,

$$\frac{\partial \Delta r_{22}}{\partial \lambda} = 0 = 6\bar{a} J_{22} \cos^2 L \sin 2(\lambda - \lambda_{22})$$

simultaneously. These minimax deviations occur (other than at the poles which are inflection points where $\Delta r_{22} = 0$) at the equator at longitudes where $\lambda = \lambda_{22}, \lambda_{22} \pm 90°, \lambda_{22} + 180°$. The absolute magnitude of these extreme deviations is

$$|\Delta r_{22}(\text{max.})| = -3\bar{a} J_{22} = 3 \times 6378 \times 1.9 \times 10^{-6} = \underline{36.4 \text{ meters}}$$

$$= \underline{119 \text{ feet}} \; .$$

This corresponds to a difference in major and minor equatorial radii of 238 feet. Compare this with Equation 97 (also see Figure 6).

c. The Deviations of the Surface of the Earth Associated with the J_{30} Term of Equation 99

The effect on the average earth sphere, whose radius is about equal to $\bar{a}$, of just the J_{30} term can be calculated when the potential of that sphere is modified by the J_{30} deviation to give

$$V = \frac{GM}{r}\left[1 - (\bar{a}/r)^3 \frac{J_{30}}{2} \left(5 \sin^3 L - 3 \sin L\right)\right] \; . \tag{100}$$

(According to Y. Kozai in Reference 7, $J_{30} = -2.6 \times 10^{-6}$.) The deviations from the average sphere will be referenced to the equator since the deviatory part of Equation 100 goes to zero there. Thus, when $L = 0$, $r = \bar{a}$ in Equation 100. The equipotential surface for Equation 100 thus has constant amount $GM/\bar{a}$. The condition equation for this equipotential surface is then

$$\frac{GM}{\bar{a}} = \frac{GM}{r}\left[1 - (\bar{a}/r)^3 \frac{J_{30}}{2} \left(5 \sin^3 L - 3 \sin L\right)\right] \; . \tag{101}$$

When Equation 101 is rearranged to solve for r, it becomes

$$(r/\bar{a})^4 - (r/\bar{a})^3 = \frac{-J_{30}}{2}\left(5\sin^3 L - 3\sin L\right) . \qquad (102)$$

Let $r = \bar{a} + \Delta r_{30}$, where $\Delta r_{30} << \bar{a}$. Then, to the first order in $\Delta r/\bar{a}$, the left side of Equation 102 becomes $\Delta r/\bar{a}$ and Equation 102 is written approximately as

$$\Delta r_{30} \doteq -\frac{\bar{a} J_{30}}{2}\left(5\sin^3 L - 3\sin L\right) . \qquad (103)$$

The zeros of Equation 103 (besides L = 0) are at L = ±50.7°. Furthermore, $\Delta r_{30} \leq 0$, $0 \leq L \leq 50.7°$ and $-50.7° \leq L \leq -90°$; $\Delta r_{30} \geq 0$ everywhere else (see Figure 7).

Extreme radial deviations are attained when

$$\frac{d(\Delta r_{30})}{dL} = -\frac{\bar{a} J_{30}}{2}\left(15\sin^2 L\cos L - 3\cos L\right) = 0 . \qquad (104)$$

The zeros of Equation 104 (besides L = 0) are at L = ±26.6° and at L = ±90°.

When the value of $\bar{a}$ in Equation 91 and J_{30} as quoted above are used, Δr extreme, calculated from Equation 103, is equal to 24 feet at L = ±26.6° and 54 feet at the poles (Figure 7).

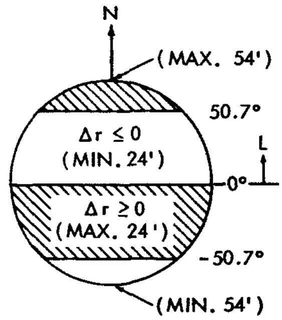


Figure 7—J_{30} deviations (Reference 7).

d. Deviations Associated with the J_{31} Term of Equation 99

The J_{31} deviatory geopotential term is

$$-J_{31}(\bar{a}/r)^3 \frac{1}{2}\left(15\sin^2 L - 3\right)\cos L\cos\left(\lambda - \lambda_{31}\right) .$$

By proceeding as in the case for the deviations giving rise to the J_{22} term, the radial deviations from the surface of the average sphere of radius $\bar{a}$ can be shown to be

$$\Delta r_{31} = -\bar{a}\frac{J_{31}}{2}\left(15\sin^2 L - 3\right)\cos L\cos\left(\lambda - \lambda_{31}\right) . \qquad (105)$$

The zeros of Equation 105 are at

L = ±90° (all longitudes)

L = ±26.6° (all longitudes)

λ = 90° + λ_{31} and 270° + λ_{31} (all latitudes) (Figure 8).

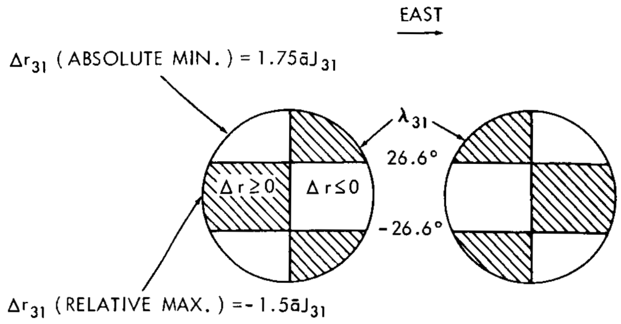


Figure 8—J_{31} deviations for $J_{31} < 0$ (Appendix A).

Extreme deviations due to J_{31} are reached when

$$\frac{\partial \Delta r_{31}}{\partial L} = \cos(\lambda - \lambda_{31})\left[-(15\sin^2 L - 3)\sin L + 30 \sin L \cos L\right] = 0 \,, \tag{106a}$$

and

$$\frac{\partial \Delta r_{31}}{\partial \lambda} = -\sin(\lambda - \lambda_{31})(15 \sin^2 L - 3)\cos L = 0 \,. \tag{106b}$$

The simultaneous solution of Equations 106a and 106b yield two sets of extreme deviation points across the average sphere. One is at the crossing of the "nodal lines" of Figure 8 where Δr is zero. These are evidently flat points of inflection for the equipotential surface due to J_{31}. The other set is at the intersection of $\lambda = \lambda_{31}$ and $180° + \lambda_{31}$ with $L = \pm 70°$ and $0°$. Four of the six extreme points of this set yield the same absolute deviation of $-1.75\,\bar{a}\,J_{31}$. The other two yield identical absolute deviations of $-3\bar{a}\,J_{31}/2$ (Figure 8).

e. Deviations Associated with the J_{32} term of Equation 99

The J_{32} deviatory geopotential term is

$$-J_{32}\,(\bar{a}/r)^3\, 15 \cos^2 L \sin L \cos 2(\lambda - \lambda_{32}) \,.$$

Deviations are calculated as in *d.* above, and

$$\Delta r_{32} = -\bar{a}\, J_{32}\, 15 \cos^2 L \sin L \cos 2(\lambda - \lambda_{31}) \tag{107}$$

with the results appearing on Figure 9 below.

f. Deviations Associated with the J_{33} Term of Equation 99

The J_{33} deviatory geopotential term is

$$-J_{33}\,(\bar{a}/r)^3\, 15 \cos^3 L \cos 3(\lambda - \lambda_{33})$$

Deviations are calculated as in *b.* above and

$$\Delta r_{33} = -\bar{a}\, J_{33}\, 15 \cos^3 L \cos 3(\lambda - \lambda_{33}) \tag{108}$$

with the results appearing in Figure 10 below.

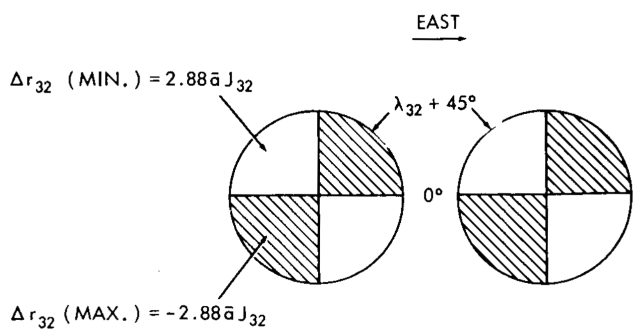


Figure 9—J_{32} deviations for $J_{32} < 0$ (Appendix A).

g. Deviations Associated with the J_{40} Term of Equation 99

The J_{40} geopotential term is

$$-J_{40}\,(\bar{a}/r)^4\left[\frac{35}{8}\sin^4 L - 30\sin^2 L + 3\right].$$

(According to Reference 7, $J_{40} = -1.84 \times 10^{-6}$.) Deviations are calculated as in *a.* above and

$$\Delta r_{40} = -\bar{a}\,J_{40}\left(\frac{35}{8}\sin^4 L - 30\sin^2 L + 3\right) \quad (109)$$

with the results appearing on Figure 11.

h. Deviations Associated with the J_{41} Term of Equation 99

The J_{41} deviatory geopotential term is

$$-J_{41}\,(\bar{a}/r)^4\left(\frac{1}{8}\right)\Big(140\sin^3 L - 60\sin L\Big)\cos L\cos\left(\lambda - \lambda_{41}\right).$$

Deviations are calculated as in *d.* above, and

$$\Delta r_{41} = -\bar{a}\,J_{41}\left(\frac{1}{8}\right)\Big(140\sin^3 L - 60\sin L\Big)\cos L\cos\left(\lambda - \lambda_{41}\right). \quad (110)$$

with the results appearing in Figure 12.

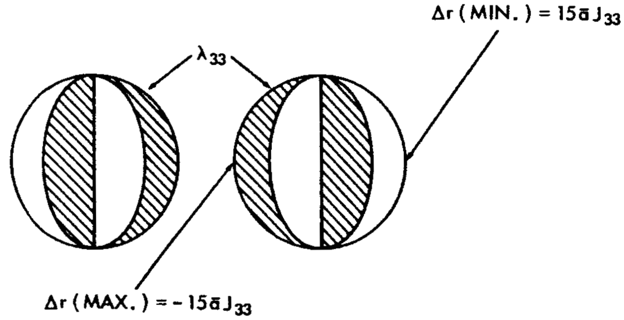


Figure 10—J_{33} deviations for $J_{33} < 0$.

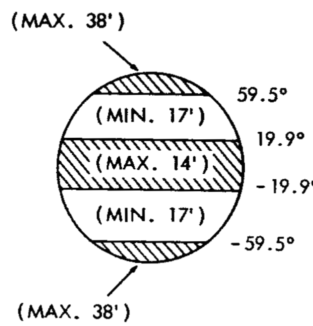


Figure 11—J_{40} deviations (Reference 7).

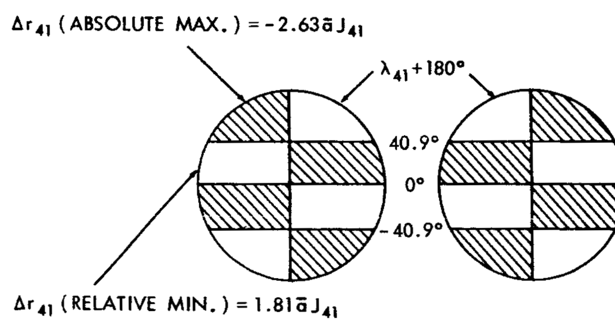


Figure 12—J_{41} deviations for $J_{41} < 0$ (Appendix A).

i. Deviations Associated with the J_{42} Term of Equation 99

The J_{42} deviatory geopotential term is

$$-J_{42}\,(\bar{a}/r)^4\left(\frac{1}{8}\right)\left(420\sin^2 L - 60\right)\cos^2 L\cos 2\left(\lambda - \lambda_{42}\right).$$

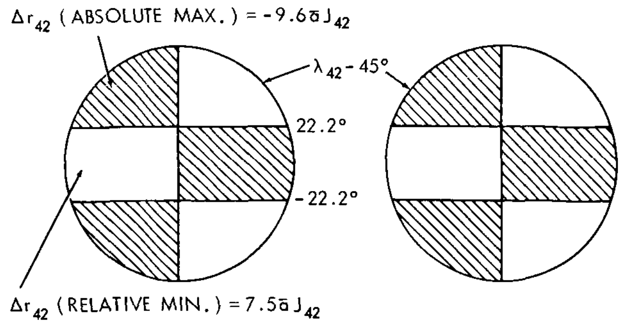


Figure 13—J_{42} deviations for $J_{42} < 0$ (Appendix A).

Deviations are calculated as in *d.* above, and

$$\Delta r_{42} = -\bar{a} J_{42}\left(\frac{1}{8}\right)\left(420 \sin^2 L - 60\right) \cos^2 L \cos 2\left(\lambda - \lambda_{42}\right) \qquad (111)$$

with the results appearing here on Figure 13.

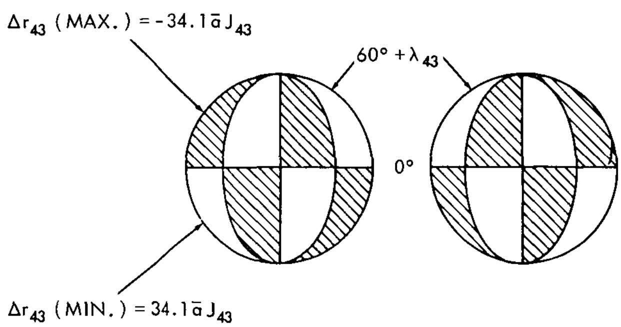


Figure 14—J_{43} deviations for $J_{43} < 0$ (Appendix A).

j. Deviations Associated with the J_{43} Term of Equation 99

The J_{43} deviatory geopotential term is

$$-J_{43}\,(\bar{a}/r)^4\left(\frac{1}{8}\right) 840 \sin L \cos^3 L \cos 3\left(\lambda - \lambda_{43}\right).$$

Deviations are calculated as in *d.* above and

$$\Delta r_{43} = -\bar{a} J_{43}\left(\frac{1}{8}\right) 840 \sin L \cos^3 L \cos 3\left(\lambda - \lambda_{43}\right) \qquad (112)$$

with the results appearing here on Figure 14.

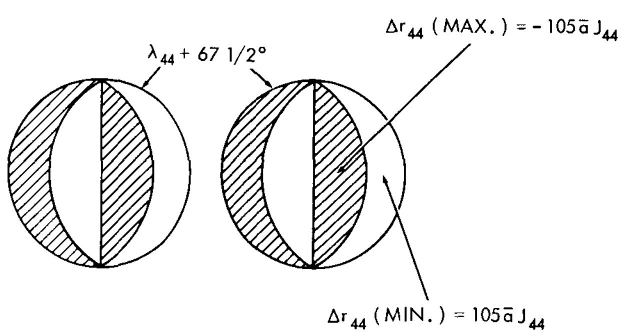


Figure 15—J_{44} deviations for $J_{44} < 0$ (Appendix A).

k. Deviations Associated with the J_{44} Term of Equation 99

The J_{44} deviatory geopotential term is

$$-J_{44}\,(\bar{a}/r)^4\left(\frac{1}{8}\right) 840 \cos^4 L \cos 4\left(\lambda - \lambda_{44}\right) \quad .$$

Deviations are calculated as in *b.* above and

$$\Delta r_{44} = -\bar{a} J_{44}\left(\frac{1}{8}\right) 840 \cos^4 L \cos 4\left(\lambda - \lambda_{44}\right) \qquad (113)$$

with the results appearing on Figure 15 above.

The complete geocentric-geographic gravity potential of Equation 99 with the zonal coefficients as reported in Reference 7 and J_{22}, λ_{22} as given in Reference 11 is given below. The longitude (λ) reference is Greenwich with positive angles to the east. The radial (r) reference is the center of mass of the model earth. The latitude (L) reference is the equatorial plane of the model earth passing through the c.m. and the north pole spin axis. Thus the spherical harmonic representation of the gravity potential of the earth through fourth order takes the form of

$$
\begin{aligned}
V_e = \frac{GM_e}{r}\Big[& 1 - \left(1082.48\times 10^{-6}\right)(\bar{a}/r)^2 \left(\tfrac{1}{2}\right)\left(3\sin^2 L - 1\right) - \left(-1.9\times 10^{-6}\right)(\bar{a}/r)^2\, 3\cos^2 L \cos 2(\lambda + 20^\circ) \\
& - \left(-2.56\times 10^{-6}\right)(\bar{a}/r)^3 \left(\tfrac{1}{2}\right)\left(5\sin^3 L - 3\sin L\right) - \left(J_{31}\right)(\bar{a}/r)^3\left(\tfrac{1}{2}\right)\cos L\left(15\sin^2 L - 3\right)\cos\left(\lambda - \lambda_{31}\right) \\
& - \left(J_{32}\right)(\bar{a}/r)^3\, 15\cos^2 L \sin L \cos 2\left(\lambda - \lambda_{32}\right) - \left(J_{33}\right)(\bar{a}/r)^3\, 15\cos^3 L\cos 3\left(\lambda - \lambda_{33}\right) \\
& - \left(-1.84\times 10^{-6}\right)(\bar{a}/r)^4\left(\tfrac{35}{8}\right)\left(\sin^4 L - 30\sin^2 L + 3\right) - \left(J_{41}\right)(\bar{a}/r)^4\left(\tfrac{1}{8}\right)\left(140\sin^3 L - 60\sin L\right)\cos\left(\lambda - \lambda_{41}\right) \\
& - \left(J_{42}\right)(\bar{a}/r)^4\left(\tfrac{1}{8}\right)\left(420\sin^2 L - 60\right)\cos^2 L\cos 2\left(\lambda - \lambda_{42}\right) - \left(J_{43}\right)(\bar{a}/r)^4\left(\tfrac{1}{8}\right) 840\sin L\cos^3 L\cos 3\left(\lambda - \lambda_{43}\right) \\
& - \left(J_{44}\right)(\bar{a}/r)^4\left(\tfrac{1}{8}\right) 840\cos^4 L\cos 4\left(\lambda - \lambda_{44}\right)\Big] , \quad (114)
\end{aligned}
$$

with $\bar{a}$ = 6378.165 km.

$$GM_e = 3.98603 \times 10^5 \text{ km}^3/\text{sec}^2 \ .$$

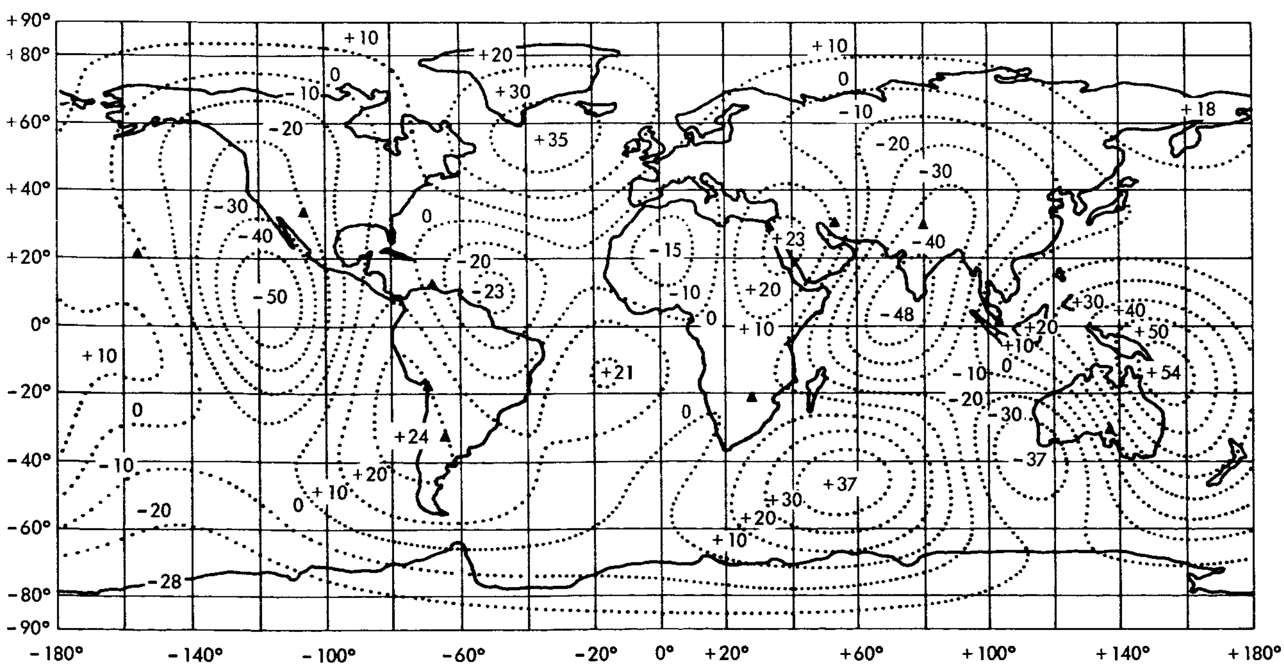


Figure 16—Satellite-Camera geoid from Izsak, 1964 (Reference 10) (Level curves of geoid heights at 10 m intervals with respect to a reference geoid of flattening 1/298.2).

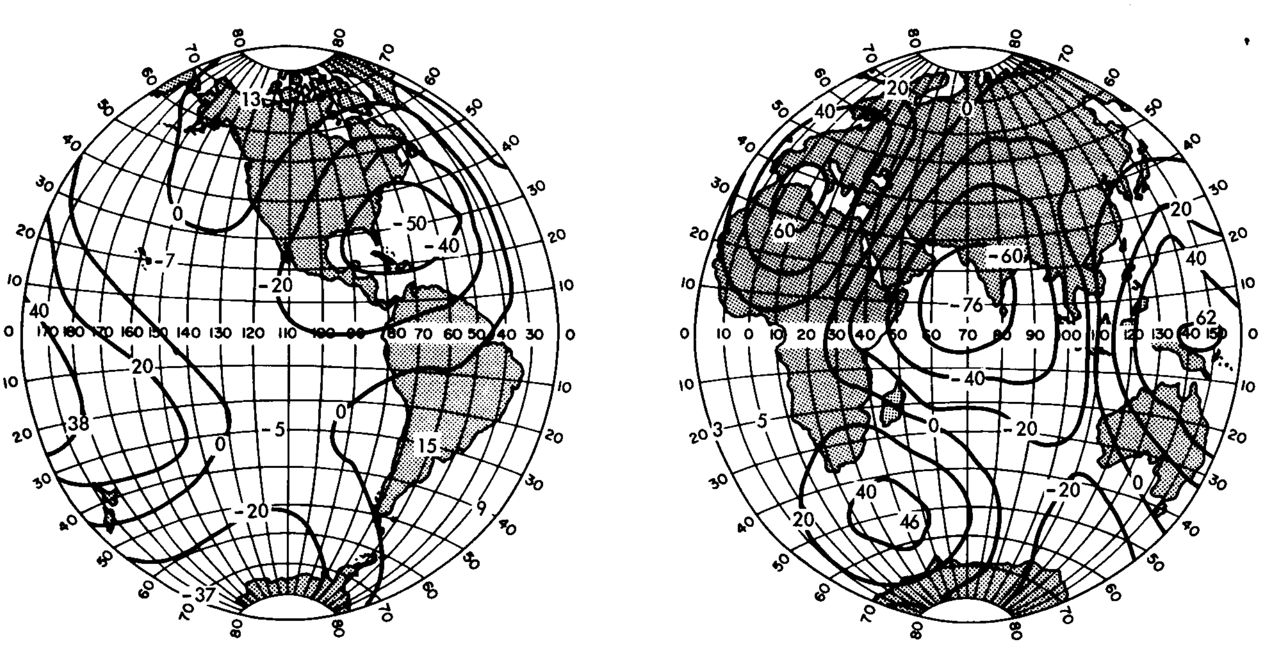


Figure 17—Satellite-Doppler geoid from Guier, 1963 (Table 1) (geoid heights in meters).

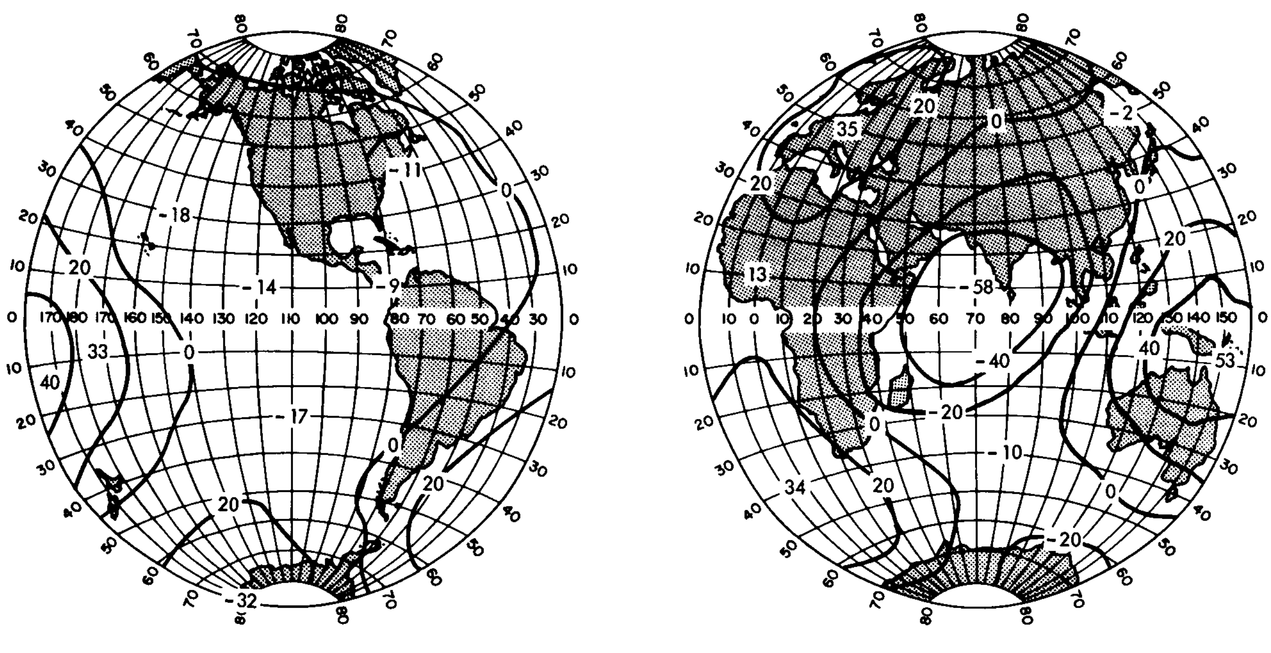


Figure 18—Satellite-Camera geoid from Kaula, 1963 (Table 1) (geoid heights in meters).

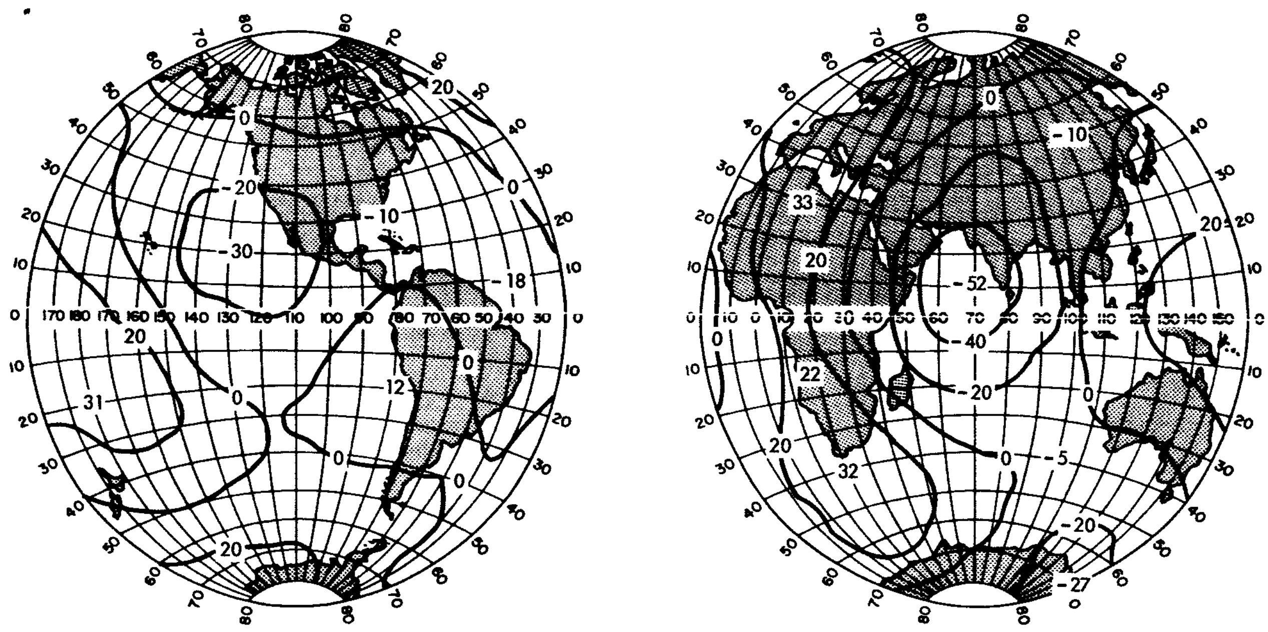


Figure 19—Satellite-Camera geoid from Izsak, 1963 (Table 1) (geoid heights in meters).

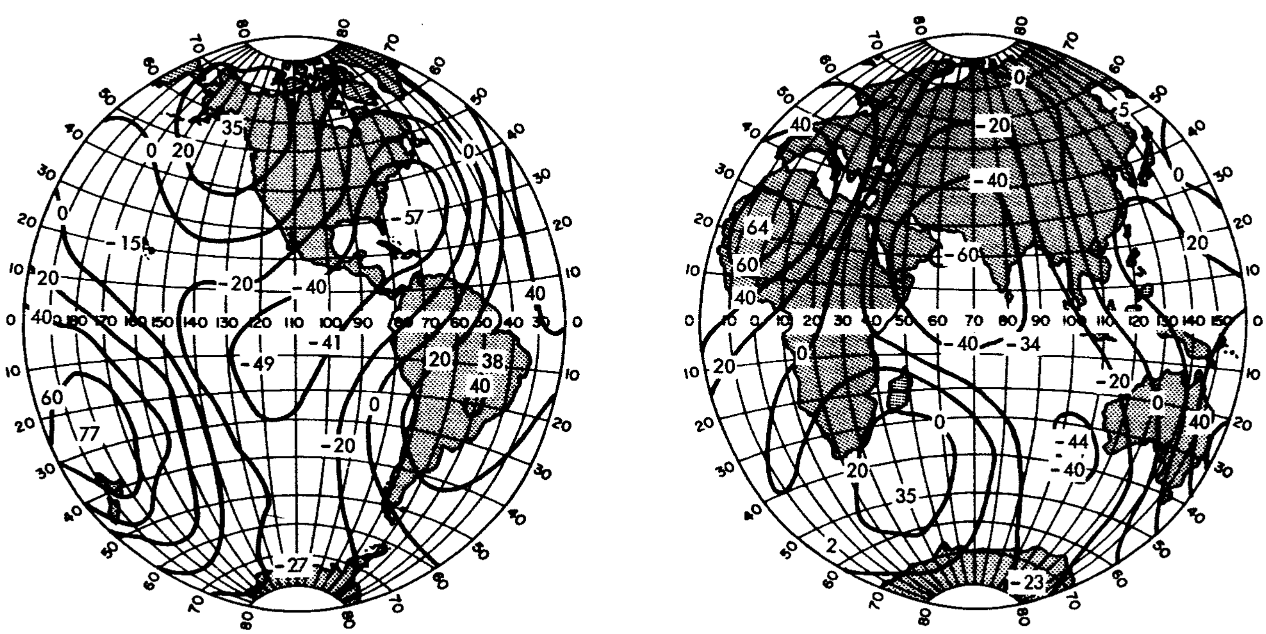


Figure 20—Satellite-Doppler geoid from Anderle and Oesterwinter, 1963 (Table 1) (geoid heights in meters).

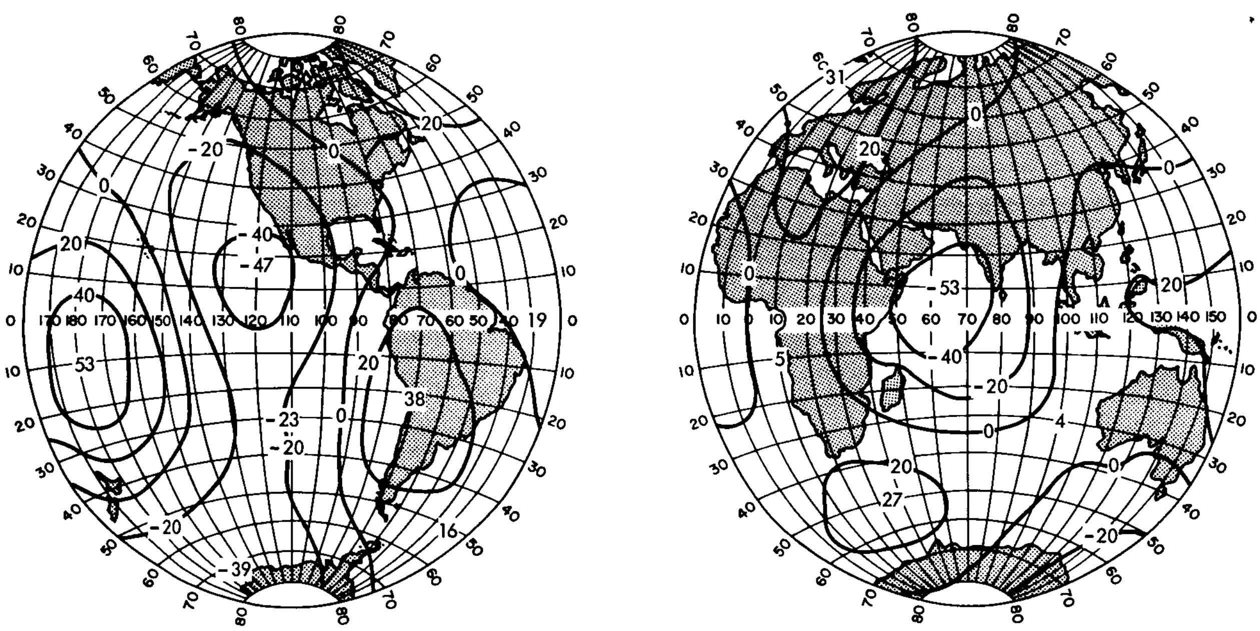


Figure 21—Satellite-Camera geoid from Kozai, 1962 (Table 1) (geoid heights in meters).

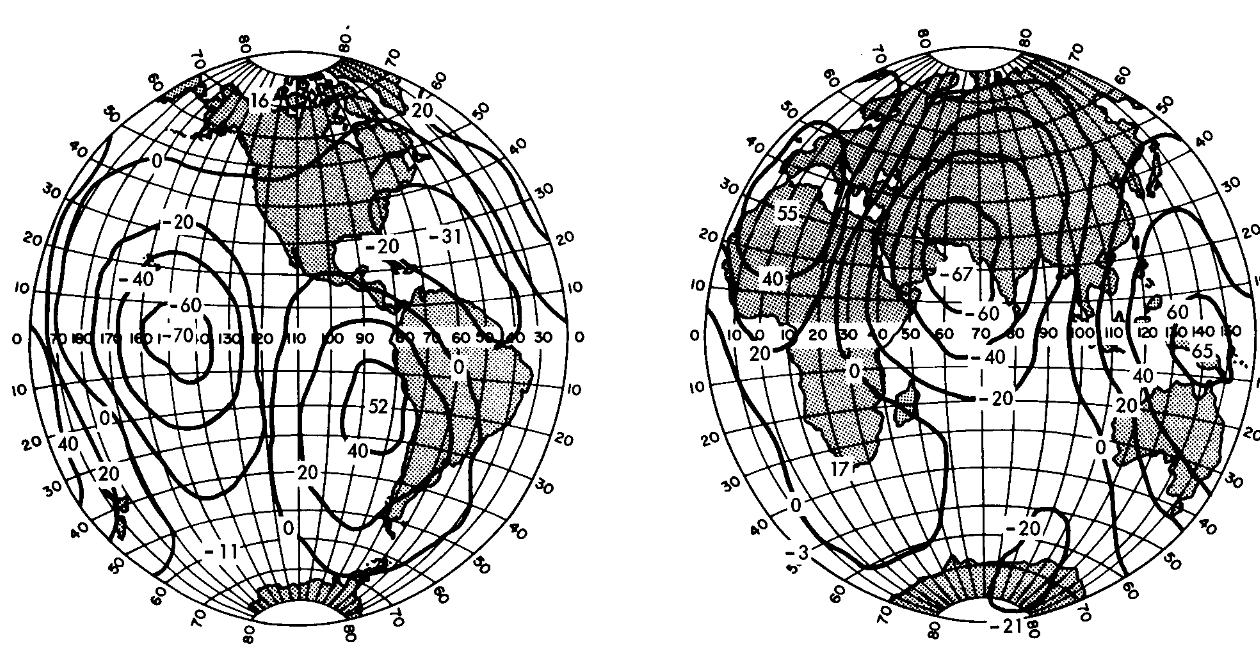


Figure 22—Gravimetric geoid from Uotila, 1962 (Table 1) (geoid heights in meters).

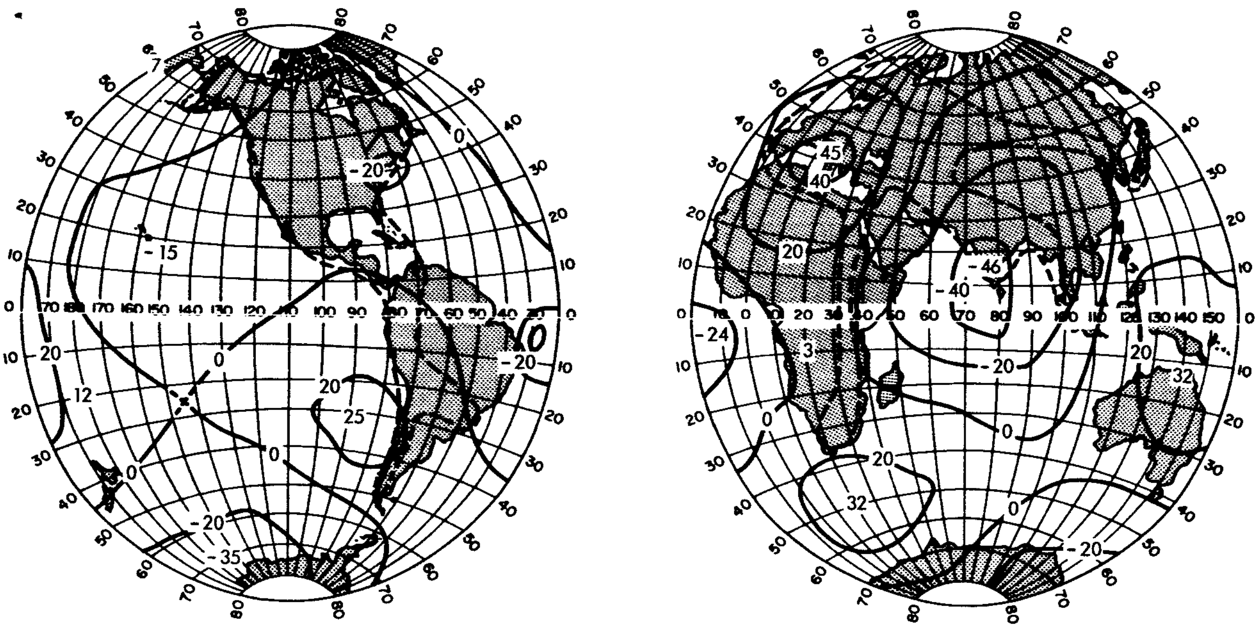


Figure 23—Astro-geodetic and gravimetric geoid from Kaula, June 1961 (Table 1) (geoid heights in meters).

VIII. THE GRAVITY FORCE FIELD OF THE EARTH THROUGH FOURTH ORDER

To display the gravity force field of the earth in geocentric coordinates referred to the earth's spin axis, we follow the general prescription that with respect to any orthogonal coordinate system

$$\vec{F}(\text{gravity}) = \nabla V_e\,(\text{gravity}) \tag{115}$$

(see Section II and References 1 and 2).

The results of taking the gradient of V_e from Equation 99 gives the force field of the earth through fourth order as

$$\vec{F} = \hat{r}F_r + \hat{\lambda}F_\lambda + \hat{L}F_L = \nabla V_e = \hat{r}\frac{\partial V_e}{\partial r} + \frac{\hat{\lambda}}{r\cos L}\frac{\partial V_e}{\partial \lambda} + \frac{\hat{L}}{r}\frac{\partial V_e}{\partial L}\ , \tag{116}$$

where

$$\begin{aligned} F_r = \frac{\mu_E}{r^2}\Bigg\{&-1 + (\bar{a}/r)^2\Big[\frac{3}{2}J_{20}\left(3\sin^2 L - 1\right) + 9J_{22}\cos^2 L\cos 2\left(\lambda - \lambda_{22}\right) \\ &+ 2(\bar{a}/r)J_{30}\left(5\sin^2 L - 3\right)(\sin L) + 6\left(R_0/r\right)J_{31}\left(5\sin^2 L - 1\right)\cos L\cos\left(\lambda - \lambda_{31}\right) \\ &+ 60(\bar{a}/r)J_{32}\cos^2 L\sin L\cos 2\left(\lambda - \lambda_{32}\right) + 60\left(R_0/r\right)J_{33}\cos^3 L\cos 3\left(\lambda - \lambda_{33}\right) \\ &+ \frac{5}{8}(\bar{a}/r)^2 J_{40}\left(35\sin^4 L - 30\sin^2 L + 3\right) + \frac{25}{2}(\bar{a}/r)^2 J_{41}\left(7\sin^2 L - 3\right)\cos L\sin L\cos\left(\lambda - \lambda_{41}\right) \\ &+ \frac{75}{2}(\bar{a}/r)^2 J_{42}\left(7\sin^2 L - 1\right)\cos^2 L\cos 2\left(\lambda - \lambda_{42}\right) + 525(\bar{a}/r)^2 J_{43}\cos^3 L\sin L\cos 3\left(\lambda - \lambda_{43}\right) \\ &+ 525(\bar{a}/r)^2 J_{44}\cos^4 L\cos 4\left(\lambda - \lambda_{44}\right)\Big]\Bigg\}\ ; \end{aligned} \tag{117}$$

$$
\begin{aligned}
F_\lambda = \frac{\mu_E}{r^2}(\bar{a}/r)^2 \Big\{ & 6J_{22}\cos L \sin 2(\lambda-\lambda_{22}) + \frac{3}{2}(\bar{a}/r) J_{31}\left[5\sin^2 L - 1\right]\sin(\lambda-\lambda_{31}) \\
& + 30(\bar{a}/r) J_{32}\cos L \sin L \sin 2(\lambda-\lambda_{32}) + 45(\bar{a}/r) J_{33}\cos^2 L \sin 3(\lambda-\lambda_{33}) \\
& + \frac{5}{2}(\bar{a}/r)^2 J_{41}\left[7\sin^2\phi - 3\right]\sin L \sin(\lambda-\lambda_{41}) + 15(\bar{a}/r)^2 J_{42}(7\sin^2 L - 1)\cos\phi \sin 2(\lambda-\lambda_{42}) \\
& + 315(\bar{a}/r)^2 J_{43}\cos^2 L \sin L \sin 3(\lambda-\lambda_{43}) \\
& + 420(\bar{a}/r)^2 J_{44}\cos^3 L \sin 4(\lambda-\lambda_{44}) \Big\} .
\end{aligned} \tag{118}
$$

$$
\begin{aligned}
F_L = \frac{\mu_E}{r^2}(\bar{a}/r)^2 \Big\{ & -3J_{20}\sin L \cos L + 6J_{22}\cos L \sin L \cos 2(\lambda-\lambda_{22}) \\
& - \frac{3}{2}(\bar{a}/r) J_{30}(5\sin^2 L - 1)\cos L + \frac{3}{2}(\bar{a}/r) J_{31}(15\sin^2 L - 11)\sin L \cos(\lambda-\lambda_{31}) \\
& + 15(\bar{a}/r) J_{32}(3\sin^2 L - 1)\cos L \cos 2(\lambda-\lambda_{32}) \\
& + 45(\bar{a}/r) J_{33}\cos^2 L \sin L \cos 3(\lambda-\lambda_{33}) - \frac{5}{2}(\bar{a}/r)^2 J_{40}(7\sin^2 L - 3)\sin L \cos L \\
& + \frac{5}{2}(\bar{a}/r)^2 J_{41}(28\sin^4 L - 27\sin^2 L + 3)\cos(\lambda-\lambda_{41}) \\
& + 30(\bar{a}/r)^2 J_{42}(7\sin^2 L - 4)\cos L \sin L \cos 2(\lambda-\lambda_{42}) \\
& + 105(\bar{a}/r)^2 J_{43}(4\sin^2 L - 1)\cos^2 L \cos 3(\lambda-\lambda_{43}) \\
& + 420(\bar{a}/r)^2 J_{44}\cos^3 L \sin L \cos 4(\lambda-\lambda_{44}) \Big\} .
\end{aligned} \tag{119}
$$

The earth gravity force field given in Equations 117 - 119 and the potential field given in Equation 99 are with respect to geocentric spherical coordinates fixed in the earth, referenced to its spin axis (the north pole) and its center of mass (Figure 24).

It is instructive to note the relative influence of the orders of earth tesseral gravity at different altitudes above the earth's surface. In general the higher one goes, the more the earth behaves as a point mass with a $1/r^2$ field. Physically, this asymptotic behaviour must occur because at increasing distances from the earth, the distances to each point in the earth from a fixed outside point become more nearly the same. The field of a single mass point in the earth is, of course, just a simple $1/r^2$ field. But in addition to this total behaviour since each order of gravity, mathematically, has a scaling factor of $(\bar{a}/r)$ with respect to the next lowest order; this means that as the distance (r) from the earth increases the lower orders of gravity play an increasingly stronger role.

This might be anticipated from the fact that the lower orders of gravity measure mass anomalies within the earth which have wider separation. At increasing distances the mass anomalies distributed with close separation, are at more nearly the same distance from the outside field point than the more widely separated anomalies of lower order.

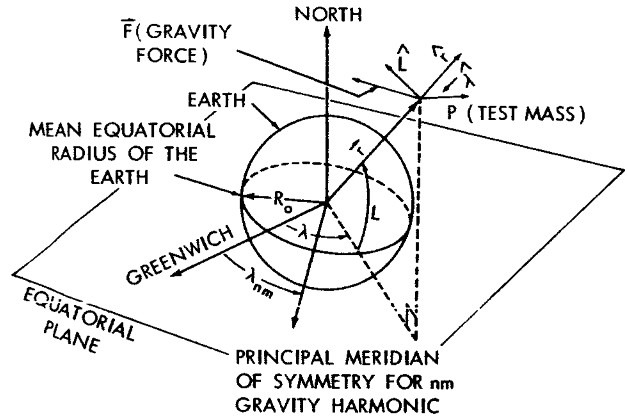


Figure 24—Earth-fixed geocentric coordinate system reference for the earth's gravity field.

Thus, these higher order gravity anomalies should have a less and less influential role in the total nonhomogeneous earth gravity field as the distance increases.

Figures 25 to 27 illustrate this phenomenon with respect to a recent geoid due to W. M. Kaula (Kaula-combined [1964] in Table 1). These graphs are a plot of the equatorial longitude force field

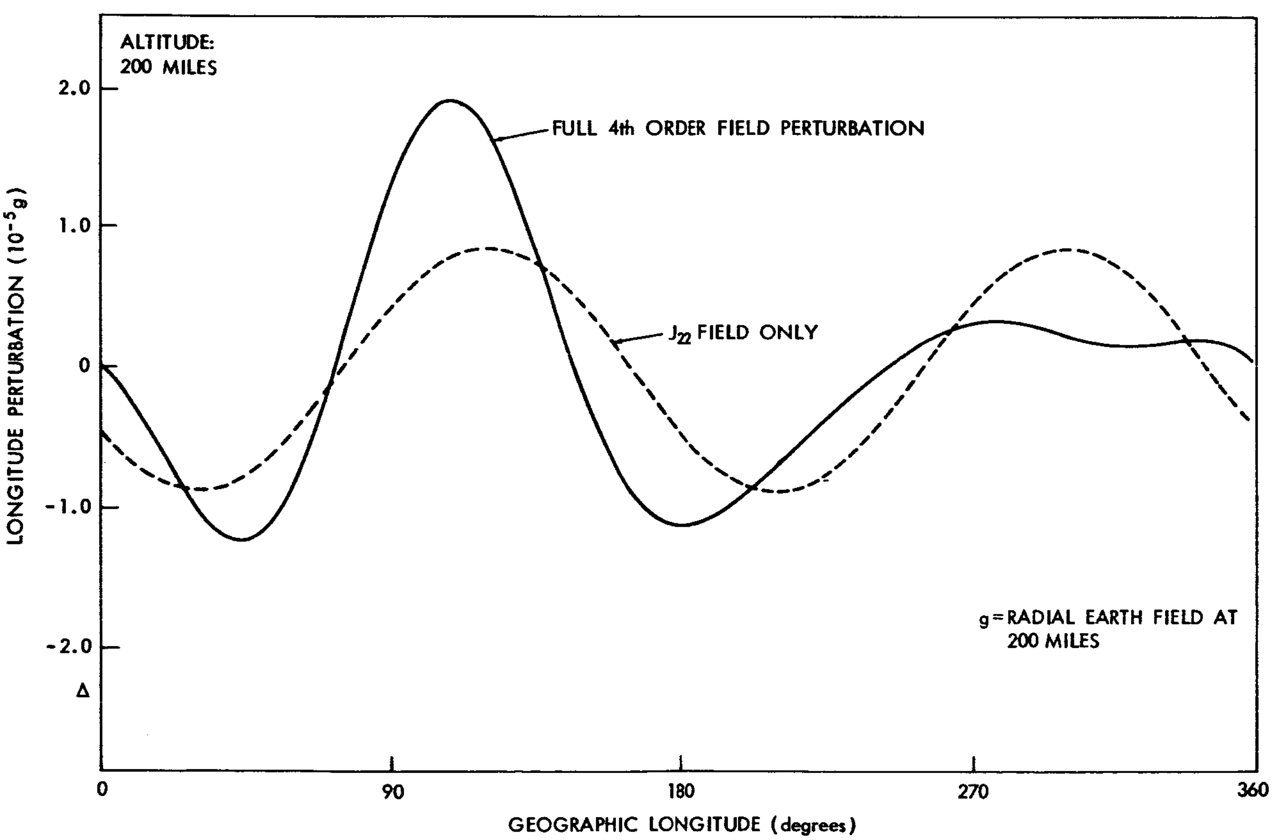


Figure 25—Equatorial longitude gravity forces at 200 miles from the composite geoid of W. M. Kaula (1964).

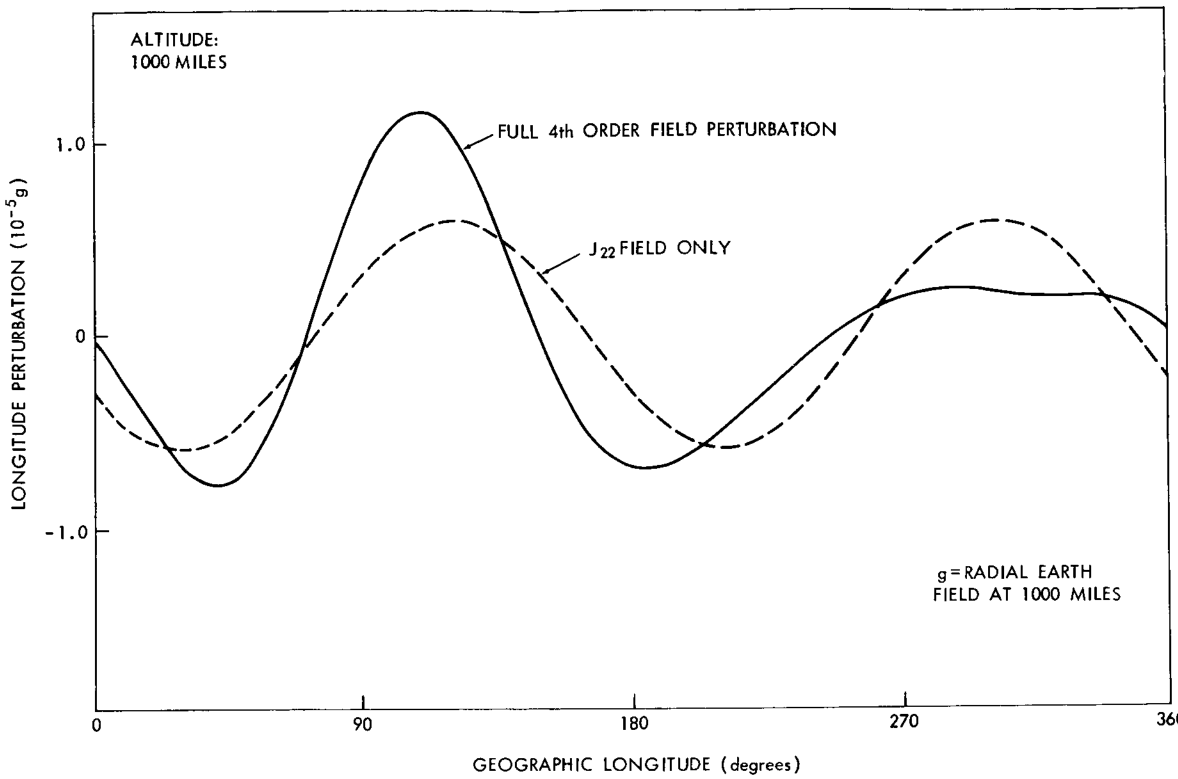


Figure 26—Equatorial longitude gravity forces at 1000 miles from the composite geoid of W. M. Kaula (1964).

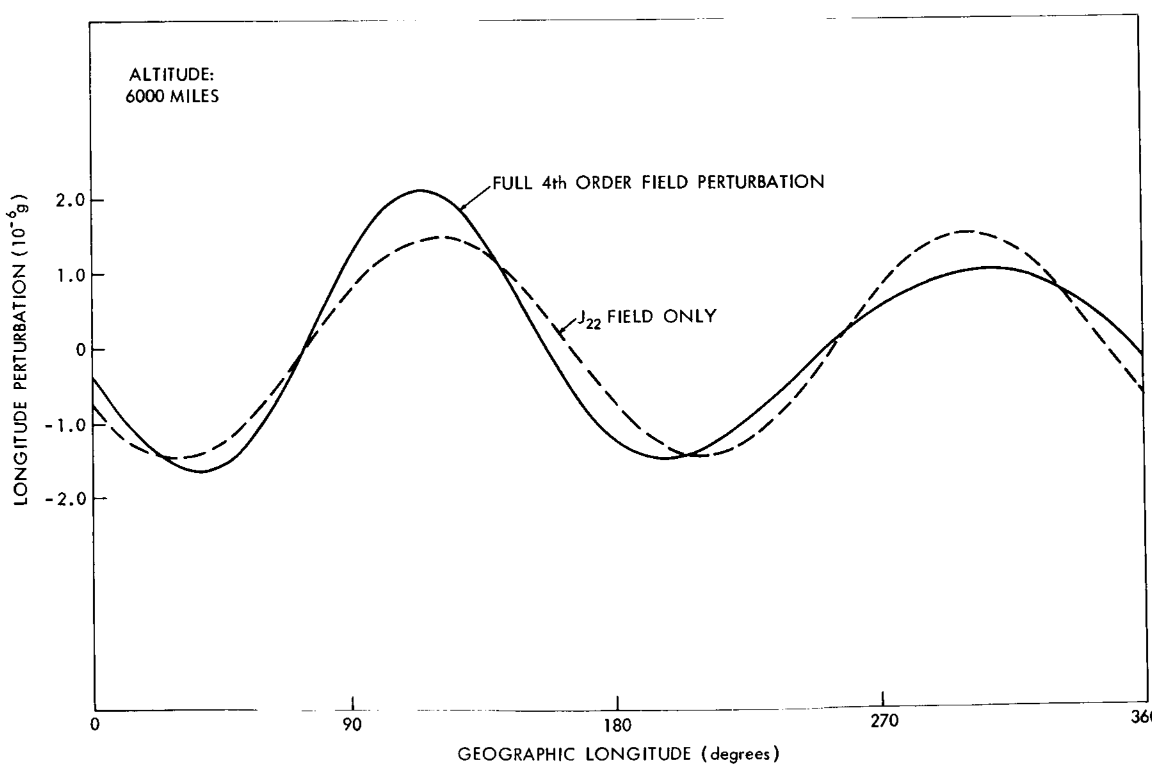


Figure 27—Equatorial longitude gravity forces at 6000 miles from the composite geoid of W. M. Kaula (1964).

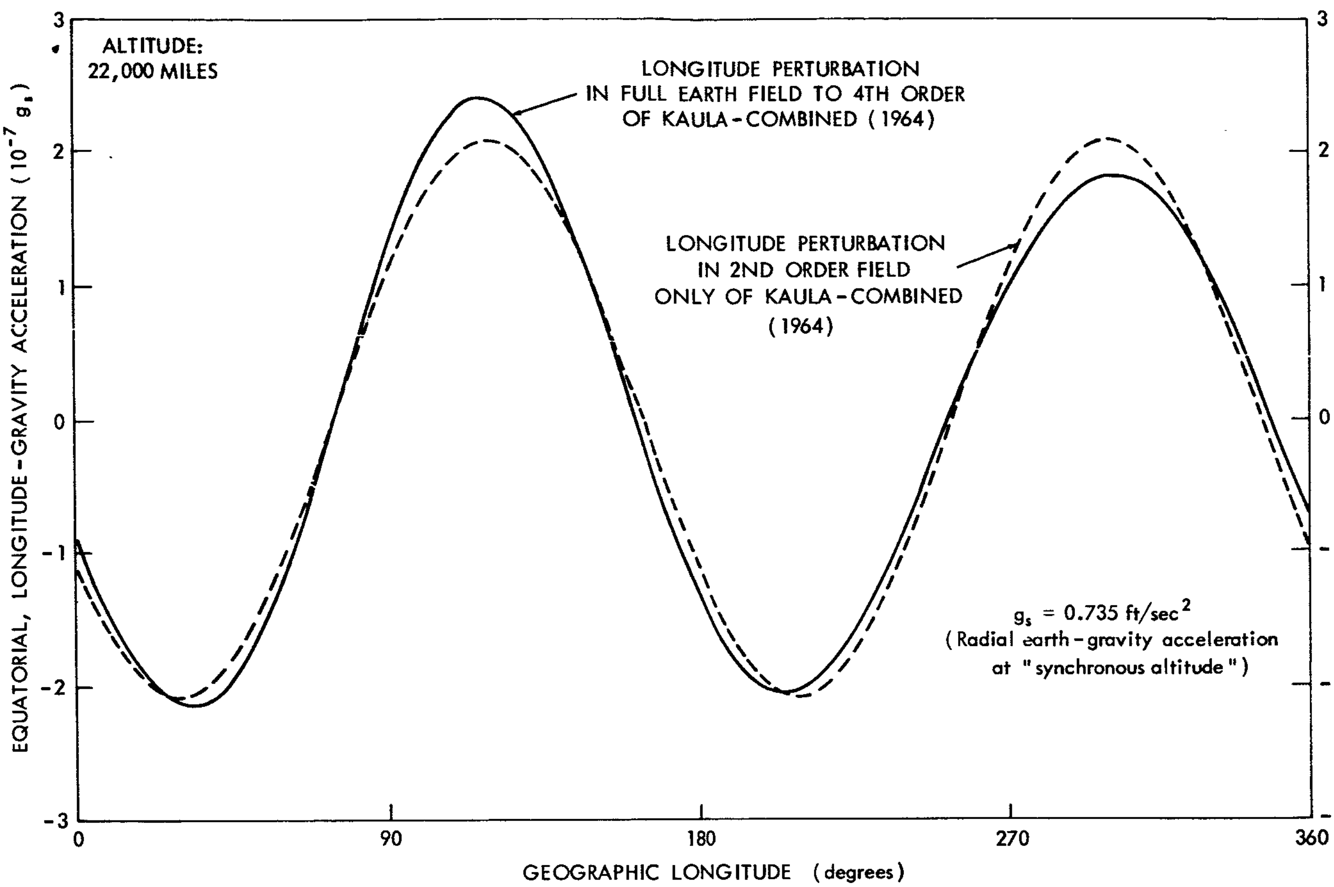


Figure 28—The earth's longitude gravity field on the equator at the synchronous altitude, according to geoid (2), (Table 1).

given by Equation 118 at various altitudes for the longitude coefficients of this geoid. They indicate, for example, that orders of longitude gravity higher than the second order can probably be safely ignored at almost all longitude locations with respect to their influence on the high altitude 24-hour satellite compared to the influence of second-order longitude gravity. On the other hand, they probably cannot be safely ignored with respect to their comparative influence on satellites orbiting below 6,000 nautical miles. Figure 28 strongly suggests that the recent determination of J_{22} from 24 hour satellite data (References 8 and 11) should be a relatively secure one.

Figure 29 illustrates the relative imprecision with which the full longitude gravity field of the earth at 24 hour altitudes was known prior to the direct measurement by its effect on the Syncom II satellite (see References 8, 11 and 15).

IX. ON THE ORDER OF MAGNITUDE OF THE EARTH GRAVITY PERTURBATION DUE TO SMALL WANDERING OF THE SPIN AND AXIS OF FIGURE OF THE EARTH

In Section VII it was pointed out that there are two consequences from the fact that the earth's spin axis is not quite a principal axis of inertia. In the first place it implies from dynamical considerations

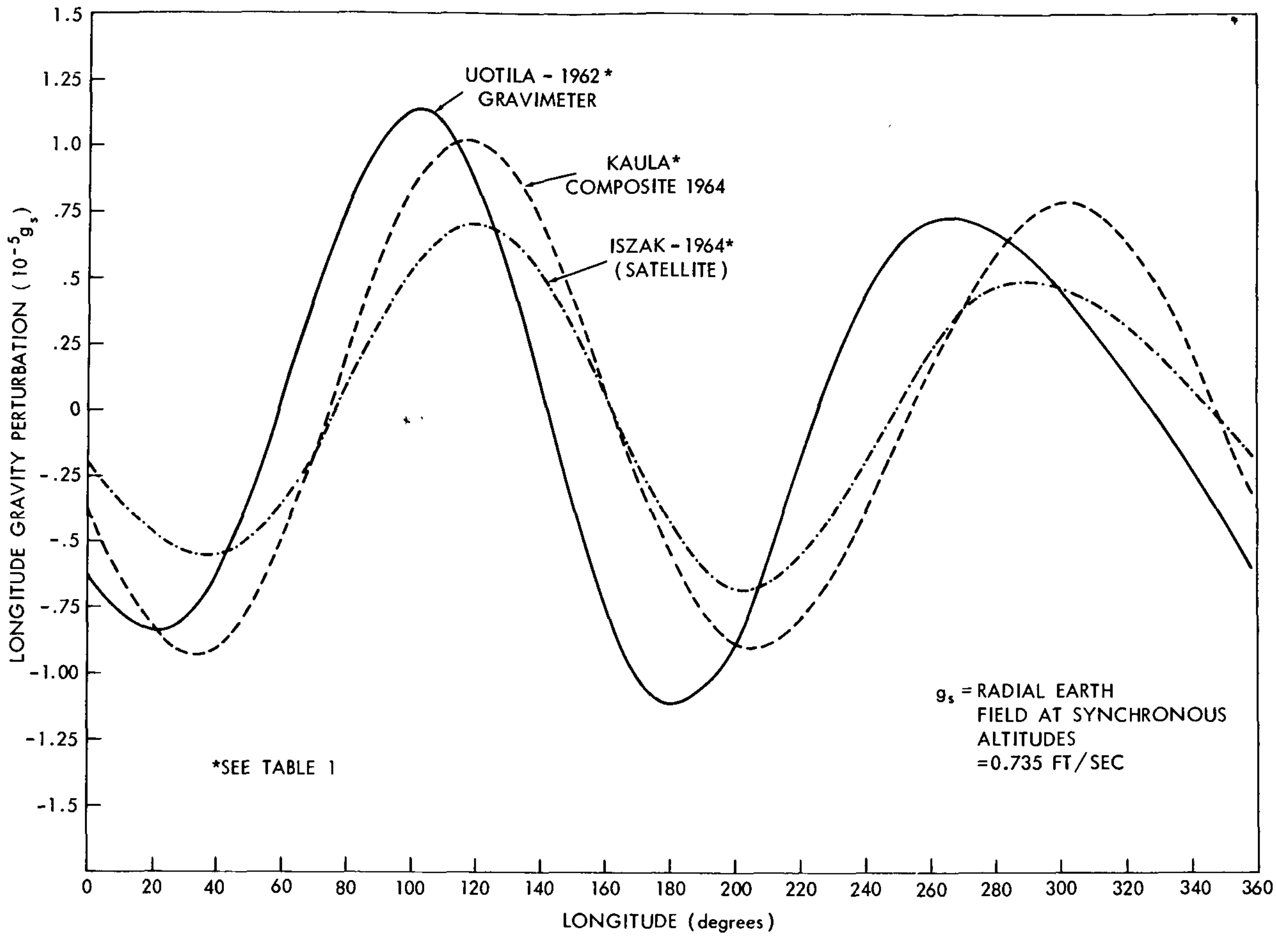


Figure 29—Full field earth longitude gravity forces (perturbations) around the equator at 24 hour altitudes (19,300 n.m.) according to three recent geoids.

that the north pole, the instantaneous spin axis, must wander with a periodic motion and with small amplitude both in inertial space and with respect to the body of the earth (Reference 5). This effect must be present even in the absence of external torques on the earth, which, of course, can be expected to exaggerate it. Such a short-period wandering of the pole (not to be confused with the long 26,000-year period of general precession) has in fact been known to astronomers for some time (Reference 5, p. 383). The largest component of it appears to have a period of about 440 days and an amplitude of about 13 feet with respect to the surface of the earth. The fact that physically, the pole is not in the same place on the earth from instant to instant means that there is a small periodic latitude variation in the stations used for determining the potential of the earth. Such a small variation in latitude cannot mean a significant change in earth potential constants between sets reduced from data at different stations at different times or the same station at different times, because these constants are by far more sensitive to the radius from the center of the earth than any other coordinate. Nevertheless, a 13 foot arc at the earth's surface is 85 feet at synchronous altitudes and within the discriminating power of Doppler range and range rate orbit determination. This *geometric* effect of the consequence of the pole's short period wandering has not heretofore been included in satellite determination programs or in geodetic applications from them. The earth constants themselves (as will be shown shortly) cannot be significantly affected by this small pole wandering. Putting it another way, physical perturbations, as distinct from the geometric effect above, from the slight wobbling of the spin axis will also be shown to be small. Nevertheless, the *geometric* effect appears to be of sufficient magnitude to be determinable from observations of satellites. This effect might be discriminated in an orbit determination program which has the capability of correcting for the latitude error (on a time basis) in the coordinates of the observation stations. Such information on the details of short-period, pole wandering would be of great value to geodesists in interpreting the physical nature and behaviour of the interior of the earth.

We now assess the magnitude of the gravity potential term which arises only when one uses the instantaneous spin axis as a coordinate reference and not the true "axis of figure" for the earth (its principal axis of inertia).

Consider in Figure 30 a polar section of the earth containing this true "axis of figure."

Instantaneously, the axis of figure of the earth X_3' is separated by γ from the polar axis X_3. It is shown below that the gravity effect of this "tipping" of the reference spin axis is proportional to the product of inertia of the earth with respect to the spin axis and the tipped equatorial axis X_2 (evidently close to zero since γ is always very small).

In Figure 30 note that dm, ξ_2, ξ_3, ξ_2' and ξ_3' are all in a plane parallel to the X_3', X_3 plane. The product of inertia of the earth with respect to the spin axis, X_3, and the tipped equatorial axis, X_2, is

$$\int_{D_e} \xi_2 \xi_3 \, dm \; .$$

From the figure it can be seen that

$$\xi_2 = \xi_2' \cos\gamma - \xi_3' \sin\gamma$$

$$\xi_3 = \xi_2' \sin\gamma + \xi_3' \cos\gamma \qquad (120)$$

since the X_2, X_3 axes are rotated by γ in the polar section with respect to X_2', X_3'.

For purposes of simplicity (and with essential accuracy) the earth is assumed to be a biaxial oblate ellipsoid in this argument so that every polar section (as in Figure 30) contains the second axis of inertia (X_2') with respect to which the moment of inertia of the earth is B.

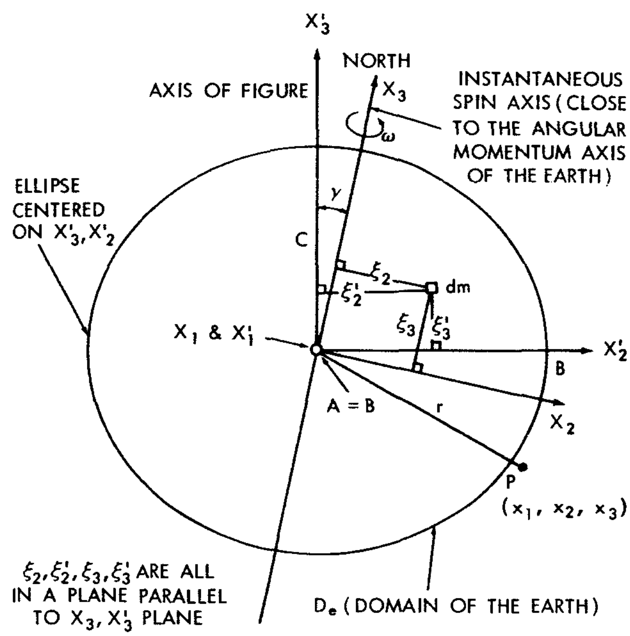


Figure 30—Polar section of the earth containing the axis of figure.

From Equation 120 the product of inertia of the earth with respect to X_2, X_3 is

$$\int_{D_e} \xi_2 \xi_3 \, dm = \int_{D_e} \left(\xi_2' \cos\gamma - \xi_3' \sin\gamma\right)\left(\xi_2' \sin\gamma + \xi_3' \cos\gamma\right) dm$$

$$= \cos\gamma \sin\gamma \left\{ \int_{D_e} (\xi_2')^2 \, dm - \int_{D_e} (\xi_3')^2 \, dm \right\}^3$$

$$+ \int_{D_e} \xi_2' \xi_3' \, dm \left\{ \cos^2\gamma - \sin^2\gamma \right\} \cdot \qquad (121)$$

But since, in this argument, X_2' is a principal axis,

$$\int_{D_e} \xi_2' \xi_3' \, dm = 0 ;$$

and by definition

$$\int_{D_e} \left[(\xi_1')^2 + (\xi_2')^2 \right] dm = C$$

and

$$\int_{D_e} \left[(\xi_3')^2 + (\xi_1')^2 \right] dm = B .$$

Equation 121 therefore reduces to

$$\int_{D_e} \xi_2 \xi_3 \, dm \;\square\; \frac{1}{2} (C - B) \sin 2\gamma . \qquad (122)$$

Additionally, it is shown below that the products of inertia with respect to axes X_1, X_2 $\left(\int_{D_e} \xi_1 \xi_2 \, dm\right)$ and X_1, X_3 $\left(\int_{D_e} \xi_1 \xi_3 \, dm\right)$ are both zero. (Not shown in Figure 30 are the X_1 and X_1' coordinates of dm which are ξ_1 and ξ_1' respectively.) These results hold for this argument because from Equation 120 and the fact that $\xi_1 = \xi_1'$,

$$\xi_1 \xi_2 = \xi_2' \xi_1' \cos\gamma - \xi_3' \xi_1' \sin\gamma$$

$$\xi_1 \xi_3 = \xi_1' \xi_2' \sin\gamma \neq \xi_1' \xi_3' \cos\gamma .$$

Since X_1', X_2', X_3' are principal axes of inertia,

$$\int_{D_e} \xi_1 \xi_2 \, dm = \cos\gamma \int_{D_e} \xi_2' \xi_1' \, dm - \sin\gamma \int_{D_e} \xi_3' \xi_1' \, dm = 0 \tag{123}$$

and

$$\int_{D_e} \xi_1 \xi_3 \, dm = \sin\gamma \int_{D_e} \xi_1' \xi_2' \, dm + \cos\gamma \int_{D_e} \xi_1' \xi_3' \, dm = 0 \, . \tag{124}$$

Other moments of inertia with respect to the X_1, X_2, X_3 axes, important in what follows, can be similarly shown to be:

$$\int_{D_e} \left[(\xi_1)^2 + (\xi_3)^2\right] dm = B + (C - B)\sin^2\gamma \quad , \tag{125}$$

$$\int_{D_e} \left[(\xi_2)^2 + (\xi_3)^2\right] dm = B \, , \tag{126}$$

$$\int_{D_e} \left[(\xi_1)^2 + (\xi_2)^2\right] dm = C - (C - B)\sin^2\gamma \quad . \tag{127}$$

We now return to the main argument. When the earth was referenced to its principal axes of inertia, it was seen that the moment of inertia of the earth about the line to the field point reduced simply to Equation 20b or

$$I_r = \left[A(x_1)^2 + B(x_2)^2 + C(x_3)^2\right] / r^2 \quad .$$

However with X_1, X_2, X_3 no longer principal axes, but rotated in the manner of Figure 30 with respect to the principal set X_1', X_2', X_3', this line moment of inertia becomes, from Equations 20a and 122-127

$$r^2 I_r = (x_1)^2 B + (x_2)^2 \left[B + (C - B)\sin^2\gamma\right] + (x_3)^2 \left[C - (C - B)\sin^2\gamma\right] - X_2 X_3 (C - B)\sin 2\gamma \, . \tag{128}$$

The moment of inertia for the biaxial earth of this simplified argument with respect to its c.m.,

$$I_0 = \frac{1}{2}(A + B + C) = \frac{1}{2}(2B + C) = \frac{1}{2}\left[2B + C\right]\left[(x_1)^2 + (x_2)^2 + (x_3)^2\right] / r^2 \, . \tag{129}$$

is still given by Equation 20c since it is independent of the coordinate reference. When the argument leading to Equation 48 is repeated, but with respect to these altered axes, Equations 128 and 129 in (47a) give

$$(x_1)^2\left[\frac{-(F_{20})_0}{2} + 3(F_{22})_0\right] + (x_2)^2\left[\frac{-(F_{20})_0}{2} - 3(F_{22})_0\right] + (x_3)^2\left[(F_{20})_0\right]$$

$$+ (x_1 x_2)\left[6(F_{22})_1\right] + (x_1 x_3)\left[2(F_{21})_0\right] + (x_2 x_3)\left[2(F_{21})_1\right]$$

$$= (x_1)^2\left[\frac{2B+C}{2} - \frac{3B}{2}\right] + (x_2)^2\left[\frac{2B+C}{2} - \frac{3}{2}\left\{B + (C-B)\sin^2\gamma\right\}\right]$$

$$+ (x_3)^2\left[\frac{2B+C}{2} - \frac{3}{2}\left\{C - (C-B)\sin^2\gamma\right\}\right] + \frac{3}{2}\left[x_2 x_3 (C-B)\sin 2\gamma\right]. \tag{130}$$

It is to be understood that the longitude reference (the x_1 axis) has not changed in the particular pole tipping of Figure 30. Since the earth is almost a biaxial ellipsoid, one can always choose the longitude reference axis arbitrarily or describe the tipping in this particular fashion. The physical effects which follow should be the same as for an arbitrary pole migration except for a longitude phase shift which would depend on the longitude as well as the latitude variation of the pole. Such a variation could be described with respect to the Greenwich meridian through the axis of figure.

The gravity constants $(F_{20})_0$, $(F_{22})_0$, $(F_{22})_1$, $(F_{21})_0$, $(F_{21})_1$ in Equation 130 thus refer to axes x_1, x_2, x_3 as in the argument of Equation 48. But now, since these axes cannot all be principal axes, we have chosen x_1 and x_2 arbitrarily to simplify the description of the effect.

Equating coefficients of linearly independent terms on the right and left sides of Equation 130 as in Equations 49 - 54 gives the following:

$$\frac{-(F_{20})_0}{2} + 3(F_{22})_0 = \frac{1}{2}(C-B) \tag{131}$$

$$\frac{-(F_{20})_0}{2} - 3(F_{22})_0 = \frac{C-B}{2} - \frac{3}{2}\sin^2\gamma\,(C-B) = \frac{1}{2}(C-B)\left(1 - 3\sin^2\gamma\right) \tag{132}$$

$$(F_{20})_0 = B - C + \frac{3}{2}\left[\sin^2\gamma\,(C-B)\right] = (B-C)\left(1 - \frac{3}{2}\sin^2\gamma\right) \tag{133}$$

$$6(F_{22})_1 = 0 \tag{134}$$

$$2(F_{21})_0 = 0 \tag{135}$$

$$2(F_{21})_1 = \frac{3}{2}(C-B)\sin 2\gamma\,. \tag{136}$$

From Equations 131 - 136 the gravity constants in terms of the moments of inertia with respect to the axis of figure and the tip angle γ are

$$\left(F_{20}\right)_0 = (B - C)\left(1 - \frac{3}{2}\sin^2\gamma\right) \tag{137}$$

$$\left(F_{22}\right)_0 = \frac{1}{4}(C - B)\sin^2\gamma \tag{138}$$

$$\left(F_{22}\right)_1 = 0 \tag{139}$$

$$\left(F_{21}\right)_0 = 0 \tag{140}$$

$$\left(F_{21}\right)_1 = \frac{3}{4}(C - B)\sin 2\gamma \tag{141}$$

Note that when $\gamma = 0$, the results of Equations 137 - 141 are the same as Equations 49 - 54 with $A = B$. In other words, the effect of a small tipping of the axis of figure with respect to the spin axis is to change J_{20} by the order of magnitude of $\sin^2\gamma$, to change J_{22} by the order of magnitude of $J_{20}\sin^2\gamma$, and to introduce an additional J_{21} longitude effect. All of these *gravity* effects are very small and in most applications should be negligible, as will be seen below.

When Equations 43 and 56 are rewritten with respect to the non-principal axis set X_1, X_2, X_3,

$$V_e \doteq \frac{F_{00}}{r} + \frac{1}{r^3}\left\{\frac{F_{20}}{2}\left(3\sin^2 L - 1\right) + \frac{3F_{21}}{2}\sin 2L\cos\left(\phi - \phi_{21}\right) + 3F_{22}\cos^2 L\cos 2\left(\phi - \phi_{22}\right)\right\}, \tag{142}$$

and

$$V_e \doteq \frac{GM_e}{r}\left\{1 - \frac{J_{20}}{2}(\bar{a}/r)^2\left(3\sin^2 L - 1\right) - \frac{3}{2}(\bar{a}/r)^2 J_{21}\sin 2L\cos\left(\phi - \phi_{21}\right)\right.$$

$$\left. - 3J_{22}(\bar{a}/r)^2\cos^2 L\cos 2\left(\phi - \phi_{22}\right)\right\}. \tag{143}$$

For the J_{21}, F_{21} coefficients, the identification of Equations 142 and 143 implies

$$\frac{3F_{21}}{2r^3} = \frac{-3(\bar{a}/r)^2 J_{21} GM_e}{2r},$$

or

$$J_{21} = \frac{-F_{21}}{GM_e\,(\bar{a})^2}. \tag{144}$$

Similarly, the identification of Equations 142 and 143 gives

$$J_{20} = \frac{-F_{20}}{GM_e\,(\bar{a})^2} \quad . \tag{145}$$

Equation 140 in (46a) implies $\phi_{21} = \pi/2$ or $3\pi/2$ radians, which implies $F_{21} = 2G(F_{21})_1/3$. This result and that of Equation 141 give

$$F_{21} = \frac{2G}{3}\left[\frac{3}{4}(C-B)\right]\sin 2\gamma \ ,$$

or from Equation 144

$$|J_{21}| = \frac{(C-B)\sin 2\gamma}{2M_e\,(\bar{a})^2} \quad . \tag{146}$$

Since $F_{20} = G(F_{20})_0$, (see Equation 46a), from Equations 145 and 137, J_{20} is given approximately by

$$J_{20} = \frac{C-B}{M_e\,(\bar{a})^2} \quad . \tag{147}$$

Combining Equations 146 and 147 gives

$$|J_{21}| = J_{20}\frac{\sin 2\gamma}{2} \doteq J_{20}\gamma \tag{148}$$

since $\gamma \ll 1$. But

$$\gamma \doteq \frac{13'}{(3960\times 5280)'} = 0.62\times 10^{-6} \quad ,$$

(Reference 6, p 383) and $J_{20} \doteq 1082.5 \times 10^{-6}$. Therefore, from Equation 148

$$|J_{21}| \doteq 0.00067 \times 10^{-6} \quad . \tag{149}$$

Earth longitude coefficients $|J_{nm}|$ (from Equation 99 for example) determined by ignoring J_{21} altogether, all appear to be greater than 0.01×10^{-6} for $n < 6$ (Reference 12). It is clear then that for orbit determinations which need inclusion of earth tesserals up to about the sixth order, the physical effects of the small periodic wandering of the north pole are entirely negligible. In fact, if J_{21} is arbitrarily set to zero, it in effect forces the latitude reference to be the true axis of figure at any instant in an orbit determination which utilizes an earth potential such as Equation 99. In the torque-free nutations of the oblate earth the spin axis is always considerably closer to the "invariable" angular momentum axis (the true inertial reference for motion with respect to the earth) than the axis of figure. Therefore, the time based correction for station latitude and longitude bias from observations of satellite motions will, assuming torque-free nutations only, give almost directly the movement of the axis of figure in inertial space.

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Appendix A

Notes on Notation

It would appear that almost every author of geodetic investigations uses a different notation or definition for the constants in the gravity potential of the earth, for example, Reference A-1. W. M. Kaula (Reference A-2) has reviewed these and has proposed a limited number of standard forms useful in the applications and easily converted from one to the other. They are all based on the form of the potential as it is written in Equation 99. As can readily be appreciated, it is important that when using a potential in this form, the earth radius ($\bar{a}$, for example) be clearly defined and stated when constants are reported.

In one form the longitude constants ($m \neq 0$) are

$$K_{nm} = -J_{nm} \ , \tag{A-1}$$

as written in Equation 99. By definition the J_{nm} ($m \neq 0$) in Equation 99 are *all* negative. With this definition the λ_{nm} are interpreted physically as in Figures 5 to 15 for both the K_{nm} and J_{nm} form for the potential. There is no logical reason why one of these forms should not be dropped. Historically (Reference A-3, for example) the J_{nm} form of Equation 99 has always been used for the zonal ($m = 0$) potential. It has only been fairly recently, to the author's knowledge, that the K_{nm} form (with $K_{nm} > 0$ by definition) has been adopted for just the longitude part of the series. The author feels that the form of the J_{nm} series valid for all n and m in Equation 99 is a reasonable unifying compromise of the historically different forms (J_n and K_{nm}) for treating zonal and longitude gravity. Again it is noted that the J_{nm}'s in the form of Equation 99 must always be negative so that the λ_{nm} in the J_{nm} form refers to the same longitudinal axis of symmetry as in the K_{nm} form for the nm harmonic. For example, if J_{22} in the potential series 99 were positive, this would mean that λ_{22} would locate the *minor* axis of the elliptical equator instead of the major axis as it does when J_{22} is negative. In the K_{nm} form of the potential it has been conventional to define K_{nm} always positive so that λ_{22} locates the *major* equatorial axis. The simple assignment, $J_{nm} = -K_{nm}$, preserves the same physical interpretation of the λ_{nm} between the two forms (Figures 5 to 15).

The third standard form proposed by Kaula is a simple expansion of the K_{nm} form for longitude-dependent gravity. When the longitude-dependent part of the general longitude gravity term is written as

$$K_{nm} \cos m\left(\lambda - \lambda_{nm}\right) = \left(k_{nm} \cos m\lambda_{nm}\right) \cos m\lambda + \left(k_{nm} \sin m\lambda_{nm}\right) \sin m\lambda \ ,$$

the two longitude constants for the nm harmonic can be defined as

$$C_{nm} = K_{nm} \cos m\lambda_{mn}$$

and

$$S_{nm} = K_{nm} \sin m\lambda_{nm} \tag{A-2}$$

With these new constants the harmonic potential series of Equation 99 is (Reference A-2).

$$V_e = \frac{\mu_E}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} (\bar{a}/r)^n P_n^m (\sin L)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right\}. \tag{A-3}$$

Alternately the K_{nm}, λ_{nm} form is

$$V_e = \frac{\mu_E}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} (\bar{a}/r)^n P_n^m (\sin L) K_{nm}\cos m\left(\lambda - \lambda_{nm}\right)\right\} \quad ; \tag{A-4}$$

and the J_{nm}, λ_{nm} form is

$$V_e = \frac{\mu_e}{r}\left\{1 - \sum_{n=1}^{\infty}\sum_{m=0}^{n} (\bar{a}/r)^n P_n^m (\sin L) J_{nm}\cos m\left(\lambda - \lambda_{nm}\right)\right\} . \tag{A-5}$$

The C_{nm}, S_{nm} form in Equation A-2 is often a convenient calculating form as the effects of the two longitude constants are linearly separated in it.

From Equations A-1 and A-2 the connections between the K_{nm}, λ_{nm}, J_{nm}, C_{nm}, and S_{nm} constants are summarized as

$$J_{nm} = -K_{nm} \quad ,$$

$$J_{nm} = -\left(C_{nm}^2 + S_{nm}^2\right)^{1/2} \quad ,$$

$$\lambda_{nm} = \frac{1}{m}\tan^{-1}\left(S_{nm}/C_{nm}\right) \quad ,$$

and

$$K_{nm} = +\left(C_{nm}^2 + S_{nm}^2\right)^{1/2} \quad . \tag{A-6}$$

In addition to Equations A-3 to A-5 Kaula (Reference A-2) proposes a form of Equation A-3 which defines the potential as

$$V_e = \frac{\mu_e}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} (\bar{a}/r)^n P_n^m (\sin L)\left[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right]\right\} , \tag{A-7}$$

where

$$(\bar{C}_{nm}, \bar{S}_{nm}) = \left[\frac{(n+m)!}{(n-m)!\,(2n+1)\,(2-\delta_m^0)}\right]^{1/2} (C_{nm}, S_{nm})$$

with

$$\delta_m^0 = 1 \text{ for } m = 0 \text{ (zonal terms)}$$

and

$$\delta_m^0 = 0 \text{ for } m \neq 0 \text{ (longitude terms)} . \qquad \text{(A-8)}$$

It can be shown (Reference A-4) that $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are coefficients of harmonics which have a mean square amplitude of 1 over the geoid for all values of m and n. Thus, comparison of the order of magnitude of the physical effects of the harmonics at any altitude is possible using $(\bar{C}_{nm}, \bar{S}_{nm})$ values modified by $(\bar{a}/\bar{r})^n$ to account for the decrease in harmonic amplitude with distance from the c.m. of the earth.

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Appendix B

List of Symbols

∇^2 differential operator signifying $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in x, y, z rectangular coordinates, for example

Q_i metric coefficient of the coordinate q_i (dimensionless)

q_i general orthogonal coordinate

ds_i small line element in the direction of increasing q_i only

r, L, λ(or ϕ) spherical coordinates of radius, latitude and longitude with respect to the mass distribution

V gravitational potential (units of energy/mass)

R, H, Φ separated variations of the potential; functions of radius, latitude and longitude, respectively

D_n^m intermediate potential constants (units of energy/mass)

P_n^m *associated Legendre function* of order n, power m

P_n^0 or P_n Legendre polynomial of order n

F_{nm} intermediate potential constants (units of energy/mass)

λ_{nm} (or ϕ_{nm}) longitude of the principal plane of symmetry corresponding to the nm harmonic of the potential of the mass distribution

G universal gravitational constant (6.673×10^{-8} dyne-cm²/gm²)

M total mass of the distribution whose potential is desired (M_e = M = Mass of the earth, in the applications in this report).

I_0 moment of inertia of the distribution about the origin

I_r moment of inertia of the distribution about the line vector $\vec{r}$ to the test point in the field

x_i or x_i' rectangular coordinates in inertial space aligned to the axes of principal moment of inertia of the mass distribution

A, B, C principal moments of inertia of the mass distribution

J_{nm} potential constant for the nm harmonic of the potential of the mass distribution (units of energy/mass)

$\bar{a}$, R_0 average of major and minor radii of the elliptical equator of the model earth

f polar flattening or oblateness coefficient of the model earth ellipsoid (dimensionless)

e eccentricity of the elliptical equator of the model earth

a, b, c half axes of the model earth ellipsoid (c is the polar radius)

R radius to the surface of the model earth ellipsoid

ω rotation rate of the earth (units of angular velocity)

C_0 constant amount of the potential at the surface of the model earth ellipsoid

α centrifugal potential constant of the earth

λ geographical longitude measured east from the Greenwich meridian

Δr deviation in radius from the surface of the average earth sphere of radius $\bar{a}$ to an equipotential surface

K_0, λ_0 intermediate potential constants

D domain of the mass distribution

ρ distance from the field point to a point mass of a mass distribution

ξ the distance from the origin of coordinates to a mass point in the domain of the mass distribution whose gravity potential is sought.

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