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FLOW FIELD COMPUTER PROGRAM
FINAL REPORT

Prepared by

G. Waldman

RESEARCH AND ADVANCED DEVELOPMENT DIVISION
AVCO CORPORATION
Wilmington, Massachusetts

Technical Report
RAD-TR-64-17
Contract NAS 9-2494

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22 May 1964

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Project Manager

Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
Houston, Texas

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ABSTRACT

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The method of integral relations has been applied to the flow over a blunt axisymmetric body at moderate angle of attack. A one-strip analysis is developed for a perfect gas or equilibrium air. A machine program has been written based on this analysis, and results of the program are presented.

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ILLUSTRATIONS

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I. INTRODUCTION

Despite certain drawbacks which will be mentioned below, Dorodnitsyn's method of integral relations has been employed in connection with a variety of problems in fluid dynamics and has generally been found satisfactory. It essentially consists of a systematic formulation of the classical integral technique in which the equations are put in integral form and the integrals are evaluated on the basis of an assumed dependence on the independent variable. An illustration of the procedure has been furnished by Dorodnitsyn[1] in the form of a solution of the linear equation

$$\frac{\partial^2}{\partial x^2}\left[(1-x)\,u\right] + \frac{\partial^2 u}{\partial y^2} = 0 \ .$$

This equation exhibits the mixed behavior of the Navier-Stokes equations for the blunt body problem, being elliptic for $x < 1$ and hyperbolic for $x > 1$. It can be written as two first-order equations:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \ , \qquad \frac{\partial}{\partial x}\left[(1-x)\,u\right] + \frac{\partial v}{\partial y} = 0 \ .$$

The boundary conditions in the elliptic region indicated in figure 1 are sufficient to specify a continuous solution in the shaded region bounded on the right by two characteristics of the system. An approximation to this solution can be developed by assuming the dependent variables have the quadratic form

$$u(x,y) = u(x,0) - \left[3\,u(x,0) + u(x,1) - 4\,u(x,1/2)\right] y$$
$$+ 2\left[u(x,0) + u(x,1) - 2u(x,1/2)\right] y^2 \ ,$$

and the same for v. Integrating the differential equations from $y = 0$ to $y = 1/2$ and from $y = 0$ to $y = 1$, and substituting the above forms for u and v, gives the following set of ordinary differential equations:

$$\frac{du(x,0)}{dx} = \frac{1 + u(x,0) - 2\left[x - 2\,v(x,1) + 4\,v(x,1/2)\right]}{1-x} \ ,$$

$$\frac{du(x,1/2)}{dx} = \frac{2u(x,1/2) - 1 - 5\,v(x,1) + 2\left[x + 2\,v(x,1/2)\right]}{2(1-x)} \ ,$$

$$\frac{dv(x,1/2)}{dx} = \frac{1}{2}\left[x - 5\,u(x,0) + 4\,u(x,1/2)\right] \ ,$$

$$\frac{dv(x,1)}{dx} = 4\left[x + u(x,0) - 2u(x,1/2)\right] \ .$$

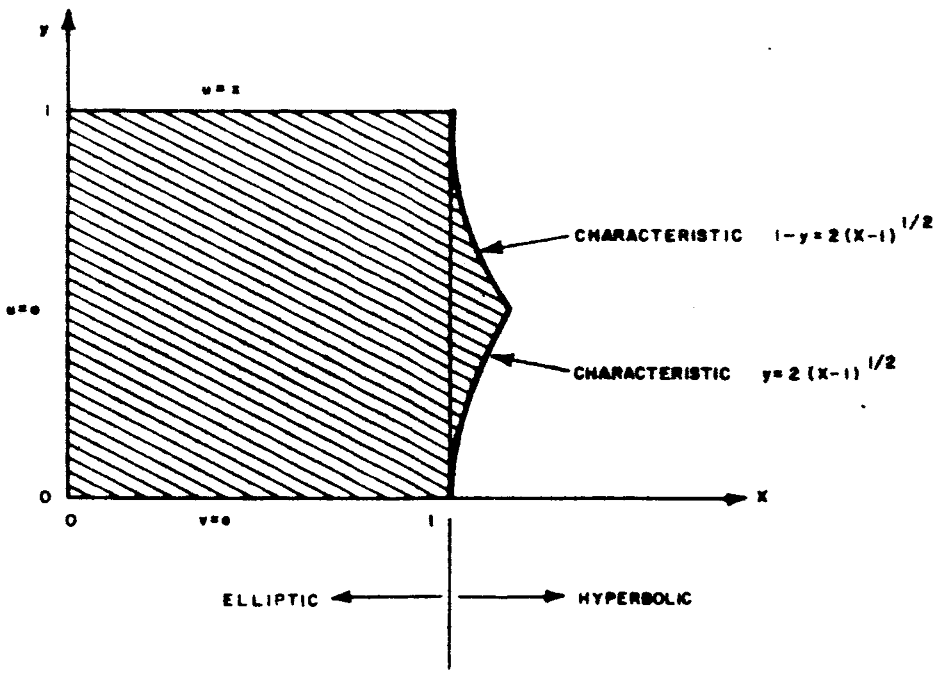

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Figure 1 BOUNDARY CONDITIONS FOR MODEL EQUATION

Here account has been taken of the boundary conditions at $y = 0$ and $y = 1$. The boundary conditions $u(0, 0) = u(0, 1/2) = 0$ are available at $x = 0$ to start the integration. The starting values $v(0, 1/2)$ and $v(0, 1)$ are dependent on the requirement that the solution be continuous through $x = 1$. A correct choice of these values assures that the numerators on the right-hand sides of the first two of the above equations will vanish at the same time as the denominators. Dorodnitsyn's numerical solution of the approximating system shows excellent agreement with the exact solution.

Consideration of the example cited above reveals the following salient points of this formulation of the method of integral relations which are common to the blunt body problem: the equations are integrated up to a number of strip boundaries which divide the region of interest, and a form of variation of the solution in the direction of integration is assumed. Singularities appear in the resulting integrated equations which are associated with the transition from an elliptic to a hyperbolic system. The starting conditions must be chosen to render the solution continuous through these singularities.

The advantage of the method of integral relations in reducing the partial differential equations to a set of ordinary differential equations is obvious. The solution is not limited to a particular region - e. g., the elliptic region - and it takes into account the influence of the elliptic region on the hyperbolic region through the singularity condition. By dividing the region into more strips and using higher-order approximating functions, the accuracy of the solution can be systematically improved. Finally, problems involving floating boundary conditions can be treated in a direct manner by suitable choice of the coordinate system.

For instance, the blunt body problem is treated in a body-oriented coordinate system. The equations of motion are integrated along the normals to the body from the body surface to the edges of a number of strips which divide the shock layer in equal intervals. A set of ordinary differential equations can then be obtained governing, say, the velocity component in the body direction evaluated at the various strip boundaries, together with the detachment distance and shock angle. These equations become singular when the velocity components become sonic. Suitable choice of the starting conditions - the stagnation point shock detachment distance and the velocities at the strip boundaries along the dividing streamline - enables the singularities to be passed through in the proper manner. The solution can then be continued in the supersonic region.

Four drawbacks are evident in this scheme. First, the singular points do not lie on the sonic line, except on the body, since the normal component of velocity does not appear. Thus the intuitively correct "choking" behavior of the solution is illusory. This objection cannot be taken seriously as a practical matter because the normal velocity component involved is generally small except at low free-stream Mach numbers. It is sufficient to recognize that for a given approximation the method of integral relations may be inaccurate at Mach numbers for which the normal extent of the shock layer is commensurate with its lateral extent.

Second, the accuracy of the method is poor in the hyperbolic region, probably due to the thickening of the shock layer downstream of the sonic line. It is therefore desirable to employ the method of characteristics in the supersonic region, starting as close as possible to the sonic line.

Third, the solution in the general neighborhood of the singularity is quite sensitive to small changes in the starting conditions, so that it becomes necessary to determine the standoff distance with useless accuracy in order that the rest of the results be acceptable. A numerical scheme for rapid convergence on the final solution is indicated.

Fourth, any set of dependent variables can be chosen to be approximated in the direction normal to the body, and any set of approximating functions can be chosen for this purpose. The choice of variables and functions is arbitary from a mathematical point of view and thus rests entirely on individual judgment. Under such circumstances it is wise to choose those dependent variables and approximating functions which are convenient to work with and which at the same time are appropriate to the physical situation so as to optimize the accuracy of the computation. The choice of coordinate system will also affect the results. The method of integral relations cannot be used automatically for any given problem with the expectation that meaningful results will be obtained; rather, close attention must be given to the suitability of various possible schemes for yielding the expected form of the solution.

Schwiderski[2] has stated some of the above drawbacks, evidently in an attempt to discredit the method of Dorodnitsyn. His arguments are based in part on mathematical and physical reasoning which is difficult to follow. At any rate, it is felt on the basis of the foregoing remarks that the method of integral relations remains on solid theoretical ground despite the criticism.

With these drawbacks in mind, the method of integral relations remains a useful tool for the solution of a number of flow problems. Its application to the blunt body problem dates from the work of Belotserkovskii[3, 4] in which power series are employed to represent the behavior of the flow variables from body to shock. Traugott[5], Holt[6], Xerikos and Anderson[7], Kennet[8], etc. have presented versions of this method. Belotserkovskii basically finds that the results for one strip - the simplest approximation, in which linear profiles are assumed - are sufficiently accurate for most engineering purposes, while for two or three strips the method has essentially converged to the desired solution. Belotserkovskii[9] has also investigated other formulations of the problem, in one of which the strips are taken in the lateral rather than normal direction. This method appears promising for low supersonic Mach number flows and reacting flows, in which the normal profiles are perhaps not well approximated by power series. It has the advantage of doing away with the sonic singularity.

Before discussing the application of the method of integral relations to flows with angles of attack, the work of Swigart[10, 11] should be mentioned. Swigart perturbs the equations of motion in the small angle of attack and at the same time develops a solution in the form of a series expansion valid near the axis of symmetry of the shock wave (i. e., near the stagnation line). The angle of attack perturbation permits the dependence on the meridional coordinate to be eliminated, and the series expansion results in a set of ordinary differential equations that are integrated inward from the given paraboloidal shock. Swigart's results for zero angle of attack agree well with experimental and other theoretical results for spherical bodies and ellipsoidal bodies of moderate eccentricity, even up to the sonic point; however, his expansion procedure would not be expected to be accurate for blunt bodies with relatively small corner radii, unless many terms in the series were retained.

Swigart observes that his computed body (dividing) streamline at angle of attack differs from the streamline that passes through the point where the shock is normal to the free stream vector -- in other words, the body entropy is not the maximum entropy in the flow field. The important question of the body entropy will be discussed in greater detail in Section 3.

The papers of Bazzhin[12] and Minailos[13] represent recent Soviet work in the field of blunt body flow at large angles of attack. Both apply the method of integral relations with one strip in the direction normal to the body, Bazzhin to the two-dimensional flat plate problem and Minailos to the problem of a yawing axisymmetric body. Vaglio-Laurin[14], in addition to his treatment of the application of the PLK method to the calculation of blunt-body flows, presents an analysis similar to Bazzhin's for asymmetric two-dimensional shapes. Both Soviet authors make allowances for the possibility that the dividing streamline does not pass through the normal point of the shock, but do not furnish the additional conditions necessary for its determination. Bazzhin summarizes the results of his calculations, which were performed with the initial assumption that the dividing streamline did pass through the normal point on the shock. For angles of attack sufficiently large ($> 30°$) that his resulting solution indicated that the initial assumption on the dividing streamline was incorrect, he notes that it was not possible to iterate the solution in order to obtain successive values of the entropy on the body.

Minailos circumvents the difficulty of describing the meridional variation of the flow variables by assuming that this variation is similar to what it would be for small angles of attack - i. e., it is sinusoidal in the meridian angle. The equations of motion can then be written in the two dimensions of the plane of symmetry, with the terms which represent the meridional flux of mass and momentum determined by the solution in the plane. Although this approach is artificial, it probably introduces no more error than the one-strip assumption normal to the body. The meridional variation can be thought of as being approximated by Fourier series in the meridian angle, just as the normal variation is approximated by power series in the distance normal to the surface. The approach of Minailos then consists of taking the leading terms in the meridional as well as the normal direction; the two approximations are consistent in this sense. If it were desired to take more than one strip in the normal direction, it would also be necessary to employ a higher approximation in the meridional direction in order to make the results consistently better, and vice versa. The sinusoidal approach is appropriate to the moderate angle of attack range; it is not a perturbation scheme limited to small angles of attack, but it would be expected to break down at very large angles of attack. The precise range of validity can only by established by comparison with experimental and other theoretical results.

Vaglio-Laurin presents a one-strip analysis which does not involve any assumption on the meridian coordinate. A finite-difference integration of the equations over the surface of the body is suggested, starting from the locus of sonic points and working inward to the stagnation point. No details on the method of integration were given and results were not reported.

The work reported below was done under Contracts NAS 9-858 and NAS 9-2494, the former under Subcontract RL-30107 from Avco Everett Research Laboratory. The former contract covered a survey of all the methods presently available for the blunt body problem, an analysis of their suitability for the determination of three-dimensional flow fields, and formulation of the method of integral relations for yawed flows when that approach appeared to be most promising. In addition, a machine program for zero yaw based on the Garabedian-Lieberstein approach[16, 17] was furnished to NASA. The latter contract has covered the development of a machine program for the three-dimensional problem.

The approach is similar to that of Minailos. The equations of motion are integrated from the shock to the body in the direction normal to the body, and a linear variation of certain flow quantities in this direction is assumed. The meridional variation is assumed to be sinusoidal. The problem then reduces to the solution of ordinary differential equations for the body velocity, detachment distance, and shock angle in the plane of symmetry of the flow. Singularities at the two sonic points (windward and leeward) on the body appear in the body velocity equation. A body-oriented asymmetric coordinate system is employed with origin at the stagnation point. The equations are integrated from the stagnation point outward in the windward and leeward directions. There are four conditions needed to start the integration: the location of the stagnation point, the shock detachment distance and shock angle at that point, and the value of the entropy on the body. The requirement that the solution be analytic in the vicinity of the sonic points essentially establishes two relations between the unknown starting conditions. Two other relations follow from the fact that the stagnation streamline must be perpendicular to the surface in order that the solution be analytic in the vicinity of the stagnation point. The solution is then developed by an iterative scheme in which two parameters - the stagnation point detachment distance and shock angle - are adjusted until the body velocity can be extended smoothly through the two sonic points.

The machine program has been written for a perfect gas and equilibrium air; the equation of state for the latter is in the form of curve fits. The output of the program includes profiles of the flow variables from the body to the shock at various subsonic and supersonic stations, computed from the assumed linearly varying functions. The iterative scheme is automatic and requires only a reasonable first guess of the two starting parameters. Comparison of the results with zero-yaw computations for a sphere indicates good agreement.

II. DIFFERENTIAL EQUATIONS

A geodesic, surface-oriented orthogonal coordinate system (r, n, θ) is employed, where n is the distance normal to the body, $n = 0$ is the body surface, r is surface distance along geodesic curves originating at the stagnation point $r = 0$. θ is the angle between the line of symmetry and a geodesic, measured on the body at the stagnation point, $\theta = 0$ is the leeward plane of symmetry, and $\theta = \pi$ is the windward plane of symmetry (figure 2). Then a general element of length ds is given by

$$ds^2 = h_1^2\, dr^2 + dn^2 + h_3^2\, d\theta^2 \quad . \tag{1}$$

Here

$$h_1 = 1 + \frac{n}{R_r} \quad . \tag{2}$$

$$h_3 = \left(1 + \frac{n}{R_\theta}\right) g \; ,$$

where $R_r\,(r, \theta)$ is the radius of curvature of the surface in the r-direction, $R_\theta\,(r, \theta)$ is the radius of curvature in the θ- direction, and $g\,(r, \theta)$ is the surface metric for the θ- coordinate. Minailos[13] uses a cylindrical coordinate system with axis along the axis of symmetry of the body; however, it is felt that the improvement in accuracy achieved with a coordinate system with origin at the stagnation point outweighs the slight additional complication (see the derivation of the cross-flow equation below).

In this coordinate system the velocity components in the (r, n, θ)- directions are denoted (u, v, w), respectively. The continuity equation and the equations of momentum in the $r, n,$ and θ directions are

$$\frac{\partial}{\partial r}(h_3 \rho u) + \frac{\partial}{\partial n}(h_1 h_3 \rho v) + \frac{\partial}{\partial \theta}(h_1 \rho w) = 0 \quad , \tag{3}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial n} + \frac{w}{h_3}\frac{\partial u}{\partial \theta} + \frac{1}{h_1 \rho}\frac{\partial p}{\partial n} = -\frac{uv}{h_1}\frac{\partial h_1}{\partial n} - \frac{uw}{h_1 h_3}\frac{\partial h_1}{\partial \theta} + \frac{w^2}{h_1 h_3}\frac{\partial h_3}{\partial r} \, , \tag{4}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial n} + \frac{w}{h_3}\frac{\partial v}{\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{u^2}{h_1}\frac{\partial h_1}{\partial n} + \frac{w^2}{h_3}\frac{\partial h_3}{\partial n} \quad , \tag{5}$$

$$\frac{u}{h_1}\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial n} + \frac{w}{h_3}\frac{\partial w}{\partial \theta} + \frac{1}{h_3 \rho}\frac{\partial p}{\partial \theta} = \frac{u}{h_1 h_3}\left(u\frac{\partial h_1}{\partial \theta} - w\frac{\partial h_3}{\partial r}\right) - \frac{vw}{h_3}\frac{\partial h_3}{\partial n} \quad . \tag{6}$$

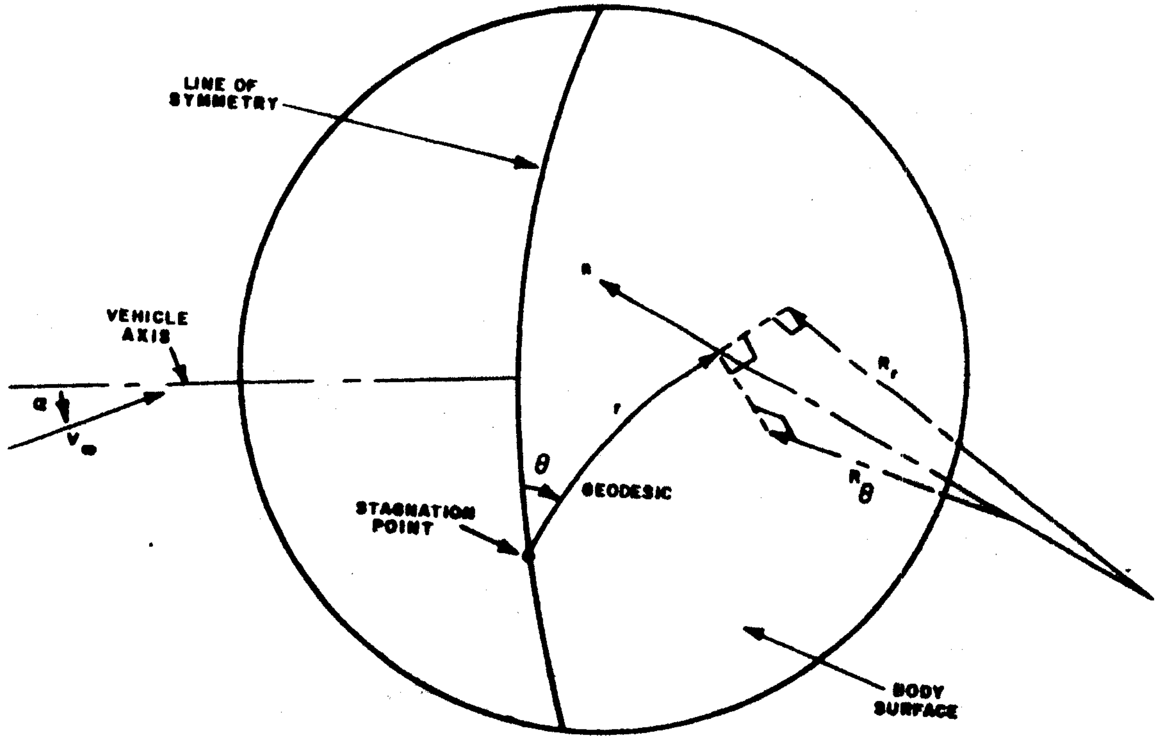

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Figure 2 CONFIGURATION AND COORDINATE SYSTEM

By making use of equation (3) and relations of the form

$$h_3 \rho u \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} (h_3 \rho u^2) - u \frac{\partial}{\partial r} (h_3 \rho u) \quad .$$

equations (4), (5), and (6) can be written in the "divergence form"

$$\frac{\partial}{\partial r} [h_3 (p + \rho u^2)] + \frac{\partial}{\partial n} (h_1 h_3 \rho u v) + \frac{\partial}{\partial \theta} (h_1 \rho u v) \tag{7}$$

$$= -\rho \left(h_3 u v \frac{\partial h_1}{\partial n} + u w \frac{\partial h_1}{\partial \theta} - w^2 \frac{\partial h_3}{\partial r} \right) + p \frac{\partial h_3}{\partial r} \quad .$$

$$\frac{\partial}{\partial r} (h_3 \rho u v) + \frac{\partial}{\partial n} [h_1 h_3 (p + \rho v^2)] + \frac{\partial}{\partial \theta} (h_1 \rho v w) \tag{8}$$

$$= \rho \left(h_3 u^2 \frac{\partial h_1}{\partial n} + h_1 w^2 \frac{\partial h_3}{\partial n} \right) + p \frac{\partial}{\partial n} (h_1 h_3) \quad ,$$

$$\frac{\partial}{\partial r} (h_3 \rho u w) + \frac{\partial}{\partial n} (h_1 h_3 \rho v w) + \frac{\partial}{\partial \theta} [h_1 (p + \rho w^2)] \tag{9}$$

$$= \rho \left(u^2 \frac{\partial h_1}{\partial \theta} - u w \frac{\partial h_3}{\partial r} - h_1 v w \frac{\partial h_3}{\partial n} \right) + p \frac{\partial h_1}{\partial \theta} \quad .$$

Before performing the integration in the n- direction, the variable n is replaced by the dimensionless variable $\zeta = n/\epsilon$, where ϵ is the value of n at the shock wave. This transformation is governed by the following equations:

$$\frac{\partial}{\partial r} = \left(\frac{\partial}{\partial r} \right)_{\zeta, \theta} - \frac{\zeta}{\epsilon} \frac{\partial \epsilon}{\partial r} \frac{\partial}{\partial \zeta} \quad ,$$

$$\frac{\partial}{\partial n} = \frac{1}{\epsilon} \frac{\partial}{\partial \zeta} \quad ,$$

$$\frac{\partial}{\partial \theta} = \left(\frac{\partial}{\partial \theta} \right)_{\zeta, \theta} - \frac{\zeta}{\epsilon} \frac{\partial \epsilon}{\partial \theta} \frac{\partial}{\partial \zeta} \quad .$$

Equations (3), (7), (8), and (9) then take the form

$$\frac{\partial P_i}{\partial r} + \frac{\partial Q_i}{\partial \zeta} + \frac{\partial R_i}{\partial \theta} = L_i \tag{10}$$

$(i = 1, 2, 3, 4)$, respectively, where

$$P_1 = h_3 \rho u, P_2 = h_3 (p + \rho u^2), P_3 = h_3 \rho u v, P_4 = h_3 \rho u w;$$

$$Q_1 = \frac{\rho}{\epsilon} \left[h_1 h_3 v - \zeta \left(h_3 u \frac{\partial \epsilon}{\partial r} + h_1 w \frac{\partial \epsilon}{\partial \theta} \right) \right] = \zeta \rho L_1 + \frac{\rho h_1 h_3 v}{\epsilon} ,$$

$$Q_2 = u Q_1 - \frac{\zeta h_3 p}{\epsilon} \frac{\partial \epsilon}{\partial r} ,$$

$$Q_3 = v Q_1 + \frac{h_1 h_3 p}{\epsilon}$$

$$Q_4 = w Q_1 - \frac{\zeta h_1 p}{\epsilon} \frac{\partial \epsilon}{\partial \theta} ; \tag{11}$$

$$R_1 = h_1 \rho w, R_2 = h_1 \rho u w, R_3 = h_1 \rho v w, R_4 = h_1 (p + \rho w^2) ;$$

$$L_1 = -\frac{\rho}{\epsilon} \left(h_3 u \frac{\partial \epsilon}{\partial r} + h_1 w \frac{\partial \epsilon}{\partial \theta} \right) ,$$

$$L_2 = u L_1 - \rho \left(h_3 u v \frac{\partial h_1}{\partial n} + u w \frac{\partial h_1}{\partial \theta} - w^2 \frac{\partial h_3}{\partial r} \right) + \epsilon p \frac{\partial}{\partial r} \left(\frac{h_3}{\epsilon} \right) ,$$

$$L_3 = v L_1 + \rho \left(h_3 u^2 \frac{\partial h_1}{\partial n} + h_1 w^2 \frac{\partial h_3}{\partial n} \right) + p \frac{\partial}{\partial n} (h_1 h_3) ,$$

$$L_4 = w L_1 + \rho \left(u^2 \frac{\partial h_1}{\partial \theta} - u w \frac{\partial h_3}{\partial r} - h_1 v w \frac{\partial h_3}{\partial n} \right) + \epsilon p \frac{\partial}{\partial \theta} \left(\frac{h_1}{\epsilon} \right) .$$

Equations (10) can now be integrated with respect to ζ from the body surface to any number of strip boundaries which divide the shock layer. For this one-strip analysis we integrate with respect to ζ from 0 (body) to 1 (shock). The coefficients P_i, R_i, L_i are assumed to have linear profiles in ζ:

$$P_i = (1 - \zeta)\, P_{i_b} + \zeta\, P_{i_s}\ , \text{etc.,} \tag{12}$$

where P_{i_b} is the value of P_i at the body and P_{i_s} is the value of P_i at the shock. The results of these integrations are

$$\frac{\partial P_{i_b}}{\partial r} + \frac{\partial P_{i_s}}{\partial r} = 2\,(Q_{i_b} - Q_{i_s}) + L_{i_b} + L_{i_s} - \frac{\partial R_{i_b}}{\partial \theta} - \frac{\partial R_{i_s}}{\partial \theta} \ . \tag{13}$$

In particular, the continuity equation becomes

$$\begin{aligned} \frac{\partial}{\partial r}\,(h_3\,\rho\,u)_b &= \left(h_3\,\rho\,\frac{\partial u}{\partial r} + h_3\,u\,\frac{\partial \rho}{\partial r} + \rho\,u\,\frac{\partial h_3}{\partial r}\right)_b = -\frac{\partial}{\partial r}\,(h_3\,\rho\,u)_s \\ &-2\,Q_{1_s} + L_{1_b} + L_{1_s} - \frac{\partial R_{1_b}}{\partial \theta} - \frac{\partial R_{1_s}}{\partial \theta} \end{aligned} \tag{14}$$

Equation (9) becomes

$$\begin{aligned} \frac{\partial}{\partial r}\,(h_3\,\rho\,u\,w)_b &= \left(h_3\,\rho\,u\,\frac{\partial w}{\partial r} + h_3\,\rho\,w\,\frac{\partial u}{\partial r} + h_3\,u\,w\,\frac{\partial \rho}{\partial r} + \rho\,u\,w\,\frac{\partial h_3}{\partial r}\right)_b \\ &= -\frac{\partial}{\partial r}\,(h_3\,\rho\,u\,w)_s - 2\,Q_{4_s} + L_{4_b} + L_{4_s} - \frac{\partial R_{4_b}}{\partial \theta} - \frac{\partial R_{4_s}}{\partial \theta} \ . \end{aligned} \tag{15}$$

The equation of conservation of stagnation enthalpy

$$h + \frac{1}{2}\,(u^2 + v^2 + w^2) = H \tag{16}$$

becomes at the body ($v_b = 0$)

$$h_b + \frac{1}{2}\,(u_b^2 + w_b^2) = H$$

or

$$\frac{\partial h_b}{\partial r} = -\left(u\,\frac{\partial u}{\partial r} + w\,\frac{\partial w}{\partial r}\right)_b \ .$$

Since the entropy is constant on the body,

$$\frac{\partial \rho_b}{\partial r} = \left(\frac{\partial \rho}{\partial h}\right)_S \frac{\partial h_b}{\partial r} = -\left[\frac{\rho}{a^2}\left(u\frac{\partial u}{\partial r} + w\frac{\partial w}{\partial r}\right)\right]_b \quad . \tag{17}$$

The quantities $\frac{\partial w_b}{\partial r}$ and $\frac{\partial \rho_b}{\partial r}$ can now be eliminated between equations (14), (15), and (17), resulting in the fundamental equation for u_b:

$$\left[h_3 \rho \left(1 - \frac{u^2}{a^2}\right) \frac{\partial u}{\partial r}\right]_b = \left(1 - \frac{w^2}{a^2}\right)_b A + \left(\frac{w}{a^2}\right)_b B \quad , \tag{18}$$

where

$$A = -\left[\frac{\rho u}{\epsilon}\frac{\partial}{\partial r}(h_3 \epsilon) + \frac{1}{\epsilon}\frac{\partial}{\partial \theta}(h_1 \epsilon \rho w)\right]_b$$

$$-\left[\epsilon \frac{\partial}{\partial r}\left(\frac{h_3 \rho u}{\epsilon}\right) + \epsilon \frac{\partial}{\partial \theta}\left(\frac{h_1 \rho w}{\epsilon}\right) + \frac{2 h_1 h_3 \rho v}{\epsilon}\right]_s \quad ,$$

$$B = -\left[\frac{1}{\epsilon}\frac{\partial}{\partial \theta}(h_1 \epsilon \rho w^2) + \rho u \left(u\frac{\partial h_1}{\partial \theta} + \frac{h_3 w}{\epsilon}\frac{\partial \epsilon}{\partial r}\right) - \frac{h_1}{\epsilon}\frac{\partial}{\partial \theta}(\epsilon p)\right]_b \tag{19}$$

$$-\left[\epsilon \frac{\partial}{\partial r}\left(\frac{h_3 \rho u w}{\epsilon}\right) + \epsilon \frac{\partial}{\partial \theta}\left(\frac{h_1 \rho w^2}{\epsilon}\right) - \rho\left\{u^2 \frac{\partial h_1}{\partial \theta} - u w \frac{\partial h_3}{\partial r}\right.\right.$$

$$\left.\left. - h_1 v w \left(\frac{\partial h_3}{\partial n} + \frac{2 h_3}{\epsilon}\right)\right\} + h_1 \epsilon \frac{\partial}{\partial \theta}\left(\frac{p}{\epsilon}\right)\right]_s \quad .$$

The second of equations (13) yields

$$\left[\epsilon \frac{\partial}{\partial r}\left(\frac{h_3 \rho u v}{\epsilon}\right)\right]_s = \left[\rho\left(h_3 u^2 \frac{\partial h_1}{\partial n} + h_1 w^2 \frac{\partial h_3}{\partial n}\right)\right.$$

$$\left. + p\left\{\frac{\partial}{\partial n}(h_1 h_3) - \frac{2 h_1 h_3}{\epsilon}\right\}\right]_b + \left[-\epsilon \frac{\partial}{\partial \theta}\left(\frac{h_1 \rho v w}{\epsilon}\right)\right. \tag{20}$$

$$\left. + \rho\left(h_3 u^2 \frac{\partial h_1}{\partial n} - \frac{2 h_1 h_3 v^2}{\epsilon} + h_1 w^2 \frac{\partial h_3}{\partial n}\right) + p\left\{\frac{\partial}{\partial n}(h_1 h_3) - \frac{2 h_1 h_3}{\epsilon}\right\}\right]_s \quad .$$

In the preceding analysis, the only assumption has been that the profiles in the n- direction are linear. The relevant equations are now written in the plane $\theta = 0, \pi$, where by symmetry $\frac{\partial p}{\partial \theta} = \frac{\partial \rho}{\partial \theta} = \ldots = 0$, $w = 0$, $\frac{\partial w}{\partial \theta} \neq 0$. Equation (18) becomes

$$\left[h_3 \rho \left(1 - \frac{u^2}{a^2}\right) \frac{du}{dr}\right]_b = -\left[\frac{\rho u}{\epsilon} \frac{d}{dr}(h_3 \epsilon) + h_1 \rho \frac{\partial w}{\partial \theta}\right]_b \tag{21}$$

$$- \left[\epsilon \frac{d}{dr}\left(\frac{h_3 \rho u}{\epsilon}\right) + h_1 \rho \frac{\partial w}{\partial \theta} + \frac{2 h_1 h_3 \rho v}{\epsilon}\right]_s ,$$

while equation (20) becomes

$$\left[\epsilon \frac{d}{dr}\left(\frac{h_3 \rho u v}{\epsilon}\right)\right]_s = \left[h_3 \rho u^2 \frac{\partial h_1}{\partial n} - p\left\{\frac{\partial}{\partial n}(h_1 h_3) + \frac{2 h_1 h_3}{\epsilon}\right\}\right]_b \tag{22}$$

$$- \left[- h_1 \rho v \frac{\partial w}{\partial \theta} + \rho \left(h_3 u^2 \frac{\partial h_1}{\partial n} - \frac{2 h_1 h_3 v^2}{\epsilon}\right)\right.$$

$$\left. + p \left\{\frac{\partial}{\partial n}(h_1 h_3) - \frac{2 h_1 h_3}{\epsilon}\right\}\right]_s .$$

It remains to evaluate the cross-flow term $\frac{\partial w}{\partial \theta}$ appearing in the above equations. This is done by assuming that the variation of the cross-flow velocity component in the circumferential direction is given with adequate accuracy by

$$w = w^{\pi/2} \sin \theta , \tag{23}$$

while the variation of the other flow quantities (enthalpy, u, v) is given by

$$h = \frac{h^o + h^\pi}{2} + \frac{h^o - h^\pi}{2} \cos \theta , \tag{24}$$

etc. Here the superscripts $o, \frac{\pi}{2}, \pi$ refer to the values of θ at which the functions are evaluated. In the surface $\theta = \pi/2$ the equation of conservation of stagnation enthalpy (12) gives

$$h^{\pi/2} + \frac{1}{2} \left[(u^{\pi/2})^2 + (v^{\pi/2})^2 + (w^{\pi/2})^2\right] = H ,$$

or, solving formally for $w^{\pi/2}$,

$$w^{\pi/2} = \left[2H - h^o - h^\pi - \left(\frac{u^o + u^\pi}{2}\right)^2 - \left(\frac{v^o + v^\pi}{2}\right)^2\right]^{1/2} .$$

The quantities h^o, h^π are eliminated by employing the enthalpy equations in the planes $\theta = 0, \pi$:

$$H - h^o = \frac{1}{2}[(u^o)^2 + (v^o)^2] ,$$

$$H - h^\pi = \frac{1}{2}[(u^\pi)^2 + (v^\pi)^2] .$$

Then $\partial w/\partial\theta$ in the planes $\theta = 0, \pi$ is determined from the equation

$$\frac{\partial w}{\partial \theta} \equiv W = w^{\pi/2} \cos\theta = \pm\left[\left(\frac{u^o - u^\pi}{2}\right)^2 + \left(\frac{v^o - v^\pi}{2}\right)^2\right]^{1/2} \tag{25}$$

where the plus sign is selected for the windward side of the stagnation point and the minus sign for the leeward side.

The significance of the above approach for the cross-flow is that the sinusoidal dependence of the flow variables which is valid for a perturbation scheme about zero angle of attack has been extended to the moderate angle of attack problem under consideration. The precise way in which this extension is made remains to a certain extent arbitrary. The usual perturbation scheme furnishes no well-defined guidelines for the treatment of larger angles of attack. Therefore the extension must be made so as to retain as many of the physical features of the problem as possible. For instance, the choice of a coordinate system will affect the validity of equation (25). The use of a coordinate system with its origin at the stagnation point insures that $\frac{\partial w}{\partial \theta}$ will vanish at the stagnation point, as it should.

Equation (21) governs the variation of u_b in the windward and leeward directions. The coefficient of the derivative is seen to vanish at the sonic points on the body. The condition must then be imposed that the right-hand side vanish at the sonic points also. The way in which this is accomplished will be discussed in section IV.

Equation (22) is essentially an implicit equation for the variation of the shock angle in the plane of symmetry. In section III, the shock conditions will be specified in terms of the shock angle, so that it will be possible to write equation (22) in explicit form. Then by using an additional equation describing the geometry of the shock, a closed system will result in which the body velocity, shock detachment distance, and shock angle can be determined by integration.

III. BOUNDARY CONDITIONS

The boundary condition at the body $v_b = 0$ has already been used. The general shock relations are written in terms of the velocity components V_{N_∞} (free stream) and V_N (behind the shock) normal to the shock wave. The conditions of continuity, normal momentum conservation, and energy conservation are:

$$\rho_s = \rho_\infty \frac{V_{N_\infty}}{V_N} \quad , \tag{26}$$

$$p_s = p_\infty + \rho_\infty V_{N_\infty} (V_{N_\infty} - V_N) \quad , \tag{27}$$

$$h_s = h_\infty + \frac{1}{2}(V_{N_\infty}^2 - V_N^2) \quad . \tag{28}$$

The condition of tangential momentum conservation yields the relations

$$u_s = u_\infty + (v_\infty - v_s) \frac{1}{h_1} \frac{\partial \epsilon}{\partial r} \quad , \tag{29}$$

$$w_s = w_\infty + (v_\infty - v_s) \frac{1}{h_3} \frac{\partial \epsilon}{\partial \theta} \quad , \tag{30}$$

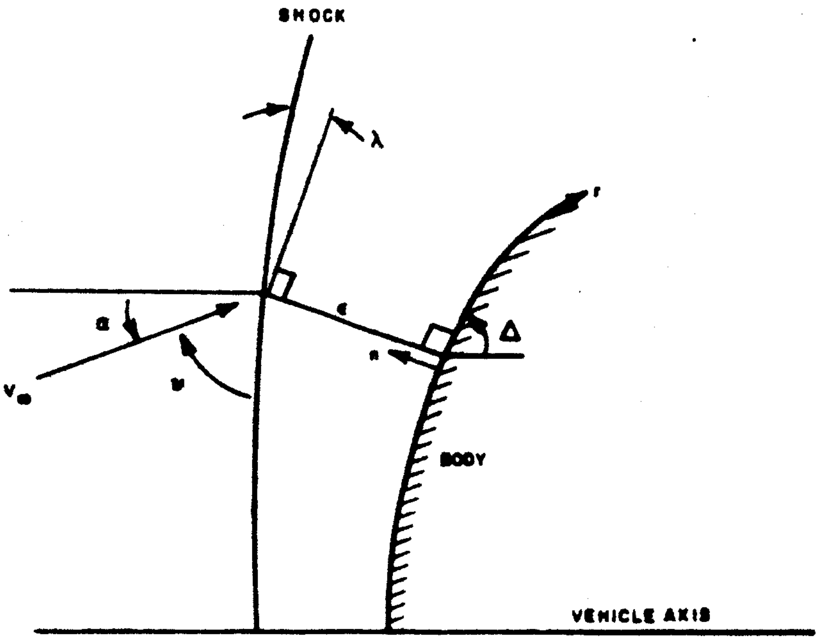
while the velocity component v_s can be related to the normal velocity component and the shock geometry by

$$v_s = - V_N \left[1 + \left(\frac{1}{h_1} \frac{\partial \epsilon}{\partial r} \right)^2 + \left(\frac{1}{h_3} \frac{\partial \epsilon}{\partial \theta} \right)^2 \right]^{-1/2}$$

$$+ \left[\left(u_\infty + \frac{v_\infty}{h_1} \frac{\partial \epsilon}{\partial r} \right) \frac{1}{h_1} \frac{\partial \epsilon}{\partial r} + \left(w_\infty + \frac{v_\infty}{h_3} \frac{\partial \epsilon}{\partial \theta} \right) \frac{1}{h_3} \frac{\partial \epsilon}{\partial \theta} \right] \left[1 + \left(\frac{1}{h_1} \frac{\partial \epsilon}{\partial r} \right)^2 + \left(\frac{1}{h_3} \frac{\partial \epsilon}{\partial \theta} \right)^2 \right]^{-1} . \tag{31}$$

Equations (26)-(31) essentially render the shock variables as functions of the shock geometry. An additional relation, the equation of state in terms of ρ, p, and h, is needed to complete the system. The system is solved in an iterative manner for equilibrium air, whose equation of state in this instance is in the form of curve fits, by requiring compatibility of equations (26)-(28) with the equation of state.

Figure 3 shows the geometry of the shock wave in the plane of symmetry. The angle between the shock wave and a line tangent to the body at the corresponding point is λ, measured positively as indicated. The angle between the shock and the free stream is ν. In specifying the shock conditions and the geometry in the plane of symmetry it is convenient to replace the velocity component u which



64-4919

Figure 3 GEOMETRY OF SHOCK WAVE IN PLANE OF SYMMETRY is positive everywhere by the component U which is equal to u in absolute value, but which is positive to the leeward of the stagnation point and negative to the windward of the stagnation point. The shock conditions then become

$$\rho_s = \frac{\rho_\infty V_\infty \sin \nu}{V_N} \quad . \tag{32}$$

$$p_s = p_\infty + \rho_\infty V_\infty \sin \nu \,(V_\infty \sin \nu - V_N) \quad , \tag{33}$$

$$h_s = h_\infty + \frac{1}{2}\,(V_\infty^2 \sin^2 \nu - V_N^2) \quad , \tag{34}$$

$$u_s = V_N \sin \lambda + V_\infty \cos \lambda \cos \nu \quad , \tag{35}$$

$$v_s = - V_N \cos \lambda + V_\infty \sin \lambda \cos \nu \quad , \tag{36}$$

where $\nu = \lambda + \Delta - \alpha$. The constants α , V_∞, ρ_∞ , p_∞, and h_∞ are input, and for a real gas V_N is determined at each point by the Newton-Raphson method so that ρ_s , p_s , and h_s are consistent with the curvefit equation of state of equilibrium air $\rho_s = \rho\,(p_s\, ,\; h_s)$.

For a perfect gas,

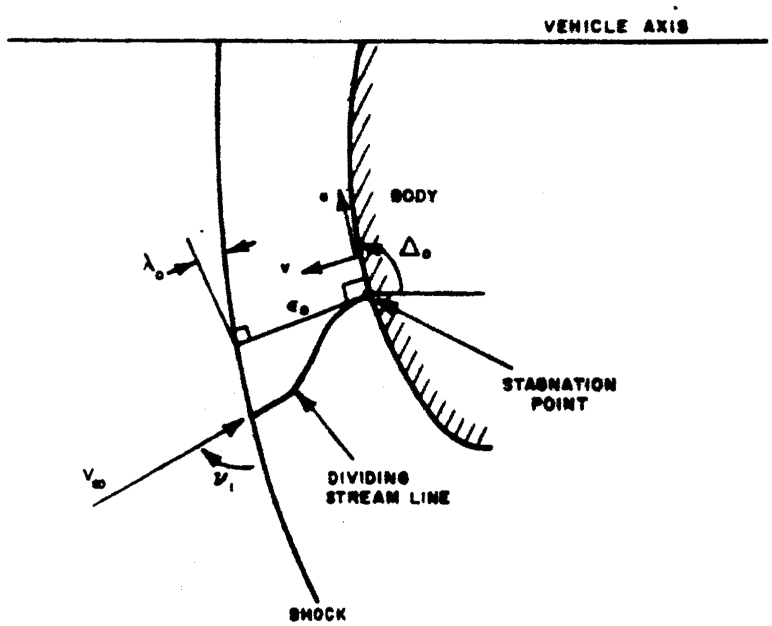
$$V_N = \frac{V_\infty \sin \nu}{\gamma + 1}\left(\frac{2}{M_\infty^2 \sin^2 \nu} + \gamma - 1\right) \quad . \tag{37}$$

Let Y be the Cartesian coordinate in the plane of symmetry indicated by figure 5. Then the shock detachment distance ϵ is related to the shock angle λ by the equation

$$\sin \Lambda \,\frac{d\epsilon}{dY} = \left(1 + \frac{\epsilon}{R_r}\right) \tan \lambda \tag{38}$$

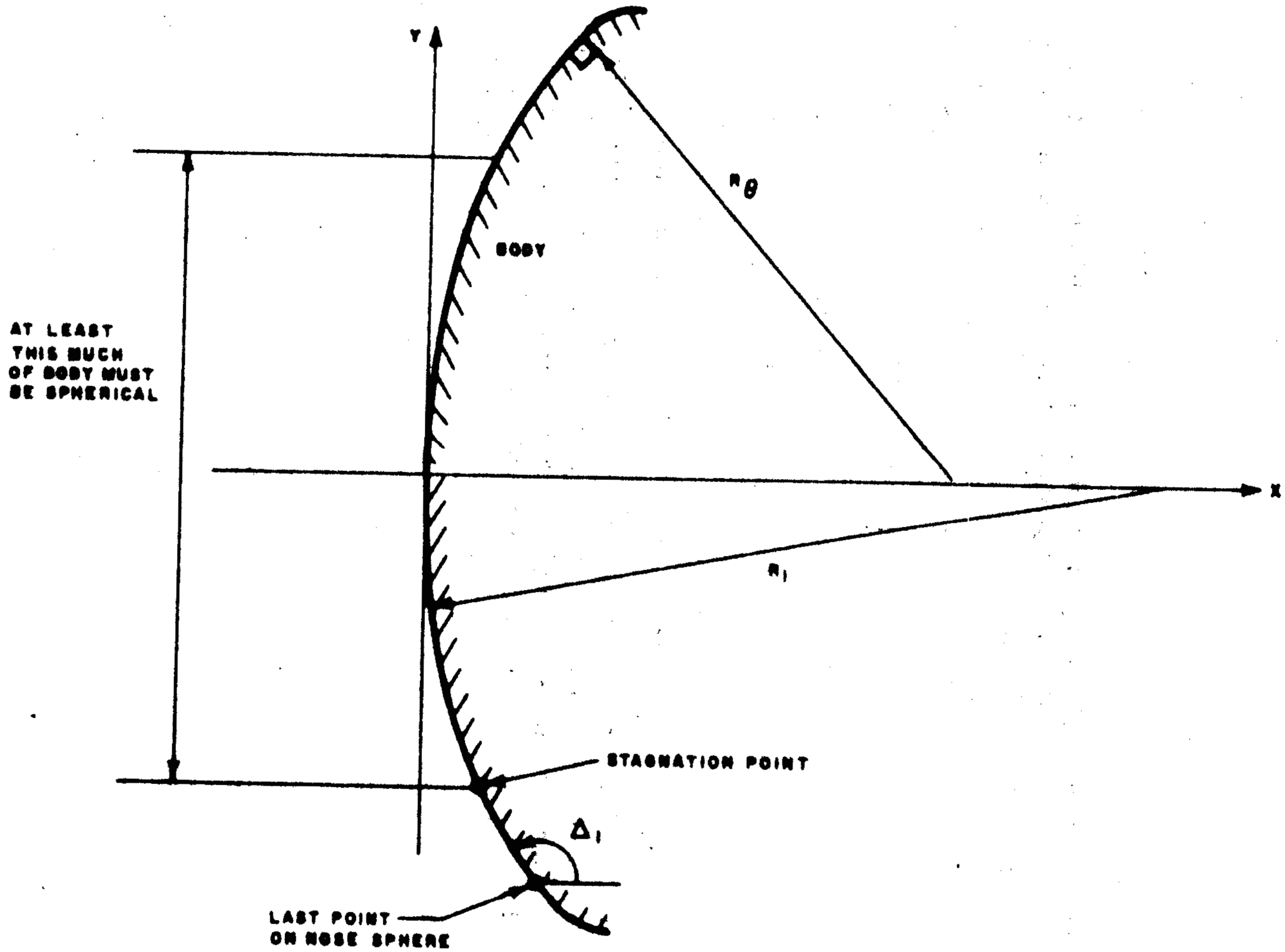
Because there are four arbitrary parameters required to start the integration and only two sonic singularity conditions as discussed in the Introduction, it is necessary to obtain additional conditions if the problem is to be well set. These additional conditions follow from an examination of the nature of the solution near the stagnation point. In the neighborhood of the stagnation point the geodesic coordinates (n, r , θ) reduce to a cylindrical coordinate system with axis normal to the body (Figure 4).

We assume that it is possible to expand the velocity components in Taylor series in r and n in the vicinity of the stagnation point:



64-4920

Figure 4 CONFIGURATION OF DIVIDING STREAMLINE



64-4921

Figure 5 GEOMETRY OF BODY IN PLANE OF SYMMETRY

$$v = f(\theta) r + c(\theta) n ,$$

$$u = a(\theta) r + d(\theta) n ,$$

$$w = b(\theta) r + e(\theta) n . \tag{39}$$

But $f(\theta) = 0$ because $v = 0$ at $n = 0$, and $c(\theta) = c$ = constant because otherwise v would not be single-valued along the n- axis.

The vorticity is given by

$$\vec{\Omega} = \left[\frac{(rw)_r - u_\theta}{r} , \quad \frac{v_\theta}{r} - w_n, \quad u_n - v_r \right] \tag{40}$$

in the (n, r , θ) directions, or

$$\vec{\Omega} = \left[2b - a_\theta + (c - d_\theta) \frac{n}{r} , \ - e, d \right] . \tag{41}$$

The momentum equation $(\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p$ gives $p_n = p_r = p_\theta = 0$ because of the vanishing velocity, and similarly the equation of conservation of stagnation enthalpy $h + 1/2\ q^2 = H$ (constant) gives $h_n = h_r = h_\theta = 0$. Hence all thermodynamic variables -- in particular, the density -- are stationary at the stagnation point by reason of the equilibrium equation of state. The continuity equation then gives

$$\rho \nabla \cdot \vec{q} + \vec{q} \cdot \nabla \rho = 0 ,$$

or, since $\nabla \rho = 0$,

$$\nabla \cdot \vec{q} = 2a + b_\theta + c + (d + e_\theta) \frac{n}{r} = 0 ;$$

thus

$$2a + b_\theta + c = 0 , \tag{42}$$

$$d + e_\theta = 0 .$$

The vorticity equation is written in the form $\nabla \times (\vec{q} \times \vec{\Omega}) = \nabla \times (\frac{1}{\rho} \nabla p)$. Since the right-hand side vanishes,

$$\nabla \times (\vec{q} \times \vec{\Omega}) = (\xi, \eta, \zeta) = 0 , \tag{43}$$

where

$$\xi = -[(2b-a_\theta)(2a+b_\theta) - (2b-a_\theta)_\theta b] - \Big[ce + (2b-a_\theta)d$$

$$+ (e-d_\theta)a + \{(e-d_\theta)b - (2b-a_\theta)e - cd\}_\theta\Big]\frac{n}{r} - [(e-d_\theta)e]_\theta\left(\frac{n}{r}\right)^2$$

$$\equiv \xi_o + \frac{n}{r}\xi_1 + \left(\frac{n}{r}\right)^2 \xi_2 \quad ,$$

$$\eta = (be)_\theta + (a+c)e + 2bd + 2e(d+e_\theta)\frac{n}{r} \equiv \eta_o + \frac{n}{r}\eta_1 \quad ,$$

$$\zeta = (2b-a_\theta)e - bd_\theta - (a+c)d + 2e(e-d_\theta)\frac{n}{r} = \zeta_o + \frac{n}{r}\zeta_1 \quad .$$

Using equation (42), $\eta_o = 0$ yields

$$bd - ae = 0 \tag{44}$$

The equation $\zeta_o = 0$ becomes

$$2b(e-d_\theta) = 0 \quad ,$$

so that

$$e - d_\theta = 0 \ . \tag{45}$$

Now $(2b - a_\theta)$ must be constant in order that the vorticity component in the n-direction (equation (41)) be single-valued. The equation $\xi_o = 0$ then implies

$$2b - a_\theta = 0 \tag{46}$$

Differentiating equation (46) with respect to θ and using the first of equations (42) to eliminate b_θ, we obtain the following differential equation for a:

$$a_{\theta\theta} + 4a + 2c = 0 \ . \tag{47}$$

Let $a(0) = A$, $b(0) = 0$ (assuming a plane of symmetry), $a(\pi/2) = B$ (A and B are the velocity gradients in two orthogonal directions away from the stagnation point on the surface). Then equation (47) has the solution

$$a = \frac{A+B}{2} + \frac{A-B}{2}\cos 2\theta \quad , \tag{48}$$

where $c = -(A + B)$ and equation (46) yields

$$b = -\frac{A - B}{2} \sin 2\theta \tag{49}$$

for the cross-flow gradient.

The two-dimensional stagnation point is represented by the special case $B = 0$. In general, the stagnation point has two axes of symmetry on the surface $(\theta = 0, \pi/2)$.

Examination of equation (45) and the second of equations (42) in the same way yields

$$d = k_1 \sin\theta - k_2 \cos\theta ,$$

$$e = -k_2 \sin\theta + k_1 \cos\theta .$$

At $\theta = 0$, equation (44) implies that $e(0) = 0 = k_1$. If in addition $B \neq 0$, then evaluation of equation (44) at $\theta = \pi/2$ gives

$$d = e \equiv 0$$

for all θ. If the stagnation point is two-dimensional, on the other hand, this conclusion cannot be drawn.

In a three-dimensional flow, therefore, there are two alternatives. First, the flow in the neighborhood of the stagnation point is analytic (i.e., can be expanded in power series). Then unless $B = 0$, all three components of the vorticity at the stagnation point are zero, and the velocity components in the neighborhood of the stagnation point have the form

$$u = a(\theta) r ,$$

$$w = b(\theta) r ,$$

$$v = cn , \tag{50}$$

where a and b are given by equations (48) and (49). The outflow velocity u and the cross flow w vanish along the axis perpendicular to the body, and the stagnation streamline is perpendicular to the surface. If $B = 0$, however (two-dimensional flow), this is not so, and the stagnation streamline subtends an angle to the surface which is dependent on the way it passes through the shock rather than on conditions near the stagnation point.

The second possibility is that the flow does not have an analytic solution near the stagnation point. This possibility is rejected in the present context because it is inconsistent with the assumption that the flow variables can be approximated by analytic functions. Hayes[18] explores this possibility in detail and arrives at several interesting conclusions. It seems questionable, however, whether an actual subsonic flow field would support the nonanalytic stagnation points which Hayes considers.

The meaning of the argument presented above in the context of the one-strip approximation being considered in this paper is that the dividing streamline becomes a straight line perpendicular to the body. This can be seen from the fact that the velocity component "u" lacks any linear dependence on n in the vicinity of the stagnation point, so that, to be consistent with the linear approximation, it must vanish identically along the stagnation point normal. In addition, the fundamental differential equation for u_b cannot be satisfied at the stagnation point unless the value of u at the shock point opposite the stagnation point vanishes.

It is therefore concluded that the initial shock angle must be such that $u_{so} = 0$, and that the body entropy has the value associated with that shock angle. The problem then essentially reduces to the determination of the proper values of two parameters, the stagnation point location Λ_o and initial shock detachment distance ϵ_o, which will allow the solution to be continued through the two sonic point singularities. The shock angle is determined at the initial point by setting the right-hand side of equation (35) equal to zero and performing the usual Newton-Raphson iteration subject to this constraint.

Vaglio-Laurin[19] originated the concept of $u_{so} = 0$ being used to determine the missing starting conditions for the one-strip method of integral relations, although his argument is different from that used above. He generalizes his argument to cover the case of N strips being taken between the body and the shock, and arrives at a streamline which is not normal to the body at the stagnation point for $N > 1$.

The above argument can readily be generalized to the case of N strips. The initial conditions are then most easily visualized if they are formulated along the dividing streamline. This curve is given by some unknown equation $r = r(n)$. For each n_i ($i = 0, 1, \ldots, N$ for N strips) there is an unknown r_i, giving a total of $(N + 1)$ unknown r_i. Also the value of n_N (standoff distance) is unknown (the other n_i can be specified as fractions of this quantity). The unknown shock angle λ at the dividing streamline determines the values of the variables at the shock together with the dividing (body) entropy. The two velocity components along the dividing streamline are related by

$$\frac{v}{u} = \frac{1}{h_1}\frac{dn}{dr} \quad .$$

and both vanish at the body. Hence there are $(N - 1)$ unknown u_i, say. All other properties along the dividing streamline can be determined from these $(2N + 2)$ unknowns by the equation of conservation of stagnation enthalpy and the equation of state. The $2N$ sonic singularity conditions determine all but two of them.

The two remaining unknowns are determined by expressing the unknown functions $r(n)$ and $u(n)$ along the dividing streamline in terms of expansions appropriate to the N-strip approach:

$$r = r_o + r_1 \frac{n}{\epsilon} + r_2 \left(\frac{n}{\epsilon}\right)^2 \cdots\cdots$$

$$u = u_1 \frac{n}{\epsilon} + u_2 \left(\frac{n}{\epsilon}\right)^2 \cdots\cdots$$

Since the streamline is perpendicular to the surface at the stagnation point and $\frac{\partial u}{\partial n} = 0$ there, we require

$$r_1 = u_1 = 0 ,$$

two relationships between the unknowns which are sufficient to yield a well-set problem.

IV. FINAL FORMULATION AND METHOD OF SOLUTION

Having introduced the Cartesian coordinate Y in the previous section (see figure 5) and the velocity component U (equal in absolute value to u but positive on the leeward side of the stagnation point and negative on the windward), we can now write the differential equations corresponding to (38), (22), and (21), respectively:

$$\sin \Lambda \, \frac{d\epsilon}{dY} = \left(1 + \frac{\epsilon}{R_r}\right) \tan \lambda \, , \tag{51}$$

$$\begin{aligned} \sin \Lambda \, \frac{d}{dY}\left[\left(\frac{1}{\epsilon} - \frac{1}{R_\theta}\right) g \rho U v\right]_s &= \frac{g}{\epsilon}\left[\frac{\rho U^2}{R_r} + \left(\frac{2}{\epsilon} + \frac{1}{R_r} - \frac{1}{R_\theta}\right) p\right]_b \\ &+ \left[\left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right) g \left\{\frac{\rho U^2}{R_r} - 2\left(\frac{1}{\epsilon} + \frac{1}{R_r}\right) \rho v^2\right\} - \left(\frac{1}{\epsilon} + \frac{1}{R_r}\right) \rho v W \right. \\ &\left. - \left(\frac{2}{\epsilon} + \frac{1}{R_r} - \frac{1}{R_\theta}\right) \frac{gp}{\epsilon}\right]_s \, , \end{aligned} \tag{52}$$

$$\begin{aligned} \left[\left(1 - \frac{U^2}{a^2}\right) g \rho \sin \Lambda \, \frac{dU}{dY}\right]_b &= -\left[\frac{\rho U}{\epsilon} \sin \Lambda \, \frac{d}{dY} (\epsilon g) - \rho W\right]_b \\ - \epsilon \left[\sin \Lambda \, \frac{d}{dY}\left\{\left(\frac{1}{\epsilon} - \frac{1}{R_\theta}\right) g \rho U\right\} + \left(\frac{1}{\epsilon} + \frac{1}{R_r}\right)\left\{2\left(\frac{1}{\epsilon} - \frac{1}{R_\theta}\right) g \rho v + \rho W\right\}\right]_s &= F \, . \end{aligned} \tag{53}$$

Here

$$h_b = H - \frac{1}{2} U_b^2 \qquad \left(H = h_\infty + \frac{1}{2} V_\infty^2\right) \, , \tag{54}$$

$$h_s = H - \frac{1}{2} (U_s^2 + v_s^2) \, . \tag{55}$$

$$\left.\begin{aligned} \rho_b &= \rho (h_b, S_b) \\ p_b &= p (h_b, S_b) \end{aligned}\right\} \quad \text{(equation of state of equilibrium air given by curve fits with } S_b = \text{constant),} \tag{56}$$

$$a_b = a (h_b, S_b) \quad \text{(curve-fit speed of sound).} \tag{57}$$

Using the shock conditions derived in the previous section, the following final equations are obtained. The derivations have been omitted.

$$\frac{d\epsilon}{dY} = \left(1 + \frac{\epsilon}{R_r}\right)\frac{\tan\lambda}{\sin\Delta} \quad , \tag{58}$$

$$\frac{d\lambda}{dY} = -\frac{D}{C}\frac{d\Delta}{dY} + \frac{1}{Cg\left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right)\sin\Delta}\left[G + g\rho Uv\sin\Delta\left(\frac{1}{\epsilon^2}\frac{d\epsilon}{dY}\right.\right.$$

$$\left.\left. + \frac{1}{R_\theta^2}\frac{dR_\theta}{dY}\right)\right] - \frac{\rho Uv}{Cg}\frac{dg}{dY} \quad , \tag{59}$$

$$\frac{dU_b}{dY} = \frac{F}{\left[\rho\left(1 - \frac{U^2}{a^2}\right)\right]_b g\sin\Delta} \quad , \tag{60}$$

where

$$C = (\rho v)_s\left[\frac{B}{A}\sin\lambda + V_N\cos\lambda - V_\infty\sin(\nu + \lambda)\right]$$

$$+ (\rho U)_s\left[-\frac{B}{A}\cos\lambda + V_N\sin\lambda + V_\infty\cos(\nu + \lambda)\right] \tag{61}$$

$$+ (Uv)_s\,\rho_\infty V_\infty\left(\frac{\cos\nu}{V_N} - \frac{B}{A}\frac{\sin\nu}{V_N^2}\right) \quad ,$$

$$D = (\rho v)_s\left[\frac{B}{A}\sin\lambda - V_\infty\sin\nu\cos\lambda\right]$$

$$- (\rho U)_s\left[\frac{B}{A}\cos\lambda + V_\infty\sin\nu\sin\lambda\right] \tag{62}$$

$$+ (Uv)_s\,\rho_\infty V_\infty\left(\frac{\cos\nu}{V_N} - \frac{B}{A}\frac{\sin\nu}{V_n^2}\right) \quad ,$$

$$A = \left[-\frac{\rho_\infty V_\infty\sin\nu}{V_N^2} + \rho_\infty V_\infty\sin\nu\left(\frac{\partial\rho}{\partial p}\right)_h + V_N\left(\frac{\partial\rho}{\partial h}\right)_p\right]_s \quad , \tag{63}$$

$$B = \frac{\rho_\infty V_\infty \cos\nu}{V_N} - \left\{ [2 V_\infty \sin\nu \cos\nu - V_N \cos\nu] \rho_\infty V_\infty \left(\frac{\partial\rho}{\partial p}\right)_h - V_\infty^2 \sin\nu \cos\nu \left(\frac{\partial\rho}{\partial h}\right)_p \right\}_s \quad (64)$$

$$G = \frac{g}{\epsilon}\left[\frac{\rho U^2}{R_r} + \left(\frac{2}{\epsilon} + \frac{1}{R_r} + \frac{1}{R_\theta}\right) p\right]_b + \left[\left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right) g \left\{\frac{\rho U^2}{R_r} - 2\left(\frac{1}{\epsilon} + \frac{1}{R_r}\right)\rho v^2\right\}\right.$$

$$\left. - \left(\frac{1}{\epsilon} + \frac{1}{R_r}\right)\rho v W - \left(\frac{2}{\epsilon} + \frac{1}{R_r} + \frac{1}{R_\theta}\right)\frac{gp}{\epsilon}\right]_s \quad , \quad (65)$$

$$F = -\left[\frac{\rho U}{\epsilon} \sin\Lambda \frac{d}{dY}(\epsilon g) + \rho W\right]_b - \epsilon\left[\sin\Lambda \left\{ g\rho U\left(\frac{1}{\epsilon^2}\frac{d\epsilon}{dY} - \frac{1}{R_\theta^2}\frac{dR_\theta}{dY}\right)\right.\right.$$

$$- \left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right)\rho U \frac{dg}{dY} + \left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right) g\left[\frac{d\lambda}{dY}\left\{\rho \frac{B}{A}\sin\lambda + \rho V_N \cos\lambda - \rho V_\infty \sin(\lambda+\nu)\right.\right.$$

$$\left.\left.\left. + U\rho_\infty V_\infty\left(\frac{\cos\nu}{V_N} - \frac{B}{A}\frac{\sin\nu}{V_N^2}\right)\right\}\right]\right\} + \left(\frac{1}{\epsilon} + \frac{1}{R_r}\right)\left\{2\left(\frac{1}{\epsilon} + \frac{1}{R_\theta}\right) g p v + \rho W\right\}\Bigg]_s \quad (66)$$

Here

$$\left(\frac{\partial\rho}{\partial p}\right)_h = -\frac{1}{\rho T}\left(\frac{\partial\rho}{\partial S}\right)_h \quad ,$$

$$\left(\frac{\partial\rho}{\partial h}\right)_p = \frac{\rho}{a^2} + \frac{1}{T}\left(\frac{\partial\rho}{\partial S}\right)_h \quad (67)$$

are given in terms of the curve-fit function $(\partial\rho/\partial S)_h$. For a perfect gas,

$$\left(\frac{\partial\rho}{\partial S}\right)_h = -\frac{\rho}{R} \quad , \quad (68)$$

where R is the gas constant.

The cross-flow W is given by

$$W = \frac{\partial w}{\partial\theta} = \pm\left[\left\{\frac{U(Y) + U(-Y + 2R_1\cos\Delta_o)}{2}\right\}^2 + \left\{\frac{v(Y) - v(-Y + 2R_1\cos\Delta_o)}{2}\right\}^2\right]^{1/2} \quad (69)$$

where the plus sign is selected for $Y < R_1 \cos \Delta_o$ and the minus sign for $Y \geq R_1 \cos \Delta_o$.

We now specify the geometric parameters appearing in the above equations. As indicated in Fig. 5, the body is assumed to be a sphere with radius R_1 at least up to the stagnation point. At the last point on the nose sphere, $\Delta = \Delta_1$. Beyond this point the body shape either may be given in tabular form or, if the input constant R_2 is specified, may be a tangent torus with corner radius R_2 .

The function $\Delta = \Delta(Y)\,[0 < \Delta < \pi,\ \Delta(-Y) = \pi - \Delta(Y)]$ is defined as follows:

for $|Y| \leq -R_1 \cos \Delta_1$, $\quad \Delta = \cos^{-1}\left(\frac{Y}{R_1}\right)$;

for $|Y| > -R_1 \cos \Delta_1$, $\Delta(Y)$ is either given in tabular form as input $(Y > 0)$,

or, if a value of the input constant R_2 is given, $\Delta = \cos^{-1}\left[\frac{Y + (R_1 - R_2)\cos \Delta_1}{R_2}\right]$ for $Y > 0$.

Here R_1, R_2, and $\Delta_1\,(\pi > \Delta_1 > \Delta_o > \pi/2)$ are input constants.

The radius of curvature R_r of the body contour in the plane of symmetry is defined as follows:

$$R_r = \begin{cases} R_1 & \text{for } |Y| \leq -R_1 \cos \Delta_1 \\ R_2 & \text{for } |Y| > -R_1 \cos \Delta_1 \text{ and } R_2 \text{ given} \\ \dfrac{-1}{\dfrac{d\Delta}{dY} \sin \Delta} & \text{for } |Y| > -R_1 \cos \Delta_1 \text{ and } R_2 \text{ not given} \end{cases} \tag{70}$$

The radius R_θ in the plane of symmetry is simply the distance from the body to the body axis of symmetry, measured along the normal to the body:

$$R_\theta = \begin{cases} R_1 & \text{for } |Y| \leq -R_1 \cos \Delta_1 \\ \dfrac{Y}{\cos \Delta} & \text{for } |Y| > -R_1 \cos \Delta_1 \end{cases} , \tag{71}$$

The metric g measures the surface distance between corresponding points on adjacent geodesics, divided by the meridian angle between them at the stagnation point. In the plane of symmetry, this becomes in terms of the coordinate Y

$$g = Y R_1 \cos \Delta_o \int_0 \frac{dY}{Y^2 \sin \Delta}$$

where the integration is carried out from the stagnation point $Y_o = R_1 \cos \Delta_o$. The integral is singular at $Y = 0$. This behavior is circumvented by noting that on the nose sphere g is equal to the cylindrical radius in a coordinate system oriented with respect to the stagnation point. The equation above can then be applied on the body contour excluding the nose sphere, resulting in the definition:

$$g = \begin{cases} R_1 \sin(\Delta_o - \Delta_1) & \text{for } |Y| \le -R_1 \cos \Delta_1 \\[2ex] \dfrac{Y \sin(\Delta_o + \Delta_1)}{\cos \Delta_1} - Y R_1 \cos \Delta_o \displaystyle\int_{-R_1 \cos \Delta_1}^{Y} \frac{dY}{Y^2 \sin \Delta} & \text{for } Y > -R_1 \cos \Delta_1 \\[2ex] \dfrac{Y \sin(\Delta_1 - \Delta_o)}{\cos \Delta_1} - Y R_1 \cos \Delta_o \displaystyle\int_{R_1 \cos \Delta_1}^{Y} \frac{dY}{Y^2 \sin \Delta} & \text{for } Y < R_1 \cos \Delta_1 \end{cases} \qquad (72)$$

Since $g_o = 0$ (the value of g at the stagnation point), the governing equation for U_b has an indeterminate form at the stagnation point. This indeterminacy must be removed by expanding locally in $Y - Y_o$. In the vicinity of the stagnation point,

$$g = \frac{Y - Y_o}{\sin \Delta_o}$$

$$U = (Y - Y_o) \frac{dU}{dY} ,$$

$$v = (Y - Y_o) \left| \frac{dv}{dY} \right| .$$

Substituting in equation (53) and dividing by $Y - Y_o$, we obtain the initial gradient of the body velocity:

$$\rho \left[\frac{dU}{dY} \right]_{b_o} = - \left[\rho \left(1 - \frac{\epsilon}{R_1} \right) \left\{ \frac{dU}{dY} \pm \frac{1}{2} \left| \frac{dv}{dY} \right| + \frac{v}{\epsilon \sin \Delta} \left(1 + \frac{\epsilon}{R_1} \right) \right\} \right]_{s_o} \qquad (73)$$

$$(+ \text{ for } Y < R_1 \cos \Delta_o, \ - \text{ for } Y \ge R_1 \cos \Delta_o) .$$

The method of solution is as follows. An initial guess is made of the unknowns Δ_o and ϵ_o. In the absence of experimental data or the like, the stagnation point location can be approximated as the point at which the body surface is perpendicular to the stream direction:

$$\Delta_o^{(1)} = \frac{\pi}{2} + \alpha . \tag{74}$$

Also the detachment distance can be approximated by the zero-yaw value for a sphere, which can be correlated fairly well by the formula

$$\epsilon_o^{(1)} = 0.715 \frac{\rho_\infty R_1}{\rho_{s_o}} \tag{75}$$

It is likely, however, that equation (75) in particular will be inaccurate for most cases of interest. Fortunately, excellent data is available for Δ_o and ϵ_o (e.g., reference 20), and at worst a series of preliminary machine program runs can be made to bracket the starting values of Δ_o and ϵ_o before the automatic iteration procedure in the program becomes effective.

The shock equations (32)-(36) are solved for $U_{s_o} = 0$ to determine the starting conditions corresponding to the initial choice of starting parameters. These values are substituted into equation (73) to determine the initial gradient of the body velocity. The differential equations (51), (52), and (53), are then integrated simultaneously for $Y \geq R_1 \cos \Delta_o, U \geq 0$ and $Y < R_1 \cos \Delta_o, U < 0$. Essentially they are considered as a set of six equations, three for the plus (leeward) direction and three for the minus (windward) direction. Positive and negative increments in Y are equal for each integration step. The first few integration steps are computed by a Runge-Kutta technique; subsequently, a predictor-corrector integration procedure is used which varies the Y-intervals according to the agreement of the integrated values with values computed by extrapolation from previous steps.

In the process of integration, the program eventually reaches two points, one in each direction, at which either $F = 0$ or $1 - \frac{U_b^2}{a_b^2} \to 0$ (whichever happens first). If $F = 0$, U_b reaches a stationary point and subsequently decreases in magnitude. If $1 - \frac{U_b^2}{a_b^2} \to 0$, U_b varies rapidly and the predictor-corrector integration routine reduces the interval between successive steps to an unacceptably small value. The location of these points is determined in the following manner:

(a) If $F = 0$ the solution is interpolated to the precise point where $F = 0$ and the quantity

$$d_1 = \left(1 - \frac{U_b^2}{a_b^2}\right)_{F = 0} \tag{76}$$

is computed at the point.

(b) If $1 - \frac{U_b^2}{a_b^2} \to 0$ (i.e., the integration interval is reduced beyond a certain point), the solution is extrapolated to the supposed point where $1 - \frac{U_b^2}{a_b^2} = 0$ and the quantity

$$d_2 = (F)\Big|_{1 - \frac{U_b^2}{a_b^2} = 0} \tag{77}$$

is computed at that point.

In practice, one of these points is reached before the other. When the first singular point is reached, the integration corresponding to the side (windward or leeward) on which the point lies is suspended. Subsequent values of Ψ, which are needed to continue the integration in the other direction, are computed by a least-squares curve fit of all the preceding values of Ψ. The integration is then continued in the other direction until the other singular point is reached, when steps (a) and (b) are repeated.

Thus each iteration corresponding to a pair of values for Λ_o and ϵ_o produces a pair of values d_{i_+} and d_{i_-} ($i = 1$ or 2), the first on the leeward side and the second on the windward. (Object: to reduce the values d_{i_+} and d_{i_-} in magnitude until the solution can be continued through the two sonic points). This is accomplished by a Newton-Raphson scheme in which successive values of d_{i_+} and d_{i_-} are estimated by numerically forming "derivatives" of these quantities with respect to Λ_o and ϵ_o. For instance, after the solution $d_{i_\pm}^{(1)}$ corresponding to the initial guess of $\Lambda_o^{(1)}$ and $\epsilon_o^{(1)}$ has been computed, two other solutions are computed, corresponding to the value

$$\begin{aligned} \Lambda_o^{(2)} &= \Lambda_o^{(1)} + \mu_\Lambda \quad , \\ \epsilon_o^{(2)} &= \epsilon_o^{(1)} \quad ; \\ \Lambda_o^{(3)} &= \Lambda_o^{(1)} \quad , \\ \epsilon_o^{(3)} &= \epsilon_o^{(1)} + \mu_\epsilon \quad . \end{aligned} \tag{78}$$

where μ_Δ and μ_ϵ are small input constants. The following derivatives are then computed:

$$\frac{\partial d_{i_\pm}}{\partial \Delta_o} = \frac{d^{(2)}_{i_\pm} - d^{(1)}_{i_\pm}}{\mu_\Delta} , \quad \frac{\partial d_{i_\pm}}{\partial \epsilon_o} = \frac{d^{(3)}_{i_\pm} - d^{(1)}_{i_\pm}}{\mu_\epsilon} . \tag{79}$$

The predicted values of $\Delta_o^{(4)}$ and $\epsilon_o^{(4)}$ are given by

$$\Delta_o^{(4)} = \Delta_o^{(1)} + \frac{f}{J}\left(d^{(1)}_{i_-} \frac{\partial d_{i_+}}{\partial \epsilon_o} - d^{(1)}_{i_+} \frac{\partial d_{i_-}}{\partial \epsilon_o}\right) , \tag{80}$$

$$\epsilon_o^{(4)} = \epsilon_o^{(1)} - \frac{f}{J}\left(d^{(1)}_{i_-} \frac{\partial d_{i_+}}{\partial \Delta_o} - d^{(1)}_{i_+} \frac{\partial d_{i_-}}{\partial \Delta_o}\right) ,$$

where

$$J = \frac{\partial d_{i_+}}{\partial \Delta_o} \frac{\partial d_{i_-}}{\partial \epsilon_o} - \frac{\partial d_{i_-}}{\partial \Delta_o} \frac{\partial d_{i_+}}{\partial \epsilon_o} ,$$

and where f is a factor less than or equal to unity. For the usual Newton-Raphson scheme, f would be set equal to unity. Convergence may be achieved more quickly, however, if f is given a fractional value. The reason is that the program must stay on the same side of the singularities from iteration to iteration - in other words, if $F = 0$ at the leeward singularity on the first iteration (case (a)), then F must go to zero before $1 - \frac{U_b^2}{a_b^2}$ goes to zero at the leeward singularity on all succeeding iterations. If this condition is not met, the Newton-Raphson scheme becomes meaningless because the function d_{i_+} which it is attempting to minimize is not smooth. Therefore if the program jumps to the other side of a singularity during an iteration, it is required to reduce the increments in Δ_o and ϵ_o systematically until it returns to the original side of the singularity. Giving f a fractional value tends to make this behavior less likely. The optimum value for f can only be determined by experimentation.

The solution is iterated until successive values of Δ_o and ϵ_o satisfy the following convergence criterion:

$$\left| \Lambda_o^{(n)} - \Lambda_o^{(n-1)} \right| \leq \kappa \, \mu_\Lambda \, ,$$
$$\left| \epsilon_o^{(n)} - \epsilon_o^{(n-1)} \right| \leq \kappa \, \mu_\epsilon \quad . \tag{81}$$

Typical values are $\mu_\Lambda = \mu_\epsilon = 10^{-7}$, $\kappa = 10$.

V. MACHINE PROGRAM

A machine program has been written for the IBM 7094 computer based on the foregoing analysis. The running time is about 5 to 10 minutes for a given body shape and angle of attack and for a perfect gas.

Figures 6 and 7 show preliminary results from this program for a perfect gas. Figure 6 is a comparison of the zero-yaw body velocity distribution at Mach 10 on a spherical body with unit radius; the comparsion curve was computed by the one-strip program of Shih et al.[21] as adapted by Springfield[22] at Avco RAD. The computed shock detachment distance was 0.12601, as compared with the value 0.12662 from the Shih program. The difference is due to a different choice of dependent variables.

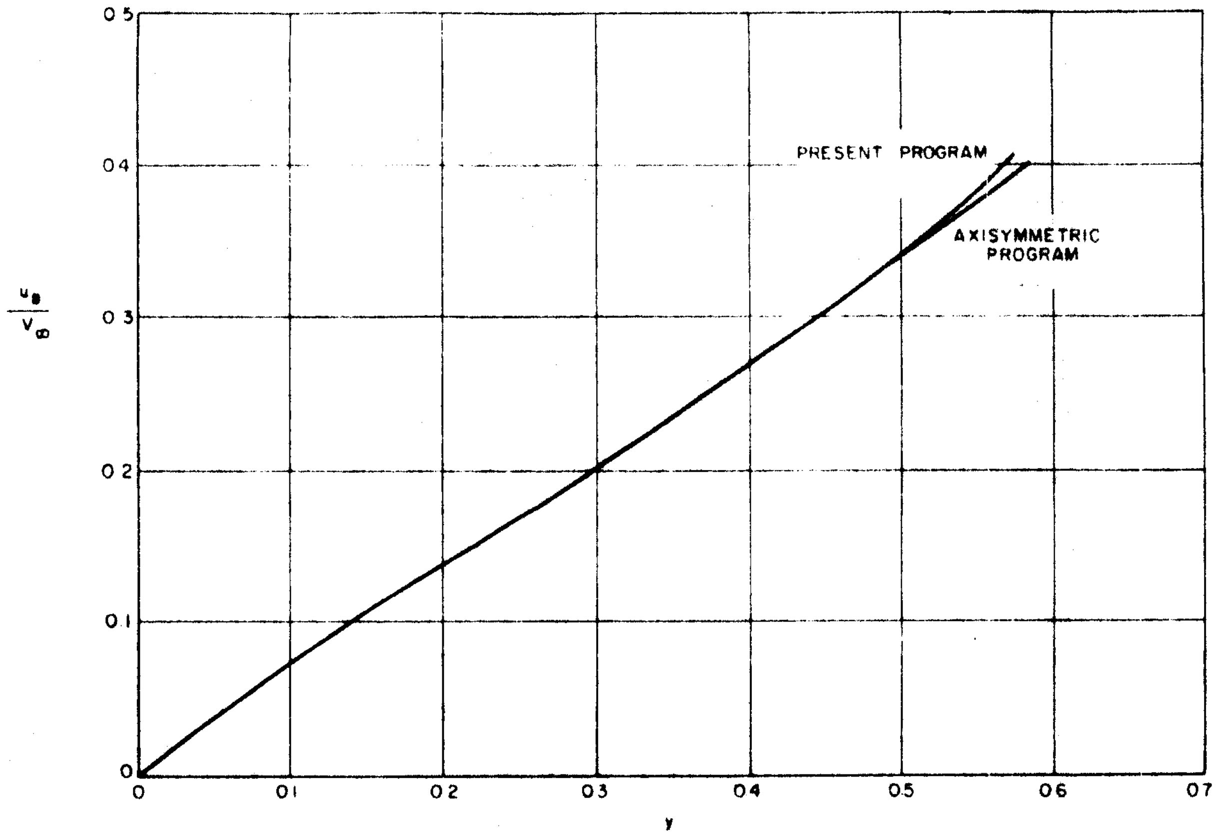
On figure 7 is plotted the body velocity distribution on the same sphere as computed at $\alpha = 0$ and $\alpha = 15$ degrees. The coordinate y_f is a wind-oriented Cartesian coordinate. Since the flow field on the sphere simply rotates with the free-stream vector, ideally the curves as plotted this way would coincide. The following values resulted from the computation: $\epsilon_0 = 0.12826$, $\Lambda_0 = 104.96$ degrees.

It is seen from figures 6 and 7 that the velocity distributions are indistinguishable except near the sonic points, where the disagreement is due to a rather loose convergence requirement on Λ_0 and ϵ_0 rather than to a deficiency in the program.

Following is a summary of the input to Program 1560. Many of the values are preset, but can be altered to suit the requirements of any particular problem.

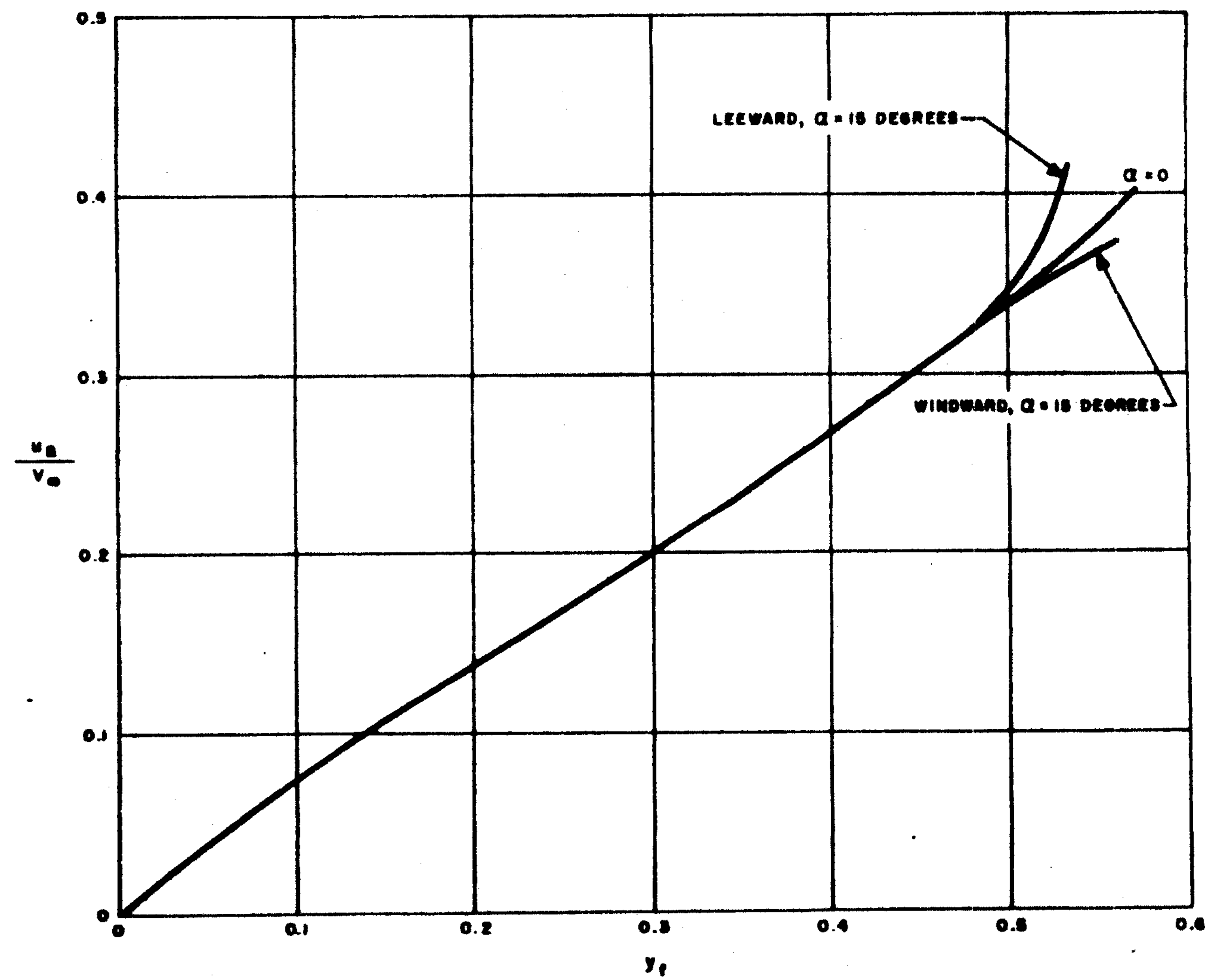
INPUT

H8	h_∞	Free-stream conditions
V8	V_∞	
RØW8	ρ_∞	
R1	R_1	
R2	R_2	



64-4922

Figure 6 BODY VELOCITY DISTRIBUTION, SPHERE, $\alpha = 0$, $M_\infty = 10$



64-5284

Figure 7 BODY VELOCITY DISTRIBUTION, SPHERE, $\alpha = 0$, AND 15°, $M_{\infty} = 10$

DI *	Λ_1	
ALPHA*	α	Angle of attack
GAMMA	γ	Ratio of specific heats
R	R	Gas constant
SOR	S_o/R	Reference (sea level) conditions
RØWO	ρ_o	Reference (sea level) conditions
HO	h_o	Reference (sea level) conditions
TO	T_o	Reference (sea level) conditions
GASØP	Gas option (0 perfect gas; 1 real gas)	
CØNV	Convergence criterion for λ_o iteration (10^{-9} for perfect gas)	
UPBND	Upper bounds for predictor-corrector	
DNBND	Lower bounds for predictor-corrector	
FACTØR	Factor to change predictor-corrector when bounds are exceeded	
EFACT	Initial value of ϵ_o (zero results in calculation from correlation)	
DELO	Initial value of Λ_o (zero causes $\Lambda_o = \pi/2 - \alpha$)	
EM	μ_ϵ - derivative change in ϵ_o for Newton-Raphson	
DM	μ_Λ - derivative change in Λ_o for Newton-Raphson	
NØLØØP	Number of Newton-Raphson iterations allowed	
XMUK	κ - factor in Newton-Raphson	
DT	Initial integration interval in y	
SQØP	Newton-Raphson square option (1 causes square of d_{i_4} to be used; 0 causes exact value to be used)	
FIXNEW	f - fraction of indicated change of Newton-Raphson which will utilized	

PRINT	Print option (0 prints only last integration; 1 prints all integrations)
N	N - Profile printed out every N^{th} integration step
L	L - (L + 1) points on each profile
*	degrees
'	radians

Aside from the parameters ϵ_o, Δ_o, λ_o, S_b, the output of the program consists of a series of profiles of the flow variables from the body to the shock. These profiles are computed during the final iteration of the program. They are based on the assumed linear profiles for the P_i:

$$P_i = (1-\zeta)\,P_{i_b} + \zeta P_{i_s} \quad .$$

The following equations result:

$$\rho U = \frac{(1-\zeta)(\rho U)_b + \zeta\left(1+\frac{\epsilon}{R_\theta}\right)(\rho U)_s}{1+\frac{\zeta\epsilon}{R_\theta}} \quad ,$$

$$p = \frac{(1-\zeta)(p+\rho U^2)_b + \zeta\left(1+\frac{\epsilon}{R_\theta}\right)(p+\rho U^2)_s}{1+\frac{\zeta\epsilon}{R_\theta}} - \rho U^2 \quad , \tag{82}$$

$$v = \frac{\zeta\left(1+\frac{\epsilon}{R_\theta}\right)(\rho U v)_s}{(1-\zeta)(\rho U)_b + \zeta\left(1+\frac{\epsilon}{R_\theta}\right)(\rho U)_s} \quad ,$$

$$W = \frac{(1-\zeta)(\rho U W)_b + \zeta\left(1+\frac{\epsilon}{R_\theta}\right)(\rho U W)_s}{(1-\zeta)(\rho U)_b + \zeta\left(1+\frac{\epsilon}{R_\theta}\right)(\rho U)_s} \quad .$$

The profiles are printed out every N^{th} integration step, where N is an input integer. At each step there are L+1 points in all on a profile, including the body and the shock, where L is another input integer. The points are equally spaced and are computed from the formula

$$\zeta = \frac{l}{L} \ , \ l = 0, 1, 2, \ldots, L \quad .$$

At each point on a profile, equations (82) are solved by a straightforward iterative scheme involving the equation of conservation of stagnation enthalpy and the equation of state.

The variables printed out are

$$U, v, W, V, \theta, p, \rho, h, S, a,$$

together with Λ, V_N, λ, ν, and ϵ, where

$$V = \sqrt{U^2 + v^2}$$

is the fluid speed and

$$\theta = \begin{cases} \Lambda + \tan^{-1} \dfrac{v}{U} & \text{for } U > 0 \\ \Lambda + \tan^{-1} \dfrac{v}{U} - \pi & \text{for } U < 0 \\ \Lambda - \pi/2 & \text{for } U = 0 \end{cases}$$

is the flow-deflection angle with respect to the vehicle axis.

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