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# COMPUTATIONAL PROCEDURE FOR VINTI'S ACCURATE REFERENCE ORBIT WITH INCLUSION OF THE THIRD ZONAL HARMONIC

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## ABSTRACT

Vinti has recently modified his spheroidal potential so as to permit exact inclusion of the effects of the third zonal harmonic of the planet's gravitational field. This corresponds to a potential fitted exactly through the third zonal harmonic and about two-thirds of the fourth. The present paper treats the method for obtaining the position and velocity coordinates of a satellite moving in the field corresponding to this accurate modified potential.

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# COMPUTATIONAL PROCEDURE FOR VINTI'S ACCURATE REFERENCE ORBIT WITH INCLUSION OF THE THIRD ZONAL HARMONIC

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## INTRODUCTION

Vinti (Reference 1) has found a gravitational potential for an axially symmetric planet in oblate spheroidal coordinates. This solution accounts for all of the second zonal harmonic and more than half of the fourth zonal harmonic. This potential, which simultaneously satisfies Laplace's equation and separates the Hamilton Jacobi equation, succeeds in reducing the problem of satellite motion to quadratures.

More recently however, Vinti (Reference 2) has generalized his potential by means of a metric preserving transformation of the associated Cartesian system. This preserves separability of the problem of orbital motion when the potential coefficients $J_{2,1}$ and $J_3$ are taken into account (Reference 3). The inclusion of $J_3$ is of considerable practical importance, permitting a more accurate treatment than that given by perturbation theory. This leads to the computing procedure for obtaining the position and velocity coordinates of a drag free satellite from a knowledge of its initial conditions.

## STATEMENT OF THE PROBLEM

If we take $r_e$ as the earth's equatorial radius, and if

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right),$$

and

$$\delta = -\frac{1}{2} r_e J_2^{-1} J_3 ,$$

then the gravitational potential

$$V = -\mu\left(\rho^2 + c^2 \eta^2\right)^{-1} (\rho + \eta\delta)$$

leads to a separability of the problem of satellite motion. Here, $\delta \approx +7$ kilometers for the earth and the above potential leads to a fit of

$$V = -\frac{\mu}{r}\left[1 - \sum_{2}^{\infty}\left(\frac{r_e}{r}\right)^n J_n P_n(\sin\theta)\right]$$

exactly through the third zonal harmonic, about two-thirds of the fourth zonal harmonic, and negligible values of order $J_2^3$ for the higher harmonics.

If we take $\rho$, $\eta$, and $\phi$ as oblate spheroidal coordinates satisfying the equations

$$\begin{aligned} x + iy &= r\cos\theta\, e^{i\phi} \\ &= \left[\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)\right]^{1/2} e^{i\phi} , \end{aligned}$$

and

$$z = r\sin\theta = \rho\eta - \delta ,$$

then $x$, $y$, and $z$ are the rectangular coordinates of a satellite in a Cartesian frame, with the origin at the center of mass of the planet. Also, $r$, $\theta$, and $\phi$ are the planetocentric distance, latitude, and right ascension.

From Vinti (Reference 4), if $\alpha_1$ is the energy of the system, $\alpha_3$ is the $z$ component of angular momentum, and $\alpha_2$ is the separation constant, then the generalized momenta are given by $p_\phi = \alpha_3$,

$$p_\rho = \pm\left(\rho^2 + c^2\right)^{-1} F^{1/2}(\rho) ,$$

and

$$p_\eta = \pm\left(1 - \eta^2\right)^{-1} G^{1/2}(\eta) .$$

Here $F(\rho)$ and $G(\eta)$ are the quartics:

$$F(\rho) = c^2\alpha_3^2 + \left(\rho^2 + c^2\right)\left(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2\right) ,$$

and

$$G(\eta) = -\alpha_3^2 + \left(1 - \eta^2\right)\left(\alpha_2^2 + 2\mu\eta\delta + 2\alpha_1 c^2\eta^2\right) .$$

The Hamilton-Jacobi function $W(\rho, \eta, \phi)$ is then

$$W = \int p_\phi \, d\phi + \int p_\rho \, d\rho + \int p_\eta \, d\eta ,$$

or

$$W = \alpha_3 \phi + \int_{\rho_1}^{\rho} \pm \left(\rho^2 + c^2\right)^{-1} F^{1/2}(\rho) \, d\rho + \int_{\eta_1}^{\eta} \pm \left(1 - \eta^2\right)^{-1} G^{1/2}(\eta) \, d\eta$$

If $\beta_1$, $\beta_2'$ and $\beta_3'$ are constants of the motion, the orbit is then given by

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} , \qquad \beta_2' = \frac{\partial W}{\partial \alpha_2} ,$$

and

$$\beta_3 = \frac{\partial W}{\partial \alpha_3} .$$

From Vinti's solution of these equations (Reference 3), together with the expressions for the generalized momenta, we can describe a computational procedure similar to that described in Reference 6.

## COMPUTATIONAL PROCEDURE

Enter the initial conditions $x_i$, $y_i$, $z_i$, $\dot{x}_i$; $\dot{y}_i$ and $\dot{z}_i$ for a time $t_i$, with the constants $\mu$, $r_e$, $J_2$, $J_3 < 0$, and

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right) \text{ and } \delta = -\frac{1}{2} r_e J_2^{-1} J_3 > 0 . \tag{1.1}$$

Compute

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2} , \qquad r_i \dot{r}_i = x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i , \tag{1.2}$$

$$\rho_i^2 = \frac{r_i^2 + 2 z_i \delta + \delta^2 - c^2}{2} \left\{ 1 + \sqrt{1 + \frac{4c^2 (z_i + \delta)^2}{(r_i^2 + 2 z_i \delta + \delta^2 - c^2)^2}} \right\} , \tag{1.3}$$

$$\rho_i^2 = \tfrac{1}{2}\left[\sqrt{(r_i^2 + 2z_i\delta + \delta^2 - c^2)^2 + 4c^2(z_i + \delta)^2} + (r_i^2 + 2z_i\delta + \delta^2 - c^2)\right]$$

$$\eta_i^2 = \frac{(z_i + \delta)^2}{\rho_i^2}, \text{ (the sign of } \eta_i = \text{sign of } (z_i + \delta) , \tag{1.4}$$

$$\dot{\rho}_i = \frac{1}{2\rho_i}\left\{(r_i \dot{r}_i + \delta \dot{z}_i) + \frac{(r_i \dot{r}_i + \delta \dot{z}_i)(r_i^2 + 2z_i \delta + \delta^2 - c^2) + 2c^2(z_i + \delta)\dot{z}}{\sqrt{(r_i^2 + 2z_i \delta + \delta^2 - c^2)^2 + 4c^2(z_i + \delta)^2}}\right\} , \tag{1.5}$$

and

$$\dot{\eta}_i = \frac{1}{2c^2\eta}\left\{-(r_i \dot{r}_i + \delta \dot{z}_i) + \frac{(r_i \dot{r}_i + \delta \dot{z}_i)(r_i^2 + 2z_i \delta + \delta^2 - c^2) + 2c^2(z_i + \delta)\dot{z}}{\sqrt{(r_i^2 + 2z_i \delta + \delta^2 - c^2)^2 + 4c^2(z_i + \delta)^2}}\right\} . \tag{1.6}$$

Compute

$$\alpha_1 = \frac{1}{2} U_i^2 - \mu(\rho_i + \eta_i \delta)(\rho_i^2 + c^2 \eta_i^2)^{-1} , \tag{1.7}$$

where

$$U_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 ,$$

$$\alpha_3 = x_i \dot{y}_i - y_i \dot{x}_i ,$$

and

$$\alpha_2^2 = (1 - \eta_i^2)^{-1}\left[(\rho_i^2 + c^2 \eta_i^2)^2 \dot{\eta}_i^2 + \alpha_3^2 - (1 - \eta_i^2)(2\alpha_1 c^2 \eta_i^2 + 2\mu\eta_i \delta)\right] .$$

Then

$$a_0 = -\frac{1}{2}\frac{\mu}{\alpha_1} , \qquad e_0 = \left(1 + \frac{2\alpha_1 \alpha_2^2}{\mu^2}\right)^{1/2} ,$$

and

$$p_0 = a_0(1 - e_0^2) . \tag{1.8}$$

## Prime Constants

Compute

$$X_D^2 = -2\alpha_1 \alpha_2^2 \mu^{-2} \quad \text{and} \quad X_D^4 , \tag{2.1}$$

$$p_0^2 \text{ and } Y_D^2 = \left(\frac{\alpha_3}{\alpha_2}\right)^2 , \tag{2.2}$$

$$K_0 = \frac{c^2}{p_0^2} \text{ and } K_0^2 , \tag{2.3}$$

$$(\rho_1 + \rho_2) = 2p_0 X_D^{-2}\left[1 - K_0 X_D^2 Y_D^2 - K_0^2 X_D^2 Y_D^2 \left(2X_D^2 - 3X_D^2 Y_D^2 - 4 + 8Y_D^2\right)\right] , \tag{2.4}$$

$$\rho_1 \rho_2 = p_0^2 X_D^{-2}\left[1 + K_0 Y_D^2 \left(X_D^2 - 4\right) - K_0^2 Y_D^2 \left(12X_D^2 - X_D^4 - 20X_D^2 Y_D^2 - 16 + 32Y_D^2 + X_D^4 Y_D^2\right)\right] , \tag{2.5}$$

$$a = \left(\frac{\rho_1 + \rho_2}{2}\right) , \tag{2.6}$$

$$g = \frac{4\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} , = \frac{\rho_1 \rho_2}{a^2} \tag{2.7}$$

$$e = \sqrt{1 - g} , \tag{2.8}$$

$$\eta_0^{-2} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2(\alpha_2^2 - \alpha_3^2)}\left\{1 + \left[1 + \frac{8\alpha_1 c^2 (\alpha_2^2 - \alpha_3^2)}{(\alpha_2^2 - 2\alpha_1 c^2)^2}\right]^{1/2}\right\} , \tag{2.9}$$

and

$$\hat{S} = (\sin^2 I) = \eta_0^2 . \tag{2.10}$$

## Mutual Constants

Compute

$$p = a(1 - e^2) , \tag{3.1}$$

$$\hat{A} = \frac{-2ac^2 (ap - c^2 \hat{S})(1 - \hat{S}) + \frac{8a^2 c^2}{p} \delta^2 \left\{1 + \frac{c^2}{ap} (3\hat{S} - 2)\right\} \hat{S}(1 - \hat{S})}{(ap - c^2)(ap - c^2 \hat{S}) + 4a^2 c^2 \hat{S} + \frac{4c^2}{p^2} \delta^2 (3ap - 4a^2 - c^2)\hat{S}(1 - \hat{S})} , \tag{3.2}$$

$$\hat{B} = (2a)^{-1}(ap - c^2)\hat{A} + c^2 , \tag{3.3}$$

$$\hat{a}_0' = a - \frac{1}{2}\hat{A} , \quad \hat{p}_0' = \frac{(\hat{B} + ap - 2\hat{A}a - c^2)}{\hat{a}_0'} , \quad \hat{\alpha}_2' = (\mu \hat{p}_0')^{1/2} , \tag{3.4}$$

$$\hat{\epsilon} = \frac{\left(\frac{2\delta}{\hat{p}_0'}\right)^2 (1 - \hat{S})\left(1 - \frac{c^2}{\hat{a}_0' \hat{p}_0'} \hat{S}\right)}{\left[1 + (c^2/\hat{a}_0' \hat{p}_0')(1 - 2\hat{S})\right]^2} , \quad \hat{U}^{-1} = 1 + \left(\frac{c^2}{\hat{a}_0' \hat{p}_0'}\right)(1 - \hat{S}) + \hat{\epsilon} , \tag{3.5}$$

$$\hat{C}_2 = \frac{c^2}{\hat{a}_0' \hat{p}_0'} \hat{U}, \qquad \hat{C}_1 = \left(1 - \frac{c^2 \hat{S}}{\hat{a}_0' \hat{p}_0'} \hat{U}\right)^{-1} \frac{2\delta}{\hat{p}_0'} \hat{U} \left(1 - \frac{c^2 \hat{U}}{\hat{a}_0' \hat{p}_0'}\right), \tag{3.6}$$

$$\hat{P} = \left(1 - \frac{c^2 \hat{S} \hat{U}}{\hat{a}_0' \hat{p}_0'}\right)^{-1} \frac{\delta}{\hat{p}_0'} \hat{U}(1 - \hat{S}), \tag{3.7}$$

$$\hat{\eta}_0' = \hat{P} + (\hat{P}^2 + \hat{S})^{1/2}, \qquad \hat{\eta}_1' = \hat{P} - (\hat{P}^2 + \hat{S})^{1/2}, \tag{3.8}$$

and

$$\hat{S}' = -\hat{\eta}_0' \hat{\eta}_1'. \tag{3.9}$$

Using $\hat{S}'$, we now repeat steps (3.2) through (3.9) to obtain the quantities *

$$A,\ B,\ a_0',\ p_0'\ \alpha_2',\ \epsilon,\ U,\ C_2,\ C_1,\ P,\ \eta_0',\ \eta_1' \text{ and } S, \tag{3.10}$$

$$b_1 = -\frac{1}{2} A, \qquad b_2 = B^{1/2}, \text{ and } \alpha_1' = -\frac{\mu}{2a_0'}. \tag{3.11}$$

Using the formulas of Reference 6 (pages twelve to fifteen) we compute

$$A_1 = (1 - e^2)^{1/2} p \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1 - e^2)^{1/2}\right]$$

and

$$A_2 = (1 - e^2)^{1/2} p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1 - e^2)^{1/2}\right], \tag{3.12}$$

$I = 0$

where $P_n(b_1/b_2)$ is the Legendre Polynomial of degree $n$, $R_n(X_s) = X_s^n P_n(X_s^{-1})$ is a ploynomial of degree $(n/2)$ in $X_s^2$, and $X_s = (1 - e^2)^{1/2}$.

If $m$ is an even integer, compute

$$D_m = D_{2i} = \sum_{n=0}^{i} (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} P_{2n}\left(\frac{b_1}{b_2}\right)$$

*Equation (3.9) for $\hat{S}'$ together with the one step iteration (3.10) has now provided for us an accurate value of the element S that was originally approximated by $\eta_0^2$ (Equation 2.10).

If m is an odd integer, compute

$$D_m = D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} P_{2n+1}\left(\frac{b_1}{b_2}\right) .$$

Then

$$A_3 = \left(1 - e^2\right)^{1/2} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2} \left[\left(1 - e^2\right)^{1/2}\right] ,$$

$$A_{11} = \frac{3}{4} \left(1 - e^2\right)^{1/2} p^{-3} e \left(- 2b_1 b_2^2 p + b_2^4\right) ,$$

$$A_{12} = \frac{3}{32} \left(1 - e^2\right)^{1/2} b_2^4 e^2 p^{-3} ,$$

$$A_{21} = \left(1 - e^2\right)^{1/2} p^{-1} e \left[b_1 p^{-1} + \left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1 + \frac{e^2}{4}\right) p^{-3} + \frac{3}{8} b_2^4 \left(4 + 3e^2\right) p^{-4}\right] ,$$

$$A_{22} = \left(1 - e^2\right)^{1/2} p^{-1} \left[\frac{e^2}{8} \left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 \left(6e^2 + e^4\right) p^{-4}\right] ,$$

$$A_{23} = \left(1 - e^2\right)^{1/2} p^{-1} \frac{e^3}{8} \left(- b_1 b_2^2 p^{-3} + b_2^4 p^{-4}\right) ,$$

$$A_{24} = \frac{3}{256} \left(1 - e^2\right)^{1/2} p^{-5} b_2^4 e^4 ,$$

$$A_{31} = \left(1 - e^2\right)^{1/2} p^{-3} e \left[2 + b_1 p^{-1} \left(3 + \frac{3}{4} e^2\right) - p^{-2} \left(\frac{1}{2} b_2^2 + c^2\right) \left(4 + 3e^2\right)\right] ,$$

$$A_{32} = \left(1 - e^2\right)^{1/2} p^{-3} \left[\frac{e^2}{4} + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2} \left(\frac{b_2^2}{2} + c^2\right)\left(\frac{3}{2} e^2 + \frac{e^4}{4}\right)\right] ,$$

$$A_{33} = \left(1 - e^2\right)^{1/2} e^3 \left[\frac{b_1}{12p} - \frac{p^{-2}}{3} \left(\frac{b_2^2}{2} + c^2\right)\right] p^{-3} ,$$

$$A_{34} = - \frac{1}{32} \left(1 - e^2\right)^{1/2} p^{-5} e^4 \left(\frac{1}{2} b_2^2 + c^2\right) ,$$

and $$Q = \left(P^2 + S\right)^{1/2} . \qquad (3.13)$$

Then through third order in $J_2$:

$$B_2 = 1 - \frac{1}{2} C_1 P + \left(\frac{3}{8} C_1^2 + \frac{1}{2} C_2\right)\left(P^2 + \frac{1}{2} Q^2\right) + \frac{9}{64} C_2^2 Q^4$$

different equation

$$- \frac{9}{8} C_1 C_2 P Q^2 + \frac{45}{128} C_1^2 C_2 Q^4 + \frac{25}{256} C_2^3 Q^6 , \tag{3.14}$$

$$B_1' = \frac{1}{2} Q^2 + P^2 - \frac{3}{4} C_1 P Q^2 + \frac{3}{2} C_2 P^2 Q^2 + \frac{3}{64}\left(4C_2 + 3C_1^2\right) Q^4$$

$$- \frac{45}{32} C_1 C_2 P Q^4 + \frac{5}{256}\left(6C_2^2 + 15 C_1^2 C_2\right) Q^6 + \frac{175}{2048} C_2^3 Q^8 , \tag{3.15}$$

$$B_3 = - \frac{1}{2} C_2 - \frac{3}{8} C_1^2 - \left(\frac{15}{16} C_1^2 C_2 + \frac{3}{8} C_2^2\right)\left(1 + \frac{1}{2} Q^2\right)$$

$$- \frac{5}{16} C_2^3 \left(1 + \frac{1}{2} Q^2 + \frac{3}{8} Q^4\right) + \frac{3}{4} C_1 C_2 P , \tag{3.16}$$

$$\zeta = \frac{P}{1 - S} , \quad \underline{h_1} = \frac{1}{2}\left(1 + C_1 - C_2\right)^{-1/2} , \quad \underline{h_2} = \frac{1}{2}\left(1 - C_1 - C_2\right)^{-1/2} , \tag{3.17}$$

$$e_2 = Q(1 - P)^{-1} , \quad e_3 = Q(1 + P)^{-1} , \tag{3.18}$$

$$2\pi\nu_1 = \left(-2\alpha_1'\right)^{1/2} \left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1} ,$$

$$2\pi\nu_2 = \alpha_2' U^{-1/2} A_2 B_2^{-1} \left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1} , \tag{3.19}$$

$$e' = \frac{ae}{a + b_1} \quad \text{where} \quad e' < e < 1 , \tag{3.20}$$

and

$$\alpha_3' = \left(\operatorname{sgn} \alpha_3\right) \alpha_2' \left(1 - \frac{S}{U}\right)^{1/2} . \tag{3.21}$$

Here $\operatorname{sgn} \alpha_3 \gtrless 0$ for a direct or retrograde orbit.

## Jacobi Constants

Compute

$$B_{11} = -2PQ + \frac{3}{8} C_1 Q^3 , \tag{4.1}$$

$$B_{12} = -\left(\frac{Q^2}{4} + \frac{1}{8} C_2 Q^4\right), \tag{4.2}$$

$$B_{13} = -C_1 Q^3/24 \ , \tag{4.3}$$

$$B_{14} = C_2 Q^4/64 \ , \tag{4.4}$$

$$B_{21} = -C_2 P Q + \frac{9}{16} C_1 C_2 Q^3 + \frac{1}{2} C_1 Q \ , \tag{4.5}$$

$$B_{22} = -\frac{1}{32}\left[\left(4C_2 + 3C_1^2\right) Q^2 + 3C_2^2 Q^4\right] , \tag{4.6}$$

$$B_{23} = -\frac{1}{16} C_1 C_2 Q^3 , \tag{4.7}$$

$$B_{24} = \frac{3}{256} C_2^2 Q^4 , \tag{4.8}$$

$$\cos\phi_i = \frac{x_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1-\eta_i^2}} ,$$

$$\sin\phi_i = \frac{y_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1-\eta_i^2}} , \tag{4.9}$$

$$h_{\rho_i}^2 = \frac{\rho_i^2 + \eta_i^2 c^2}{\rho_i^2 + c^2} , \qquad h_{\eta_i}^2 = \frac{\rho_i^2 + \eta_i^2 c^2}{1 - \eta_i^2} , \tag{4.10}$$

$$\cos E_i = \frac{1}{e}\left(1 - \frac{\rho_i}{a}\right) ,$$

$$\sin E_i = \frac{\dot{\rho}_i h_{\rho_i}^2 \left(\rho_i^2 + c^2\right)}{ae\sqrt{-2\alpha_1'\left(\rho_i^2 + A\rho_i + B\right)}} , \tag{4.11}$$

$$\sin\psi_i = \frac{\eta_i - P}{Q} ,$$

$$\cos\psi_i = \frac{\dot{\eta}\left(\rho^2 + c^2\eta^2\right)}{Q\alpha_2'}\sqrt{\frac{u}{\left(1 + C_1\eta - C_2\eta^2\right)}} , \tag{4.12}$$

$$\cos v_i = \frac{\cos E_i - e}{1 - e\cos E_i} ,$$

$$\sin v_i = \frac{(1-e^2)^{1/2}\sin E_i}{1 - e\cos E_i} , \tag{4.13}$$

$$\sin n v_i \quad \text{for} \quad n = 2,\ 3,\ 4 ,$$

$$\sin n\psi_i \quad \text{for} \quad n = 2,\ 4 \tag{4.14}$$

and

$$\cos 3\psi_i .$$

Then

$- \sin \psi$

$$\cos E_{2i}' = \frac{e_2 + \cos\cancel{\left(\psi_i + \frac{\pi}{2}\right)}}{1 + e_2\cos\cancel{\left(\psi_i + \frac{\pi}{2}\right)}} ,$$

$- \sin \psi$

$\cos \psi$

$$\sin E_{2i}' = \frac{(1-e_2^2)^{1/2}\sin\left(\psi_i + \frac{\pi}{2}\right)}{1 + e_2\cos\left(\psi_i + \frac{\pi}{2}\right)}$$

$- \sin \psi$

$\sin \psi$

$$\cos E_{3i}' = \frac{e_3 + \cos\cancel{\left(\psi_i - \frac{\pi}{2}\right)}}{1 + e_3\cos\left(\psi_i - \frac{\pi}{2}\right)} ,$$

$\sin \psi$

$- \cos \psi$

$$\sin E_{3i}' = \frac{(1-e_3^2)^{1/2}\sin\left(\psi_i - \frac{\pi}{2}\right)}{1 + e_3\cos\left(\cancel{\psi_i} - \frac{\pi}{2}\right)} , \tag{4.15}$$

$$\chi_{0i} = \frac{E_{2i}'}{2\sqrt{1-2\zeta}} + \frac{E_{3i}'}{2\sqrt{1+2\zeta}} ,$$

$$\chi_{1i} = \frac{E_{2i}'}{2\sqrt{1-2\zeta}} - \frac{E_{3i}'}{2\sqrt{1+2\zeta}} , \tag{4.16}$$

$$\beta_1 = (-2\alpha_1')^{-1/2}\left[b_1 E_i + a(E_i - e\sin E_i) + A_1 v_i + A_{11}\sin v_i + A_{12}\sin 2v_i\right]$$

$$+ \frac{c^2 U^{1/2}}{\alpha_2'}\left[B_1'\ \psi_i + B_{11}\cos\psi_i + B_{12}\sin 2\psi_i + B_{13}\cos 3\psi_i + B_{14}\sin 4\psi_i\right] - t_i , \tag{4.17}$$

$$\beta_2 = -\alpha_2'\left(-2\alpha_1'\right)^{-1/2}\left[A_2 v_i + \sum_{n=1}^{4} A_{2n}\sin n v_i\right] + U^{1/2}\left[\psi_i B_2 + B_{21}\cos\psi_i\right.$$

$$\left. + B_{22}\sin 2\psi_i + B_{23}\cos 3\psi_i + B_{24}\sin 4\psi_i\right], \qquad (4.18)$$

$$\beta_3 = \phi_i + c^2\alpha_3'\left(-2\alpha_1'\right)^{-1/2}\left[A_3 v_i + \sum_{j=1}^{4} A_{3j}\sin j v_i\right] - \frac{\alpha_3'}{\alpha_2'}U^{1/2}\left\{(1-S)^{-1/2}\left[\left(h_1+h_2\right)\chi_{0i}\right.\right.$$

$$\left.\left. + \left(h_1-h_2\right)\chi_{1i}\right] + B_3\psi_i - \frac{3}{4}C_1C_2Q\cos\psi_i + \frac{3}{32}C_2^2Q^2\sin 2\psi_i\right\}, \qquad (4.19)$$

$$\ell_0 = 2\pi\nu_1\left(\beta_1 - \frac{c^2\beta_2 B_1'}{\alpha_2' B_2}\right), \qquad (4.20)$$

and

$$\ell_0 + g_0 = 2\pi\nu_2\left[\beta_1 + \left(\frac{\beta_2}{\alpha_2'}\right)\left(a + b_1 + A_1\right)A_2^{-1}\right]. \qquad (4.21)$$

## Orbit Generator

Compute

$$M_s = \ell_0 + 2\pi\nu_1 t$$

and

$$\psi_s = \ell_0 + g_0 + 2\pi\nu_2 t. \qquad (5.1)$$

By Newton-Raphson iteration we solve $M_s + E_0 - e'\sin\left(M_s + E_0\right) = M_s$ where $M_s + E_0 = \mathcal{E}$; therefore,

$$\mathcal{E} = \mathcal{E}_{n+1} = \mathcal{E}_n - \frac{\left[\mathcal{E}_n - e'\sin\mathcal{E}_n - M_s\right]}{\left(1 - e'\cos\mathcal{E}_n\right)} - \frac{1}{2}\left[\frac{\mathcal{E}_n - e'\sin\mathcal{E}_n - M_s}{1 - e'\cos\mathcal{E}_n}\right]^2\left[\frac{e'\sin\mathcal{E}_n}{1 - e'\cos\mathcal{E}_n}\right] \qquad (5.2)$$

and $\mathcal{E}_n = M_s$ initially.

Compute

$1-e$

$$\cos v' = (\cos\mathcal{E} - e)(1 - e\cos\mathcal{E})^{-1}, \qquad 1$$

$$\sin v' = \left(1 - e^2\right)^{1/2}(1 - e\cos\mathcal{E})^{-1}\sin\mathcal{E}, \qquad \sin v = 0 \qquad (5.3)$$

$1-e^2$

$$v_0 = v' - M_s \ , \tag{5.4}$$

$$\psi_0 = \alpha_2' \left(-2\alpha_1'\right)^{-1/2} U^{-1/2} A_2 B_2^{-1} v_0 \ , \tag{5.5}$$

$$M_1 = -\left(a + b_1\right)^{-1} \left[\left(A_1 + c^2 A_2 B_1' B_2^{-1}\right) v_0 + \frac{c^2}{\alpha_2'} \left(-2\alpha_1'\right)^{1/2} U^{1/2} B_{12} \sin\left(2\psi_s + 2\psi_0\right)\right] , \tag{5.6}$$

$$E_1 = \frac{M_1}{1 - e' \cos\left(M_s + E_0\right)} - \frac{e'}{2} \frac{M_1^2 \sin\left(M_s + E_0\right)}{\left[1 - e' \cos\left(M_s + E_0\right)\right]^3} \ , \tag{5.7}$$

$$\cos v'' = \left[\cos\left(\mathcal{E} + E_1\right) - e\right]\left[1 - e \cos\left(\mathcal{E} + E_1\right)\right]^{-1},$$

$$\sin v'' = \left(1 - e^2\right)^{1/2} \left[1 - e \cos\left(\mathcal{E} + E_1\right)\right]^{-1} \sin\left(\mathcal{E} + E_1\right) \tag{5.8}$$

$$v_1 = v'' - \left(v_0 + M_s\right) = \left(v'' - v'\right), \tag{5.9}$$

$$\psi_1 = -B_{22} B_2^{-1} \sin\left(2\psi_s + 2\psi_0\right) + \alpha_2' \left(-2\alpha_1'\right)^{-1/2} U^{-1/2} B_2^{-1} \left[A_2 v_1 + A_{21} \sin\left(M_s + v_0\right) + A_{22} \sin\left(2M_s + 2v_0\right)\right]$$
$$- B_{21} B_2^{-1} \cos\left(\psi_s + \psi_0\right), \tag{5.10}$$

$$M_2 = -\left(a + b_1\right)^{-1} \Big[A_1 v_1 + A_{11} \sin\left(M_s + v_0\right) + A_{12} \sin\left(2M_s + 2v_0\right) + \frac{c^2}{\alpha_2'} \left(-2\alpha_1'\right)^{1/2} U^{1/2} \Big\{B_1' \psi_1$$
$$+ B_{11} \cos\left(\psi_s + \psi_0\right) + 2B_{12} \psi_1 \cos\left(2\psi_s + 2\psi_0\right) + B_{13} \cos\left(3\psi_s + 3\psi_0\right) + B_{14} \sin\left(4\psi_s + 4\psi_0\right)\Big\}\Big] , \tag{5.11}$$

$$E_2 = \frac{M_2}{1 - e' \cos\left(M_s + E_0 + E_1\right)} \ , \tag{5.12}$$

$$\cos v''' = \left[\cos\left(\mathcal{E} + E_1 + E_2\right) - e\right]\left[1 - e \cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1},$$

$$\sin v''' = \left(1 - e^2\right)^{1/2} \left[1 - e \cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1} \sin\left(\mathcal{E} + E_1 + E_2\right) \ , \tag{5.13}$$

$$v_2 = v''' - \left(v_0 + M_s + v_1\right) = \left(v''' - v''\right) , \tag{5.14}$$

and

$$\psi_2 = -B_2^{-1} \left[-B_{21} \psi_1 \sin\left(\psi_s + \psi_0\right) + 2B_{22} \psi_1 \cos\left(2\psi_s + 2\psi_0\right) + B_{23} \cos\left(3\psi_s + 3\psi_0\right) + B_{24} \sin\left(4\psi_s + 4\psi_0\right)\right]$$
$$+ \alpha_2' U^{-1/2} \left(-2\alpha_1'\right)^{-1/2} B_2^{-1} \Big[A_2 v_2 + A_{21} v_1 \cos\left(M_s + v_0\right) + 2A_{22} v_1 \cos\left(2M_s + 2v_0\right)$$
$$+ A_{23} \sin\left(3M_s + 3v_0\right) + A_{24} \sin\left(4M_s + 4v_0\right)\Big] . \tag{5.15}$$

Then

$$E = \mathcal{E} + E_1 + E_2 \ ,$$

$$v = M_s + v_0 + v_1 + v_2 \ ,$$

and

$$\psi = \psi_s + \psi_0 + \psi_1 + \psi_2 \ . \tag{5.16}$$

Next

$$\cos E_2' = \frac{e_2 + \cos\left(\psi + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi + \frac{\pi}{2}\right)} \ ,$$

Cos ψ Cos π/2 − sin ψ sin π/2

$$\sin E_2' = \frac{\left(1 - e_2^2\right)^{1/2} \sin\left(\psi + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi + \frac{\pi}{2}\right)} \ ,$$

$$\cos E_3' = \frac{e_3 + \cos\left(\psi - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi - \frac{\pi}{2}\right)} \ ,$$

$$\sin E_3' = \frac{\left(1 - e_3^2\right)^{1/2} \sin\left(\psi - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi - \frac{\pi}{2}\right)} \ , \tag{5.17}$$

$$\chi_0 = \frac{E_2'}{2\sqrt{1 - 2\zeta}} + \frac{E_3'}{2\sqrt{1 + 2\zeta}} \ ,$$

$$\chi_1 = \frac{E_2'}{2\sqrt{1 - 2\zeta}} - \frac{E_3'}{2\sqrt{1 + 2\zeta}} \ , \tag{5.18}$$

$$\rho = a(1 - e\cos E) \ , \qquad \eta \doteq P + Q\sin\psi \ ,$$

$$\phi = \beta_3 - c^2 \alpha_3' \left(-2\alpha_1'\right)^{-1/2} \left[A_3 v + \sum_{j=1}^{4} A_{3j} \sin jv\right] + \frac{\alpha_3'}{\alpha_2'} U^{1/2} \left\{(1-S)^{-1/2} \left[\left(h_1 + h_2\right) \chi_0 + \left(h_1 - h_2\right) \chi_1\right] + B_3 \psi - \frac{3}{4} C_1 C_2 Q \cos\psi + \frac{3}{32} C_2^2 Q^2 \sin 2\psi\right\} , \quad (5.19)$$

$$h_\rho^2 = \frac{\rho^2 + \eta^2 c^2}{\rho^2 + c^2} , \qquad h_\eta^2 = \frac{\rho^2 + \eta^2 c^2}{1 - \eta^2} ,$$

$$h_\phi^2 = \left(\rho^2 + c^2\right)\left(1 - \eta^2\right) , \quad (5.20)$$

$$\dot\rho = \frac{ae \sqrt{-2\alpha_1' \left(\rho^2 + A\rho + B\right)}}{h_\rho^2 \left(\rho^2 + c^2\right)} \sin E ,$$

$$\dot\eta = \frac{Q\alpha_2' \sqrt{1 + C_1 \eta - C_2 \eta^2} \cos\psi}{\sqrt{u} \left(\rho^2 + c^2 \eta^2\right)} \quad (5.21)$$

$$X = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \cos\phi ,$$

$$Y = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \sin\phi ,$$

$$Z = \rho\eta - \delta ,$$

$$\dot X = X\left(\frac{\rho\dot\rho}{\rho^2 + c^2} - \frac{\eta\dot\eta}{1 - \eta^2}\right) - \frac{Y \cdot \alpha_3'}{h_\phi^2} ,$$

$$\dot Y = Y\left(\frac{\rho\dot\rho}{\rho^2 + c^2} - \frac{\eta\dot\eta}{1 - \eta^2}\right) + \frac{X \cdot \alpha_3'}{h_\phi^2} ,$$

and

$$\dot Z = \rho\dot\eta + \eta\dot\rho . \quad (5.22)$$

## CONCLUSION

The accuracy of the orbit itself as a solution for the given potential is carried out through terms of the third order in $J_2$, the coefficient of the second zonal harmonic. Its accuracy however, and thus that of the secular terms, may be increased at will. Periodic terms are carried through the second order, but their accuracy may also be increased. In order to carry this kind of accuracy most perturbation methods presently in use would become far too cumbersome and impractical for orbit computational purposes.

An advantage of accounting for $J_3$ in this way is the absence of small denominators in $e$ or $\sin I$ that occur in certain perturbation theories. Thus, one can easily compute polar orbits and circular equatorial orbits.

This program, similar in structure to the previous Vinti accurate intermediary orbit requires a relatively small number of storage locations throughout the entire computing procedure. Recent tests on the IBM 7094 have indicated that this latter Vinti method enjoys a rather rapid computational speed and it is indicated through preliminary comparisons that it is faster than other methods which use perturbation techniques to compute orbits. Electronic timing on the Theoretical Division computer shows that this program can compute 3000 minute vector points (time, x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$, 3000 times per minute of IBM 7094 computer operation) with simultaneous production of BCD tape output (Reference 6).

The residual fourth harmonic term (Reference 5) has been programmed and is presently being tested as is the $J_{2,1}$ tesseral harmonic term. In addition, the orbit differential correction program written for the Vinti accurate intermediary orbit theory (Reference 7), will need only slight modification to account for the element $S$ which replaces $\eta_0 = \sin I$ in Vinti's new theory. Consequently it is expected to go just as rapidly, producing a set of mean elements of even greater accuracy with which to predict satellite orbits over a longer interval of time. In addition, comparison with a very accurate, double precision numerical integration for polar and near polar (89° inclination) and equatorial cases over a five day period indicate there is no loss of accuracy from the general or intermediate inclination cases. The agreement between the Vinti program and the numerical integration was always six to eight decimal places. The Vinti method computed in single precision or eight decimal places only.

It has been shown (Reference 7) that the orbit differential correction program for the Vinti accurate intermediary orbit theory converges rapidly and with great accuracy. This latter program, tested on the Relay II Satellite with an eccentricity of 0.23597617, and Satellite ANNA with an eccentricity of 0.00671710 gave the following results:

Using radio direction cosine observation data, and for seventy equations of condition of the Relay II Satellite extending over a five hour arc following injection into orbit, the program converged on the third iteration to a standard deviation of fit criterion of $2.7 \times 10^{-3}$ within thirty seconds. Using Smithsonian Astrophysical Observatory optical data from the Anna Satellite extending over an arc of seventy-five hours following injection, and with twenty equations of condition, the program converged on the second iteration to a standard deviation of fit of $0.2 \times 10^{-3}$. For seven equations of condition of Satellite ANNA covering the first forty-five hours following injection, the Vinti orbit differential correction program converged on the second iteration to a standard deviation of fit criterion of $0.04 \times 10^{-3}$. All tests were conducted using the IBM 7094 Mod. I electronic digital computer. The orbit differential correction program utilizing the new Vinti equations of motion, now carries the additional accuracy inherent in the new method, and should merit even tighter tolerances for the standard deviation of fit.

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