NASA TECHNICAL NOTE

NASA TN D-3316

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THE DRIFT OF AN INCLINED-ORBIT 24-HOUR SATELLITE IN AN EARTH GRAVITY FIELD THROUGH FOURTH ORDER

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1966

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Springfield, Virginia 22151 – Price $2.50

ABSTRACT

This report extends previous investigations of 24-hour satellites by considering the long-term drift of such satellites with orbits having any inclination but small eccentricity in an earth gravity field through fourth order.

It is demonstrated that the drift regime for the inclined-orbit satellite is the same as for the equatorial satellite, modified only by an "inclination factor" distinct for each relevant gravity harmonic and in each case less than unity for non-zero inclinations.

In particular, it is shown that for the circular orbit satellite through fourth-order earth gravity, only the longitude gravity harmonics J_{22}, J_{31}, J_{33}, J_{42}, and J_{44} can have long-term (secular) effects on the orbit. Except for very high inclinations, the 24-hour drift regime, as influenced by presently determined earth gravity, is dominated by the effect of the J_{22} harmonic (that associated with the ellipticity of the earth's equator).

For the longitude gravity harmonics, J_{31} and J_{42}, certain "non-resonant" inclinations are shown to exist. Twenty-four-hour satellites having orbits with these inclinations will experience negligible long-term accelerated drift due to either J_{31} or J_{42}.

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INTRODUCTION

In recent years there have been many studies of the perturbations of a 24-hour satellite due to the high-order gravity field of the earth as well as the gravity fields of the sun and moon (References 1, 2, 3, 4, 5 and 6). Since the period of this satellite is almost the same as the earth's rotation period, its mean geographic position may be virtually unchanged for many months. It is evident that even very small longitude mass inhomogeneities within the earth may be able to cause appreciable longitude drift of this satellite, which can respond in the same direction to these distant mass concentrations over a long period of time. This phenomenon is often referred to as "resonance" in the literature. While use of this description can cause confusion if one visualizes the action only in geographic space, it is an accurate one for the action in inertial space and it will be retained.

A virtually fixed station in the sky with almost every place on earth permanently visible from one of three such equatorial stations is an obvious advantage to worldwide communications and navigation. Previous studies of the equatorial 24-hour satellite (References 2 and 5) have shown that the long-term east-west station-keeping requirement for this satellite is controlled by second-order earth-longitude gravity which is associated with earth-equatorial ellipticity.

The first operational 24-hour communications satellite, Syncom II, launched in July 1963, has an orbit with an inclination of about 33°. The study in Reference 6 shows the drift regime of such a satellite in a second-order longitude-gravity field to be essentially unchanged from that of the equatorial satellite in the second-order field. Syncom II has now (Spring, 1965) drifted almost around the earth in over a year. Comparison of observations on its drift in this time with the gravity drift theory of Reference 6 has already revealed the ellipticity of the earth's equator to a greater precision than ever before (References 6 and 7). The advantages of the use of freely drifting 24-hour satellites for geodetic purposes have thus already been dramatically demonstrated.

This study is designed to provide a simple theoretical framework for the use of such satellites of any inclination for basic investigations of the complex gravity field of the earth to high order.

SUMMARY OF PREVIOUS INVESTIGATIONS

In Reference 5 it was shown that in the vicinity of a momentarily stationary longitude λ_0 the longitude-drift acceleration of a 24-hour, circular orbit, equatorial satellite in an earth gravity field to fourth order is given by

$$\ddot{\lambda}_0 = \frac{-3(2\pi)^2}{\omega_e^2 a_s}\Big\{F_{22}\sin 2(\lambda_0 - \lambda_{22}) + F_{31}\sin(\lambda_0 - \lambda_{31}) + F_{33}\sin 3(\lambda_0 - \lambda_{33})$$

$$+ F_{42}\sin 2(\lambda_0 - \lambda_{42}) + F_{44}\sin 4(\lambda_0 - \lambda_{44})\Big\} \text{ rad/(sid. day)}^2 \quad , \qquad (1)$$

where

$$\left.\begin{aligned}
F_{22} &= \frac{6\mu_E}{a_s^2}(R_0/a_s)^2 J_{22} \,, \\
F_{31} &= \frac{-3\mu_E}{2a_s^2}(R_0/a_s)^3 J_{31} \,, \\
F_{33} &= \frac{45\mu_E}{a_s^2}(R_0/a_s)^3 J_{33} \,, \\
F_{42} &= \frac{-15\mu_E}{a_s^2}(R_0/a_s)^4 J_{42} \,, \\
F_{44} &= \frac{420\mu_E}{a_s^2}(R_0/a_s)^4 J_{44} \,.
\end{aligned}\right\} \qquad (2)$$

The constants are defined as follows: ω_e is the rotation rate of the earth (2π radians/sidereal day); a_s is the "synchronous" semi-major axis of the satellite; μ_E is the Gaussian constant of the earth; R_0 is the mean equatorial radius of the earth; J_{nm} and λ_{nm} are earth gravity constants in the spherical harmonic expansion of gravity for a general mass distribution. The J_{nm} ($m \neq 0$) for the earth are very small dimensionless constants, of the order of 10^{-6} or less. The J_{nm} give the magnitude of longitude-dependent gravity forces. The λ_{nm} give the phase angle of the dependence. For a truly spherical earth, every J_{nm} would be zero (Reference 8). The $F_{nm}\sin m(\lambda_0 - \lambda_{nm})$ terms in Equation 1 are the earth-gravity longitude perturbation forces (on the equator) for the particular harmonic, *nm*. Equation 1 is a statement of the proportionality of these longitude driving forces and the resultant drift acceleration for the 24-hour equatorial satellite. It is not a self-evident statement because the coordinate system r, λ, ϕ is not inertial but is rotating at the earth rate with respect to inertial space.

In Reference 6 the orbit-averaged longitude drift of a 24-hour, inclined circular-orbit satellite in a second-order longitude-dependent earth field was found to satisfy the equation

$$\ddot{\lambda} = A_{22} \sin 2(\lambda - \lambda_{22}) \ , \tag{3}$$

where

$$A_{22} = -72\pi^2 (R_0/a_s)^2 J_{22} \left(\frac{\cos^2 i + 1}{2}\right) \text{rad/(sid. day)}^2 \ , \tag{4}$$

and the gravity constant λ_{22} locates the longitude of the major axis of the hypothesized elliptical equator of the earth. Equations 1 and 3 are compatible since A_{22} is equal to

$$\frac{3(2\pi)^2 F_{22}}{\omega_e^2 a_s}$$

for $i = 0$, noting that

$$\frac{\mu_E}{a_s^3} = \omega_e^2$$

defines the semi-major axis of the circular-orbit earth satellite with a period of exactly one day. The strict derivation of Equation 3 applied only to a 24-hour satellite in the vicinity of a stationary configuration whose ascending node (or mean daily geographic longitude) was located at λ. However, even for the long period and wide longitude excursion of libratory drift which follows from the regime established by the "pendulum equation," Equation 3, the near-stationary conditions assumed in its derivation are not seriously violated (Reference 6). Thus, for the entire libratory regime, Equation 3 can be expected to hold sufficiently well with λ representing the daily geographic position of the ascending node. In fact, numerically integrated Syncom II trajectories in the presence of sun, moon and earth zonal gravity have shown that the drift regime established by the simple formula, Equation 3, with λ interpreted as the daily position of the ascending node of the satellite, is a fully adequate representation of the actual long-term drift within the small "noise limits" of these higher order gravity perturbations (References 6 and 7). These trajectories have included some with eccentricities as high as 0.0012. Drift rates have been as high as 0.8 degrees/day, which is twice the maximum allowable for this 24-hour satellite in a libratory regime (Reference 6). R. R. Allan* presumes that Equation 3 may fairly represent the 24-hour drift regime for eccentricities as high as 0.3 if λ is taken as the mean longitude of ascending and decending nodes which, for perigees away from the equator, is a better measure of the mean daily geographic longitude than either nodal position alone.

*Private communication.

At any rate, for a sufficiently wide specification of orbital elements to be practical, Equation 3 has been shown to establish the correct second-order longitude drift regime with λ interpreted as the mean daily geographic longitude of the satellite (or nodal longitude for the near-circular inclined-orbit satellite). In a second-order earth-longitude gravity field, Equation 1 is merely a special case of Equation 3 for zero inclination. For the second-order case at least, the long term trajectories in References 6 and 7 have shown it is proper to replace λ_0 in Equation 1 by λ, the general longitude position of the 24-hour satellite in the drift. Thus for second-order gravity drift of an inclined 24-hour satellite the relevant longitude acceleration regime is

$$\ddot{\lambda} = -\frac{3}{a_s}\left\{F_{22}\, F(i)_{22} \sin 2\left(\lambda - \lambda_{22}\right)\right\} \text{ rad/(sid. day)}^2 , \tag{5}$$

where

$$F(i)_{22} \doteq \frac{\cos^2 i + 1}{2} . \tag{6}$$

$F(i)_{22}$ is an "inclination factor" which may be applied to the zero inclination regime to get the proper acceleration regime for the inclined satellite. It is not at all obvious, even for a circular satellite orbit, that one can apply such a simple factor, independent of the longitude, to modify the equatorial regime. In Reference 6, Equation 6 was found to be a good approximation of the correct factor for $i < 45°$.

In the first section of the present report it is shown that for the inclined circular orbit of the 24-hour satellite, the second-order factor, which is exact for all inclinations, is

$$F(i)_{22} = \frac{(1 + \cos i)^2}{4} . \tag{7}$$

It is also shown there, rigorously, that for third and fourth-order as well as second-order longitude gravity, such exact inclination factors exist to modify the higher order equatorial regimes of Equation 1.

The modification (on an orbit-averaging basis) gives the complete orbit-averaged drift regime of the inclined-orbit satellite (to the fourth order) as

$$\ddot{\lambda} = -\frac{12\pi^2}{g_s} \sum_{n=2}^{4} \sum_{\substack{m=1 \\ \text{For } n-m \text{ even}}}^{n} \left[F_{nm}\, F(i)_{nm} \sin m\left(\lambda - \lambda_{nm}\right)\right] \text{ rad/(sid. day)}^2 , \tag{8}$$

where

$$g_s \doteq \omega_e^2 a_s \doteq \mu_E / a_s^2 \; .$$

Equation 8 (without the inclination factors) was first proposed as a good approximation for the second order drift regime of the equatorial 24-hour satellite by Frick and Garber (Reference 2), and for the higher order regimes by R. R. Allan (Reference 4). Earlier than Allan, in Reference 5, the author limited his investigation of equatorial 24-hour drift in a fourth-order field to the vicinity of the initial longitude placement of the satellite and derived Equation 1 which is seen to be a special case of Equation 8.

DERIVATION OF THE LONG-TERM ORBIT-AVERAGED DRIFT OF A 24-HOUR, NEARLY CIRCULAR, INCLINED-ORBIT SATELLITE IN A FOURTH-ORDER EARTH GRAVITY FIELD

In Reference 6 the technique of orbit averaging the disturbing force along the track of a 24-hour satellite to obtain the relevant long-term (secular) drift motion was proposed for the inclined-orbit satellite. While the energy-changing disturbing force on the equatorial satellite due to the gravity of the earth is constant over an orbit, it is, in general, quite a complicated function of the instantaneous orbit position of the satellite if the orbit plane has the slightest inclination. It can be anticipated, then, that for the inclined-orbit case the exact drift equations of motion for the 24-hour satellite will contain many small, time-varying "functions" of daily period, introducing non-linearities which will be difficult to deal with. Since it is desired at the start to arrive at a theory of long-term (i.e., greater than one day) drift motion, it is natural (since the magnitudes of the short period terms are small) to seek a perturbation technique which, in a first approximation, smooths out these short period effects. The orbit-averaged perturbation force technique chosen in Reference 6 seemed a natural one to complement the idea used there of dealing directly with the daily perturbation of the two-body energy of the satellite. Computation of the mean daily effect for second-order longitude gravity was not too difficult. As a first approximation to the exact drift motion, numerical integrations over many months proved the long-term theory derived in this manner to be an excellent approximation of reality, even in the presence of a more complicated gravity field with much stronger effects of daily period. Nevertheless, the technique of direct orbit averaging of perturbation forces has its drawbacks. The chief one is computational. It leads immediately to definite integrals (over 2π) involving the anomaly θ of the satellite and the inclination i. (The eccentricity is not involved since only the first-order perturbation of a circular orbit is sought.) These are cumbersome to reduce to simple closed forms. Nevertheless, it is easy to show which integrals, no matter how complicated, will orbit average to zero and which will yield non-zero, secular, long-term effects. The cumbersome integrals yielding non-zero mean daily effects can, at worst, be left as they are and numerically integrated for any inclination to yield the proper "inclination factor" corresponding to that secular effect. It will be found that in every harmonic case (through fourth-order) the inclination integral can be directly reduced to yield a factor equivalent to one computed for this harmonic by an entirely different method* (Reference 9). Though no necessary equivalence condition has been found between the two theories for the development of first-order secular terms of the earth's gravity disturbing function, it is presumed on the basis of the exact agreement of these through fourth-order that such inclination factors will be equivalent in the two theories to all orders of earth gravity.

*Kaula, W. M., private communication.

Proceeding on the basis of the orbit-averaging theory established by Equation 10 in Reference 6, the long-term longitudinal drift acceleration of a 24-hour, inclined, nearly circular-orbit satellite is given by

$$\ddot{\lambda} = \frac{-12\pi^2 \overline{F}}{\mu_E / a_s^2} \text{ rad/(sid. day)}^2 , \tag{9}$$

where

$$\overline{F} = \frac{1}{2\pi} \int_0^{2\pi} F_T(\theta, i)\, d\theta = \overline{F_T(\theta)} . \tag{10}$$

F_T is the tangential or along-track perturbation force acting on the satellite at an angle θ from its ascending node. Similarly, the orbit-averaged rate of change of the semi-major axis of the satellite orbit, coupled with Equation 9, is given by Equation 11 in Reference 6 as

$$\dot{a} = \frac{4\pi a_s^3 \overline{F}}{\mu_E} \text{ length units/sid. day} . \tag{11}$$

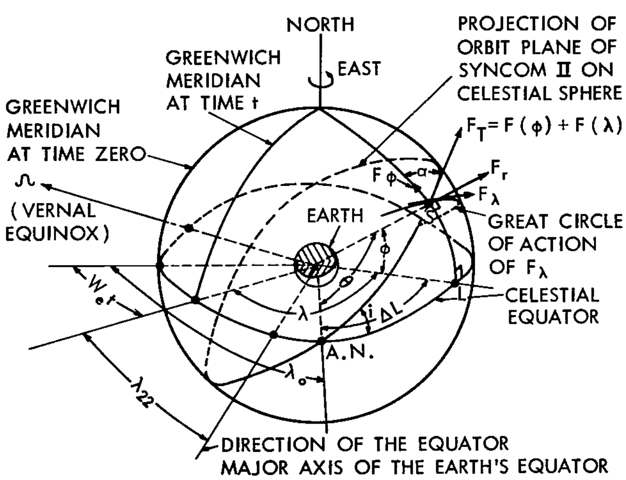

Figure 1—Position of a 24-hour satellite with a near-circular orbit with respect to the earth and the celestial sphere.

Figure 1 shows the position of the 24-hour satellite with respect to the earth and the celestial sphere. F_r, F_ϕ and F_λ, earth-gravity perturbing forces in the radial, latitude, and longitude directions, are assumed to be acting on the satellite at S (see Appendix A). As long as the orbit is nearly circular, F_r will make a negligible contribution to F_T. The tangential perturbing force will then be close to the circumferential perturbing force and we can write

$$F_T \doteq F_\phi \cos\alpha + F_\lambda \sin\alpha , \tag{12}$$

where F_ϕ and F_λ are the latitude and longitude components of the gravity force on the satellite and α is the angle between a meridian plane and the orbit plane when the satellite is θ distant from its ascending node. Writing out Equation 12 in its full fourth-order-gravity form will involve the sum of contributions to F_T from every earth harmonic (zonal and tesseral) up to and including J_{44} (see Appendix A).

We will now show that the zonal terms of this series (terms in J_{20}, J_{30} and J_{40}) give zero orbit-averaged effects; i.e., $\overline{F}(\text{zonal}) \equiv 0$.

The only force component of zonal gravity capable of having an energy-changing effect on the satellite is the latitude one, if the orbit is circular.

Orbit-Averaged Effect of J_{20}

From Appendix A, the relevant term is

$$F_{\phi,20} = \frac{\mu_E \left(\frac{R_0}{a_s}\right)^2}{a_s^2} \left\{-3\, J_{20} \sin\phi \cos\phi\right\} \quad . \tag{13}$$

But from the spherical trigonometry of the orbit on the celestial sphere (Figure 1), the following relations must hold:

$$\left.\begin{aligned} \sin\phi &= \sin i \sin\theta \;, \\ \Delta L &= \tan^{-1}(\cos i \cos\theta) \\ \cos\alpha &= \frac{\tan\phi}{\tan\theta} \quad . \end{aligned}\right\} \tag{14}$$

Substitution of Equations 14 and 13 into Equation 12 gives

$$\begin{aligned} F_{T,20} &= \frac{-3\mu_E \left(\frac{R_0}{a_s}\right)^2}{a_s^2} J_{20} \;\frac{\tan\phi \sin\phi \cos\phi}{\tan\theta} \\ &= k_{20} \frac{\sin^2\phi \cos\theta}{\sin\theta} = \frac{k_{20}}{2} \sin^2 i \sin 2\theta \;, \end{aligned} \tag{15}$$

where K_{20} is a constant over the circular orbit assumed. But the integral of Equation 15 over an orbit $(0 \le \theta \le 2\pi)$ is zero, or

$$\overline{F_{T,20}} = \frac{1}{2\pi}\int_0^{2\pi} F_{T,20}\, d\theta = 0 \quad . \tag{16}$$

Thus, as long as the orbit remains circular, the J_{20} term will produce no change in the energy of the satellite over periods of time which are multiples of one day. It must be noted at this point that this first-order (zero eccentricity) result for J_{20} may not be sufficient in all cases to guarantee negligible long-term energy contributions compared to those of longitude gravity for actual orbits of small eccentricity. While the effects of small eccentricity can generally be ignored as second-order in the theory of the tesseral perturbations, they will be about 1000 times larger in the theory to the same order for J_{20} due to the relatively large oblateness of the earth compared to the other mass anomalies (see DISCUSSION).

Orbit-Averaged Effect of J_{30}

From Appendix A the relevant term is

$$F_{\phi,30} = k_{30}\left\{\left(5\sin^2\phi - 1\right)\cos\phi\right\} , \tag{17}$$

where k_{30} is a constant for the circular-orbit satellite. As in Equation 15, we now get

$$F_{T,30} = F_{\phi,30}\cos\alpha = k_{30}\frac{\tan\phi\cos\phi}{\tan\theta}\left(5\sin^2\phi - 1\right)$$

$$= k_{30}\sin i\cos\theta\left(5\sin^2 i\sin^2\theta - 1\right) . \tag{18}$$

Since $\int_0^{2\pi}\cos\theta\, d\theta = 0$, the orbit-averaged tangential force from Equation 18 is

$$\overline{F_{T,30}} = \frac{5k_{30}\sin^3 i}{2\pi}\int_0^{2\pi}\sin^2\theta\cos\theta\, d\theta = 0 , \tag{19}$$

since $\sin^2\theta\cos\theta$ is an odd function with respect to $\theta = \pi/2$ and $3\pi/2$.

Orbit-Averaged Effect of J_{40}

From Appendix A the relevant term is

$$F_{\phi,40} = k_{40}\sin\phi\cos\phi\left\{7\sin^2\phi - 3\right\} \cdot \tag{20}$$

As before,

$$F_{T,40} = F_{\phi,40}\cos\alpha = k_{40}\sin\phi\cos\phi\left(7\sin^2\phi - 3\right)\frac{\tan\phi}{\tan\theta}$$

$$= \frac{k_{40}\sin^2\phi\cos\theta}{\sin\theta}\left(7\sin^2\phi - 3\right) = \frac{k_{40}\sin^2 i\sin 2\theta}{2}\left(7\sin^2 i\sin^2\theta - 3\right) .$$

But since $\int_0^{2\pi}\sin 2\theta\, d\theta = 0$,

$$\overline{F_{T,40}} = \frac{1}{2\pi}\int_0^{2\pi}F_{T,40}\, d\theta = \frac{7k_{40}\sin^4 i}{4\pi}\int_0^{2\pi}\sin^2\theta\sin 2\theta\, d\theta = 0 , \tag{21}$$

because $\sin^2\theta\sin 2\theta$ is an odd function with respect to $\theta = \pi$.

It is conjectured that similar zero results can be expected to hold for *all* higher zonal gravity forces due to their latitude symmetry. The longitude harmonics of the gravity field of the earth will now be treated in a similar manner.

Orbit-Averaged Effect of J_{22}

Since geographic longitude dependence is involved in all the longitude effects, it will be necessary to develop certain trigonometric relationships involving the longitude excursion in the "figure 8" 24-hour ground track (see Figure 3, Reference 6), as a function of nodal argument θ.

In Reference 6 (see also Figure 1), it is shown that the excursion $\Delta\lambda$ from the ascending equator crossing (at λ_0 geographical longitude) in the "figure 8" track is given by:

$$\Delta\lambda = \Delta L - \omega_{et} = \tan^{-1}(\cos i \tan\theta) - \theta \quad . \tag{22}$$

Thus from Equation 22 the following identities can be seen to hold:

$$\cos\Delta\lambda = \cos\theta\cos\left[\tan^{-1}(\cos i\tan\theta)\right] + \sin\theta\sin\left[\tan^{-1}(\cos i\tan\theta)\right]$$

$$= \frac{\cos^2\theta + \sin^2\theta\cos i}{\left(\cos^2\theta + \sin^2\theta\cos^2 i\right)^{1/2}} = \frac{1-\sin^2\theta(1-\cos i)}{\left(1-\sin^2\theta\sin^2 i\right)^{1/2}} \; ; \tag{23}$$

$$\sin\Delta\lambda = \cos\theta\sin\left[\tan^{-1}(\cos i\tan\theta)\right] - \sin\theta\cos\left[\tan^{-1}(\cos i\tan\theta)\right]$$

$$= \frac{\cos\theta\cos i\tan\theta - \sin\theta}{\left(1+\cos^2 i\tan^2\theta\right)^{1/2}} = \frac{\sin 2\theta(\cos i - 1)}{2\left(1-\sin^2\theta\sin^2 i\right)^{1/2}} \; ; \tag{24}$$

$$\cos 2\Delta\lambda = 1 - 2\sin^2\Delta\lambda = 1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)} \; ; \tag{25}$$

$$\sin 2\Delta\lambda = 2\sin\Delta\lambda\cos\Delta\lambda = \frac{\left[1-\sin^2\theta(1-\cos i)\right]\left[\sin 2\theta(\cos i - 1)\right]}{1-\sin^2\theta\sin^2 i}$$

$$= \frac{\sin 2\theta(\cos i - 1)}{\cos i + 1} + \frac{\cos i}{(\cos i + 1)}\left\{\frac{\sin 2\theta(\cos i - 1)}{1-\sin^2\theta\sin^2 i}\right\} \; ; \tag{26}$$

$$\cos 3\Delta\lambda = \cos(2\Delta\lambda + \Delta\lambda) = \cos 2\Delta\lambda \cos\Delta\lambda - \sin 2\Delta\lambda \sin\Delta\lambda$$

$$= \left\{1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta \sin^2 i\right)}\right\} \left\{\frac{1 - \sin^2\theta(1-\cos i)}{\left(1-\sin^2\theta\sin^2 i\right)^{1/2}}\right\}$$

$$- \left[1-\sin^2\theta(1-\cos i)\right] \left[\frac{\sin 2\theta(\cos i - 1)}{1 - \sin^2\theta\sin^2 i}\right] \left[\frac{\sin 2\theta(\cos i - 1)}{2\left(1-\sin^2\theta\sin^2 i\right)^{1/2}}\right] \quad (27)$$

$$\sin 3\Delta\lambda = \sin(2\Delta\lambda + \Delta\lambda) = \sin 2\Delta\lambda \cos\Delta\lambda + \cos 2\Delta\lambda \sin\Delta\lambda.$$

$$= \frac{\left[1-\sin^2\theta(1-\cos i)\right]\left[\sin 2\theta(\cos i - 1)\right]\left[1-\sin^2\theta(1-\cos i)\right]}{\left(1-\sin^2\theta\sin^2 i\right)\left(1-\sin^2\theta\sin^2 i\right)^{1/2}}$$

$$+ \left\{1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{2\left(1-\sin^2\theta\sin^2 i\right)^{1/2}}\right\}; \quad (27a)$$

$$\cos 4\Delta\lambda = \cos 2(2\Delta\lambda) = 1 - 2\sin^2 2\Delta\lambda$$

$$= 1 - 2\left\{\frac{\sin 2\theta(\cos i - 1)}{\cos i + 1} + \frac{\cos i}{\cos i + 1}\left[\frac{\sin 2\theta(\cos i - 1)}{1 - \sin^2\theta\sin^2 i}\right]\right\}^2$$

$$= 1 - 2\,\frac{\left[1-\sin^2\theta(1-\cos i)\right]^2\left[\sin 2\theta(\cos i - 1)\right]^2}{\left[1-\sin^2\theta\sin^2 i\right]^2}; \quad (28)$$

$$\sin 4\Delta\lambda = \sin 2(2\Delta\lambda) = 2\sin 2\Delta\lambda \cos 2\Delta\lambda$$

$$= 2\left\{1 - \sin^2\theta(1-\cos i)\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{\left(1-\sin^2\theta\sin^2 i\right)}\right\}\left\{1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\right\}. \quad (29)$$

Continuing with the evaluation of the along-track force due to the longitude and latitude contributions from J_{22} (see Appendix A), we have

$$F_{T,22} = F_{\lambda,22}\sin\alpha + F_{\phi,22}\cos\alpha$$

$$= k_{22}\left\{\sin 2\left(\lambda - \lambda_{22}\right)\left[\sin\alpha\cos\phi\right] + \cos 2\left(\lambda-\lambda_{22}\right)\left[\cos\alpha\sin\phi\cos\phi\right]\right\}$$

where

$$k'_{22} = 6\,J_{22}\,\frac{\mu_E}{(a_s)^2}\,(R_0/a_s)^2$$

When the orbit geometry relations (from Figure 1):

$$\lambda = \lambda_0 + \Delta\lambda\ , \tag{29a}$$

$$\sin\alpha = \frac{\cos i}{\cos\phi}\ , \tag{29b}$$

$$\cos\alpha = \frac{\tan\phi}{\tan\theta}\ , \tag{29c}$$

and

$$\sin\phi = \sin i \sin\theta \tag{29d}$$

are introduced into this form, the tangential force on the 24-hour satellite due to J_{22} is

$$F_{T,22} = k_{22}\left\{\sin 2\left(\lambda_0 - \lambda_{22}\right)\left[\cos i \cos 2\Delta\lambda - \frac{\sin 2\theta \sin^2 i \sin 2\Delta\lambda}{2}\right]\right.$$
$$\left. + \cos 2\left(\lambda_0 - \lambda_{22}\right)\left[\cos i \sin 2\Delta\lambda + \frac{\sin 2\theta \sin^2 i \cos 2\Delta\lambda}{2}\right]\right\}\ . \tag{30}$$

When the longitude excursion relations, Equations 25 and 26, are used, Equation 30 becomes

$$F_{T,22} = k_{22}\left[\sin 2\left(\lambda_0 - \lambda_{22}\right)\left\{\left[1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta \sin^2 i\right)}\right]\cos i\right.\right.$$
$$\left. - \frac{\sin^2 i}{2}\left[\sin^2 2\theta(\cos i - 1)\right]\left[\frac{1 - \sin^2\theta(1 - \cos i)}{1 - \sin^2\theta\sin^2 i}\right]\right\}$$
$$+ \cos 2\left(\lambda_0 - \lambda_{22}\right)\left\{\cos i\left[1 - \sin^2\theta(1 - \cos i)\right]\frac{\left[\sin 2\theta(\cos i - 1)\right]}{1 - \sin^2\theta\sin^2 i}\right.$$
$$\left.\left. + \frac{\sin^2 i \sin 2\theta}{2}\left[1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right]\right\}\right] \tag{31}$$

When the factor of $\cos 2(\lambda_0 - \lambda_{22})$ in Equation 31 is examined, it is noted that it is an *odd* function with respect to $\theta = \pi$. Therefore, the orbit-averaged $(0 \le \theta \le 2\pi)$ contribution of the $\cos 2(\lambda_0 - \lambda_{22})$ term in the forcing function is zero. It is this result, which will be found to be valid for all the higher order tesserals, which enables one to write the relevant long-term drift equations for the inclined-orbit 24-hour satellite as a simple (inclination-factored) modification of the equations for the equatorial satellite.

Thus, only the factor of $\sin 2(\lambda_0 - \lambda_{22})$ in Equation 31 need be considered, or

$$\overline{F_{T,22}} = k_{22} \sin 2(\lambda_0 - \lambda_{22})\, \overline{F(\theta, i)}_{22} , \tag{31a}$$

where

$$\begin{aligned}
F(\theta, i)_{22} &= \cos i - \sin^2 2\theta \left\{ \frac{(1-\cos i)^2 \cos i + \sin^2 i(\cos i - 1) + \sin^2 i \sin^2 \theta (1-\cos i)^2}{2(1-\sin^2\theta \sin^2 i)} \right\} \\
&= \cos i - \sin^2 2\theta \left\{ \frac{(1-\cos i)\left[\cos i(1-\cos i) - \sin^2 i + \sin^2 i \sin^2\theta(1-\cos i)\right]}{2(1-\sin^2\theta\sin^2 i)} \right\} \\
&= \cos i - \sin^2 2\theta (1-\cos i) \left\{ \frac{\cos i - \cos^2 i - \sin^2 i + \sin^2 i \sin^2 \theta (1-\cos i)}{2(1-\sin^2\theta \sin^2 i)} \right\} \\
&= \cos i + \frac{\sin^2 2\theta(1-\cos i)^2}{2} \frac{\left[1-\sin^2 i \sin^2\theta\right]}{1-\sin^2 i \sin^2\theta} = \frac{\cos i + \sin^2 2\theta (1-\cos i)^2}{2} .
\end{aligned} \tag{32}$$

Thus, the orbit-averaged along-track perturbation force due to J_{22} is (from Equations 31a and 32)

$$\begin{aligned}
\overline{F_{T,22}}(\theta) &= k_{22} \sin 2(\lambda_0 - \lambda_{22})\, \overline{F(\theta, i)}_{22} \\
&= \left[k_{22} \sin 2(\lambda_0 - \lambda_{22})\right] \frac{1}{2\pi} \int_0^{2\pi} \left[\cos i + (1-\cos i)^2 \frac{\sin^2 2\theta}{2}\right] d\theta \\
&= k_{22} \sin 2(\lambda_0 - \lambda_{22}) \left[\cos i + \frac{(1-\cos i)^2}{4}\right] \\
&= k_{22} \frac{\sin 2(\lambda_0 - \lambda_{22})}{4} \left[4\cos i + 1 - 2\cos i + \cos^2 i\right] \\
&= k_{22} \frac{\sin 2(\lambda_0 - \lambda_{22})}{4} \left[1 + 2\cos i + \cos^2 i\right] \\
&= k_{22} \frac{\sin 2(\lambda_0 - \lambda_{22})}{4} (1+\cos i)^2 .
\end{aligned} \tag{33}$$

It is seen that the orbit-averaged, energy-changing perturbation force on an inclined-orbit 24-hour satellite due to J_{22} is just $F(i)_{22}$ times that on the equatorial satellite (Equations 1 and 2, or A-4) at the same mean longitude λ_0, where (from Equation 33)

$$F(i)_{22} = \frac{(1 + \cos i)^2}{4} . \tag{34}$$

Orbit-Averaged Effect of J_{31}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{31}, is (see Appendix A)

$$F_{T,31}(\theta) = F_{\lambda,31} \sin\alpha + F_{\phi,31} \cos\alpha$$

$$= k_{31} \left\{ \sin(\lambda - \lambda_{31}) \left[5 \sin^2\phi - 1\right] \sin\alpha + \cos(\lambda - \lambda_{31}) \left[15 \sin^2\phi - 11\right] \sin\phi \cos\alpha \right\} \tag{35}$$

where

$$k_{31} = \frac{3}{2} (R_0/a_s)^3 J_{31} . \tag{35a}$$

Introduction of the orbit geometry relations, Equations 29a through 29d, causes Equation 35 to become

$$\begin{aligned} F_{T,31}(\theta) = k_{31} \Bigg\{ & \sin(\lambda_0 - \lambda_{31}) \left[\frac{\left(5 \sin^2\theta \sin^2 i - 1\right) \cos i \cos\Delta\lambda}{\left(1 - \sin^2 i \sin^2\theta\right)^{1/2}} \right. \\ & \left. - \frac{\left(15 \sin^2 i \sin^2\theta - 11\right) \left(\sin^2 i \sin 2\theta\right) \sin\Delta\lambda}{\left(1 - \sin^2 i \sin^2\theta\right)^{1/2}} \right] \\ & + \cos(\lambda_0 - \lambda_{31}) \left[\frac{\left(5 \sin^2\theta \sin^2 i - 1\right) \cos i \sin\Delta\lambda}{\left(1 - \sin^2 i \sin^2\theta\right)^{1/2}} \right. \\ & \left. + \frac{\left(15 \sin^2 i \sin^2\theta - 11\right) \left(\sin^2 i \sin 2\theta\right) \cos\Delta\lambda}{2\left(1 - \sin^2 i \sin^2\theta\right)^{1/2}} \right] \Bigg\} . \end{aligned} \tag{36}$$

Introducing the expressions for the longitude excursion in the "figure 8", Equations 23 and 24, causes Equation 36 to become

$$F_{T,31}(\theta) = k_{31}\left\{\sin\left(\lambda_0-\lambda_{31}\right)\left[\frac{\cos i\,\{1-\sin^2\theta\,[1-\cos i]\}\left(5\sin^2\theta\sin^2 i-1\right)}{1-\sin^2 i\sin^2\theta} - \frac{\sin^2 i\sin^2 2\theta(\cos i-1)\left(15\sin^2\theta\sin^2 i-11\right)}{4\left(1-\sin^2\theta\sin^2 i\right)}\right]\right.$$

$$\left. + \cos\left(\lambda_0-\lambda_{31}\right)\left[\frac{\cos i\left(5\sin^2\theta\sin^2 i-1\right)\sin 2\theta(\cos i-1)}{2\left(1-\sin^2\theta\sin^2 i\right)} + \frac{\sin^2 i\sin 2\theta\left(15\sin^2 i\sin^2\theta-11\right)\{1-\sin^2\theta(1-\cos i)\}}{2\left(1-\sin^2\theta\sin^2 i\right)}\right]\right\}. \quad (37)$$

Once again it is seen that the factor of $\cos\left(\lambda_0-\lambda_{31}\right)$ in Equation 37 is an odd function about $\theta = \pi$ so that its orbit-averaged effect over $0<\theta<2\pi$ is zero. When only the relevant term $\sin\left(\lambda_0-\lambda_{31}\right)$ term is retained, Equation 37 becomes

$$F_{T,31}(\theta) = \frac{k_{31}\sin\left(\lambda_0-\lambda_{31}\right)}{4\left(1-\sin^2\theta\sin^2 i\right)}\left\{4\cos i\left[1-\sin^2\theta(1-\cos i)\right]\left[5\sin^2\theta\sin^2 i-1\right] - \sin^2 i\sin^2 2\theta(\cos i-1)\left[15\sin^2\theta\sin^2 i-11\right]\right\} \quad (37a)$$

$$= \frac{k_{31}\sin\left(\lambda_0-\lambda_{31}\right)}{4\left(1-\sin^2\theta\sin^2 i\right)}\left\{\frac{4\cos i}{1+\cos i}\left[\cos i+\left(1-\sin^2\theta\sin^2 i\right)\right]\left[-\left(1-\sin^2\theta\sin^2 i\right)+4\sin^2\theta\sin^2 i\right] - \sin^2 i\sin^2 2\theta(\cos i-1)\left[-10\left(1-\sin^2 i\sin^2\theta\right)+5\sin^2 i\sin^2\theta-1\right]\right\} \quad (37b)$$

$$= k_{31}\sin\left(\lambda_0-\lambda_{31}\right)\left\{\overbrace{\frac{-4\cos i\left(1-\sin^2\theta\sin^2 i\right)}{4(1+\cos i)} - \frac{4\cos^2 i}{4(1+\cos i)} + \frac{16\cos i\sin^2\theta\sin^2 i}{4(1+\cos i)} + \frac{10\sin^2 i\sin^2 2\theta(\cos i -}{4}}^{A(\theta)}\right.$$

$$\left. + \frac{\dfrac{16\cos^2 i\sin^2 i\sin^2\theta}{1+\cos i} - \sin^2 i\sin^2 2\theta(\cos i-1)\left[5\sin^2\theta\sin^2 i-1\right]}{4\left(1-\sin^2\theta\sin^2 i\right)}\right\} \quad (37c)$$

$$= k_{31}\sin\left(\lambda_0-\lambda_{31}\right)\left\{A(\theta) + \frac{16\cos^2 i\sin^2 i\sin^2\theta+\sin^4 i\sin^2 2\theta\left(5\sin^2\theta\sin^2 i-1\right)}{4(1+\cos i)\left(1-\sin^2\theta\sin^2 i\right)}\right\} \quad (37d)$$

$$= k_{31} \sin \lambda_0 - \lambda_{31} \left\{ A(\theta) + \frac{16 \cos^2 i \sin^2 i \sin^2 \theta - \sin^4 i \sin^2 2\theta \left[\left(1 - \sin^2 \theta \sin^2 i\right) - 4 \sin^2 \theta \sin^2 i\right]}{4(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \right\} \tag{37e}$$

$$= k_{31} \sin \left(\lambda_0 - \lambda_{31}\right) \left\{ \overbrace{\frac{-4 \cos i \left(1 - \sin^2 \theta \sin^2 i\right) - 4 \cos^2 i + 16 \cos i \sin^2 \theta \sin^2 i - 11 \sin^4 i \sin^2 2\theta}{4(1 + \cos i)}}^{B(\theta)} + \frac{16 \cos^2 i \sin^2 i \sin^2 \theta + 4 \sin^6 i \sin^2 2\theta \sin^2 \theta}{4(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \right\} \tag{37f}$$

$$= k_{31} \sin \left(\lambda_0 - \lambda_{31}\right) \left\{ B(\theta) + \frac{\sin^2 \theta \sin^2 i \left[4\left(1 - \sin^2 i\right) + \sin^4 i \sin^2 2\theta\right]}{(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \right\} \tag{37g}$$

$$= k_{31} \sin \left(\lambda_0 - \lambda_{31}\right) \left\{ B(\theta) + \frac{\sin^2 \theta \sin^2 i \left[4 - 4 \sin^2 i \left(1 - \sin^2 i \sin^2 \theta\right) - 4 \sin^4 \theta \sin^4 i\right]}{(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \right\} \tag{37h}$$

$$= k_{31} \sin \left(\lambda_0 - \lambda_{31}\right) \left\{ \overbrace{B(\theta) - \frac{4 \sin^2 \theta \sin^4 i}{1 + \cos i}}^{C(\theta)} + \frac{4 \sin^2 \theta \sin^2 i \left[1 - \sin^4 \theta \sin^4 i\right]}{(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \right\} \tag{37i}$$

$$= k_{31} \sin \left(\lambda_0 - \lambda_{31}\right) \left\{ \overbrace{C(\theta) + \frac{4 \sin^2 \theta \sin^2 i \left(1 + \sin^2 \theta \sin^2 i\right)}{1 + \cos i}}^{D(\theta)} \right\} . \tag{37j}$$

Reducing $D(\theta)$ in Equation 37j gives

$$D(\theta) = -\frac{4 \sin^2 \theta \sin^2 i (4 \cos i + \cos i) - 4 \cos i (1 + \cos i) - 16 \sin^2 \theta \sin^4 i - 11 \sin^4 i \sin^2 2\theta + 16 \sin^2 \theta \sin^2 i \left(1 + \sin^2 \theta \sin^2 i\right)}{4(1 + \cos i)}$$

$$= \frac{\sin^2 \theta \sin^2 i (5 \cos i + 4)}{(1 + \cos i)} - \cos i - \frac{\sin^4 i}{4(1 + \cos i)} \left\{11 \sin^2 2\theta - 16 \sin^4 \theta + 16 \sin^2 \theta\right\} . \tag{37k}$$

From Equations 37j and 37k the orbit-averaged tangential perturbing force on the inclined-orbit 24-hour satellite, due to J_{31} is

$$\overline{F_{T,31}} = \frac{1}{2\pi}\int_0^{2\pi} F_{T,31}(\theta)\,d\theta = k_{31}\sin(\lambda_0-\lambda_{31})\left\{\frac{1}{2\pi}\int_0^{2\pi} D(\theta)\,d\theta\right\}$$

$$= k_{31}\sin(\lambda_0-\lambda_{31})\left\{-\cos i + \frac{\sin^2 i(5\cos i+4)}{2(1+\cos i)} - \frac{\sin^4 i}{4(1+\cos i)}\left[\frac{11}{2} - 6 + \frac{16}{2}\right]\right\}$$

$$= k_{31}\sin(\lambda_0-\lambda_{31})\left\{-\cos i + \frac{(1-\cos i)(5\cos i+4)}{2} - \frac{15(1-\cos i)\sin^2 i}{8}\right\}$$

$$= k_{31}\sin(\lambda_0-\lambda_{31})\left\{-\cos i + \frac{5\cos i}{2} - \frac{5}{2}\left(1-\sin^2 i\right) + 2 - 2\cos i - \frac{15}{8}\sin^2 i + \frac{15}{8}\sin^2 i\cos i\right\}$$

$$= -k_{31}\sin(\lambda_0-\lambda_{31})\left\{\overbrace{\underbrace{\frac{1+\cos i}{2} - \frac{5\sin^2 i(1+3\cos i)}{8}}}^{F(i)_{31}}\right\} . \tag{38}$$

But $-k_{31}\sin(\lambda_0-\lambda_{31}) = F_{31}\sin 2(\lambda_0-\lambda_{31})$, which is the tangential perturbation force on the 24-hour equatorial satellite at longitude λ_0. Thus, the orbit-averaged force on a 24-hour inclined circular orbit satellite whose ascending node is at λ_0 is just $F(i)_{31}$ times the force on the equatorial 24-hour satellite at λ_0, due to J_{31}, where

$$F(i)_{31} = \frac{1+\cos i}{2} - \frac{5\sin^2 i}{8}(1+3\cos i) . \tag{39}$$

Orbit-Averaged Effect of J_{32}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{32}, is (see Appendix A)

$$F_{T,32}(\theta) = F_{\lambda,32}\sin\alpha + F_{\phi,32}\cos\alpha$$

$$= k_{32}\left\{\sin 2(\lambda-\lambda_{32})\left[2\sin\alpha\sin\phi\cos\phi\right] + \cos 2(\lambda-\lambda_{32})\left[\cos\alpha\left(3\sin^2\phi-1\right)\cos\phi\right]\right\} , \tag{40}$$

where

$$k_{32} = 15\frac{\mu_E}{(a_s)^2}\left(R_0/a_s\right)^3 J_{32} .$$

Introducing the orbit geometry relations, Equations 29a through 29d, causes Equation 40 to become

$$F_{T,32}(\theta) = k_{32}\Big\{\sin 2(\lambda_0 - \lambda_{32})\left[2\cos i \sin i \sin\theta \cos 2\Delta\lambda - \cos\theta \sin i\left(3\sin^2 i \sin^2\theta - 1\right)\sin 2\Delta\lambda\right]$$

$$+ \cos 2(\lambda_0 - \lambda_{32})\left[2\cos i \sin i \sin\theta \sin 2\Delta\lambda + \cos\theta \sin i\left(3\sin^2 i \sin^2\theta - 1\right)\cos 2\Delta\lambda\right]\Big\}. \quad (41)$$

When Equations 25 and 26 are used, Equation 41 now becomes

$$F_{T,32} = k_{32}\Bigg\{\sin 2(\lambda_0 - \lambda_{32})\Bigg[\sin 2i \sin\theta\left\{1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\right\} - \sin i\cos\theta\left(3\sin^2 i\sin^2\theta - 1\right)$$

$$\cdot \frac{\{1-\sin^2\theta(1-\cos i)\}\ \{\sin 2\theta(\cos i - 1)\}}{1-\sin^2\theta\sin^2 i}\Bigg]$$

$$+ \cos 2\left(\lambda_0 - \lambda_{32}\right)\Bigg[\sin 2i\sin\theta\,\frac{\{1-\sin^2\theta(1-\cos i)\}\ \{\sin 2\theta(\cos i - 1)\}}{1-\sin^2\theta\sin^2 i}$$

$$+ \cos\theta\sin i\left(3\sin^2 i\sin^2\theta - 1\right)\left\{1 - \frac{\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\right\}\Bigg]\Bigg\}. \quad (42)$$

The factor of $\sin 2(\lambda_0 - \lambda_{32})$ in Equation 42 is an odd function with respect to $\theta = \pi$. Thus it has a zero orbit-averaged effect. The factor of $\cos 2(\lambda_0 - \lambda_{32})$ in Equation 42 is an odd function with respect to $\theta = \pi/2$ and $3\pi/2$. The orbit-averaged effect of this term is also zero. Therefore, the orbit-averaged contribution to the longitude acceleration of the inclined-orbit 24-hour satellite due to the tesseral J_{32} is zero, as it is at all orbit anomalies, identically, for the equatorial satellite.

Orbit-Averaged Effect of J_{33}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{33} is (see Appendix A)

$$F_{T,33} = F_{\lambda,33}\sin\alpha + F_{\phi,33}\cos\alpha$$

$$= k_{33}\left\{\sin 3\left(\lambda - \lambda_{33}\right)\sin\alpha\cos^2\phi + \cos 3\left(\lambda - \lambda_{33}\right)\cos\alpha\cos^2\phi\sin\phi\right\}, \quad (43)$$

where

$$k_{33} = 45\,\frac{\mu_E}{(a_s)^2}\left(R_0/a_s\right)^3 J_{33}\,.$$

Introducing the orbit geometry relations, Equations 29a through 29d, causes Equation 43 to become

$$F_{T,33}(\theta) = k_{33}\left\{\sin 3(\lambda_0 - \lambda_{33})\left[\cos i\left(1 - \sin^2\theta\sin^2 i\right)^{1/2}\cos 3\Delta\lambda - \frac{\sin^2 i}{2}\sin 2\theta\left(1 - \sin^2\theta\sin^2 i\right)^{1/2}\sin 3\Delta\lambda\right]\right.$$

$$\left. + \cos 3(\lambda_0 - \lambda_{33})\left[\cos i\left(1 - \sin^2\theta\sin^2 i\right)^{1/2}\sin 3\Delta\lambda + \frac{\sin^2 i}{2}\sin 2\theta\left(1 - \sin^2\theta\sin^2 i\right)^{1/2}\cos 3\Delta\lambda\right]\right\}. \quad (44)$$

Using the longitude excursion relations, Equations 27 and 27a, in Equation 44 gives

$$F_{T,33}(\theta) = k_{33}\left\{\sin 3(\lambda_0 - \lambda_{33})\left[\cos i\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\left\{1 - \sin^2\theta(1 - \cos i)\right\}\right.\right.$$

$$- \cos i\left\{1 - \sin^2\theta(1 - \cos i)\right\}\left\{\frac{\sin^2 2\theta(\cos i - 1)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}$$

$$- \frac{\sin^2 i}{2}\sin 2\theta\left\{\frac{\left[1 - \sin^2\theta(1 - \cos i)\right]\left[\sin 2\theta(\cos i - 1)\right]\left[1 - \sin^2\theta(1 - \cos i)\right]}{\left(1 - \sin^2\theta\sin^2 i\right)}\right\}$$

$$\left. - \frac{\sin^2 i\sin 2\theta}{2}\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{2}\right\}\right]$$

$$+ \cos 3(\lambda_0 - \lambda_{33})\left[\cos i\left\{\frac{\left[1 - \sin^2\theta(1 - \cos i)\right]\left[\sin 2\theta(\cos i - 1)\right]\left[1 - \sin^2\theta(1 - \cos i)\right]}{\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\right.$$

$$+ \cos i\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{2}\right\}$$

$$+ \frac{\sin^2 i\sin 2\theta}{2}\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\left\{1 - \sin^2\theta(1 - \cos i)\right\}$$

$$\left.\left. - \frac{\sin^2 i\sin 2\theta}{2}\left\{\frac{\left[1 - \sin^2\theta(1 - \cos i)\right]\left[\sin 2\theta(\cos i - 1)\right]\left[\sin 2\theta(\cos i - 1)\right]}{2\left(1 - \sin^2\theta\sin^2 i\right)}\right\}\right]\right\}. \quad (45)$$

Once again the factor of $\cos 3(\lambda_0 - \lambda_{33})$ above is an odd function with respect to $\theta = \pi$, so that it orbit-averages to zero.

Rewriting Equation 44 and including only the relevant $\sin 3(\lambda_0 - \lambda_{33})$ term gives

$$
\begin{aligned}
F_{T,33}(\theta) = {} & k_{33} \sin 3(\lambda_0 - \lambda_{33}) \Bigg\{ \cos i \left[1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \right] \left[1 - \sin^2 \theta (1 - \cos i) \right] \\
& - \cos i \left[1 - \sin^2 \theta (1 - \cos i) \right] \frac{\left[\sin^2 2\theta (\cos i - 1)^2 \right]}{2\left(1 - \sin^2 \theta \sin^2 i\right)} - \frac{\sin^2 i}{2} \frac{\left[1 - \sin^2 \theta (1 - \cos i) \right]^2 \sin^2 2\theta (\cos i - 1)}{\left(1 - \sin^2 \theta \sin^2 i\right)} \\
& - \frac{\sin^2 i}{4} \left[\sin^2 2\theta (\cos i - 1) \right] \left[1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \right] \Bigg\}
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
= {} & k_{33} \sin 3(\lambda_0 - \lambda_{33}) \Bigg\{ \overbrace{\cos i \left[1 - \sin^2 \theta (1 - \cos i) \right] - \frac{\sin^2 i}{4} \left[\sin^2 2\theta (\cos i - 1) \right]}^{A_1(\theta)} \\
& + \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \Bigg[- \cos i \left\{ 1 - \sin^2 \theta (1 - \cos i) \right\} + \frac{\sin^2 i}{2} \left\{ \sin^2 2\theta (\cos i - 1) \right\} \\
& - \cos i \left\{ 1 - \sin^2 \theta (1 - \cos i) \right\} - \frac{\sin^2 i}{(\cos i - 1)} \left\{ 1 - \sin^2 \theta (1 - \cos i) \right\}^2 \Bigg] \Bigg\}
\end{aligned}
\tag{46a}
$$

$$
\begin{aligned}
= {} & k_{33} \sin 3(\lambda_0 - \lambda_{33}) \Bigg\{ A_1(\theta) + \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \Bigg[\frac{-2 \cos i}{(1 + \cos i)} \left\{ \cos i + 1 - \sin^2 \theta \sin^2 i \right\} \\
& + \frac{\sin^2 i \sin^2 2\theta (\cos i - 1)}{4} + \frac{\left\{ \cos i + \left(1 - \sin^2 \theta \sin^2 i\right) \right\}^2}{1 + \cos i} \Bigg] \Bigg\}
\end{aligned}
\tag{46b}
$$

$$
\begin{aligned}
= {} & k_{33} \sin 3(\lambda_0 - \lambda_{33}) \Bigg\{ A_1(\theta) + \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \Bigg[\left\{ \frac{\cos i + \left(1 - \sin^2 \theta \sin^2 i\right)}{1 + \cos i} \right\} \Big\{ -2 \cos i \\
& + \left[\cos i + \left(1 - \sin^2 \theta \sin^2 i\right) \right] \Big\} + \frac{\sin^2 i}{4} \sin^2 2\theta (\cos i - 1) \Bigg] \Bigg\}
\end{aligned}
\tag{46c}
$$

$$
\begin{aligned}
= {} & k_{33} \sin 3(\lambda_0 - \lambda_{33}) \Bigg\{ \overbrace{A_1(\theta) + \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 + \cos i)} \left[1 - \cos i - \sin^2 \theta \sin^2 i \right]}^{B_1(\theta)} + \frac{\sin^2 2\theta (1 - \cos i)^2}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \\
& \left[\frac{\cos i}{1 + \cos i} \left\{ - \cos i + \left(1 - \sin^2 \theta \sin^2 i\right) \right\} + \sin^2 i \sin^2 \theta \left(1 - \sin^2 \theta\right) (1 - \cos i) \right] \Bigg\}
\end{aligned}
\tag{46d}
$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \overbrace{B_1(\theta) + \frac{\sin^2 2\theta (1-\cos i)^2 \cos i}{2(1+\cos i)}}^{C_1(\theta)} + \frac{\sin^2 2\theta(1-\cos i)^2}{2(1+\cos i)(1-\sin^2\theta \sin^2 i)} \right.$$

$$\left. \left[-\cos^2 i - \sin^4 i \sin^2 \theta \left(1 - \sin^2 \theta\right)\right] \right\} \quad (46e)$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ C_1(\theta) - \frac{\sin^2 2\theta (1-\cos i)^2}{2(1+\cos i)\left(1-\sin^2\theta \sin^2 i\right)} \right.$$

$$\left. \left[1 - \left(1 - \sin^2\theta \sin^2 i\right) \sin^2 i - \sin^4 i \sin^4 \theta\right] \right\} \quad (46f)$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \overbrace{C_1(\theta) - \frac{\sin^2 2\theta(1-\cos i)^2 \left(1+\sin^2\theta \sin^2 i\right)}{2(1+\cos i)} + \frac{\sin^2 i \sin^2 2\theta (1-\cos i)^2}{2(1+\cos i)}}^{D_1(\theta)} \right\} \quad (46g)$$

Thus, the orbit-averaged tangential force due to J_{33} is

$$\overline{F_{T,33}(\theta)} = \frac{1}{2\pi}\int_0^{2\pi} F_{T,33}(\theta)\, d\theta = k_{33} \sin 3(\lambda_0 - \lambda_{33}) \frac{1}{2\pi}\int_0^{2\pi} D_1(\theta)\, d\theta$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \cos i \left[1 - \frac{(1-\cos i)}{2}\right] - \frac{\sin^2 i}{8}(\cos i - 1) + \frac{(1-\cos i)^3}{4(1+\cos i)} \right.$$

$$- \frac{(1-\cos i)^2 \sin^2 i}{8(1+\cos i)} + \frac{(1-\cos i)^2 \cos i}{4(1+\cos i)} - \frac{(1-\cos i)^2}{4(1+\cos i)}$$

$$\left. - \frac{\sin^2 i (1-\cos i)^2}{8(1+\cos i)} + \frac{\sin^2 i(1-\cos i)^2}{4(1+\cos i)} \right\} \quad (47)$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{\cos i}{2}(1+\cos i) + \frac{(1-\cos i)^2(1+\cos i)}{8} \right.$$

$$\left. + (1-\cos i)^3 \left[\frac{1}{4(1-\cos i)}\right] + \frac{(1-\cos i)^2 \cos i}{4(1+\cos i)} - \frac{(1-\cos i)^2}{4(1+\cos i)} \right\} \quad (47a)$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{\cos i}{2} (1 + \cos i) + \frac{(1 - \cos i)^2}{8(1 + \cos i)} \left[(1 + \cos i)^2 + 2(1 - \cos i) - 2 + 2 \cos i\right] \right\} \tag{47b}$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{\cos i(1 + \cos i)}{2} + \frac{(1 - \cos i)^2 (1 + \cos i)^2}{8(1 + \cos i)} \right\}$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{1 + \cos i}{8} \left[4 \cos i + (1 - \cos i)^2\right] \right\}$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{1 + \cos i}{8} \left[4 \cos i + 1 - 2 \cos i + \cos^2 i\right] \right\}$$

$$= k_{33} \sin 3(\lambda_0 - \lambda_{33}) \left\{ \frac{(1 + \cos i)^3}{8} \right\} . \tag{47c}$$

But $k_{33} \sin 3(\lambda_0 - \lambda_{33}) = F_{31} \sin 3(\lambda_0 - \lambda_{33})$, which is the tangential perturbation force on the 24-hour equatorial satellite at longitude λ_0. Thus, the orbit-averaged perturbing force on a 24-hour, inclined-circular orbit satellite whose ascending node is at λ_0 is just $F(i)_{33}$ times the perturbing force on the equatorial 24-hour satellite at λ_0, due to J_{33}, where

$$F(i)_{33} = \frac{(1 + \cos i)^3}{8} . \tag{48}$$

Orbit-Averaged Effect of J_{41}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{41}, is (see Appendix A)

$$F_{T,41}(\theta) = F_{\lambda,41} \sin \alpha + F_{\phi,41} \cos \alpha$$

$$= k_{41} \left\{ \sin (\lambda - \lambda_{44}) \sin \alpha \left[7 \sin^2 \phi - 3\right] \sin \phi + \cos (\lambda - \lambda_{41}) \cos \alpha \left[28 \sin^4 \phi - 27 \sin^2 \phi + 3\right] \right\}, \tag{49}$$

where

$$k_{41} = \frac{5}{2} \frac{\mu_E}{(a_s)^2} (R_0/a_s)^4 J_{41} .$$

Introducing the orbit geometry relations, Equations 29a through 29d, causes Equation 49 to take the form

$$F_{T,41}(\theta) = k_{41} \Bigg\{ \sin(\lambda_0 - \lambda_{41}) \left[\frac{\cos i \left(7 \sin^2\theta \sin^2 i - 3\right) \sin\theta \sin i \cos\Delta\lambda}{\left(1 - \sin^2\theta \sin^2 i\right)^{1/2}} \right.$$

$$\left. - \frac{\sin i \cos\theta \left(28 \sin^4\theta \sin^4 i - 27 \sin^2\theta \sin^2 i + 3\right) \sin\Delta\lambda}{\left(1 - \sin^2\theta \sin^2\right) i^{1/2}} \right]$$

$$+ \cos(\lambda_0 - \lambda_{41}) \left[\frac{\cos i \left(7 \sin^2\theta \sin^2 i - 3\right) \sin\theta \sin i \sin\Delta\lambda}{\left(1 - \sin^2\theta \sin^2 i\right)^{1/2}} \right.$$

$$\left. + \sin i \cos\theta \frac{\left(28 \sin^4\theta \sin^4 i - 27 \sin^2\theta \sin^2 i + 3\right)}{\left(1 - \sin^2\theta \sin^2 i\right)^{1/2}} \cos\Delta\lambda \right] \Bigg\} \quad . \tag{50}$$

Next the longitude excursion relations, Equations 23 and 24, in Equation 50 give

$$F_{T,41}(\theta) = k_{41} \quad \sin(\lambda_0 - \lambda_{41}) \left[\frac{\cos i \left(7 \sin^2\theta \sin^2 i - 3\right) \sin\theta \sin i \left\{1 - \sin^2\theta (1 - \cos i)\right\}}{\left(1 - \sin^2\theta \sin^2 i\right)} \right.$$

$$\left. - \frac{\sin i \cos\theta \left(28 \sin^4\theta \sin^4 i - 27 \sin^2\theta \sin^2 i + 3\right) \sin 2\theta (\cos i - 1)}{2\left(1 - \sin^2\theta \sin^2 i\right)} \right]$$

$$+ \cos(\lambda_0 - \lambda_{41}) \left[\frac{\cos i \left(7 \sin^2\theta \sin^2 i - 3\right) \sin\theta \sin i \sin 2\theta (\cos i - 1)}{2\left(1 - \sin^2\theta \sin^2 i\right)} \right.$$

$$\left. + \frac{\sin i \cos\theta \left(28 \sin^4\theta \sin^4 i - 27 \sin^2\theta \sin^2 i + 3\right)}{\left(1 - \sin^2\theta \sin^2 i\right)} \left\{1 - \sin^2\theta (1 - \cos i)\right\} \right] \Bigg\} \quad \cdot \tag{51}$$

The first term of the factor of $\sin(\lambda_0 - \lambda_{41})$ in Equation 51 is an odd function about $\theta = \pi$. Thus, this term orbit averages to zero. The second term of the factor of $\sin(\lambda_0 - \lambda_{41})$ in Equation 51 also is an odd function about $\theta = \pi$ as it is controlled by $\sin 2\theta \cos\theta = 2 \sin\theta \cos^2\theta$, which is odd about $\theta = \pi$. The first term of the factor of $\cos(\lambda_0 - \lambda_{41})$ in Equation 51 is an odd function about $\theta = \pi/2$ and $3\pi/2$, as it is controlled by $\sin\theta \sin 2\theta = 2 \sin^2\theta \cos\theta$, which is odd about $\theta = \pi/2$ and $3\pi/2$. Therefore, this term orbit averages to zero. The second term of the factor of $\cos(\lambda_0 - \lambda_{41})$ in Equation 51 is also an odd function about $\theta = \pi/2$ and $3\pi/2$. Thus, the orbit average of the entire forcing function $F_{T,41}(\theta)$ is zero;

$$\overline{F_{T,41}(\theta)} = \frac{1}{2\pi} \int_0^{2\pi} F_{T,41}(\theta)\, d\theta = 0 \ .$$

Thus, there can be no long-term contribution to the longitude drift of a 24-hour, inclined, circular-orbit satellite due to this gravity harmonic.

Orbit-Averaged Effect of J_{42}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{42}, is (see Appendix A)

$$\begin{aligned} F_{T,42}(\theta) &= F_{\lambda,42} \sin\alpha + F_{\phi,42} \cos\alpha \\ &= k_{42} \Big\{ \sin 2(\lambda - \lambda_{42}) \sin\alpha (7 \sin^2\phi - 1) \cos\phi \\ &\qquad + \cos 2(\lambda - \lambda_{42}) \cos\alpha (7 \sin^2\phi - 4)\, 2 \cos\phi \sin\phi \Big\} , \end{aligned} \tag{52}$$

where

$$k_{42} = 15 \frac{\mu_E}{(a_s)^2} (R_0/a_s)^4 J_{42} .$$

Introducing the orbit geometry relations, Equations 29a through 29d, causes Equation 52 to become

$$\begin{aligned} F_{T,42}(\theta) = k_{42} \Big\{ &\sin 2(\lambda_0 - \lambda_{42}) \Big[\cos i (7 \sin^2\theta \sin^2 i - 1) \cos 2\Delta\lambda - \sin^2 i \sin^2 2\theta (7 \sin^2\theta \sin^2 i - 4) \sin 2\Delta\lambda \Big] \\ &+ \cos 2(\lambda_0 - \lambda_{42}) \Big[\cos i (7 \sin^2\theta \sin^2 i - 1) \sin 2\Delta\lambda \\ &+ \sin^2 i \sin 2\theta (7 \sin^2\theta \sin^2 i - 4) \cos 2\Delta\lambda \Big] \Big\} . \end{aligned} \tag{53}$$

Using the longitude excursion relations, Equations 25 and 26, makes Equation 52 become

$$\begin{aligned} F_{T,42}(\theta) = k_{42} \Bigg\{ &\sin 2(\lambda_0 - \lambda_{42}) \Bigg[\cos i (7 \sin^2\theta \sin^2 i - 1) \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \\ &- \sin^2 i \sin 2\theta (7 \sin^2\theta \sin^2 i - 4) \frac{\{1 - \sin^2\theta (1 - \cos i)\}\{\sin 2\theta (\cos i - 1)\}}{(1 - \sin^2\theta \sin^2 i)} \Bigg] \\ &+ \cos 2(\lambda_0 - \lambda_{42}) \Bigg[\frac{\cos i (7 \sin^2\theta \sin^2 i - 1)}{(1 - \sin^2\theta \sin^2 i)} \{1 - \sin^2\theta (1 - \cos i)\}\{\sin 2\theta (\cos i - 1)\} \\ &+ \sin^2 i \sin 2\theta (7 \sin^2\theta \sin^2 i - 4) \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \Bigg] \Bigg\} . \end{aligned} \tag{54}$$

The factor of $\cos 2(\lambda_0 - \lambda_{42})$ in Equation 54 is an odd function about $\theta = \pi$. Thus, it orbit averages to zero and has no relevance on the long term drift of the satellite.

Considering only the relevant $\sin 2(\lambda_0 - \lambda_{42})$ term, Equation 54 becomes

$$F_{T,42}(\theta) = k_{42}\left\{\sin 2(\lambda_0 - \lambda_{42})\left[\overbrace{\underbrace{\cos i\left(7\sin^2\theta\sin^2 i - 1\right)}}^{A_2(\theta)}\right.\right.$$

$$\left.\left. - \frac{\cos i\left(7\sin^2\theta\sin^2 i - 1\right)\sin^2 2\theta\left(1-\cos i\right)^2 - 2\sin^2 i\sin^2 2\theta\left(\cos i - 1\right)\left(7\sin^2\theta\sin^2 i - 4\right)\left\{1-\sin^2\theta\left(1-\cos i\right)\right\}}{2\left(1-\sin^2\theta\sin^2 i\right)}\right]\right\} \quad (55)$$

$$= k_{42}\sin 2(\lambda_0 - \lambda_{42})\, A_2(\theta)$$

$$+ k_{42}\sin 2(\lambda_0 - \lambda_{42})\left\{-\frac{\cos i\left(7\sin^2\theta\sin^2 i - 1\right)\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\right.$$

$$\left. + \frac{2\sin^2 i\sin^2 2\theta(1-\cos i)^2\left[\left(7\sin^2\theta\sin^2 i - 1\right) - 3\right]\left[1-\sin^2\theta(1-\cos i)\right]}{2\left(1-\sin^2\theta\sin^2 i\right)(1-\cos i)}\right\} \quad (55a)$$

$$= k_{42}\sin 2(\lambda_0 - \lambda_{42})\, A_2(\theta)$$

$$+ \frac{k_{42}\sin 2(\lambda_0 - \lambda_{42})}{2\left(1-\sin^2\theta\sin^2 i\right)}\left\{\sin^2 2\theta(1-\cos i)^2\left(7\sin^2\theta\sin^2 i - 1\right)\left[-\cos i + \frac{2\sin^2 i}{(1-\cos i)}\left\{1-\sin^2\theta(1-\cos i)\right\}\right]\right.$$

$$\left. - 6\sin^2 i\cdot\sin^2 2\theta(1-\cos i)\left[1-\sin^2\theta(1-\cos i)\right]\right\} \quad (55b)$$

$$= k_{42}\sin 2(\lambda_0 - \lambda_{42})\, A_2(\theta) + k_{42}\sin 2(\lambda_0 - \lambda_{42})\left\{\overbrace{\underbrace{\sin^2 2\theta(1-\cos i)^2\left(7\sin^2\theta\sin^2 i - 1\right)}}^{B_2(\theta)}\right.$$

$$\left. + \frac{\cos i\sin^2 2\theta(1-\cos i)^2\left(7\sin^2\theta\sin^2 i - 1\right) - 6\sin^2 i\sin^2 2\theta(1-\cos i)\left[1-\sin^2\theta(1-\cos i)\right]}{2\left(1-\sin^2\theta\sin^2 i\right)}\right\} \quad (55c)$$

$$= k_{42}\sin 2(\lambda_0 - \lambda_{42})\left[A_2(\theta) + B_2(\theta)\right]$$

$$+ \frac{k_{42}\sin 2(\lambda_0 - \lambda_{42})\sin^2 2\theta(1-\cos i)^2}{2\left(1-\sin^2\theta\sin^2 i\right)}\left\{\cos i\left(7\sin^2\theta\sin^2 i - 1\right) - \frac{6\sin^2 i\left[\cos i + \left(1-\sin^2\theta\sin^2 i\right)\right]}{(1-\cos i)(1+\cos i)}\right\} \quad (55d)$$

$$= k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left[A_2(\theta) + B_2(\theta)\right]$$

$$+ \frac{k_{42} \sin 2(\lambda_0 - \lambda_{42}) \sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \left\{ \cos i \left[-\left(1 - \sin^2\theta \sin^2 i\right) + 6 \sin^2\theta \sin^2 i \right] - 6\left[\cos i + \left(1 - \sin^2\theta \sin^2 i\right)\right] \right\} \tag{55e}$$

$$= k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left[A_2(\theta) + B_2(\theta)\right]$$

$$+ \frac{k_{42} \sin 2(\lambda_0 - \lambda_{42}) \sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \left\{ \left[1 - \sin^2\theta \sin^2 i\right] \left[(-6 - \cos i) - 6 \cos i\right] \right\} \tag{55f}$$

$$= k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left\{ A_2(\theta) + B_2(\theta) \overbrace{- \frac{\sin^2 2\theta (1 - \cos i)^2 \left[6 + 7\cos i\right]}{2}}^{C_2(\theta)} \right\} \tag{55g}$$

Thus, the orbit-averaged tangential force due to J_{42} is (from Equation 55g)

$$\overline{F_{T,42}(\theta)} = \frac{1}{2\pi} \int_0^{2\pi} F_{T,42}(\theta)\, d\theta = k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \left[A_2(\theta) + B_2(\theta) + C_2(\theta)\right] d\theta$$

$$= k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left(\frac{1}{2\pi}\right) \left\{ -2\pi \cos i + \pi \cos i \left(7 \sin^2 i\right) - \pi (1 - \cos i)^2 + 7 \sin^2 i (1 - \cos i)^2 \int_0^{2\pi} \sin^2 2\theta \sin^2\theta \, d\theta - \frac{\pi}{2} (1 - \cos i)^2 (6 + 7\cos i) \right\} \tag{56}$$

$$= k_{42} \sin 2(\lambda_0 - \lambda_{42}) \left\{ \cos i \left(\frac{7 \sin^2 i}{2} - 1 \right) - \frac{(1 - \cos i)^2}{2} + \frac{7 \sin^2 i}{4} (1 - \cos i)^2 - \frac{(1 - \cos i)^2}{4} (6 + 7 \cos i) \right\} \tag{56a}$$

$$= \frac{k_{42} \sin 2(\lambda_0 - \lambda_{42})}{4} \left\{ 14 \cos i \left(1 - \cos^2 i\right) - 4 \cos i - 2\left(1 - 2\cos i + \cos^2 i\right) + 7\left(1 - \cos^2 i\right)\left(1 - 2\cos i + \cos^2 i\right) - \left(1 - 2\cos i + \cos^2 i\right)(6 + 7\cos i) \right\} \tag{56b}$$

$$= -\frac{k_{42}\sin 2(\lambda_0-\lambda_{42})}{4}\{-14\cos i + 14\cos^3 i + 4\cos i + 2 - 4\cos i + 2\cos^2 i - 7$$

$$+ 14\cos i - 7\cos^2 i + 7\cos^2 i - 14\cos^3 i + 7\cos^4 i + 6 - 12\cos i$$

$$+ 6\cos^2 i + 7\cos i - 14\cos^2 i + 7\cos^3 i\} \quad (56c)$$

$$= -\frac{k_{42}\sin 2(\lambda_0-\lambda_{42})}{4}\{1 - 5\cos i - 6\cos^2 i + 7\cos^3 i + \cos^4 i\} \quad (56d)$$

$$= -\frac{k_{42}\sin 2(\lambda_0-\lambda_{42})}{4}\left\{1 + 2\cos i + \cos^2 i - 7\cos i\left(1 + \cos i - \cos^2 i - \cos^3 i\right)\right\} \quad (56e)$$

$$= -k_{42}\sin 2(\lambda_0-\lambda_{42})\left\{\frac{(1+\cos i)^2}{4} - \frac{7\cos i\left(1-\cos^2 i\right)(1+\cos i)}{4}\right\} \quad (56f)$$

$$= -k_{42}\sin 2(\lambda_0-\lambda_{42})\left\{\frac{(1+\cos i)^2}{4} - \frac{7\sin^2 i\cos i(1+\cos i)}{4}\right\} . \quad (56g)$$

But $-k_{42}\sin 2(\lambda_0-\lambda_{42}) = F_{42}\sin 2(\lambda_0-\lambda_{42})$ is the tangential perturbation force on the 24-hour equatorial satellite at longitude λ_0. Thus, the orbit-averaged perturbing force on a 24-hour, inclined circular orbit satellite whose ascending node is at λ_0 is just $F(i)_{42}$ times the perturbing force on the equatorial 24-hour satellite at λ_0, due to J_{42}, where

$$F(i)_{42} = \frac{(1+\cos i)^2}{4} - \frac{7\sin^2 i\cos i(1+\cos i)}{4} \quad (57)$$

Orbit-Averaged Effect of J_{43}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{43}, is (see Appendix A)

$$F_{T,43}(\theta) = F_{\lambda,43}\sin\alpha + F_{\phi,43}\cos\alpha$$

$$= k_{43}\left\{\sin 3(\lambda-\lambda_{43})\sin\alpha\cos^2\phi\sin\phi + \cos 3(\lambda-\lambda_{43}) \cdot \frac{\cos\alpha\left(4\sin^2\phi - 1\right)\cos^2\phi}{3}\right\} , \quad (58)$$

where

$$k_{43} = 315 \frac{\mu_E}{(a_s)^2} (R_0 \, a_s)^4 J_{43} .$$

Introducing the orbit geometry relations, Equations 29a through 29d, causes Equation 58 to become

$$\begin{aligned} F_{T,43}(\theta) = k_{43} \Bigg\{ & \sin 3(\lambda_0 - \lambda_{43}) \Bigg[\cos i \, (1 - \sin^2\theta \sin^2 i)^{1/2} \sin\theta \sin i \cos 3\Delta\lambda \\ & - \frac{\sin i \cos\theta}{3} (4 \sin^2\theta \sin^2 i - 1)(1 - \sin^2\theta \sin^2 i)^{1/2} \sin 3\Delta\lambda \Bigg] \\ & + \cos 3(\lambda_0 - \lambda_{43}) \Bigg[\cos i \, (1 - \sin^2\theta \sin^2 i)^{1/2} \sin\theta \sin i \sin 3\Delta\lambda \\ & + \frac{\sin i \cos\theta}{3} (4 \sin^2\theta \sin^2 i - 1)(1 - \sin^2\theta \sin^2 i)^{1/2} \cos 3\Delta\lambda \Bigg] \Bigg\} \end{aligned} \tag{59}$$

Using the longitude excursion relations, Equations 27 and 27a, causes Equation 58 to become

$$\begin{aligned} F_{T,43}(\theta) = k_{43} \Bigg\{ & \sin 3(\lambda_0 - \lambda_{43}) \Bigg[\cos i \sin i \sin\theta \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \left\{ 1 - \sin^2\theta (1 - \cos i) \right\} \\ & - \cos i \sin i \sin\theta \left\{ 1 - \sin^2\theta (1 - \cos i) \right\} \left\{ \frac{\sin^2 2\theta (\cos i - 1)^2}{(1 - \sin^2\theta \sin^2 i)} \right\} \\ & - \frac{\sin i \cos\theta}{3} (4 \sin^2\theta \sin^2 i - 1) \left\{ 1 - \sin^2\theta (1 - \cos i) \right\}^2 \frac{\sin 2\theta (\cos i - 1)}{(1 - \sin^2\theta \sin^2 i)} \\ & - \frac{\sin i \cos\theta}{6} (4 \sin^2\theta \sin^2 i - 1) \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \sin 2\theta (\cos i - 1) \Bigg] \\ & + \cos 3(\lambda_0 - \lambda_{43}) \Bigg[\cos i \sin i \sin\theta \left\{ 1 - \sin^2\theta (1 - \cos i) \right\}^2 \frac{\sin 2\theta (\cos i - 1)}{(1 - \sin^2\theta \sin^2 i)} \\ & + \frac{\cos i \sin i \sin\theta}{2} \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \sin 2\theta (\cos i - 1) \\ & + \frac{\sin i \cos\theta}{3} (4 \sin^2\theta \sin^2 i - 1) \left\{ 1 - \frac{\sin^2 2\theta (1 - \cos i)^2}{2(1 - \sin^2\theta \sin^2 i)} \right\} \left\{ 1 - \sin^2\theta (1 - \cos i) \right\} \\ & - \frac{\sin i \cos\theta}{6} (4 \sin^2\theta \sin^2 i - 1) \left\{ 1 - \sin^2\theta (1 - \cos i) \right\} \frac{\sin^2 2\theta (\cos i - 1)^2}{(1 - \sin^2\theta \sin^2 i)} \Bigg] \Bigg\} \end{aligned} \tag{60}$$

The first two terms of the factor of $\sin 3(\lambda_0 - \lambda_{43})$ in Equation 60 are odd functions about $\theta = \pi$. The third and fourth terms of the factor of $\sin 3(\lambda_0 - \lambda_{43})$ in Equation 60 are also odd functions about $\theta = \pi$, since they are controlled by $\sin 2\theta \cos\theta = 2\cos^2\theta \sin\theta$ which is "odd" about $\theta = \pi$. All the terms of the factor of $\cos 3(\lambda_0 - \lambda_{43})$ in Equation 60 are odd functions about $\theta = \pi/2$ and $3\pi/2$. The first and second terms are controlled by $\sin\theta \sin 2\theta = 2\sin^2\theta\cos\theta$, and these are "odd" about $\theta = \pi/2$ and $3\pi/2$. Thus, the orbit average of $F_{T,43}(\theta)$ is zero and there can be no long-term contribution to the longitude drift of a 24-hour, inclined, circular-orbit satellite due to this gravity harmonic.

Orbit-Averaged Effect of J_{44}

The tangential force on the 24-hour inclined-orbit satellite at θ from its ascending node, due to J_{44}, is (see Appendix A)

$$
\begin{aligned}
F_{T,44}(\theta) &= F_{\lambda,44}\sin\alpha + F_{\phi,44}\cos\alpha \\
&= k_{44}\left\{\sin 4(\lambda - \lambda_{44})\sin\alpha\cos^3\phi + \cos 4(\lambda - \lambda_{44})\cos\alpha\cos^3\phi\sin\phi\right\} \,, \qquad (61)
\end{aligned}
$$

where

$$
k_{44} = 420\,\frac{\mu_E}{(a_s)^2}\,(R_0/a_s)^4\,J_{44}
$$

Introducing the orbit geometry relations, Equations 29a through 29d causes Equation 61 to become

$$
\begin{aligned}
F_{T,44}(\theta) = k_{44}\Bigg\{&\sin 4(\lambda_0 - \lambda_{44})\left[\cos i\,(1 - \sin^2\theta\sin^2 i)\cos 4\Delta\lambda - \frac{\sin^2 i}{2}\sin 2\theta\,(1 - \sin^2\theta\sin^2 i)\sin 4\Delta\lambda\right] \\
&+ \cos 4(\lambda_0 - \lambda_{44})\left[\cos i\,(1 - \sin^2\theta\sin^2 i)\sin 4\Delta\lambda + \frac{\sin^2 i}{2}\sin 2\theta\,(1 - \sin^2\theta\sin^2 i)\cos 4\Delta\lambda\right]\Bigg\} \,. \qquad (62)
\end{aligned}
$$

Using the longitude excursion relations, Equations 28 and 29, causes Equation 62 to become

$$
\begin{aligned}
F_{T,44}(\theta) = k_{44}\Bigg\{&\sin 4(\lambda_0 - \lambda_{44})\Bigg[\cos i\,(1 - \sin^2\theta\sin^2 i)\left\{1 - \frac{2\left[1 - \sin^2\theta(1 - \cos i)\right]^2\left[\sin 2\theta(\cos i - 1)\right]^2}{(1 - \sin^2\theta\sin^2 i)^2}\right\} \\
&- \sin^2 i\sin 2\theta\left(1 - \sin^2\theta\sin^2 i\right)\left\{1 - \sin^2\theta(1 - \cos i)\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{(1 - \sin^2\theta\sin^2 i)}\right\}\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2(1 - \sin^2\theta\sin^2 i)}\right\}\Bigg] \\
&+ \cos 4(\lambda_0 - \lambda_{44})\Bigg[\cos i\,(1 - \sin^2\theta\sin^2 i)\,2\left\{1 - \sin^2\theta(1 - \cos i)\right\}\left\{\frac{\sin 2\theta(\cos i - 1)}{(1 - \sin^2\theta\sin^2 i)}\right\}\left\{1 - \frac{\sin^2 2\theta(1 - \cos i)^2}{2(1 - \sin^2\theta\sin^2 i)}\right\} \\
&+ \frac{\sin^2 i\sin 2\theta}{2}\,(1 - \sin^2\theta\sin^2 i)\left\{1 - \frac{2\left[1 - \sin^2\theta(1 - \cos i)\right]^2\left[\sin 2\theta\,(\cos i - 1)\right]^2}{(1 - \sin^2\theta\sin^2 i)^2}\right\}\Bigg]\Bigg\} \,. \qquad (63)
\end{aligned}
$$

The factor of $\cos 4(\lambda_0 - \lambda_{44})$ in Equation 63 is an odd function about $\theta = \pi$. Thus, it orbit averages to zero and has no relevance on the long term drift of the satellite.

Considering only the relevant $\sin 4(\lambda_0 - \lambda_{44})$ term, Equation 63 becomes

$$
\begin{aligned}
F_{T,44}(\theta) = {} & k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ \overbrace{\cos i \left(1 - \sin^2 \theta \sin^2 i\right) - \sin^2 i \sin 2\theta \left[1 - \sin^2 \theta (1 - \cos i)\right] \sin 2\theta (\cos i - 1)}^{A'(\theta)} \right\} \\
& + \frac{k_{44} \sin 4(\lambda_0 - \lambda_{44})}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \Big\{ -4 \cos i \left[1 - \sin^2 \theta (1 - \cos i)\right]^2 \left[\sin 2\theta (\cos i - 1)\right]^2 \\
& \qquad - \sin^2 i \sin^4 2\theta (1 - \cos i)^3 \left[1 - \sin^2 \theta (1 - \cos i)\right] \Big\}
\end{aligned}
\tag{63a}
$$

$$
= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ A'(\theta) + \frac{\left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2 \left[-4 \cos i \left\{1 - \sin^2 \theta (1 - \cos i)\right\} + \sin^2 i \sin^2 2\theta (\cos i - 1)\right]}{2\left(1 - \sin^2 \theta \sin^2 i\right)} \right\}
\tag{63b}
$$

$$
= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ A'(\theta) + \frac{2\left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2 \left[-\frac{\cos i}{1 + \cos i} \left\{\cos i + \left(1 - \sin^2 \theta \sin^2 i\right)\right\} - \frac{\sin^4 i}{1 + \cos i} \sin^2 \theta \left(1 - \sin^2 \theta\right)\right]}{\left(1 - \sin^2 \theta \sin^2 i\right)} \right\}
\tag{63c}
$$

$$
\begin{aligned}
= {} & k_{44} \sin 4(\lambda_0 - \lambda_{44}) \Bigg\{ A'(\theta) - \overbrace{\frac{2 \cos i \left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2}{1 + \cos i}}^{B'(\theta)} \\
& - \frac{2\left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2}{(1 + \cos i)\left(1 - \sin^2 \theta \sin^2 i\right)} \left[1 - \sin^2 i \left(1 - \sin^2 i \sin^2 \theta\right) - \sin^4 i \sin^4 \theta\right] \Bigg\}
\end{aligned}
\tag{63d}
$$

$$
\begin{aligned}
= {} & k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left[A'(\theta) + B'(\theta)\right] \\
& + \frac{k_{44} \sin 4(\lambda_0 - \lambda_{44})}{(1 + \cos i)} \Bigg\{ \overbrace{2 \sin^2 i \left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2}^{C'(\theta)} \\
& \qquad \overbrace{- 2\left[1 - \sin^2 \theta (1 - \cos i)\right] \left[\sin 2\theta (\cos i - 1)\right]^2 \left[1 + \sin^2 \theta \sin^2 i\right]}^{D'(\theta)} \Bigg\} .
\end{aligned}
\tag{63e}
$$

Thus, from Equation 63e

$$\overline{F_{T,44}(\theta)} = k_{44}\sin 4(\lambda_0-\lambda_{44})\ \left\{\overline{A'(\theta)}+\overline{B'(\theta)}+\overline{C'(\theta)}+\overline{D'(\theta)}\right\}$$

$$= k_{44}\sin 4(\lambda_0-\lambda_{44})\left\{\cos i - \frac{\cos i\sin^2 i}{2} - \sin^2 i(\cos i - 1)\left(\frac{1}{2}\right) - \sin^2 i(1-\cos i)^2\left(\frac{1}{4}\right)\right.$$

$$+\frac{2\cos i(1-\cos i)(\cos i - 1)^2\left(\frac{1}{4}\right)}{(1+\cos i)} - \frac{2\cos i(\cos i - 1)^2\left(\frac{1}{2}\right)}{(1+\cos i)}$$

$$+\frac{2\sin^2 i(\cos i - 1)^2\left(\frac{1}{2}\right)}{(1+\cos i)} - \frac{2\sin^2 i(\cos i-1)^2(1-\cos i)\left(\frac{1}{4}\right)}{(1+\cos i)}$$

$$-\frac{(\cos i - 1)^2}{(\cos i + 1)} + \left(\frac{1}{4}\right)\left[\frac{2(1-\cos i)(\cos i - 1)^2}{(1+\cos i)} - \frac{2\sin^2 i(\cos i - 1)^2}{(1+\cos i)}\right]$$

$$\left.+\frac{(1-\cos i)\sin^2 i(\cos i - 1)^2\left(\frac{5}{16}\right)}{(1+\cos i)}\right\} \tag{64}$$

$$= k_{44}\sin 4(\lambda_0-\lambda_{44})\left\{\frac{\cos i}{2}\left(2-\sin^2 i\right) + \frac{\sin^2 i}{2}(1-\cos i)\left[1-\frac{(1-\cos i)}{2}\right]\right.$$

$$+\frac{\cos i(\cos i - 1)^2}{1+\cos i}\left[\frac{1-\cos i}{2} - 1\right] + \frac{\sin^2 i(\cos i - 1)^2}{1+\cos i}\left[1-\frac{(1-\cos i)}{2}\right]$$

$$\left.-\frac{(\cos i - 1)^2}{\cos i + 1}\left[1-\frac{(1-\cos i)}{2}\right] + \frac{\sin^2 i}{2} - \frac{5\sin^2 i}{16}(1-\cos i)\right\} \tag{64a}$$

$$= k_{44}\sin 4(\lambda_0-\lambda_{44})\left\{\frac{\cos i}{2}\left(1+\cos^2 i\right) + \frac{\sin^2 i}{4}(1-\cos i)(1+\cos i)\right.$$

$$+\frac{(\cos i - 1)^2}{1+\cos i}\left[-\frac{\cos i}{2}(1+\cos i) + \frac{\sin^2 i}{2}(1+\cos i) - \frac{(1+\cos i)}{2}\right.$$

$$\left.\left.-\frac{\sin^2 i}{2} + \frac{5\sin^2 i(1-\cos i)}{16}\right]\right\} \tag{64b}$$

$$= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ \frac{\cos i}{2} \left(1 + \cos^2 i\right) + \frac{\left(1 - \cos^2 i\right)^2}{4} \right.$$

$$\left. + \frac{(\cos i - 1)^2}{2} \left[-(1 + \cos i) + \left(1 - \cos^2 i\right) - \frac{(\cos i - 1)^2 (1 - \cos i)}{16} (3 + 5 \cos i) \right] \right\} \qquad (64c)$$

$$= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ \frac{\cos i}{2} + \frac{\cos^3 i}{2} + \frac{1}{4} - \frac{\cos^2 i}{2} + \frac{\cos^4 i}{4} - \frac{\cos^3 i}{2} + \cos^2 i \right.$$

$$\left. - \frac{\cos i}{2} - \frac{\cos^4 i}{2} + \cos^3 i - \frac{\cos^2 i}{2} - \frac{(3 + 5 \cos i)}{16} \left(1 - 3 \cos i + 3 \cos^2 i - \cos^3 i\right) \right\} \qquad (64d)$$

$$= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left\{ \frac{1}{16} + \frac{4 \cos i}{16} + \frac{6 \cos^2 i}{16} + \frac{4 \cos^3 i}{16} + \frac{\cos^4 i}{16} \right\} \qquad (64e)$$

$$= k_{44} \sin 4(\lambda_0 - \lambda_{44}) \left[\frac{(1 + \cos i)^4}{16} \right] . \qquad (64f)$$

But $k_{44} \sin 4(\lambda_0 - \lambda_{44}) = F_{44} \sin 4(\lambda_0 - \lambda_{42})$, is the tangential perturbation force on the 24-hour equatorial satellite at longitude λ_0. Thus, the orbit-averaged perturbing force on a 24-hour, inclined circular orbit satellite whose ascending node is at λ_0 is just $F(i)_{44}$ times the perturbing force on the equatorial 24-hour satellite at λ_0, due to J_{44}, where

$$F(i)_{44} = \frac{(1 + \cos i)^4}{16} . \qquad (65)$$

Summary

The orbit-averaged, long-term drift of the inclined circular orbit, 24-hour satellite in an earth gravity field through fourth order is derivable from

$$\ddot{\lambda} = \frac{-\left(12\pi^2\right)}{g_s} \sum_{n=2}^{4} \sum_{m=1}^{n} F_{nm} \sin m\left(\lambda - \lambda_{nm}\right) F(i)_{nm}, \ \text{rad/(sid. day)}^2 , \qquad (66)$$

For n−m, even

where the F_{nm} are given in Equation 2, and

$$F(i)_{22} = \frac{(1 + \cos i)^2}{4} , \qquad (67a)$$

$$F(i)_{31} = \frac{(1+\cos i)}{2} - \frac{5\sin^2 i(1+3\cos i)}{8} \quad , \tag{67b}$$

$$F(i)_{33} = \frac{(1+\cos i)^3}{8} \quad , \tag{67c}$$

$$F(i)_{42} = \frac{(1+\cos i)^2}{4} - \frac{7\sin^2 i \cos i(1+\cos i)}{4} \quad , \tag{67d}$$

$$F(i)_{44} = \frac{(1+\cos i)^4}{16} \quad , \tag{67e}$$

and

$$\dot{a} = \frac{(4\pi\, a_s)}{g_s} \sum_{n=2}^{4} \sum_{m=1}^{n} F_{nm} \sin m(\lambda - \lambda_{nm})\, F(i)_{nm}, \quad \frac{\text{length units}}{\text{sid. day}} \quad . \tag{68}$$

For n−m, even

FIRST INTEGRAL OF THE LONG-TERM LONGITUDE DRIFT

The "trajectory solution" of λ and a as a function of time for the 24-hour satellite drift in the fourth-order gravity field, stemming from Equations 66 and 68, cannot be expressed in a closed form with elementary functions. However a first "energy integral" of Equation 66 can be found which will prove valuable in the applications.

The common separation technique used in Reference 5 yields:

$$\ddot{\lambda} = \frac{d\dot{\lambda}}{dt} = \frac{d(\dot{\lambda})^2}{2\dot{\lambda}\, dt} = \frac{d(\dot{\lambda})^2}{2d\lambda} \quad . \tag{69}$$

Substitution of Equation 69 into Equation 66 enables one to separate variables giving

$$d(\dot{\lambda})^2 = \frac{-2\left(12\pi^2\right)}{g_s} \sum \sum F_{nm} \sin m(\lambda - \lambda_{nm})\, F(i)_{nm}\, d\lambda \quad . \tag{70}$$

Integration of Equation 70 under the assumption that the only variables are λ, the geographic longitude of the ascending node of the satellite, and $\dot{\lambda}$, its drift rate, gives

$$(\dot{\lambda})^2 = C_0 + \frac{\left(24\pi^2\right)}{g_s} \sum \sum \frac{F_{nm}}{m} \cos m(\lambda - \lambda_{nm})\, F(i)_{nm}, \quad [\text{rad/sid. day}]^2 \quad . \tag{71}$$

In the applications, C_0 in Equation 71 can be considered to be the mean long-term squared drift rate of the 24-hour satellite in the drift period under consideration. C_0 is evidently a function of the initial elements of the orbit of the satellite as well as all the earth gravity constants and the initial positions of the sun and moon (see the section, COMPARISON OF THE THEORY WITH NUMERICALLY INTEGRATED TRAJECTORIES).

SIMPLE CLOSED-FORM SOLUTION TO THE LONGITUDE DRIFT OF A 24-HOUR SATELLITE FOR PERIODS CLOSE TO SYNCHRONOUS

If the period of a 24-hour satellite is sufficiently close to synchronous, Equation 66 describes the motion of the mean geographic longitude of this satellite of small orbit eccentricity, due to earth gravity through fourth order (see DISCUSSION).

With the definitions of the constants A_{nm} given in Equations 76, 85, 94, 102, and 105, Equation 66 becomes

$$\ddot{\lambda} = \sum_{\substack{n=2 \\ n-m,\ \text{even}}}^{4} \sum_{m=1}^{n} A_{nm} \sin m\left(\lambda - \lambda_{nm}\right) . \tag{71a}$$

If the problem is restricted to small excursions in mean longitude from an initial 24-hour satellite configuration, Equation 71a may be integrated twice to give the long-term drift explicitly in a simple closed form.

Let this small excursion $\Delta\lambda$ be given from an initial configuration at mean longitude (or longitude of the ascending equator crossing) λ_0, so that

$$\lambda = \lambda_0 + \Delta\lambda . \tag{71b}$$

Providing $m\Delta\lambda$ is sufficiently small, when Equation 71b is substituted into Equation 71a and it is expanded, the small excursion, long-term acceleration due to fourth-order earth gravity becomes

$$\Delta\ddot{\lambda} = \sum\sum A_{nm} \sin m\left(\lambda_0 - \lambda_{nm}\right) + \sum\sum A_{nm}\, m\Delta\lambda \cos m\left(\lambda_0 - \lambda_{nm}\right) . \tag{71c}$$

With Equation 71c rearranged, the differential equation of long-term longitude drift for small excursions is

$$\Delta\ddot{\lambda} + \Delta\lambda\left[-\sum\sum A_{nm}\, m \cos m\left(\lambda_0 - \lambda_{nm}\right)\right] = \sum\sum A_{nm} \sin m\left(\lambda_0 - \lambda_{nm}\right) . \tag{71d}$$

A trial solution to the homogeneous equation of Equation 71d is

$$\Delta\lambda_h = C \sin \omega t + D \cos \omega t \quad . \tag{71e}$$

Equation 71e satisfies the homogeneous equation of Equation 71d only if

$$\omega = + \left[-\sum\sum A_{nm} \, m \cos m \left(\lambda_0 - \lambda_{nm} \right) \right]^{1/2} \quad . \tag{71f}$$

The particular solution to Equation 71d is

$$\Delta\lambda_p = \frac{\sum\sum A_{nm} \sin m \left(\lambda_0 - \lambda_{nm} \right)}{\sum\sum A_{nm} \, m \cos m \left(\lambda_0 - \lambda_{nm} \right)} \quad . \tag{71g}$$

The complete solution to Equation 71d is thus

$$\begin{aligned} \Delta\lambda &= \Delta\lambda_h + \Delta\lambda_p \\ &= C \sin \omega t + D \cos \omega t - \frac{\sum\sum A_{nm} \sin m \left(\lambda_0 - \lambda_{nm} \right)}{\sum\sum A_{nm} \, m \cos m \left(\lambda_0 - \lambda_{nm} \right)} \quad . \end{aligned} \tag{71h}$$

An initial value problem corresponding to Equation 71d may be specified by giving the initial drift rate $\Delta\dot{\lambda}_0$ (small) at time $t = 0$, when $\Delta\lambda = 0$.

Thus, let

$$\Delta\lambda = 0, \qquad \text{when} \qquad t = 0,$$

and

$$\Delta\dot{\lambda} = \Delta\dot{\lambda}_0, \qquad \text{when} \qquad t = 0.$$

This specification in the complete solution, Equation 71h, determines the coefficients C and D:

$$D = \frac{\sum\sum A_{nm} \sin m \left(\lambda_0 - \lambda_{nm} \right)}{\sum\sum A_{nm} \, m \cos m \left(\lambda_0 - \lambda_{nm} \right)} \quad ,$$

and

$$C = \frac{\Delta\dot{\lambda}_0}{\omega} \quad .$$

The small excursion from an initial near 24-hour configuration at λ_0 and $t = 0$, is thus given from Equation 71h as

$$\Delta\lambda = \frac{\Delta\dot{\lambda}_0}{\omega} \sin \omega t + \frac{\sum\sum A_{nm} \sin m(\lambda_0 - \lambda_{nm})}{\sum\sum A_{nm}\, m \cos m(\lambda_0 - \lambda_{nm})} [\cos \omega t - 1] \quad . \tag{71i}$$

It may now be asked under what conditions will the mean longitude excursion of a 24-hour satellite be negligibly small for all time under the influence of earth gravity through fourth order. Such potentially stable mean longitude positions for the 24-hour satellite are to be found at those longitudes λ_0, where (from Equation 71i)

$$\sum_{\substack{n=2 \\ n-m,\ \text{even}}}^{4} \sum_{m=1}^{n} A_{nm} \sin m(\lambda_0 - \lambda_{nm}) = 0 \quad . \tag{71j}$$

For the equatorial 24-hour satellite, the general stability criterion for 24-hour satellites, Equation 71j, was first stated in Reference 5. There it was shown that for the "real earth" there are four such points around the equator, where long term stability (on the order of years) can be achieved for this satellite in the presence of earth gravity through fourth order. Of these four longitudes, two are dynamically stable and two are only statically so. The first two points fall at the two relative minima of the longitude drift potential for the 24-hour equatorial satellite. The two longitudes only statically stable fall at the two relative maxima of the longitude drift potential (see Figure 5). In Reference 6 it was shown that all of these stability points, in fact the entire drift regime for the very high 24-hour equatorial satellite of the "real earth," are strongly dominated by the second order tesseral harmonic J_{22} that is associated with the ellipticity of the equator of the earth. A comparison of the relevant inclination factors $F(i)_{nm}$, Equations 67, will show that equatorial ellipticity dominates the drift regime due to the earth of the inclined 24-hour satellite to an even greater degree for all inclinations less than 90°. However, for the inclined satellite, the total strength (for example the absolute difference between its relative maxima and minima) of this potential declines roughly in proportion to $(1 + \cos i)^2/4$. Thus, for highly inclined 24-hour satellites, sun and moon perturbations may have appreciable influence on long-term 24-hour satellite drift.

Returning to Equation 71i, it may be seen that the criterion for the type of long term stability for satellite placements near λ_0 with zero initial drifts, where $\sum\sum A_{nm} \sin m(\lambda_0 - \lambda_{nm}) \doteq 0$, is established by the character of the "frequency" term ω at the placement longitude λ_0. If $-\sum\sum A_{nm}\, m \cos m(\lambda_0 - \lambda_{nm})$ at the "stable" longitudes λ_0 (from Equation 71j) is greater than zero, then ω (from Equation 71f) is a real number and the character of the long term drift in the neighborhood of λ_0 is harmonic or self-limiting. These longitudes are the *dynamically* stable ones, and they have been shown (References 1 and 5) to be at or near the longitudes of the minor equatorial axes for the "real earth" $(\lambda_{22} \pm 90°)$. On the other hand, if $-\sum\sum A_{nm}\, m \cos m(\lambda_0 - \lambda_{nm})$ is less than zero at or near a "stable" longitude λ_0, ω is an imaginary number and the character of the long term drift in the neighborhood of λ_0 will be exponentially divergent. These longitudes are the merely *statically* stable ones and they have been shown (References 1 and 5) to be at or near the longitudes of the major equatorial axis for the "real earth" (λ_{22}) .

As to the excursion range of validity of Equation 71i, numerical evaluation of the exact elliptic integral of J_{22}-drift (Reference 6, Appendix D), in comparison with Equation 71i indicates it will predict excursions to the order of $m\Delta\lambda = 20°$, with an order of accuracy of about 1%. Since J_{22} dominates the drift of the 24-hour satellite at almost all longitudes, this implies that Equation 71i is generally useful for predicting long term excursions of the 24-hour satellite from an initial mean longitude λ_0 up to about 10 degrees, providing $\Delta\dot{\lambda}_0$ is sufficiently small (see DISCUSSION on this latter point), and providing sun and moon perturbations are negligible.

To illustrate the accuracy of Equation 71i for predicting long term high order drift in the neighborhood of the major equatorial axis of the earth where the influence of J_{22} is small, a comparison of Equation 71i with a numerically integrated third order earth, sun, and moon gravity drift is presented in Table 4, Case 2. The earth gravity field used is a recent weighted average estimate due to W. M. Kaula (Appendix A).

The coefficients and initial orbit parameters in Table 4, Case 2, give the following:

$$A_{22} = 20.7949 \times 10^{-6}, \text{ rad/sid. day}^2 ,$$

$$A_{31} = -.254774 \times 10^{-6}, \text{ rad/sid. day}^2 ,$$

$$A_{33} = 2.14239 \times 10^{-6}, \text{ rad/sid. day}^2 , \tag{71k}$$

$$\sum\sum A_{nm}\, m \cos m(\lambda_0 - \lambda_{nm}) = 35.5169 \times 10^{-6}, \text{ rad/sid. day}^2 , \tag{71l}$$

$$\sum\sum A_{nm} \sin m(\lambda_0 - \lambda_{nm}) = -6.4100 \times 10^{-6} \text{ rad/sid. day}^2 . \tag{71m}$$

Equation 71l substituted in Equation 71f gives

$$\omega = i\left(5.95961 \times 10^{-3} \text{ rad/sid. day}\right) .$$

Thus Equation 71i becomes [noting $\cos i\alpha t = \cosh \alpha t$, $\sin i\alpha t = i \sinh \alpha t$]

$$\Delta\lambda = + \frac{.02289^\circ/\text{solar day} \sinh\left[5.95961 \times 10^{-3}\, t\right]}{5.95961 \times 10^{-3}\, \text{rad/sid. day}} - .180477 \left\{\cosh\left[5.95961 \times 10^{-3}\, t\right] - 1\right\}$$

$$= (.0668525) \sinh\left[5.97593 \times 10^{-3}\, t\right] - .180477 \left\{\cosh\left[5.97593 \times 10^{-3}\, t\right] - 1\right\} \text{ radians} , \qquad (71n)$$

with t (in the final form of Equation 71n) in units of solar days. Equation 71h is evaluated in the last column of Table 4. The discrepancy between the theoretical and the numerically integrated drift illustrates the relatively large influence solar-lunar gravity can have on the drift of a 24-hour satellite near the longitude of the axes of the elliptical equator of the earth (see Figure 5b).

COMPARISON OF THE THEORY WITH NUMERICALLY INTEGRATED TRAJECTORIES

Tables 1, 2, 3 and 4 and Figures 2, 3, 4 and 5 give data from numerically integrated 24-hour particle trajectories about the earth in various perturbing gravity fields.

The trajectories were computed utilizing an earth gravity field as represented in Appendix A. The Gaussian gravity constant of the earth used for all the trajectories was

$$\mu_E = 3.9860319 \times 10^5 \text{ km}^3/\text{sec}^2 .$$

The mean equatorial radius of the earth used for all the trajectories was

$$R_0 = 6378.165 \text{ km} .$$

Tables 1, 2 and 3 give trajectories in earth gravity perturbing fields due to the tesseral harmonics J_{22}, J_{31} and J_{33} respectively, acting alone on a 24-hour satellite of zero and 60 degree orbit inclination.

The J_{nm} longitude harmonic constants were chosen larger than they are in reality for these cases to emphasize the effects of the perturbations in a reasonable period of time.

The additional earth gravity constants used in these trajectories were (Tables 1, 2, and 3; illustrated in Figures 2, 3 and 4)

$$\left.\begin{aligned} J_{22} &= -6.0 \times 10^{-6} , & \lambda_{22} &= -21.0^\circ , \\ J_{31} &= -100.0 \times 10^{-6} , & \lambda_{31} &= -156.0^\circ , \\ J_{33} &= -10.0 \times 10^{-6} , & \lambda_{33} &= -36.0^\circ . \end{aligned}\right\} \qquad (72)$$

Table 1

24-hour Satellite Drift (J_{22} perturbation only).
($J_{22} = -6.0 \times 10^{-6}$, $\lambda_{22} = -21.0°$)

INJECTION CONDITIONS: CASE 1
Inclination: 0.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: -.00566°/day at -66.00567°

INJECTION CONDITIONS: CASE 2
Inclination: 60.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: .00322°/day at -66.00323°

Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)	Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)
0.00000	-66.00001			0.00000	-66.00001		
(1.00000)	(-66.00567)	-.00566	-.00566		(-66.00323)	-.00322	-.00322
2.00000	-66.01132			1.99454	-66.00644		
13.9687	-66.54857			13.96261	-66.30873		
(14.9635)	(-66.63223)	-.08409			(-66.35585)	-.04724	
15.9583	-66.71588			15.95742	-66.40297		
29.9271	-68.51610			29.92199	-67.41438		
(30.9271)	(-68.68971)	-.17361			(-67.51189)	-.09775	
31.9271	-68.86331			31.91706	-67.60939		
43.8958	-71.40683			43.88826	-69.03966		
(44.8958)	(-71.65802)	-.25119			(-69.18039)	-.14106	
45.8958	-71.90921			45.88360	-69.32112		
57.8750	-75.37480			57.85623	-71.27551		
(58.8698	(-75.69993)	-.32682	-.32701		(-71.45938)	-.18428	-.18520
59.8646	-76.02505			59.85180	-71.64325		

Table 2

24-hour Satellite Drift (J_{31} perturbation only).
($J_{31} = -100.0 \times 10^{-6}$, $\lambda_{31} = -156°$)

INJECTION CONDITIONS: CASE 1
Inclination: 0.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: -.00358°/day, at -66.00359°

INJECTION CONDITIONS: CASE 2
Inclination: 60.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: .00146°/day at -65.99855°

Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)	Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)
0.00000	-66.00001			0.00000	-66.00001		
	(-66.00359)	-.00358	-.00358		(-65.99855)	.00146	.00146
2.00000	-66.00716			1.99451	-65.99709		
13.96875	-66.34588			13.96135	-65.85450		
	(-66.39862)	-.05302			(-65.83231)	.02226	
15.95833	-66.45136			15.95578	-65.81011		
29.92708	-67.58556			29.91624	-65.33211		
	(-67.69498)	-.10942			(-65.28606)	.04619	
31.92708	-67.80440			31.91051	-65.24000		
43.89583	-69.40947			43.87586	-64.56675		
	(-69.56747)	-.15883			(-64.50049)	.06645	
45.88542	-69.72547			45.87005	-64.43423		
57.86458	-71.92084			57.83472	-63.51238		
	(-72.12761)	-.20785	-.20814		(-63.42540)	.08724	.08769
59.85417	-72.33437			59.82879	-63.33842		

Table 3

24-hour Satellite Drift (J_{33} perturbation only).
$\left(J_{33} = -10.0 \times 10^{-6},\ \lambda_{33} = -36.0°\right)$

INJECTION CONDITIONS: CASE 1
Inclination: 0.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: -.01066°/day, at -66.01067°

Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)
0.00000	-66.00001		
	(-66.01067)	-.01066	-.01066
2.00000	-66.02133		
13.99687	-67.03684		
	(-67.19582)	-.15898	
15.99687	-67.35480		
29.93750	-70.75079		
	(-71.07700)	-.32621	
31.93750	-71.40321		
43.91667	-76.14489		
	(-76.60462)	-.46213	
45.90625	-77.06434		
57.89583	-83.30173		
	(-83.87285)	-.57112	-.57160
59.89583	-84.44396		

INJECTION CONDITIONS: CASE 2
Inclination: 60.0°
Eccentricity: $.149 \times 10^{-7}$
Initial Drift Rate: -.00454°/day, at -66.00454°

Time (days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Drift Rate (°/day)	Theoretical Drift Rate (°/day)
0.00000	-66.00001		
	(-66.00454)	-.00454	-.00454
1.99455	-66.00906		
13.96297	-66.43768		
	(-66.50455)	-.06704	
15.95790	-66.57142		
29.92364	-68.00616		
	(-.68.14449)	-.13865	
31.91893	-68.28281		
43.89178	-70.30889		
	(-70.50875)	-.20030	
45.88742	-70.70861		
57.86230	-73.46010		
	(-73.71939)	-.25981	-.25953
59.85826	-73.97868		

Table 4

24-hour Satellite Drift in a Third Order Earth, Sun and Moon Gravity Field.
$\left(J_{22} = -1.51 \times 10^{-6},\ \lambda_{22} = -15.5°,\ J_{31} = -1.51 \times 10^{-6},\ \lambda_{31} = 0.0°,\ J_{33} = -.149 \times 10^{-6},\ \lambda_{33} = 22.8°\right)$

INITIAL CONDITIONS: CASE 1
Inclination: 32.8°
Eccentricity: $.45 \times 10^{-7}$, $\left(a_s/R_0\right) = 6.6107211)$
Longitude: -66.00001°
Longitude Rate: .00554°/day

Time (Solar days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Numerically Calculated Longitude Rate (°/day)	Theoretical Longitude Rate (°/day) 4.
1.0	-65.99449	.00554	.00544
24.9	-66.12636	-.01827	
50.85	-66.91847	-.04323	
74.8	-68.24578	-.066799	
100.75	-70.34043	-.093496	-.09415
124.7	-72.86641	-.11874	
148.6	-75.94402	-.13992	-.13970

INITIAL CONDITIONS: CASE 2
Inclination: 32.8°
Eccentricity: $.745 \times 10^{-8}$, $\left(a_s/R_0\right) = 6.6107208$
Longitude: -22.47719°
Longitude Rate: .02289°/day

Time (Solar days from injection)	Numerically Calculated Longitude of the Ascending Equator Crossing (degrees)	Theoretical Longitude (degrees)
0.00000	-22.47719	-22.47719
22.93488	-22.00060	-22.048
48.86178	-21.67153	-21.79
74.78919	-21.55145	-21.76
98.72270	-21.62043	-21.94
122.65685	-21.89455	-22.32
148.58637	-22.47249	-22.97

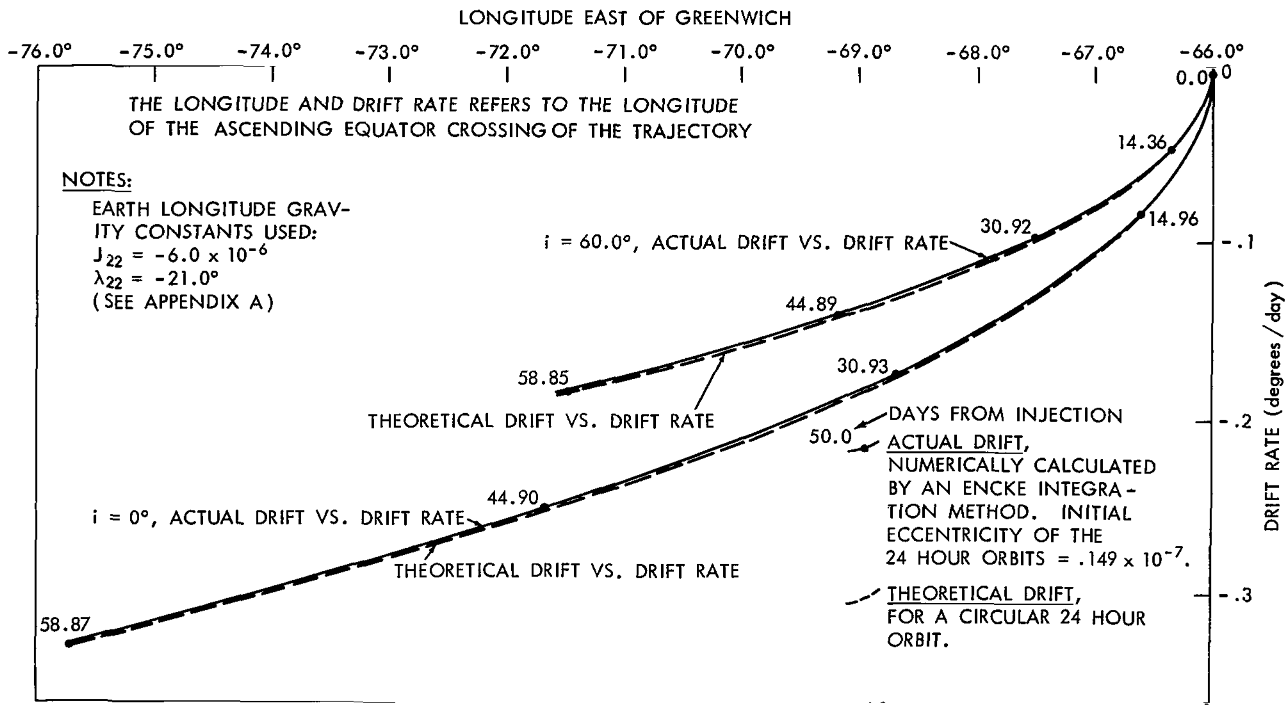

Figure 2—24-hour satellite drift due to J_{22} only.

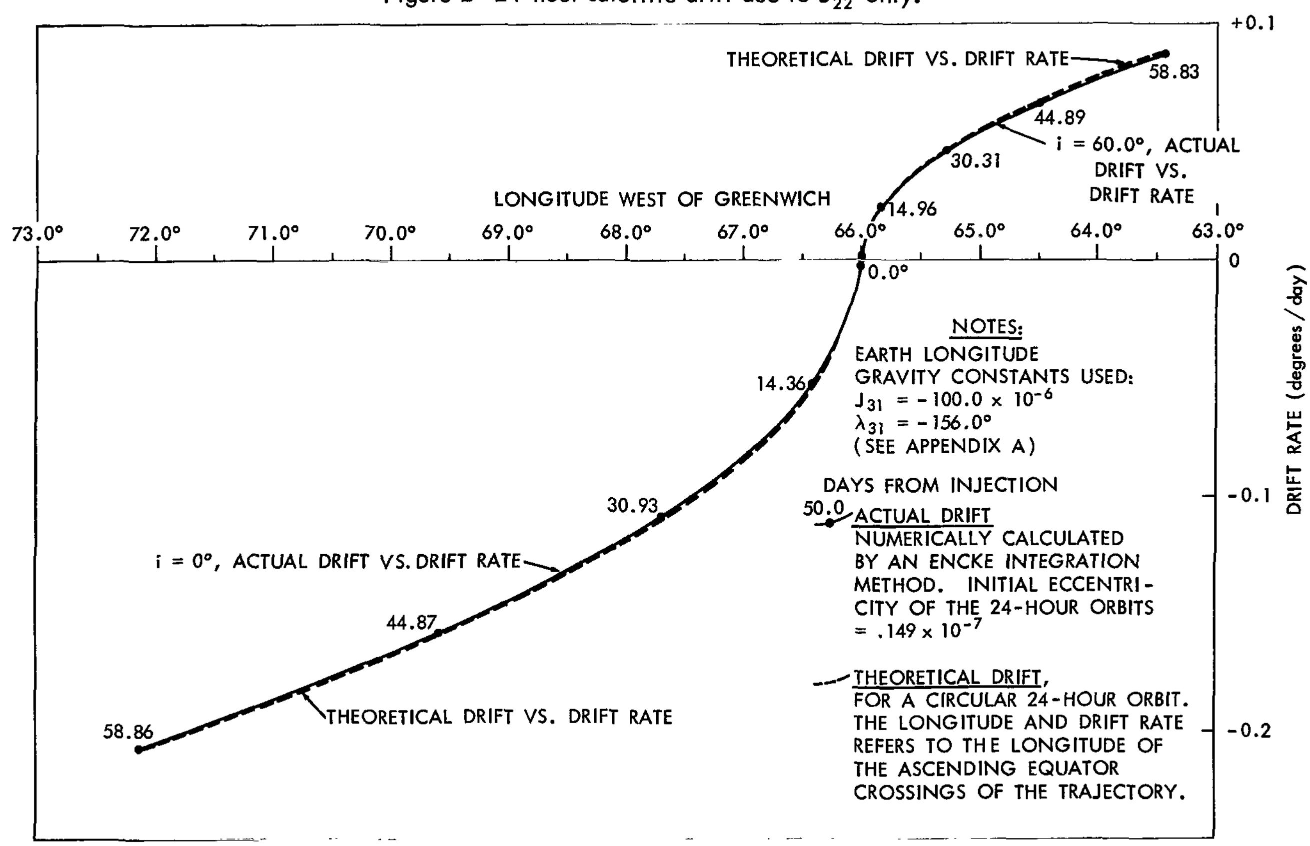

Figure 3—24-hour satellite drift due to J_{31} only.

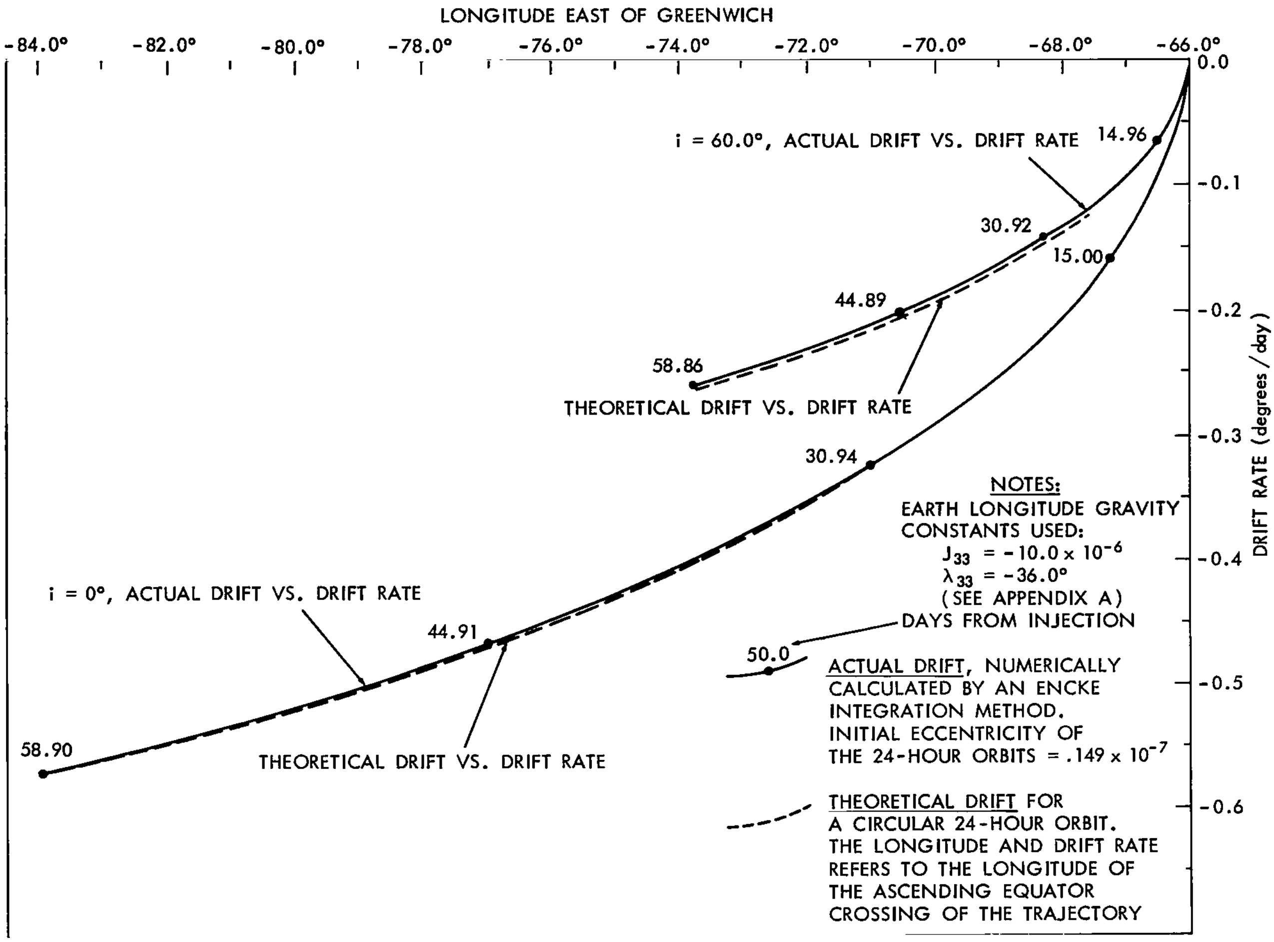

Figure 4—24-hour satellite drift due to J_{33} only.

The "synchronous" semi-major axis a_s, assumed in the theory to be constant for all the orbits of these trajectories, is taken as the semi-major axis at "injection" of the 24-hour satellite, and is

$$a_s = 42,164.27 \text{ km} .$$

Thus, in the theory, the constant (R_0/a_s) is

$$(R_0/a_s) = \left(\frac{6378.165}{52164.27}\right) = .1512694 , \tag{73}$$

the constant $(R_0/a_s)^2$ is

$$(R_0/a_s)^2 = .02288243 , \tag{74}$$

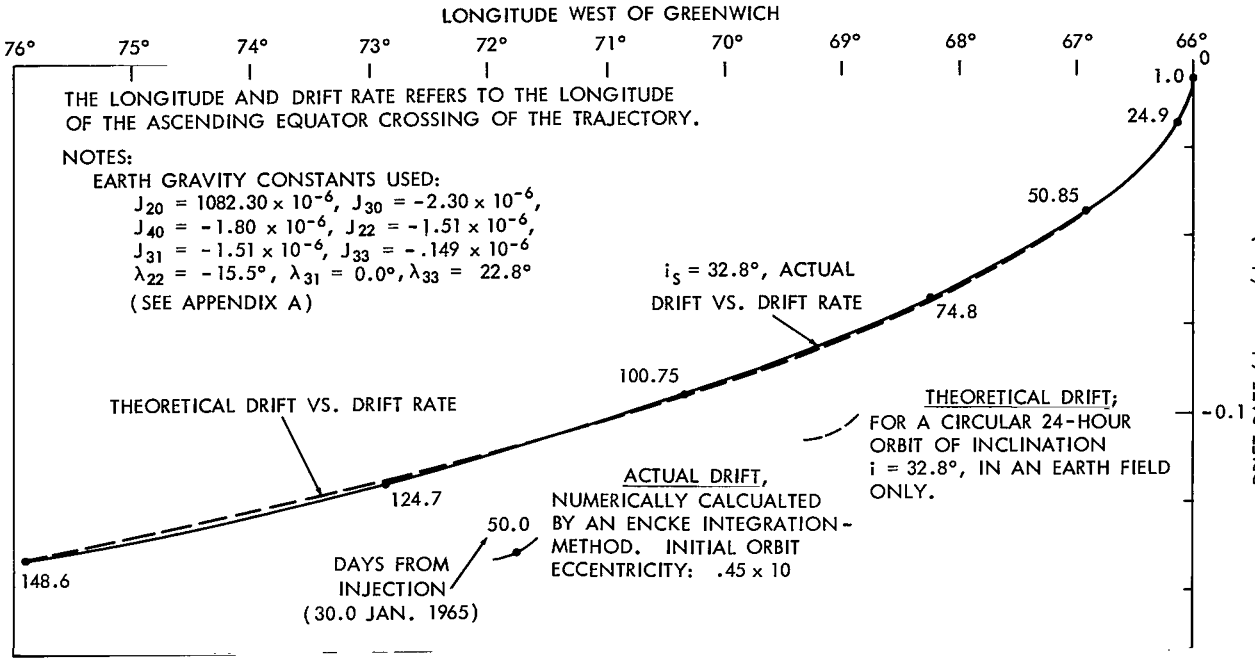

Figure 5a—24-hour satellite drift in a third order earth, sun and moon gravity field.

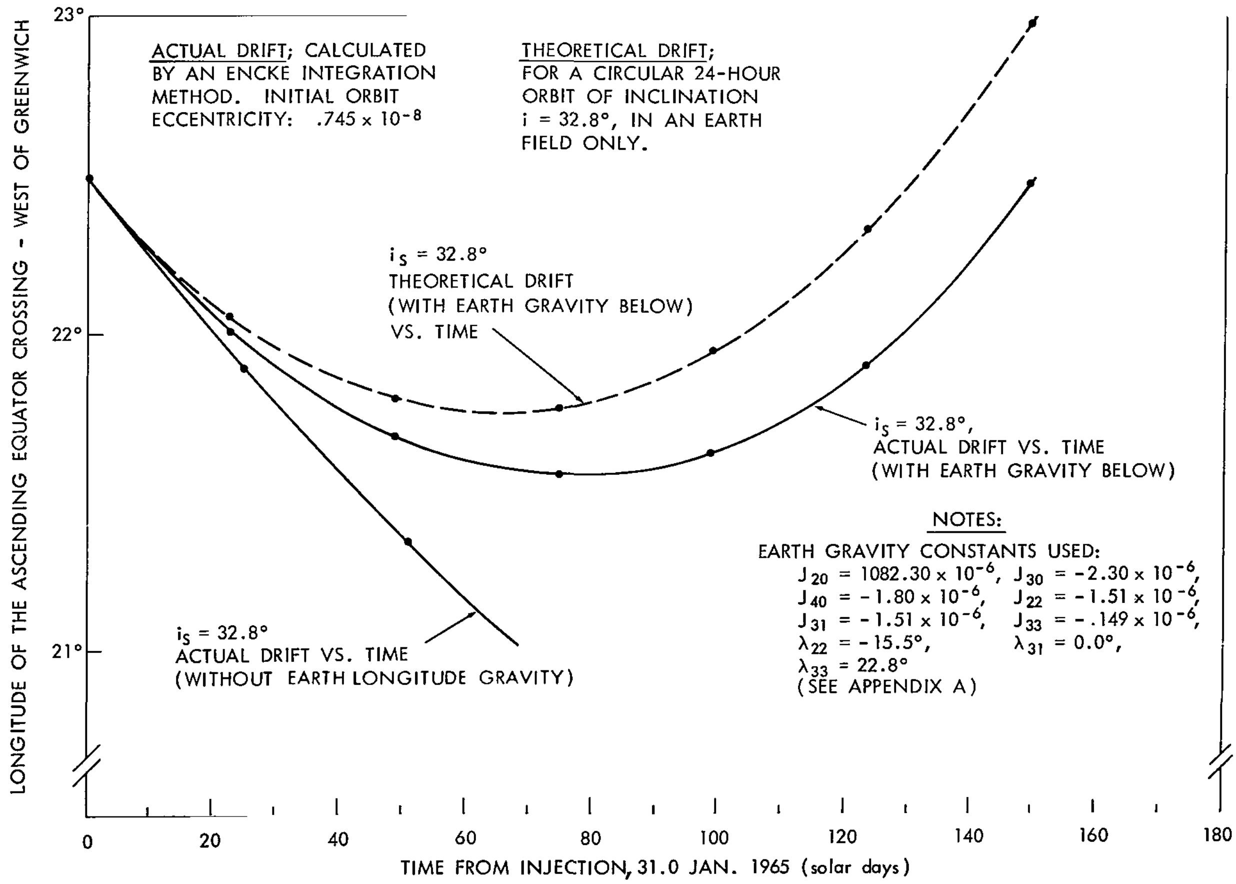

Figure 5b—24-hour satellite drift in a third order earth, sun and moon gravity field.

and the constant $(R_0/a_s)^3$ is

$$(R_0/a_s)^3 = .003461411 \ . \tag{75}$$

The Theory for 24-Hour Satellite Drift in a "J_{22}-only" Perturbation Field

When Equations 71, 2 and 67a are combined and it is noted from Equation 8 that $g_s = \mu_E/(a_s)^2$, Equation 71 becomes for "J_{22}-only" drift

$$(\dot{\lambda})^2 = C_0 - A_{22}\cos 2(\lambda - \lambda_{22}) \ , \tag{76}$$

where

$$A_{22} = -72\pi^2 (R_0/a_s)^2 J_{22} \frac{(\cos i_s + 1)^2}{4} \quad (\text{rad/sid. day})^2 \ . \tag{77}$$

(Compare with Reference 6, Equation 36.)

The corresponding longitude acceleration during "J_{22}-only" drift is given from Equation 65 as

$$\ddot{\lambda} = A_{22}\sin 2(\lambda - \lambda_{22}) \ . \tag{77a}$$

The coefficient A_{22} in Equation 77a is always positive since all the tesseral J_{nm}'s in the gravity potential form of Appendix A are arbitrarily assigned as negative numbers. Equation 77a establishes a one-dimensional potential regime for the long term movement of the mean longitude of the circular orbit 24-hour satellite. This potential has a shape derivable from the right side of Equation 77a which is the effective "force" moving the longitude of the satellite. For a one-dimensional potential, the force is given as the first derivative of the potential. The one-dimensional (longitude) potential energy of the satellite (the negative of the potential) associated with the longitude perturbation expressed by Equation 77a is thus (assigning the potential constant as zero)

$$(\text{P.E.})_{\lambda_{22}} = +\frac{A_{22}}{2}\cos 2(\lambda - \lambda_{22}) \tag{77b}$$

(see Figure 6)

The longitude potential energy of the satellite S from Equation 77b thus has an absolute and a relative maximum at $\lambda = \lambda_{22}$, and at $\lambda = \lambda_{22} + 180°$. These are evidently points of unstable equilibrium where the longitude perturbation force from Equation 77a is momentarily zero. The potential energy due to J_{22} and λ_{22} (from Equation 77b) has an absolute and a relative minimum

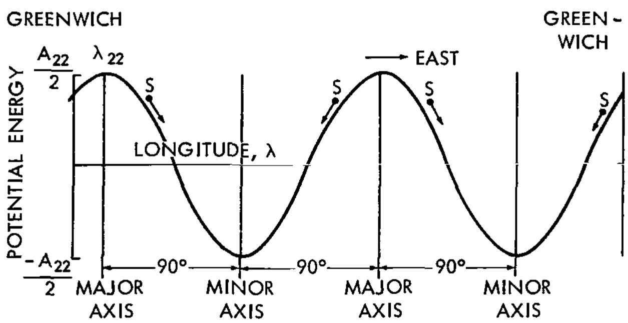

Figure 6—24-hour satellite longitude-drift potential energy and drift regimes due to J_{22}.

at $\lambda - \lambda_{22} = \pm 90°$, or $\lambda = \pm 90° + \lambda_{22}$, which are at longitudes lying on the extensions of the equatorial minor axis. These are evidently the only points of stable equilibrium for the placement of 24-hour satellites considering only the J_{22} perturbation. The "pendulum" dynamics associated with Equation 77a has been discussed previously in Reference 6 in greater detail.

If the initial longitude drift rate of the satellite is $\dot{\lambda}_0$, with the ascending equator crossing at λ_0, then C_0 in Equation 76 is evaluated from

$$(\dot{\lambda}_0)^2 = C_0 - A_{22} \cos 2(\lambda_0 - \lambda_{22}) ,$$

giving

$$C_0 = (\dot{\lambda}_0)^2 + A_{22} \cos 2(\lambda_0 - \lambda_{22}) . \tag{78}$$

Substitution of Equation 78 into Equation 76 gives the theoretical drift rate of the 24-hour satellite at any subsequent longitude λ, as

$$\dot{\lambda} = \pm \left\{ (\dot{\lambda}_0)^2 + A_{22} \left[\cos 2(\lambda_0 - \lambda_{22}) - \cos 2(\lambda - \lambda_{22}) \right] \right\}^{1/2} \text{rad/sid. day} . \tag{79}$$

(The sign of the root in Equation 79 is to be taken according to the sense of the drift with respect to the longitude where $\dot{\lambda} = 0$ for the libratory regime (Reference 6).)

From Equations 72, 74 and 77, A_{22} is

$$A_{22}(i = 0°) = 97.56311 \times 10^{-6} (\text{rad/sid. day})^2 \tag{80}$$

and

$$A_{22}(i = 60°) = 97.56311 \times 10^{-6} \times \frac{\left(1 + \frac{1}{2}\right)^2}{4}$$

$$= 54.87925 \times 10^{-6} (\text{rad/sid. day})^2 . \tag{81}$$

For the case of the equatorial 24-hour satellite in Table 1, (Case 1)

$$\lambda_0 = -66.00567 \ ,$$

$$(\dot{\lambda}_0)^2 = \left[\frac{.00566°/\text{solar day}}{57.29578°/\text{rad}} \times .9972696 \ \frac{\text{solar day}}{\text{sidereal day}}\right]^2$$

$$= 97.05385 \times 10^{-10} \ (\text{rad/sid. day})^2 \ . \qquad (82)$$

Substitution of Equations 72, 80 and 82 into Equation 79 gives the theoretical drift rate for the equatorial 24-hour satellite in Table 1, Case 1 as

$$\dot{\lambda} = -\left\{97.054 \times 10^{-10} + 97.563 \times 10^{-6} \left[-.00020 - \cos 2 \ \lambda + 21.0°)\right]\right\}^{1/2} \ (\text{rad/sid. day}). \qquad (83)$$

See Table 1, Case 1 for an evaluation of Equation 83 at the respective ascending equator-crossing longitudes listed for the numerically calculated drift (see also Figure 2).

For the case of the 60.0° inclined-orbit 24-hour satellite in Table 1, the theoretical drift formula corresponding to Equation 83 is

$$\dot{\lambda} = -\left\{31.412 \times 10^{-10} + 54.879 \times 10^{-6} \left[-.00011 - \cos 2(\lambda + 21.0°)\right]\right\}^{1/2} \ (\text{rad/sid. day}) \qquad (84)$$

(see Table 1, Case 2 and Figure 2). These cases make it clear that the theory for the drift of a 24-hour circular orbit satellite due to J_{22}, summarized in Equations 76 or 79, is essentially an accurate one. In the "Discussion" the probable limits of accuracy of the theory with respect to eccentricity and period are stated more precisely.

The Theory for 24-Hour Satellite Drift in a "J_{31}-only" Perturbation Field

Combining Equations 71, 2 and 67b as in the first part of this section causes Equation 71 to become for "J_{31}-only" drift

$$(\dot{\lambda})^2 = C_0 - 2 A_{31} \cos(\lambda - \lambda_{31}) \ , \qquad (85)$$

where

$$A_{31} = 18\pi^2 (R_0/a_s)^3 \ J_{31} \left\{\frac{1 + \cos i_s}{2} - \frac{5 \sin^2 i_s}{8} \ (1 + 3 \cos i_s)\right\} \ (\text{rad/sid. day})^2 \ . \qquad (86)$$

The corresponding longitude acceleration during "J_{31}-only" drift, is given from Equation 66 as

$$\ddot{\lambda} = A_{31} \sin(\lambda - \lambda_{31}) \quad . \tag{86a}$$

Equation 86a establishes a one-dimensional potential energy for the longitude excursions of the 24-hour satellite with respect to the J_{31} perturbation. As before, this energy function is given as the negative of the first integral of the right hand side of Equation 86a (ignoring the integration constant):

$$(\text{P.E.})_{\lambda_{31}} = A_{31} \cos(\lambda - \lambda_{31}) \quad . \tag{86b}$$

(see Figures 7 and 8).

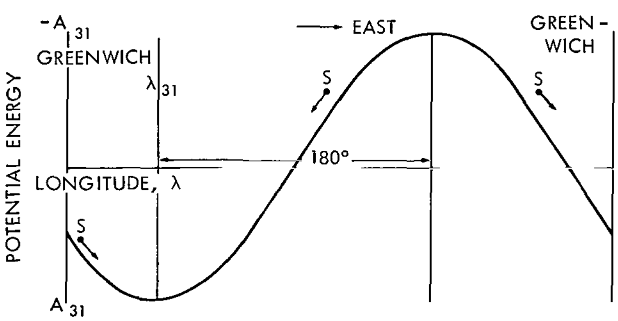

Figure 7—24-hour satellite longitude-drift potential energy and drift regimes due to J_{31} ($A_{31} < 0$, $-41° < i < 41°$, $|i| > 95.1°$).

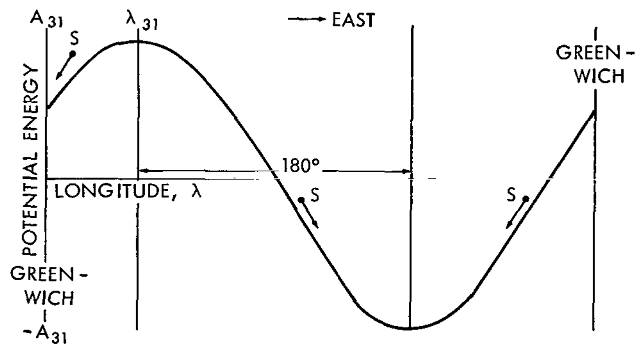

Figure 8—24-hour satellite longitude-drift potential energy and drift due to J_{31} ($A_{31} > 0$, $41° < |i| < 95.1°$).

Depending on the inclination of the 24-hour satellite, two oppositely directed libratory longitude drift regimes are possible due to J_{31} (see "NONRESONANT" INCLINATIONS FOR THE EARTH GRAVITY HARMONICS J_{31} AND J_{42}). In the first, for $|i| < 41°$ and i between $\pm 180°$ and $\pm 95.1°$, stable long-term librations can occur about $\lambda = \lambda_{31}$. In the second regime, for i between $\pm 41°$ and $\pm 95.1°$, stable long-term librations can occur about $\lambda = \lambda_{31} + 180°$ (see Figures 7 and 8 and Appendix A). At the critical inclinations $i = \pm 41°$, $\pm 95.1°$ and $\pm 180°$, of course, there is no long-term drift perturbation at any longitude due to J_{31} since $A_{31} = 0$ for these inclinations.

If the initial longitude drift rate of the satellite is $\dot{\lambda}_0$ with the ascending equator crossing at λ_0, then C_0 in Equation 85 is evaluated as in Equation 78 to yield, for "J_{31}-only" drift,

$$(\dot{\lambda}) = \pm \left\{ (\dot{\lambda}_0)^2 + 2 A_{31} \left[\cos(\lambda_0 - \lambda_{31}) - \cos(\lambda - \lambda_{31}) \right] \right\}^{1/2} (\text{rad/sid. day}) \tag{87}$$

(The sign of the root in Equation 87 is established from the sense of the drift with respect to the longitude where $\dot{\lambda} = 0$ for the libratory regime.)

From Equations 72, 75 and 85, $2A_{31}$ is

$$2\,A_{31}\,(i = 0°) = -122.9859 \times 10^{-6}\ (\text{rad/sid. day})^2 \ , \tag{88}$$

and

$$2\,A_{31}\,(i = 60°) = 51.88468 \times 10^{-6}\ (\text{rad/sid. day})^2 \ . \tag{89}$$

For the case of the equatorial 24-hour satellite in Table 2, (Case 1)

$$\lambda_0 = -66.00359°$$

and

$$(\dot{\lambda}_0)^2 = \left[\frac{(.00358) \times (.9972696)}{57.29578}\right]^2 = 38.82808 \times 10^{-10}\,(\text{rad/sid. day})^2 \ . \tag{90}$$

Substitution of Equations 72, 88 and 90 into Equation 87 gives the theoretical drift rate for the equatorial 24-hour satellite in Table 2, Case 1 as

$$\dot{\lambda} = -\left\{38.828 \times 10^{-10} - 122.99 \times 10^{-6}\ \left[.00006 - \cos(\lambda + 156.0°)\right]\right\}^{1/2} (\text{rad/sid. day}) \ . \tag{91}$$

See Table 2, Case 1 and Figure 3 for an evaluation of Equation 91 at the maximum longitude drift excursion of the satellite in the numerically computed trajectory.

For the case of the 60.0° inclined-orbit, 24-hour satellite in Table 2, the theoretical drift formula corresponding to Equation 91 is

$$\dot{\lambda} = +\left\{6.458 \times 10^{-10} + 51.885 \times 10^{-6}\ \left[-.00003 - \cos(\lambda + 156.0°)\right]\right\}^{1/2} (\text{rad/sid. day}) \tag{92}$$

(see Table 2, Case 2, and Figure 3).

The Theory for 24-Hour Satellite Drift in a "J_{33}-only" Perturbation Field

Combining Equations 71, 2 and 67c causes Equation 71 to become for "J_{33}-only" drift

$$(\dot{\lambda})^2 = C_0 - \frac{2\,A_{33}}{3}\cos 3(\lambda - \lambda_{33}) \ , \tag{93}$$

where

$$A_{33} = -540\pi^2 (R_0/a_s)^3 J_{33} \frac{\left[\cos i_s + 1\right]^3}{8} \text{ (rad/sid. day)}^2 . \tag{94}$$

The corresponding longitude acceleration during "J_{33}-only drift" is given from Equation 66 as

$$\ddot{\lambda} = A_{33} \sin 3(\lambda - \lambda_{33}) . \tag{94a}$$

Equation 94a establishes a one-dimensional potential energy for the longitude excursions of the 24-hour satellite with respect to the J_{33} perturbation. As before, this is given as the negative of the first integral of the right hand side of Equation 94a (ignoring the integration constant):

$$(\text{P.E.})_{\lambda_{33}} = \frac{A_{33}}{3} \cos 3(\lambda - \lambda_{33}) . \tag{94b}$$

(see Figure 9).

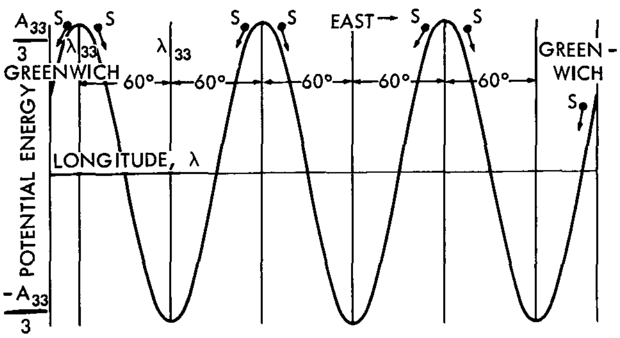

Figure 9—24-hour satellite longitude-drift potential energy and drift regimes due to J_{33}.

Since A_{33} is always positive, the libratory longitude drift regime that can be established by J_{33} for the 24-hour satellite has the form shown in Figure 9. Stable librations can take place only about the three longitudes $\lambda = \lambda_{33} \pm 60°$ and $\lambda = \lambda_{33} + 180°$, though momentarily no long term acceleration is experienced at $\lambda = \lambda_{33}$ and $\lambda = \lambda_{33} \pm 120°$ (see also Appendix A).

If the initial longitude drift rate of the satellite is $\dot{\lambda}_0$ with the ascending equator crossing at λ_0, then C_0 in Equation 93 is evaluated as in Equation 78 to yield for "J_{33}-only" drift

$$\dot{\lambda} = \pm \left\{ (\dot{\lambda}_0)^2 + \frac{2 A_{33}}{3} \left[\cos 3(\lambda_0 - \lambda_{31}) - \cos 3(\lambda - \lambda_{31}) \right] \right\}^{1/2} \text{ (rad/sid. day)} . \tag{95}$$

(The sign of the root in Equation 95 is established from the sense of the drift with respect to the longitude where $\dot{\lambda} = 0$ for the libratory regime.)

From Equations 72, 75 and 94, A_{33} is

$$\frac{2 A_{33}}{3} (i = 0°) = 122.9859 \times 10^{-6} \text{ (rad/sid. day)}^2 \tag{96}$$

and

$$\frac{2A_{33}}{3}(i = 60.0°) = 51.88468 \times 10^{-6} \text{ (rad/sid. day)}^2 \quad . \tag{97}$$

For the case of the equatorial 24-hour satellite in Table 2, (Case 1)

$$\lambda_0 = -66.01067°$$

and

$$(\dot{\lambda}_0)^2 = \left[\frac{.01066 \times .9972696}{57.29578}\right]^2 = 3.442661 \times 10^{-8} \text{ (rad/sid. day)}^2 \quad . \tag{98}$$

Thus substitution of Equations 72, 96 and 98 into Equation 95 gives the theoretical drift rate for the equatorial 24-hour satellite in Table 3, Case 1, as

$$\dot{\lambda} = -\left\{3.4427 \times 10^{-8} + 122.99 \times 10^{-6}\left[-.00056 - \cos 3(\lambda + 36.0°)\right]\right\}^{1/2} \text{(rad/sid. day)}. \tag{99}$$

See Table 3, Case 1 and Figure 4 for an evaluation of Equation 99 at the maximum longitude drift excursion of the satellite in the numerically computed trajectory.

For the case of the 60.0° inclined-orbit, 24-hour satellite in Table 3, the theoretical drift formula corresponding to Equation 99 is

$$\dot{\lambda} = -\left\{7.9022 \times 10^{-9} + 51.885 \times 10^{-6}\left[-.00024 - \cos 3(\lambda + 36.0°)\right]\right\}^{1/2} \text{(rad/sid. day)}. \tag{100}$$

(see Table 3, Case 2 and Figure 4).

The Theory for 24-Hour Satellite Drift in a "J_{42}-only" Perturbation Field

Combining Equations 71, 2 and 67d causes Equation 71 to become for "J_{42}-only" drift

$$(\dot{\lambda})^2 = C_0 - A_{42}\cos 2(\lambda - \lambda_{42}) \quad , \tag{101}$$

where

$$A_{42} = 180\pi^2 (R_0/a_s)^4 J_{42}\left\{\frac{(1 + \cos i)^2}{4} - \frac{7\sin^2 i \cos i(1 + \cos i)}{4}\right\} \quad . \tag{102}$$

The corresponding longitude acceleration during "J_{42}-only drift" is given from Equation 66 as

$$\ddot{\lambda} = A_{42} \sin 2(\lambda - \lambda_{42}) \quad . \tag{103}$$

Equation 103 establishes a one-dimensional potential energy for the longitude excursions of the 24-hour satellite with respect to the J_{42} perturbation. As before, this energy function is given as the negative of the first integral of the right hand side of Equation 103 (ignoring the integration constant):

$$(\text{P.E.})_{\lambda_{42}} = \frac{A_{42}}{2} \cos 2(\lambda - \lambda_{42}) \quad . \tag{104}$$

(see Figures 10 and 11).

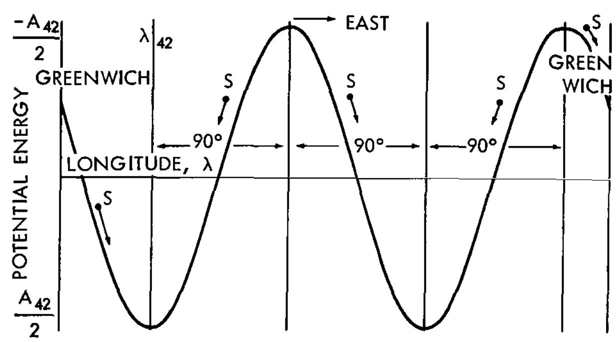

Figure 10—24-hour satellite longitude-drift potential energy and drift regimes due to J_{42} ($A_{42} < 0$, $-34.1° < i < 34.1°$, $|i| > 80.25°$).

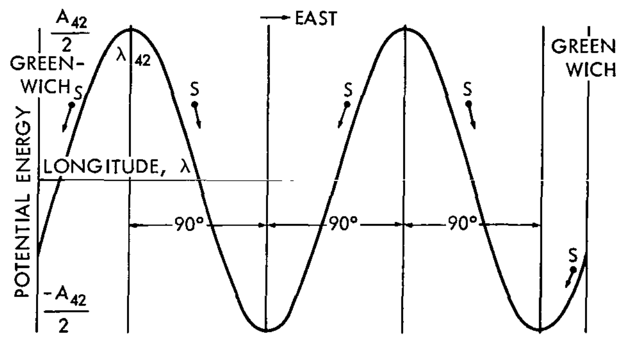

Figure 11—24-hour satellite longitude-drift potential energy and drift regimes due to J_{42} ($A_{42} > 0$, $34.1° < |i| < 80.25°$).

Depending on the inclination of the 24-hour satellite, two oppositely directed libratory longitude drift regimes are possible due to J_{42} (also see "NONRESONANT INCLINATIONS FOR THE EARTH GRAVITY HARMONICS J_{31} AND J_{42}"). In the first, for $|i| < 34.1°$ and $|i| > 80.25°$, stable long term librations can occur about $\lambda = \lambda_{42}$. In the second regime, for $34.1° < |i| < 80.25°$, stable long-term librations can occur about $\lambda = \lambda_{42} \pm 90°$ (see Figures 10 and 11 and Appendix A). At the critical inclinations $i = \pm 34.1°$, $\pm 80.25°$ and $\pm 180.0°$, there is no long term drift perturbation at any longitude due to J_{42}, since $A_{42} = 0$ for these inclinations.

The Theory for 24-Hour Satellite Drift in a "J_{44}-only" Perturbation Field

Combining Equations 71, 2 and 67e causes Equation 71 to become for "J_{44}-only" drift

$$(\dot{\lambda})^2 = C_0 - \frac{A_{44}}{2} \cos 4(\lambda - \lambda_{44}) \quad , \tag{104}$$

where

$$A_{44} = -5040\pi^2 (R_0/a_s)^4 J_{22} \frac{(1 + \cos i)^4}{16} \quad . \tag{105}$$

The corresponding longitude acceleration during "J_{44}-only" drift is given from Equation 66 as

$$\ddot{\lambda} = A_{44} \sin 4(\lambda - \lambda_{44}) \quad . \tag{106}$$

Equation 106 establishes a one-dimensional potential energy for the longitude excursions of the 24-hour satellite with respect to the J_{44} perturbation. As before, this energy function is given as the negative of the first integral of the right hand side of Equation 106 (ignoring the integration constant):

$$(\text{P.E.})_{\lambda_{44}} = \frac{A_{44}}{4} \cos 4(\lambda - \lambda_{44}) \tag{107}$$

(see Figure 12).

Since A_{44} is always positive, the libratory longitude drift regime that can be established by J_{44} for the 24-hour satellite has the form shown in Figure 12. Stable librations can take place only about the four longitudes $\lambda = \lambda_{44} \pm 45°$ and $\lambda = \lambda_{44} \pm 135°$. Momentarily, no long term acceleration is experienced at the four longitudes $\lambda = \lambda_{44}$, $\lambda = \lambda_{44} \pm 90°$, and $\lambda = \lambda_{44} + 180°$ (see Appendix A).

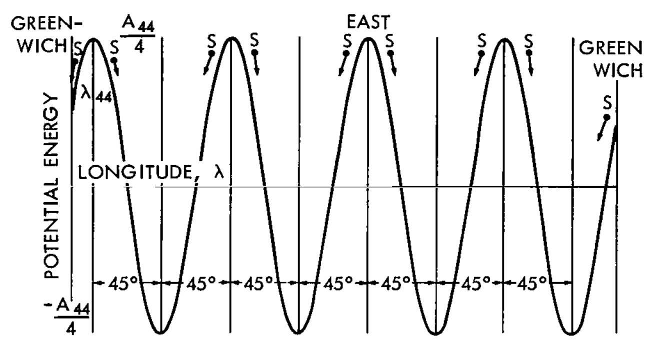

Figure 12—24-hour satellite longitude-drift potential energy and drift regimes due to J_{44}.

It is noted that the actual longitude potential energy function of the earth for the 24-hour satellite will be a superposition of Figures 6 through 12. The exact form of this total potential energy will depend on the relative proportions of the actual longitude gravity constants of the earth. It is evident from the studies in References 6, 1 and 10 that J_{22} dominates the actual longitude drift potential energy for the 24-hour satellite of the earth.

Comparison of a Theoretical and Numerically Integrated 24-Hour Drift Trajectory in a Third Order Earth, Solar and Lunar Gravity Field

In Table 4, Case 1 are tabulated the results of a numerically integrated 24-hour satellite trajectory in a third order earth, solar and lunar gravity field. The longitude earth field used is a recent weighted average estimate from W. M. Kaula (Appendix A).

The theoretical phase plane trajectory of the long term drift of a 24-hour satellite in a full earth field alone, is derivable from Equation 71. Let $\dot{\lambda}_0$ be the initial small drift rate of such a satellite at a time when its mean longitude is at λ_0. Then from Equation 71 the constant C_0 is evaluated as

$$C_0 = (\dot{\lambda}_0)^2 + \frac{(-24\pi^2)}{g_s} \sum \sum \frac{F_{nm}}{m} \cos m(\lambda_0 - \lambda_{nm}) F(i)_{nm} \quad . \tag{108}$$

Substitution of Equation 108 into Equation 71 gives the drift rate as a function of longitude:

$$\dot{\lambda} = \left\{ (\dot{\lambda}_0)^2 + \frac{24\pi^2}{g_s} \sum \sum \frac{F_{nm}}{m} F(i)_{nm} \left[\cos m(\lambda - \lambda_{nm}) - \cos m(\lambda_0 - \lambda_{nm}) \right] \right\}^{1/2} \quad . \tag{109}$$

In terms of the A_{nm} gravity-orbit constants, Equation 109 can be rewritten as

$$\dot{\lambda} = \left\{ (\dot{\lambda}_0)^2 + \sum \sum \frac{2 A_{nm}}{m} \left[\cos m(\lambda_0 - \lambda_{nm}) - \cos m(\lambda - \lambda_{nm}) \right] \right\}^{1/2} \quad . \tag{110}$$

The A_{nm}'s for the trajectory in Table 4, Case 1 are the same as 71k. For these constants and the given initial conditions, the theoretical trajectory of Table 4, Case 1 is

$$\dot{\lambda} = \left\{ -4.2556 \times 10^{-6} - \sum \sum \frac{2 A_{nm}}{m} \cos m(\lambda - \lambda_{nm}) \right\}^{1/2} \text{(rad/sid. day)} \tag{111}$$

(see Table 4, Case 1).

It is noted that the maximum deviation of the actual from the theoretical phase plane trajectory in this case is of the order of 0.5% and is due to long term effects of the sun and moon (see Figure 5a).

"NONRESONANT" INCLINATIONS FOR THE EARTH GRAVITY HARMONICS J_{31} AND J_{42}

Equations 67a-67e make it evident that at every inclination the total long term drift regime is due to different proportions (functionally independent) of the relevant gravity harmonics. In general, it can be noted that all the harmonics higher than the second have reduced influence with respect to the J_{22} harmonic for increasing inclinations (at least for prograde orbits). For the "mixed tesserals" J_{31} and J_{42}, which are strongly latitude-sensitive, a large part of the smoothing-down effect is due to the latitude excursion of the inclined orbit. For the higher spatial frequency sectorial tesseral harmonics J_{33} and J_{44}, the effect of the daily longitude excursion is to smooth down their orbit-averaged contributions with respect to that of the J_{22} harmonic. (See the pictorial representations of the potential in Appendix A.)

These facts make it possible, for example, to distinguish between the effects of the gravity harmonics J_{22} and J_{42} by utilizing two 24-hour satellites of reasonably different inclinations. With a single 24-hour satellite, this would not be possible because these harmonics have the same spatial frequency m. Of special interest for geodetic satellites are those inclinations for which the "tesserals," (J_{nm} with $n - m$ even, $n \neq m$) have zero inclination factors $F(i)_{nm}$. Satellites having these special inclinations and periods commensurate with the rotation period of the earth, such as the 24-hour satellite, can have no secular perturbations of the semi-major axis due to that particular mixed tesseral harmonic. It is conjectured that there can always be found at least one "nonresonant" inclination for each of these harmonics for which $F(i)_{nm} = 0$. Secular perturbation is defined as a non-zero along track force which is orbit averaged over the synodic period of the satellite with respect to the earth's rotation. Utilization of these special inclinations would simplify the difficult problem of separating the effects of the individual longitude harmonics in determining the exact shape of the geoid from observations on a limited number of earth satellites.

"Nonresonant" Inclinations of J_{31}

In Equation 67b let $x = \cos i$. Then the condition for a nonresonant 24-hour satellite due to J_{31} is established when

$$F(i)_{31} = 0 = \frac{1+x}{2} - \frac{5\left(1-x^2\right)}{8}(1+3x) = 4 + 4x - 5 + 5x^2 + 15x^3 - 15x \quad . \tag{112}$$

Equation 112 reduces to

$$(x+1)\left(15x - 10x - 1\right) = 0 \quad . \tag{113}$$

The three roots of Equation 113 (giving the critical inclinations for J_{31}) are

$$x_1 = -1 (i = \pm 180^\circ)$$

$$x_2 = .755 (i = \pm 41.0^\circ)$$

$$x_3 = -.089 (i = \pm 95.1^\circ) \quad . \tag{114}$$

"Nonresonant" Inclinations of J_{42}

In Equation 67d let $x = \cos i$. Then the condition for a nonresonant 24-hour satellite due to J_{42} is established when

$$F(i)_{42} = 0 = (1+x)^2 - 7\left(1-x^2\right)(x)(1+x) \quad , \tag{115}$$

Equation 115 reduces to

$$(1+x)^2 \left(7x^2 - 7x + 1\right) = 0 \ . \tag{116}$$

The four roots of Equation 116 (giving the critical inclinations for J_{42}) are

$$x_1, x_2 = -1(i = \pm 180°) \ ,$$

$$x_3 = .828(i = \pm 34.1°) \ ,$$

$$x_4 = .172(i = \pm 80.25°) \ . \tag{117}$$

DISCUSSION

The exact theory of 24-hour satellite drift due to earth gravity presented in this report and summarized by Equations 66, 68 and 71 rests on three fundamental assumptions. The first is that the orbit of the satellite is always perfectly circular. It should be evident that any eccentricity in the orbit will give rise to greater errors in this theory for inclined satellites than for equatorial ones. For the equatorial satellite of up to moderate eccentricity (i.e., $e \leq .2$), the only change indicated in the simple theory would be to use the mean daily longitude position of the satellite as the characteristic longitude λ of the satellite. In fact, this is also the indicated theory improvement change for the equatorial satellite of appreciable drift rate (i.e. $\dot{\lambda} > 2°$/day) since the perturbation method relies fundamentally on orbit averaging the earth gravity effects with respect to geographical longitude and latitude. The situation with respect to eccentricity for inclined-orbit satellites is more severe. This is not merely because of the greater difficulty in locating the orbit averaged longitude location of the inclined-eccentric orbit 24-hour satellite. Difficult as this may be for perigee locations distant from the equator, the essential phenomenon which might cause distortion of the simple circular-inclined orbit theory applied to orbits with eccentricity is the possible introduction of significant energy perturbations to the orbit due to zonal earth gravity.

W. M. Kaula* believes that long term effects on the semi-major axis of a satellite due to zonal gravity must be negligible to at least the second order in the eccentricity. Assuming contributions to orbit-averaged energy from zonal gravity proportional to the third power of the eccentricity, it is easy to see that these will give rise to effects of the order of magnitude of the longitude effects only when the eccentricity is of the order of magnitude of 0.1. This is because the magnitude of the dominant zonal gravity force is of the order of 1000 times that of the strongest tesseral force at a given altitude.

The second assumption is that the semi-major axis of the 24-hour satellite is unchanged in the drift from a perfectly synchronous orbit. In Reference 6 it was shown that over the widest

*Private communication.

possible longitude libration for a 24-hour satellite the expected maximum change of semi-major axis is only of the order of 0.1%.

The third assumption (related to the second) is that the period of the orbit remains synchronous. This assumption enters the theory in two places. One is in the calculation of the orbit-averaged effect of the harmonics. These effects are assumed to be referable to a single constant longitude location over each orbit, an assumption valid only for an orbit with the earth synchronous period. The second place is in the assignment of the units "sidereal day" for the basic orbit-averaged drift Equations 9 and 11 and all subsequent drift equations. The proper time unit in these orbit averaged equations is the period of the orbit $2\pi/\omega_s$, where ω_s is the mean motion (mean angular rate) of the satellite in its orbit. Let $\omega' = |\omega_s - \omega_e|$ measure the difference of the actual mean motion of the satellite from the earth rate. Then it is evident that the error arising from the assumption of a constant earth period unit in the drift equations is of the order of ω'/ω_e. It can be shown that the first source of error in the calculation of the orbit averaged effects (referred to a single longitude) for the longitude harmonic J_{qp}, in the case of the nonsynchronous satellite of mean motion ω', is of order $p\omega'/\omega_e$.

Thus to limit errors in the simple drift equations for the second order harmonic J_{22} and the fourth J_{42} to one part in a hundred, $2\omega'/\omega_e \leq 10^{-2}$. Since the drift rate for the near-synchronous satellite is given approximately as $360\,\omega'/\omega_e$ degrees/day, the limiting drift rate with respect to these harmonics is 1.8 degrees/day. With respect to the third harmonic, J_{33}, it is 1.2 degrees/day. With respect to the fourth order harmonic, J_{44}, it is .9 degrees/day. With respect to the third order harmonic, J_{31}, the limiting drift rate is 3.6 degrees/day. For drift rates in excess of these limits, the simple theory given in this report may still hold with reasonable accuracy if the longitude λ is redefined as the mean geographic longitude traversed in the ground track over an orbit (i.e. the mean of successive ascending equator crossings for a nearly circular orbit). In addition, the time unit of the drift equations should be redefined in terms of ω_s.

Many numerical trajectories made in conjunction with geodetic analysis of the motions of the Syncom II satellite (References 6 and 7) have confirmed the adequacy, for these purposes, of the simple, "first-order" perturbation drift equations presented herein (at least for the J_{22} effect) with unmodified λ, for eccentricities up to .0012 and drift rates as high as 1 degree/day.

CONCLUSIONS

1. The first order long-term longitude drift of a 24-hour satellite due to earth gravity through fourth order has been shown to be strictly analogous to a mathematical pendulum for each of the relevant tesseral harmonics J_{22}, J_{31}, J_{33}, J_{42} and J_{44}.

2. Under the perturbation of the relevant gravity harmonic J_{nm}, there are potentially m stable longitude positions around the equator about which long-term longitude librations of the 24-hour satellite can occur.

3. The actual perturbed drift of the 24-hour earth satellite due to high-order earth gravity will, to first order, be a linear sum of the separate harmonic effects in a proportion depending on the magnitude of the actual gravity coefficients.

4. For the "real earth" the long term drift of the 24-hour satellite is undoubtedly dominated by the second order tesseral effect (J_{22}). Terms of third order are probably next most influential on the 24-hour satellite, dominated by the effect of J_{33}.

5. The simple, first order pendulum drift theory due to earth gravity presented herein is valid for 24-hour circular orbit satellites of all inclinations and should give a good approximation to the actual long term drift for nearly synchronous period satellites having an eccentricity of up to order 0.1 and/or drift rates of up to the order of 1 degree/day. The theory can be expected to give the poorest accuracy for predicting drift in the immediate vicinity of the longitudes of the axes of the earth's elliptical equator under the additional perturbing influence of the sun and moon.

6. The drift perturbation for each relevant gravity harmonic is strongest on the 24-hour equatorial satellite, and for the "tesserals," J_{31} and J_{42}, becomes zero at all mean longitude locations for certain critical orbit inclinations.

7. For geodetic purposes, use of one or more *inclined* orbit 24-hour satellites (such as Syncom II) as well as equatorial 24-hour satellites gives promise of defining most efficiently gravity terms in the earth's potential to third and fourth order.

(Manuscript received June 3, 1965)

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Appendix A

Earth's Gravity Field as a Series of Spherical Harmonic Components. Comparison of Recent Longitude Gravity Fields

The gravity potential used as the basis for this study is the exterior potential of the earth derived in Reference A-1 for geocentric spherical coordinates referenced to the spin axis and center of mass of the earth. The infinite series of spherical harmonics is truncated after J_{44}. The zonal harmonic constants J_{20}, J_{30} and J_{40} used in the section on the comparison of the theory with numerically integrated trajectories are taken from Reference A-2.

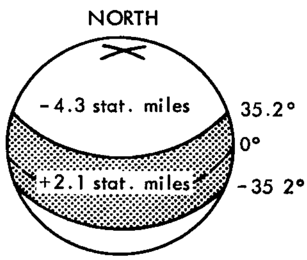
The earth equatorial radius R_0 used in this study is

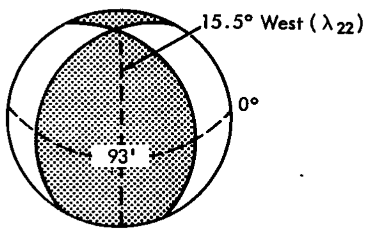
$$R_0 = 6378.165 \text{ km} .$$

The Gaussian gravity constant of the earth used is

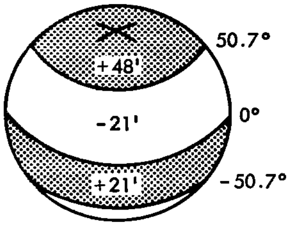
$$\mu_E = 3.986032 \times 10^5 \text{ km}^3/\text{sec}^2 .$$

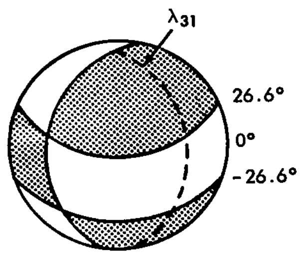
The longitude harmonic constants used in the simulation studies are those of Kaula-Combined (1964) as listed in Table A-1. These correspond to the values shown on the "longitude geoids" below. The gravity potential of the earth (to fourth order, probably sufficient to account for all significant perturbations on a 24-hour satellite) may be illustrated (following Reference A-3 with the zonal constants of Reference A-2) as follows, from Kaula-combined (1964):



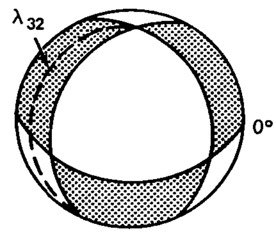


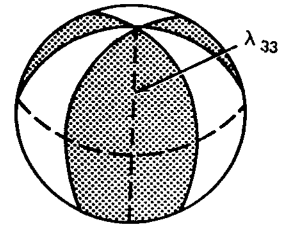
$$V_E = \frac{\mu_E}{r}\left\{1 - \frac{J_{20} R_0^2}{2r^2}\left(3\sin^2\phi - 1\right) - 3J_{22}\frac{R_0^2}{r^2}\cos^2\phi\cos 2\left(\lambda - \lambda_{22}\right)\right.$$



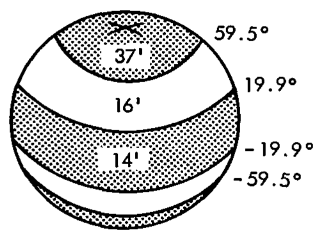


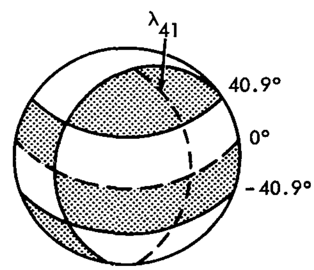
$$- \frac{J_{30} R_0^3}{2r^3}\left(5\sin^3\phi - 3\sin\phi\right) - \frac{J_{31} R_0^3}{2r^3}\cos\phi\left(15\sin^2\phi - 3\right)\cos\left(\lambda - \lambda_{31}\right)$$



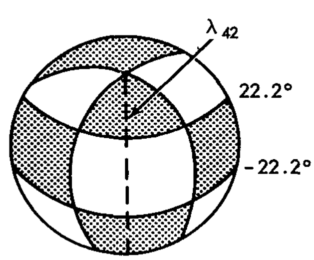


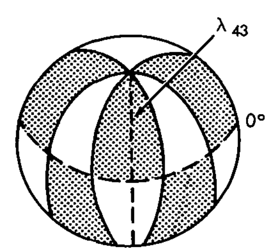
$$- 15J_{32}\,\frac{R_0^3}{r^3}\,\cos^2\phi\,\sin\phi\,\cos 2\left(\lambda - \lambda_{32}\right) \; - \; 15J_{33}\,\frac{R_0^3}{r^3}\,\cos^3\phi\,\cos 3\left(\lambda - \lambda_{33}\right)$$



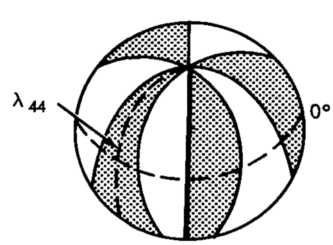


$$- \frac{J_{40}\,R_0^4}{8r^4}\left(35\sin^4\phi - 30\sin^2\phi + 3\right) \; - \; \frac{J_{41}\,R_0^4}{8r^4}\left[140\sin^3\phi - 60\sin\phi\right]\cos\phi\,\cos\left(\lambda - \lambda_{41}\right)$$





$$- \frac{J_{42}\,R_0^4}{8r^4}\left[420\sin^2\phi - 60\right]\cos^2\phi\,\cos 2\left(\lambda - \lambda_{42}\right) \; - \; \frac{J_{43}\,R_0^4}{8r^4}\,840\sin\phi\,\cos^3\phi\,\cos 3\left(\lambda - \lambda_{43}\right)$$



$$- \frac{J_{44}\,R_0^4}{8r^4}\,840\cos^4\phi\,\cos 4\left(\lambda - \lambda_{44}\right)\Bigg\} \; . \tag{A-1}$$

The earth-gravity field (per unit test mass) whose potential is Equation A-1 is given as the gradient of Equation A-1 or

$$\vec{F} = \hat{r}F_r + \hat{\lambda}F_\lambda + \hat{\phi}F_\phi = \nabla V_E = \hat{r}\frac{\partial V_E}{\partial r} + \frac{\hat{\lambda}}{r\cos\phi}\frac{\partial V_E}{\partial \lambda} + \frac{\hat{\phi}}{r}\frac{\partial V_E}{\partial \phi} , \tag{A-2}$$

$$\begin{aligned} F_r = \frac{\mu_E}{r^2}\Big\{-1 &+ (R_0/r)^2 \Big[3/2J_{20}\,(3\sin^2\phi - 1) + 9J_{22}\cos^2\phi\cos 2(\lambda - \lambda_{22}) \\ &+ 2(R_0/r)\,J_{30}\,(5\sin^2\phi - 3)\,(\sin\phi) + 6(R_0/r)\,J_{31}\,(5\sin^2\phi - 1)\cos\phi\cos(\lambda - \lambda_{31}) \\ &+ 60(R_0/r)\,J_{32}\cos^2\phi\sin\phi\cos 2(\lambda - \lambda_{32}) + 60(R_0/r)\,J_{33}\cos^3\phi\cos 3(\lambda - \lambda_{33}) \\ &+ 5/8(R_0/r)^2 J_{40}\,(35\sin^4\phi - 30\sin^2\phi + 3) \\ &+ 25/2(R_0/r)^2 J_{41}\,(7\sin^2\phi - 3)\cos\phi\sin\phi\cos(\lambda - \lambda_{41}) \\ &+ 75/2(R_0/r)^2 J_{42}\,(7\sin^2 - 1)\cos^2\phi\cos 2(\lambda - \lambda_{42}) \\ &+ 525(R_0/r)^2 J_{43}\cos^3\phi\sin\phi\cos 3(\lambda - \lambda_{43}) + 525(R_0/r)^2 J_{44}\cos^4\phi\cos 4(\lambda - \lambda_{44})\Big]\Big\} . \end{aligned} \tag{A-3}$$

$$\begin{aligned} F_\lambda = \frac{\mu_E}{r^2}(R_0/r)^2\Big\{&6J_{22}\cos\phi\sin 2(\lambda - \lambda_{22}) + 3/2(R_0/r)\,J_{31}\,[5\sin^2\phi - 1]\sin(\lambda - \lambda_{31}) \\ &+ 30(R_0/r)\,J_{32}\cos\phi\sin\phi\sin 2(\lambda - \lambda_{32}) + 45(R_0/r)\,J_{33}\cos^2\phi\sin 3(\lambda - \lambda_{33}) \\ &+ 5/2(R_0/r)^2 J_{41}\,[7\sin^2\phi - 3]\sin\phi\sin(\lambda - \lambda_{41}) + 15(R_0/r)^2 J_{42}(7\sin^2\phi - 1)\cos\phi\sin 2(\lambda - \lambda_{42}) \\ &+ 315(R_0/r)^2 J_{43}\cos^2\phi\sin\phi\sin 3(\lambda - \lambda_{43}) \\ &+ 420(R_0/r)^2 J_{44}\cos^3\phi\sin 4(\lambda - \lambda_{44})\Big\} . \end{aligned} \tag{A-4}$$

$$\begin{aligned} F_\phi = \frac{\mu_E}{r^2}(R_0/r)^2\Big\{&-3J_{20}\sin\phi\cos\phi + 6J_{22}\cos\phi\sin\phi\cos 2(\lambda - \lambda_{22}) \\ &- 3/2(R_0/r)\,J_{30}\,(5\sin^2\phi - 1)\cos\phi + 3/2(R_0/r)\,J_{31}\,(15\sin^2\phi - 11)\sin\phi\cos(\lambda - \lambda_{31}) \\ &+ 15(R_0/r)\,J_{32}\,(3\sin^2\phi - 1)\cos\phi\cos 2(\lambda - \lambda_{32}) \\ &+ 45(R_0/r)\,J_{33}\cos^2\phi\sin\phi\cos 3(\lambda - \lambda_{33}) - 5/2(R_0/r)^2 J_{40}\,(7\sin^2\phi - 3)\sin\phi\cos\phi \\ &+ 5/2(R_0/r)^2 J_{41}\,(28\sin^4\phi - 27\sin^2\phi + 3)\cos(\lambda - \lambda_{41}) \\ &+ 30(R_0/r)^2 J_{42}\,(7\sin^2\phi - 4)\cos\phi\sin\phi\cos 2(\lambda - \lambda_{42}) \\ &+ 105(R_0/r)^2 J_{43}\,(4\sin^2\phi - 1)\cos^2\phi\cos 3(\lambda - \lambda_{43}) \\ &+ 420(R_0/r)^2 J_{44}\cos^3\phi\sin\phi\cos 4(\lambda - \lambda_{44})\Big\} . \end{aligned} \tag{A-5}$$

The actual sea level surface of the earth is to be visualized through Equation A-1 as a sphere of radius 6378 km, around which is superimposed the sum of the separate spherical harmonic deviations illustrated. To these static gravity deviations, of course, must be added a centrifugal earth-rotation potential at the surface of the earth to get the true sea level surface (Reference A-1).

Table A-1 gives recent estimates from a wide variety of sources of longitude gravity constants in the earth's potential.

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Table A-1

Longitude Coefficients in the Earth's Gravity Potential $\left\{V_E = \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[1 - (R_0/r)^n P_n^m (\sin\phi) J_{nm} \cos m(\lambda - \lambda_{nm})\right]\right\}^1$, As Reported 1915-1964[2].

Longitude Geoid[7]	J_{22}	λ_{22}	J_{31}	λ_{31}	J_{32}	λ_{32}	J_{33}	λ_{33}	J_{41}	λ_{41}	J_{42}	λ_{42}	J_{43}	λ_{43}	J_{44}	λ_{44}
(1) Wagner (1964)[3]	-1.7×10^{-6}	-19°														
(2) Kaula-Combined (1964)[6]	-1.51	-15.5	-1.51×10^{-6}	0.0°	$-.102 \times 10^{-6}$	0.0°	$-.149 \times 10^{-6}$	22.8°	$-.465 \times 10^{-6}$	-136.0°	$-.163 \times 10^{-6}$	37.0°	$-.061 \times 10^{-6}$	- 1.9°	$-.0053 \times 10^{-6}$	35.8°
(3) Izsak (1964)[4]	-1.00	-17.0	- .934	-15.5	-.116	19.0	- .173	38.0	- .949	-146.0	-.074	47.5	-.024	- 3.9	-.0206	25.3
(4) Kaula (1964)[4]	-1.77	-18.2	-2.12	- 5.4	-.379	10.5	- .105	23.1	- .263	-239.0	-.117	42.3	-.0473	15.0	-.0104	14.5
(5) Anderle and Oesterwinter (1963)[3]	-2.09	-14.1							- .773	-140.6	-.287	34.7	-.163	- 4.3		
(6) Guier (1963)[3]	-1.80	-10.4	-1.77	6.3	-.286	- 2.6	- .204	24.1	- .73	-141.0	-.273	38.6	-.0791	- .7	-.0102	35.0
(7) Kaula (Sept. 1963)[4]	-1.51	-18.1	-1.65	5.3	-.144	46.4	- .145	15.8	- .471	-228.0	-.078	44.2	-.0265	22.6	-.0038	23.3
(8) Izsak (July 1963)[4]	-1.05	-11.2	-1.1	3.2	-.20	-21.8	- .14	20.0	- .43	-132.1	-.13	37.0	-.026	11.5	-.019	14.8
(9) Kaula (May 1963)[4]	-1.4	-21.5	-1.6	- 1.9	-.15	35.8	- .156	18.5	- .53	-233.7	-.12	44.5	-.019	10.7	-.0038	23.3
(10) Cohen (May 1963)[3]	-2.08	-14.1							- .775	-159.0	-.288	34.6	-.162	- 4.3		
(11) Kaula (Jan. 1963)[4]	-1.62	-21.4	-1.81	- 3.57	-.145	6.6	- .112	37.6	- .479	-245.5	-.072	47.7	-.0988	5.9	-.0132	28.4
(12) Uotila (1962)[5]	-1.52	-36.5	- .685	-81.0	-.409	- 5.2	- .398	19.5	- .238	-127.0	-.211	14.6	-.082	- 9.3	-.0142	- 2.6
(13) Kozai (Oct. 1962)[4]	-1.2	-26.4	-1.9	4.6	-.14	-16.8	- .10	42.6	- .52	-122.5	-.062	65.2	-.035	0.5	-.031	14.9
(14) Newton (April 1962)[3]	-2.15	-10.9							-2.53	-189.1						
(15) Newton (Jan. 1962)[3]	-4.16	-11.0														
(16) Kozai (June 1961)[4]	-2.32	-37.5	-3.21	22.0	-.41	31.0	-1.91	51.3	- .262	-196.5	-.168	54.0	-.044	-13.0	-.054	50.3
(17) Kaula (June 1961)[6]	- .55	-13.3	-1.19	20.6	-.33	- .9	- .21	22.6	- .617	-166.0	-.14	21.1	-.031	- .5	-.008	26.4
(18) Izsak (Jan. 1961)[4]	-5.35	-33.2														
(19) Kaula (1961)[5]	-1.68	-38.5							-1.15	- 13.0						
(20) Krassowski (1961?)	-5.53	15.0														
(21) Kaula (1959)[5]	- .62	-20.9	- .98	55.4	-.11	13.3	- .19	14.3	- .46	-132.3	-.081	48.6	-.01	-30.0	-.02	22.5
(22) Jeffreys (1959)[5]	-4.17	0.0														
(23) Uotila (1957)[5]	-3.5	- 6.0														
(24) Zhongolovitch (1957)[5]	-5.95	- 7.7	-2.21	-25.7	-.628	-26.4	- .54	13.0	- .78	-149.1	-.080	45.0	-.051	- 3.8	-.0224	15.9
(25) Subbotin (1949)[5]	-5.5															
(26) Niskanen (1945)[5]	-7.67	- 4.0														
(27) Jeffreys (1942)[5]	-4.1	0.0	-2.1	0.0	-.66	0.0	- .24	33.3								
(28) Heiskanen (1928)[5]	-6.34	0.0														
(29) Heiskanen (1924)[5]	-9.0	18.0														
(30) Helmert (1915)[5]	-6.0	-17.0														

[1] r is the radial distance of the field point to the center of mass of the earth, μ the earth's Gaussian gravity constant $\doteq 3.9860 \times 10^{20}$ cm^3/sec^2, R_0 the mean equatorial radius of the earth $\doteq$ 6378.2 km. ϕ is the geocentric latitude of the field point. λ is the geographic longitude of the field point. $J_{21} \doteq 0$, since the polar axis is very nearly a principal axis of inertia for the earth. $P_n^m(\sin\phi) = \cos^m \phi \sum_{t=0}^{K} T_{nmt} \sin^{n-m-2t}\phi$, where K is the integer part of $(n-m)/2$ and $T_{nmt} = \frac{(-1)^t (2n-2t)!}{2^n t! (n-t)! (n-m-2t)!}$ (See Kaula, 1964). The longitude coefficients are those for which $m \neq 0$.

[2] The J_{nm}'s and λ_{nm}'s in this table, except in one or two instances, have been converted from the original author's set of gravity coefficients. The blanks indicate the author did not consider that particular harmonic in fitting an earth potential to the observed data. In one or two instances, noted below, the author reported tesseral coefficients to higher order than the fourth.

[3] Satellite - Doppler geoid

[4] Satellite-camera geoid.

[5] Surface-gravimetric geoid.

[6] Combined astro-geodetic, gravimetric and satellite geoid.

[7] Detailed information on references for the geoids listed below given on following page.

References for Geoids Listed in Table A-1

Geoid	References
(1)	"Determination of the Ellipticity of the Earth's Equator from Observations on the Drift of the Syncom II Satellite" TN D-2759, May 1965.
(2)	Private communication from W. M. Kaula (July 1964).
(3),(4)	Private communication from W. M. Kaula (July 1964).
(5)	Private communication from W. M. Kaula (July 1964).
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(7)	Kaula, W. M., "Improved Geodetic Results from Camera Observations of Satellites," J. Geophys. Res. 68(18):5183-5190, September 15, 1963.
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(20)	"Passive Dynamics in the Space Flight," Bureau of Naval Weapons Paper by J. D. Nicolaides and M. M. Macomber, 1962.
(21)	Kaula, W. M., "Statistical and Harmonic Analysis of Gravity," J. Geophys. Res. 64(12): 2401-2421, December 1959.
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(24)	Zhongolovich, I. D., "Potential of the Terrestrial Attraction," Akad. Nauk SSSR Inst. Theoret. Astron. 6(8):505-523, 1957 (In Russian).
(25)	Subbotin, M. F., Kurs Nebesnoi Mekhaniki, Vol. 3, 2d ed., Moscow: Gostekhizdat, 1949.
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Appendix B

Symbols Used

A_{nm}	The long term 24-hour satellite longitude-drift driving function of the gravity harmonic J_{nm}
a, a_s	The instantaneous semi-major axis, and a near-synchronous semi-major axis of the 24-hour earth satellite assumed to be constant in a particular drift regime
F	A gravity force per unit mass, acting on a 24-hour satellite
F_T	The tangential (or circumferential) component of a gravity force acting on a 24-hour near circular orbit satellite
$\bar{F}$	An orbit-averaged gravity force
$F(i)_{nm}$	The inclination factor of the longitude drift driving function A_{nm} for the 24-hour satellite
g_s	The radial acceleration of the earth's gravity field at the altitude of the synchronous satellite ($\sim$.7355 ft./sec.^{2})
i, i_s	The inclination of the orbit plane with respect to the equator and the inclination of the near synchronous satellite's orbit plane assumed to be constant in a particular drift regime
J_{nm}, λ_{nm}	Spherical harmonic constants (order n, power m) expressing magnitude and phase angle of that component of the earth's gravity potential
R_0	The mean equatorial radius of the earth ($\sim$6378.2 km)
α	The angle between a meridian plane through the satellite's position and the orbit plane
$(\ )_0$	Time zero quantity of the argument ( ).
θ	The argument from the ascending node to the 24-hour satellite's position (defined in Figure 1)
λ, r, ϕ	Geographic longitude, geocentric radius and geocentric latitude of the 24-hour satellite position. In the text λ also refers to the daily longitude position of the ascending equator crossing, or the orbit-averaged, mean daily longitude position of the 24-hour satellite
μ_E	The earth's Gaussian gravity constant (3.98603×10^5 km^3/sec^2)
ω, ω_e	A circular frequency, and the earth's sidereal rotation rate ($.7292115 \times 10^{-4}$ rad./sec.)

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