ON DIAMETERS OF GRAPHS AND DIGRAPHS

by

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ON DIAMETERS OF GRAPHS AND DIGRAPHS (29pp.)

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The object of this thesis is to investigate certain extremal properties of the elongation diameter of connected graphs and the diameter of strongly connected digraphs.

Let us partition the vertex set V of a graph of order n into two subsets V_1 and V_2 such that through each vertex in V_1 there are exactly n-1 edges (to the other n-1 vertices in V), while the number of edges through any vertex in V_2 is strictly less than n-1. Clearly $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

A graph of order n on the set V is said to be exact k-complete, $k \geq 1$, if the subset V_1 has k vertices, and none of the vertices of the subset V_2 are joined by means of edges to each other; such a graph is denoted by G(n,k). A graph of order n on the set V is said to be plus k-complete if it contains an exact k-complete graph and one vertex in V_2 is joined by means of edges to d of the remaining n-k-1 vertices in V_2, where $1 \leq d < n-k-1$.

Let $G_{n,N}$ be the class of all connected graphs with n vertices and N edges. Let $C_{n,N}$ denote the class of all exact k-complete graphs ($k \geq 1$) and plus k-complete graphs. We denote the elongation diameter of a graph G by e(G). Some properties of the elongation diameter and of k-complete graphs yield the following:

Theorem: If G belongs to $C_{n,N}$ then $e(G) \leq e(G')$ where G' belongs to $G_{n,N} - C_{n,N}$.

An analogous problem for digraphs is known as Bratton's problem: Construct a strongly connected digraph $\Gamma_{n,N}$ with n vertices and N edges in which the diameter of the digraph is at a minimum.

While a complete solution to this problem has not been found, we have been able to obtain at least an interesting side result:

Theorem: Let the digraph $\Gamma_{n,N}$ be such that $\delta(\Gamma_{n,N}) \geq \frac{r+2}{2}$ where $\delta(\Gamma)$ denotes the diameter of the digraph Γ and r denotes the rank of the incidence matrix isomorphic to the digraph $\Gamma_{n,N}$, then $2\,\delta(\Gamma_{n,N}) \geq 2n - N + 2$.

Acknowledgments

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Table of Contents

Introduction and Summary

In this thesis we are mainly concerned with an investigation of the properties (a) of those connected graphs which have a fixed number of edges and vertices and whose elongation diameter is a minimum; and (b) of those finite strongly connected digraphs which have a fixed number of diedges and vertices and whose diameter is a minimum.

Finite connected graphs and finite strongly connected directed graphs enjoy special properties which make them applicable as mathematical models in such areas as communication networks and group dynamics. There are numerous examples of such applications; for instance, see Bhargava [2], Busacker and Saaty [4], or Flament [7]. Our purpose here is to look into one of these special properties of graphs, namely the notion of diameter and elongation diameter of a graph. These concepts arise in discussions of paths of maximum and minimum length between pairs of vertices in a graph.

Moon [11] has considered problems dealing with the diameter of a graph from the viewpoint of finding the degree of each vertex given the number of vertices and the maximum diameter. However, we consider the problem of constructing graphs and digraphs with a fixed number of edges and vertices in which the diameter or elongation diameter, as the case may be, is a minimum.

The first chapter of this thesis is devoted to some basic definitions and relevant notations from the theory of graphs and directed graphs. Also, some results from the theory of incidence matrices which are used in our work are presented.

In the second chapter we introduce a new class of graphs C, which is a subset of the class of all connected graphs G, whose elements have the property of minimum elongation diameter. This class $C = C' \cup C''$, where C' is the class of all exact k-complete graphs and C'' is the class of all plus k-complete graphs.

In Chapter III, we summarize the previous results related to Bratton's problem, that is, construction of a graph with n vertices and N edges whose diameter is a minimum, and we state Bratton's conjecture. While a complete solution of this problem has not been found we have been able to obtain at least an interesting side result: Let the digraph Γ be such that $\delta(\Gamma) \geq \frac{r+2}{2}$, where $\delta(\Gamma)$ denotes the diameter of digraph Γ and r denotes the rank of the incidence matrix isomorphic to the digraph Γ, then $2\delta(\Gamma) \geq 2n - N + 2$.

Finally, in the appendix we give the earliest and latest stage in which a random directed graph can belong to each of the connectivity classifications.

Chapter I

Preliminaries

This chapter consists of some relevant definitions and notations from the theory of graphs and directed graphs, and from the theory of 0-1 matrices that are used throughout this thesis. These are standard definitions and they can be found in Berge [1], and Bhargava [2], and Ore [12].

1.1 <u>Graphs</u>

Let V_n be a finite set consisting of n elements, denoted by v_i, $i = 1, 2, \ldots, n$, and let E be the set of all possible lines joining all the pairs of distinct points belonging to the set V_n. Clearly, the number of elements in E is $\frac{n(n-1)}{2}$.

<u>Definition 1.1.1</u>: The pair (V_n, E_N), $\emptyset \subseteq F_N \subseteq E$, is called a finite <u>graph</u> of order n, and is denoted by $G_{n,N} = G(V_n, F_N)$, where n is the number of elements in V_n and N is the number of elements in E_N. The elements of V are called the <u>vertices</u> of the graph $G_{n,N}$, and the elements of F_N are called the <u>edges</u> of the graph $G_{n,N}$. Whenever n and N are fixed and no confusion can arise, we drop the subscripts and denote $G_{n,N} = G(V,F)$ simply by G.

All the graphs, $G_{n,N}$ with which we are dealing are assumed to be without multiple edges and loops (that is edges of the type (v_i, v_i)). Thus, the number of edges in $G_{n,N}$ varies from zero (no edges at all, in which case the graph is called a <u>null graph</u>) to $n(n-1)/2$ (all possible edges, in which case the graph is called a <u>complete graph</u>).

<u>Definition 1.1.2</u>: A <u>path</u> is a sequence of edges,

$$e_i = (v_{i-1}, v_i), \quad i = 1, 2, \ldots, m$$

where e_{i-1} and e_i always have a common vertex and each e_i appears once and only once (vertices may appear more than once). The vertex v_o is called the <u>initial vertex</u> while the vertex v_m is called the <u>terminal vertex</u> of the path; all other vertices of the path are called <u>intermediate vertices</u>. The length of a path, $\{e_i, i = 1, 2, \ldots, m\}$ is m, the number of edges between the initial and terminal vertices.

<u>Definition 1.1.3</u>: If $v_o = v_m$ the path is said to be <u>cyclic</u>.

<u>Definition 1.1.4</u>: A path is said to be a <u>simple path</u> if none of its vertices are transversed more than once.

<u>Definition 1.1.5</u>: A <u>complete cycle</u> (often called a Hamilton line) is a cycle which passes through all the vertices of a graph.

<u>Definition 1.1.6</u>: A graph G is said to be <u>connected</u> if for every pair, (v_i, v_j), there exists a path with v_i as the initial vertex and v_j as the terminal vertex.

For finite connected graphs G, the lengths of paths between any two vertices are positive integers. Consequently, there exist paths of minimum and maximum lengths.

<u>Definition 1.1.7</u>: The length of the shortest simple path between v_i and v_j is called the <u>distance</u> of v_i and v_j, and

is denoted by $d(v_i,v_j)$; the <u>diameter</u> is the maximum of $d(v_i,v_j)$ over all pairs (v_i,v_j), $i \neq j$, and is denoted by $\delta(G_{n,N})$.

<u>Definition 1.1.8</u>: The length of the longest simple path between v_i and v_j is called the <u>elongation</u> of v_i and v_j, and is denoted by $e(v_i,v_j)$; the <u>elongation diameter</u> is the maximum of $e(v_i,v_j)$ over all pairs (v_i,v_j), $i \neq j$, and is denoted by $e(G_{n,N})$.

It is obvious that for any (v_i,v_j), $i \neq j$,

$$e(v_i,v_j) = e(v_j,v_i) \text{ and } d(v_i,v_j) = d(v_j,v_i). \qquad 1.1.1$$

Also,

$$0 \leq d(v_i,v_j) \leq \delta(G_{n,N}) \leq n-1, \text{ and} \qquad 1.1.2$$
$$0 \leq e(v_i,v_j) \leq e(G_{n,N}) \leq n-1.$$

<u>Definition 1.1.9</u>: The <u>local degree</u> of a vertex v_i is the number of edges through that vertex, and is denoted by $\rho(v_i)$.

<u>Definition 1.1.10</u>: A graph $G'(V',F')$ is said to be a <u>subgraph</u> of $G(V,F)$ if V' is a subset of V and F' is a subset of F.

1.2 <u>Directed Graphs</u>

Let V_n be a finite set consisting of n elements, denoted by v_i, $i = 1, 2,\ldots,n$, and let D be the set of all ordered pairs of points $\langle v_i,v_j\rangle$, $i \neq j$, belonging to the set V. Clearly, the number of elements in E is $n(n-1)$.

<u>Definition 1.2.1</u>: The pair (V_n,C_N), $\emptyset \subseteq C_N \subseteq D$, is called a finite <u>directed graph</u> (or simply <u>digraph</u>) of order n, and

is denoted by $\Gamma_{n,N}$, where n is the number of elements in V_n and N is the number of elements in C_N. The elements of V_n are called the vertices of the digraph $\Gamma_{n,N}$, and the elements of C_N are called the directed edges (or simply diedges) of $\Gamma_{n,N}$. A diedge $e = <v_i, v_j>$ has v_i as its initial vertex and v_j as its terminal vertex. Whenever n and N are fixed and no confusion can arise, we drop the subscripts and denote $\Gamma_{n,N} = \Gamma(V,C)$ simply by Γ.

All the digraphs, $\Gamma_{n,N}$, with which we are concerned are assumed to be without multiple diedges and loops. Thus, the number of diedges in $\Gamma_{n,N}$ may vary from zero (the null digraph) to n(n-1) (the complete digraph).

Definition 1.2.2: A directed path (or simply dipath) is a sequence of diedges,

$$e_i = <v_{i-1}, v_i> ; \quad i = 1, 2, \ldots, m$$

where the terminal vertex of e_{i-1} is always the initial vertex of e_i and each e_i, $i = 1, 2, \ldots, m$, appears once and only once. The vertex v_o is the initial vertex while the vertex v_m is the terminal vertex of the dipath; all other vertices of the dipath are called intermediate vertices. The length of a dipath $\{e_i, i = 1, 2, \ldots, m\}$ is m, the number of edges between the initial and terminal vertices.

Definition 1.2.3: If $v_o = v_m$ the dipath is said to be cyclic.

Definition 1.2.4: A dipath is said to be a simple dipath if none of its vertices are transversed more than once.

In order to discuss connectedness in a digraph $\Gamma_{n,N}$, a classification system for the digraph based on strength of connectedness is necessary, and is described below.

Definition 1.2.5: A digraph is strongly connected if for every ordered pair $\langle v_i, v_j \rangle$, $i \neq j$, there is a dipath from v_i to v_j and a dipath from v_j to v_i.

Definition 1.2.6: A digraph is unilaterally connected if for every ordered pair $\langle v_i, v_j \rangle$, $i \neq j$, there is a dipath from v_i to v_j or a dipath from v_j to v_i, or both.

Definition 1.2.7: A digraph is weakly connected if for every ordered pair $\langle v_i, v_j \rangle$, $i \neq j$, there is a sequence of edges, ignoring the direction, from v_i to v_j.

Definition 1.2.8: A digraph is disconnected if it is not even weakly connected.

Since a digraph which is strong is also unilateral, and a digraph which is unilateral is also weak, this classification system is not mutually exclusive. However, a mutually exclusive system can be obtained in an obvious manner, namely, a digraph is said to be

(a) type S_3 if it is strong,

(b) type S_2 if it is unilateral but not strong,

(c) type S_1 if it is weak but not unilateral, and

(d) type S_0 if it is not even weak.

In order to discuss distances, elongations, diameters and elongation diameters in a digraph $\Gamma_{n,N}$, we must be assured of a dipath from v_i to v_j for every ordered pair $\langle v_i, v_j \rangle$, that is, $\Gamma_{n,N}$ must be strongly connected.

For finite strongly connected digraphs $\Gamma_{n,N}$, the lengths of dipaths from one vertex to another are positive integers; as a consequence, there exist dipaths of maximum and minimum length.

Definition 1.2.9: The length of the shortest simple dipath from v_i to v_j is the distance of v_i and v_j, and is denoted by $d(v_i,v_j)$; the diameter is the maximum of $d(v_i,v_j)$ over all $\langle v_i,v_j\rangle$, $i \neq j$, and is denoted by $\delta(\Gamma_{n,N})$.

Definition 1.2.10: The length of the longest simple dipath from v_i to v_j is called the elongation of v_i and v_j, and is denoted by $e(v_i,v_j)$; the elongation diameter is maximum of $e(v_i,v_j)$ over all $\langle v_i,v_j\rangle$, $i \neq j$, and is denoted by $e(\Gamma_{n,N})$. We remark that inequalities like 1.1.2 for graphs hold for digraphs also.

Definition 1.2.11: A digraph $\Gamma'(V',C')$ is said to be a subdigraph of $\Gamma(V,C)$, if V' is a subset of V and C' is a subset of C.

1.3 Incidence Matrices

Definition 1.3.1: Associated with every graph $G_{n,N}$, is an $n \times n$ matrix, $M(G)$, whose entries are defined as follows:

$$\text{For } i \neq j, \qquad m_{ij} = \begin{cases} 1 \text{ if } (v_i,v_j) \text{ belongs to } F \\ 0 \text{ if } (v_i,v_j) \text{ does not belong to } F. \end{cases}$$

For $i = j$, we define $m_{ii} = 0$.

Since $(v_i,v_j) = (v_j,v_i)$, we note that $M(G)$ is symmetric with respect to the main diagonal. Such a matrix is called the incidence matrix of the graph $G_{n,N}$.

Definition 1.3.2: In a similar manner, we associate with every digraph $\Gamma_{n,N}$ the n×n matrix, M(Γ), whose entries are defined as follows:

$$\text{For } i \neq j, \quad m_{ij} = \begin{cases} 1 \text{ if } \langle v_i, v_j \rangle \text{ belongs to } C \\ 0 \text{ if } \langle v_i, v_j \rangle \text{ does not belong to } C. \end{cases}$$

For i = j, we define $m_{ii} = 0$.

Such a matrix is called the incidence matrix of the digraph $\Gamma_{n,N}$.

To indicate the $(i,j)^{th}$ entry of the matrix M^n, the n^{th} power of M, we shall employ the symbol m^n_{ij}. The following result is true for incidence matrices associated with a graph as well as a digraph.

Lemma 1.3.1: In M^n, the entry $m^n_{ij} = c$ if and only if there are c distinct paths of length n from v_i to v_j, $i \neq j$, and where the c distinct paths of length n are not necessarily simple paths.

For proof, we refer to Luce and Perry [8].

Definition 1.3.3: The rank of a matrix is the order of the largest nonzero determinant which is obtainable by the possible deletion of rows and columns from the matrix.

Chapter II

Elongation Diameters

Let $\mathcal{G}_{n,N}$ be the class of all connected finite graphs with n vertices and N edges, that is,

$$\mathcal{G}_{n,N} = \{G_{n,N} \mid G_{n,N} \text{ is connected; } n, N \text{ fixed}\},$$

where $n \geq 2$ and $n-1 \leq N \leq \frac{n(n-1)}{2}$.

Let e(G) denote the elongation diameter of G, where G is a graph belonging to $\mathcal{G}_{n,N}$.

The problem we consider in this chapter essentially consists of finding a subset $\mathcal{G}' \subseteq \mathcal{G}_{n,N}$, such that if G' belongs to $\mathcal{G}'$ and G belongs to $\mathcal{G}_{n,N}$, then $e(G') \leq e(G)$.

In other words we wish to find a graph (or collection of graphs) which is connected, and has fixed number of vertices and edges, and has minimum elongation diameter.

2.1 Minimum Elongation Diameter

Let us partition the vertex set V of a graph of order n into two subsets V_1 and V_2 such that through each vertex in V_1 there are exactly n-1 edges (to the other n-1 vertices in V), while the number of edges through any vertex in V_2 is strictly less than n-1. Clearly, $V_1 \cup V_2 = V$, and $V_1 \cap V_2 = \emptyset$.

Definition 2.1.1: A graph G of order n on set V is said to be exact k-complete, $k \geq 1$, if the subset V_1 has k vertices and none of the vertices in the subset V_2 are joined by means of an edge to each other; such a graph is denoted by $G(n,k)$. We note that an exact k-complete graph has $N = kn - \frac{k(k+1)}{2}$ edges. We call the points in the subset V_1 complete vertices of the graph G and those in subset V_2 the incomplete vertices of the graph $G(n,k)$. See Figure 1 for an example of such a graph.

Definition 2.1.2: A graph G of order n on set V is said to be plus k-complete if it has an exact k-complete graph $G(n,k)$ as a subgraph and there is exactly one incomplete vertex which is joined by means of edges to d of the remaining $n-k-1$ incomplete vertices, and $1 \leq d < n-k-1$; such a graph is denoted by $G(n,k^+)$. See Figure 2 for an example of such a graph.

Definition 2.1.3: A graph G is said to be k-complete if it is either exact k-complete or plus k-complete.

Figure 1. Example of an exact k-complete graph; $n=7$, $k=2$, $N=11$, $V_1=\{a,b\}$, $V_2=\{c,d,e,f,g\}$.

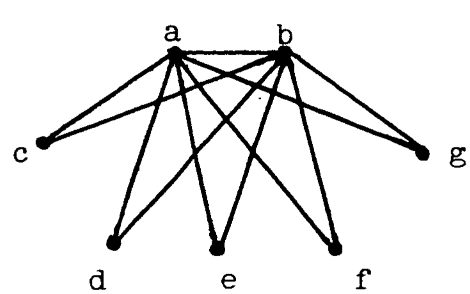
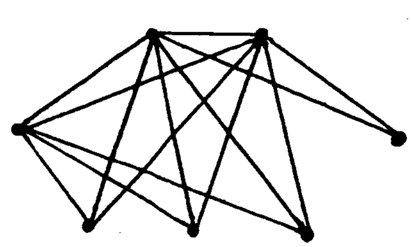


Figure 2. Example of plus k-complete graph; $n=7$, $k=2$, $N=14$.



To avoid having to make trivial exceptions later, let us assume $n \geq 3$.

Consider the following classes of graphs:

$C'_{n,N} = \{G_{n,N} \mid G_{n,N}$ is exact k-complete; $1 \leq k < n\}$,

$C''_{n,N} = \{G_{n,N} \mid G_{n,N}$ is plus k-complete; $1 \leq k < n-1\}$, and

$C_{n,N} = \{G_{n,N} \mid G_{n,N}$ is k-complete; $1 \leq k < n-1\}$.

We note that $C_{n,N} \subseteq G_{n,N}$, $C'_{n,N} \cap C''_{n,N} = \emptyset$, and

$$C'_{n,N} \cup C''_{n,N} = C_{n,N}.$$

Theorem 2.1.1: For any graph $G_{n,N}$ belonging to $C_{n,N}$ $e(G_{n,N}) \leq e(G'_{n,N})$ where $G'_{n,N}$ belongs to $G_{n,N} - C_{n,N}$.

The proof of this theorem requires several results regarding the exact k-complete graphs and plus k-complete graphs; these are stated and proved in the form of Lemmas 2.1.2 through 2.1.5 and Corollary 2.1.6; the proof of the theorem then follows.

We have already seen that if G(n,k) is an exact k-complete graph and V_1 and V_2 denote the set of complete vertices and the set of incomplete vertices respectively, that $V_1 \cup V_2 = V$, the set of vertices of G(n,k), and $V_1 \cap V_2 = \emptyset$.

Let P(i,j) denote the path of maximum length between v_i and v_j; let this path consist of r+1 vertices, v_{i0}, $v_{i1}, \ldots, v_{ir}$ with $v_{i0} = v_i$ and $v_{ir} = v_j$ so that the length of path is r. Let us represent P(i,j) by the sequence

$$\{v_{i\alpha}, \ \alpha = 0, 1, \ldots, r\}.$$

Lemma 2.1.2: For an exact k-complete graph, $G(n,k)$,

$$P(i,j) = \{v_{i\alpha} \mid v_{i\alpha} \in W \text{ for } \alpha = 0,2,\ldots,2m \text{ and } v_{i\alpha} \in \overline{W} \text{ for } \alpha = 1,3,\ldots,2m+1\},$$

where $W = V_1$ or V_2 and $r = 2m$ or $2m+1$; unless $e(v_i,v_j)=n-1$.

Proof: The above path from v_i to v_j is such that no two consecutive vertices in the path belong to the same subset W of vertices. We call such a path an alternating path.

Assume the elongation, $P(i,j) = \{v_{i\alpha}, \alpha = 0,1,2,\ldots,r\}$, is not an alternating path, that is, both $v_{i\alpha-1}$ and $v_{i\alpha}$ belong to V_1 for some $\alpha = 1,2,\ldots,r$; we need not consider both $v_{i\alpha-1}$ and $v_{i\alpha}$ belonging to V_2, because this is not possible in an exact k-complete graph. Since $v_{i\alpha-1}$ and $v_{i\alpha}$ are both complete vertices, this implies the existence of a path segment from $v_{i\alpha-1}$ to $v_{i\alpha}$ through some v_ℓ belonging to V_2 unless, of course, V_2 has been exhausted by this time; in which case $k > \frac{n}{2}$ and $e(v_i,v_j) = n-1$. Thus the path through v_ℓ, that is $\{v_{i\alpha}, \alpha = 0,1,\ldots,i-1,\ell,i,\ldots,r\}$ is longer which is a contradiction.

Lemma 2.1.3: If $G(n,k)$ is an exact k-complete graph, $k \leq [\frac{n}{2}]$ then $e(G(n,k)) = 2k$; if $k > [\frac{n}{2}]$ then $e(G(n,k)) = n-1$.

Proof: By Lemma 2.1.2 we know the path constituting the elongation diameter is alternating; between the initial and terminal vertices we have as many vertices as possible.

If at any stage the elongation is $n-1$, which is the maximum length of any simple path in $G(n,k)$, then this is necessarily the elongation diameter.

Since the set of vertices of an exact k-complete graph consists of two disjoint sets, namely, V_1 and V_2, we need only consider three types of paths of maximum length:

(i) between v_i and v_j both belonging to V_1, in which case $e(v_i,v_j) = 2k-2$, because v_i and v_j have only one issuing edge,

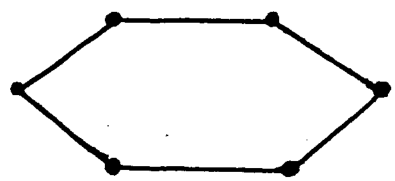
ii) between v_i and v_j both belonging to V_2, in which case $e(v_i,v_j) = 2k$, because each of the k vertices of V_1 have two issuing edges, and

iii) between v_i belonging to V_1 and v_j belonging to V_2, in which case $e(v_i,v_j) = 2k-1$, because v_j has only one issuing edge.

If $e(G(n,k)) = \max (v_i,v_j)$ over all (v_i,v_j), $i \neq j$, is less than n-1, then it is evident this implies $e(G(n,k)) = 2k$. From this we note that $k > [\frac{n}{2}]$ gives $e(G(n,k)) = n-1$, where the brackets indicate the greatest integer not exceeding n/2.

The graph in Figure 3 shows that nothing can be said in the way of a converse to Lemma 2.1.3; in this example we have e(G) = n-1 and yet the number of edges is only N = n.

Figure 3. Example of a graph with n=6, N=6, and e(G) = 5.



<u>Lemma 2.1.4</u>: If $G_{n,N}$ contains an exact k-complete graph

$G(n,k)$ with N-1 edges, then $e(G_{n,N}) \leq 2k+1$.

Proof: See Figure 4 for an example of the graph described in this lemma.

Let $G(n,k,N-1)$ denote an exact k-complete graph with N-1 edges. If $e\{G(n,k,N-1)\} = n-1$, then $e(G_{n,N})$ is obviously n-1.

Suppose $e\{G(n,k,N-1)\} < n-1 = 2k$.

Let $\{e_i = (v_{i-1},v_i),\ i = 1,2,\ldots,2k\}$ represent the elongation diameter. Since $v_o \in V_2$ is an arbitrary element of V_2, we let the additional edge be from v_o to $v_{2k+1} \in V_2$, where $v_{2k+1} \neq v_i$, $i = 1,2,\ldots,2k$. Now e_{2k+1} increases $e\{G(n,k,N-1)\}$ by one. From $v_{2k+1} \in V_2$, there are only edges to set of complete vertices V_1, but all the complete vertices have been transversed. Thus $e(G_{n,N}) = 2k+1$.

Again Figure 3 shows that the converse cannot hold; $e(G) = n-1$ tells us nothing about the number of edges of G.

Lemma 2.1.5: If $G_{n,N}$ contains an exact k-complete graph $G(n,k)$ with N-2 edges, then $e(G_{n,N}) \leq 2k+2$.

Proof: See Figure 5 for an example of the graph described in this lemma.

Let $G(n,k,N-2)$ denote an exact k-complete graph with N-2 edges. If $e\{G(n,k,N-2)\} = n-2$, then by Lemma 2.1.4 we have $e(G_{n,N}) = n-1$.

If $e\{G(n,k,N-2)\} = n-1$, then $e(G_{n,N})$ is obviously n-1.

Suppose $e\{G(n,k,N-2)\} < n-2 = 2k$. By Lemma 2.1.4 $e\{G(n,k^+,N-1)\} = 2k+1$. Let $\{e_i = (v_{i-1},v_i),\ i = 1,2,\ldots,2k+1\}$

denote the elongation diameter of $G(n,k^+,N-1)$. At this stage we have a plus k-complete graph whose vertex set V consists of three disjoint sets:

a) V_1 where $\rho(v) = n-1$ for $v \varepsilon V_1$,

b) V_2 where $\rho(v) = k$ for $v \varepsilon V_2$, and

c) V_2' where $\rho(v) = k+1$ for $v \varepsilon V_2'$.

We note that V_1 consists of k vertices, V_2 consists of n-k-2 vertices, and V_2' consists of two vertices.

Now the last edge, $e_{2k+2} = (v_{2k+1}, v_{2k+2})$ is such that

i) $v_{2k+1} \varepsilon V_2$ and $v_{2k+2} \varepsilon V_2$ or

ii) $v_{2k+1} \varepsilon V_2'$ and $v_{2k+2} \varepsilon V_2$.

We need not consider other cases, because the elements of V_1 have all possible edges through them.

i) Since v_o is an arbitrary element of V_2, we let $v_o = v_{2k+1}$. Now $e_{2k+2} = (v_{2k+1}, v_{2k+2})$, where v_{2k+2} belongs to V_2, increases $e\big(G(n,k^+,N-1)\big)$ by one. Since all the elements of V_1 have been transversed, we cannot go beyond v_{2k+2} belonging to V_2.

ii) In this case $e_{2k+2} = (v_{2k+1}, v_{2k+2})$ where $v_{2k+1} \varepsilon V_2'$ and $v_{2k+2} \varepsilon V_2$; e_{2k+2} increases $e\big(G(n,k^+,N-1)\big)$ by one. From v_{2k+2} we can go no further because all vertices of V_1 have been transversed.

Therefore, $e(G_{n,N}) = e\big(G(n,k,N-2)\big) + 2$. We note again from Figure 3 that nothing can be said in the way of a converse; it is also evident that a third lemma in the same line as the previous two does not exist.

Figure 4. Example of a connected graph $G_{n,N}$ with a subgraph $G(n,k,N-1)$ which is exact k-complete; n=7, k=2, N=12.

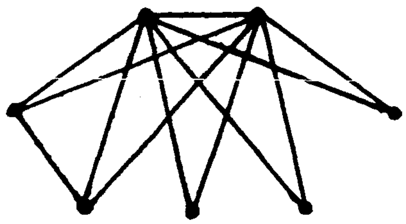
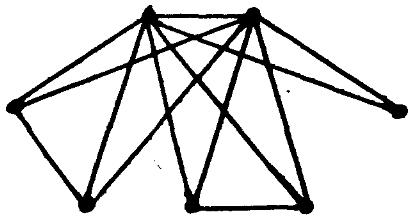


Figure 5. Example of a connected graph $G_{n,N}$ with a subgraph $G(n,k,N-2)$ which is exact k-complete; n=7, k=2, N=13.



Corollary 2.1.6: If $G_{n,N}$ belongs to $C_{n,N}$, then

$$e(G_{n,N}) = \begin{cases} \text{at most } 2k+1 \text{ if } N = kn - \frac{k(k+1)}{2} + 1 \\ \text{at most } 2k+2 \text{ if } kn - \frac{k(k+1)}{2} + 1 < N < (k+1)n - \frac{(k+1)(k+2)}{2} \end{cases}$$

We are now ready to complete the proof of Theorem 2.1.1.

The technique involved in proving this theorem is induction on k. We first prove that the elements of $C'_{n,N}$ have minimum elongation diameter. We note that it is impossible for $e(G_{n,N}) = 1$ and have $G_{n,N}$ connected. Therefore, the smallest elongation diameter of any graph $G_{n,N}$ is $e(G_{n,N}) = 2$.

We prove Theorem 2.1.1 for an exact 1-complete graph, that is, $G(n,1)$ belongs to $C'_{n,n-1}$ if and only if $e(G(n,1)) = 2$.

Let $e(G_{n,n-1}) = 2$. This means there is a path of length two, $\{e_i = (v_{i-1}, v_i),\ i = 1,2\}$. All additional edges must be added to v_1; thus $G_{n,n-1}$ belongs to $C'_{n,n-1}$.

If $G(n,1)$ belongs to $C'_{n,n-1}$, it is evident $e\big(G(n,1)\big) = 2$. Let us assume that an exact k-complete graph $G(n,k)$ with N_1 edges has $e\big(G(n,k)\big)$ at a minimum. We want to show that an exact k+1-complete graph $G(n,k+1)$ with N_2 edges has $e\big(G(n,k+1)\big)$ at a minimum. Here, $N_1 = kn - \frac{k(k+1)}{2}$ and $N_2 = (k+1)n - \frac{(k+1)(k+2)}{2}$.

In the second part of Lemma 2.1.5, we show that any two edges added to $G(n,k)$ increases $e\big(G(n,k)\big)$ by two, but we can add n-k edges to $G(n,k)$ in going to $G(n,k+1)$, which belongs to $C'_{n,N}$ and we increase $e\big(G(n,k)\big) = 2k$ to $e\big(G(n,k+1)\big) = 2(k+1)$.

Therefore, an exact k+1-complete graph has a minimum elongation diameter provided an exact k-complete graph has a minimum elongation diameter. Thus far, we have shown that the elements of $C'_{n,N}$ have minimum elongation diameter.

The first part of this theorem together with Lemma 2.1.6 tells us that the elements of $C''_{n,N}$ have minimum elongation diameter for $kn - \frac{k(k+1)}{2} < N < (k+1)n - \frac{(k+1)(k+2)}{2}$.

<u>Corollary 2.1.7</u>: If $N \geq \begin{cases} (3n^2 - 2n)/8 & \text{if } n \text{ is even} \\ (3n^2 - 4n + 1)/8 & \text{if } n \text{ is odd,} \end{cases}$

then $e(G_{n,N}) = n-1$ for all $G_{n,N}$.

<u>Proof</u>: The corollary follows directly from noting that

$$k \geq \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd,} \end{cases} \quad \text{gives } e(G_{n,N}) = n-1.$$

2.2 Minimum Diameters

Let $\mathcal{G}_{n,N}$ be the class of all connected graphs with n vertices and N edges. The problem of finding the subset of $\mathcal{G}_{n,N}$ in which the diameter is a minimum can be solved by the following two lemmas:

Lemma 2.2.1: $\delta(G_{n,N}) = 1$ if and only if $N = \frac{n(n-1)}{2}$, that is, $G_{n,N}$ is complete.

Proof: Let $e(G_{n,N}) = 1$. This means an arbitrary v_o has n-1 edges through it; thus a complete graph.

If $G_{n,N}$ is complete, it is clear that $\delta(G_{n,N}) = 1$.

Let $\mathcal{R}_{n,N}$ denote the classes of finite graphs of order n, with $N \neq \frac{n(n-1)}{2}$ that have G(n,1) belonging to $C'_{n,n-1}$ as a subgraph, that is,

$$\mathcal{R}_{n,N} = \{G_{n,N} \mid G(n,1) \subseteq G_{n,N};\ N \neq \frac{n(n-1)}{2}\}.$$

Lemma 2.2.2: If $G_{n,N}$ belongs to $\mathcal{R}_{n,N}$ then $\delta(G_{n,N})$ is a minimum.

Chapter III

Bratton's Problem

Let $\mathcal{D}_{n,N}$ be the class of all strongly connected digraphs with n vertices and N diedges, both n and N fixed; that is, $\mathcal{D}_{n,N} = \{\Gamma_{n,N} \mid \Gamma_{n,N}$ strongly connected; n,N fixed$\}$.

We denote the diameter of a digraph Γ by $\delta(\Gamma_{n,N})$ or $\delta(\Gamma)$.

This chapter considers a problem analogous to the one we solved in the previous chapter, but in terms of digraphs now. The problem consists of finding a subset, $\mathcal{D}' \subseteq \mathcal{D}_{n,N}$, such that if Γ' belongs to $\mathcal{D}'$ and Γ belongs to $\mathcal{D}_{n,N}$ then $\delta(\Gamma') \leq \delta(\Gamma)$.

In other words we wish to find all strongly connected digraphs with n vertices and N edges, for which the diameter of the digraph is as small as possible. We note here that this problem is posed in Ore [12] and is called Bratton's problem.

For convenience, let us write δ in place of the more explicit symbols $\delta(\Gamma_{n,N})$ or $\delta(\Gamma)$.

First of all, we notice that we are concerned with those N for which $n \leq N \leq n(n-1)$. The lower bound here is due to the fact that Γ is strongly connected, and the upper bound comes from the fact that Γ has no multiple edges or loops.

We note that $1 \leq \delta \leq n-1$. Again the lower bound is due to the fact that Γ is strongly connected, and the upper bound comes from the fact that the diameter is a simple dipath, that is, any vertex is transversed at most once.

Bratton's problem can be interpreted in the following manner: "Given Γ a strongly connected digraph with n vertices and N edges, n and N fixed, find as sharp a lower bound for the diameter as possible."

3.1 <u>δ equal to 1 or 2</u>

Since δ is always greater than zero, let us consider the smallest possible diameter, that is, $\delta = 1$.

<u>Lemma 3.1.1</u>: $\delta = 1$ if and only if $\Gamma_{n,N}$ is complete.

<u>Proof</u>: The proof of this lemma is exactly identical to the proof of Lemma 2.2.1 except we work in terms of digraphs now. This lemma completely exhausts the case $\delta = 1$.

<u>Definition 3.1.1</u>: A <u>node</u> is a vertex with more than two edges through it (either incident or emanating).

<u>Definition 3.1.2</u>: An <u>antinode</u> is a vertex with exactly two edges through it.

<u>Definition 3.1.3</u>: A <u>rosette</u> is a strongly connected digraph with exactly one node. Since a rosette is strongly connected, all other vertices will be antinodes with one incident edge and one emanating edge. A rosette on n vertices with a maximum number of edges, that is 2(n-1) edges, is denoted by Γ^* throughout the rest of this chapter and is called a <u>star digraph</u>. The <u>center</u> of a rosette is the vertex that

is the node.

Now we consider the case $\delta = 2$.

Lemma 3.1.2: If Γ^* is a subdigraph of $\Gamma_{n,N}$, $N \neq n(n-1)$, then $\delta(\Gamma_{n,N}) = 2$.

Proof: Consider Γ^* a subdigraph of $\Gamma_{n,N}$. Since $N \neq n(n-1)$, we cannot have $\delta = 1$, and the fact that Γ^* is a subdigraph assures us that $\delta = 2$.

It would be desirable to prove the converse to the above lemma in order to completely exhaust the case of $\delta = 2$; however, it remains an open problem.

3.2 Bratton's Conjecture

For the sake of completeness we list a few results which have been previously obtained by others.

Luce [9] has been able to show the following results for general δ, but these results are in the same direction as Lemma 3.1.2.

Lemma 3.2.1: (Luce) For any n, N and δ satisfying

$$2 \leq \delta \leq n-1, \tag{3.2.1}$$

$$2n = \delta + N, \tag{3.2.2}$$

and

$$n \leq N \leq n(n-1) \tag{3.2.3}$$

there exists $\Gamma_{n,N}$ with diameter δ.

Lemma 3.2.2: (Luce) For any n, N and δ satisfying 3.2.1 and

$$2n \leq \delta + N, \tag{3.2.4}$$

and

$$n \leq N \leq n(n-1) \tag{3.2.5}$$

there exists $\Gamma_{n,N}$ with diameter δ.

The proof of Lemma 3.2.1 involves taking δ of the N edges and constructing on δ vertices a simple cycle. From any vertex v_o belonging to the above cycle, construct a star graph with the remaining vertices and v_o as the center.

Now, the Γ^* subdigraph has

$$N_o = 2(n - \delta),$$

and the simple cycle has

$$N_1 = \delta.$$

Therefore, $N = N_o + N_1 = 2(n - \delta) + \delta = 2n - \delta$, or

$$2n = \delta + N.$$

Lemma 3.2.2 follows directly by noting that the addition of edges does not increase the diameter.

Definition 3.2.1: A branch is a dipath in which the initial and terminal vertices are nodes.

Lemma 3.2.3: (Berge) The number of branches β of a rosette is N-n+1. For proof we refer to Berge [1].

Let q be the quotient of n-1 by N-n+1, and let r be the remainder.

Let

$$\Delta(N,n) = \begin{cases} 2q & \text{if } r = 0 \\ 2q+1 & \text{if } r = 1 \\ 2q+2 & \text{if } r \geq 2. \end{cases}$$

Lemma 3.2.4: (Berge) The maximum diameter of a rosette with n vertices and N edges is 2n-N; the minimum diameter of a rosette with n vertices and N edges is $\Delta(N,n)$. For proof we refer to Berge [1].

Intuitively, the rosette with diameter $\Delta(N,n)$ would seem to be the digraph with minimum diameter for arbitrary n and N;

that this is not so can be seen by the following counter-example which can be found in Berge [1]. Consider the following two digraphs.

Figure 6. Digraph with $n=15$, $N=18$, $\delta=7$.

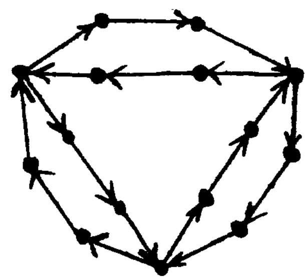
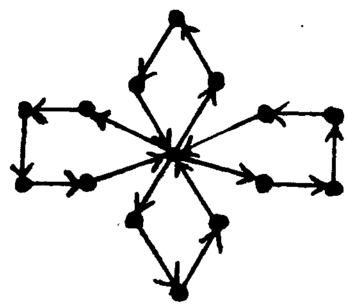


Figure 7. Rosette with $n=15$, $N=18$, $\delta=8$.



In Figure 7 we have a rosette with $n=15$ and $N=18$, and in Figure 6 we have a digraph with $n=15$ and $N=18$ and a diameter that is smaller than the diameter of the rosette. Therefore, a rosette does not necessarily have the minimum diameter.

Let us now state Bratton's conjecture: "Every strongly connected digraph $\Gamma_{n,N}$ with n vertices and N edges has a diameter which is greater than or equal to $\Delta(N,n) - 1$."

We now present a partial solution to Bratton's problem. Below we express the diameter of the digraph in Figure 6 in terms of n and N alone rather than in terms of $\Delta(N,n)$.

$$\delta(\Gamma_{n,N}) = n - \frac{N}{2} + 1 \text{ for } \Gamma_{n,N} \text{ in Figure 6.}$$

The above equality and inspection of digraphs with small n and N leads us to believe that a solution to Bratton's problem can be obtained by showing

$$2\delta + N \geq 2n + 2 \qquad 3.2.6$$

is true for all δ, n and N.

However, the inequality 3.2.6 can be shown to be false; for example, a rosette with minimum diameter and n=25 and N=41. In this case the diameter of a rosette is $\Delta(25,41) = 4$, and $2\delta + N = 2(4) + 41 = 49$ is not greater than $2n + 2 = 52$.

Luce [10] has shown a similar inequality to be true, namely,

$$r + N \geq 2n \qquad 3.2.7$$

for all r, N and n, where r is the rank of the incidence matrix $M(\Gamma_{n,N})$ associated with $\Gamma_{n,N}$.

It would be desirable to determine the relationship between 3.2.6 and 3.2.7, that is, the relationship between r and $2\delta - 2$. We want to determine for what δ's the inequality 3.2.6 is correct.

<u>Lemma 3.2.5</u>: Let r be the rank of the incidence matrix $M(\Gamma)$ associated with the digraph Γ; then $r \geq \delta$.

<u>Proof</u>: Let Γ have diameter δ. This implies there exists some dipath $e_i = (v_{i-1}, v_i)$, $i = 1,2,\ldots,\delta$, of length δ in the digraph Γ. Consider $M(\Gamma)$ with a $(\delta+1)\times(\delta+1)$ sub-determinant:

$$\begin{array}{c} \\ v_o \\ v_1 \\ v_2 \\ \vdots \\ v_{\delta-1} \\ v_\delta \end{array}
\begin{array}{c} \begin{array}{ccccc} v_o & v_1 & v_2 & \cdots & v_\delta \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

We can see the $\delta \times \delta$ sub-determinant

$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & \cdots & v_\delta \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_\delta \end{matrix} & \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{array}$$

is non-zero; therefore, the rank will be at least δ.

At this point, we note that the rank r is always less than or equal to n, and no other upper bound can be placed on r for arbitrary n and N.

From inspection of 3.2.7, we see that the inequality 3.2.6 holds when $r \leq 2\delta - 2$.

These results can be summarized in the following theorem. Let $P_{n,N} = \{\Gamma_{n,N} | \Gamma_{n,N}$ is strongly connected; $\delta(\Gamma_{n,N}) \geq \frac{r+2}{2}\}$.

<u>Theorem 3.2.6</u>: If Γ belongs to $P_{n,N}$, then $\delta(\Gamma) \geq n - \frac{N}{2} + 1$.

Here, we have found a class of subdigraphs of $\mathcal{D}_{n,N}$ for which we have a lower bound of $n - \frac{N}{2} + 1$ for δ.

Appendix

In this section we present four lemmas which give the first and last stage in which a random digraph (defined below) can belong to each of the connectivity classifications (see Definition 1.2.10). We remark that these results were derived for this thesis independently but it was found later that most of these results were already obtained in a paper by Cartwright and Harary [5] using a slightly different approach.

Let V be a finite set consisting of n elements v_i, $i = 1, 2, \ldots, n$.

Definition 4.1: We define a random digraph as follows:

At stage t=1, pick out one of the n(n-1) possible diedges, denoting this diedge by e_1.

At stage t=2, pick out one of the remaining n(n-1) diedges, denoting this edge by e_2.

Continuing this process, at stage t=k+1, pick one diedge out of the remaining n(n-1) - k diedges; this diedge is picked with probability $\frac{1}{n(n-1)-k}$.

Denote by $R_{n,N}$ the random digraph consisting of vertices v_i, $i = 1, 2, \ldots, n$, and diedges e_i, $i = 1, 2, \ldots, N$.

Lemma 4.1: If $R_{n,N}$ is of type S_0, then $0 \le t \le (n-1)(n-2)$.

Proof: The earliest stages at which $R_{n,N}$ is disconnected is obviously t=0. The latest stage, $t = (n-1)(n-2)$, is due to the fact that a complete digraph on n-1 vertices has (n-1)(n-2) diedges.

<u>Lemma 4.2</u>: If $R_{n,N}$ is of type S_1, then $n-1 \le t \le (n-1)(n-2)$.

<u>Proof</u>: We need n-1 diedges to become type S_1; before this we have at least one isolated vertex.

The last stage at which a random digraph is of type S_1 is $t = (n-1)(n-2)$. We have two vertices, v_i and v_j, for which there does not exist a dipath from v_i and v_j or from v_j to v_i. Therefore, we construct a complete digraph on (n-2) vertices, and from v_i and v_j we construct all possible issuing edges, thus

$$\begin{aligned} t &= (n-2)(n-3) + 2(n-2) \\ &= n^2 - 3n + 2 \\ &= (n-1)(n-2) \end{aligned}$$

<u>Lemma 4.3</u>: If $R_{n,N}$ is of type S_2, then $n-1 \le t \le (n-1)^2$.

<u>Proof</u>: We need n-1 diedges, transversed in the proper manner, for $R_{n,N}$ to be of type S_2. $(n-1)^2$ is obtained from noting that we can construct a complete digraph on n-1 vertices and have one vertex with all possible issuing edges; thus

$$\begin{aligned} t &= (n-1)(n-2) + (n-1) \\ &= (n-1)(n-1) \\ &= (n-1)^2 \end{aligned}$$

<u>Lemma 4.4</u>: If $R_{n,N}$ is of type S_3, then $n \le t \le n(n-1)$.

<u>Proof</u>: Here, we need the n diedges to assure us a dipath between every pair of vertices. The upper bound, n(n-1), is due to the fact that a complete digraph has n(n-1) edges.

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