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LINEAR THEORY OF MICROPOLAR VISCOELASTICITY

by

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Abstract

The present paper is concerned with the construction of a linear theory of viscoelasticity for micropolar solids. The constitutive equations of strain and micro-rotation rate dependent materials, stress and couple stress rate dependent materials and continuous memory dependent micropolar elastic solids and fluids are obtained and the thermodynamic restrictions are studied. An indeterminate couple stress theory is also derived as a special case.

Author

1. Introduction

The present paper is concerned with the development of a theory of micropolar viscoelasticity as an extension of the micropolar elasticity and micropolar fluids developed by Eringen and Suhubi [1], [2] and Eringen [3] to [6]. General theories given in [1] to [4] are believed to have applications in the understanding of mechanical behavior of materials with granular and fiberous structure and anisotropic fluids and fluids carrying additives. The special theories of micropolar elasticity and fluids [5], [6] may find their uses in a special class of solid materials and fluids in which the micro-rotational motions and inertia are important. Mechanics of fluids and solids made of dumbbell molecules should be governed by these theories.

The micropolar elasticity is extended here to construct linear constitutive theories for the micropolar materials that possess internal friction. We consider only materials possessing microisotropy. In Section 2 we present the resume of basic equations of micropolar theory of elasticity and fluids. In Sections 3 and 4 we give rate dependent theories of microvisco-elasticity generalizing the classical Voigt-Kelvin theory, the Maxwell theory and the general rate theory. Section 5 is devoted to linear memory dependent micropolar materials. Thermodynamic restrictions are studied. In Section 6 we make passage to micro-polar viscoelastic fluids. An indeterministic couple stress theory is obtained as a result of constrained motion of the micro-polar viscoelasticity (Section 7).

2. Basic Equations of Micropolar Theory

The theories of micropolar elasticity and micropolar fluids are based on the following balance laws, [2], [5], [6],

$$\frac{\partial \rho}{\partial t} + (\rho\, v_k)_{,k} = 0 \quad \text{(mass)} \tag{2.1}$$

$$t_{k\ell,k} + \rho(f_\ell - \dot{v}_\ell) = 0 \quad \text{(momentum)} \tag{2.2}$$

$$m_{rk,r} + \epsilon_{k\ell r}\, t_{\ell r} + \rho(\ell_k - j\, \dot{\nu}_k) = 0 \quad \text{(moment of momentum)} \tag{2.3}$$

$$\rho\, \dot{\epsilon} = t_{k\ell}\, d_{\ell k} + \epsilon_{k\ell r}\, t_{k\ell}\, (\omega_r - \nu_r) + m_{k\ell}\, \nu_{\ell,k} + q_{k,k} + \rho\, h \quad \text{(energy)} \tag{2.4}$$

where

ρ = mass density	v_k = velocity vector
$t_{k\ell}$ = stress tensor	f_k = body force
$m_{k\ell}$ = couple stress	ℓ_k = body couple
j = micro-inertia	ν_k = micro-rotation velocity (gyration vector)
ϵ = internal energy density	q_k = heat vector, directed outward the body
ω_k = vorticity vector	h = heat source

Throughout the paper we employ the rectangular coordinates x_k $(k = 1,2,3)$ and the cartesian tensor notation. Accordingly the repeated indices are summed over the range $(1,2,3)$ and the free indices take the values 1, 2, 3. A superposed dot indicates the material derivative and an index following a comma the partial differentiation, e.g.,

$$\dot{v}_k = \frac{\partial v_k}{\partial t} + v_{k,\ell}\, v_\ell \quad , \quad v_{k,\ell} = \frac{\partial v_k}{\partial x_\ell}$$

The triple indexed $\epsilon_{k\ell r}$ is the permutation symbol which has the values

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = 1$$

otherwise zero.

In what follows two other vectors are introduced, the displacement vector u_k and the micro-rotation vector φ_k. In terms of these we have the kinematical relations

$$\begin{aligned} v_k &= \dot{u}_k \quad , \quad \nu_k = \dot{\varphi}_k \\ \omega_k &= \frac{1}{2}\, \epsilon_{k\ell m}\, v_{m,\ell} \quad , \quad d_{k\ell} = \frac{1}{2}\,(v_{k,\ell} + v_{\ell,k}) \end{aligned} \tag{2.5}$$

The properties of micropolar materials are restricted according to the entropy inequality

$$\begin{aligned} \rho\,\Gamma = &-\frac{\rho}{\theta}\,(\dot{\psi} + \eta\,\dot{\theta}) + \frac{1}{\theta}\, t_{k\ell}\, d_{\ell k} + \frac{1}{\theta}\,\epsilon_{k\ell r}\, t_{k\ell}\,(\omega_r - \nu_r) \\ &+ \frac{1}{\theta}\, m_{k\ell}\, \nu_{\ell,k} + \frac{q_k \theta_{,k}}{\theta^2} \geq 0 \end{aligned} \tag{2.6}$$

which is assumed to be valid for all independent processes. In (2.6) θ is the temperature, η is the entropy and ψ is the free energy defined by

$$\psi \equiv \epsilon - \theta \eta \tag{2.7}$$

The surface tractions $\underset{\sim}{t}_{(\underset{\sim}{n})}$ and surface couples $\underset{\sim}{m}_{(\underset{\sim}{n})}$ acting at a point $\underset{\sim}{x}$ on the surface $\mathcal{S}$ of the body $\mathcal{V} + \mathcal{S}$ are calculated by

$$t_{(\underset{\sim}{n})k} = t_{\ell k}\, n_\ell \quad , \quad m_{(\underset{\sim}{n})k} = m_{\ell k}\, n_\ell \tag{2.8}$$

Equations of balance are supplemented by the constitutive equations appropriate to each material medium. For two such media they are given below:

(a) Linear isotropic micropolar elasticity

$$t_{k\ell} = \lambda e_{rr}\, \delta_{k\ell} + (2\mu + \kappa)\, e_{k\ell} + \kappa\, \epsilon_{k\ell m}\, (r_m - \varphi_m) \tag{2.9}$$

$$m_{k\ell} = \alpha\, \varphi_{r,r}\, \delta_{k\ell} + \beta\, \varphi_{k,\ell} + \gamma\, \varphi_{\ell,k} \tag{2.10}$$

(b) Linear micropolar fluids

$$t_{k\ell} = (-\pi + \lambda_v d_{rr})\, \delta_{k\ell} + (2\mu_v + \kappa_v)\, d_{k\ell} + \kappa_v\, \epsilon_{k\ell m}\, (\omega_m - \nu_m) \tag{2.11}$$

$$m_{k\ell} = \alpha_v\, \nu_{r,r}\, \delta_{k\ell} + \beta_v\, \nu_{k,\ell} + \gamma_v\, \nu_{\ell,k} \tag{2.12}$$

where

$$e_{k\ell} \equiv \frac{1}{2}(u_{k,\ell} + u_{\ell,k}) \quad , \quad r_k \equiv \frac{1}{2}\epsilon_{k\ell m}\, u_{m,\ell} \tag{2.13}$$

are respectively the infinitesimal strain tensor and the rotation vector. Eringen [5], [6] has shown that the new elastic moduli λ, μ, κ, α, β, γ and viscosities λ_v, μ_v, κ_v, α_v, β_v and γ_v are subject to the restrictions

$$\begin{aligned} &0 \leq 3\lambda + 2\mu + \kappa \quad , \quad 0 \leq \mu \quad , \quad 0 \leq \kappa \\ &0 \leq 3\alpha + 2\gamma \quad , \quad -\gamma \leq \beta \leq \gamma \quad , \quad 0 \leq \gamma \end{aligned} \tag{2.14}$$

$$\begin{aligned} &0 \leq 3\lambda_v + 2\mu_v + \kappa_v \quad , \quad 0 \leq \mu_v \quad , \quad 0 \leq \kappa_v \\ &0 \leq 3\alpha_v + 2\gamma_v \quad , \quad -\gamma_v \leq \beta_v \leq \gamma_v \quad , \quad 0 \leq \gamma_v \end{aligned} \tag{2.15}$$

of which (2.14) is necessary and sufficient for the non-negative internal energy and (2.15) is necessary and sufficient for the entropy inequality (2.6) to be satisfied for all independent processes.

3. Micropolar Viscoelasticity Involving the Time Rate of Strain and Micro-rotation

In this section we obtain a set of constitutive equations for a class of linear viscoelastic materials in which stress and couple stress depend on the strain measures, micro-rotation, micro-rotation gradient and their first order time rates. These correspond to the generalizations of the Kelvin-Voigt solids. A linear theory of isotropic microelastic material may be constructed by a linear combination of linear isotropic micropolar elasticity and micropolar fluids in the same manner as those of the classical Kelvin-Voigt theory of viscoelasticity. However, we proceed through the thermodynamical considerations.

An inspection of (2.9) to (2.12) reveals that the following constitute a set of objective variables to construct the theory:

$$\epsilon_{k\ell} = e_{k\ell} + \epsilon_{k\ell m}(r_m - \varphi_m)$$

$$\Delta_{k\ell} = d_{k\ell} + \epsilon_{k\ell m}(\omega_m - \nu_m) = \dot{\epsilon}_{k\ell} \tag{3.1}$$

$$\varphi_{k,\ell} \quad , \quad \nu_{k,\ell} \quad , \quad \theta$$

We define microisotropic viscoelastic solid type 1 by the following constitutive equations:*

$$t_{k\ell} = F_{k\ell}(\epsilon_{rs}, \Delta_{rs}, \varphi_{r,s}, \nu_{r,s}, \theta)$$

$$m_{k\ell} = M_{k\ell}(\epsilon_{rs}, \Delta_{rs}, \varphi_{r,s}, \nu_{r,s}, \theta) \tag{3.2}$$

$$q_k = H_k(\epsilon_{rs}, \Delta_{rs}, \varphi_{r,s}, \nu_{r,s}, \theta)$$

---

*The present method can be used to obtain the general nonlinear theory, provided we replace $\epsilon_{k\ell}$ by the finite strain measure.

$$\psi = \Psi(\epsilon_{rs}, \Delta_{rs}, \varphi_{r,s}, \nu_{r,s}, \theta)$$

$$\eta = N(\epsilon_{rs}, \Delta_{rs}, \varphi_{r,s}, \nu_{r,s}, \theta)$$

where $F_{k\ell}$, $M_{k\ell}$ are tensor-valued functions, H_k is vector-valued and ψ and N are scalar-valued functions of the variables listed. Note that (3.2) obeys the rule of equipresence. Substitution of (3.2) into (2.6) gives

$$-\frac{\rho}{\theta}\left[\frac{\partial\Psi}{\partial\epsilon_{rs}}\Delta_{rs} + \frac{\partial\Psi}{\partial\Delta_{rs}}\dot{\Delta}_{rs} + \frac{\partial\Psi}{\partial\varphi_{r,s}}\dot{\varphi}_{r,s} + \frac{\partial\Psi}{\partial\nu_{r,s}}\dot{\nu}_{r,s} + \frac{\partial\Psi}{\partial\theta}\dot{\theta}\right.$$

$$\left. + \eta\,\dot{\theta}\right] + \frac{1}{\theta}t_{k\ell}\Delta_{k\ell} + \frac{1}{\theta}m_{k\ell}\nu_{\ell,k} + \frac{q_k\theta_{,k}}{\theta^2} \geq 0 \qquad (3.3)$$

This inequality must be satisfied for all independent variations of the variables $\dot{\Delta}_{rs}$, $\dot{\nu}_{r,s}$, $\dot{\theta}$, $\theta_{,k}$, $\Delta_{r,s}$ and $\nu_{\ell,k}$. It is linear in the first four of these variables. Hence the coefficients of these variables must vanish, i.e.,

$$\frac{\partial\Psi}{\partial\Delta_{rs}} = \frac{\partial\Psi}{\partial\nu_{r,s}} = 0 \qquad (3.4)$$

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \qquad q_k = 0 \qquad (3.5)$$

The remaining terms may be arranged into

$$\frac{1}{\theta}\left(-\rho\frac{\partial\Psi}{\partial\epsilon_{k\ell}} + t_{k\ell}\right)\Delta_{k\ell} + \frac{1}{\theta}\left(-\rho\frac{\partial\Psi}{\partial\varphi_{k,\ell}} + m_{\ell k}\right)\dot{\varphi}_{k,\ell} \geq 0 \quad (3.6)$$

We decompose the stress and couple stress into non-dissipative and dissipative parts as follows:

$$t_{k\ell} = {}_E t_{k\ell}(\epsilon_{k\ell}, \varphi_{k,\ell}, \theta) + {}_D t_{k\ell}(\epsilon_{k\ell}, \varphi_{k,\ell}, \Delta_{k\ell}, \nu_{k,\ell}, \theta)$$

$$m_{k\ell} = {}_E m_{k\ell}(\epsilon_{k\ell}, \varphi_{k,\ell}, \theta) + {}_D m_{k\ell}(\epsilon_{k\ell}, \varphi_{k,\ell}, \Delta_{k\ell}, \nu_{k,\ell}, \theta) \tag{3.7}$$

In view of (3.4) we have

$$\psi = \Psi(\epsilon_{k\ell}, \varphi_{k,\ell}, \theta) \tag{3.8}$$

Inequality (3.6) is linear both in $\Delta_{k\ell}$ and $\dot{\varphi}_{k,\ell}$. The necessary and sufficient condition for (3.6) to be satisfied for all possible values of these variables is

$${}_E t_{k\ell} = \rho \frac{\partial \Psi}{\partial \epsilon_{k\ell}} \tag{3.9}$$

$${}_E m_{\ell k} = \rho \frac{\partial \Psi}{\partial \varphi_{k,\ell}} \tag{3.10}$$

and

$${}_D t_{k\ell}\, \Delta_{k\ell} + {}_D m_{\ell k}\, \nu_{k,\ell} \geq 0 \tag{3.11}$$

Hence we have proved

Theorem 1. The isotropic microviscoelastic solid defined by (3.2) is thermodynamically admissible if and only if the free energy ψ is independent of the rate variables $\Delta_{k\ell}$, $\nu_{k,\ell}$; the stress and couple stress are given by (3.7) with their non-dissipative parts given by (3.9) and (3.10) and the dissipative parts satisfy the inequality (3.11).

<u>Theorem</u> 2. <u>If</u> $_D t$ <u>and</u> $_D m$ <u>are continuous functions of their arguments then they vanish with</u> $\Delta_{k\ell}$ <u>and</u> $\nu_{k,\ell}$, i.e.,

$$
\begin{aligned}
{}_D t\,(\epsilon_{k\ell}\,,\ \varphi_{k,\ell}\,,\ 0\,,\ 0\,,\ \theta) &= 0 \\
{}_D m\,(\epsilon_{k\ell}\,,\ \varphi_{k,\ell}\,,\ 0\,,\ 0\,,\theta) &= 0
\end{aligned}
\qquad (3.12)
$$

To prove the first of these we set $\nu_{k,\ell} = 0$ and take all $\Delta_{k\ell} = 0$ except Δ_{11} . Then according to (3.11) ${}_D t_{11}\,\Delta_{11} \geq 0$. If $\Delta_{11} > 0$ then ${}_D t_{11} > 0$. If $\Delta_{11} < 0$ then ${}_D t_{11} < 0$. Since ${}_D t_{11}$ is continuous in Δ_{11} then ${}_D t_{11} = 0$ when $\Delta_{11} = 0$. The argument is the same for other components of ${}_D t_{k\ell}$. To prove $(3.12)_2$ we follow the same method except that this time we set $\Delta_{k\ell} = 0$.

Several remarks are in order:

(a) The stress and couple stress, each, are composed of a purely <u>microelastic part</u> and a <u>dissipative part</u>. The elastic parts are derivable from a potential (3.9), (3.10). The dissipative energy, according to (3.11), must be non-negative.

(b) According to (3.12) the dissipative stress and couple stress contain no purely elastic effect.

(c) The free energy, according to (3.8), is independent of the rates of deformations and microdeformations.

(d) The heat vanish. This, of course, is the consequence of not including the thermal gradients among the constitutive variables.

Since we are dealing with the microisotropic solid the function ψ is further restricted. In fact, it must be a function of the joint invariants of the tensors

$$\epsilon_{(k\ell)} \quad , \quad \epsilon_{[k\ell]} \quad , \quad \varphi_{(k,\ell)} \quad , \quad \varphi_{[k,\ell]}$$

where a parenthesis enclosing the indices indicates the symmetric part and a bracket the antisymmetric part, e.g.,

$$\epsilon_{(k\ell)} \equiv \frac{1}{2}(\epsilon_{k\ell} + \epsilon_{\ell k}) \quad , \quad \epsilon_{[k\ell]} \equiv \frac{1}{2}(\epsilon_{k\ell} - \epsilon_{\ell k})$$

Invariants of such tensors are well-known, cf. Spencer [7], and therefore we can construct the elastic parts of the couple stress and the stress tensor. Since we are concerned with the linear theory, we may proceed in a simple way as follows. Write

$$\rho\,\psi = A_o + A_{k\ell}\,\epsilon_{k\ell} + \frac{1}{2} A_{k\ell mn}\,\epsilon_{k\ell}\,\epsilon_{mn} + B_{k\ell}\,\varphi_{k,\ell}$$
$$\frac{1}{2} B_{k\ell mn}\,\varphi_{k,\ell}\,\varphi_{m,n} + C_{k\ell mn}\,\epsilon_{k\ell}\,\varphi_{m,n}$$

where A_o , $A_{k\ell}$, $A_{k\ell mn}$, $B_{k\ell}$, . . . are functions of θ only. Since φ_k is an axial vector, it can be shown that $C_{k\ell mn} \equiv 0$ for otherwise upon the reflection of the material axes ψ would not remain invariant. For the isotropic tensors $A_{k\ell}$, $A_{k\ell mn}$, ... , we have

$$A_{k\ell} = A\delta_{k\ell} \quad , \quad A_{k\ell mn} = A_1\delta_{k\ell}\,\delta_{mn} + A_2\delta_{km}\,\delta_{\ell n} + A_3\delta_{kn}\,\delta_{\ell m}$$

Consequently

$$\rho\,\psi = A_o + A\epsilon_{kk} + B\varphi_{k,k} + \frac{1}{2}(A_1\epsilon_{kk}\,\epsilon_{\ell\ell} + A_2\epsilon_{k\ell}\,\epsilon_{k\ell} + A_3\epsilon_{k\ell}\,\epsilon_{\ell k})$$
$$+ \frac{1}{2}(B_1\varphi_{k,k}\,\varphi_{\ell,\ell} + B_2\varphi_{k,\ell}\,\varphi_{k,\ell} + B_3\varphi_{k,\ell}\,\varphi_{\ell,k}) \qquad (3.13)$$

Upon substituting this into (3.9) and (3.10) we obtain

$$\begin{aligned} \overset{t}{E}_{k\ell} &= A\delta_{k\ell} + A_1\epsilon_{rr}\,\delta_{k\ell} + A_2\epsilon_{k\ell} + A_3\epsilon_{\ell k} \\ \overset{m}{E}_{k\ell} &= B\delta_{k\ell} + B_1\varphi_{r,r}\,\delta_{k\ell} + B_2\,\varphi_{\ell,k} + B_3\varphi_{k,\ell} \end{aligned} \tag{3.14}$$

For vanishing initial stress and couple stress we have $A = B = 0$. Upon using $(3.1)_1$ in $(3.14)_1$, we see that equations (3.14) go into (2.9) and (2.10) if we set

$$\begin{aligned} A_1 &= \lambda \quad , \quad A_2 = \mu + \kappa \quad , \quad A_3 = \mu \\ B_1 &= \alpha \quad , \quad B_2 = \gamma \quad , \quad B_3 = \beta \end{aligned} \tag{3.15}$$

With these the free energy (3.13) reads

$$\begin{aligned} \rho\,\psi = {} & A\epsilon_{kk} + B\varphi_{k,k} + \frac{1}{2}[\lambda\,\epsilon_{kk}\,\epsilon_{\ell\ell} + (\mu+\kappa)\,\epsilon_{k\ell}\,\epsilon_{k\ell} \\ & + \mu\,\epsilon_{k\ell}\,\epsilon_{\ell k}] + \frac{1}{2}(\alpha\,\varphi_{k,k}\,\varphi_{\ell,\ell} + \beta\,\varphi_{k,\ell}\,\varphi_{\ell,k} + \gamma\,\varphi_{k,\ell}\,\varphi_{k,\ell}) \end{aligned} \tag{3.16}$$

or

$$\begin{aligned} \rho\,\psi = {} & Ae_{kk} + B\varphi_{k,k} + \frac{1}{2}[\lambda\,e_{kk}\,e_{kk} + (2\mu+\kappa)\,e_{k\ell}\,e_{k\ell}] \\ & + \kappa(r_k-\varphi_k)(r_k-\varphi_k) + \frac{1}{2}(\alpha\,\varphi_{k,k}\,\varphi_{\ell,\ell} + \beta\,\varphi_{k,\ell}\,\varphi_{\ell,k} + \gamma\varphi_{k,\ell}\varphi_{k,\ell}) \end{aligned} \tag{3.17}$$

This result is identical to that given in [5] with the exception of the initial stress terms containing A and B which were excluded in [5]. Both $\underset{\sim}{D}^{t}$ and $\underset{\sim}{D}^{m}$ are isotropic functions. Therefore, the dissipative stress and couple stress, linear in their constitutive variables, may be expressed as

$$ {}_{D}t_{k\ell} = (a + a_1 \Delta_{rr}) \delta_{k\ell} + a_2 \Delta_{k\ell} + a_3 \Delta_{\ell k} + a_4 \nu_{k,\ell} + a_5 \nu_{\ell,k} $$

$$ {}_{D}m_{k\ell} = (b + b_1 \nu_{r,r}) \delta_{k\ell} + b_2 \nu_{k,\ell} + b_3 \nu_{\ell,k} + b_4 \Delta_{k\ell} + b_5 \Delta_{\ell k} $$

where in view of (3.12) $a = b = 0$ and a_k , b_k are functions of θ only.

Since $\underset{\sim}{\nu}$ is an axial vector while ${}_{D}\underset{\sim}{t}$ is an absolute tensor the condition of isotropy requires that we further set $a_4 = a_5 = 0$. Similarly ${}_{D}\underset{\sim}{m}$ is a relative tensor of degree 1 while $\underset{\sim}{\Delta}$ is an absolute tensor. The full isotropy condition in this case implies that $b_4 = b_5 = 0$. If we write

$$ a_1 = \lambda_v \quad , \quad a_2 = \mu_v + \kappa_v \quad , \quad a_3 = \mu_v $$

$$ b_1 = \alpha_v \quad , \quad b_2 = \beta_v \quad , \quad b_3 = \gamma_v $$

we get the constitutive equations (2.11) (excluding the pressure term $-\pi$) and (2.12) of the micropolar fluids.

The thermodynamic inequality (3.11) can now be used to prove the inequalities (2.15) (cf. [6]).

The constitutive equations of the class of viscoelastic solids considered here are therefore given by

$$ t_{k\ell} = \lambda\epsilon_{rr} \delta_{k\ell} + (\mu+\kappa) \epsilon_{k\ell} + \mu\epsilon_{\ell k} + \lambda_v \Delta_{rr} \delta_{k\ell} + (\mu_v+\kappa_v) \Delta_{k\ell} + \mu\Delta_{\ell k} \tag{3.18} $$

$$m_{k\ell} = \alpha\varphi_{r,r}\,\delta_{k\ell} + \beta\varphi_{k,\ell} + \gamma\varphi_{\ell,k} + \alpha_v\nu_{r,r}\,\delta_{k\ell} + \beta_v\nu_{k,\ell} + \gamma_v\nu_{\ell,k} \qquad (3.19)$$

Since we are dealing with the linear theory we can further write $\Delta_{k\ell} = \dot{\epsilon}_{k\ell}$ and $\nu_{k,\ell} = \dot{\varphi}_{k,\ell}$.

Equations (3.18) and (3.19) are the final forms of the linear constitutive equations of this class of isotropic micropolar viscoelastic solids.

The basic difference of the micropolar theory from the corresponding Kelvin-Voigt theory is apparent in two counts:

(i) Here the stress also depends upon the micro-rotation and microgyration and their time rates and it is not a symmetric tensor and, of course,

(ii) the couple stress is totally absent in the classical theory.

4. Microviscoelasticity Involving Stress and Couple Stress Rates

A constitutive theory of microviscoelasticity can be constructed generalizing the Maxwell solid of the classical theory. Thus

$$(1 + a \frac{\partial}{\partial t})\, t_{rr}\, \delta_{k\ell} + (1 + b \frac{\partial}{\partial t})\, t_{k\ell} = \lambda_v\, \Delta_{rr}\, \delta_{k\ell} + (\mu_v + \kappa_v)\, \Delta_{k\ell} + \mu\, \Delta_{\ell k} \quad (4.1)$$

$$(1 + p \frac{\partial}{\partial t})\, m_{rr}\, \delta_{k\ell} + (1 + 2q \frac{\partial}{\partial t})\, m_{k\ell} = \alpha_v\, \nu_{r,r}\, \delta_{k\ell} + \beta_v\, \nu_{k,\ell} + \gamma_v\, \nu_{\ell,k} \quad (4.2)$$

where a , b , p and q are functions of temperature θ also. Thermodynamics of the Maxwell solid has not been constructed to date. We do not expect to present one for the present theory. We note here the fundamental departure from the classical Maxwell solid, namely the presence of couple stress and its time rate.

Further generalizations (3.18), (3.19) and (4.1), (4.2) involving higher order time rates of $\underset{\sim}{t}$, $\underset{\sim}{\Delta}$, $\underset{\sim}{m}$ and $\underset{\sim}{\nu}$ are not difficult to construct. To this end we first introduce

$$t_{k\ell} = -p\, \delta_{k\ell} + \bar{t}_{k\ell} \quad , \quad \epsilon_{k\ell} = -\epsilon\, \delta_{k\ell} + \bar{\epsilon}_{k\ell}$$

$$m_{k\ell} = m\, \delta_{k\ell} + \bar{m}_{k\ell} \quad , \quad \varphi_{k,\ell} = \varphi_{r,r}\, \delta_{k\ell} + \bar{\varphi}_{k,\ell} \quad (4.3)$$

such that

$$p \equiv -\frac{1}{3} t_{kk} \quad , \quad \epsilon \equiv -\frac{1}{3} \epsilon_{kk} \quad , \quad m \equiv \frac{1}{3} m_{kk} \quad ,$$

$$\bar{t}_{kk} = \bar{\epsilon}_{kk} = \bar{m}_{kk} = \bar{\varphi}_{k,k} = 0 \quad . \tag{4.4}$$

Now write constitutive equations of the form

$$P\, p = Q\, \epsilon \quad , \quad R\, \bar{t}_{k\ell} = S\, \bar{\epsilon}_{k\ell} + T\, \bar{\epsilon}_{\ell k}$$

$$F\, m = H\, \varphi_{r,r} \quad , \quad L\, \bar{m}_{k\ell} = M\, \bar{\varphi}_{k,\ell} + N\, \bar{\varphi}_{\ell,k} \tag{4.5}$$

where P , Q , R , S , T , F , H , . . . are differential operators of the form

$$P \equiv \sum_{k=0}^{p} p_k(\theta) \frac{\partial^k}{\partial t^k} \quad , \quad Q \equiv \sum_{k=0}^{q} q_k(\theta) \frac{\partial^k}{\partial t^k} \quad , \quad \ldots \tag{4.6}$$

The order of the differential operators depends on the singularities allowable in the stress and couple stress fields. The discussion of certain simple cases, e.g., the standard solid involving only the time rates of first order, is not difficult.

5. Memory Dependent Linear Isotropic Micropolar Materials

Constitutive equations of micropolar materials, whose behavior at time t is influenced by the past history of motion, can be formulated in a similar fashion to those of the classical Boltzmann-Volterra theory [8]. In view of the fact that we are dealing with isotropic materials we need not distinguish the material and spatial coordinates. Obeying the rule of equipresence the constitutive equations of non-heat conducting, memory dependent isotropic micropolar elastic materials may be expressed as

$$
\begin{aligned}
t_{k\ell} &= \mathop{F_{k\ell}}_{\tau=0}^{\infty} [\epsilon_{rs}(\tau) \;,\; \varphi_{r,s}(\tau) \;,\; \theta(\tau)] \\
m_{k\ell} &= \mathop{M_{k\ell}}_{\tau=0}^{\infty} [\epsilon_{rs}(\tau) \;,\; \varphi_{r,s}(\tau) \;,\; \theta(\tau)] \\
q_k &= \mathop{H_k}_{\tau=0}^{\infty} [\epsilon_{rs}(\tau) \;,\; \varphi_{r,s}(\tau) \;,\; \theta(\tau)] \qquad (5.1) \\
\psi &= \mathop{\Psi}_{\tau=0}^{\infty} [\epsilon_{rs}(\tau) \;,\; \varphi_{r,s}(\tau) \;,\; \theta(\tau)] \\
\eta &= \mathop{N}_{\tau=0}^{\infty} [\epsilon_{rs}(\tau) \;,\; \varphi_{r,s}(\tau) \;,\; \theta(\tau)]
\end{aligned}
$$

where the constitutive functional $\underset{\sim}{F}$ is an absolute tensor-valued functional of the absolute second order tensor function $\underset{\sim}{\epsilon}$, the gradient of an axial vector function $\underset{\sim}{\varphi}$ and the absolute scalar $\theta(\tau)$ in the full range $\tau = o$ to ∞ . The functional $M_{k\ell}$ is a second order axial tensor-valued function; H_k is an absolute vector functional and Ψ and N are absolute scalar-valued functionals of the same argument functions as $\mathfrak{F}$. The invariance under the time shift implies that these functionals are of the form

$$t_{k\ell} = \mathop{F_{k\ell}}_{\tau=0}^{\infty} [\epsilon_{rs}(t-\tau) , \varphi_{r,s}(t-\tau) , \theta(t-\tau)] \tag{5.2}$$

The requirement of form-invariance to arbitrary rigid motions of the spatial frame of reference implies that these functionals are isotropic. The consequence of these restrictions are studied below.

The axiom of thermodynamic admissibility

$$-\frac{\rho}{\theta}(\dot{\psi}+\eta\dot{\theta}) + \frac{1}{\theta} t_{k\ell} \Delta_{k\ell} + \frac{1}{\theta} m_{k\ell} \nu_{\ell,k} + \frac{1}{\theta^2} q_k \theta_{,k} \geq 0 \tag{5.3}$$

imposes restrictions on the form of the constitutive functionals. To investigate this we introduce, for convenience, the difference histories

$$\begin{aligned} \bar{\epsilon}_{rs}(\tau) &\equiv \epsilon_{rs}(t-\tau) - \epsilon_{rs}(t) \\ \bar{\varphi}_{r,s}(\tau) &\equiv \varphi_{r,s}(t-\tau) - \varphi_{r,s}(t) \\ \bar{\theta}(\tau) &\equiv \theta(t-\tau) - \theta(\tau) \end{aligned} \tag{5.4}$$

The constitutive equations can now be written in the form

$$t_{k\ell} = \mathop{F_{k\ell}}_{\tau=0}^{\infty} [\bar{\epsilon}_{rs}(\tau) , \bar{\varphi}_{r,s}(\tau) ; \epsilon_{rs}(t) , \varphi_{r,s}(t) , \theta(t)] \tag{5.5}$$

where we also assumed that the temperature memory is negligible.

This latter requirement can be relaxed trivially. We are interested only in a linear rate theory. To avoid clumsy formalism of functionals we assume that the constitutive functionals are sufficiently smooth to allow a "power series" representation. Thus a second order approximation to the free energy ψ is

$$\begin{aligned}\psi = {} & \psi_o[\epsilon_{rs}(t), \varphi_{r,s}(t), \theta(t)] + \int_0^\infty A_{k\ell}(s)\, \bar{\epsilon}_{k\ell}(s)ds \\ & + \int_0^\infty B_{k\ell}(s)\, \bar{\varphi}_{k,\ell}(s)ds + \int_0^\infty\!\!\int_0^\infty A_{k\ell mn}(s_1,s_2)\, \bar{\epsilon}_{k\ell}(s_1)\bar{\epsilon}_{mn}(s_2)ds_1ds_2 \\ & + \int_0^\infty\!\!\int_0^\infty B_{k\ell mn}(s_1,s_2)\, \bar{\varphi}_{k,\ell}(s_1)\, \bar{\varphi}_{m,n}(s_2)ds_1ds_2 \\ & + \int_0^\infty\!\!\int_0^\infty C_{k\ell mn}(s_1,s_2)\, \bar{\epsilon}_{k\ell}(s_1)\, \bar{\varphi}_{m,n}(s_2)\, ds_1ds_2\end{aligned} \tag{5.6}$$

where $A_{k\ell}$, $B_{k\ell}$, $A_{k\ell mn}$, $B_{k\ell mn}$ and $C_{k\ell mn}$ are in general functions of s , $\epsilon_{k\ell}(t)$, $\varphi_{k,\ell}(t)$ and $\theta(t)$, and

$$\begin{aligned}A_{k\ell mn}(s_1,s_2) &= A_{mnk\ell}(s_2,s_1) \\ B_{k\ell mn}(s_1,s_2) &= B_{mnk\ell}(s_2,s_1) \\ C_{k\ell mn}(s_1,s_2) &= C_{mnk\ell}(s_2,s_1)\end{aligned} \tag{5.7}$$

are isotropic tensors. Since ψ and $\underset{\sim}{\epsilon}$ are absolute tensors while $\bar{\underset{\sim}{\varphi}}$ is an axial vector, the form-invariance of ψ to the reflection of material axes requires that $C_{k\ell mn} \equiv 0$. Now substitute (5.6) into (5.3)

$$
\begin{aligned}
&-\frac{\rho}{\theta}\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}+\frac{1}{\theta}\Big\{t_{k\ell}-\rho\frac{\partial\psi_o}{\partial\epsilon_{k\ell}}+\rho\int_o^\infty A_{k\ell}(s)ds \\
&\quad+2\rho\iint_{o\,o}^{\infty\,\infty} A_{k\ell mn}(s_1,s_2)\,\bar{\epsilon}_{mn}(s_2)\,ds_1ds_2\Big\}\Delta_{k\ell} \\
&\quad+\frac{1}{\theta}\Big\{m_{\ell k}-\rho\frac{\partial\psi_o}{\partial\varphi_{k,\ell}}+\rho\int_o^\infty \bar{B}_{k\ell}(s)ds \\
&\quad+2\rho\iint_{oo}^{\infty\,\infty} B_{k\ell mn}(s_1,s_2)\,\bar{\varphi}_{m,n}(s_2)ds_2\Big\}\nu_{k,\ell} \qquad (5.8)\\
&\quad+\frac{\rho}{\theta}\int_o^\infty\Big[A_{k\ell}(s)\frac{d\,\epsilon_{k\ell}(t-s)}{ds}+B_{k\ell}(s)\frac{d\,\varphi_{k,\ell}(t-s)}{ds}\Big]ds \\
&\quad+\frac{\rho}{\theta}\iint_{o\,o}^{\infty\,\infty}\Big\{A_{k\ell mn}(s_1,s_2)\Big[\frac{d\epsilon_{k\ell}(t-s_1)}{ds_1}\bar{\epsilon}_{mn}(s_2)+\bar{\epsilon}_{k\ell}(s_1)\frac{d\epsilon_{mn}(t-s_2)}{ds_2}\Big] \\
&\quad+B_{k\ell mn}(s_1,s_2)\Big[\frac{d\varphi_{k,\ell}(t-s_1)}{ds_1}\bar{\varphi}_{m,n}(s_2)+\bar{\varphi}_{k,\ell}(s_1)\frac{d\varphi_{m,n}(t-s_2)}{ds_2}\Big]\Big\}ds_1ds_2 \\
&\quad+\frac{q_k\,\theta_{,k}}{\theta^2}\ \geq\ 0
\end{aligned}
$$

This inequality is linear in $\dot{\theta}$ and $\theta_{,k}$ and it cannot be maintained for arbitrary variations of these variables unless their coefficient vanishes, identically, i.e.

$$\eta = -\frac{\partial\psi}{\partial\theta}\quad,\quad q_k = 0 \qquad (5.9)$$

It can be shown that for any history $\epsilon_{k\ell}(s)$ and $\varphi_{k,\ell}(s)$ there exist sufficiently near histories $\epsilon_{k\ell}(s+)$, $\varphi_{k,\ell}(s+)$ such that $\dot{\epsilon}_{k\ell}(t) \equiv \Delta_{k\ell}$ and $\dot{\varphi}_{k,\ell}(t) \equiv \nu_{k,\ell}$ are arbitrary. Consequently, the necessary and sufficient conditions for (5.8) to be valid for all independent processes are

$$t_{k\ell} = \rho \frac{\partial\psi_o}{\partial\epsilon_{k\ell}} - \rho \int_o^\infty A_{k\ell}(s)ds \; -2\rho \int_o^\infty\!\!\int_o^\infty A_{k\ell mn}(s_1,s_2)\; \bar{\epsilon}_{mn}(s_2)ds_1 ds_2 \tag{5.10}$$

$$m_{\ell k} = \rho \frac{\partial\psi_o}{\partial\varphi_{k,\ell}} - \rho \int_o^\infty B_{k\ell}(s)ds \; -2\rho \int_o^\infty\!\!\int_o^\infty B_{k\ell mn}(s_1,s_2)\; \bar{\varphi}_{m,n}(s_2)ds_1 ds_2 \tag{5.11}$$

and

$$\begin{aligned}
&\frac{\rho}{\theta}\int_o^\infty \Big[A_{k\ell}(s)\, \frac{d\,\epsilon_{k\ell}(t-s)}{ds} + B_{k\ell}(s)\, \frac{d\,\varphi_{k,\ell}(t-s)}{ds}\Big]ds \\
&\quad + 2\,\frac{\rho}{\theta}\int_o^\infty\!\!\int_o^\infty \Big[A_{k\ell mn}(s_1,s_2)\; \bar{\epsilon}_{k\ell}(s_1)\, \frac{d\,\epsilon_{mn}(t-s_2)}{ds_2} \\
&\quad + B_{k\ell mn}(s_1,s_2)\; \bar{\varphi}_{k,\ell}(s_1)\, \frac{d\varphi_{m,n}(t-s_2)}{ds_2}\Big]\, ds_1 ds_2 \;\geq\; 0
\end{aligned} \tag{5.12}$$

This last inequality further implies that

$$A_{k\ell} = B_{k\ell} = 0 \tag{5.13}$$

$$\begin{aligned}
&\frac{\rho}{\theta}\int_o^\infty\!\!\int_o^\infty \Big[a_{k\ell mn}(s_1,s_2)\; \dot{\bar{\epsilon}}_{k\ell}(s_1)\; \dot{\bar{\epsilon}}_{mn}(s_2) \\
&\quad + b_{k\ell mn}(s_1,s_2)\; \dot{\bar{\varphi}}_{k,\ell}(s_1)\; \dot{\bar{\varphi}}_{m,n}(s_1)\Big]ds_1 ds_2 \;\geq\; 0
\end{aligned} \tag{5.14}$$

where we used by-part integrations and wrote

$$a_{k\ell mn}(s_1,s_2) \equiv \int_{s_1}^{\infty} A_{k\ell mn}(s_3,s_2)ds_3$$

$$b_{k\ell mn}(s_1,s_2) \equiv \int_{s_1}^{\infty} B_{k\ell mn}(s_3,s_2)ds_3 \tag{5.15}$$

$$\dot{\epsilon}_{k\ell}(s_1) \equiv \left.\frac{d\,\epsilon_{k\ell}(s)}{ds}\right|_{s = t - s_1}$$

If we now introduce the notation

$$\tilde{A}_{k\ell mn}(s) = -\int_{s}^{\infty} ds_2 \int_{o}^{\infty} A_{k\ell mn}(s_1,s_2)ds_1$$

$$\tilde{B}_{k\ell mn}(s) = -\int_{s}^{\infty} ds_2 \int_{o}^{\infty} B_{k\ell mn}(s_1,s_2)ds_1 \tag{5.16}$$

The constitutive equations (5.10) and (5.11) may be written as

$$t_{k\ell} = {}_E t_{k\ell} + {}_D t_{k\ell} \quad , \quad m_{k\ell} = {}_E m_{k\ell} + {}_D m_{k\ell} \tag{5.17}$$

where

$$_E t_{k\ell} \equiv \rho\,\frac{\partial\psi_o}{\partial\epsilon_{k\ell}} \quad , \quad {}_E m_{\ell k} \equiv \rho\,\frac{\partial\psi_o}{\partial\varphi_{k,\ell}} \tag{5.18}$$

are the elastic part of the stress and the couple stress and

$$_D t_{k\ell} \equiv 2\rho\int_{o}^{\infty} \tilde{A}_{k\ell mn}(s)\,\dot{\epsilon}_{mn}(s)ds$$

$$_D m_{\ell k} \equiv 2\rho\int_{o}^{\infty} \tilde{B}_{mnk\ell}(s)\,\dot{\varphi}_{m,n}(s)ds \tag{5.19}$$

are the dissipative parts.

For the linear theory, as shown in Section 3, (5.18) leads to

$$
\begin{aligned}
E^t_{k\ell} &= \lambda\, \epsilon_{rr}\, \delta_{k\ell} + (\mu + \kappa)\, \epsilon_{k\ell} + \mu\, \epsilon_{\ell k} \\
E^m_{k\ell} &= \alpha\, \varphi_{r,r}\, \delta_{k\ell} + \beta\, \varphi_{k,\ell} + \gamma\, \varphi_{\ell,k}
\end{aligned}
\tag{5.20}
$$

Upon putting

$$
\begin{aligned}
2\rho\, A_{k\ell mn} &= A_1(s_1,s_2)\delta_{k\ell}\delta_{mn} + A_2(s_1,s_2)\delta_{km}\delta_{\ell n} + A_3(s_1,s_2)\delta_{kn}\delta_{\ell m} \\
2\rho\, B_{k\ell mn} &= B_1(s_1,s_2)\delta_{k\ell}\delta_{mn} + B_2(s_1,s_2)\delta_{km}\delta_{\ell n} + B_3(s_1,s_2)\delta_{kn}\delta_{\ell m}
\end{aligned}
\tag{5.21}
$$

and

$$
\begin{aligned}
\lambda_v(s) &\equiv -\int_s^\infty ds_2 \int_0^\infty A_1(s_1,s_2)ds_1 \\
\mu_v(s) + \kappa_v(s) &\equiv -\int_0^\infty ds_2 \int_0^\infty A_2(s_1,s_2)ds_1 \\
\mu_v(s) &\equiv -\int_0^\infty ds_2 \int_0^\infty A_3(s_1,s_2)ds \\
a_k(s_1,s_2) &= \int_{s_1}^\infty A_k(s_3,s_2)ds_3 \\
b_k(s_1,s_2) &= \int_{s_1}^\infty B_k(s_3,s_2)ds_3
\end{aligned}
\tag{5.22}
$$

the dissipative parts of the stress and couple stress take the forms

$$
{}_D t_{k\ell} = \int_0^\infty [\lambda_v(s)\, \Delta_{rr}(t-s)\, \delta_{k\ell} + [\mu_v(s) + \epsilon_v(s)]\, \Delta_{k\ell}(t-s) + \mu_v(s)\, \Delta_{\ell k}(t-s)]ds
$$

(5.23)

$$
{}_D m_{k\ell} = \int_0^\infty [\alpha_v(s)\, \nu_{r,r}(t-s)\, \delta_{k\ell} + \beta_v(s)\, \nu_{k,\ell}(t-s) + \gamma_v(s)\, \nu_{\ell,k}(t-s)]ds
$$

The free energy ψ takes the form

$$
\psi = {}_E\psi + {}_D\psi \tag{5.24}
$$

where $\rho\, {}_E\psi$ is identical to (3.17) and

$$
\begin{aligned}
\rho\, {}_D\psi = \frac{1}{2} \int_0^\infty\!\!\int_0^\infty [& A_1(s_1,s_2)\, \bar{\epsilon}_{kk}(s_1)\, \bar{\epsilon}_{\ell\ell}(s_2) + A_2(s_1,s_2)\, \bar{\epsilon}_{k\ell}(s_1)\, \bar{\epsilon}_{k\ell}(s_2) \\
& + A_3(s_1,s_2)\, \bar{\epsilon}_{k\ell}(s_1)\, \bar{\epsilon}_{\ell k}(s_2) + B_1(s_1,s_2)\, \bar{\varphi}_{k,k}(s_1)\bar{\varphi}_{\ell,\ell}(s_2) \\
& + B_2(s_1,s_2)\, \bar{\varphi}_{k,\ell}(s_1)\, \bar{\varphi}_{k,\ell}(s_2) \\
& + B_3(s_1,s_2)\, \bar{\varphi}_{k,\ell}(s_1)\, \bar{\varphi}_{\ell,k}(s_2)]ds_1 ds_2
\end{aligned} \tag{5.25}
$$

and the dissipation inequality (5.14) becomes

$$\frac{1}{2\theta}\int_0^\infty\int_0^\infty [a_1(s_1,s_2)\,\dot{\epsilon}_{kk}(s_1)\,\dot{\epsilon}_{\ell\ell}(s_2) + a_2(s_1,s_2)\,\dot{\epsilon}_{k\ell}(s_1)\,\dot{\epsilon}_{k\ell}(s_2)$$

$$+ a_3(s_1,s_2)\,\dot{\epsilon}_{k\ell}(s_1)\,\dot{\epsilon}_{\ell k}(s_2)$$

$$+ b_1(s_1,s_2)\,\dot{\varphi}_{k,k}(s_1)\,\dot{\varphi}_{\ell,\ell}(s_2) + b_2(s_1,s_2)\,\dot{\varphi}_{k,\ell}(s_1)\,\dot{\varphi}_{k,\ell}(s_2)$$

$$+ b_3(s_1,s_2)\,\dot{\varphi}_{k,\ell}(s_1)\,\dot{\varphi}_{\ell,k}(s_2)]ds_1 ds_2 \geq 0 \tag{5.26}$$

Compatible with this approximation, for the temperature, we also take

$$\eta = -\frac{\partial_E \psi}{\partial\theta} \tag{5.27}$$

Equations (5.17), (5.20) and (5.23) are the final forms of the stress and couple stress for the linear theory of isotropic micropolar viscoelasticity. The dissipation inequality (5.26) places restrictions on the viscous moduli appearing in (5.23), since both a_K, b_K and λ_v, μ_v, κ_v, α_v, β_v and γ_v are expressed in terms of the original six viscous moduli A_K and B_K (see Eq. 5.22).

The above results may be summarized in the form of

Theorem 3. Isotropic micropolar viscoelastic materials defined by (5.1) that are linear in the past history of motion are thermodynamically admissible if and only if (a) the free energy ψ has the form (5.24), (5.25); (b) the stress and couple stress are given by (5.17), (5.20) and (5.23); (c) these materials are non-heat conducting and possess entropy given by (5.9) and (d) the

<u>memory functions are restricted by the entropy inequality</u> (5.26).

Corresponding to Theorem 2 we also note that the dissipative parts of <u>stress, couple stress and the free energy vanish with the strain measures</u> $\dot{\epsilon}_{k\ell}$ <u>and</u> $\dot{\varphi}_{k,\ell}$. The difference between the micropolar viscoelasticity and the classical Boltzmann-Volterra theory of viscoelasticity arise from the non-symmetrical nature of stress and the entirely new concept of memory dependent couple stress. Both are affected by the memory of past global motions and the micro-rotations. It is also clear that the classical Boltzmann-Volterra theory of viscoelasticity is obtained as a special case of the present theory.

6. Micropolar Viscoelastic Fluids

The results of Section 5 can be used to obtain the constitutive equations of viscoelastic fluids. For this all we need is to replace $\epsilon_{k\ell}(t)$ by ρ^{-1} and drop $\varphi_{k,\ell}(t)$ from the expression of ψ_o. It then follows that

$$
\begin{aligned}
{}_E t_{k\ell} &= -\pi(t)\,\delta_{k\ell} \\
{}_E m_{k\ell} &= 0
\end{aligned}
\tag{6.1}
$$

where $\pi(t)$ is the pressure defined by

$$
\pi(t) = -\frac{\partial \psi_o}{\partial \rho^{-1}} \tag{6.2}
$$

The expressions (5.23) of ${}_D t$ and ${}_D m$ remain unchanged. For the incompressible fluids one replaces $\pi(t)$ by an unknown pressure $p(t)$ and adjoins an equation of incompressibility by

$$
v_{k,k} = 0 \tag{6.3}
$$

7. Indeterministic Couple Stress Theory

An indeterminate couple stress theory can be obtained by placing internal constraints on the micropolar motion. For this, as we have shown earlier [5], all we need is to set

$$r_k = \varphi_k \tag{7.1}$$

where $\mathbf{r}$ is the rotation vector defined by

$$r_k = \frac{1}{2} \epsilon_{k\ell m} v_{m,\ell} \tag{7.2}$$

Using (7.1) in $(5.20)_1$ and $(5.23)_1$ we find that

$$t_{k\ell} = \lambda e_{rr} \delta_{k\ell} + (2\mu + \kappa) e_{k\ell} + \int_0^\infty [\lambda_v(s) \dot{e}_{rr}(t-s) + [2\mu_v(s) + \kappa_v(s)]\dot{e}_{k\ell}(t-s)]ds \tag{7.3}$$

which are identical to the stress constitutive equations of the classical Boltzmann-Volterra theory of viscoelasticity provided we replace μ by $\mu + \frac{\kappa}{2}$ and μ_v by $\mu_v + \kappa_v/2$. Using (7.3) in (2.3) we see that the displacement $\mathbf{u}$ drops out so that (2.3) determines the body couple $\boldsymbol{\ell}$. With this viewpoint the present theory goes into the classical theory.

There is, however, another interpretation: suppose that in (7.3) $t_{k\ell}$ is understood to be the symmetric part $t_{(k\ell)}$ of the stress tensor where

$$t_{(k\ell)} = \frac{1}{2}(t_{k\ell} + t_{\ell k})$$

and that the skew-symmetric part

$$t_{[k\ell]} = \frac{1}{2}(t_{k\ell} - t_{\ell k})$$

of the stress is calculated through (2.3), i.e.,

$$t_{[k\ell]} + \frac{1}{2} \epsilon_{rk\ell} m_{nr,n} - \rho(\ell_{[k\ell]} - j \ddot{u}_{[k,\ell]}) = 0 \qquad (7.4)$$

where

$$\ell_{[k\ell]} \equiv -\epsilon_{k\ell m} \ell_m \qquad (7.5)$$

Substituting (7.2) into $(5.20)_2$ and $(5.23)_2$ we get

$$m_{k\ell} = \frac{\beta}{2} \epsilon_{kmn} u_{n,m\ell} + \frac{\gamma}{2} \epsilon_{\ell mn} u_{n,mk} + \frac{1}{2}\int_0^\infty [\beta_v(s) \epsilon_{kmn} \dot{u}_{n,mk}(t-s)$$

$$+ \gamma_v(s) \epsilon_{\ell mn} \dot{u}_{n,mk}(t-s)]ds \qquad (7.6)$$

Through (7.4) and (7.6) we have

$$t_{[k\ell]} = \frac{\gamma}{4} \nabla^2 (u_{k,\ell} - u_{\ell,k}) + \rho(\ell_{[k\ell]} - j \ddot{u}_{[k,\ell]}) \qquad (7.7)$$

Using this we calculate

$$t_{k\ell,k} = t_{(k\ell),k} + t_{[k\ell],k}$$

where for the first term on the right we employ (7.3). Equations (2.2) now give

$$
\begin{aligned}
&(\lambda + \mu + \frac{\kappa}{2} + \frac{\gamma}{4}\nabla^2)\, u_{k,\ell k} + (\mu + \frac{\kappa}{2} - \frac{\gamma}{4}\nabla^2)\, u_{\ell,kk} \\
&\quad + \int_0^\infty \{[\lambda_v(s) + \mu_v(s) + \frac{\kappa_v(s)}{2} + \frac{\gamma_v(s)}{4}\nabla^2]\dot{u}_{k,\ell k}(t-s) \\
&\quad + [\mu_v(s) + \frac{\kappa_v(s)}{2} - \frac{\gamma_v(s)}{4}\nabla^2]\dot{u}_{\ell,kk}(t-s)\}ds \qquad (7.8) \\
&\quad + \frac{\rho}{2}(\ell_{k\ell,k} - \ell_{\ell k,k}) + \rho f_\ell = \rho(1-\frac{j}{2}\nabla^2)\ddot{u}_\ell + \frac{\rho j}{2}\ddot{u}_{k,\ell k}
\end{aligned}
$$

where ∇^2 is the Laplacian operator in rectangular coordinates.

Equations (7.8) are the field equations of the indeterminate couple stress theory of viscoelasticity. We do not place, however, much faith in the indeterminate couple stress theory.

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