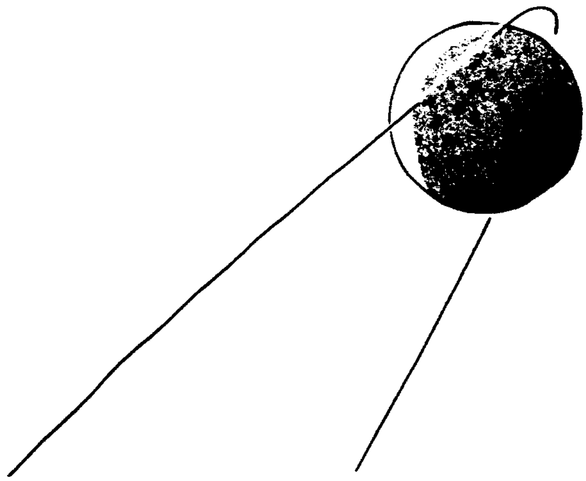

OPTIMAL STOCHASTIC CONTROL THEORY APPLIED TO INTERPLANETARY GUIDANCE

BY

JAMES F. JORDAN, JR.

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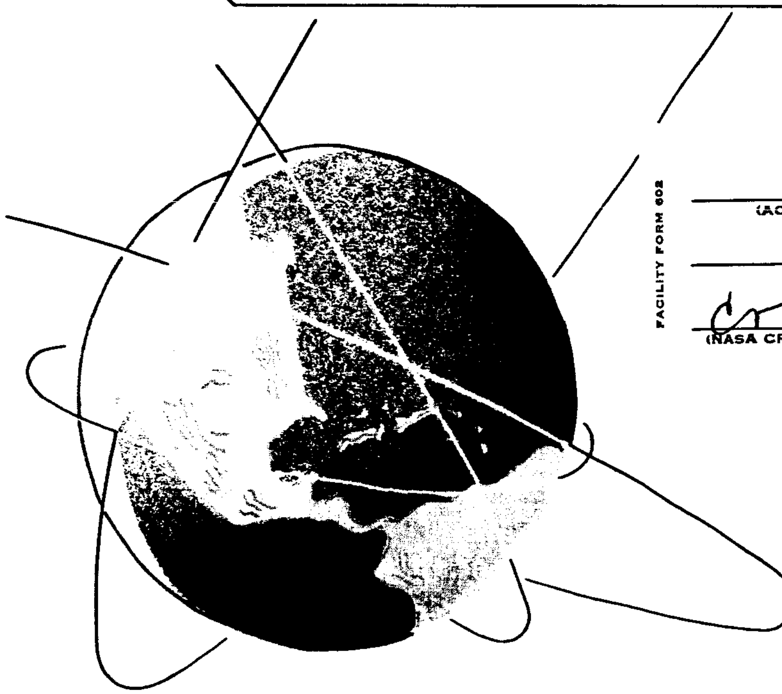

ENGINEERING MECHANICS RESEARCH LABORATORY

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PREFACE

As engineers have conceived and designed space missions from Earth to other planets in the solar system, they have found it necessary to develop new areas of technology and utilize techniques from these areas in conjunction with classical scientific methods. For example, in the study of the guidance of outer space vehicles, engineers are faced with two primary problems: (1) that of determining the motion of a space vehicle under the gravitational influence of the surrounding celestial bodies, and (2) that of determining a method for guiding the spacecraft such that the given mission objectives are met in the best manner. The first of these problems can be handled with the classical methods of celestial mechanics, but the solution to the second problem requires concepts from the relatively new field of optimal control theory.

The results of space missions which have been performed at the time of this writing indicate that there is a third problem in space guidance (3), which is as important as the other two. The problem is that of guiding a space vehicle accurately in the presence of disturbances, acting on the spacecraft, which do not obey strict deterministic laws. The existance of such disturbances is indicated by the inability of engineers in predicting accurately spacecraft trajectories in past space missions.

One method of dealing with such random disturbances is to model the behavior of a disturbance as a stochastic process, and determine the statistics of the process by experiments made a priori to the space flight.

The techniques of optimal control theory and celestial mechanics can then be used along with the probabilistic concepts from the theory of stochastic processes in order to design a space vehicle guidance procedure which takes into account the expected effects of the disturbance process on the spacecraft. A control procedure developed in this manner is called an optimal stochastic control.

The purpose of this dissertation is to analyze the general space guidance problem (1, 2, 3) and develop an optimal stochastic control program for interplanetary spacecraft guidance. It is hoped that the investigation is a reasonable integration of the disciplines of stochastic processes, optimal control theory, and celestial mechanics, into one research effort.

This dissertation could not have been realized without innumerable contributions from several persons. The author wishes to thank Dr. B. D. Tapley of The University of Texas for supervising the research and making many helpful suggestions regarding the manuscript preparation. He also wishes to thank Dr. L. G. Clark, Dr. P. L. Odell, and Dr. E. J. Prouse for serving on the dissertation committee. The author is indebted to E. L. Davis, Jr., and E. H. Brock of the Manned Spacecraft Center for providing an academic environment in which the numerical studies could be performed. He is also indebted to Dr. J. M. Lewallen for his suggestions regarding the numerical work. The author would like to express his gratitude to R. D. Witty of Lockheed Electronics Company for his dedicated assistance with the computer programs and would also like to express his gratitude to J. Rodriquez of The University of Texas for his help with the trajectory simulation. The author would like to thank C. G. Pfeiffer of the Jet Propulsion Laboratory for his helpful suggestions during the initial phases of the investigation.

Finally, the author would like to express his appreciation of his parents for their understanding and patience during the time of his graduate studies.

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CHAPTER 1

INTRODUCTION

Preliminary Remarks

During recent years there has been a remarkable growth of interest in problems associated with the optimal control of nonlinear dynamic systems. A great part of the motivation toward developing techniques in control theory lies in its important applications to space guidance theory. Since space missions, in general, require the use of great amounts of energy and require long times of travel, the necessity of performing spacecraft guidance manuvers in an optimal manner is paramount. Most of the effort which has gone into the development of control theory as a tool for the astrodynamicist has been concerned with deterministic dynamic models. This assumption may prove to be too idealistic for reasonable applicability of optimal control theory to space guidance problems. The purpose of this work is to examine the effects of noise on a nonlinear dynamic system, to extend the variational techniques of optimal deterministic control theory to the control of a stochastic dynamic system, and to apply the results to a simulated interplanetary transfer guidance problem. The theoretical results will be derived for a general nonlinear multidimensional dynamic system. In the notation used, vector quantities will be subscripted, and repeated subscripts will imply summation unless stated otherwise. Numerical results are obtained for the example problem by using a digital computer.

Deterministic Control Theory

Deterministic control theory is concerned with the control of dynamic systems whose motion is described by the set of nonlinear differential equations

$$\dot{x}_i(t) = f_i(x, u, t) \qquad i = 1, \dots n \tag{1.1}$$

The $x_i(t)$ are components of the n dimensional state of the system and the $u_i(t)$, $i = 1, \dots, m$ are components of the m dimensional control, where $m \leq n$. If the control $u_i(t)$ is to be optimal, in some sense, over some time interval, $t_0 \leq t \leq t_f$, a performance index functional of the state and control, i.e.,

$$I[u] = \int_{t_0}^{t_f} f_{n+1}(x, u, t)\, dt \tag{1.2}$$

must be extremized subject to the constraints that Equation (1.1) be satisfied at all points in time for $t_0 \leq t \leq t_f$, and that the state $x_i(t)$ be specified at t_0 and t_f, i.e.,

$$\begin{aligned} x_i(t_0) &= x_{i0} \\ x_i(t_f) &= x_{if} \end{aligned} \tag{1.3}$$

The extremization of the functional given in Equation (1.2) may be carried out by a calculus of variations technique. A formal development of the variational method as applied to the optimal control problem is presented in Appendix A.

Stochastic Control Theory

Stochastic control theory is concerned with control of dynamic systems which in some sense are random. The motion of such a system can be described by the following set of differential equations

$$\dot{x}_i(t) = f_i(x(t), u(t), \eta(t), t) \tag{1.4}$$

where $\eta_i(t)$ may be a multidimensional random process which could be caused by one or more of the following phenomena:

a. unknown parameters in the dynamic model

b. unpredictable external disturbances

c. random noise in the controls

d. uncertainties in initial conditions.

If the control $u_i(t)$ is to be optimal, then it is desired that the control be selected to extremize the functional given in Equation (1.2). However, due to the presence of the noise $\eta_i(t)$ in the equations of motion (1.4), the functional given in Equation (1.2) is a random quantity, whose value depends on the particular noise function which is manifested during the time interval $t_0 \le t \le t_f$. Since it is not possible to predict the value of the functional given in Equation (1.2) before the occurrence of the noise, a control which extremizes the functional cannot be realized a priori. It is therefore desirable that the control be selected to extremize some deterministic quantity associated with the performance index functional. Several authors, among them Kushner (Ref. 1), Lass (Ref. 2), Wonham (Ref. 3), and Tung (Ref. 4), have suggested that the control be selected to extremize the statistical average, or the expected value, of the functional given in Equation (1.2), i.e.,

$$I[u] = E \int_{t_0}^{t_f} f_{n+1}(x, u, t)\, dt \tag{1.5}$$

where E is the expected value operator, and the expected value is taken with respect to the random process $\eta_i(t)$. The functional given in Equation (1.5) can be thought of as the average of the functional given in Equation (1.2) over a great number of trials. It is reasonable that the control which extremizes an average over many trials will yield an approximate extremal in a particular case.

Previous studies have been made (see Wonham (Ref. 3), and Tung (Ref. 4)) in which an optimal stochastic control is computed by means of the dynamic programming method. The condition which the optimal control must satisfy takes the form of a partial differential equation which is very difficult to solve. Kushner (Ref. 5, 6, 7, 8), and Lass (Ref. 2) have presented a calculus of variations approach for determining the optimal stochastic control, which is analogous to the deterministic calculus of variations method. Kushner (Ref. 1) has applied the approach to a nonlinear control problem in which additive external noise occurs in the dynamic process at discrete points in time.

The Problem To Be Studied

In this study, stochastic systems which contain small continuous additive noise in the controls, as well as small uncertainties in the initial conditions, will be considered. The conditions which the control must satisfy for optimality of the functional given in Equation (1.5) are derived by using the stochastic variational approach. The variation of the functional given in Equation (1.5) is carried out with the constraints that the equation of motion (1.4) must be satisfied at all points of time in the controlling interval, and that the expected value of the state, $x_i(t)$, is specified at t_0 and t_f, i.e.,

$$E\,[x_i(t_0)] = x_{i0}$$

$$E\,[x_i(t_f)] = x_{if} \tag{1.6}$$

Application To A Space Guidance Problem

The example picked to illustrate the theory is that of a continuously thrusting ion-engine space vehicle, traveling on a minimum time Earth-to-Mars transfer. The state of the system consists of the position and velocity coordinates of the spacecraft, and the controls are the magnitude of the engine thrust per unit mass and the thrust orientation angle. The thrust/mass magnitude is considered as a control in the sense that it is a parameter in the forcing function of the equations of motion. The thrust orientation angle is a true control in the sense that it can be varied to guide the spacecraft. It should be noted that noise is assumed to occur in the thrust/mass magnitude and/or thrust orientation angle. The vehicle model is simplified to a point mass, and the equations of motion exclude all effects other than those due to the engine and the gravitational attraction of the sun. The orbital planes of the Earth and Mars are assumed to coincide, and the spacecraft trajectory as well as the noise errors are assumed to occur in that plane. Therefore the analysis is carried out in two dimensions.

Outline of the Study

In Chapter 2, a model for the disturbing noise is developed, and its applicability to the controls of a space vehicle is discussed. The main difference between the noise model assumed in this work and the noise model used in previous studies is that for this problem noise which is autocorrelated in time will be considered. It is felt that time correlated noise is more representative of physical phenomena than uncorrelated or "white" noise.

Chapter 3 is concerned with the effect of autocorrelated noise on an optimal deterministic trajectory. The effects are examined by deriving differential equations which describe the time histories of the means and standard deviations of the state errors resulting from the perturbing noise. These means and standard deviations are computed for the Earth-Mars transfer problem, and the results are compared with the results obtained by taking averages over several Monte Carlo simulated trajectories.

In Chapter 4 the optimal stochastic control is found by extremizing a functional of the type given in Equation (1.5) by applying a stochastic calculus of variations technique. The solution takes the form of an expected value over the necessary conditions which result from the variational problem. Then the stochastic solution is expanded about the deterministic necessary conditions and a corrective optimal control program is derived. Since the perturbing noise is assumed small, the expansions are carried out only to second order. The results obtained by applying the control program to the Earth-Mars transfer are presented at the end of the chapter.

In Chapter 5, the problem of finding the optimal stochastic control, conditioned on information about the state of the system gained during flight, is treated. A range-rate type observation, which contains additive error noise, is made at discrete points in time, and conditional means of the system state components are computed. The scheme includes the computation of conditional means of the noise occurring in the system at the times of observations, as well as the computation of conditional means of the state components. An optimal control correction is made after each observation. This closed-loop control scheme reduces the standard deviations of the state components while increasing the degree of optimality of the control.

A summary of the results and a list of possible extensions to this work appear in Chapter 6.

CHAPTER 2

FORMULATION OF THE NOISE MODEL

Characteristics of the Perturbing Noise

Consider the dynamic system which obeys the differential equations of motion

$$\dot{x}_i = f_i(x, u, \eta, t) \qquad i = 1, \ldots, n \tag{2.1}$$

The $\eta_i(t)$ are components of multidimensional additive noise in the controls $u_i(t)$. In order to analyze, in a precise manner, the behavior of a dynamic system such as that described by Equation (2.1), some of the statistical properties of the noise $\eta_i(t)$ must be known. Since, in the case of noise occurring in the controls of continuously thrusting space vehicle, these properties are not known, certain intuitive assumptions about $\eta_i(t)$ must be made. It is desirable that $\eta_i(t)$ possess the following properties:

1. $\eta_i(t)$ should possess a unimodal bell-shaped probability density function. This implies that small values of the noise are expected to occur more often than large values.
2. $\eta_i(t)$ should be unbiased, i.e., the statistical average of the noise should tend to zero.
3. $\eta_i(t)$ should be autocorrelated in time. This is desirable since some control noise could be internally generated by mechanical failures.
4. $\eta_i(t)$ should be a stationary process. This implies that the variance of the noise is expected to remain constant in time.

The Ornstein-Uhlenbeck Stochastic Process

A stochastic process which fits the preceding description was introduced by Ornstein and Uhlenbeck as a model for the velocity of a particle undergoing a Brownian motion (see Ref. 9). Let $\eta(t)$ be a scalar example of this process. The statistical properties of the Ornstein Uhlenbeck (O.U.) process are defined by the following relations:

1. The probability density function is

$$f(\eta(t)) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{\eta(t)}{\sigma}\right]^2} \tag{2.2}$$

where σ is the standard deviation of the process. From Equation (2.2), it follows that $f(\eta(t))$ is unimodal and bell-shaped.

2. $$E[\eta(t)] = \int_{-\infty}^{\infty} \eta(t)\, f(\eta(t))\, dt = 0 \tag{2.3}$$

The O. U. process is unbiased.

3. $$E[\eta(t_1)\,\eta(t_2)] = R(t_1, t_2) = \sigma^2 e^{-\beta|t_2-t_1|} \tag{2.4}$$

The process is exponentially autocorrelated in time, and since $R(t_1, t_2)$ depends only on the time difference (t_2-t_1), $\eta(t)$ is stationary.

It will be instructive to examine further the properties of the O. U. process and its effect on a simple linear dynamic system. The O. U. process obeys a Langevin equation of the following type

$$\dot{\eta}(t) + \beta\eta(t) = w(t) \tag{2.5}$$

where $w(t)$ is Gaussian white noise, that is,

$$E\,[w(t)\,] = 0 \tag{2.6}$$

$$E\,[w(t_1)\,w(t_2)\,] = Q\,\delta(t_2-t_1) \tag{2.7}$$

where Q is the variance of $w(t)$ and $\delta(t_2-t_1)$ is the Dirac delta function. A solution of the Langevin equation can be written, in terms of a stochastic integral, in the form

$$\eta(t) = e^{-\beta(t-t_0)}\,\eta(t_0) + \int_{t_0}^{t} e^{-\beta(t-\tau)}\,w(\tau)\,d\tau \tag{2.8}$$

It will be helpful to digress for a moment from the current line of reasoning, in order to develop an important property of the stochastic integral. It is known (see Ref.10) that if x is a random variable distributed according to the density function $f(x)$ and $g(x)$ is some function of x, then

$$E\,[g(x)\,] = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx \tag{2.9}$$

If then $g(t)$ is some functional of the random process, $x(\tau)$ say

$$g(t) = \int_{0}^{t} h(x(\tau))\,d\tau \tag{2.10}$$

then $g(t)$ depends on the entire function $x(\tau)$, $0 \leq \tau \leq t$, i.e.,

$$g(t) = g(x(\tau_1), x(\tau_2), \ldots, x(\tau_i), \ldots) \tag{2.11}$$

where τ_i runs over all points in time. The expected value of $g(t)$ can then take the form

$$E\,[g(t)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^t h(x(\tau)d\tau f(x(\tau_1), x(\tau_2), \ldots)dx(\tau_1)dx(\tau_2)\ldots \tag{2.12}$$

Now, if the integration process is visualized as the limit of a sum, Equation (2.12) can be expressed as follows:

$$E[g(t)] = \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} \lim_{n\to\infty} \sum_{i=1}^{n} h(x(\tau_i))\Delta\tau_i f(x(\tau_1),\ldots)dx(\tau_1)\ldots \tag{2.13}$$

Taking the summation outside of the integration over the random variables $x(\tau_1)$, $x(\tau_2)$, ..., will lead to

$$E[g(t)] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} h(x(\tau_i))f(x(\tau_1),\ldots)dx(\tau_1)\ldots \right] \Delta\tau_i \tag{2.14}$$

Now, on converting the summation back to an integral, the following result is obtained

$$E\,[g(t)] = \int_0^t E\, h(x(\tau))\, d\tau \tag{2.15}$$

Thus the expected value operator and the stochastic integral commute. This property will be used extensively in discussions given in Chapters

3, 4, and 5. It should also be noted that if Equation (2.15) is differentiated with respect to t the following expression is obtained:

$$\frac{d}{dt} E[g(t)] = E\, h(x(t)) = E\left[\frac{dg}{dt}\right] \tag{2.16}$$

Thus the expected value operator and the derivative commute.

Returning to Equation (2.8), and using the notation

$$E(\cdot) = \overline{(\cdot)}$$

it follows that

$$\overline{\eta}(t) = e^{-\beta(t-t_0)}\overline{\eta}(t_0) + \int_{t_0}^{t} e^{-\beta(t-\tau)}\overline{w}(\tau)\, d\tau \tag{2.17}$$

Now since $\overline{w}(\tau) = 0$ Equation (2.7) reduces to

$$\overline{\eta}(t) = e^{-\beta(t-t_0)}\overline{\eta}(t_0) \tag{2.18}$$

Thus, if for any O.U. process $\overline{\eta}(t_0) = 0$, then $\overline{\eta}(t) = 0$ for all $t \geq t_0$.

Now consider the autocorrelation properties of the O. U. process. Note that $\eta(t_1)\eta(t_2)$ can be expressed as follows:

$$\begin{aligned}\eta(t_1)\eta(t_2) = {} & e^{-\beta(t_2-t_0)} e^{-\beta(t_1-t_0)} \eta(t_0)\eta(t_0) + \\ & \int_{t_0}^{t_1} e^{-\beta(t_1-\tau)} \eta(t_0)w(\tau)d\tau + \int_{t_0}^{t_2} e^{-\beta(t_2-\rho)} \eta(t_0)w(\rho)d\rho + \\ & \int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-\beta(t_1-\tau)} e^{-\beta(t_2-\rho)} w(\rho)w(\tau)\, d\rho\, d\tau\end{aligned} \tag{2.19}$$

By taking the expected value of Equation (2.19) and imposing the condition that $E\,(w(\tau)n(t_0)) = 0$, the following expression is obtained.

$$E\,[n(t_1)n(t_2)] = R(t_1, t_2) = e^{-\beta(t_1-t_0)-\beta(t_2-t_0)} R(t_0,t_0) +$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-\beta(t_1-\tau)-\beta(t_2-\rho)} Q\,\delta(t_2-t_1)\,d\tau d\rho \qquad (2.20)$$

Carrying out the integration of Equation (2.20) leads to the following expression,

$$R(t_1,t_2) = \frac{Q}{2\beta}\,e^{-\beta|t_2-t_1|} + \left[\,R(t_0,t_0) - \frac{Q}{2\beta}\,\right] e^{-\beta(t_1+t_2)+2\beta t_0} \qquad (2.21)$$

Stationarity of the process, i.e., $R(t_1,t_2) = R(|t_2-t_1|)$ requires that $R(t_0,t_0) = \frac{Q}{2\beta}$ hence

$$R(t_1,t_2) = \frac{Q}{2\beta}\,e^{-\beta|t_2-t_1|} \qquad (2.22)$$

In view of Equation (2.4), Equation (2.22) leads to

$$R(t,t) = \sigma^2 = \frac{Q}{2\beta} \qquad (2.23)$$

Sample functions of the O. U. process can be generated with the aid of a normal random number generator. Consider the statistics of $n(t)$ when $n(t_0)$ is known. It follows then, that

$$E\,[n(t)|n(t_0)] = e^{-\beta(t-t_0)} n(t_0) \qquad (2.24)$$

and

$$E\,[(\eta(t) - E(\eta(t)|\eta(t_0)))^2|\eta(t_0)\,] = \int_{t_0}^{t}\int_{t_0}^{t} e^{-\beta(t-\tau)}e^{-\beta(t-\rho)}Q\,\delta(\tau-\rho)d\rho d\tau \tag{2.25}$$

Carrying out the integration will lead to

$$\sigma^2|\eta(t_0) = \frac{Q}{2\beta}\left[1-e^{-2\beta(t-t_0)}\right] \tag{2.26}$$

Hence, the conditional density function is given as follows (see Ref. 9)

$$f(\eta(t)|\eta(t_0)) = \frac{1}{\sqrt{2\pi}\;\sigma|\eta(t_0)}\,e^{-\frac{1}{2}\left[\frac{\eta(t)-\overline{\eta(t)}|\eta(t_0)}{\sigma|\eta(t_0)}\right]^2} \tag{2.27}$$

If the output of a random number generator, x_i, is independently distributed according to the density function

$$f(x_i) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x_i)^2} \tag{2.28}$$

Then a sample function $\eta(t)$ can be discretely generated by the recursion relation

$$\eta(t_{i+1}) = x_{i+1}\,\sigma\,\sqrt{1-e^{-2\beta(t_{i+1}-t_i)}} + \eta(t_i)\,e^{-\beta(t_{i+1}-t_i)} \tag{2.29}$$

where $\eta(t_0) = x_0\sigma$. Figure 1 illustrates such a sample function, where the numerical results were generated with Equation (2.29) for the values $\sigma = 1$ and $\beta = .01$.

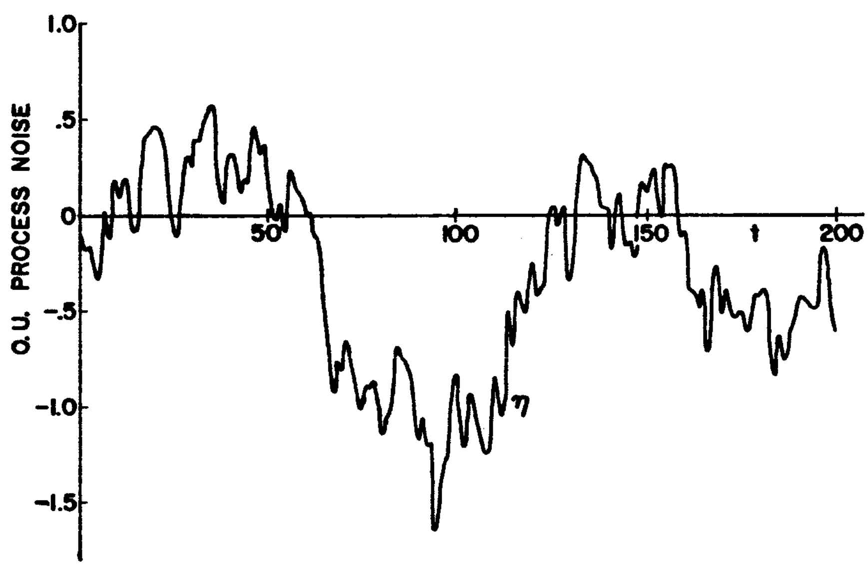

Figure 1. A Simulation of the Ornstein-Uhlenbeck Stochastic Process

(σ = 1, β = .01)

Application To A Simple Linear Dynamic System

The effect on a dynamic system of a random noise process such as the O. U. process can be shown by the following example. Consider the following system of equations, in which a particle of unit mass undergoes one dimensional motion under the influence of an O. U. process acceleration. Such motion is described by the following equations,

$$\dot{v} = \eta(t)$$
$$\dot{x} = v \tag{2.30}$$

where the initial conditions are specified as

$$v(t_0) = 0$$
$$x(t_0) = 0$$
$$t_0 = 0$$

The solutions for $v(t)$ and $x(t)$ can be expressed as integrals which depend on the stochastic forcing function, i.e.,

$$\begin{aligned} v(t) &= \int_0^t \eta(\tau)\, d\tau \\ x(t) &= \int_0^t \int_0^\tau \eta(\rho)\, d\rho\, d\tau \end{aligned} \tag{2.31}$$

The expected values of Equations (2.31) can be written as follows:

$$\begin{aligned} E\,[v(t)] &= \int_0^t E(\eta(\tau))\, d\tau = 0 \\ E\,[x(t)] &= \int_0^t \int_0^\tau E(\eta(\rho))\, d\rho\, d\tau = 0 \end{aligned} \tag{2.32}$$

Consider now

$$E\,[v(t_1)v(t_2)] = E\int_0^{t_1}\int_0^{t_2} n(\rho)n(\tau)\,d\rho\,d\tau \tag{2.33}$$

If it is assumed that $t_2 > t_1$ then Equation (2.33) can be expressed as

$$E\,[v(t_1)v(t_2)] = \int_0^{t_2}\left[\int_0^{\tau}\sigma^2 e^{-\beta(\tau-\rho)}\,d\rho + \int_{\tau}^{t_1}\sigma^2 e^{-\beta(\rho-\tau)}\,d\rho\right]d\tau \tag{2.34}$$

Now, carrying out the integration in Equation (2.34) leads to the following expression

$$E\,[v(t_1)v(t_2)] = \frac{\sigma^2}{\beta^2}\left[e^{-\beta t_2} + e^{-\beta t_1} + e^{\beta(t_1+t_2)} + 2\beta\max(t_1,t_2) - 2\right] \tag{2.35}$$

The variance of $v(t)$ is

$$E\,[v(t)^2] = \frac{2\sigma^2}{\beta^2}\left[e^{-\beta t} + \beta t - 1\right] \tag{2.36}$$

In a similar manner, the following expressions can be obtained

$$E\,[v(t)n(t)] = \frac{\sigma^2}{\beta}\left[1 - e^{-\beta t}\right] \tag{2.37}$$

$$E\,[x(t)v(t)] = \frac{\sigma^2}{\beta^3}\left[1-\beta t + \frac{1}{2}(\beta t)^2 - e^{-\beta t}\right] \tag{2.38}$$

$$E\,[x(t)x(t)] = \frac{2\sigma^2}{\beta^4}\left[e^{-\beta t} + \frac{1}{6}(\beta t)^3 - \frac{1}{2}(\beta t)^2 + \beta t - 1\right] \tag{2.39}$$

the standard deviations of $v(t)$ and $x(t)$ are now defined respectively by

$$\sigma_v = \left(E\,[v(t)^2]\right)^{\frac{1}{2}} \tag{2.40}$$

$$\sigma_x = \left(E\,[x(t)^2]\right)^{\frac{1}{2}}$$

Figures 2 and 3 show the velocity and displacement histories which result as a consequence of the O. U. acceleration process shown in Figure 1. The standard deviations are shown also in the figures. It should be noted that while the mean values of $v(t)$ and $x(t)$, given in Equation (2.32), are zero for all time, the standard deviations increase without bound.

Summary

The motivation for this chapter lies in the justification for selecting the O. U. process as the noise process to be dealt with in the following chapters. The process is seen to satisfy the intuitive criteria designated for random disturbing phenomena, and appears to have a reasonable effect on a simple physical system. It should be noted that by adjusting the parameter β in Equation (2.4) one can simulate near-white noise (in the case of large β) and noise which is constant in time (small β). This flexibility increases the desirability of the O. U. process model.

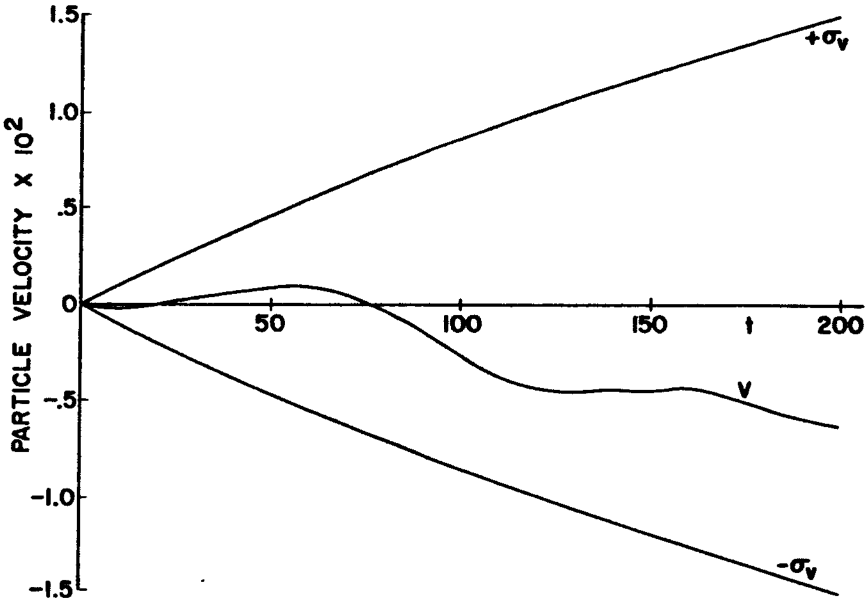

Figure 2. Velocity Time History Resulting from an O.U. Process Acceleration

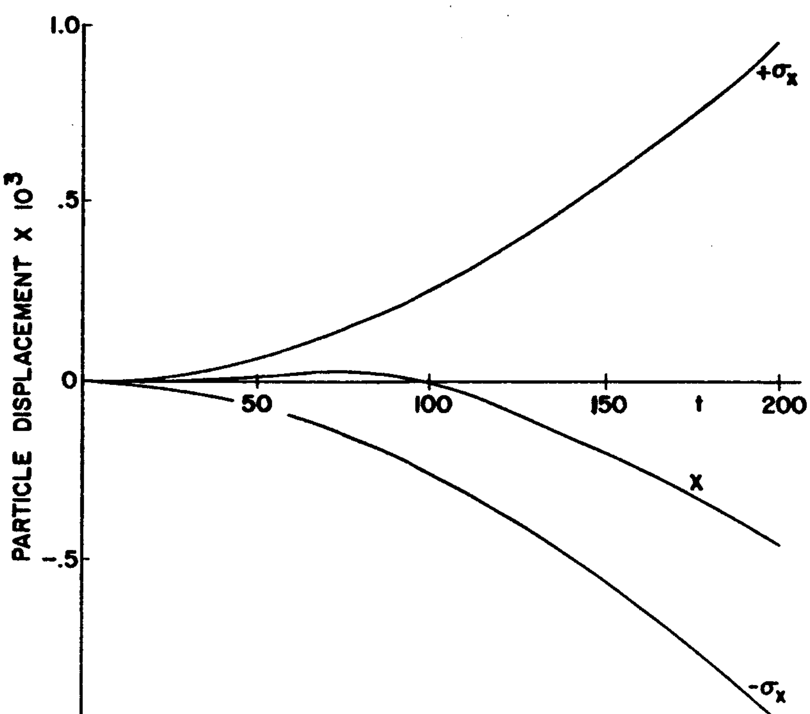

Figure 3. Displacement Time History Resulting from an O.U. Process Acceleration

CHAPTER 3

THE EFFECTS OF NOISE ON AN OPTIMAL DETERMINISTIC TRAJECTORY

Theoretical Development

The next step in the study of optimal stochastic control is to look at the effects of a perturbing noise such as the O. U. process on a multidimensional nonlinear dynamic system. Consider the solution of an optimal deterministic control problem which can be written in the form

$$\dot{x}_i^* = f_i(x^*, u^*, t) \tag{3.1}$$

where the $*$ designates the optimal deterministic trajectory. Suppose that the controls $u_i^*(t)$ are perturbed by an additive multidimensional version of the O. U. process developed in Chapter 2, i.e., the actual control input to the system is:

$$u_i(t) = u_i^*(t) + \eta_i(t) \tag{3.2}$$

where

$$E[\eta_i] = 0 \tag{3.3}$$

$$E[\eta_j(t_1)\,\eta_k(t_2)] = \sigma_{jk}^2\, e^{-\beta_{jk}|t_2-t_1|} \qquad (j,\ k \text{ not summed})$$

The coefficient σ_{jk}^2, $j = 1,\ldots, m$, $k = 1, \ldots, m$ is a covariance component of the multidimensional $\eta_i(t)$ process, and β_{jk} is the time correlation coefficient associated with the σ_{jk}^2 component of the covariance. It is assumed that the multidimensional noise is not cross-correlated, i.e., $\sigma_{ij} = 0$ for $i \neq j$.

The state, resulting from the noisy control $u_i(t)$, obeys the differential equation

$$\dot{x}_i = f_i(x, u, t) \tag{3.4}$$

An ensemble of stochastic trajectories is implied by Equation (3.4). Consider the Taylor series expansion of one of these trajectories about the optimal deterministic trajectory described by Equation (3.1).

$$\begin{aligned} \dot{x}_i &= f_i(x^*, u^*, t) + \frac{\partial f_i^*}{\partial x_j}(x_j - x_j^*) + \frac{\partial f_i^*}{\partial u_j}(u_j - u_j^*) \\ &+ \frac{1}{2}\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_j - x_j^*)(x_k - x_k^*) + \frac{\partial^2 f_i^*}{\partial x_j \partial u_k}(x_j - x_j^*)(u_k - u_k^*) \\ &+ \frac{1}{2}\frac{\partial^2 f_i^*}{\partial u_j \partial u_k}(u_j - u_j^*)(u_k - u_k^*) + \ldots \end{aligned} \tag{3.5}$$

Now introduce the notation

$$x_j - x_j^* = \delta x_j$$

$$\frac{\partial f_i^*}{\partial x_j} = f_{ix_j} \qquad \frac{\partial f_i^*}{\partial u_j} = f_{iu_j}$$

$$\frac{\partial^2 f_i^*}{\partial x_j \partial x_k} = f_{ix_jx_k} \quad \frac{\partial^2 f_i^*}{\partial x_j \partial u_k} = f_{ix_ju_k} \quad \frac{\partial^2 f_i^*}{\partial u_j \partial u_k} = f_{iu_ju_k} \tag{3.6}$$

By rewriting Equation (3.5) and subtracting out Equation (3.1), the following result is obtained

$$\delta\dot{x}_i = f_{ix_j}\,\delta x_j + f_{iu_j}(u_j - u_j^*) + \frac{1}{2} f_{ix_jx_k}\,\delta x_j\,\delta x_k + f_{ix_ju_k}\,\delta x_j (u_k - u_k^*) + \frac{1}{2} f_{iu_ju_k}(u_j - u_j^*)(u_k - u_k^*) + \ldots \tag{3.7}$$

Substitution of equations (3.2) into Equations (3.7) leads to

$$\delta\dot{x}_i = f_{ix_j}\,\delta x_j + f_{iu_j}\,\eta_j + \frac{1}{2} f_{ix_jx_k}\,\delta x_j\,\delta x_k + f_{ix_ju_k}\,\delta x_j \eta_k + \frac{1}{2} f_{iu_ju_k}\,\eta_j \eta_k + \ldots \tag{3.8}$$

Making use of linear system theory (see Ref. 11), a solution to Equation (3.8) may be written in the form

$$\delta x_i(t) = \Phi_{ij}(t,t_0)\,\delta x_i(t_0) + \int_{t_0}^{t} \Phi_{ij}(t,\tau) \left\{ f_{ju_k}(\tau)\,\eta_k(\tau) + \frac{1}{2} f_{jx_kx_\ell}\,\delta x_k\,\delta x_\ell + f_{jx_ku_\ell}\,\delta x_k\,\eta_\ell + \frac{1}{2} f_{ju_ku_\ell}\,\eta_k \eta_\ell + \ldots \right\} d\tau \tag{3.9}$$

where $\delta x_i(t_0)$ is the initial state deviation, and the coefficient $\Phi_{ij}(t,t_0)$ is an element of the so-called state transition matrix. Φ_{ij} satisfies the conditions that

$$\dot{\Phi}_{ij}(t,t_0) = f_{ix_k}(t)\, \Phi_{kj}(t,t_0)$$

$$\Phi_{ij}(t_0,t_0) = \delta_{ij} \begin{array}{l} = 1, \quad i = j \\ = 0, \quad i \neq j \end{array} \tag{3.10}$$

by resubstituting Equation (3.9) in for δx_k and δx_ℓ in Equation (3.9) the following expression is obtained.

$$\delta x_i(t) = \Phi_{ij}(t,t_0)\, \delta x_j(t_0) + \int_{t_0}^{t} \Phi_{ij}(t,\tau) \Bigg\{ f_{ju_k} \eta_k +$$

$$\frac{1}{2} f_{jx_kx_\ell} \left[\Phi_{km}(\tau,t_0)\, \delta x_m(t_0) + \int_{t_0}^{\tau} \Phi_{kp}(\tau,\rho)\, f_{pu_m} \eta_m d\rho \right]$$

$$\left[\Phi_{\ell n}(\tau,t_0)\, \delta x_n(t_0) + \int_{t_0}^{\tau} \Phi_{\ell q}(\tau,\S)\, f_{qu_n} \eta_n\, d\S \right] + \tag{3.11}$$

$$f_{jx_ku_\ell} \left[\Phi_{km}(\tau,t_0)\, \delta x_m(t_0) + \int_{t_0}^{\tau} \Phi_{kp}(\tau,\rho)\, f_{pu_m} \eta_m\, d\rho \right] \eta_\ell + \frac{1}{2} f_{ju_ku_\ell} \eta_k \eta_\ell \Bigg\} d\tau$$

As stated in the introduction, the analysis will be carried out under the assumption that the variance in the perturbing noise is small. Since $\eta_i(t)$ is assumed to be Gaussian, and $E\,[\eta_i(t)] = 0$, it follows that

$$E\,[\eta_i(t)^{2j-1}] = 0 \qquad j = 1, 2, \ldots.$$

$$E\,[\eta_i(t)^{2j}] = 1.3 \ldots. (2j-1)\, [E(\eta_i(t)^2)]^j \tag{3.12}$$

The first condition implies that, for a normal distribution, all odd moments about the mean vanish, while the second condition implies that all even moments about the mean can be expressed in terms of positive powers of the variance. Hence, in the following discussions, all moments of $\eta_i(t)$ higher than the 2nd are assumed small enough to be neglected.

By taking the expected value of Equation (3.11) and by requiring that the perturbing noise is uncorrelated with any uncertainties in the initial state, i.e.,

$$E\,[\delta x_i(t_0)\;\eta_j(t)\,] \;=\; 0 \tag{3.13}$$

the following expression is obtained.

$$\delta\bar{x}_i(t) \;=\; \Phi_{ij}(t,t_0)\;\delta x_j(t_0) \;+$$

$$\int_{t_0}^{t} \Phi_{ij}(t,\tau)\left\{\frac{1}{2} f_{jx_kx_\ell}\,M_{k\ell}(\tau,\tau) + f_{jx_ku_\ell}h_{k\ell}(\tau,\tau) + \frac{1}{2}f_{ju_ku_\ell}R_{k\ell}(\tau,\tau)\right\} d\tau \tag{3.14}$$

where

$$h_{ij}(t,t) \;=\; E\,[\delta x_i(t)\;\eta_j(t)] = \int_{t_0}^{t}\Phi_{ik}(t,\tau)f_{ku_m}(\tau)R_{mj}(t,\tau)\;d\tau \tag{3.15}$$

and

$$M_{ij}(t,t) = E\,[\delta x_i(t)\delta x_j(t)] \;=\; \Phi_{ik}(t,t_0)\;\Phi_{j\ell}(t,t_0)\;E\,[\delta x_k(t_0)\delta x_\ell(t_0)\,] \;+$$

$$\int_{t_0}^{t}\int_{t_0}^{t}\Phi_{ik}(t,\rho)\Phi_{j\ell}(t,s)f_{ku_m}(\rho)f_{\ell u_n}(s)R_{mn}(\rho,s)\;ds\;d\rho \tag{3.16}$$

It should be noted that Equation (3.14) is a very important result. It can be seen that if $R_{ij}(\rho,\tau) \neq 0$, or if $M_{ij}(t_0,t_0) \neq 0$, then, in general $\delta\bar{x}_i(t) = 0$. This is true only for nonlinear dynamic systems where the second partial derivatives of f_i do not vanish. In the case of a linear dynamic system, $\delta\bar{x}_i(t)$ will vanish unless $\delta\bar{x}_i(t_0)$ is nonzero.

In order to solve the set of Equations (3.14), (3.15), and (3.16), by conventional numerical integration methods, the equations will be converted back to differential form. By making use of the Leibnitz rule, i.e.,

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,\tau)d\tau = f(t,h(t))\frac{dh}{dt} - f(t,g(t))\,\frac{dg}{dt} + \int_{g(t)}^{h(t)} \frac{\partial f(t,\tau)}{dt}\,d\tau \qquad (3.17)$$

Equation (3.14) can be differentiated to obtain

$$\delta\dot{\bar{x}}_i = \dot{\Phi}_{ij}(t,t_0)\;\delta x_j(t_0) + \Phi_{ij}(t,t)\;\Big[\frac{1}{2} f_{jx_k x_\ell}\; M_{k\ell}(t,t) \;+$$

$$f_{jx_k u_\ell}\; h_{k\ell}(t,t) + \frac{1}{2} f_{iu_k u_\ell}\; R_{k\ell}(t,t)\Big] \;+\; \int_{t_0}^{t} \Phi_{ij}(t,\tau)\Big\{\frac{1}{2} f_{jx_\ell x_k} M_{k\ell}(\tau,\tau)$$

$$+\; f_{jx_\ell u_k}\, h_{k\ell}\,(\tau,\tau) + \frac{1}{2} f_{ix_\ell u_k}\; R_{k\ell}(\tau,\tau)\Big\}\; d\tau \qquad (3.18)$$

By substituting Equation (3.10) into Equation (3.18) the following expression can be written

$$\delta\dot{\bar{x}}_i(t) = f_{ix_k}\left[\Phi_{kj}(t,t_0)\,\delta\bar{x}_j(t_0) + \int_{t_0}^{t}\Phi_{kj}(t,\tau)\left\{\frac{1}{2}f_{jx_\ell x_m}M_{\ell m} + \right.\right.$$

$$\left.\left. f_{jx_\ell u_m}h_{\ell m} + \frac{1}{2}f_{ju_\ell u_m}R_{\ell m}\right\}d\tau\right] + \frac{1}{2}f_{ix_jx_k}M_{jk}(t,t) +$$

$$f_{ix_ju_k}h_{jk}(t,t) + \frac{1}{2}f_{iu_ju_k}R_{jk}(t,t) \qquad (3.19)$$

and by substituting Equation (3.14) into Equation (3.19), the following differential equation for $\delta\bar{x}_i(t)$ is obtained.

$$\delta\dot{\bar{x}}_i = f_{ix_j}\,\delta\bar{x}_j + \frac{1}{2}f_{ix_jx_k}M_{jk} + f_{ix_ju_k}h_{jk} + f_{iu_ju_k}R_{jk} \qquad (3.20)$$

From Equation (3.20) it is seen that, under the small noise restriction, $\delta\bar{x}_i(t)$ obeys a forced linear differential equation in which the forcing functions involve the covariance components of the state and noise.

In a similar manner, Equation(3.16)can be differentiated to obtain

$$\dot{M}_{ij}(t,t) = \dot{\Phi}_{ik}(t,t_0)\ \Phi_{j\ell}(t,t_0)\ M_{k\ell}(t_0,t_0)\ +$$

$$\Phi_{ik}(t,t_0)\ \dot{\Phi}_{j\ell}(t,t_0)\ M_{k\ell}(t_0,t_0)\ +$$

$$\Phi_{ik}(t,t)\ f_{ku_m}(t) \int_{t_0}^{t} \Phi_{j\ell}(t,\S)\ f_{\ell u_n}\ R_{mn}(t,\S)\ d\S\ +$$

$$\int_{t_0}^{t} \Phi_{ik}(t,\rho)\ f_{ku_m}(\rho)\ R_{mn}(\rho,t)\ d\rho\ \Phi_{j\ell}(t,t)\ f_{\ell u_n}(t)\ +$$

$$\int_{t_0}^{t} \dot{\Phi}_{ik}(t,\rho)\ f_{ku_m}(\rho) \int_{t_0}^{t} \Phi_{j\ell}(t,\S)\ f_{\ell u_m}(\S)\ R_{mn}(\rho,\S)\ d\S d\rho\ +$$

$$\int_{t_0}^{t} \Phi_{ik}(t,\rho)\ f_{ku_m}(\rho) \int_{t_0}^{t} \dot{\Phi}_{j\ell}(t,\S)\ f_{\ell u_n}(\S)\ R_{mn}(\rho,\S)\ d\S d\rho \qquad (3.21)$$

Equation (3.21) reduces to

$$\dot{M}_{ij} = f_{ix_k}\ M_{kj} + M_{ik}\ f_{jx_k} +$$

$$f_{iu_k}\ h_{kj} + h_{ik}\ f_{ju_k} \qquad (3.22)$$

By differentiating the expression given in Equation (3.15) the following result is obtained

$$\dot{h}_{ij}(t,t) = \Phi_{ik}(t,t)\ f_{ku_m}(t)\ R_{mj}(t,t) + \int_{t_0}^{t} \Phi_{ik}(t,\tau)\ f_{ku_m}(\tau)\ R_{mj}(t,\tau)\ d\tau\ +$$

$$\int_{t_0}^{t} \Phi_{ik}(t,\tau)\ f_{ku_m}(\tau)\ \dot{R}_{mj}(t,\tau)\ d\tau \qquad (3.23)$$

By recalling Equation (3.4), i.e., $R_{ij}(t,\tau) = \sigma_{ij}^{\,2}\, e^{-\beta_{ij}(t-\tau)}$, the derivative of $R_{ij}(t,\rho)$ can be computed as follows

$$\dot{R}_{ij}(t,\tau) = -\beta_{ij}\,\sigma_{ij}^{\,2}\, e^{-\beta_{ij}(t-\tau)} \qquad i,\ j \text{ not summed} \qquad (3.24)$$

Since $\sigma_{ij} = 0$, if $i \neq j$,

$$\dot{R}_{ij}(t,\tau) = -\beta_{ik}\, R_{kj}(t,\tau) = -R_{ik}\beta_{kj} \qquad (3.25)$$

Hence, after substituting Equation (3.25) into the expression given in Equation (3.23), the differential equation for $h_{ij}(t,t)$ can be written as follows,

$$\dot{h}_{ij} = f_{ix_k}\, h_{kj} + f_{iu_k}\, R_{kj} - h_{ik}\,\beta_{kj} \qquad (3.26)$$

The set of Equations (3.19), (3.21), and (3.26) fully describes the behavior of the expected value or "mean" deviation from the optimal deterministic trajectory which obeys Equation (3.1). Equations (3.20), (3.22), and (3.26), can be directly integrated in terms of specified initial conditions

$$\begin{aligned} \delta\overline{x}_i(t_0) &= 0 \\ M_{ij}(t_0,t_0) &= M_{ij0} \\ h_{ij}(t_0,t_0) &= 0 \end{aligned} \qquad (3.27)$$

<u>Application To A Space Guidance Problem</u>

The results derived in the previous section will now be applied to the example space guidance problem discussed in the introduction. Consider a point-mass spacecraft undergoing a minimum time transfer from Earth to Mars under the influence of the gravitational field of the Sun and a continuously operating low-thrust ion engine. The geometry of such a system is illustrated in Figure 4. The state equations of the transfer trajectory, in polar coordinates are

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + a \sin\alpha$$

$$\dot{v} = -\frac{uv}{r} + a \cos\alpha$$

$$\dot{r} = u$$

$$\dot{\theta} = \frac{v}{r} \qquad (3.28)$$

where:

$$a = \frac{T}{m_0 - \dot{m}(t-t_0)}$$

r is the Sun-spacecraft distance

is the angle made with the Sun-spacecraft line with the Sun-Earth line at launch

u is the velocity component along the Sun-spacecraft line

v is the velocity component perpendicular to the Sun-spacecraft line

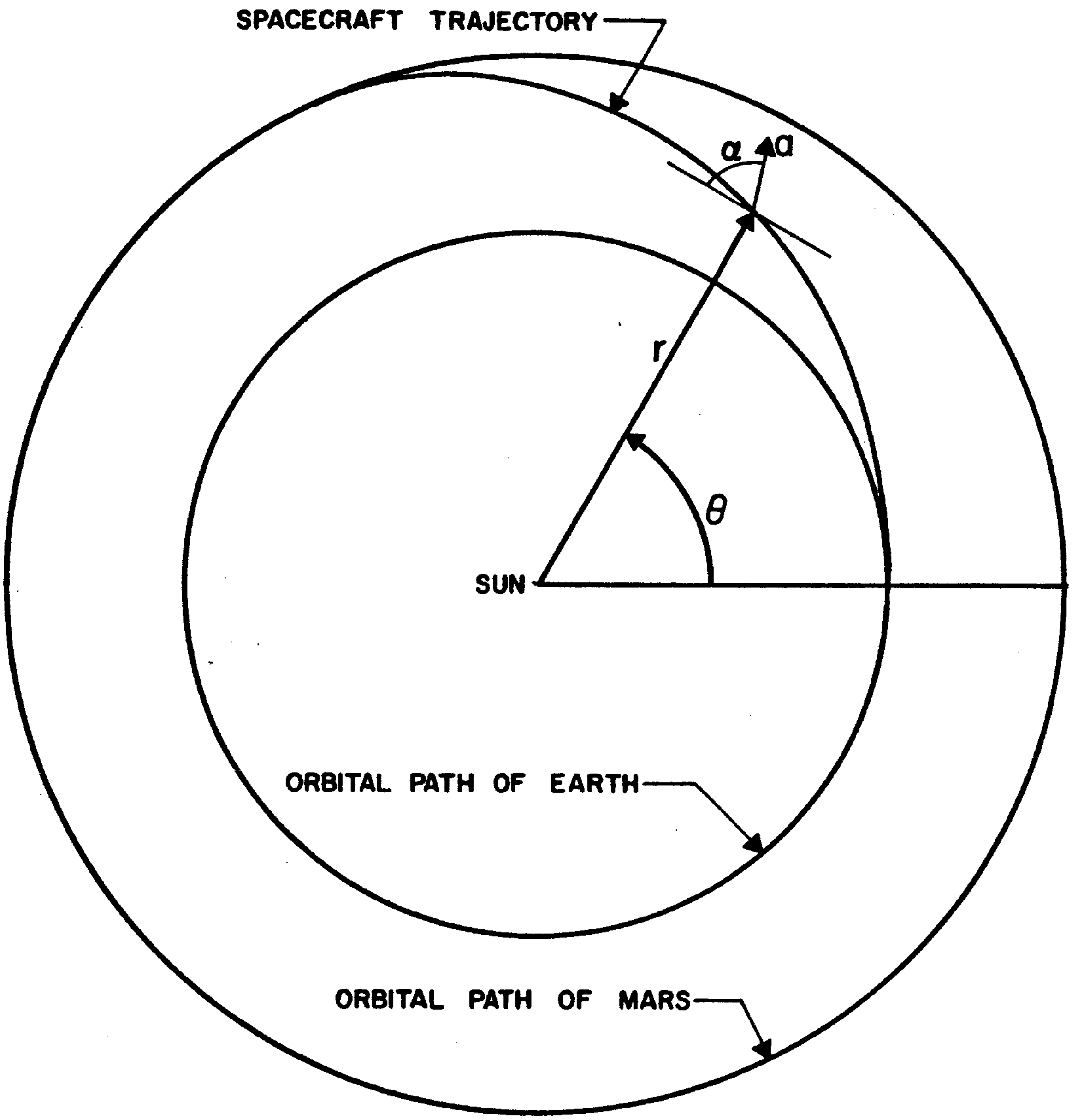

Figure 4. Earth-Mars Transfer Geometry

μ is the solar gravitational constant

T is the thrusting force magnitude

m_0 is the initial spacecraft mass

$\dot{m}$ is the mass flow rate and

α is the engine thrust direction angle, measured from a perpendicular to the Sun-spacecraft line.

The thrusting force of the engine, T, is held to a specified constant, and the control which is used to bring the terminal system state to coincide with that of Mars is α, the thrust direction angle. A calculus of variations approach is used to find the deterministic thrust direction program $\alpha(t)$, which completes the transfer in minimum time. The solution to the variational problem is presented in Appendix A. The solution to this optimization problem is used as the optimal deterministic trajectory, about which the mean values of δu, δv, δr, $\delta\theta$, and their respective standard deviations, σ_u, σ_v, σ_r, and σ_θ, are computed.

The purpose of examining the characteristics of the mean deviations is to determine whether or not a stochastic control will help appreciably to satisfy the terminal conditions of the transfer. An analysis of the standard deviations will give some indication of the dispersion of the possible occurring stochastic trajectories.

Matrix formulations of Equations (3.20), (3.22), and (3.26), applied to the Earth-Mars transfer, appear in Appendix B. These equations have been numerically integrated forward in time for several combinations of values of the following parameters.

a. σ_a, the standard deviation of noise occurring in the thrust/mass magnitude

b. σ_α, the standard deviation of noise occurring in the thrust orientation angle

c. $T_a = \frac{1}{\beta_a}$, the correlation time of the noise occurring in the thrust/mass magnitude

d. $T_\alpha = \frac{1}{\beta_\alpha}$, the correlation time of the noise occurring in the thrust orientation angle

e. σ_0, the standard deviation of the error in an initial state component. It should be noted that errors in the individual components of the initial state are assumed equal and are not cross-correlated, i.e., $\sigma_{u0} = \sigma_{v0} = \sigma_{r0} = \sigma_{\theta 0} = \sigma_0$.

The results are shown in Figures 5 through 18. The figures labeled a show the time histories of the mean deviations from the optimal deterministic trajectory, $\delta\bar{u}$, $\delta\bar{v}$, $\delta\bar{r}$, and $\delta\bar{\theta}$. The figures labeled b show the time histories of the standard deviations, i.e.,

$$\sigma_u = (M_{11} - \delta\bar{u}^2)^{\frac{1}{2}} \qquad \sigma_r = (M_{33} - \delta\bar{r}^2)^{\frac{1}{2}}$$

$$\sigma_v = (M_{22} - \delta\bar{v}^2)^{\frac{1}{2}} \qquad \sigma_\theta = (M_{44} - \delta\bar{\theta}^2)^{\frac{1}{2}} \tag{3.29}$$

The means, $\delta\bar{u}$, $\delta\bar{v}$, $\delta\bar{r}$, $\delta\bar{\theta}$, and standard deviations, σ_u, σ_v, σ_r, σ_θ, are computed and shown on the plots in the following system of units,

unit of distance = radius of Earth's orbit (1AU)

unit of velocity = velocity of Earth (1 V_E)

unit of mass = initial spacecraft mass (1 m_0)

The remaining constants used in the computation are listed in Appendix C. Although the plots of the time histories of the means and standard deviations are presented in the above units, i.e., 1AU for position components, and $1V_E$ for velocity components, the time scale is presented in days. The parameters of interest in the following respective plots are:

In Figures 5 through 7, σ_a ranges from .02T to .05T

$\sigma_\alpha = 0$

T_a = 1 Day

$\sigma_0 = 0$

In Figures 8 through 10, T_a ranges from 10 days to 1000 days.

σ_a = .02T

$\sigma_\alpha = 0$

$\sigma_0 = 0$

In Figures 11 through 13, σ_α ranges from 1° to 3°

$\sigma_a = 0$

T_α = 1 day

$\sigma_0 = 0$

In Figures 14 through 16, T_a ranges from 10 days to 1000 days

$\sigma_a = 0$

$\sigma_\alpha = 1°$

$\sigma_0 = 0$

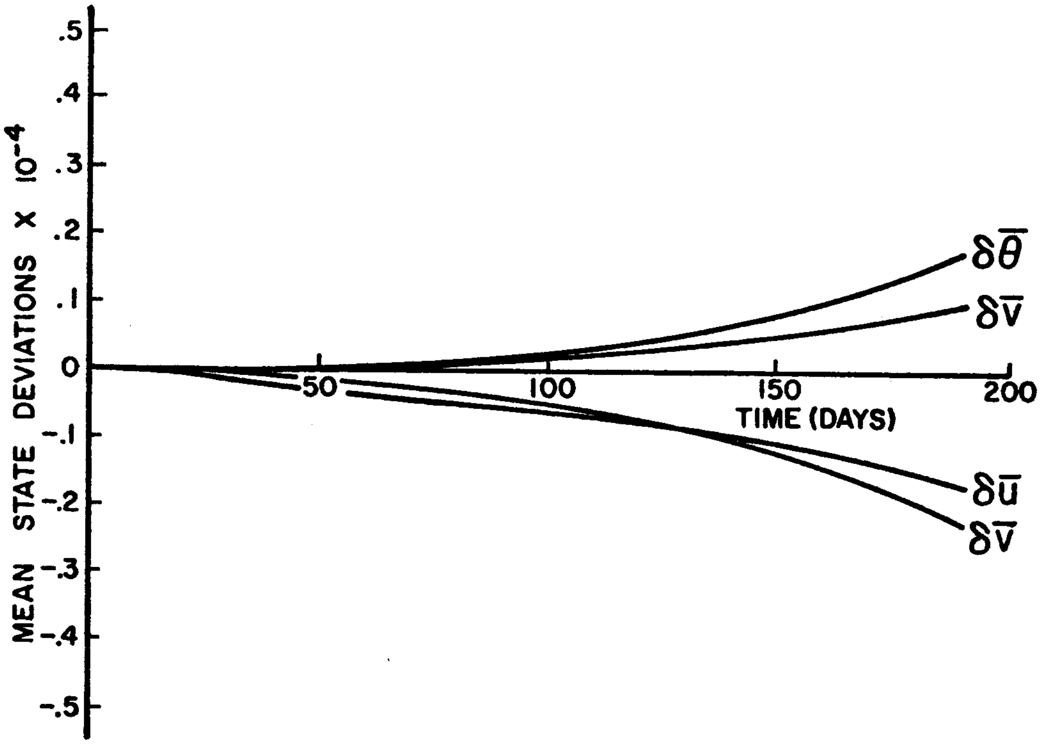

Figure 5a. Mean State Deviation Time Histories

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

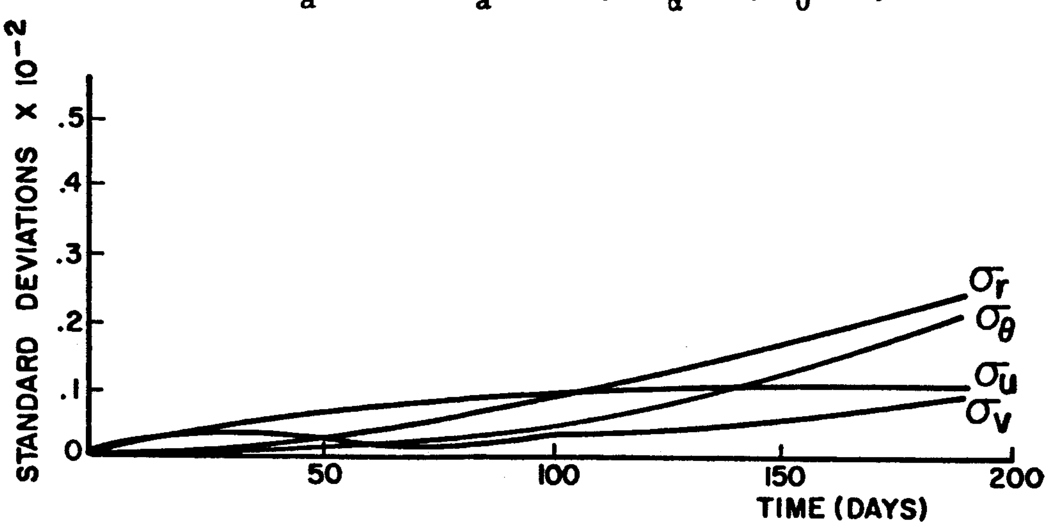

Figure 5b. Standard Deviation Time Histories

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

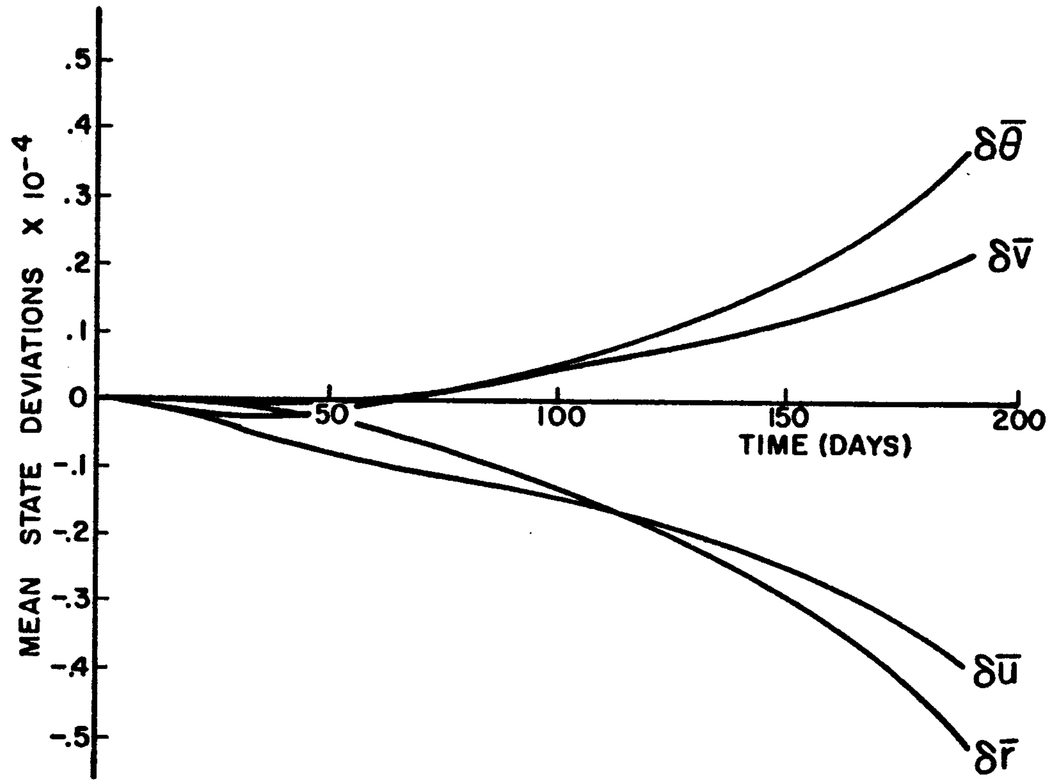

Figure 6a. Mean State Deviation Time Histories

(σ_a = .03T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

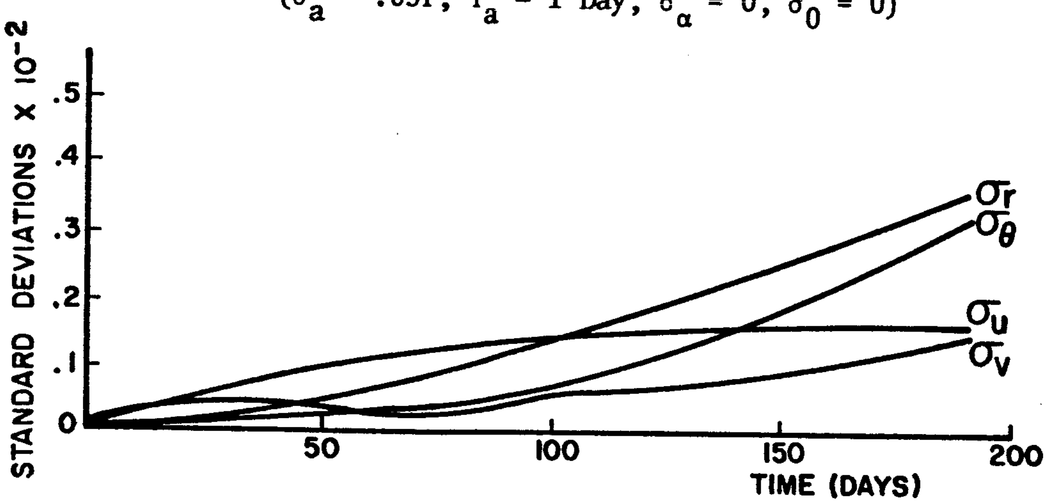

Figure 6b. Standard Deviation Time Histories

(σ_a = .03T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

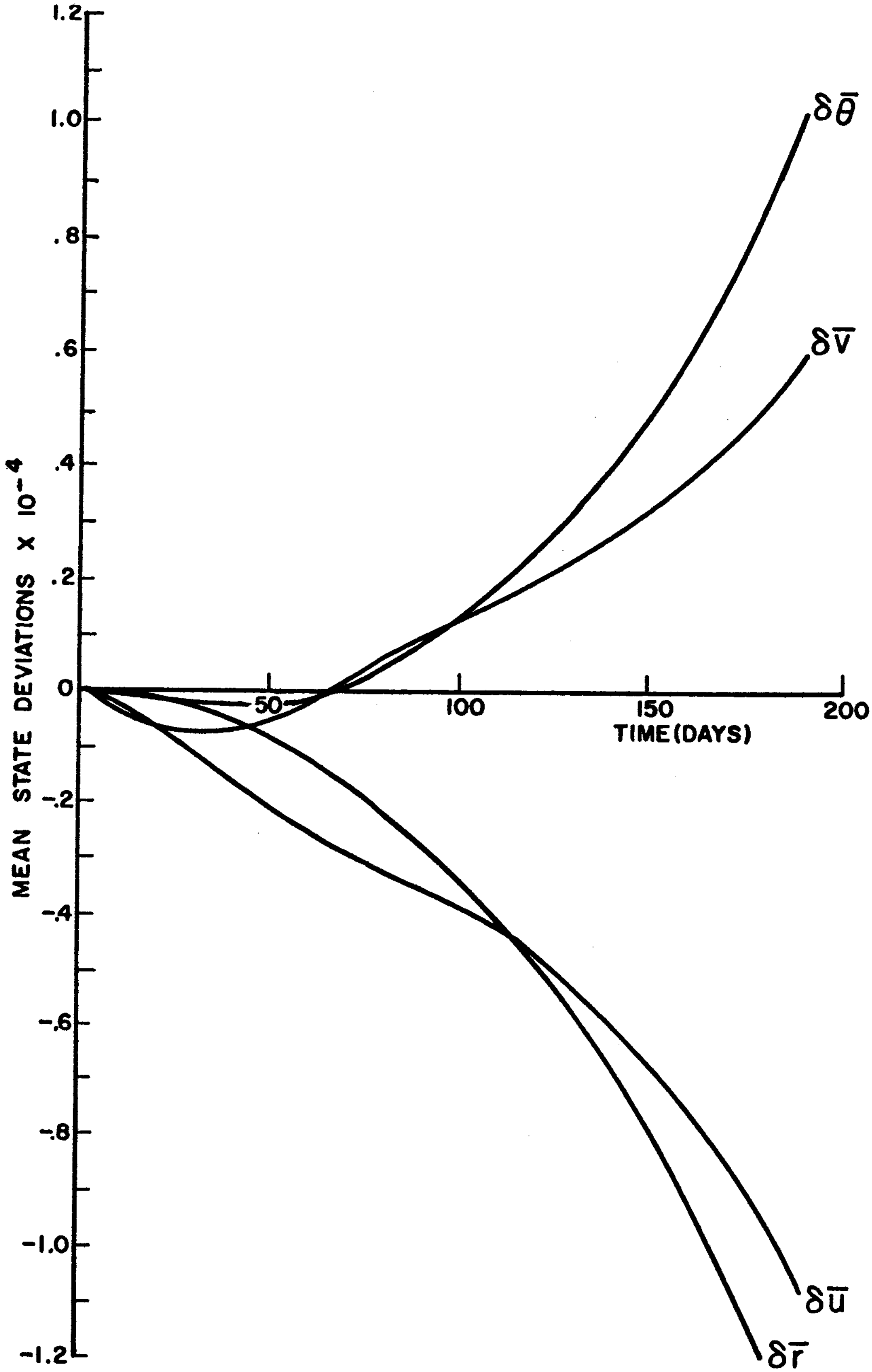

Figure 7a. Mean State Deviation Time Histories
(σ_a = .05T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

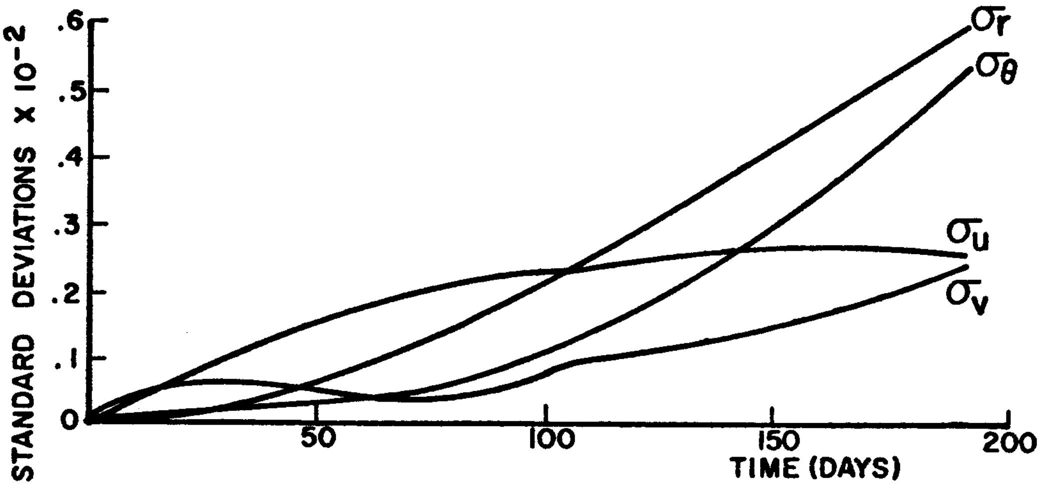

Figure 7b. Standard Deviation Time Histories

(σ_a = .05T, T_a = 1 Day, σ_α = 0, σ_0 = 0)

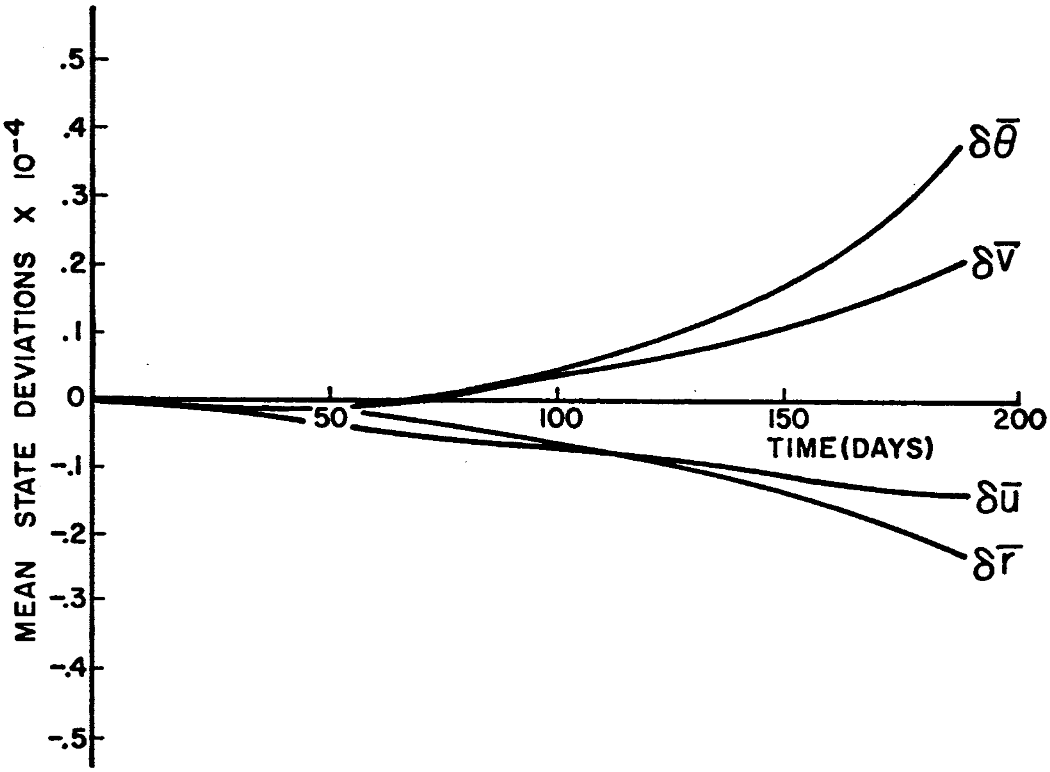

Figure 8a. Mean State Deviation Time Histories

(σ_a = .02T, T_a = 10 Days, σ_α = 0, σ_0 = 0)

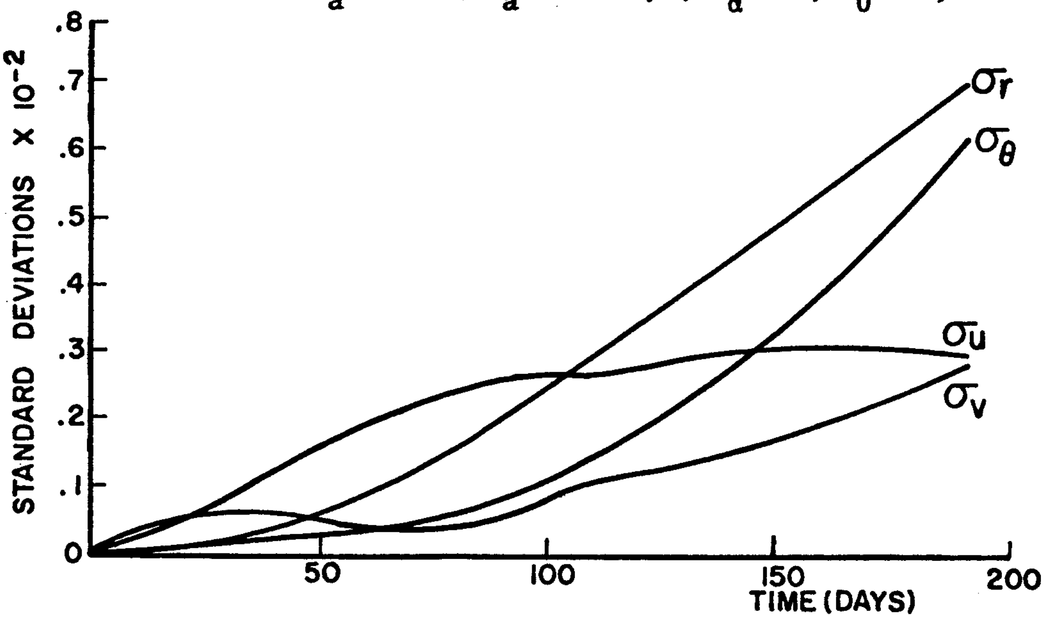

Figure 8b. Standard Deviation Time Histories

(σ_a = 02T, T_a = 10 Days, σ_α = 0, σ_0 = 0)

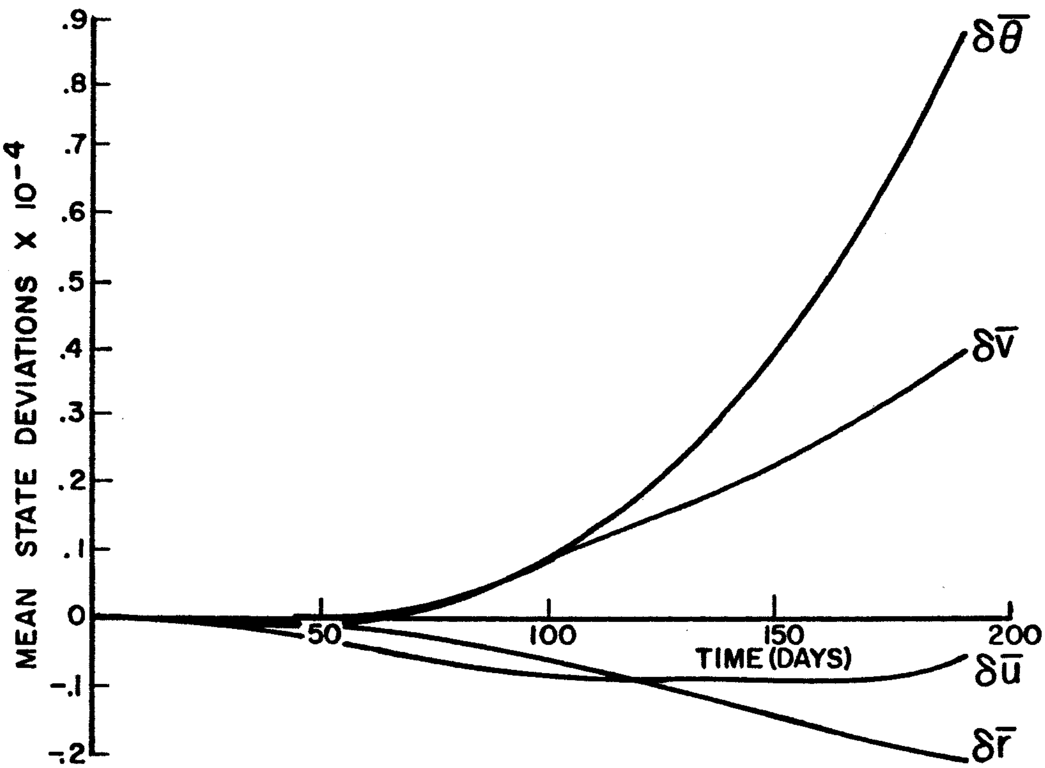

Figure 9a. Mean State Deviation Time Histories

(σ_a = .02T, T_a = 100 Days, σ_α = 0, σ_0 = 0)

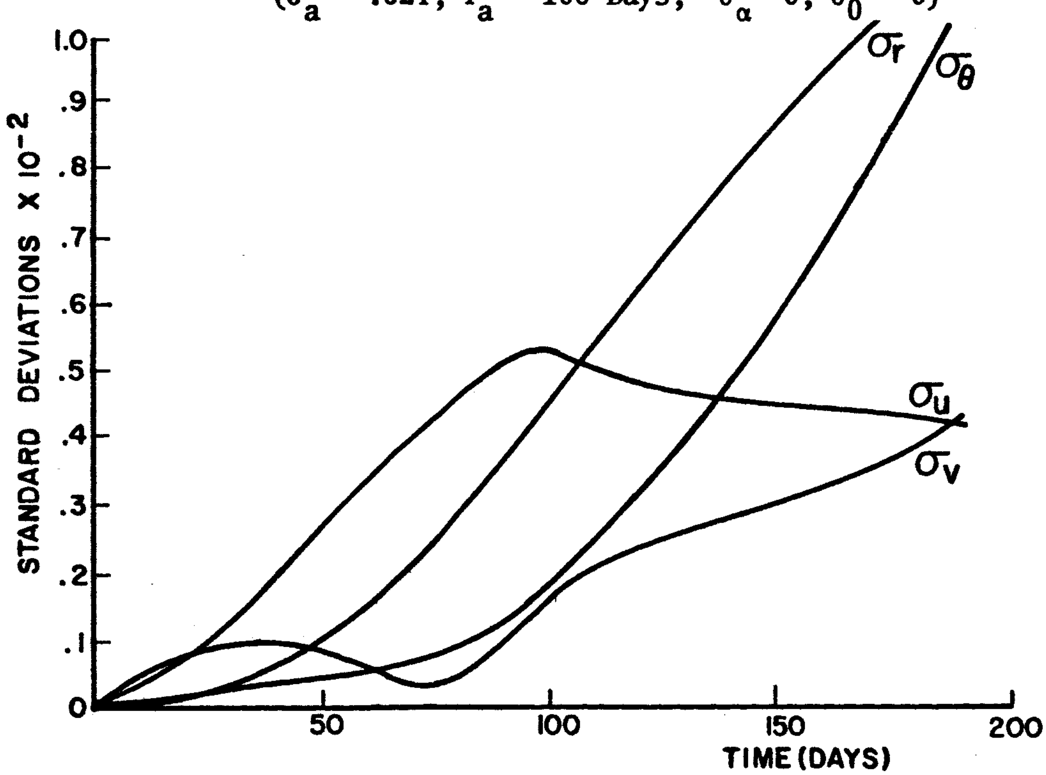

Figure 9b. Standard Deviation Time Histories

(σ_a = .02T, T_a = 100 Days, σ_α = 0, σ_0 = 0)

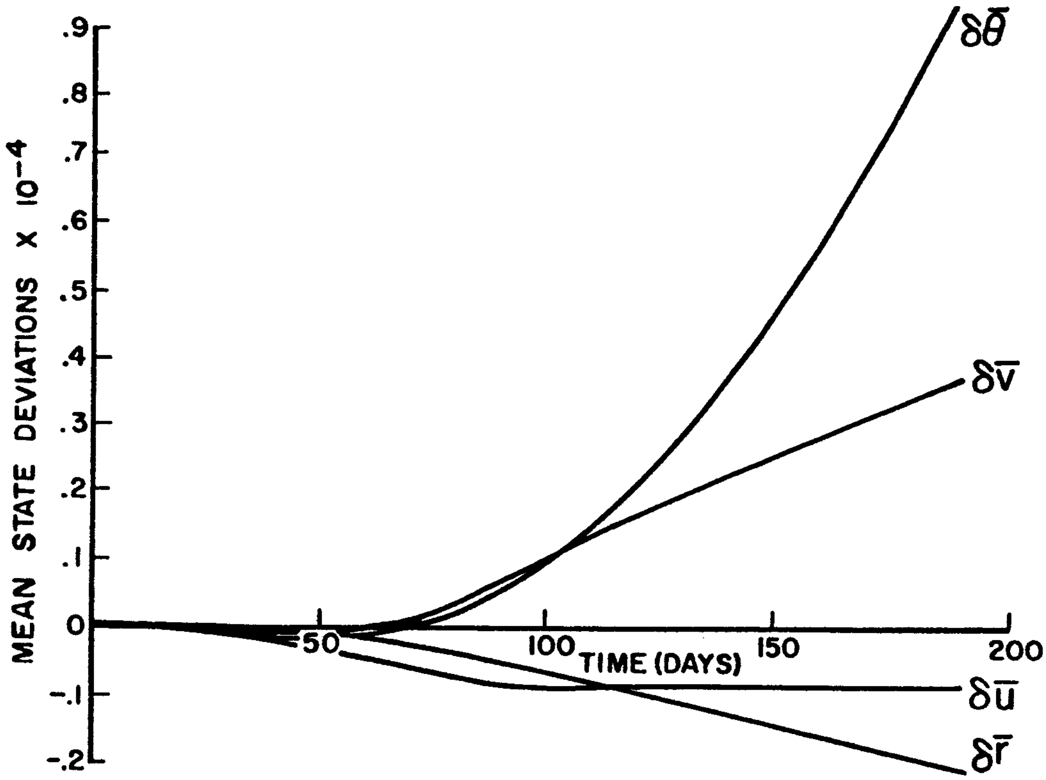

Figure 10a. Mean State Deviation Time Histories
(σ_a = .02T, T_a = 1000 Days, σ_α = 0, σ_0 = 0)

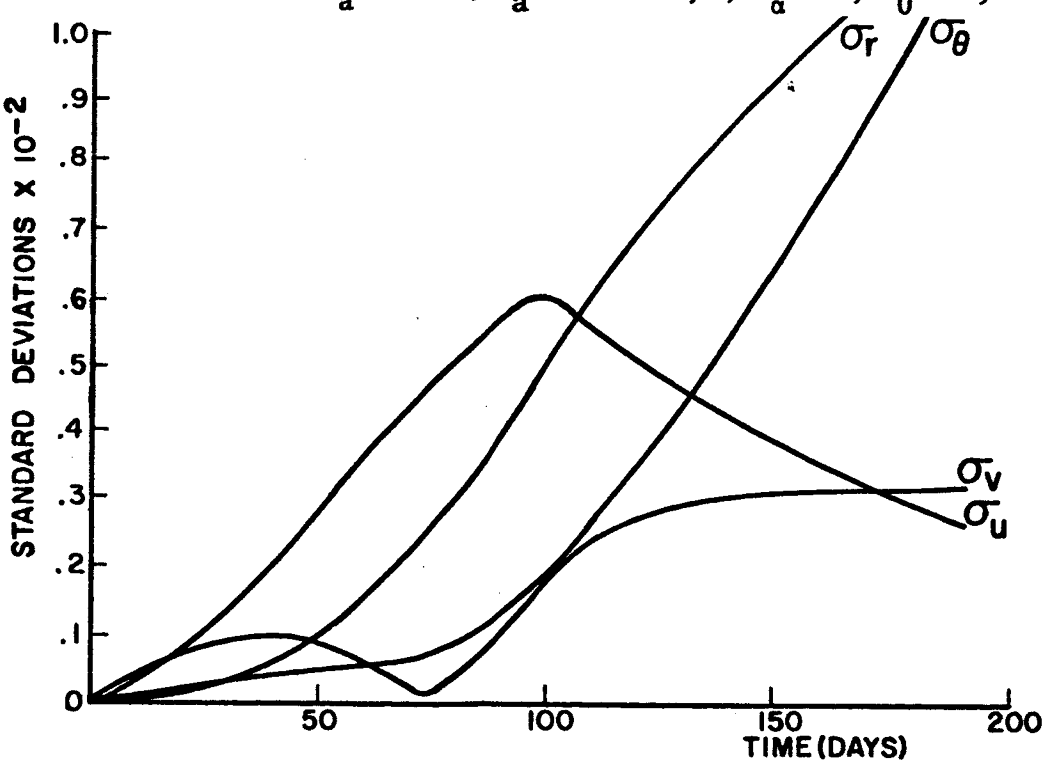

Figure 10b. Standard Deviation Time Histories
(σ_a = .02T, T_a = 1000 Days, σ_α = 0, σ_0 = 0)

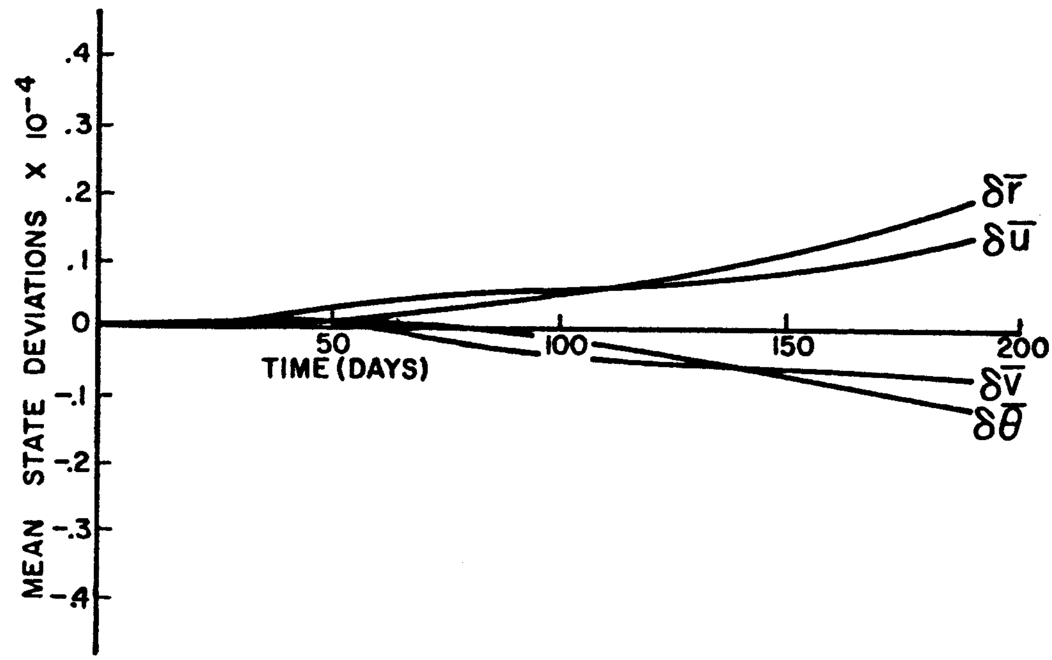

Figure 11a. Mean State Deviation Time Histories

(σ_a=0,σ_α = 1°, T_α = 1 Day, σ_0 = 0)

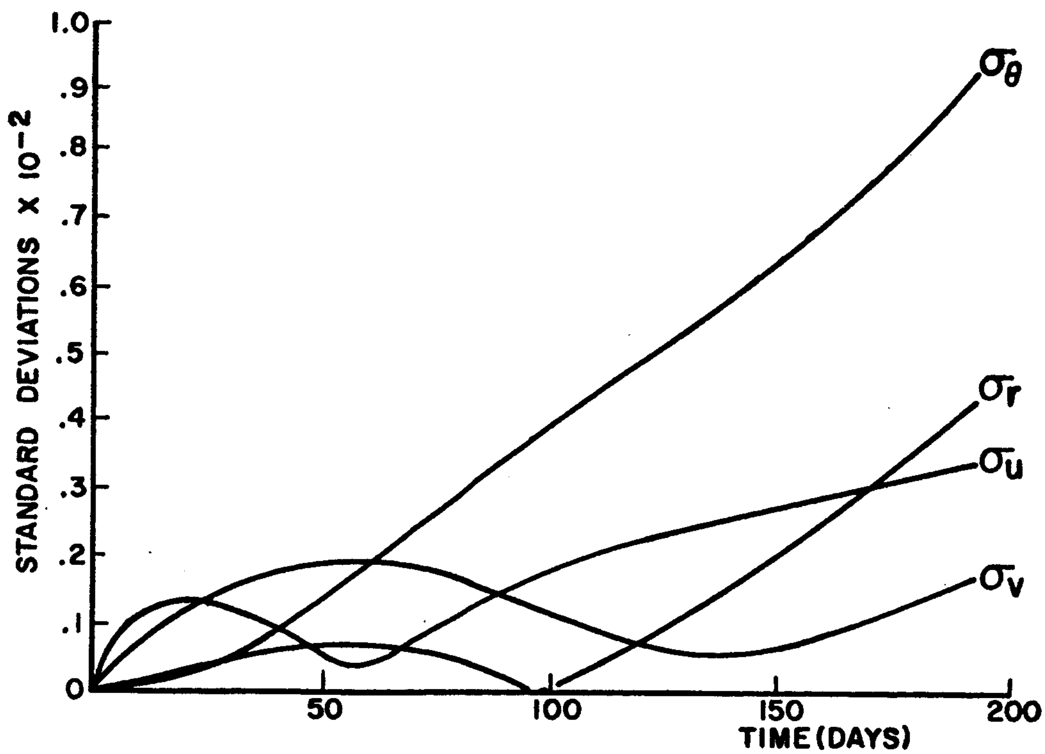

Figure 11b. Standard Deviation Time Histories

(σ_a= 0, σ_α=1°, T_α= 1 Day, σ_0= 0)

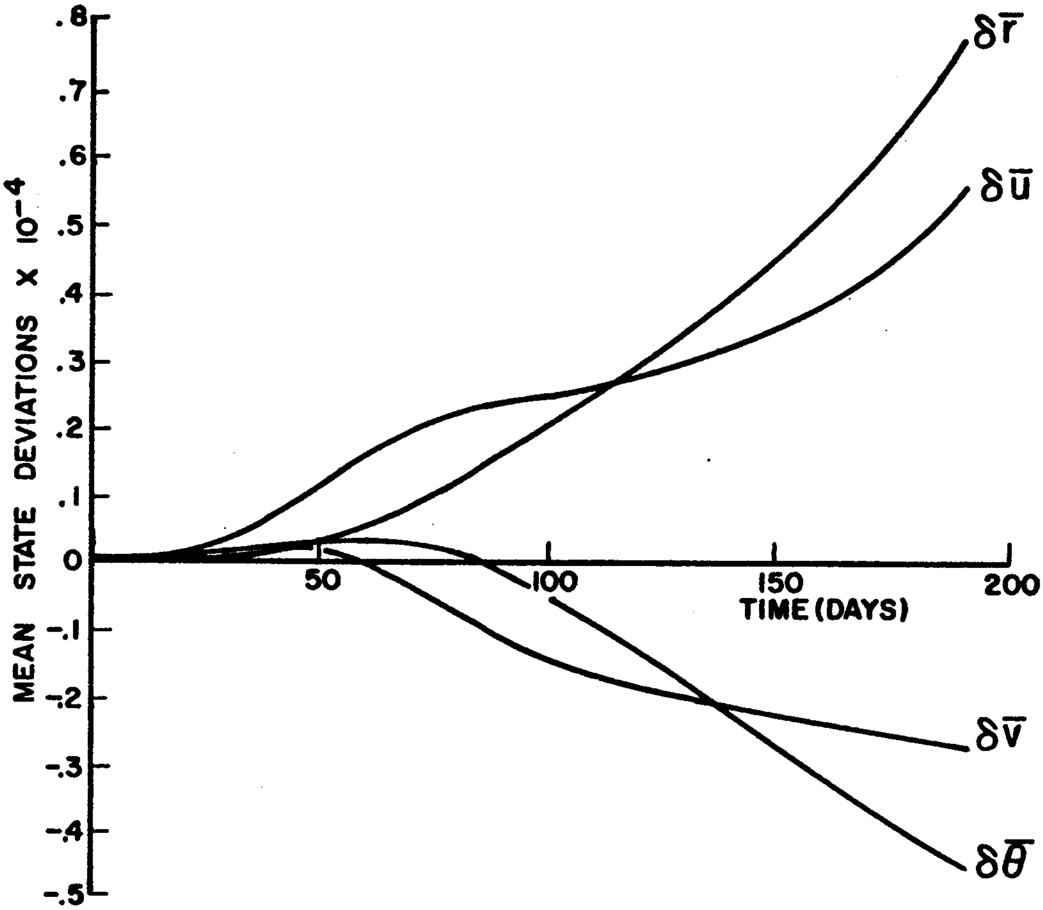

Figure 12a. Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 2°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

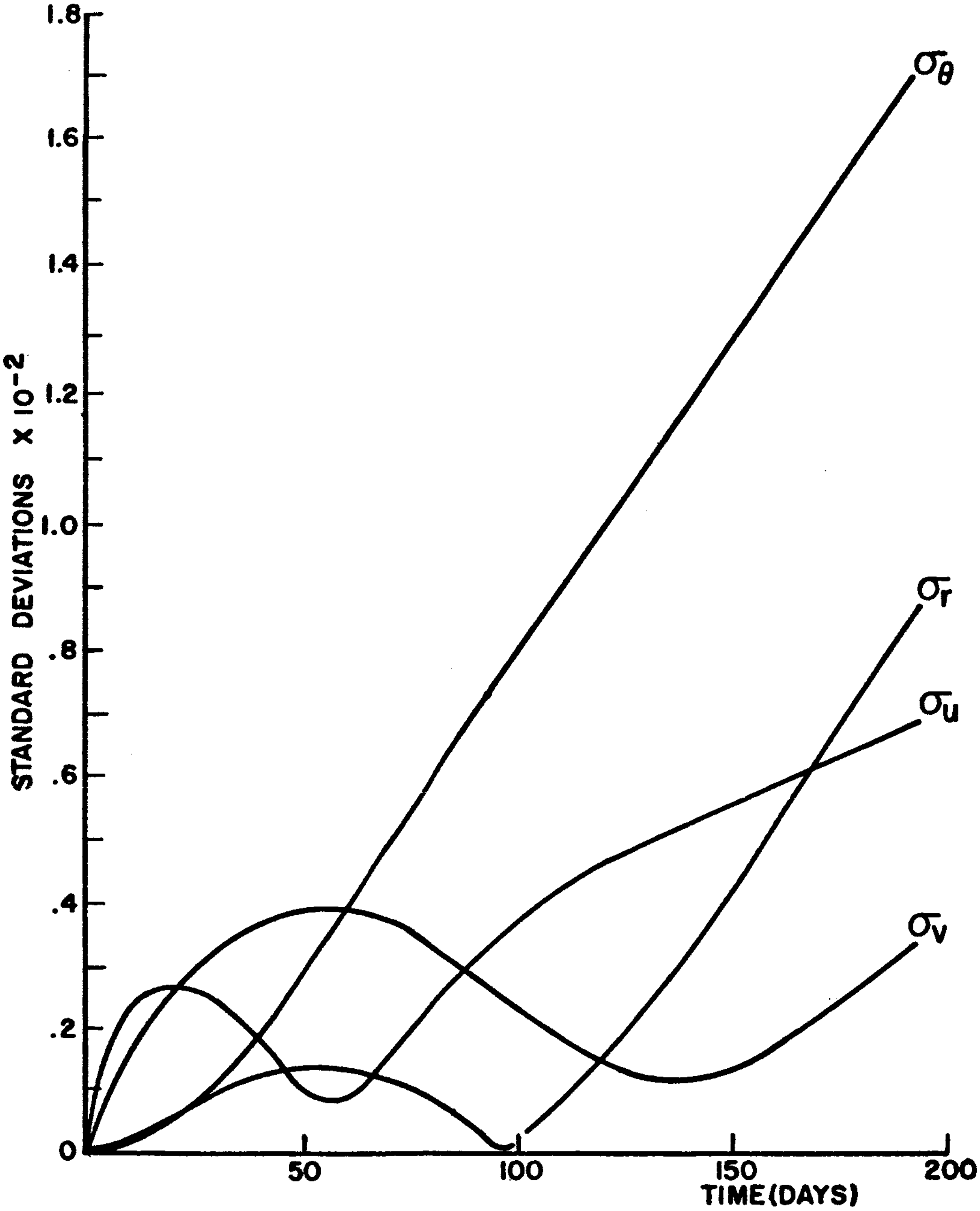

Figure 12b. Standard Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 2°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

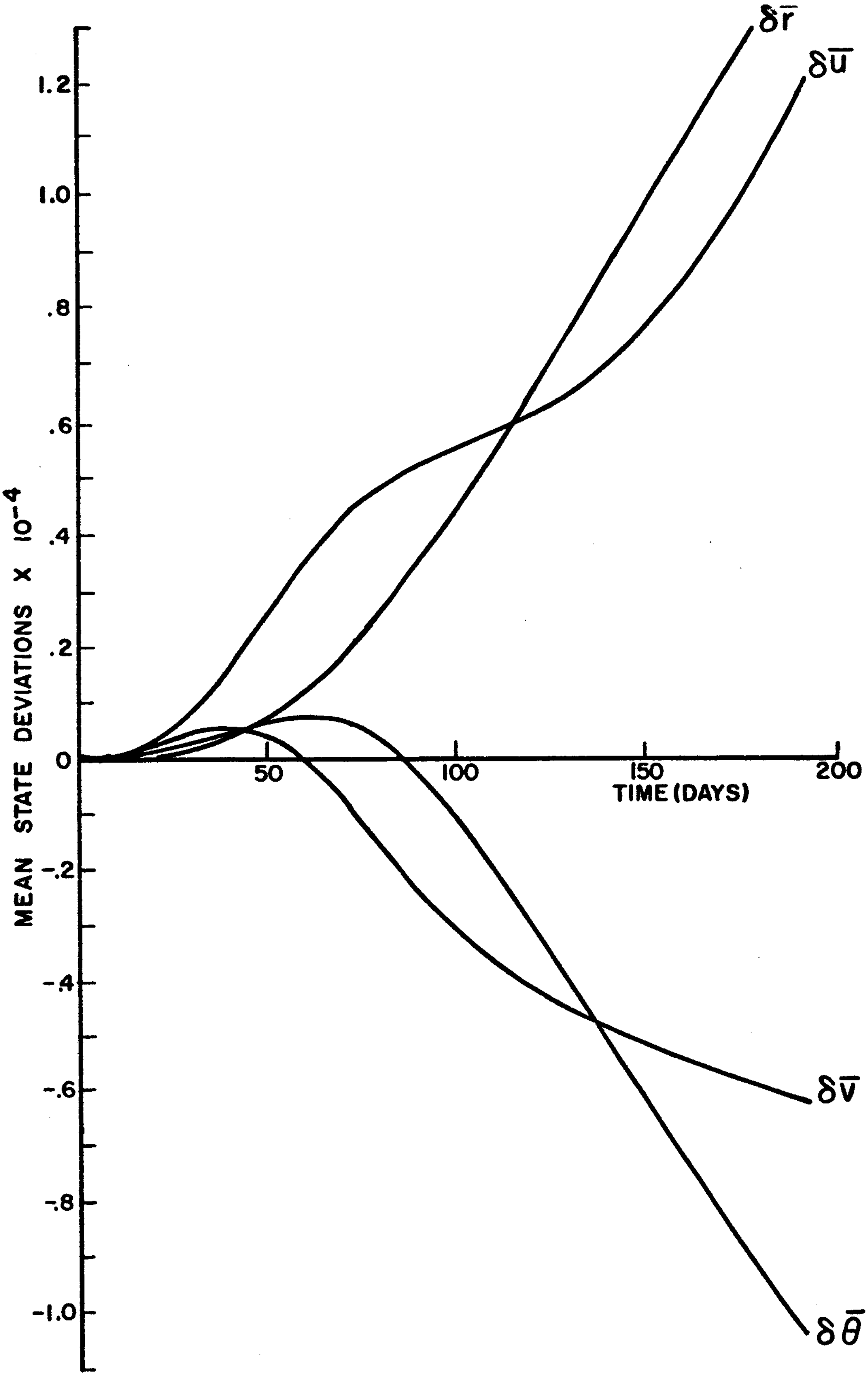

Figure 13a. Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 3°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

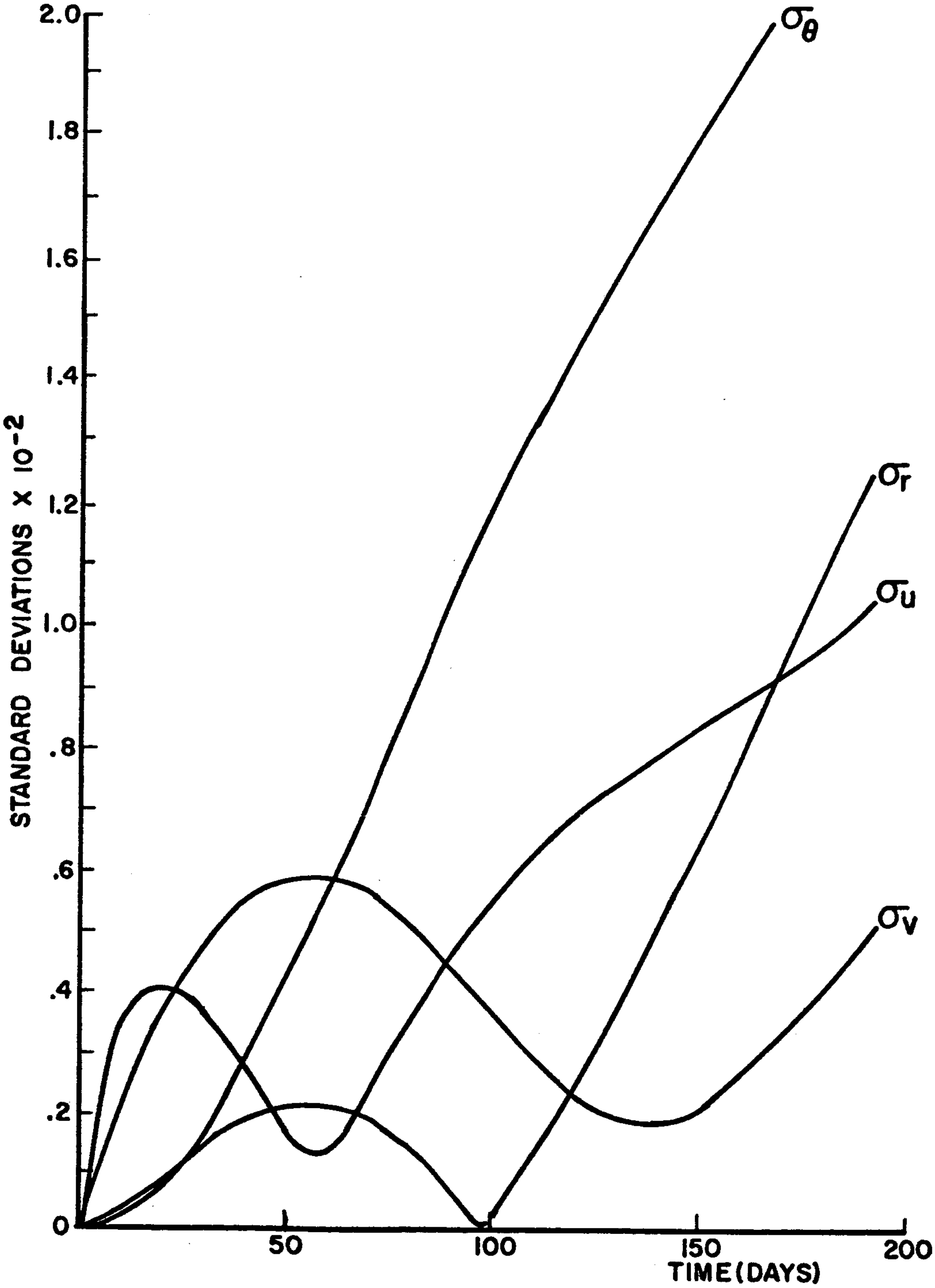

Figure 13b. Standard Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 3°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

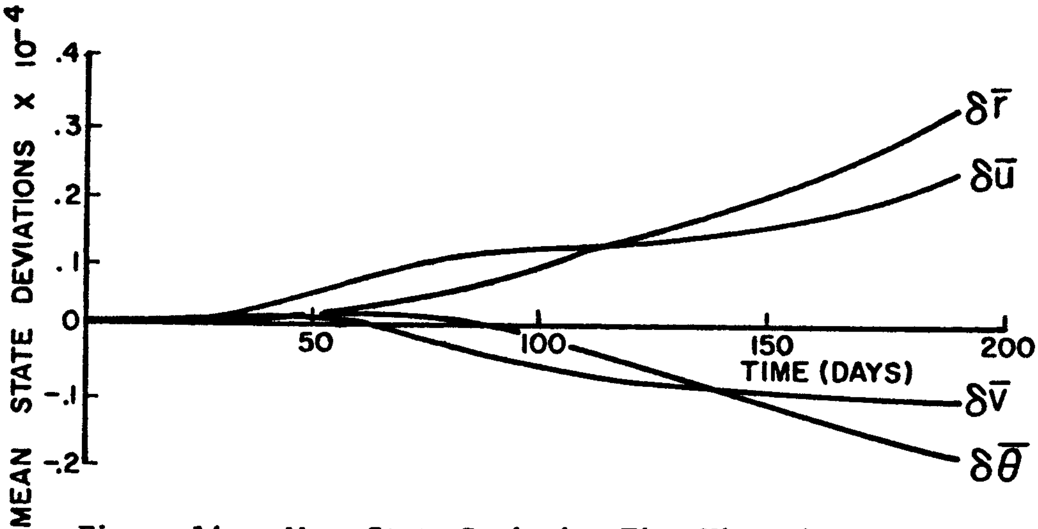

Figure 14a. Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 10$ Days, $\sigma_0 = 0$)

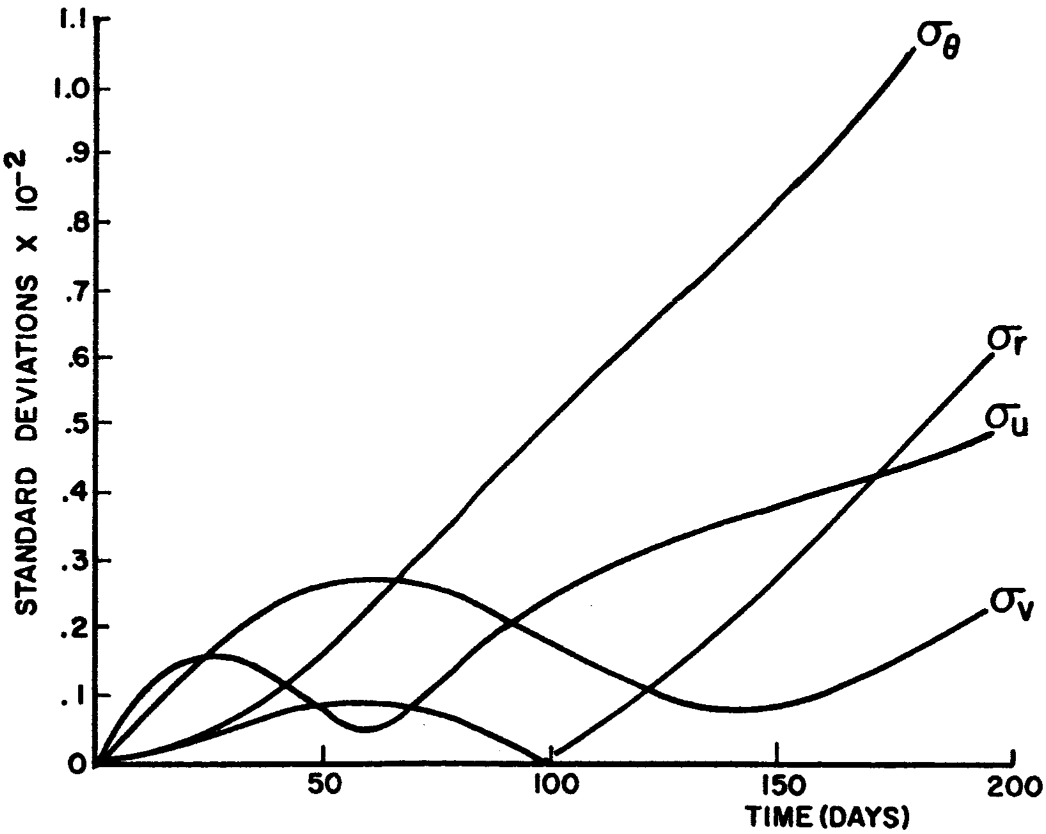

Figure 14b. Standard Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 10$ Days, $\sigma_0 = 0$)

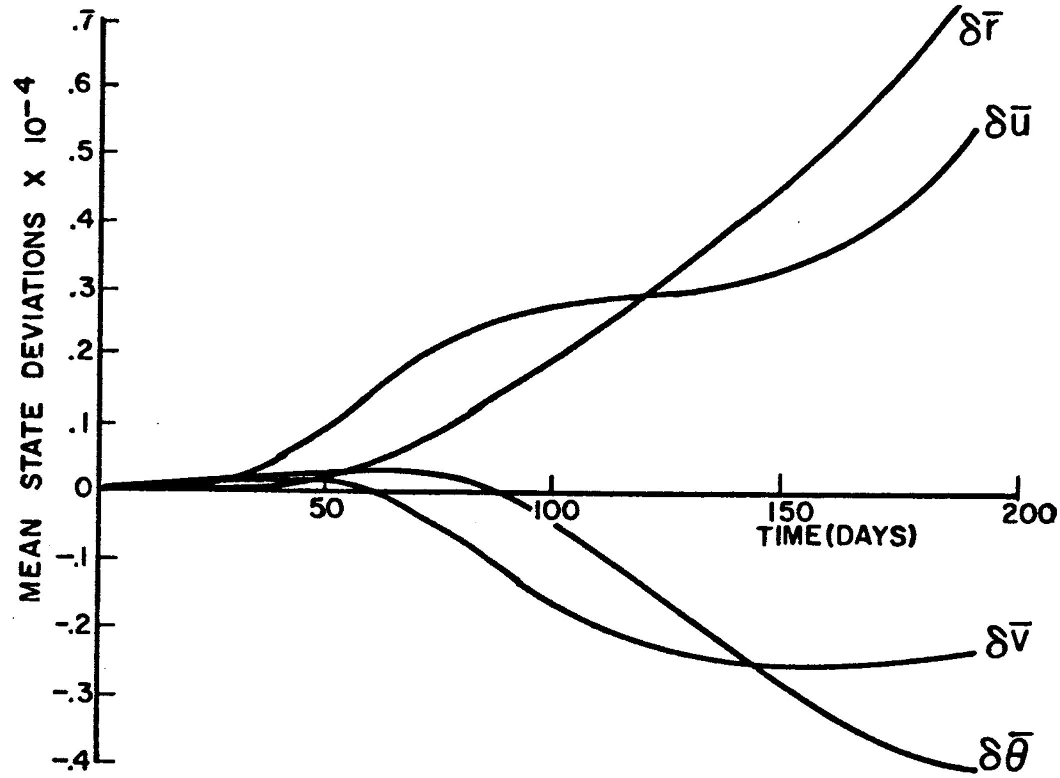

Figure 15a. Mean State Deviation Time Histories
($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 100$ Days, $\sigma_0 = 0$)

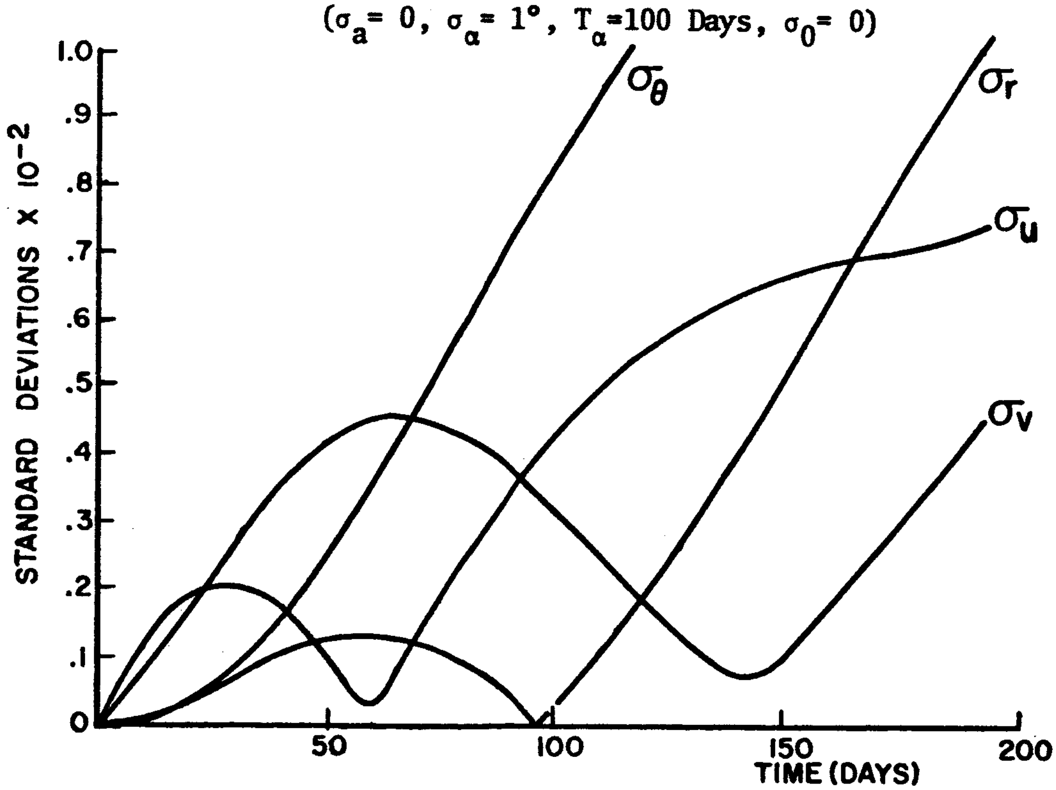

Figure 15b. Standard Deviation Time Histories
($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 100$ Days, $\sigma_0 = 0$)

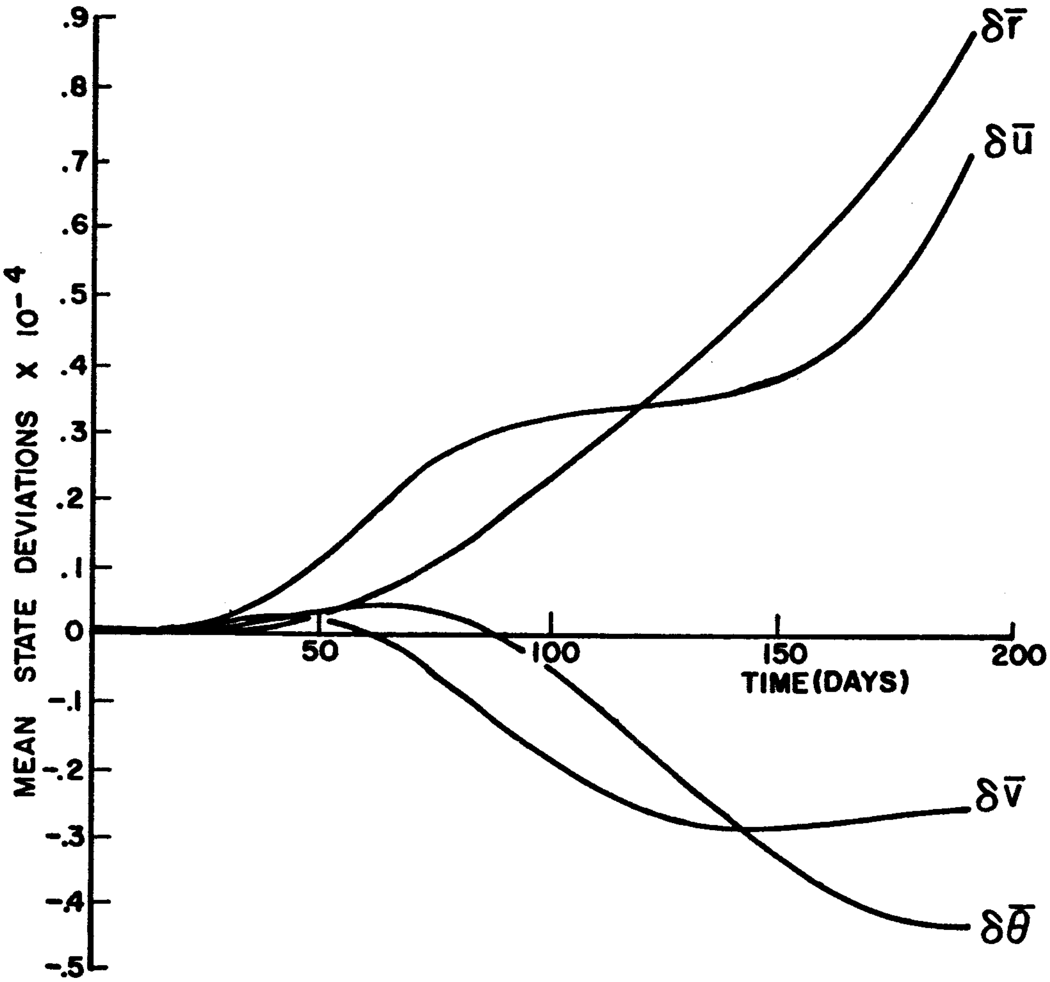

Figure 16a. Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 1000$ Days, $\sigma_0 = 0$)

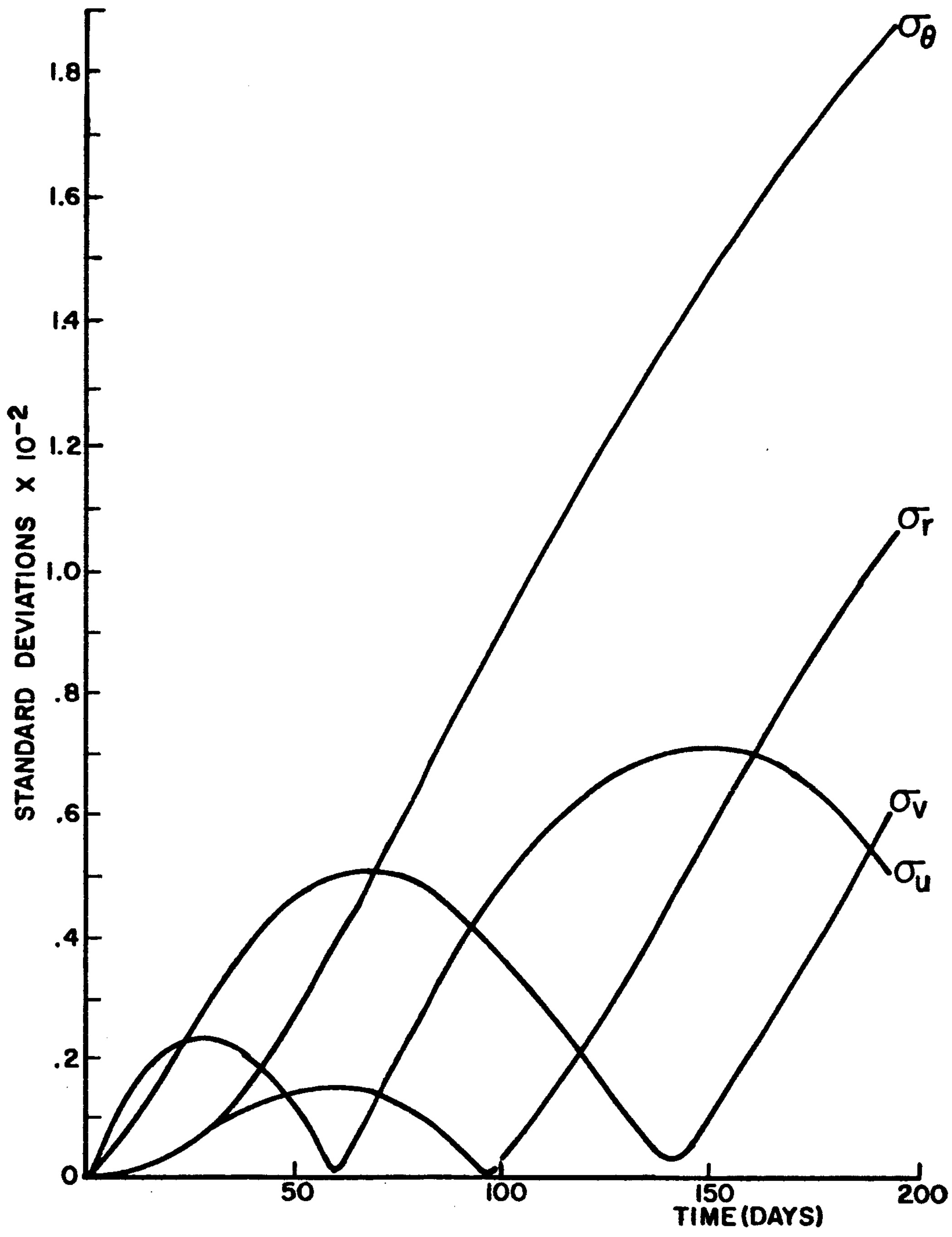

Figure 16b. Standard Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = 1°$, $T_\alpha = 1000$ Days, $\sigma_0 = 0$)

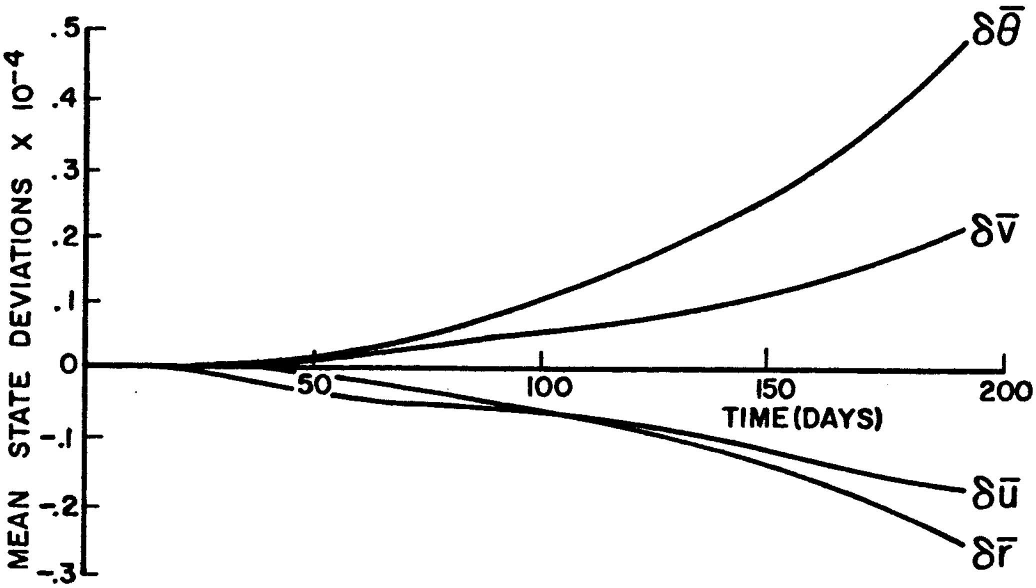

Figure 17a. Mean State Deviation Time Histories ($\sigma_a = .02T$, T_a = 1 Day, $\sigma_\alpha = 0$, $\sigma_0 = 1 \times 10^{-3}$)

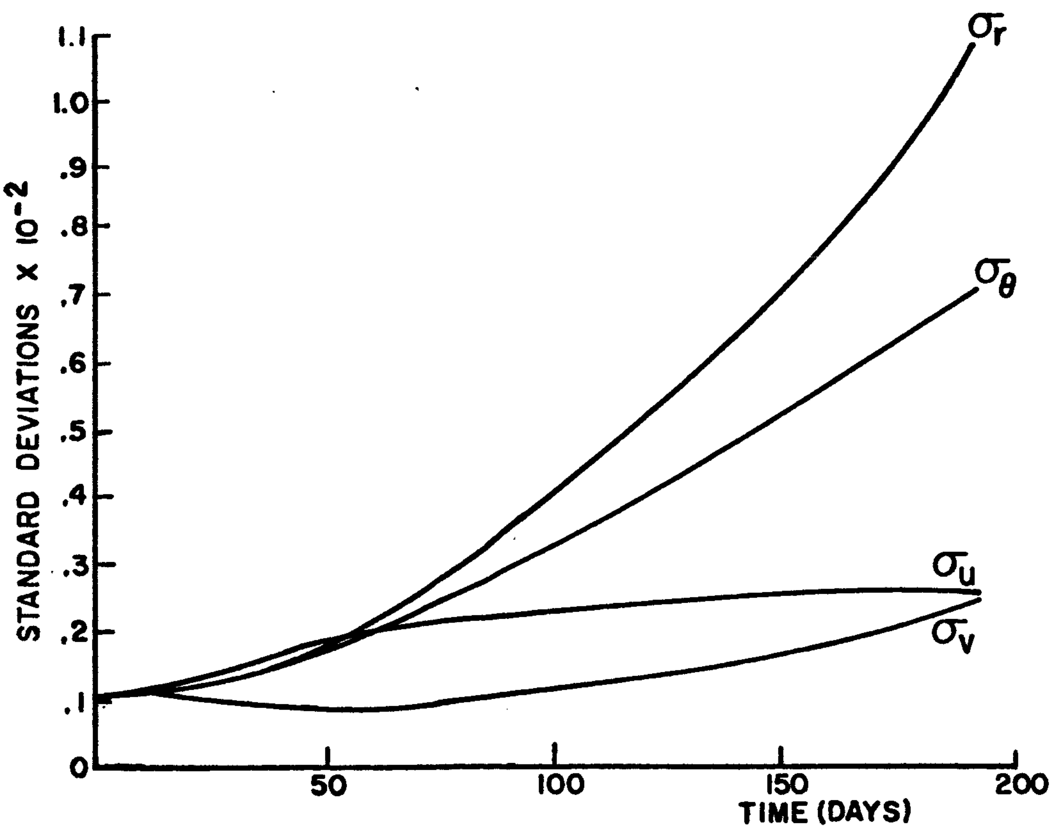

Figure 17b. Standard Deviation Time Histories ($\sigma_a = .02T$, T_a = 1 Day, $\sigma_\alpha = 0$, $\sigma_0 = 1 \times 10^{-3}$)

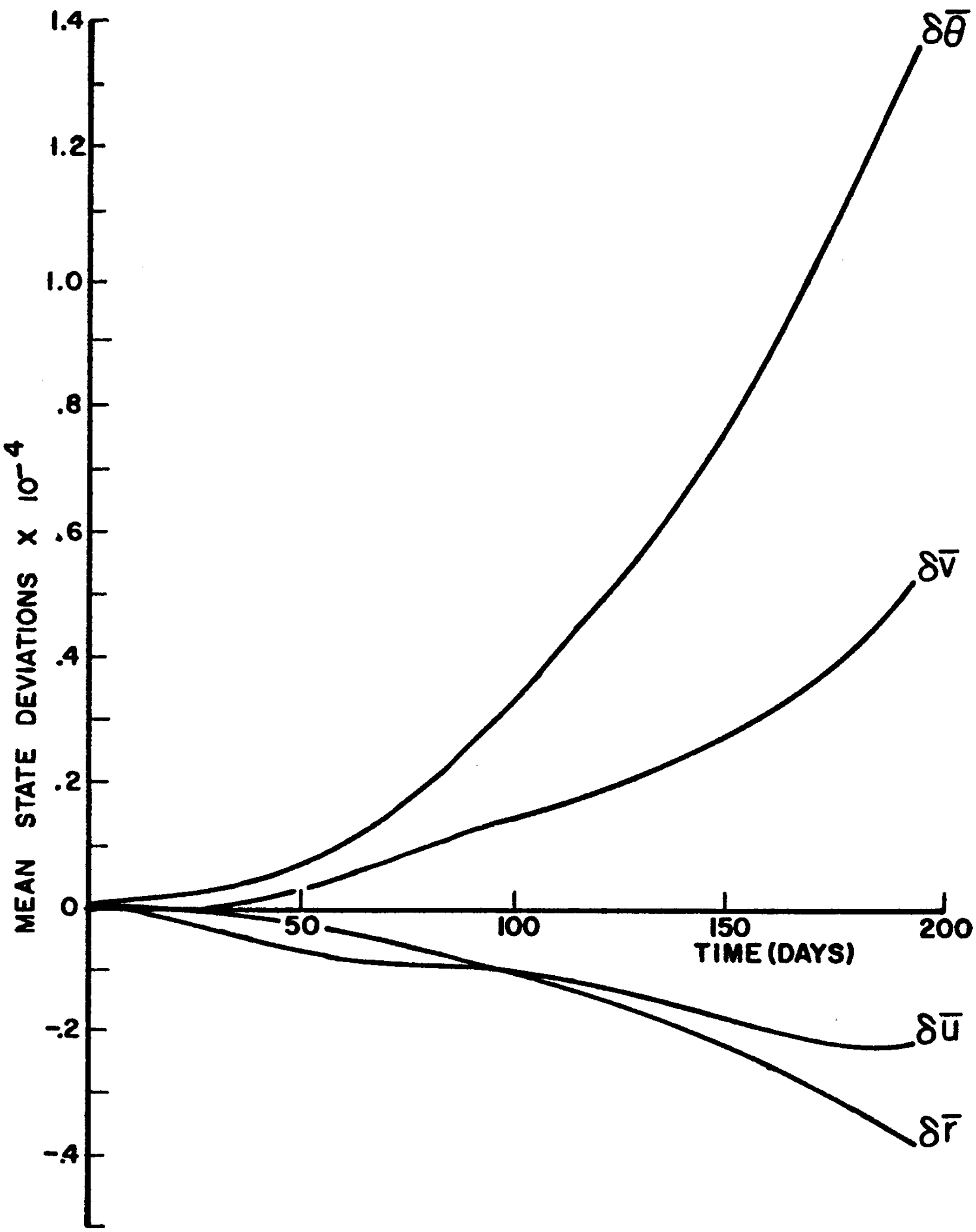

Figure 18a. Mean State Deviation Time Histories (σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 2 x 10^{-3})

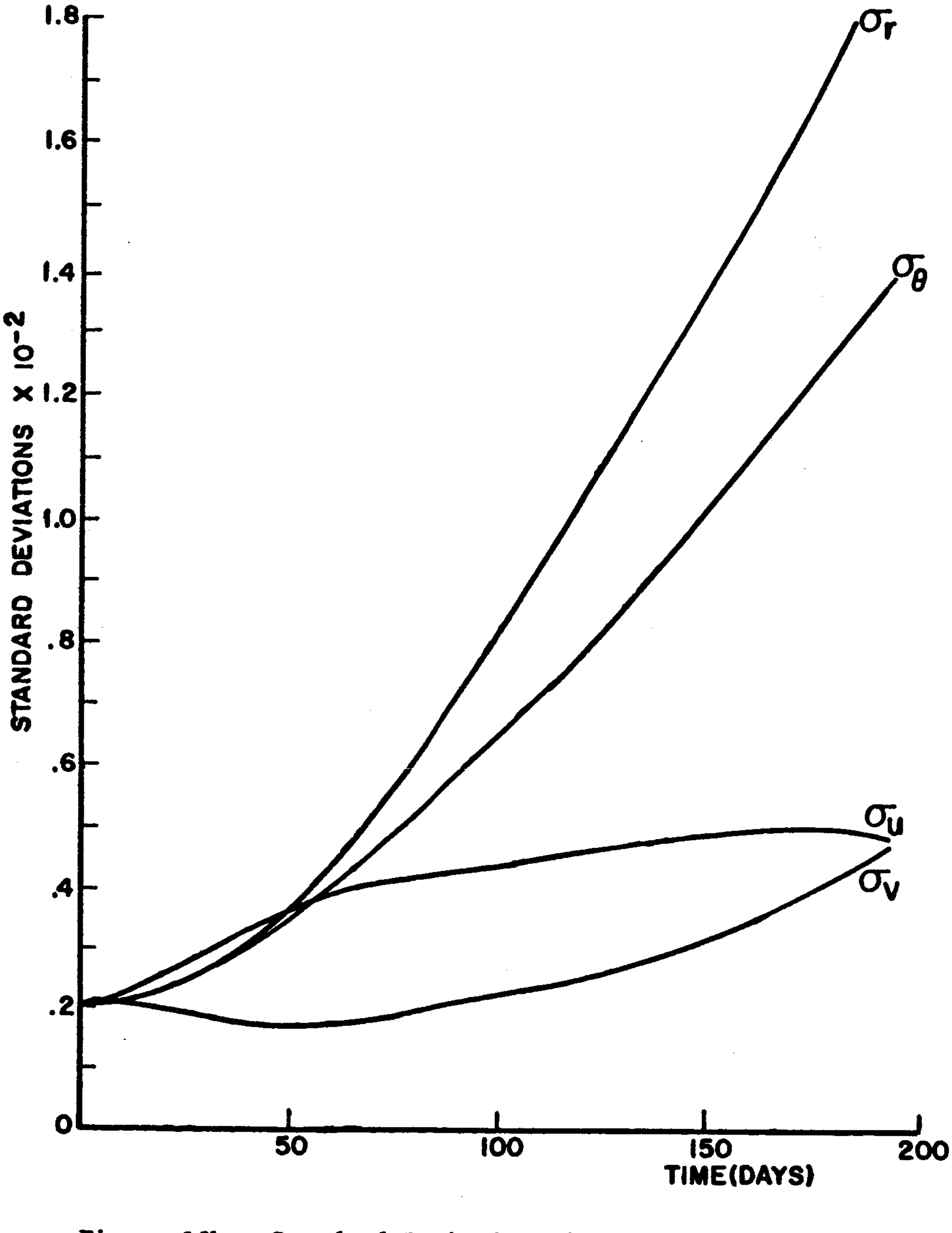

Figure 18b. Standard Deviation Time Histories (σ_a = .02T, T_a = 1 Day, σ_α = 0, $\sigma_0 = 2 \times 10^{-3}$)

In Figures 17 and 18, σ_0 ranges from 1×10^{-3} to 2×10^{-3}

$\sigma_a = .02T$

$T_a = 1$ day

$\sigma_\alpha = 0$

The plots in Figures 5 through 18 illustrate several important facts regarding the effects of noise on nonlinear deterministic optimal trajectories. The primary trends shown in the figures are summarized as follows,

1. The occurrence of noise in the equations of motion always implies that the mean trajectory will differ from the deterministic trajectory. This is a consequence of the nonlinearity of the equations of motion. The plots demonstrate that $\delta\bar{u}$, $\delta\bar{v}$, $\delta\bar{r}$, and $\delta\bar{\theta}$, are in general non-zero if noise occurs in either the thrust/mass magnitude or its direction.
2. The mean deviations $\delta\bar{u}$, $\delta\bar{v}$, $\delta\bar{r}$, and $\delta\bar{\theta}$, are all seen to increase as the perturbing noise standard deviation increases. See Figures 5a through 7a and Figures 11a through 13a. A tentative conclusion is that the larger the perturbing noise is, the larger the average deviation from the deterministic trajectory will be.
3. The mean deviations are also seen to increase with increasing correlation time. See Figures 8a through 10a and Figures 14a through 16a.
4. The standard deviations of the state, σ_u, σ_v, σ_r, and σ_θ, all increase with both increasing noise standard deviation and increasing noise correlation time. See Figures 5b through 16b.

5. Both the means and standard deviations of the state grow larger with increasing initial state uncertainties. See Figures 17 and 18.
6. The effect of the nonlinearity of the system is shown on the standard deviation plots, Figures 5b through 18b, especially for the case of noise occurring in the thrust orientation angle. Unlike the standard deviation histories in Figures 2 and 3, which show a monotonic increase of the standard deviations with time, the values of σ_u, σ_v, σ_r, and σ_θ are seen to show oscillatory tendencies. See especially Figures 11b through 16b.
7. The effect of the optimality of the deterministic trajectory is shown on the standard deviation plots especially in the case of large noise correlation times. For instance in Figure 10b, σ_u is seen to decrease after the rapid thrust direction change in the optimal deterministic trajectory. See Figure A.2.

Simulation of Stochastic Trajectories

In order that the effects of noise in the controls of a spacecraft on an Earth-Mars trajectory can be examined further, several sample trajectories are integrated using values for the perturbing noise generated from a random number generator. In particular, the sample trajectories are generated with noise occurring in the thrust/mass magnitude.

The equations which are integrated forward are the perturbed versions of Equations (3.28).

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + (a+\eta_a)\ \sin\alpha$$

$$\dot{v} = -\frac{uv}{r} + (a+\eta_a)\ \cos\alpha$$

$$\dot{r} = u$$

$$\dot{\theta} = \frac{v}{r} \tag{3.30}$$

where η_a is the sample of noise occurring in the thrust/mass magnitude. The values for η_a are generated recursively for the numerical integration by the formula

$$\eta_a(t_{i+1}) = x_{i+1}\sigma_a\sqrt{1-e^{-2\beta_a(t_{i+1}-t_i)}} + \eta_a(t_i)e^{-\beta_a(t_{i+1}-t_i)} \tag{3.31}$$

where the x_i are generated from a normal random number generator.

The components of the deterministic state, found by integrating Equations (3.28), are subtracted from the components of the sample trajectory state computed from the integration of Equations (3.30). The resulting components of the sample state deviation for one of the simulated trajectories are presented in Figure 19a. The corresponding sample perturbing noise, generated with the relation given in Equation (3.31), is plotted in Figure 19b.

The noise parameters for the sample trajectory in Figure 19 are listed here.

$\sigma_a = .05T \qquad \sigma_\alpha = 0 \qquad \sigma_0 = 0$

$T_a = 1$ day

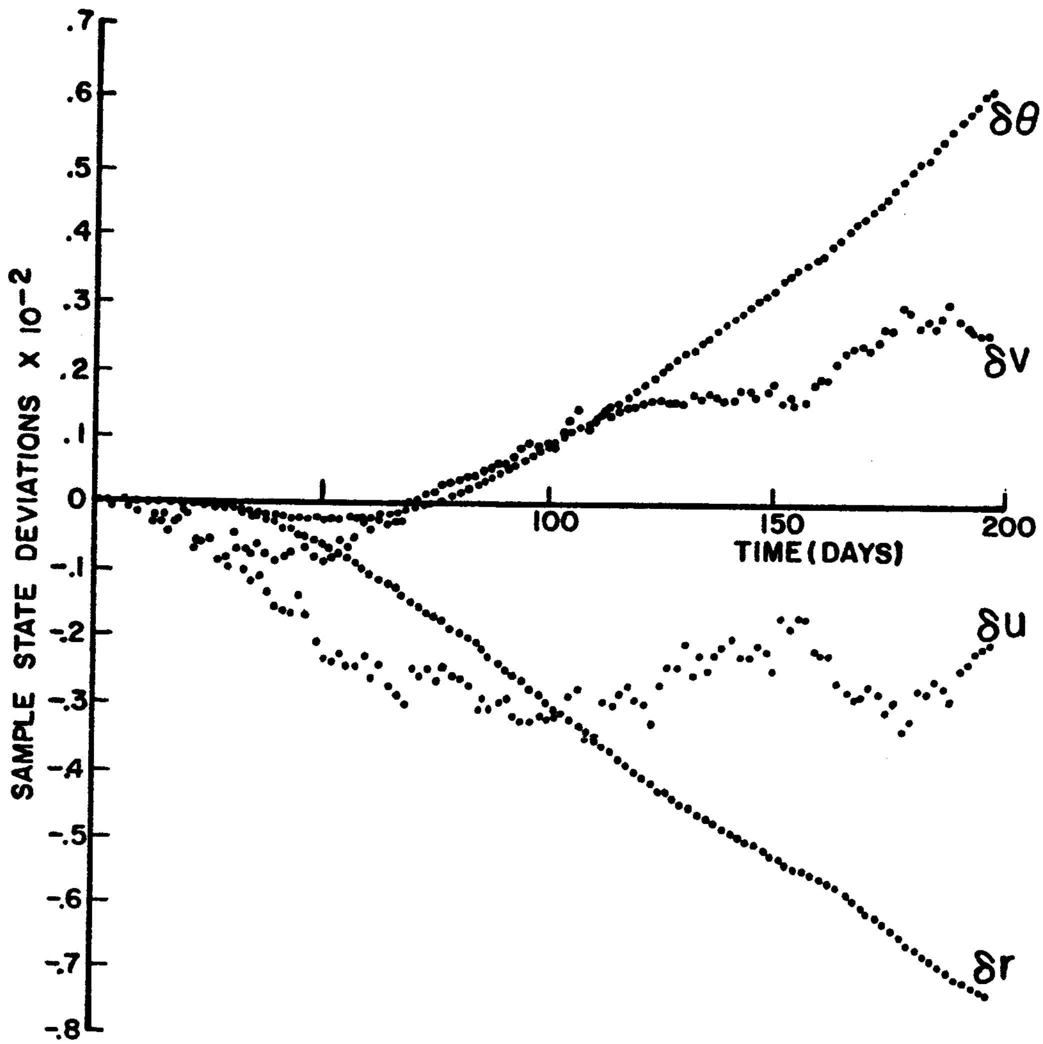

Figure 19a. Simulated State Deviation Time Histories

(σ_a= .05T, T_a= 1 Day, σ_α= 0, σ_0 = 0)

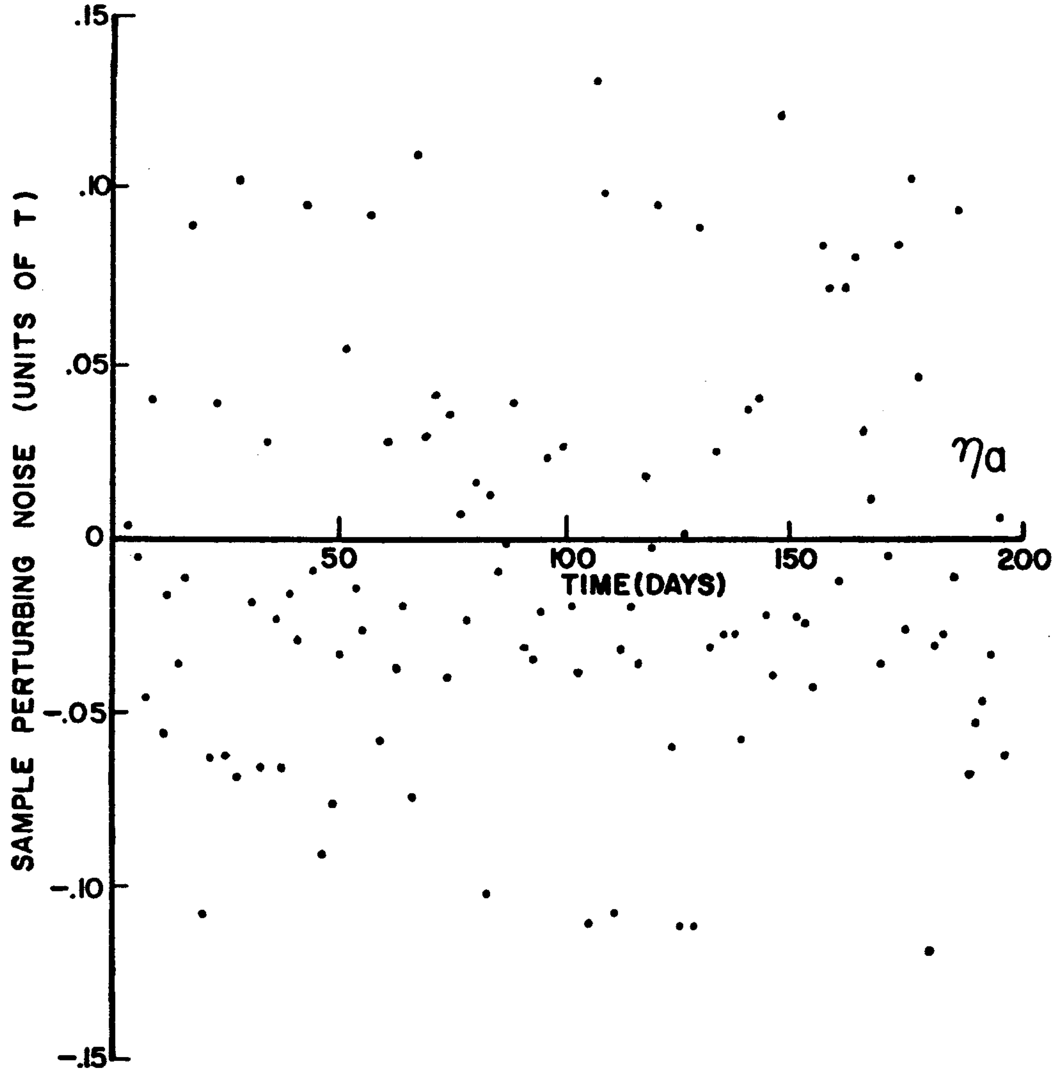

Figure 19b. Simulated O.U. Process Noise Occurring the Thrust/Mass Magnitude ($\sigma_a = .05T$, $T_a = 1$ Day)

Ten sample trajectories, with these same noise parameters have been used to generate sample means and standard deviations of the state, which can be compared to the theoretical means and standard deviations appearing in Figures 7a and 7b respectively.

The sample means are computed from the following formulas

$$\overline{\delta u} = \frac{\sum_{1}^{N} \delta u_i}{N}$$

$$\overline{\delta v} = \frac{\sum_{1}^{N} \delta v_i}{N}$$

$$\overline{\delta r} = \frac{\sum_{1}^{N} \delta r_i}{N}$$

$$\overline{\delta \theta} = \frac{\sum_{1}^{N} \delta \theta_i}{N} \tag{3.32}$$

with $N = 10$. The sample standard deviations are computed from the formulas

$$\sigma_u = \left[\frac{\sum_{1}^{N} (\delta u_i - \overline{\delta u})^2}{N-1}\right]^{\frac{1}{2}}$$

$$\sigma_v = \left[\frac{\sum_{1}^{N} (\delta v_i - \overline{\delta v})^2}{N-1}\right]^{\frac{1}{2}}$$

$$\sigma_r = \left[\frac{\sum_{1}^{N} (\delta r_i - \overline{\delta r})^2}{N-1}\right]^{\frac{1}{2}}$$

$$\sigma_\theta = \left[\frac{\sum_{1}^{N} (\delta \theta_i - \overline{\delta \theta})^2}{N-1}\right]^{\frac{1}{2}} \tag{3.33}$$

The time histories of the sample means and standard deviations of the state are shown in Figures 20a and 20b respectively. The sample mean and standard deviation of the noise η_a are computed with the formulas

$$\bar{\eta}_a = \frac{\sum_1^N \eta_{ai}}{N}$$

$$\sigma_a = \left[\frac{\sum_1^N (\eta_{ai} - \bar{\eta}_a)^2}{N-1}\right]^{\frac{1}{2}} \qquad (3.34)$$

The time histories of $\bar{\eta}_a$ and σ_a appear in Figures 20c and 20d, respectively. Since the sample mean and standard deviation of the noise η_a show a large dispersion about the theoretical values of $\bar{\eta}_a$ and σ_a, respectively, it can be concluded that many more trajectories would have to be included in the averaging in order to find close agreement between the sample means and standard deviations of the state, and their theoretical counterparts. However, the time histories of the sample standard deviations in Figure 20b are seen to resemble the theoretical standard deviation time histories for the same noise parameters, shown in Figure 7b.

Summary

The main reason for examining the effects of perturbing noise on an optimal deterministic trajectory is to determine if there is sufficient reason for developing a stochastic control, or, in other words, if there is sufficient reason for developing a control which

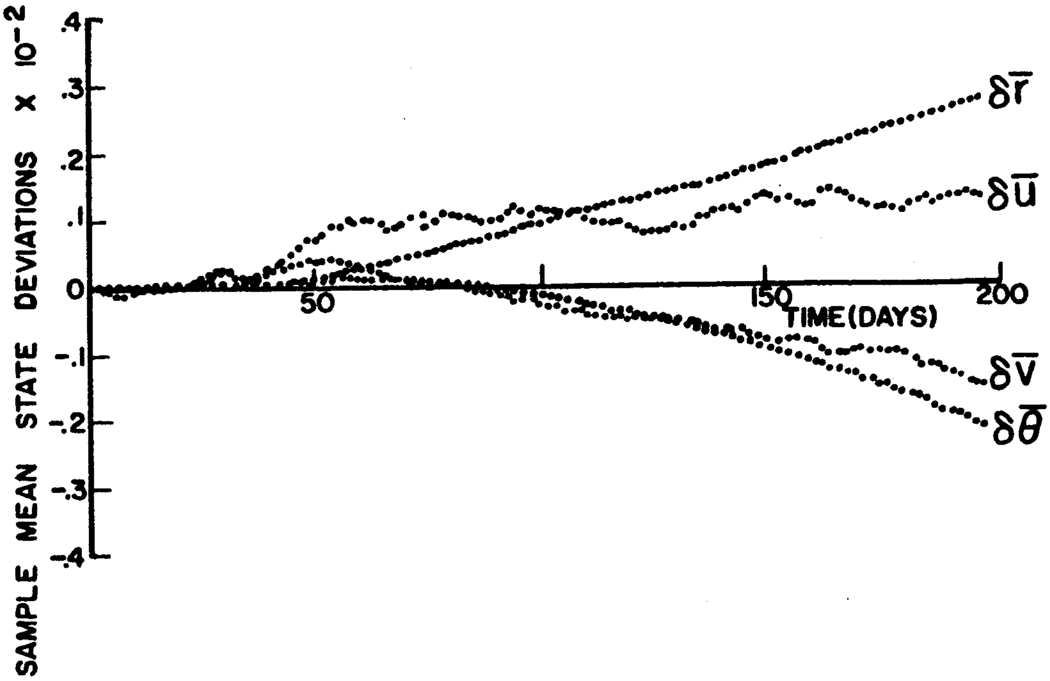

Figure 20a. Sample Mean State Deviation Time Histories

(σ_a= .05T, T_a= 1 Day, σ_α= 0, σ_0= 0, N = 10)

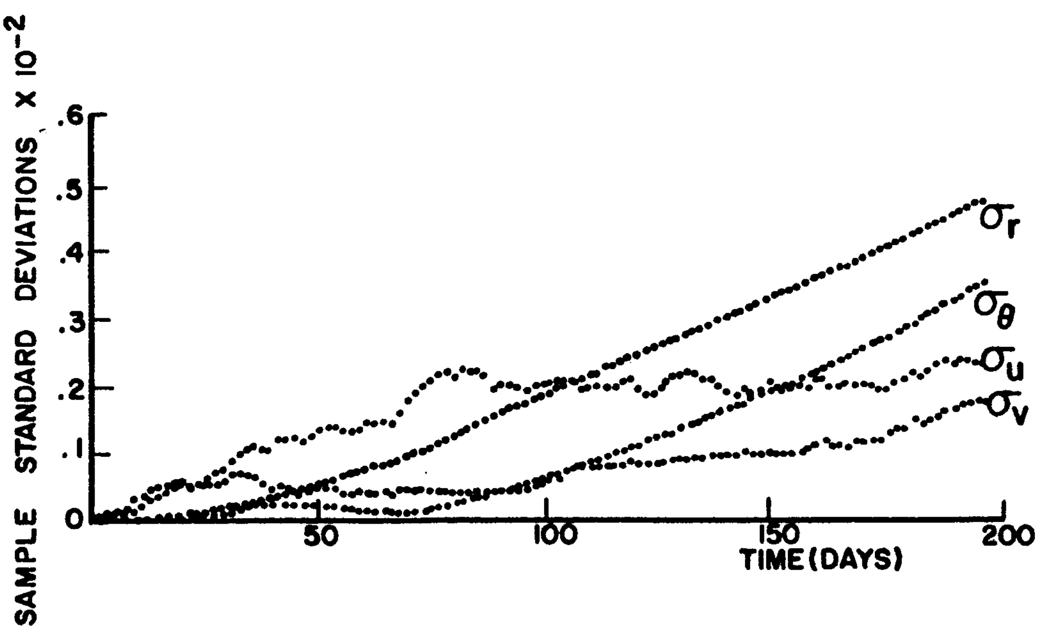

Figure 20b. Sample Standard Deviation Time Histories

(σ_a= .05T, T_a = 1 Day, σ_α= 0, σ_0= 0, N = 10)

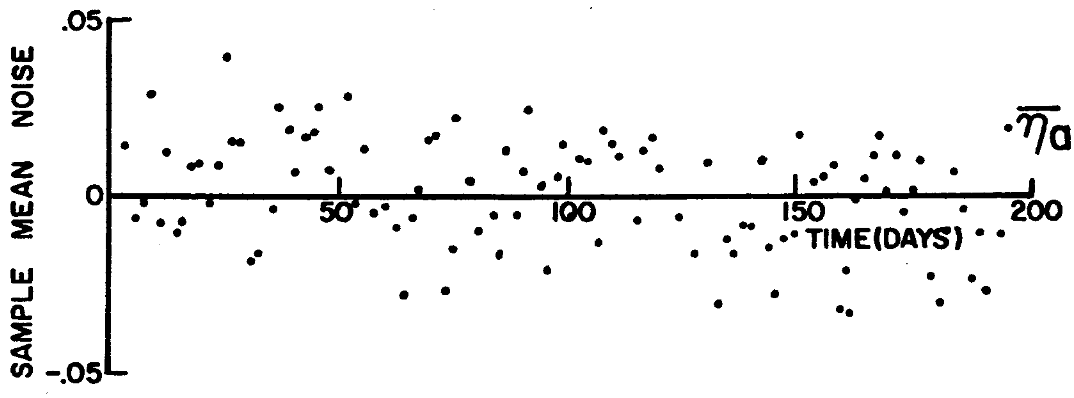

Figure 20c. Sample O.U. Process Noise Mean

(σ_a = .05T, T_a = 1 Day, N = 10)

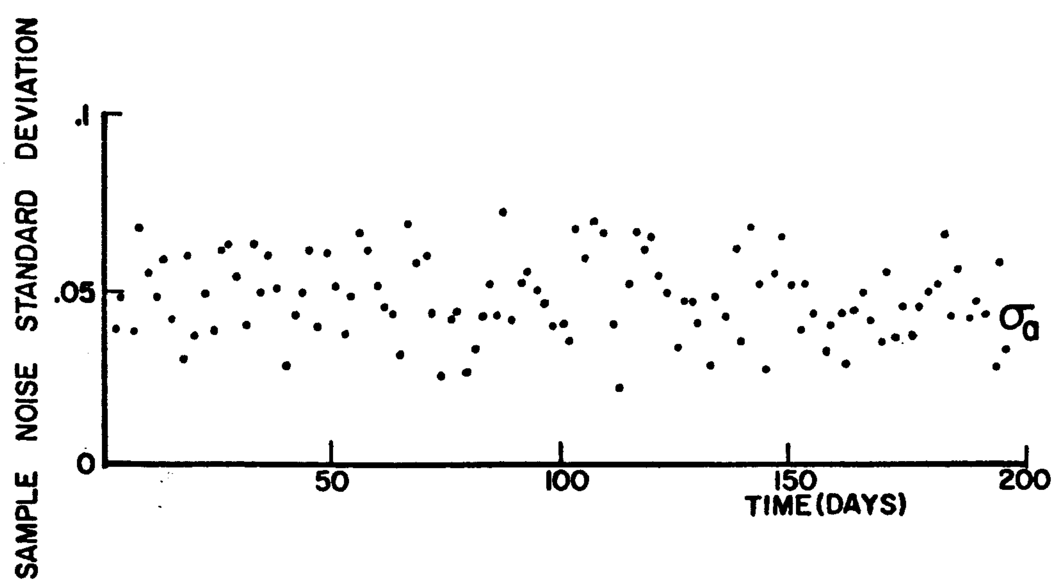

Figure 20d. Sample O.U. Process Noise Standard Deviation

(σ_a = .05T, T_a = 1 Day, N = 10) compensates for the expected effects of the perturbing noise. The theoretical results demonstrate that for the case of a nonlinear system the mean trajectory will always differ from the deterministic trajectory. Hence a stochastic control, as defined in the introduction, will bring the final state closer to the terminal conditions in an average sense. It should also be noted that in the case of the Earth-Mars transfer, the standard deviations are large compared with the mean deviations. This is true of all dynamic systems which are not too highly nonlinear. In such systems a method for updating the control program during the controlling interval is necessary in order to achieve a high degree of satisfaction of the terminal constraints.

CHAPTER 4

THE STOCHASTIC CALCULUS OF VARIATIONS APPLIED TO OPTIMAL STOCHASTIC CONTROL

Theoretical Development

The results of Chapter 3 indicate two important facts about the effect of small perturbing noise on an optimal deterministic trajectory. First, the mean of the ensemble of possible random trajectories differs from the deterministic trajectory, and second, the standard deviation of the state ensemble, in general, increases throughout the controlling interval $t_0 \leq t \leq t_f$. Both of these characteristics indicate an inadequacy of an optimal deterministic control for randomly perturbed dynamic systems. The first of the difficulties can be overcome by the determination of a control program which compensates for the expected effects of the perturbing noise on the system state. Such a control will be called optimal stochastic control. This chapter is devoted to the derivation of an optimal stochastic control procedure. The procedure is determined by utilizing a stochastic calculus of variations method which is analogous to the methods used in the deterministic calculus of variations.

In the theory of optimal deterministic control, (see Appendix A), the following set of differential equations is considered

$$\dot{x}_i = f_i(x,u,t) \qquad i = 1, \ldots, n \tag{4.1}$$

A set of controls $u_i(t)$, $i = 1, \ldots m$, is sought such that

$$I[u] = \int_{t_0}^{t_f} f_{n+1}(x,u,t)\, dt \tag{4.2}$$

is an extremum, subject to constraints at the initial and final times of the form

$$x_i(t_0) = x_{i0} \tag{4.3}$$

$$x_i(t_f) = x_{if}$$

For the optimal stochastic control problem, the following set of stochastic differential equations is considered

$$\dot{x}_i = f_i(x,u,\eta,t) \qquad i = 1, \ldots n \tag{4.4}$$

where, in the present study, $\eta_i(t)$, $i = 1, \ldots, m$, is additive noise perturbing the controls $u_i(t)$. The initial conditions for the x_i's are specified. That is,

$$E[x_i(t_0)] = x_{i0} \tag{4.5}$$

The functional to be extremized and the constraints at the final time cannot be formulated in a deterministic manner because of the presence of the noise in the differential equations. Hence, a set of nonrandom controls $u_i(t)$ is sought such that

$$I[u] = E\int_{t_0}^{t_f} f_{n+1}(x,u,t)\,dt \tag{4.6}$$

is an extremum, subject to the constraint that Equation (4.5) be satisfied at the specified initial time t_0, and that at the unspecified terminal time t_f

$$E[x_i(t_f)] = x_{if} \tag{4.7}$$

Also, the differential equations (4.1) must be satisfied at all points of time along the trajectory. The constraints given in Equation (4.5) and (4.7) are adjoined to the functional by means of unknown sets of constants μ_i and ν_i respectively, and the equations of motion are adjoined to the functional by means of a set of stochastic Lagrange multipliers $p_i(\eta,t)$. The constrained extremal value of $I[u]$ can be found by extremizing

$$J[u] = \nu_i[E(x_i(t_f) - x_{if})] + \mu_i[E(x_i(t_0) - x_{i0})] + E\int_{t_0}^{t_f} f_{n+1}(x,u,t)dt + E\int_{t_0}^{t_f} p_i(\dot{x}_i - f_i(x,u,\eta,t))\,dt \tag{4.8}$$

Since ν_i, μ_i, x_{if}, and x_{i0}, are deterministic quantities, the functional expressed in Equation (4.8) can be written as follows

$$J[u] = E\left[\nu_i(x_i(t_f)-x_{if}) + \mu_i(x_i(t_0)-x_{i0}) + \int_{t_0}^{t_f} f_{n+1}+p_i(\dot{x}_i-f_i)dt\right] \tag{4.9}$$

The expectation is taken over the adjoined differential equations of constraint,(4.1),so that the control can be found in terms of deterministic quantities, i.e., expectations over functions of the noise, instead of in terms of functions of the noise itself. The method of adjoining the differential equations of motion to the functional $I[u]$ with a stochastic Lagrange multiplier which depends on the noise

$n_i(t)$ has been suggested by Lass (Ref. 2), and has been discussed by Kushner (Ref. 1, 2).

A generalized stochastic Hamiltonian can be defined by

$$H(x,p,u,n,t) = p_i f_i - f_{n+1} \tag{4.10}$$

In view of Equation (4.10), the functional given in Equation (4.9) can be written as follows

$$J[u] = E\left[\nu_i(x_i(t_f) - x_{if}) + \mu_i(x_i(t_0) - x_{i0}) + \int_{t_0}^{t_f} (p_i\dot{x}_i - H)\, dt\right] \tag{4.11}$$

Now assume that the set of controls that extremizes $J[u]$ is $u_j'(t)$. Assume the correct values of ν_i, μ_i, and t_f, are ν_i', μ_i', and t_f'. Let a resulting trajectory, for a realizable sample of the $n_i(t)$ process, be $x_i'(t)$ with resulting Lagrange multipliers $p_i'(t)$. Then consider neighboring trajectories of the form (see Appendix A.)

$$\begin{aligned} x_i &= x_i' + \varepsilon\delta x_i & \mu_i &= \mu_i' + \varepsilon\delta\mu_i \\ u_i &= u_i' + \varepsilon\delta u_i & \nu_i &= \nu_i' + \varepsilon\delta\nu_i \\ p_i &= p_i' + \varepsilon\delta p_i & t_f &= t_f' + \varepsilon\delta t_f \end{aligned} \tag{4.12}$$

where the δx_i's, δu_i's, δp_i's, $\delta\nu_i$'s $\delta\mu_i$'s, and δt_f are arbitrary independent quantities, which in particular are independent of the noise $n_i(t)$. The constant ε is an independent parameter, which is also noise independent. Note that the functional $J[u]$ is a function of ε. The condition necessary for optimality of the control $u_i'(t)$ can be stated as follows

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \delta J = 0 \tag{4.13}$$

By making use of the commutative property of the derivative and the expected value operator, as shown in Chapter 2, δJ can be expressed as follows

$$\delta J = E\Bigg[\nu_i \frac{dx_i(t_f)}{d\varepsilon} + \frac{d\nu_i}{d\varepsilon}(x_i(t_f) - x_{if}) + \mu_i \frac{dx_i(t_0)}{d\varepsilon} + \frac{d\mu_i}{d\varepsilon}(x_i(t_0)-x_{i0}) + (p_i\dot{x}_i - H)\Big|_{t_f} \frac{dt_f}{d\varepsilon} + \int_{t_0}^{t_f} p_i \frac{d\dot{x}_i}{d\varepsilon} + \frac{dp_i}{d\varepsilon}\dot{x}_i - H_{x_i}\frac{dx_i}{d\varepsilon} - H_{p_i}\frac{dp_i}{d\varepsilon} - H_{u_i}\frac{du_i}{d\varepsilon}\, dt\Bigg] \tag{4.14}$$

Equation (4.14) reduces, in a manner like that of the deterministic problem in Appendix A, to the following expression

$$\delta J = E\Bigg[(\nu_i + p_i(t_f))(\delta x_i(t_f) + \dot{x}_i(t_f)\delta t_f) - H(t_f)\delta t_f + \delta\nu_i(x_i(t_f) - x_{if}) + (\mu_i - p_i(t_0))\delta x_i(t_0) + \delta\mu_i(x_i(t_0) - x_{i0}) - \int_{t_0}^{t_f} (\dot{p}_i + H_{x_i})\delta x_i + (H_{p_i} - \dot{x}_i)\delta p_i + H_{u_j}\delta u_j\, dt\Bigg] \tag{4.15}$$

Since the arbitrary functions δx_i, δu_i, δp_i, $\delta \mu_i$, $\delta \nu_i$, and δt_f are independent of the noise $n_i(t)$, δJ can be written in the following manner

$$\delta J = E(\nu_i+p_i(t_f))(\delta x_i(t_f)+\dot{x}_i(t_f)\delta t_f) - E[H(t_f)]\ \delta t_f$$

$$+ \delta\nu_i\ E[x_i(t_f)-x_{if}] + E(\mu_i-p_i(t_0))\delta x_i(t_0) + \delta\mu_i\ E[x_i(t_0)\ -x_{i0}]$$

$$- \int_{t_0}^{t_f} E[\dot{p}_i+H_{x_i}]\delta x_i + E[H_{p_i}-\dot{x}_i]\delta p_i + E[H_{u_i}]\ \delta u_i\ dt \qquad (4.16)$$

By the fundamental Lemma of the calculus of variations, the arbitrary nature of the terms δx_i, δu_i, δp_i, $\delta \mu_i$, $\delta \nu_i$, and δt_f, imply that their coefficients vanish identically. Thus the conditions necessary for the set x_i', u_i', p_i', $_i'$, $_i'$, t_f' to be an extremal solution are

$$E[\dot{x}_i - H_{p_i}] = 0 \qquad (4.17)$$

$$E[\dot{p}_i + H_{x_i}] = 0 \qquad (4.18)$$

$$E[H_{u_i}] = 0 \qquad (4.19)$$

at all points of time in the controlling interval $t_0 \leq t \leq t_f$,

$$E[x_i(t_0)-x_{i0} = 0 \qquad (4.20)$$

$$E[p_i(t_0-\mu_i] = 0 \qquad (4.21)$$

at the initial time t_0, and

$$E[x_i(t_f)-x_{if}] = 0 \qquad (4.22)$$

$$E[p_i(t_f)+\nu_i] \quad = \quad 0 \tag{4.23}$$

$$E[H(t_f)] \quad = \quad 0 \tag{4.24}$$

at the terminal time t_f.

The Equations (4.17), (4.18), and (4.19), with end conditions given in Equations (4.20), (4.21), (4.22), (4.23), and (4.24), theoretically yield an optimal control which takes in to account the expected effects of the perturbing noise on the state and Lagrange multipliers. It should be noted that although the control procedure derived from the solution of Equations (4.17), (4.18), and (4.19) is referred to as an optimal stochastic control, the control procedure is a nonrandom function of time, based on an a priori knowledge of the statistical behavior of the noise in the controls of the dynamic system.

In general the preceding equations are very difficult to evaluate, since the probability density functions necessary for the computation of the expected values are not readily available. Recall that in Chapter 3, an approximate differential equation, which describes the motion of the mean of a stochastic ensemble of trajectories, was developed in terms of the mean deviation from a deterministic trajectory. The differential equation for the mean deviation was developed by expanding the differential equation governing the stochastic trajectory about the differential equation governing the deterministic trajectory. Expected values were taken over the terms of the expansion in order to yield a differential equation for the mean deviation. The equation for the mean deviation was found to be driven by covariance components of the state deviation and the perturbing noise.

A similar procedure will be employed here in order to evaluate the necessary conditions of the optimal stochastic control problem. The conditions necessary for stochastic optimality, Equations (4.17) through (4.24), will be expanded about the solutions to the deterministic optimal control problem, derived in Appendix A. The resulting necessary conditions will be differential equations for the mean deviations of the state and the Lagrange multipliers. The differential equations will be driven by covarinace components of state, noise, and Lagrange multipliers, and by δu_i, the difference between the optimal stochastic control and the optimal deterministic control.

The deterministic necessary conditions, derived in Appendix A, will be stated here for the reader's convenience. The conditions are

$$\dot{x}_i^* - H_{p_i}^* = 0 \tag{4.25}$$

$$\dot{p}_i^* + H_{x_i}^* = 0 \tag{4.26}$$

$$H_{u_i}^* = 0 \tag{4.27}$$

at all points of time in the controlling interval $t_0 \leq t \leq t_f$,

$$x_i^*(t_0) = x_{i0} \tag{4.28}$$

$$p_i^*(t_0) = \mu_i^* \tag{4.29}$$

at the specified initial time t_0, and

$$x_i^*(t_f) = x_{if} \tag{4.30}$$

$$p_i^*(t_f) = -\nu_i^* \tag{4.31}$$

$$H^*(t_f) = 0 \tag{4.32}$$

at the unspecified terminal time t_f.

Consider first a Taylor expansion of the terms in Equation (4.17) about the deterministic solution given in Equation (4.25). The expansion can be written as follows

$$E[\delta\dot{x}_i + \dot{x}_i^* - f_i(x^*,u^*,t) - f_{ix_j}\delta x_j - f_{iu_j}(\delta u_j+\eta_j) - \frac{1}{2} f_{ix_jx_k}\delta x_j\delta x_k - f_{ix_ju_k}\delta x_j(\delta u_k+\eta_k) - \frac{1}{2} f_{iu_ju_k}(\delta u_j+\eta_j)(\delta u_k+\eta_k)] = 0 \tag{4.33}$$

In view of Equation (4.25), the following expression can be obtained.

$$E[\delta\dot{x}_i - f_{ix_j}\delta x_j - f_{iu_j}(\delta u_j+\eta_j) - \frac{1}{2} f_{ix_jx_k}\delta x_j\delta x_k - f_{ix_ju_k}\delta x_j(\delta u_k+\eta_k) - \frac{1}{2} f_{iu_ju_k}(\delta u_j+\eta_j)(\delta u_k+\eta_k)] = 0 \tag{4.34}$$

By making use of the commutative property of the derivative and the expectation operator, the Equation (4.34) reduces to

$$\delta\dot{\bar{x}}_i = f_{ix_j}\,\delta\bar{x}_j + f_{iu_j}\,\delta u_j + \frac{1}{2} f_{ix_jx_k} M_{jk} +$$

$$+\ f_{ix_ju_k}(h_{jk}+\delta\bar{x}_j\delta u_k) + \frac{1}{2} f_{iu_ju_k}\,(\delta u_j\delta u_k+R_{jk}) \qquad (4.35)$$

where, following the definition used in Chapter 3, M_{ij} is defined as follows

$$M_{ij} = E[\delta x_i\delta x_j] \qquad (4.36)$$

It should be noted that for a given control deviation $\delta u_i(t)$ and a given sample of the $\eta_i(t)$ process, the actual state deviation will obey the differential equation

$$\delta\dot{x}_i = f_{ix_j}\,\delta x_j + f_{iu_j}(\delta u_j+\eta_j) + \frac{1}{2} f_{ix_jx_k}\,\delta x_j\delta x_k +$$

$$f_{ix_jx_k}\,\delta x_j(\delta u_k+\eta_k) + \frac{1}{2} f_{iu_ju_k}\,(\delta u_j+\eta_j)(\delta u_k+\eta_k) \qquad (4.37)$$

By differentiating Equation (4.36), it is seen that M_{ij} obeys a differential equation of the form

$$M_{ij} = E[\delta\dot{x}_i\delta x_j] + E[\delta x_i\delta\dot{x}_j] \qquad (4.38)$$

By substituting Equation (4.37) into Equation (4.38), and neglecting terms of higher order than the second, the following expression can be derived

$$\dot{M}_{ij} = f_{ix_k} M_{kj} + M_{ik} f_{jx_k} + f_{iu_k}(\delta u_k \delta\overline{x}_j + h_{kj}) +$$

$$(\delta\overline{x}_i \delta u_k + h_{ik}) f_{ju_k} \tag{4.39}$$

where h_{ij} is defined as follows

$$h_{ij} = E[\delta x_i \eta_j] \tag{4.40}$$

From differentiating equation (4.40) with respect to time, it is seen that h_{ij} obeys the differential Equation (3.26), i.e.,

$$\dot{h}_{ij} = f_{ix_k} h_{kj} + f_{iu_k} R_{kj} - h_{ik} \beta_{kj} \tag{4.41}$$

Now consider the expansion of Equation (4.18) about the necessary condition of the deterministic problem, given in Equation (4.26). The expansion can be written as follows

[illegible] $+H_{x_i}{}^{*} + H_{x_i x_j} \delta x_j + H_{x_i u_j}(\delta u_j + \eta_j) +$

$$H_{x_i p_j} \delta p_j + \tfrac{1}{2} H_{x_i x_j x_k} \delta x_j \delta x_k + \tfrac{1}{2} H_{x_i u_j u_k} (\delta u_j + \eta_j)(\delta u_k + \eta_k) +$$

$$\tfrac{1}{2} H_{x_i p_j p_k} \delta p_j \delta p_k + H_{x_i x_j u_k} \delta x_j (\delta u_k + \eta_k) +$$

$$H_{x_i x_j p_k} \delta x_j \delta p_k + H_{x_i u_p p_k} (\delta u_j + \eta_j)\, \delta p_k] = 0 \tag{4.42}$$

Since H is linear in p_i, the terms which contain second derivatives of H with respect to p_i vanish. Subtracting the deterministic necessary condition, given in Equation (4.26), from Equation (4.42) leads to the following expression

$$\begin{aligned}\delta\dot{\bar{p}}_i = & -H_{x_i x_j}\delta\bar{x}_j - H_{x_i u_j}\delta u_j - H_{x_i p_j}\delta\bar{p}_j - \\ & \tfrac{1}{2}H_{x_i x_j x_k}M_{jk} - \tfrac{1}{2}H_{x_i u_j u_k}(\delta u_j \delta u_k + R_{jk}) - \\ & H_{x_i x_j u_k}\delta\bar{x}_j\delta u_k - H_{x_i x_j u_k}h_{jk} - \\ & H_{x_i x_j p_k}N_{jk} - H_{x_i u_j p_k}(\delta u_j \delta\bar{p}_k + f_{kj}) \end{aligned} \qquad (4.43)$$

where

$$N_{ij} = E[\delta x_i \delta p_j] \qquad (4.44)$$

and

$$f_{ij} = E[\delta p_i n_j] \qquad (4.45)$$

It should be noted that for a given control $\delta u_i(t)$ and a given sample of the noise process $n_i(t)$, the actual stochastic Lagrange multipliers, $\delta p_i(t)$, satisfy the differential equation

$$\delta\dot{p}_i = -H_{x_i x_j}\delta x_j - H_{x_i u_j}(\delta u_j+\eta_j) - H_{x_i p_j}\delta p_j -$$

$$\frac{1}{2}H_{x_i x_j x_k}\delta x_j \delta x_k - \frac{1}{2}H_{x_i u_j u_k}(\delta u_j+\eta_j)(\delta u_k+\eta_k) -$$

$$H_{x_i x_j u_k}\delta x_j(\delta u_k+\eta_k) - H_{x_i x_j p_k}\delta x_j \delta p_k -$$

$$H_{x_i u_j p_k}(\delta u_j+\eta_j)\delta p_k \tag{4.46}$$

Differential equations for N_{ij} and f_{ij} can be derived in a manner similar to the manner in which Equations (4.39) and (4.41) were derived. The resulting expressions are

$$\dot{N}_{ij} = f_{ix_k}N_{kj} - N_{ik}f_{kx_j} + f_{iu_k}(\delta u_k \delta\bar{p}_j + f_{jk}) -$$

$$M_{ik}H_{x_k x_j} - (\delta\bar{x}_i \delta u_k + h_{ik})H_{x_j u_k} \tag{4.47}$$

and

$$\dot{f}_{ij} = -H_{x_i x_k}h_{kj} - H_{x_i u_k}R_{kj} - H_{x_i p_k}f_{kj} - f_{ik}\beta_{kj} \tag{4.48}$$

Finally, consider the expansion of the stochastic optimality condition given in Equation (4.19) about the optimality condition of the deterministic control problem, given in Equation (4.27). The expansion can be written as follows

$$E[H_{u_i}^{*}+H_{u_ix_j}\delta x_j+H_{u_iu_j}(\delta u_j+\eta_j)+H_{u_ip_j}\delta p_j \;+$$

$$\frac{1}{2}H_{u_iu_ju_k}(\delta u_j+\eta_j)(\delta u_k+\eta_k)+\frac{1}{2}H_{u_ix_jx_k}\delta x_j\delta x_k \;+$$

$$\frac{1}{2}H_{u_ip_jp_k}\delta p_j\delta p_k+H_{u_ix_ju_k}\delta x_j(\delta u_k+\eta_k) \;+$$

$$H_{u_ix_jp_k}\delta x_j\delta p_k+H_{u_iu_jp_k}(\delta u_j+\eta_j)\delta p_k] = 0 \tag{4.49}$$

Since H is linear in p_i, and the deterministic quantity $H_{u_i}^{*}$ satisfies Equation (4.27), Equation (4.49) reduces to the following expression

$$H_{u_ix_j}\delta\overline{x}_j+H_{u_iu_j}\delta u_j+H_{u_ip_j}\delta\overline{p}_j \;+$$

$$\frac{1}{2}H_{u_iu_ju_k}(\delta u_j\delta u_k+R_{jk})+\frac{1}{2}H_{u_ix_jx_k}M_{jk} \;+$$

$$H_{u_ix_ju_k}(\delta\overline{x}_j\delta u_k+h_{jk})+H_{u_ix_jp_k}N_{jk} \;+$$

$$H_{u_iu_jp_k}(\delta u_j\delta\overline{p}_k+f_{kj}) = 0 \tag{4.50}$$

The set of Equations (4.35), (4.39), (4.41), (4.43), (4.47), (4.48), and (4.50), describes the behavior of the first and second order moments of the deviations of the state and Lagrange multipliers from their respective deterministic values. The constraints which must be satisfied at the initial and final times can be derived by expanding the stochastic conditions at the end points, i.e., Equations (4.20), (4.21), (4.22), (4.23), and (4.24), about their deterministic analogues, Equations (4.28), (4.29), (4.30), (4.31), and (4.32), respectively.

If the stochastic constraint given in Equation (4.20) is expanded about the deterministic constraint given in Equation (4.28), the following condition is obtained

$$\delta\bar{x}_i(t_0) = 0 \tag{4.51}$$

The expansion of the constraint given in Equation (4.21) about the constraint given in Equation (4.29) leads to the condition

$$\delta\bar{p}_i(t_0) = -\delta\nu_i \tag{4.52}$$

The stochastic condition given in Equation (4.22) can be expanded about the deterministic condition given in Equation (4.30) in the following manner

$$E[x_i^*(t_f) + \delta x_i(t_f) - x_{if}] = 0 \tag{4.53}$$

The term $x_i^*(t_f)$ can be approximated by $x_i^*(t_f^*) + \dot{x}_i^*(t_f - t_f^*)$. By substituting this approximation into Equation (4.53), and subtracting out the deterministic terms, the following constraint can be derived

$$\dot{x}_i^*(t_f^*)\delta\bar{t}_f + \delta\bar{x}_i = 0 \tag{4.54}$$

The expansion of Equation (4.23) about the deterministic condition given in Equation (4.31) leads to the condition

$$\delta\overline{p}_i(t_f) = \delta\nu_i \tag{4.55}$$

Finally, the condition given in Equation (4.24) can be expanded about the deterministic solution given in Equation (4.32), in the following manner

$$E[H^*(t_f) + H_{x_i}\delta x_i(t_f) + \frac{1}{2}H_{x_ix_j}\delta x_i\delta x_j +$$

$$H_{p_i}\delta p_i(t_f) + \frac{1}{2}H_{p_ip_j}\delta p_i\delta p_j + H_{u_i}(\delta u_i+\eta_i) +$$

$$\frac{1}{2}H_{u_iu_j}(\delta u_i+\eta_i)(\delta u_j+\eta_j) + H_{x_iu_j}\delta x_i(\delta u_j+\eta_j) +$$

$$H_{x_ip_j}\delta x_i\delta p_j + H_{p_iu_j}\delta p_i(\delta u_j+\eta_j)] = 0 \tag{4.56}$$

$H^*(t_f)$ can be approximated by $H^*(t_f^*) + H^*(t_f-t_f^*)$, and Equation (4.56) reduces to the following expression

$$\dot{H}^*(t_f^*)\delta\overline{t}_f + H_{x_i}\delta\overline{x}_i(t_f) + \frac{1}{2}H_{x_ix_j}M_{ij}(t_f) +$$

$$H_{p_i}\delta\overline{p}_i(t_f) + H_{u_j}\delta u_i(t_f) + \frac{1}{2}H_{u_iu_j}(\delta u_i(t_f)\delta u_j(t_f)+R_{jk}(t_f)) +$$

$$H_{x_iu_j}(\delta\overline{x}_i(t_f)\delta u_j(t_f) + h_{ij}(t_f)) + H_{x_ip_j}N_{ij}(t_f) +$$

$$H_{p_iu_j}f_{ij}(t_f) = 0 \tag{4.57}$$

In theory, the stochastic optimality condition given in Equation (4.50) can be solved for $\delta u_j(t)$, and the solution can be substituted into the remaining differential equations in order to eliminate the control from the analysis. The differential equations, i.e., Equations (4.35), (4.39), (4.41),(4.43), (4.47), and (4.48), then form $2[n+n^2+nm]$ equations involving $\delta\overline{x}_i$, $\delta\overline{p}_i$, M_{ij}, N_{ij}, h_{ij}, and f_{ij}. These equations must satisfy the $2n+1$ terminal constraints given by Equations (4.54), (4.55), and (4.57), at the final time t_f, and must also satisfy the $2n$ conditions given by Equations (4.51) and (4.52) at the initial time t_0. In addition the equations must satisfy the following specified initial conditions at t_0

$$M_{ij}(t_0, t_0) = M_{ij0}$$

$$N_{ij}(t_0, t_0) = 0$$

$$h_{ij}(t_0, t_0) = 0$$

$$f_{ij}(t_0, t_0) = 0 \tag{4.58}$$

The set of differential equations and the end conditions form a two point boundary value problem with split end conditions, which can be solved by a number of existing numerical methods for the remaining unspecified end conditions $\delta\nu_i$, $\delta\mu_i$, and $\delta\overline{t}_f$. (See Ref. 12) The solution to the boundary value problem will theoretically yield the optimal time histories of $\delta\overline{x}_i(t)$ and $\delta\overline{p}_i(t)$, from which the optimal stochastic control deviation $\delta u_j(t)$ can be found. The approximate solutions to the original stochastic necessary conditions can be stated theoretically as follows

$$E[x_i(t)] = x_i^*(t) + \delta\bar{x}_i(t)$$

$$E[p_i(t)] = p_i^*(t) + \delta\bar{p}_i(t)$$

$$u_j(t) = u_j^*(t) + \delta u_j(t) \tag{4.59}$$

Application To The Space Guidance Problem

The results derived in the previous sections of this chapter will now be applied to the low-thrust Earth-Mars transfer problem studied in Chapter 3. It is seen from the curves at the end of Chapter 3 that the standard deviations of the state associated with a stochastic ensemble of randomly perturbed trajectories are in general much larger than the respective mean deviations from the deterministic trajectory. This is indicative of a dynamic system which is not too highly nonlinear. In such systems, the optimal stochastic control correction $\delta u_i(t)$, which corrects the deterministic control in such a manner to take into account the expected effects of the noise on the state, is expected to have a smaller effect on the system than the noise itself. This leads to the assumption that the control deviation derived previously in this chapter, is much smaller than the standard deviation of the perturbing noise, i.e.,

$$\delta u_i(t) < < \sigma_{ii} \qquad \text{i not summed} \tag{4.60}$$

The assumption given in Equation (4.60) will be incorporated into the differential equations, when applied to the interplanetary transfer problem, by neglecting all second order terms containing $\delta u_i(t)$, that is, by neglecting all terms containing the products $\delta u_i \delta u_j$, $\delta u_i \delta \overline{x}_j$, and $\delta u_i \delta \overline{p}_j$.

It should also be noted that the equations of motion of the interplanetary transfer, i.e., Equations (3.28), fall into a class of differential equations which can be separated into the following form

$$\dot{x}_i = f_i{}'(x,t) + f_i{}''(u,t) \tag{4.61}$$

It also should be noted that the functional to be extremized, I[u], which can be written as follows

$$I[u] = E \int_{t_0}^{t_f} 1 \, dt \tag{4.62}$$

falls into a class of functionals which can be written in the form

$$I[u] = E \int_{t_0}^{t_f} f_{n+1}{}'(x,t) + f_{n+1}{}''(u,t) \, dt \tag{4.63}$$

The generalized Hamiltonian for a variational problem involving a functional of the type given in Equation (4.63) and differential equations of the type given in Equation (4.61) can be written in the following manner

$$H = p_i \, f_i' - f_{n+1}' + p_i \, f_i'' - f_{n+1}'' \tag{4.64}$$

By grouping the terms in Equation (4.64) in a proper manner, the generalized Hamiltonian, under the restrictions given in Equation

(4.61) and (4.63), becomes separable in the state and control, i.e.,

$$H = H'(x,t) + H''(u,t) \tag{4.65}$$

Thus the cross partial derivatives of f_i and H with respect to the state and control vanish, i.e.,

$$\begin{aligned} f_{ix_j u_k} &= 0 \\ H_{x_i u_k} &= 0 \\ H_{u_i x_j x_k} &= 0 \\ H_{u_i u_j x_k} &= 0 \\ H_{u_i x_j p_k} &= 0 \end{aligned} \tag{4.66}$$

In view of the assumption imposed by Equation (4.60) and the conditions given by Equation (4.66), the differential equations to be applied to the interplanetary transfer are the following

$$\begin{aligned} \delta\dot{\bar{x}}_i &= f_{ix_j}\delta\bar{x}_j + f_{iu_j}\delta u_j + \frac{1}{2} f_{ix_j x_k} M_{jk} + \frac{1}{2} f_{iu_j u_k} R_{jk} \\ \delta\dot{\bar{p}}_i &= - H_{x_i x_j}\delta x_j - H_{x_i p_j}\delta\bar{p}_j - \frac{1}{2} H_{x_i x_j x_k} M_{jk} - H_{x_j x_j p_k} N_{jk} \\ \dot{M}_{ij} &= f_{ix_k} M_{kj} + M_{ik} f_{jx_k} + f_{iu_k} h_{kj} + h_{ik} f_{ju_k} \\ \dot{N}_{ij} &= f_{ix_k} N_{kj} - N_{ik} f_{kx_j} + f_{iu_k} f_{kj} - M_{ik} H_{x_k x_j} \\ \dot{h}_{ij} &= f_{ix_k} h_{kj} + f_{iu_k} R_{kj} + h_{ik}\beta_{kj} \\ \dot{f}_{ij} &= - H_{x_i x_k} h_{kj} - H_{x_i p_j} f_{kj} - f_{ik}\beta_{kj} \end{aligned} \tag{4.67}$$

with the optimality condition

$$H_{u_iu_j}\,\delta u_j + H_{u_ip_j}\,\delta p_j + \frac{1}{2} H_{u_iu_ju_k}\,R_{jk} + H_{u_iu_jp_k}\,f_{kj} = 0 \qquad (4.68)$$

Matrix formulations of the terms in Equations (4.67) and (4.68), applied to the Earth-Mars transfer, appear in Appendix B. The optimal corrective control, $\delta\alpha(t)$, and the resulting mean state deviations $\delta\bar{u}$, $\delta\bar{v}$, $\delta\bar{r}$, and $\delta\bar{\theta}$, have been computed for several cases, and the results are illustrated in Figures 21 through 31. The plots labeled a illustrate the time histories of the mean deviations of the state from the deterministic trajectory. The plots labeled b show the corrective optimal control $\delta\alpha(t)$. $\delta\alpha$ is plotted in degrees.

The parameters of interest in the following respective figures are:

For Figure 21,	$\sigma_a = .05T$
	$T_a = 1$ day
	$\sigma_\alpha = 0$
	$\sigma_0 = 0$
For Figures 22 through 24,	T_a ranges from 10 days to 1000 days
	$\sigma_a = .02T$
	$\sigma_\alpha = 0$
	$\sigma_0 = 0$
For Figure 25,	$\sigma_\alpha = 1°$
	$T_\alpha = 1$ day
	$\sigma_a = 0$
	$\sigma_0 = 0$

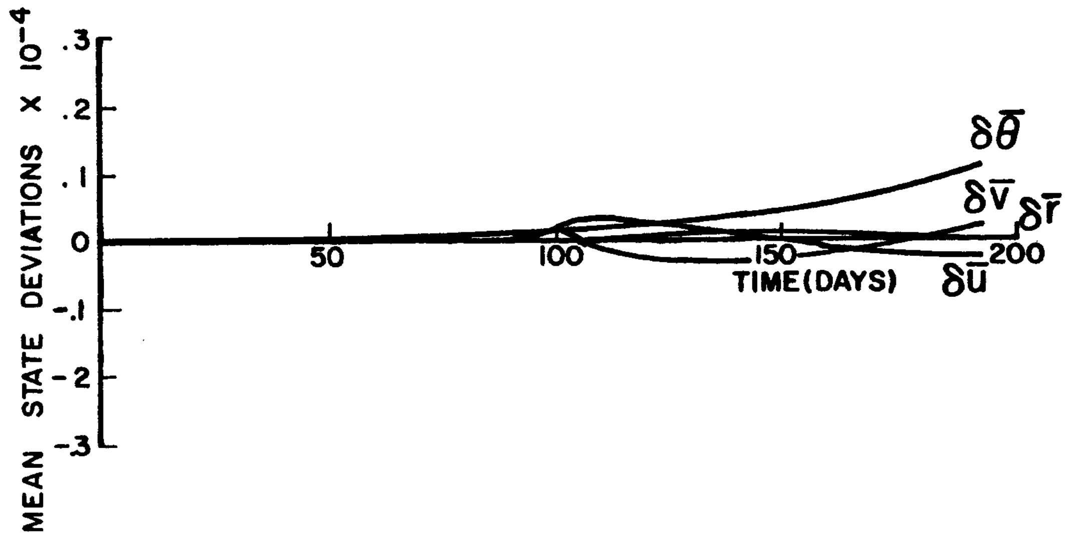

Figure 21a. Optimal Mean State Deviation Time Histories

(σ_a= .05T, T_a= 1 Day, σ_α= 0, σ_0= 0)

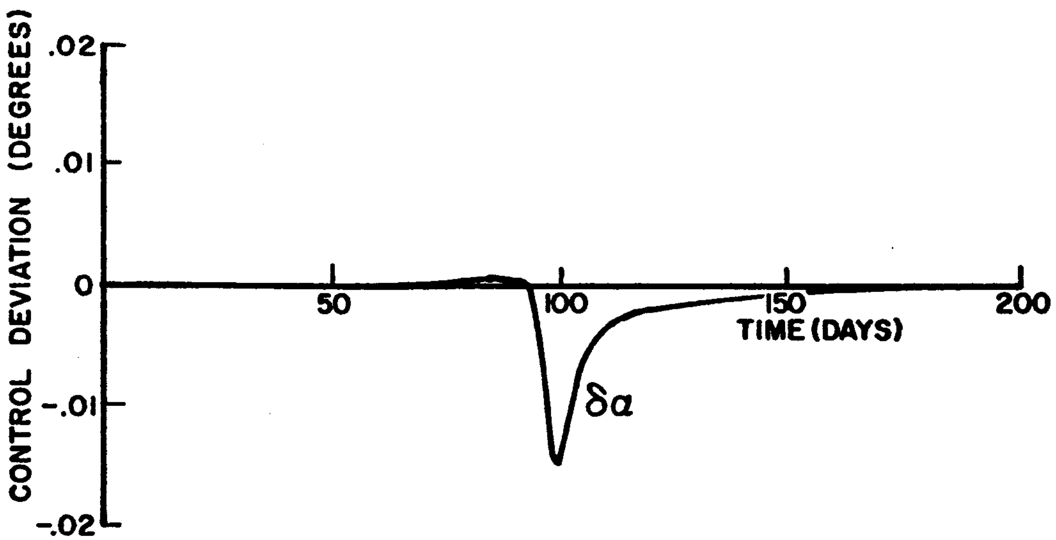

Figure 21b. Optimal Control Deviation Time History

(σ_a= .05T, T_a= 1 Day, σ_α= 0, σ_0= 0)

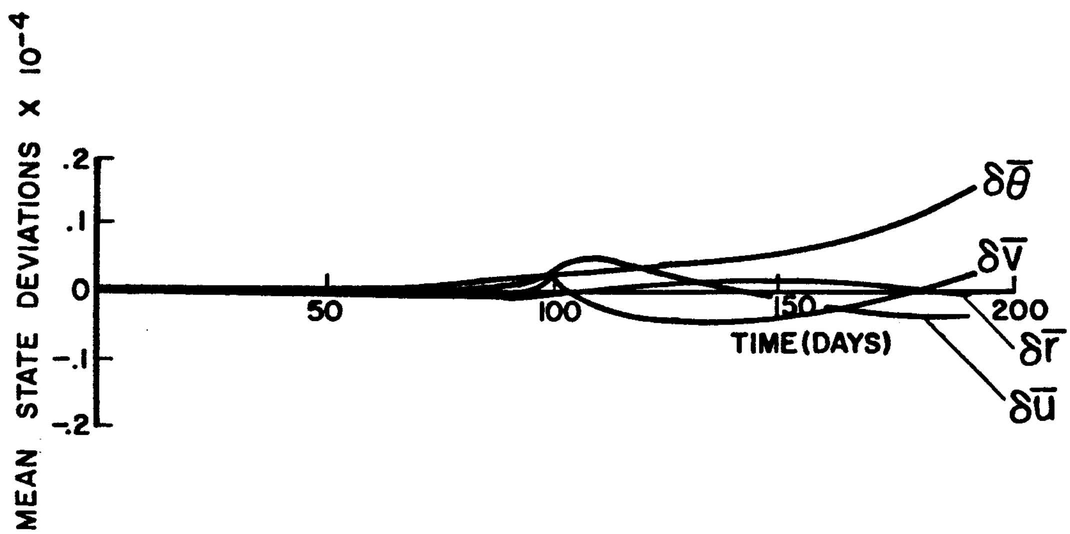

Figure 22a. Optimal Mean State Deviation Time Histories

(σ_a= .02T, T_a= 10 Days, σ_α= 0, σ_0= 0)

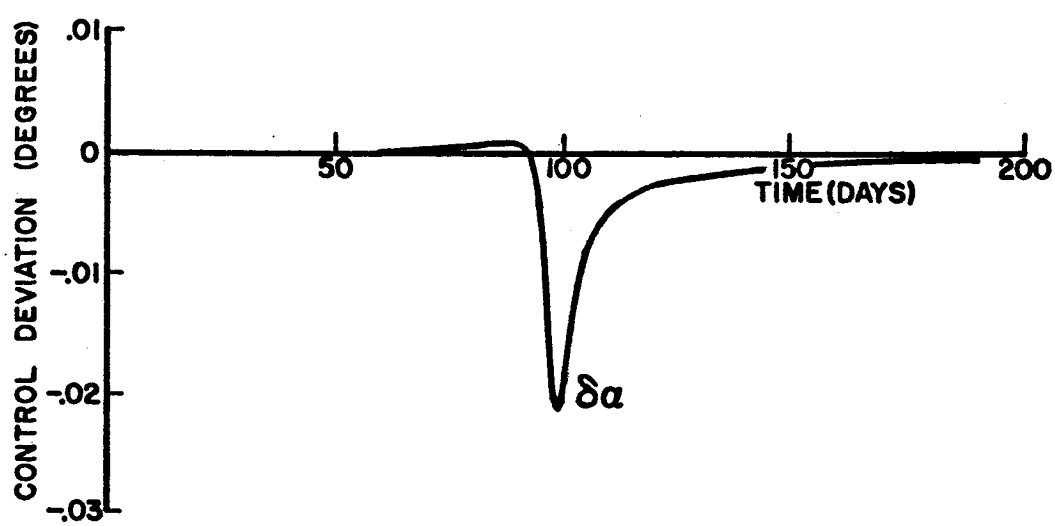

Figure 22b. Optimal Control Deviation Time History

(σ_a= .02T, T_a= 10 Days,σ_α= 0, σ_0= 0)

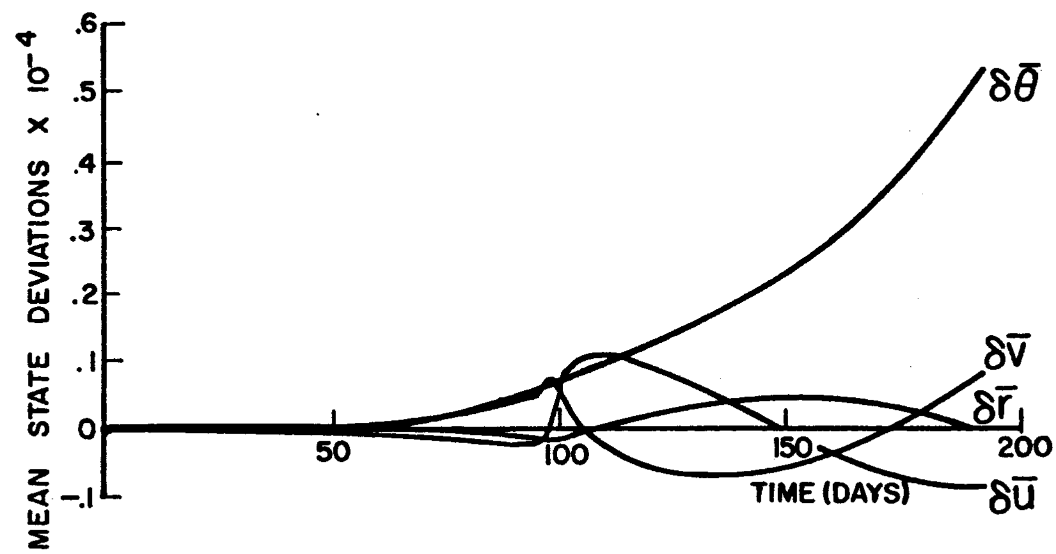

Figure 23a. Optimal Mean State Deviation Time Histories

(σ_a= .02T, T_a= 100 Days, σ_α= 0, σ_0= 0)

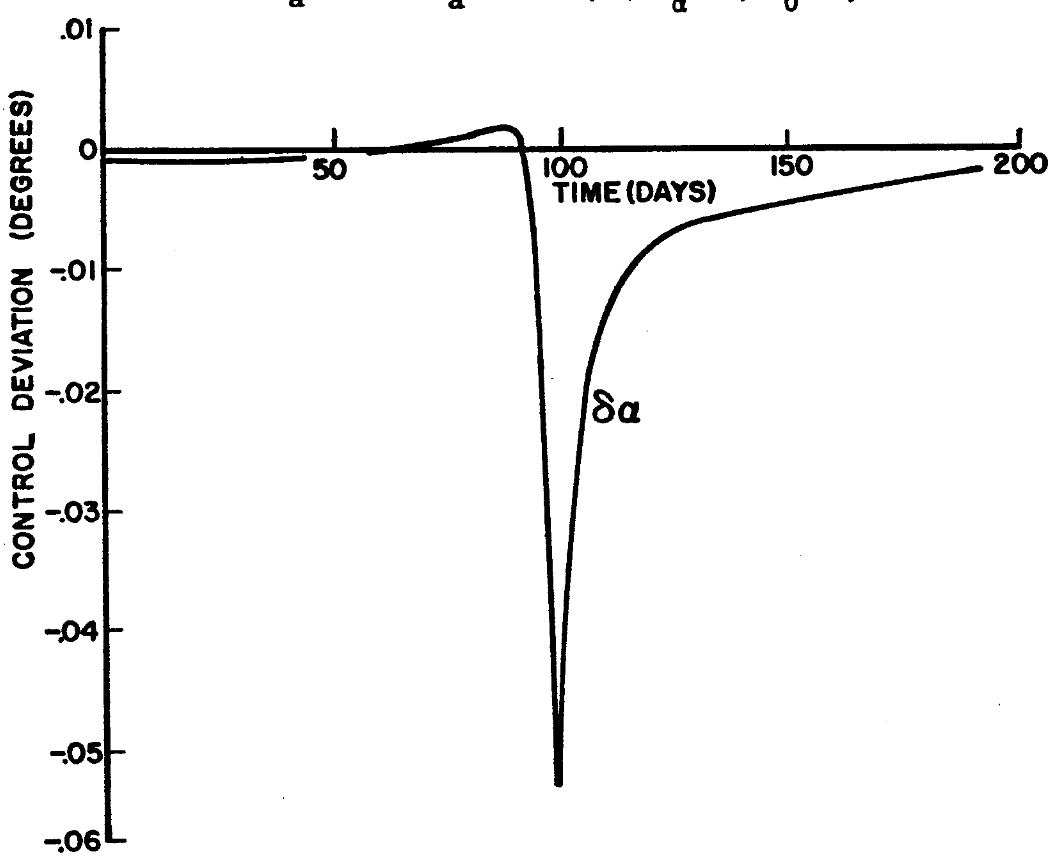

Figure 23b. Optimal Control Deviation Time History

(σ_a= .02T, T_a= 100 Days, σ_α= 0, σ_0= 0)

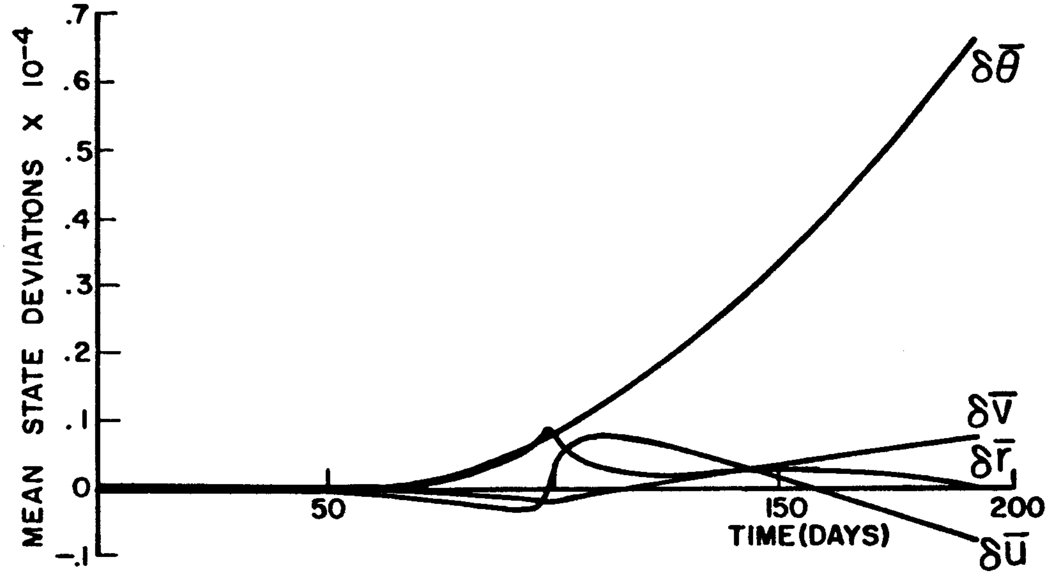

Figure 24a. Optimal Mean State Deviation Time Histories

(σ_a= .02T, T_a= 1000 Days, σ_α= 0, σ_0= 0)

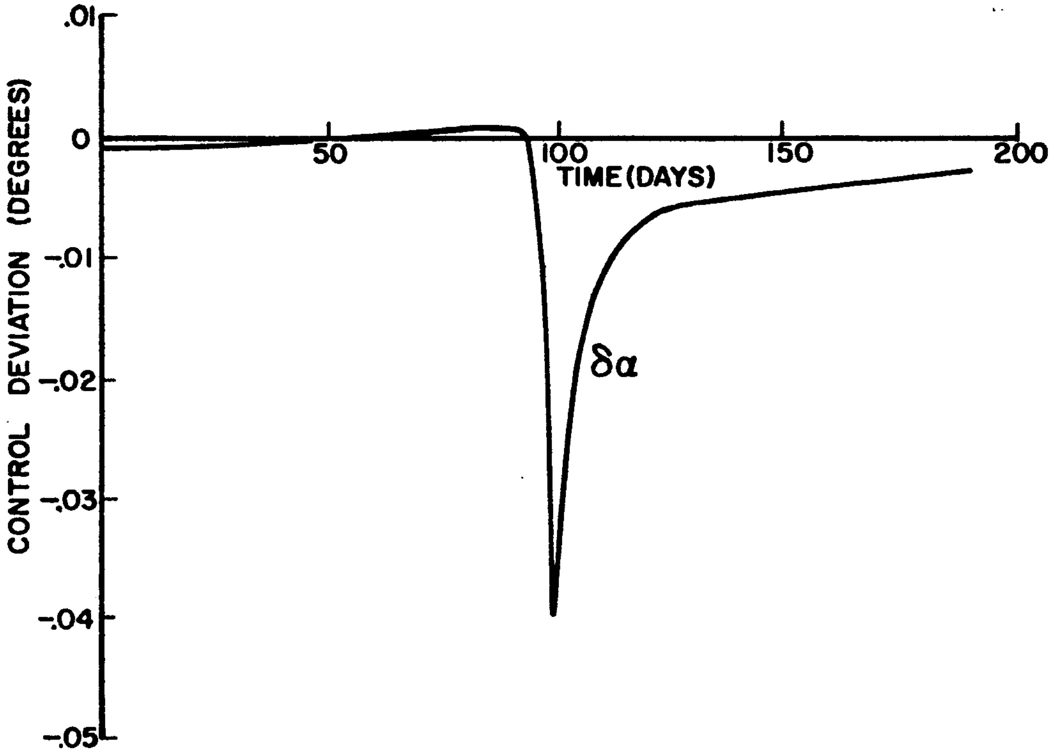

Figure 24b. Optimal Control Deviation Time History

(σ_a= .02T, T_a= 1000 Days, σ_α= 0, σ_0= 0)

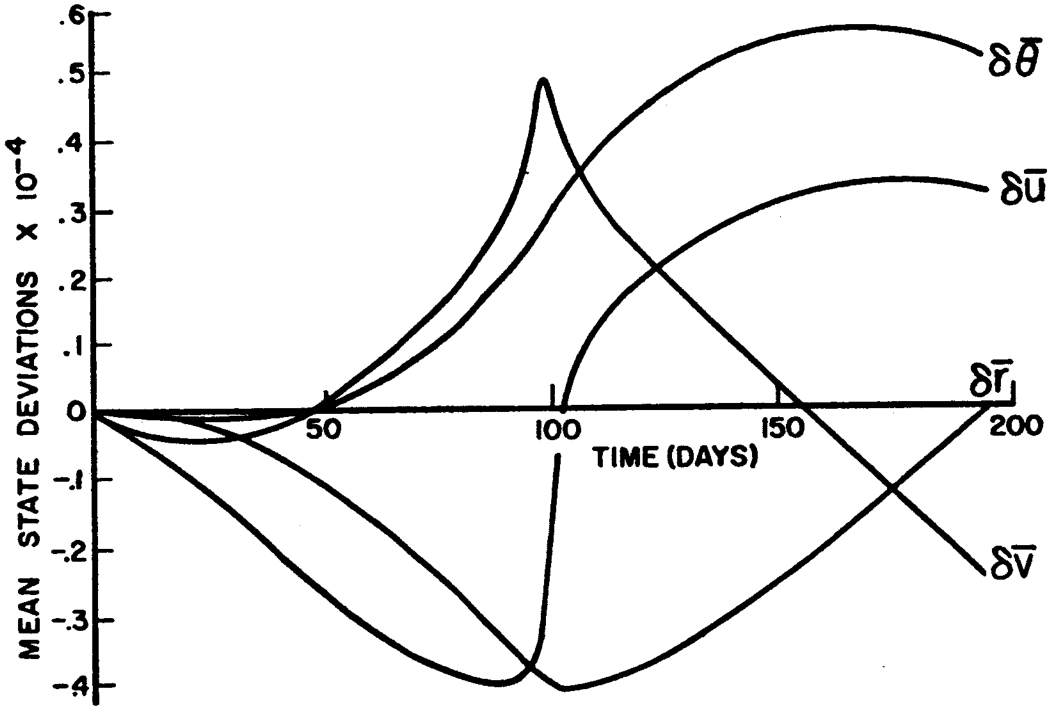

Figure 25a. Optimal Mean State Deviation Time Histories ($\sigma_a = 0$, $\sigma_\alpha = 1°$, T = 1 Day, $\sigma_0 = 0$)

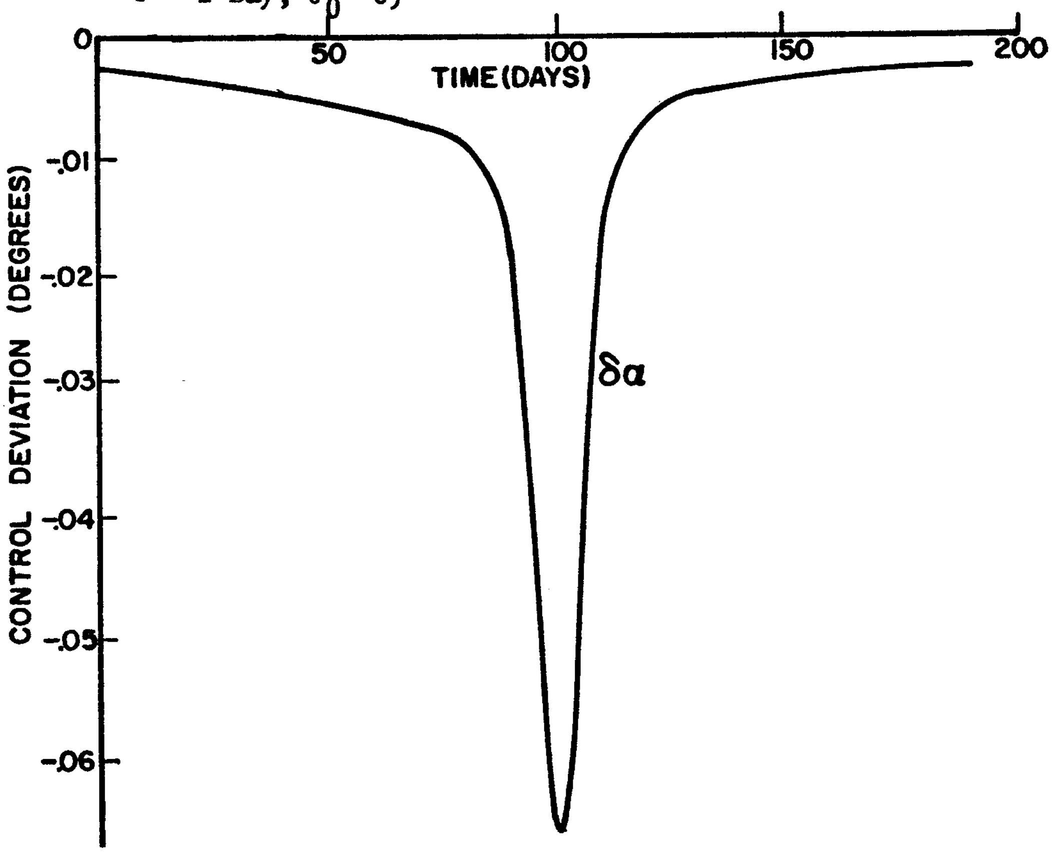

Figure 25b. Optimal Control Deviation Time History

($\sigma_a = 0$, $\sigma_\alpha = 1°$, T_α = 1 Day, $\sigma_0 = 0$)

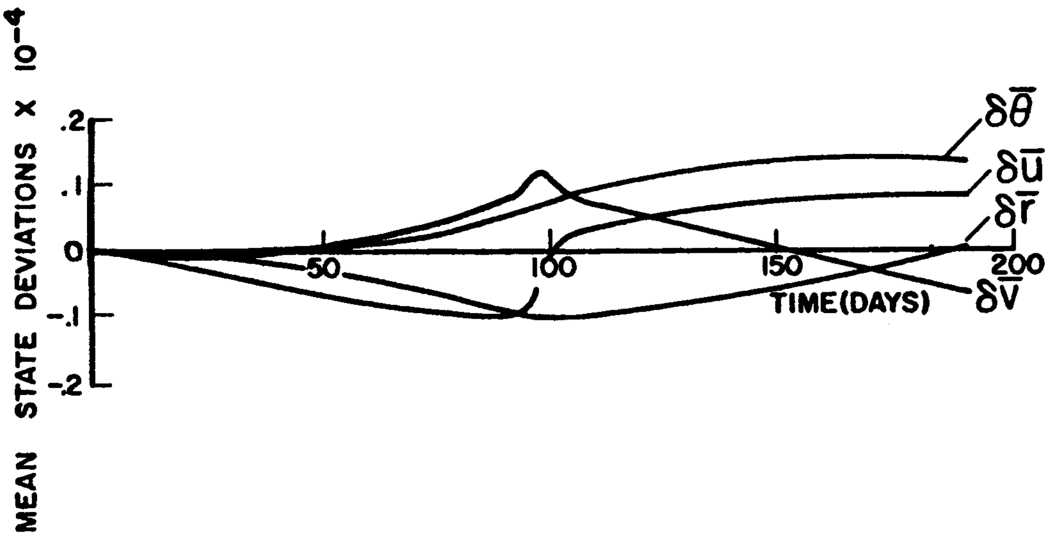

Figure 26a. Optimal Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

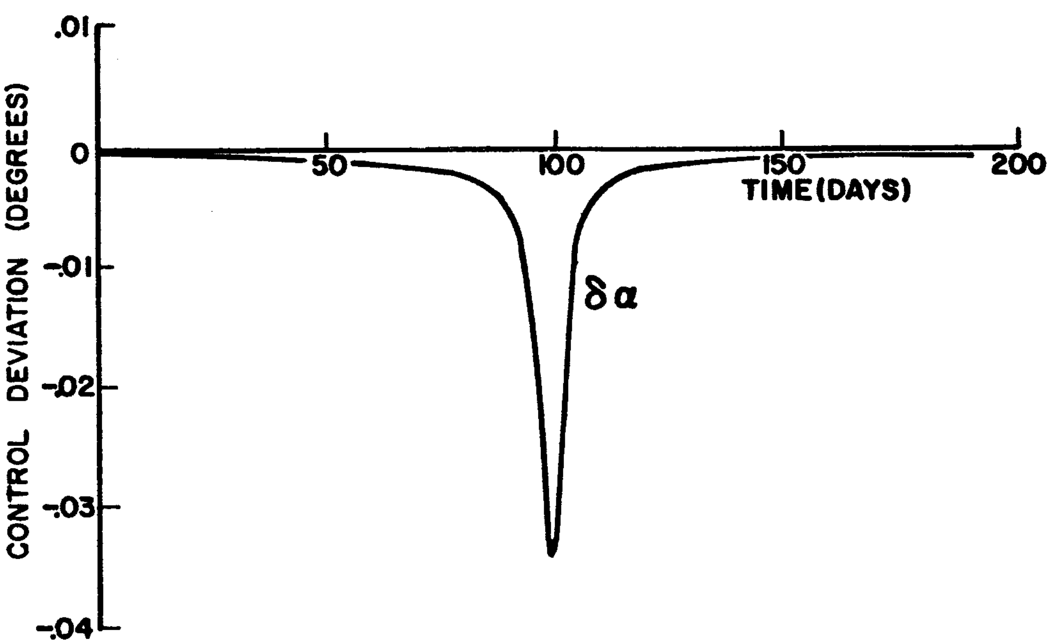

Figure 26b. Optimal Control Deviation Time History

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 1$ Day, $\sigma_0 = 0$)

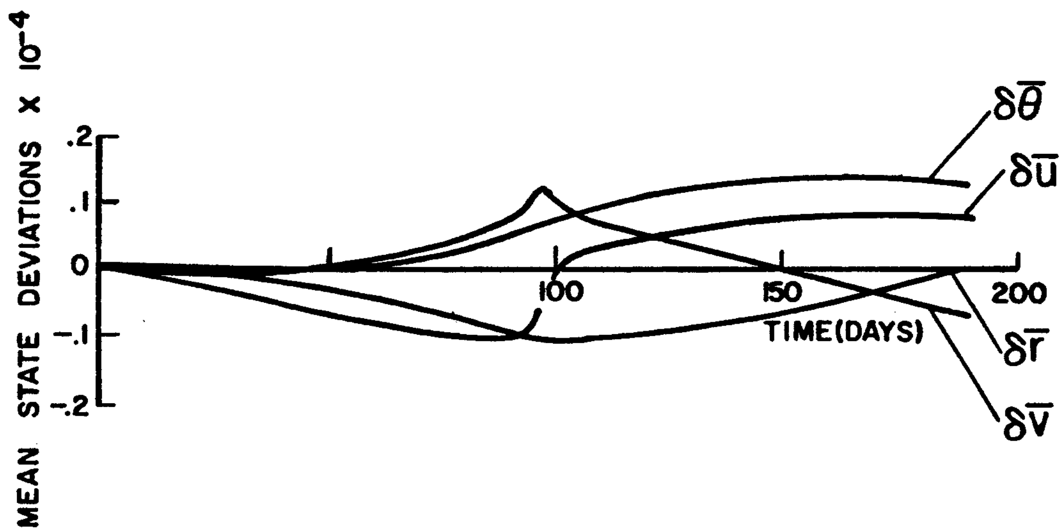

Figure 27a. Optimal Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 10$ Days, $\sigma_0 = 0$)

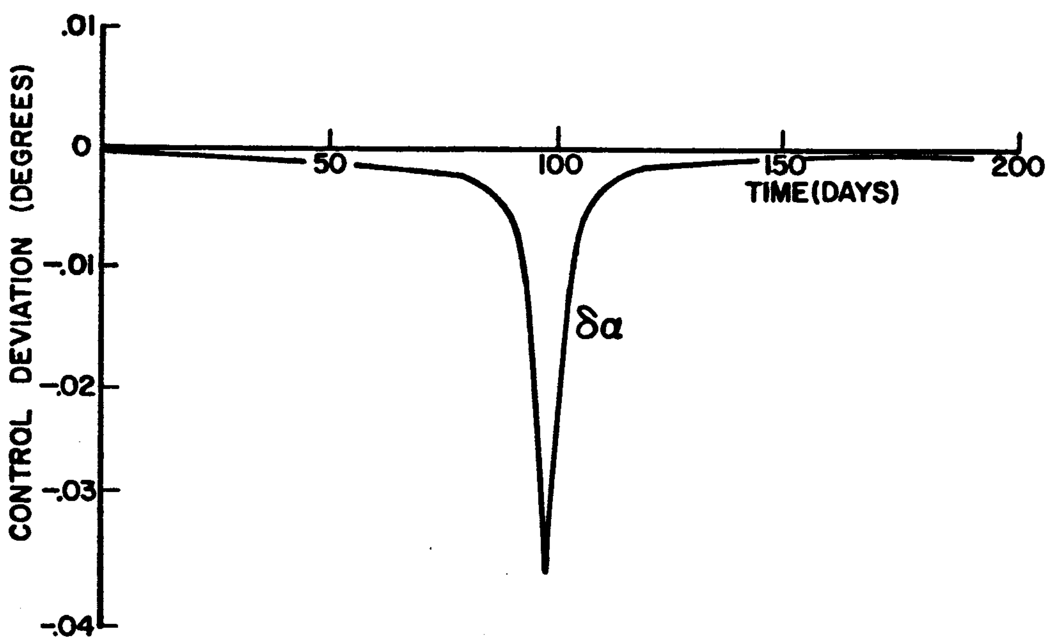

Figure 27b. Optimal Control Deviation Time History

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 10$ Days, $\sigma_0 = 0$)

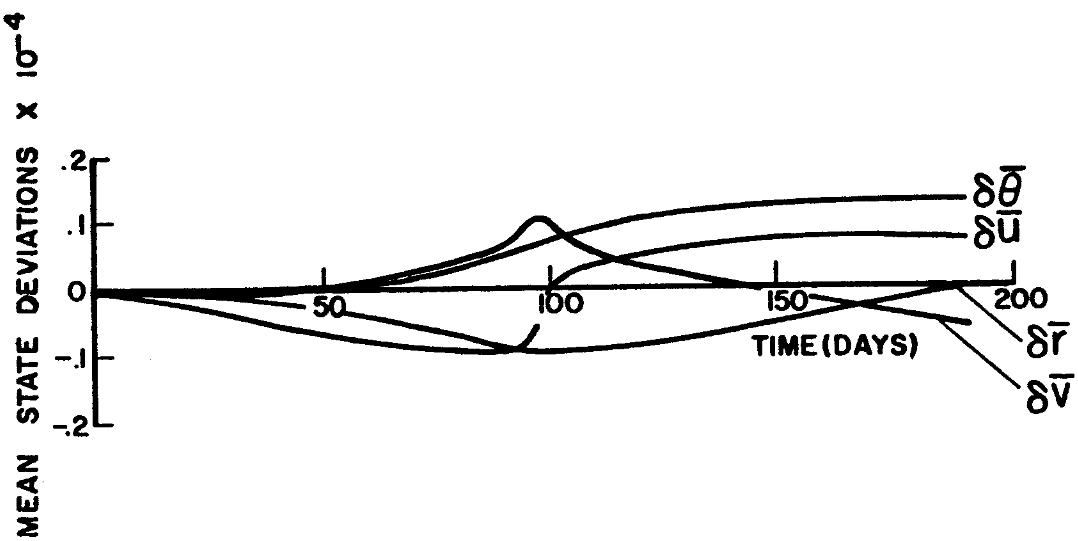

Figure 28a. Optimal Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 100$ Days, $\sigma_0 = 0$)

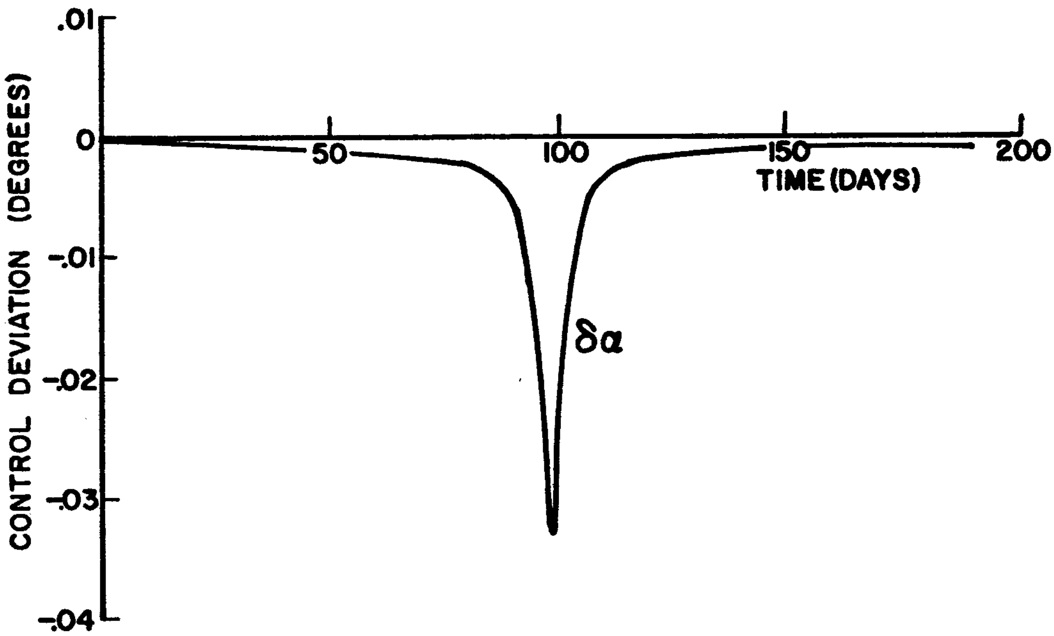

Figure 28b. Optimal Control Deviation Time History

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 100$ Days, $\sigma_0 = 0$)

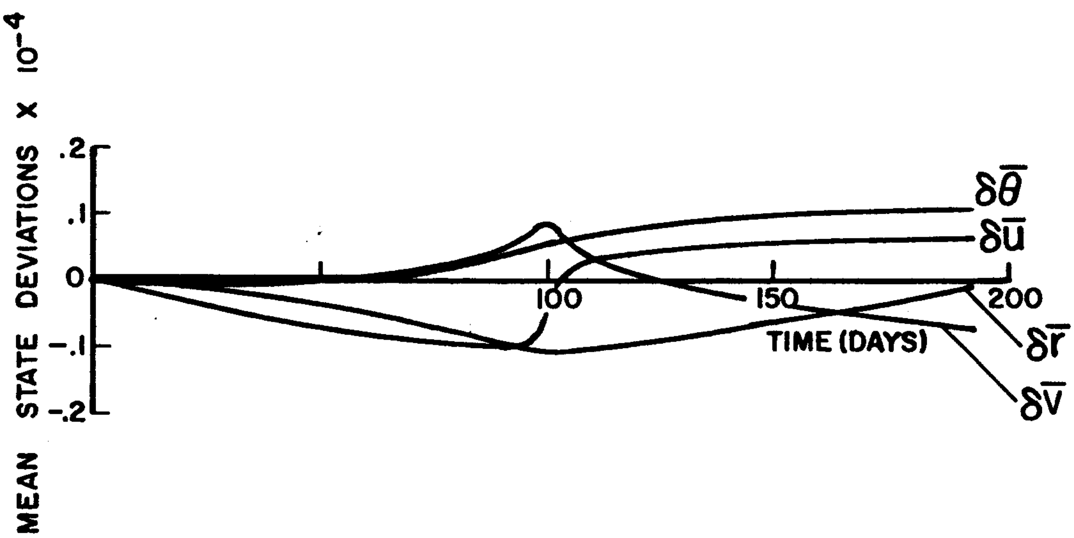

Figure 29a. Optimal Mean State Deviation Time Histories

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 1000$ Days, $\sigma_0 = 0$)

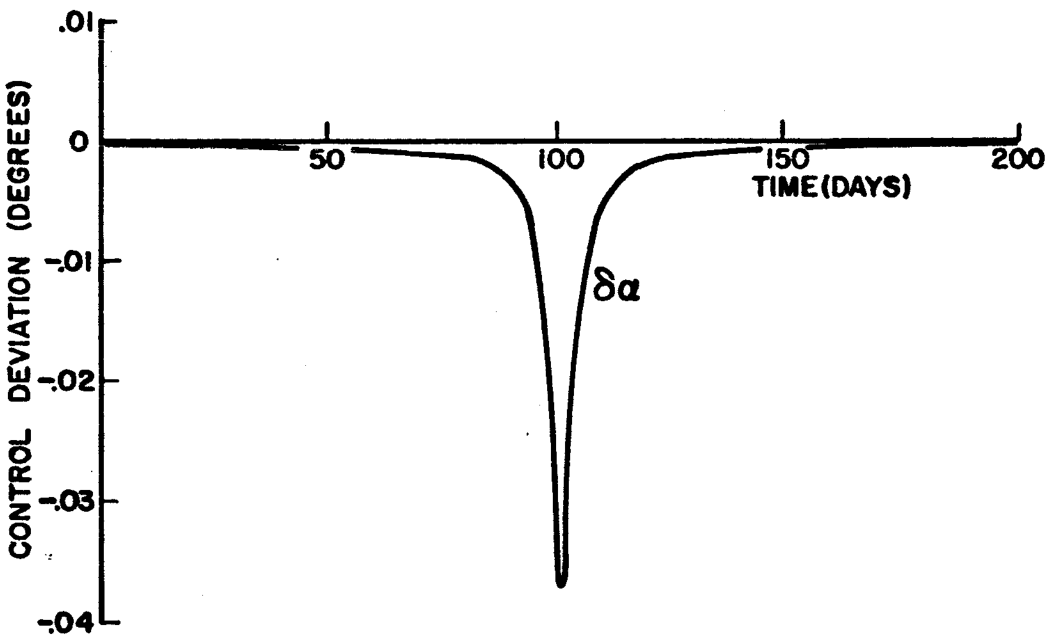

Figure 29b. Optimal Control Deviation Time History

($\sigma_a = 0$, $\sigma_\alpha = ½°$, $T_\alpha = 1000$ Days, $\sigma_0 = 0$)

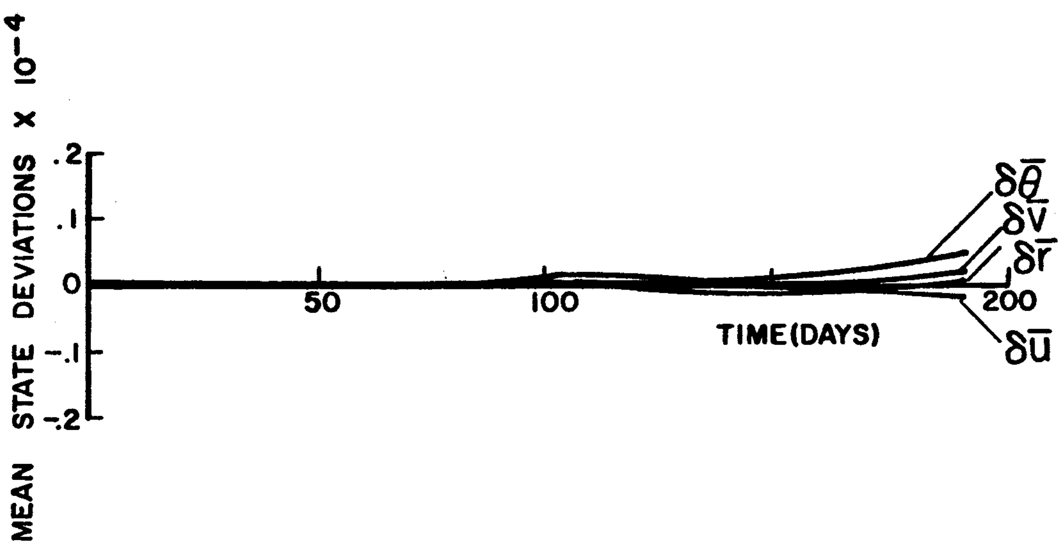

Figure 30a. Optimal Mean State Deviation Time Histories

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 1 x 10^{-3})

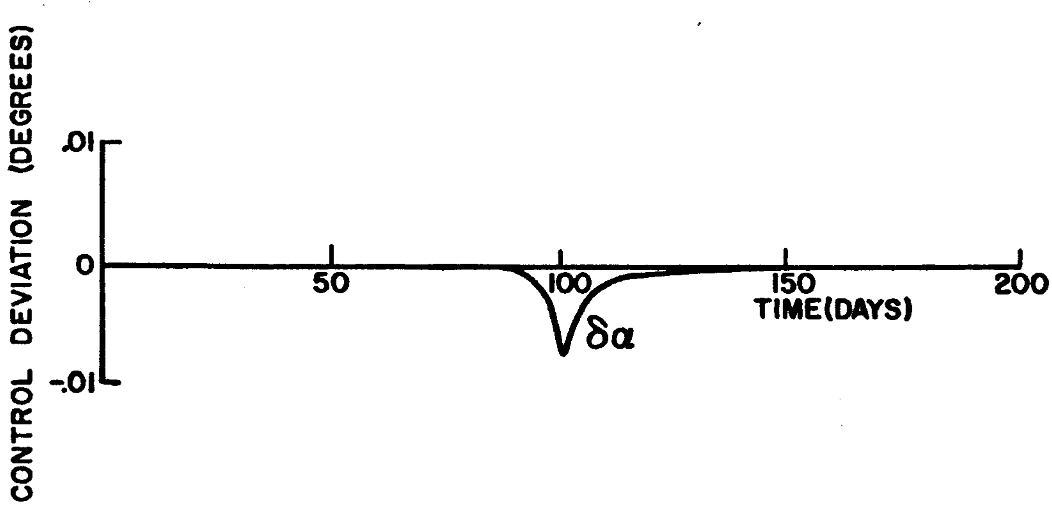

Figure 30b. Optimal Control Deviation Time History

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 1 x 10^{-3})

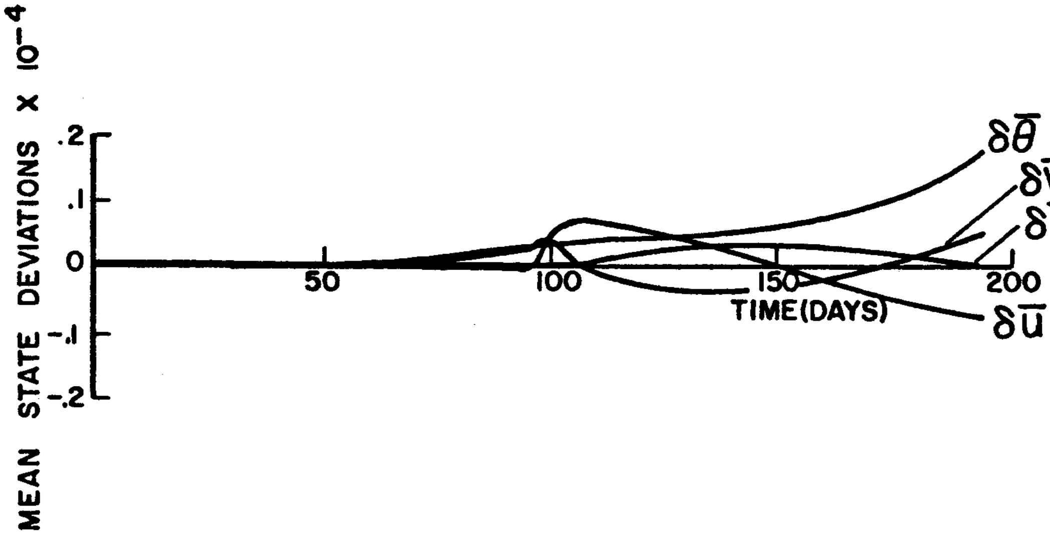

Figure 31a. Optimal Mean State Deviation Time Histories

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 2 x 10^{-3})

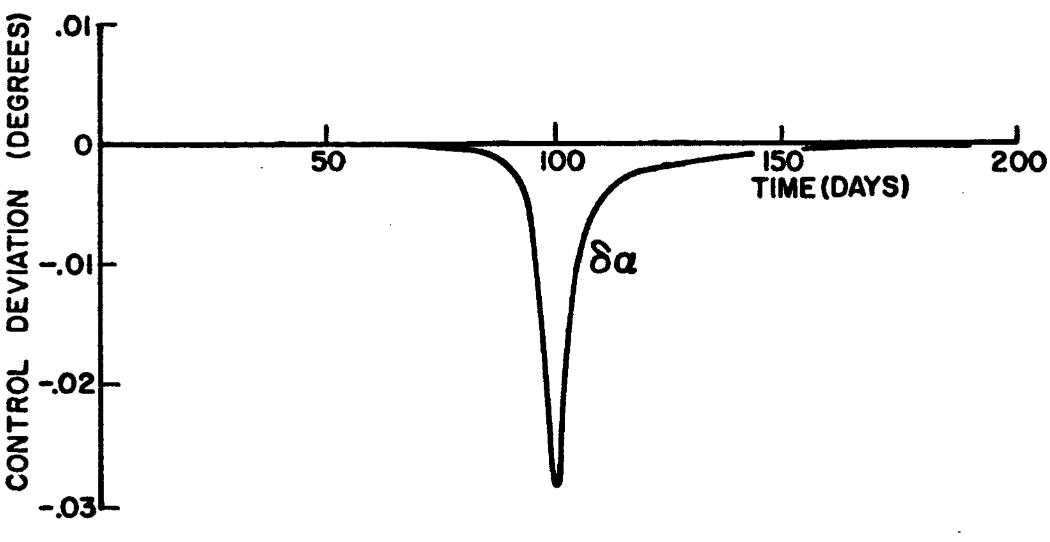

Figure 31b. Optimal Control Deviation Time History

(σ_a = .02T, T_a = 1 Day, σ_α = 0, σ_0 = 2 x 10^{-3})

For Figures 26 through 29, T_α ranges from 1 day to 1000 days

$\sigma_\alpha = \frac{1}{2}^\circ$

$\sigma_a = 0$

$\sigma_0 = 0$

For Figures 30 through 31, σ_0 ranges from 1×10^{-3} to 2×10^{-3}

$\sigma_a = .02T$

$T_a = 1$ day

$\sigma_\alpha = 0$

Several characteristics of the optimal stochastic control and resulting mean trajectories can be seen in the figures. The important characteristics can be summarized as follows:

1. The optimal stochastic control angle $\alpha(t)$, is approximately equal to the deterministic control angle $\alpha^*(t)$, except in the region of rapid change of the control angle $\alpha^*(t)$. (See Figure A.2). The figures show that the stochastic control $\alpha(t)$ lags slightly behind the deterministic control $\alpha^*(t)$ during the region of rapid change.
2. In all cases the control deviation $\delta\alpha(t)$ is much less than the standard deviation of the perturbing noise. This characteristic adds some justification for neglecting the second order terms containing the control correction $\delta\alpha(t)$.
3. The mean state deviations are seen to undergo peaks in the region of rapid change in the control angle $\alpha^*(t)$. This characteristic is a consequence of the nature of the control deviation, which also exhibits a peak in this region. It should be pointed out that the mean states are controlled so as to satisfy the same terminal constraints (rendezvous

with Mars) as the deterministic state is controlled to satisfy in Appendix A. For this reason the mean state deviations do not all tend to zero at the terminal time. For instance, at the time when the mean state satisfies the terminal conditions, the deterministic state may have not yet reached the terminal state computed in Appendix A. In this case, a nonzero $\delta\bar{x}_i(t_f)$ will occur at the final time.

The effect of noise on the final time is illustrated in Figure 32, where the final time deviation is plotted versus the standard deviations of the noise. It is seen that for noise occurring in the thrust orientation angle $\alpha^*(t)$, the final time increases with increasing standard deviation of the noise. For noise occurring in the thrust/mass magnitude, the final time decreases with increasing standard deviation of the noise. However, the change in final time for noise in the thrust/mass magnitude is very slight.

The optimal control developed in this chapter appears to have the properties which are desired of a control which must guide a dynamic system in the presence of noise. Loosely speaking, the stochastic control developed here guides the mean of the ensemble of stochastic trajectories to the terminal conditions, while extremizing the expected value of a performance index functional of the type given in Equation (4.2). It should be noted that although the non-random control developed here does the "best" job possible in an average, or expected value, sense. The standard deviations of the

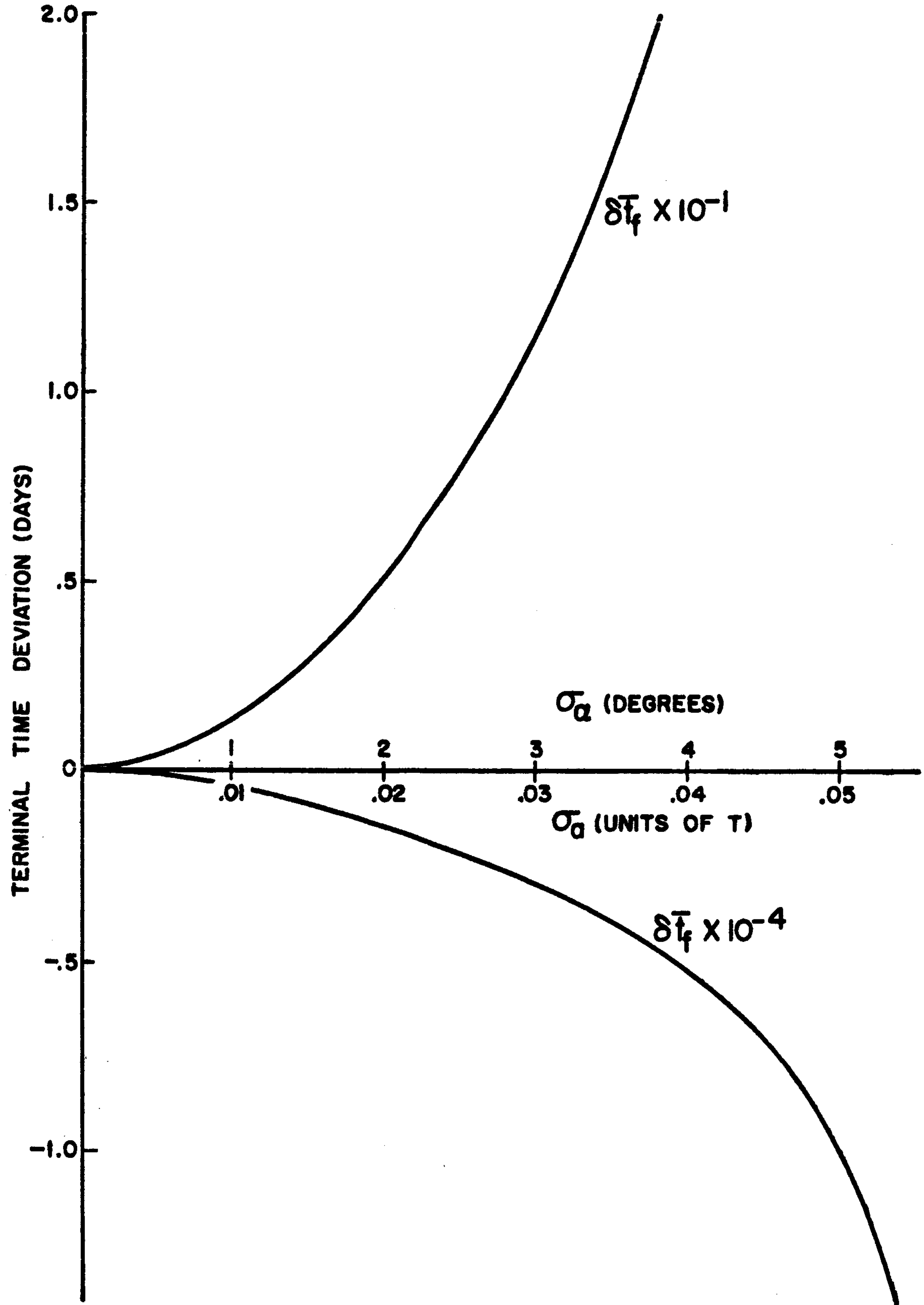

Figure 32. Terminal Time Deviation vs. Perturbing Noise Standard Deviation: For Upper Plot, δt_f vs. σ_α (T_α= 1 Day, σ_a= 0, σ_0= 0), For Lower Plot, δt_f vs. σ_a (T_a= 1 Day, σ_α = 0, σ_0 = 0)

state at the final time are not appreciable smaller than they were in the case of the deterministic control. This indicates that in order to achieve a creditable degree of state accuracy at the final time, some information about the perturbations which actually occur must be utilized by the controller to update the control during the controlling interval.

CHAPTER 5

OPTIMAL STOCHASTIC CONTROL CONDITIONED ON DISCRETE OBSERVATIONS OF THE PROCESS

Introduction

In Chapter 4 the optimal stochastic control problem was solved by obtaining the conditions necessary for the functional given in Equation (4.9), i.e.,

$$J[u] = E\left[\nu_i(x_i(t_f)-x_{if})+\mu_i(x_i(t_0)-x_{i0})+\int_{t_0}^{t_f} f_{n+1}(x,u,t)+p_i(\dot{x}_i-f_i)\,dt\right] \tag{5.1}$$

to be an extremum, where the expected value of the functional is defined as follows

$$E(\cdot) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\cdot)\; f(\xi,\eta(t_1),\eta(t_2),\ldots)\,d\xi\, d\eta(t_1)\ldots \tag{5.2}$$

The function $f(\xi,\eta(t_1),\eta(t_2),\ldots)$ is the joint probability density function of the entire noise process $\eta_i(t)$ in the region $t_0 \le t \le t_f$ and the initial state uncertainty errors ξ_i, i=1, ..., n. The necessary conditions take the following form

$$\begin{aligned} E[\dot{x}_i-f_i] &= 0 \\ E[\dot{p}_i+H_{x_i}] &= 0 \\ E[H_{u_i}] &= 0 \end{aligned} \tag{5.3}$$

at each point of time in the controlling interval $t_0 \le t \le t_f$,

$$E[x_i(t_0)] = x_{i0}$$

$$E[p_i(t_0)] = \mu_i \tag{5.4}$$

at the initial time t_0, and

$$E[x_i(t_f)] = x_{if}$$

$$E[p_i(t_f)] = -\nu_i$$

$$E[H(t_f)] = 0 \tag{5.5}$$

at the terminal time t_f.

The optimal control, which is derived as a solution to Equations (5.3) with end conditions given by Equations (5.4) and (5.5), could be called an expected value, or "mean" value control, since this control procedure drives the expected value of a stochastic ensemble of trajectories to satisfy the deterministic end conditions and, in so doing, extremizes the expected value of some performance index functional. The optimal stochastic control procedure developed in Chapter 4 is better than the optimal deterministic control procedure derived in Appendix A in the sense that an average over a stochastic ensemble is controlled, rather than a deterministic idealization. However, the noise in general has a far greater effect on the dynamic system than that which can be compensated for by any control program based on a priori noise statistics. This is illustrated in Figures 5 through 18 at the end of Chapter 3, where it is seen that in general the standard deviations are much larger than the corresponding mean deviations. It should also be noted that the Monte Carlo simulated trajectory illustrated in Figure 19, exhibits state deviations which are much greater than the corresponding mean deviations shown in Figure 7A. Since the expected, or mean, deviations are used as a basis for deriving the optimal control in Chapter 4, (i.e., the control essentially guides the mean), it can be concluded that, in general, the implementation of a nonrandom control based on a priori statistics of the noise process will not insure that the terminal constraints will be met satisfactorily.

This chapter is devoted to the derivation of an optimal stochastic control which incorporates information gained during the controlling interval into the control program. The information about the process is in the form of observations of some function of the state, which are made at discrete points in time, and which are available to the controller with no time lag. The control is essentially designed to guide the expected value of the state, conditioned on the observations, to satisfy original terminal constraints, while extremizing the conditional mean of the original performance index functional $I[u]$.

Suppose there exists a multidimensional function of the state of the dynamic system

$$z_i = g_i(x,t) \qquad i = 1, \ldots, p \tag{5.6}$$

where $p \leq n$. In addition, suppose that the controller has available sample values of the function

$$y_i(t) = g_i(x,t) + \varepsilon_i(t) \tag{5.7}$$

at discrete instances of time $t_1, t_2, t_3, \ldots, t_k, \ldots, t_N$, where $\varepsilon_i(t_k)$ is a normal random observation error with the following a priori statistics

$$E[\varepsilon_i(t_k)] = 0 \qquad k = 1,2,\ldots N \tag{5.8}$$

$$E[\varepsilon_i(t_k)\varepsilon_j(t_\ell)] = \rho_{ij}{}^2 \, \delta_{k\ell} \tag{5.9}$$

$$E[\varepsilon_i(t_k) \, \delta x_j(t_\ell)] = 0 \tag{5.10}$$

The optimal stochastic control procedure in the presence of these "observations" can be updated or corrected after each particular observation is made available to the controller.

Theoretical Development

A method for updating the optimal stochastic program after an observation value is made available to the controller is presented in the following presentation. Consider the case in which k observations have been made at times $t_1, t_2, \ldots,$ and t_k, respectively, and the controller has updated the control program at the times $t_1, t_2, \ldots, t_{k-1}$, in accordance with the information gained by the previous observations. The optimal control for the time segment $t_k \leq t \leq t_{k+1}$, where t_{k+1} is the time of the next observation, can be found by extremizing the functional

$$J[u] = E\Big[\nu_i(x_i(t_f) - x_{if}) + \mu_i(x_i(t_k) - \hat{x}_i(t_k)) + \int_{t_k}^{t_f} f_{n+1} + p_i(\dot{x}_i - f_i)\,dt \;\Big|\; y(t_k), y(t_{k-1}), \ldots, y(t_1)\Big] \tag{5.11}$$

where

$$E\left[(\cdot) \;\middle|\; y(t_k), y(t_{k-1}), \ldots\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\cdot) f(\xi, \eta, \varepsilon \mid y(t_k), \ldots) \, d\xi \, d\eta \, d\varepsilon \tag{5.12}$$

and $\hat{x}_i(t_k)$ is the conditional mean of $x_i(t_k)$ given the values of the observations made at $t_1, t_2, \ldots, t_k$, i.e.,

$$\hat{x}_i(t_k) = E\left[x_i(t_k) \;\middle|\; y(t_k), y(t_{k-1}), \ldots, y(t_1)\right] \tag{5.13}$$

The function $f(\xi, \eta, \varepsilon \mid y(t_k), y(t_{k-1}), \ldots)$ is the joint conditional probability density function of the noise process $\eta_i(t)$, the initial uncertainty errors ξ_i, and the observation errors ε_i.

By carrying out the variation of the functional given in Equation (5.11) in the same manner as the variation of the functional was carried out in Chapter 4, the following set of necessary conditions can be obtained

$$E[\dot{x}_i - f_i \mid y(t_k), y(t_{k-1}), \ldots] = 0$$

$$E[\dot{p}_i + H_{x_i} \mid y(t_k), y(t_{k-1}), \ldots] = 0$$

$$E[H_{u_i} \mid y(t_k), y(t_{k-1}), \ldots] = 0 \tag{5.14}$$

at all points of time in the controlling interval $t_0 \le t \le t_f$,

$$E[x_i(t_k) \mid y(t_k), \ldots] = \hat{x}_i(t_k)$$

$$E[p_i(t_k) \mid y(t_k), \ldots] = \mu_i \tag{5.15}$$

at the observation time t_k, and

$$E[x_i(t_f) \mid y(t_k),\ldots] = x_{if}$$

$$E[p_i(t_f) \mid y(t_k),\ldots] = -\nu_i$$

$$E[H(t_f) \mid y(t_k),\ldots] = 0 \tag{5.16}$$

at the final time t_f.

It should be noted that although the control for the interval $t_k \le t \le t_{k+1}$, which is based on the values of the observations made at t_1, t_2, ..., t_k, is computed under the assumption that it will drive the conditional mean state to the terminal conditions, specified by Equation (5.16), the control procedure will actually be replaced by an updated control procedure after each new observation is made available to the controller. For instance, at the time of the next observation t_{k+1}, the control for the interval $t_{k+1} \le t \le t_{k+2}$ will be computed on the basis of the values of the observations made at t_1, t_2,..., t_k, t_{k+1}.

The conditional necessary conditions given by Equations (5.14), can be expanded about the optimal deterministic solution given by Equations (4.25), (4.26), and (4.27), to obtain the set of differential equations which describe the behavior of the first and second moments of the state and Lagrange multiplier deviations. They are as follows

$$\delta\dot{\bar{x}}_i = f_{ix_j}\,\delta\bar{x}_j + f_{iu_j}(\delta u_j + \bar{\eta}_j) + \frac{1}{2} f_{ix_jx_k} M_{jk} +$$

$$\frac{1}{2} f_{iu_ju_k}(\delta u_j \delta u_k + \delta u_j \bar{\eta}_k + \bar{\eta}_j \delta u_k + R_{jk}) + f_{ix_ju_k}(\delta\bar{x}_j \delta u_k + h_{jk}) \tag{5.17}$$

$$\delta\dot{\bar{p}}_i = -H_{x_ix_j}\,\delta\bar{x}_j - H_{x_iu_j}(\delta u_j+\bar{\eta}_j) - H_{x_ip_j}\,\delta\bar{p}_j -$$

$$\frac{1}{2}H_{x_ix_jx_k}\,M_{jk} - \frac{1}{2}H_{x_iu_ju_k}(\delta u_j\delta u_k+\delta u_j\bar{\eta}_k+\bar{\eta}_j\delta u_k+R_{jk}) -$$

$$H_{x_ix_ju_k}(\delta\bar{x}_j\delta u_k+h_{jk}) - H_{x_ix_jp_k}\,N_{jk} -$$

$$H_{x_iu_jp_k}\,(\delta u_j\,\bar{p}_k + f_{kj}) \tag{5.18}$$

$$\dot{M}_{ij} = f_{ix_k}\,M_{kj} + M_{ik}\,f_{jx_k} + f_{iu_k}\,(\delta u_k\delta\bar{x}_j+h_{jk}) +$$

$$(\delta\bar{x}_i\;u_k+h_{ik})\;f_{ju_k} \tag{5.19}$$

$$\dot{N}_{ij} = f_{ix_k}\,N_{kj} - N_{ik}\,f_{kx_j} + f_{iu_k}(\delta u_k\delta\bar{p}_j+f_{jk})$$

$$- M_{ik}\,H_{x_kx_j} - (\delta\bar{x}_i\delta u_k+h_{ik})\,H_{x_ju_k} \tag{5.20}$$

$$\dot{h}_{ij} = f_{ix_k}\,h_{kj} + f_{iu_k}(\delta u_k\bar{\eta}_j+R_{kj}) - h_{ik}\beta_{kj} \tag{5.21}$$

$$\dot{f}_{ij} = -H_{x_ix_k}h_{kj} - H_{x_iu_k}\,(\delta u_k\bar{\eta}_j+R_{kj}) - H_{x_ip_k}\,f_{kj} - f_{ik}\beta_{kj} \tag{5.22}$$

$$H_{u_ix_j}\,\delta\bar{x}_j + H_{u_iu_j}\,(\delta u_j+\bar{\eta}_j) + H_{u_ip_j}\,\delta\bar{p}_j +$$

$$\frac{1}{2}H_{u_iu_ju_k}\,(\delta u_j\delta u_k+\bar{\eta}_j\delta u_k+\delta u_j\bar{\eta}_k + R_{jk}) +$$

$$\frac{1}{2}H_{u_ix_jx_k}\,M_{jk} + H_{u_ix_ju_k}\,(\delta\bar{x}_j\delta u_k + h_{jk}) +$$

$$H_{u_ix_jp_k}\,N_{jk} + H_{u_iu_jp_k}\,(\delta u_j\delta\bar{p}_k + f_{kj}) = 0 \tag{5.23}$$

where

$$\overline{\cdot}(t) = E[\cdot(t) \mid y(t_k), y(t_{k-1}), \ldots] \qquad t > t_k \qquad (5.24)$$

The following definitions were used in the preceding expressions

$$M_{ij} = E[\delta x_i \delta x_j \mid y(t_k), \ldots] \qquad (5.25)$$

$$h_{ij} = E[\delta x_i \eta_j \mid y(t_k), \ldots] \qquad (5.26)$$

$$N_{ij} = E[\delta x_i \delta p_j \mid y(t_k), \ldots] \qquad (5.27)$$

$$f_{ij} = E[\delta p_i \eta_j \mid y(t_k), \ldots] \qquad (5.28)$$

$$R_{ij} = E[\eta_i \eta_j \mid y(t_k), \ldots] \qquad (5.29)$$

If the conditional expected terminal constraints, i.e., Equations (5.16), are expanded about the deterministic terminal conditions, i.e., Equations (4.30), (4.31), and (4.32), the following expressions relating $\delta\overline{x}_i(t_f)$, $\delta\overline{p}_i(t_f)$, $\delta u_i(t_f)$ and $\delta\overline{t}_f$ are derived.

$$\dot{x}_i^*(t_f^*)\delta\overline{t}_f + \delta\overline{x}_i(t_f) = 0 \qquad (5.30)$$

$$\delta\overline{p}_i(t_f) = -\delta\nu_i \qquad (5.31)$$

$$\dot{H}^*(t_f^*)\delta\overline{t}_f + H_{x_i}\delta\overline{x}_i + \frac{1}{2} H_{x_i x_j} M_{ij} + H_{p_i}\delta\overline{p}_i +$$

$$H_{u_i}(\delta u_i + \overline{\eta}_i) + \frac{1}{2} H_{u_i u_j}(\delta u_i \delta u_j + \delta u_i \overline{\eta}_j + \delta u_j \overline{\eta}_i + R_{ij}) +$$

$$H_{x_i u_j}(\delta\overline{x}_i \delta u_j + h_{ij}) + H_{x_i p_j} N_{ij} + H_{p_i u_j}(\delta p_i \delta u_j + f_{ij}) = 0 \qquad (5.32)$$

The initial conditions at the time t_k for the variables in Equations (5.17) through (5.23) can be listed in the following manner

$$
\begin{aligned}
E[\delta x_i(t_k) \,|\, y(t_k),\ldots] &= \hat{\delta x}_i(t_k) \\
E[\delta p_i(t_k) \,|\, y(t_k),\ldots] &= \hat{\delta p}_i(t_k) \\
E[\eta_i(t_k) \,|\, y(t_k),\ldots] &= \hat{\eta}_i(t_k) \\
E[\delta x_i \delta x_j \,|\, y(t_k),\ldots] &= \hat{M}_{ij}(t_k) \\
E[\delta x_i \delta p_j \,|\, y(t_k),\ldots] &= \hat{N}_{ij}(t_k) \\
E[\eta_i \eta_j \,|\, y(t_k),\ldots] &= \hat{R}_{ij}(t_k) \\
E[\delta x_i \eta_j \,|\, y(t_k),\ldots] &= \hat{h}_{ij}(t_k) \\
E[\delta p_i \eta_j \,|\, y(t_k),\ldots] &= \hat{f}_{ij}(t_k)
\end{aligned}
\tag{5.33}
$$

where the quantities on the right hand side of Equations (5.33) are conditional mean values, which are computed on the basis of the observations made at times $t_1,\ldots, t_k$, that is,

$$
\hat{\cdot}(t_k) = E[\cdot(t_k) \,|\, y(t_k), y(t_{k-1}),\ldots] \tag{5.34}
$$

Once the initial conditions, i.e., the quantities on the right hand side of Equations (5.33) are computed on the basis of the observations $y_i(t_1),\ldots, y_k(t_k)$, Equations (5.17) through (5.23) form a two-point boundary value problem with split end conditions at the terminal time t_f and the observation time t_k. The boundary value problem can be solved for the corrected values of $\delta p_i(t_k) = \delta\mu_i$, which determine the updated optimal control program, and the corrected value of the terminal time deviation $\delta\bar{t}_f$.

It should be noted that the quantities R_{ij} and $\bar{\eta}_j$ in the differential equations are determined throughout the controlling interval $t_k \le t \le t_f$ in terms of $\hat{R}_{ij}(t_k)$ and $\hat{\eta}_j(t_k)$ by the following generalizations of Equations (2.20) and (2.19) respectively.

$$R_{ij}(t) = \sigma_{ij}^{\,2} + \left[\hat{R}_{ij}(t_k) - \sigma_{ij}^{\,2}\right] e^{-\beta_{ij}(t-t_k)} \quad \text{i,j not summed}$$

$$\bar{\eta}_i(t) = e^{-\beta_{ij}(t-t_k)}\hat{\eta}_i(t_k) \qquad (5.35)$$

The corrective control program, initiated at the observation time t_k, is designed to guide the conditional mean of the ensemble of stochastic trajectories, given the observation values $y_i(t_k)$, $y_i(t_{k-1})$, ... $y_i(t_1)$, to the terminal constraints. It is assumed that at the time of the last observation t_{k-1} a control was initiated on the basis of the observations made up to that time, i.e., $y_i(t_{k-1})$, $y_i(t_{k-2}),\ldots, y_i(t_1)$, and that the conditional means of the process, based on that control program, are available to the controller at t_k. Thus the controller has available the following quantities, prior to the observation at t_k

$$E[\delta x_i(t_k) \mid y(t_{k-1}),\ldots] = \delta\bar{x}_i(t_k)$$

$$E[\delta p_i(t_k) \mid y(t_{k-1}),\ldots] = \delta\bar{p}_i(t_k)$$

$$E[\eta_i(t_k) \mid y(t_{k-1}),\ldots] = \bar{\eta}_i(t_k)$$

$$E[\delta x_i \delta x_j \mid y(t_{k-1}),\ldots] = M_{ij}(t_k)$$

$$E[\delta x_i \delta p_j \mid y(t_{k-1}),\ldots] = N_{ij}(t_k)$$

$$E[\eta_i \eta_j \mid y(t_{k-1}),\ldots] = R_{ij}(t_k)$$

$$E[\delta x_i \eta_j \mid y(t_{k-1}),\ldots] = h_{ij}(t_k)$$

$$E[\delta p_i \eta_j \mid y(t_{k-1}),\ldots] = f_{ij}(t_k) \qquad (5.36)$$

where the bar designates the conditional expected value at the time t given the values of the observations made prior to t, i.e.,

$$\bar{\cdot}(t) = E[\cdot(t) \mid y(t_{k-1}), y(t_{k-2})\ldots] \qquad t_{k-1}<t\leq t_k \qquad (5.37)$$

The remaining task is the developing of a technique for computing the initial conditions for the boundary value problem given in Equations (5.33) in terms of the observation values $y_i(t_k)$, $y_i(t_{k-1}),\ldots,$ $y(t_1)$, and the previously computed conditional moments given in Equations (5.36).

In order to simplify the notation, define the $2n+m$ dimensional generalized state deviation variable $\delta\chi_i(t)$, the $(2n+m)^2$ dimensional generalized second moment $\mathcal{M}_{ij}$, and the generalized covariance $\mathcal{P}_{ij}$, in the following manner

$$\delta\chi_i = \delta x_i \quad , \quad i = 1, \ldots, n$$

$$\delta\chi_i = \delta p_{i-n} \quad , \quad i = n+1, \ldots, 2n$$

$$\delta\chi_i = \eta_{i-2n} \quad , \quad i = 2n+1, \ldots, 2n+m$$

$$M_{ij} = E[\delta\chi_i \ \delta\chi_j]$$

$$P_{ij} = E[(\delta\chi_i - \delta\overline{\chi}_i)(\delta\chi_j - \delta\overline{\chi}_j)] \tag{5.38}$$

It should be noted that $\delta\chi_i(t_k)$ and the observations $y_i(t_k)$, $y_i(t_{k-1})$, ..., $y_i(t_1)$ are jointly distributed random variables which possess some joint probability density function. It is recognized that $\delta\hat{\chi}_i(t_k)$ is some function of $y_i(t_k)$, $y_i(t_{k-1})$, ..., $y_i(t_1)$, that is to say

$$\delta\hat{\chi}_i(t_k) = G_i[y(t_k), y(t_{k-1}), \ldots] \tag{5.39}$$

In order to determine the function G_i, Equation (5.39) will be expanded about the deterministic value of $y_i(t_k)$ i.e.,

$$y_i^*(t_k) = g_i(x^*, t_k) \tag{5.40}$$

The expansion will be carried out to include only linear terms in $y_i(t_k)$, that is, quadratic and higher order terms will be neglected in the analysis. The expansion can be expressed in the following manner

$$\hat{\delta\chi}_i(t_k) = G_i[y^*(t_k), y(t_{k-1}),\ldots] +$$

$$G_{iy_j}(y^*(t_k), y(t_{k-1}),\ldots)\,[y_j(t_k)-y_j^*(t_k)] \qquad (5.41)$$

where

$$G_{iy_j} = \frac{\partial G_i^*}{\partial y_j}$$

The coefficients G_i and G_{iy_j} can be determined in terms of the known quantities given in Equations (5.36) and covariance components of the observation error ρ_{ij}^2 with the aid of the following theorem from probability theory. The theorem, which is proved in Appendix D, can be stated in the following manner

$$E[F(y(t_k))\ \delta\chi_i(t_k)\ \Big|\ y(t_{k-1}), y(t_{k-2}),\ldots] =$$

$$E\Big[[F(y(t_k))E(\delta\chi_i(t_k)\ \Big|\ y(t_k),\ldots)]\ \Big|\ y(t_{k-1}), y(t_{k-2}),\ldots\Big]$$

$$(5.42)$$

Consider an application of the theorem given in Equation (5.42) for the case in which $F(y(t_k)) = 1$. For this special case, the theorem can be stated in the following manner

$$E[\delta\chi_i(t_k)\ \Big|\ y(t_{k-1}),\ldots] = E[E(\delta\chi_i(t_k)\ \Big|\ y(t_k)\ldots)\Big|y(t_{k-1})\ldots]$$

$$(5.43)$$

If the expression given in Equation (5.41) is substituted into the right side of Equation (5.43), the following result is obtained

$$E[\delta\chi_i(t_k)\Big|y(t_{k-1})\ldots] = E\Big[G_i+G_{iy_j}(y_j(t_k)-y_j^*(t_k))\Big|y(t_{k-1}),\ldots\Big]$$

$$(5.44)$$

Equation (5.44) reduces to the following expression

$$\delta\bar{x}_i(t_k) = G_i + G_{iy_j}[\bar{y}_j(t_k) - y_j^*(t_k)] \tag{5.45}$$

By substituting the result obtained in Equation (5.45) back into Equation (5.41), the following expression for $\delta\hat{x}_i(t_k)$ is obtained

$$\delta\hat{x}_i(t_k) = \delta\bar{x}_i(t_k) + G_{iy_j}[y_j(t_k) - \bar{y}_j(t_k)] \tag{5.46}$$

Now consider an application of the theorem given in Equation (5.42) in which $F(y(t_k)) = y_k(t_k) - y_k^*(t_k) = \delta y_k(t_k)$. For this special case, Equation (5.42) can be stated in the following manner

$$E[\delta y_k(t_k)\delta x_i(t_k) \,|\, y(t_{k-1})\ldots] =$$

$$E\left[[\delta y_k(t_k)\; E(\delta x_i(t_k) \,|\, y(t_k),\ldots)] \,\middle|\, y(t_{k-1}),\ldots\right] \tag{5.47}$$

If the expression given in Equation (5.46) is substituted into the right side of Equation (5.47), the following result is obtained

$$E[\delta y_k \delta x_i \,|\, y(t_{k-1})\ldots] = \delta\bar{y}_k\delta\bar{x}_i +$$

$$G_{iy_j}E[\delta y_k(y_j - \bar{y}_j) \,|\, y(t_{k-1}),\ldots] \tag{5.48}$$

By rearranging the terms in Equation (5.48), the following expression is obtained

$$E[(\delta y_k - \delta\bar{y}_k)(\delta x_i - \delta\bar{x}_i) \,|\, y(t_{k-1})\ldots] =$$

$$G_{iy_j}E[(\delta y_k - \delta\bar{y}_k)(\delta y_j - \delta\bar{y}_j) \,|\, y(t_{k-1})\ldots] \tag{5.49}$$

where

$$\delta\bar{y}_i = \bar{y}_i - y_i^*$$

In order to formulate the terms involving δy_i in Equation (5.49), in terms of known quantities, consider an expansion of Equation (5.7) about the deterministic value of the state x_i^*. The expansion can be written as follows

$$\delta y_i + y_i^*(t) = g_i(x^*,t) + \frac{\partial g_i^*}{\partial x_j}\,\delta x_j + \frac{1}{2}\frac{\partial^2 g_i^*}{\partial x_j \partial x_k}\,\delta x_j \delta x_k + \ldots + \varepsilon_i \tag{5.50}$$

By subtracting out Equation (5.6) from the expression given in Equation (5.50), and by using the following notation, i.e.,

$$\frac{\partial g_i^*}{\partial x_j} = g_{ix_j} \qquad\qquad \frac{\partial^2 g_i^*}{\partial x_j\, x_k} = g_{ix_jx_k} \tag{5.51}$$

the expansion given in Equation (5.50) can be written as follows

$$\delta y_i = g_{ix_j}\,\delta x_j + \frac{1}{2} g_{ix_jx_k}\,\delta x_j \delta x_k + \varepsilon_i \tag{5.52}$$

Equation (5.52) can be written in terms of the generalized state deviations $\delta \chi_i$ as follows

$$\delta y_i = g_{i\chi_j}\delta\chi_j + \frac{1}{2} g_{i\chi_j\chi_k}\,\delta\chi_j\delta\chi_k + \varepsilon_i \tag{5.53}$$

with the restriction that

$$g_{i\chi_j} = 0 \quad , \qquad j > n$$

$$g_{i\chi_j\chi_k} = 0 \quad , \qquad j > n \text{ or } k > n \tag{5.54}$$

The expected value of δy_i can be expressed as follows

$$\delta\bar{y}_i = g_{i\chi_j}\delta\bar{\chi}_j + \frac{1}{2} g_{i\chi_j\chi_k} M_{jk} \tag{5.55}$$

By substituting Equation (5.53) and Equation (5.55) into the Equation (5.49), and neglecting terms in $\delta\chi_i$ of higher order than the second, the following expression is derived,

$$g_{k\chi_j} E[(\delta\chi_i-\delta\bar{\chi}_i)(\delta\chi_j-\delta\bar{\chi}_j+\varepsilon_j) \mid y(t_{k-1})\ldots] =$$

$$G_{iy_j} g_{k\chi_\ell} g_{j\chi_m} E[(\delta\chi_\ell-\delta\bar{\chi}_\ell+\varepsilon_\ell)(\delta\chi_m-\delta\bar{\chi}_m+\varepsilon_m) \mid y(t_{k-1})\ldots]$$

Now by recalling Equations (5.8), (5.9), and (5.10), i.e., that

$$E[\varepsilon_i(t_k)] = 0$$

$$E[\varepsilon_i(t_k)\varepsilon_j(t_k)] = \rho_{ij}^{\ 2}$$

$$E[\varepsilon_i(t_k)\delta x_j(t_k)] = 0$$

the expected value operation can be carried out in Equation (5.56), and the following result is obtained

$$P_{ij}g_{k\chi_j} = G_{iy_j}[g_{k\chi_\ell} g_{j\chi_m} P_{\ell m}+\rho_{kj}^{\ 2}] \tag{5.57}$$

In order to solve for the coefficient G_{iy_j}, the quantity B_{ij} is defined by the following expression

$$[g_{k\chi_i} g_{\ell\chi_j} P_{ij} + \rho_{k\ell}^{2}] B_{km} = \delta_{\ell m} \tag{5.58}$$

By multiplying Equation (5.57) by the quantity B_{kn}, the following result is obtained

$$P_{ij} g_{k\chi_j} B_{kn} = G_{iy_j} \delta_{jn} = G_{iy_n} \tag{5.59}$$

Substitution of the expression for G_{iy_n}, given in Equation (5.59), into the expression for $\delta\hat{\chi}_i(t)$, given in Equation (5.46), leads to the following result,

$$\delta\hat{\chi}_i(t_k) = \delta\bar{\chi}_i(t_k) + P_{i\ell} g_{k\chi_\ell} B_{kj} [\delta y_j(t_k) - \delta\bar{y}_j(t_k)] \tag{5.60}$$

where $\delta\bar{y}_j$ is defined in Equation (5.55).

Equation (5.60) relates the conditional mean of the generalized state deviation, given the observations $y_i(t_k)$, $y_i(t_{k-1})$,... to the known conditional mean of the state deviation, given the previous observations $y(t_{k-1})$, $y(t_{k-2})$, ... and the observation deviation value $\delta y_i(t_k)$. It is interesting to note that the previous observation values $\delta y_i(t_{k-1})$, $\delta y_i(t_{k-2})$, ... $\delta y_i(t_1)$ are not contained explicitly in Equation (5.60), but are implicitly contained in the value of $\delta\bar{\chi}_i(t_k)$.

The components of $\hat{\delta x_i}(t_k)$ break down into the components of $\hat{\delta x_i}(t_k)$, $\hat{\delta p_i}(t_k)$, and $\hat{n}_i(t_k)$, as shown in the definitions given in Equations (5.38). The remainder of the quantities given in Equations (5.33), i.e., $\hat{M}_{ij}$, $\hat{N}_{ij}$, $\hat{h}_{ij}$, $\hat{f}_{ij}$, and $\hat{R}_{ij}$, can be computed with equations derived in the following discussion. Consider the identity

$$E[(\delta\chi_i-\hat{\delta\chi_i})(\delta\chi_j-\hat{\delta\chi_j}) \mid y(t_{k-1}),\ldots] =$$

$$E[(\delta\chi_i-\hat{\delta\chi_i})(\delta\chi_j-\hat{\delta\chi_j}) \mid y(t_{k-1}),\ldots] \tag{5.61}$$

Substitution of Equation (5.60) into the right hand side of Equation (5.61) leads to the following expression

$$E[(\delta\chi_i-\hat{\delta\chi_i})(\delta\chi_j-\hat{\delta\chi_j}) \mid y(t_{k-1})\ldots] =$$

$$E[(\delta\chi_i-\delta\bar{\chi}_i)(\delta\chi_j-\delta\bar{\chi}_j) \mid y(t_{k-1})\ldots] -$$

$$E[(\delta\chi_i-\delta\bar{\chi}_i)P_{j\ell}g_{k\chi_\ell}B_{km}(\delta y_m-\delta\bar{y}_m) \mid y(t_{k-1})\ldots] -$$

$$E[(\delta\chi_j-\delta\bar{\chi}_j)P_{i\ell}g_{k\chi_\ell}B_{km}(\delta y_m-\delta\bar{y}_m) \mid y(t_{k-1})\ldots] +$$

$$E[P_{i\ell}g_{k\chi_\ell}B_{km}(\delta y_m-\delta\bar{y}_m)P_{jp}g_{qx_p}B_{qn}(\delta y_n-\delta\bar{y}_n) \mid y(t_{k-1})\ldots] \tag{5.62}$$

By substituting Equations (5.53) and (5.60) into Equation (5.62), the following expression can be obtained,

$$E[(\delta\chi_i-\hat{\delta\chi_i})(\delta\chi_j-\hat{\delta\chi_j}) \mid y(t_{k-1})\ldots] =$$

$$P_{ij}-P_{i\ell}g_{k\chi_\ell}B_{km}g_{m\chi_n}P_{jn} \tag{5.63}$$

Consider for a moment the correlation between the error in the conditional mean and the observation deviation, i.e.,

$$E[(\delta x_i - \delta\hat{x}_i)(\delta y_j) \mid y(t_{k-1}) \ldots] =$$

$$E[(\delta x_i - \delta\overline{x}_i)(\delta y_j) \mid y(t_{k-1}) \ldots] - P_{i\ell} g_{kx_\ell} B_{km} E[(\delta y_m - \delta\overline{y}_m)\delta y_j \mid y(t_{k-1}) \ldots] \tag{5.64}$$

After carrying out the expected value operation, Equation (5.64) reduces to

$$E[(\delta x_i - \delta\hat{x}_i)\ \delta y_j \mid y(t_{k-1}) \ldots] =$$

$$P_{i\ell} g_{jx_\ell} - P_{i\ell} g_{kx_\ell}\ \delta_{kj} = 0 \tag{5.65}$$

It is seen from Equation (5.65) that the error in the conditional mean, $(\delta x_i(t_k) - \delta\hat{x}_i(t_k))$ is uncorrelated through the second order with the observation $\delta y_i(t_k)$. It will be assumed that this lack of correlation is sufficient to imply that the following identity is valid to second order

$$E[(\delta x_i - \delta\hat{x}_i)(\delta x_j - \delta\hat{x}_j) \mid y(t_k), y(t_{k-1}), \ldots y(t_1)] =$$

$$E[(\delta x_i - \delta\hat{x}_i)(\delta x_j - \delta\hat{x}_j) \mid y(t_{k-1}), y(t_{k-2}), \ldots y(t_1)] \tag{5.66}$$

By incorporating Equation (5.65) into Equation (5.63), the following expression for the corrected generalized covariance is obtained

$$\hat{P}_{ij} = P_{ij} - P_{i\ell} g_{m\chi_\ell} B_{mn} g_{n\chi_k} P_{jk} \tag{5.67}$$

The generalized second moment from the deterministic trajectory is derived from the following expression

$$\hat{M}_{ij}(t_k) = \hat{P}_{ij}(t_k) - \delta\hat{x}_i(t_k)\ \delta\hat{x}_j(t_k) \tag{5.68}$$

The components of $\hat{M}_{ij}$ can be broken down into the quantities $\hat{M}_{ij}$, $\hat{N}_{ij}$, $\hat{h}_{ij}$, $\hat{f}_{ij}$, and $\hat{R}_{ij}$.

Equations (5.60) and (5.68) yield the conditional means and second moments which are used as initial conditions in the two-point boundary value problem. The solution to this problem will yield the corrected optimal stochastic control for the interval $t_k \leq t < t_{k+1}$. This corrected control could be called a "conditional mean" control, since the control essentially guides the conditional mean, given a set of observations, to satisfy the original deterministic terminal constraints and, in so doing, extremizes the conditional mean of the performance index functional.

Since the equation which updates the conditional mean at the time of observation t_k, i.e., Equation (5.60), contains explicitly only the value of the present observation $y_i(t_k)$, the scheme can be used recursively at all of the observation times $t_1, t_2, \ldots, t_N$.

<u>Application To The Earth-Mars Transfer Problem</u>

The results of this chapter will now be applied to the Earth-Mars transfer problem. Since for this application the Hamiltonian $H(x,u,p,t)$ and the function $f_i(x,u,t)$ are both separable in the state and control, the conditional differential equations, given in Equations (5.17) and (5.18), and the conditional optimality condition, i.e., Equation (5.23), reduce to the following system of equations

$$\delta\dot{\bar{x}}_i = f_{ix_j}\delta\bar{x}_j + f_{iu_j}(\delta u_j+\bar{\eta}_j) + \frac{1}{2} f_{ix_jx_k}M_{jk}$$

$$+ \frac{1}{2} f_{iu_ju_k}(\delta u_j\delta u_k+\delta u_j\bar{\eta}_k + \bar{\eta}_j\delta u_k+R_{jk})$$

$$\delta\dot{\bar{p}}_i = -H_{x_ix_j}\delta\bar{x}_j - H_{x_ip_j}\delta p_j - \frac{1}{2} H_{x_ix_jx_k}M_{jk} -$$

$$H_{x_jx_jp_k}N_{jk} \tag{5.69}$$

$$H_{u_iu_j}(\delta u_j+\bar{\eta}_j) + H_{u_ip_j}\delta\bar{p}_j + \frac{1}{2} H_{u_iu_ju_k}(\delta u_j\delta u_k$$

$$+ \delta u_j\bar{\eta}_k + \bar{\eta}_j\delta u_k +R_{jk}) + H_{u_iu_jp_k}(\delta u_j\delta\bar{p}_k+f_{jk}) = 0$$

It is shown in Appendix E that the generalized covariance P_{ij} obeys the following differential equation

$$\dot{P}_{ij} = \Gamma_{ik}P_{kj} + P_{ik}\Gamma_{jk} \tag{5.70}$$

where Γ_{ij} is a $(2n+m)^2$ dimensional quantity which is defined by the following relations

$$
\begin{aligned}
\Gamma_{ij} &= f_{ix_j} && , \quad i = 1,\ldots,n \\
& && \quad\ j = 1,\ldots, n \\
\Gamma_{ij} &= 0 && , \quad i = 1,\ldots\ n \\
& && \quad\ j = n+1,\ldots, 2n \\
\Gamma_{ij} &= f_{iu_{j-2n}} && , \quad i = 1,\ldots, n \\
& && \quad\ j = 2n+1,\ldots, 2n+m \\
\Gamma_{ij} &= - H_{x_{i-n}x_j} && , \quad i = n+1, \ldots, 2n \\
& && \quad\ j = 1, \ldots, n \\
\Gamma_{ij} &= - H_{x_{i-n}p_{j-n}} && , \quad i = n+1, \ldots, 2n \\
& && \quad\ j = n+1, \ldots, 2n \\
\Gamma_{ij} &= - H_{x_{i-n}u_{j-2n}} && , \quad i = n+1, \ldots, 2n \\
& && \quad\ j = 2n+1, \ldots, 2n+m \\
\Gamma_{ij} &= 0 && , \quad i = 2n+1,\ldots, 2n+m \\
& && \quad\ j = 1, \ldots, n \\
\Gamma_{ij} &= 0 && , \quad i = 2n+1, \ldots, 2n+m \\
& && \quad\ j = n+1, \ldots, 2n \\
\Gamma_{ij} &= -\beta_{i-2n\ j-2n} && , \quad i = 2n+1,\ldots, 2n+m \\
& && \quad\ j = 2n+1, \ldots, 2n+m
\end{aligned}
\tag{5.71}
$$

It is therefore convenient to integrate Equation (5.70) instead of the set of Equations (5.19), (5.20), (5.21), and (5.22), and then obtain the quantities M_{ij}, N_{ij}, h_{ij}, and f_{ij}, at each point in time from the relation

$$M_{ij} = P_{ij} + \delta\overline{x}_i\delta\overline{x}_j$$

Matrix formulations of the terms in Equations (5.69) and (5.70), for the example problem considered here, are given in Appendix B.

For the Earth-Mars transfer problem, the observational information is in the form of the time rate of change of the position vector of the spacecraft relative to the Earth. The values of this observable, referred to here as range rate, are available to the controller at discrete points in time. Assuming that the Earth moves in a circular orbit about the Sun, the distance from the Earth to the spacecraft is defined by the following expression

$$\rho = [r^2+R^2-2rR\cos(\theta-\omega(t-t_0))]^{\frac{1}{2}} \tag{5.72}$$

where R is the orbital radius of the Earth, and ω is the angular velocity of the Earth about the Sun. The rate of change of the Earth-spacecraft distance is accordingly defined by the following expression

$$\dot{\rho} = \frac{ru-Ru\cos(\theta-\omega(t-t_0)) + rR(\frac{v}{r}-\omega)\sin(\theta-\omega(t-t_0))}{\rho} \tag{5.73}$$

The function z_i, in Equation (5.6), becomes the scalar variable $\dot{\rho}$ for the Earth-Mars transfer problem. Thus the scalar observation $y(t_k)$ is given by the following relation

$$y(t_k) = \dot{\rho}(t_k) + \varepsilon(t_k) \tag{5.74}$$

where

$$E[\varepsilon(t_k)] = 0$$

$$E[\varepsilon(t_k)^2] = \sigma_\varepsilon^{\ 2} \tag{5.75}$$

Equations (5.55), (5.60), and (5.67), are applied ot the interplanetary transfer problem so that the control can be updated at the observation times. Matrix formulations of the terms in these equations are given in Appendix B.

The corrective control scheme, based on range-rate observations made at discrete points in time, is applied to two Monte Carlo simulated trajectories. The first trajectory considered is the example trajectory presented in Chapter 3, Figure 19. The parameters of interest for the stochastic trajectory are $\sigma_a = .05T$, $T_a = 1$ day, $\sigma_\alpha = 0$, and $\sigma_0 = 0$. The time histories of the mean state deviations and the standard deviations of an ensemble of trajectories possessing the preceding noise parameters are shown in Chapter 3, Figure 7. The a priori optimal stochastic control program, which is computed in the absence of any observational information, and the resulting optimal mean state deviations for the same ensemble of trajectories is shown in Chapter 4, Figure 21.

A series of observation-correction operations are made on the trajectory in Figure 19, and the results are presented in Figure 33. The standard deviation of the error in each observation is assumed to be $10^{-3}V_E$, where V_E is the velocity of the Earth. An outline of the steps of the recursive observation-correction scheme, with appropriate referrals to the figures and discussions of the interesting characteristics of the results, is given in the subsequent presentation.

a. An observation is made 30 days after the time of initiation of the transfer. The conditional mean state deviations, given the observation, are illustrated in Figure 33a. Discontinuities occur in the curves at the time of observation and indicate the change from a priori mean state deviations to conditional mean state deviations. Note that no control correction has been made.

b. The conditional standard deviations of the state components, given the observation, are given in Figure 33b. Note the discontinuity at the time the observation is made. A comparison of Figure 33b with Figure 7b in Chapter 3 shows that the standard deviations are smaller after the observation is made, than the corresponding standard deviations in the case in which no observation is made. The smaller standard deviation indicates the controller's increased knowledge of the actual state history in the time interval after the observation.

c. A control correction is made on the basis of the observation value, and the resulting mean deviations are illustrated in Figure 33c. Note the peaks in the mean deviations at the time of rapid change of the thrust direction angle $\alpha(t)$.

d. The optimal control deviation is illustrated in Figure 33d. Note the discontinuity at the time of the observation, and the peak at the time of rapid change in the thrust direction angle.

e. The resulting state deviations of the sample trajectory are presented in Figure 33e. It can be seen, by comparing the

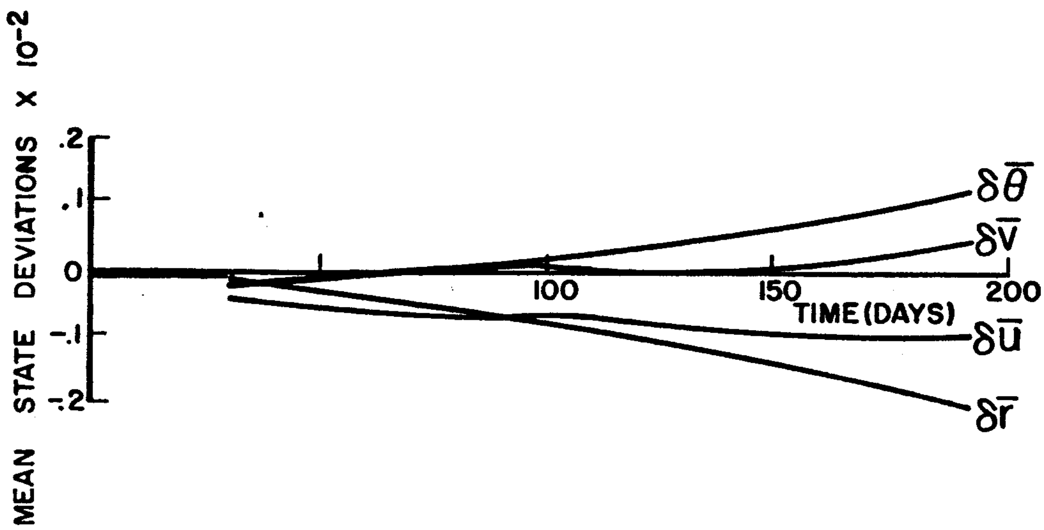

Figure 33a. Conditional Mean State Deviation Time Histories (Observation at $t = 30$ Days, $\sigma_\varepsilon = 10^{-3}$) Trajectory Parameters: ($\sigma_a = .05T$, $T_a = 1$ Day, $\sigma_\alpha = 0$, $\sigma_0 = 0$)

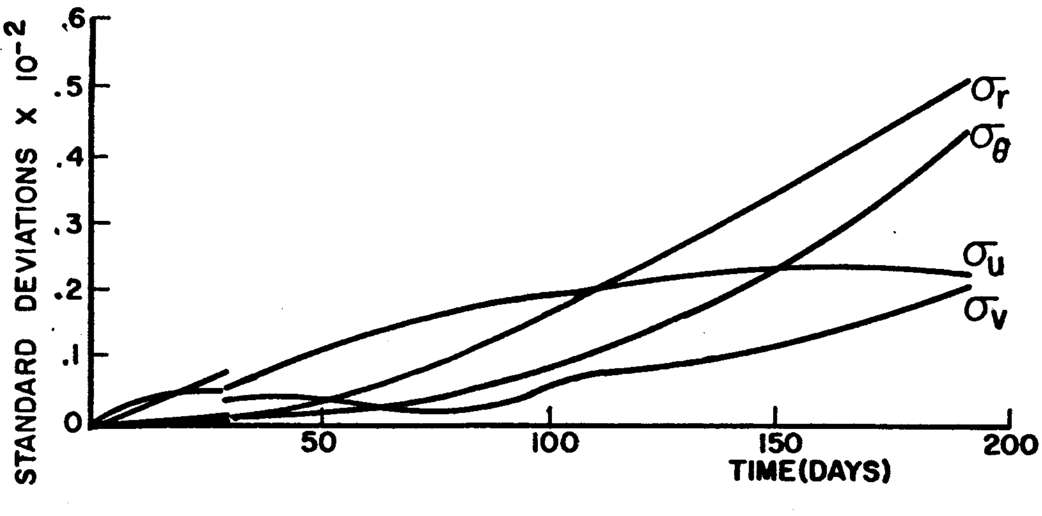

Figure 33b. Conditional Standard Deviation Time Histories (Observation at $t = 30$ Days, $\sigma_\varepsilon = 10^{-3}$)

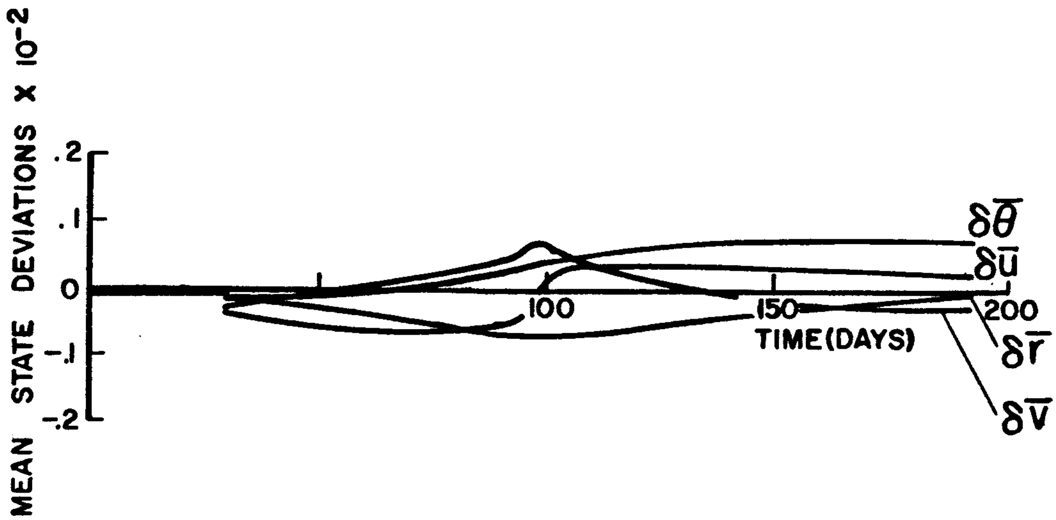

Figure 33c. Optimal Conditional Mean State Deviation Time Histories (Control Correction at t = 30 Days, $\sigma_\varepsilon = 10^{-3}$)

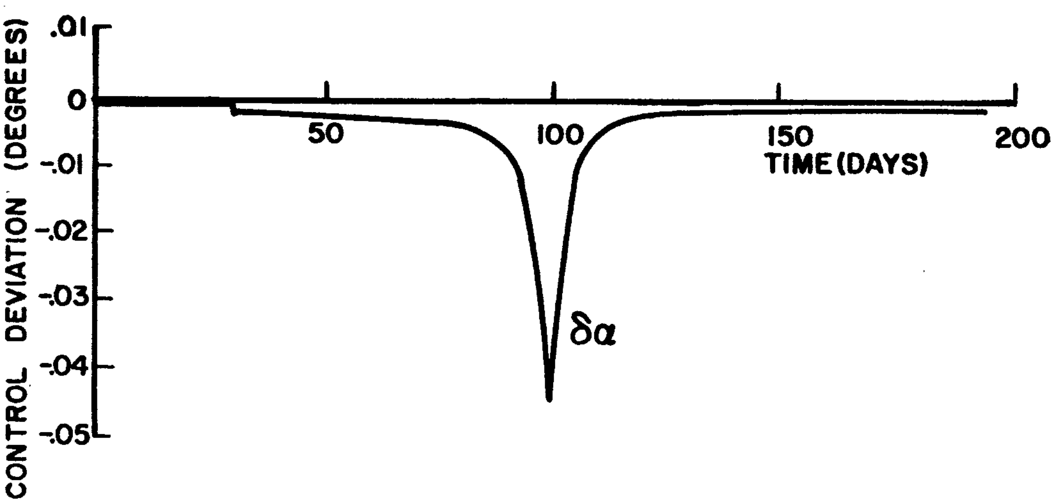

Figure 33d. Optimal Corrective Control Deviation Time History (Control Correction at t = 30 Days; $\sigma_\varepsilon = 10^{-3}$)

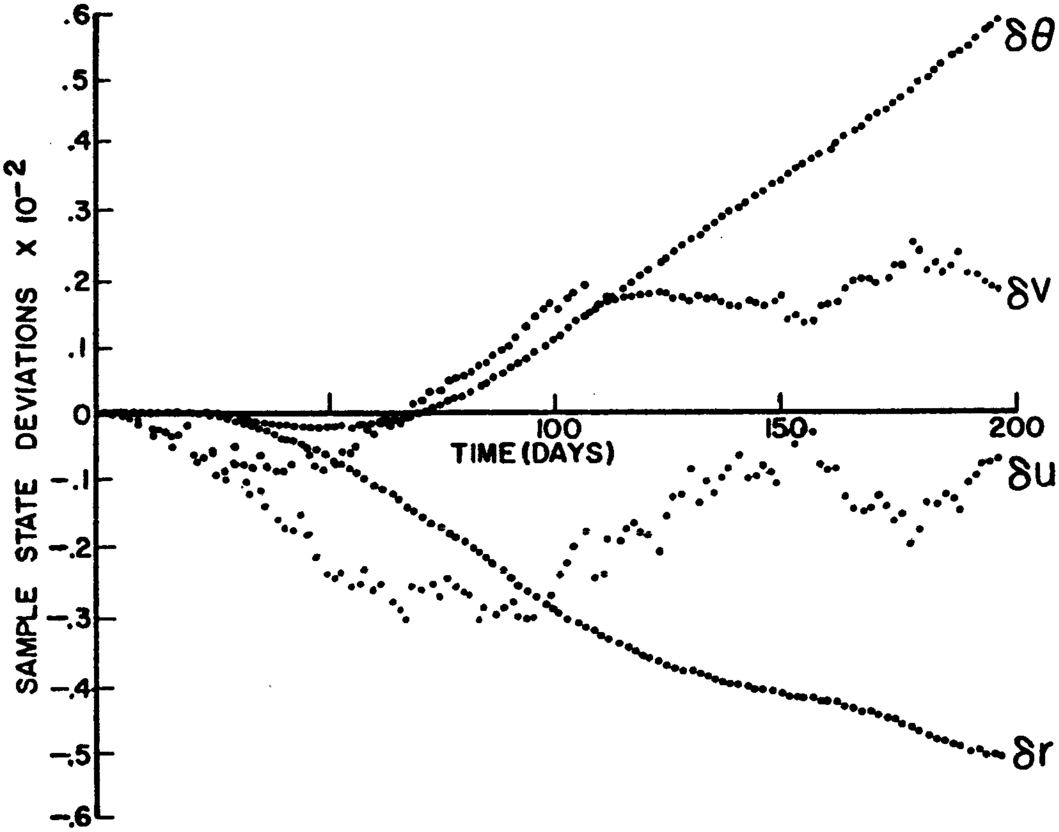

Figure 33e. Simulated State Deviation Time Histories

(Control Correction at t = 30 Days, $\sigma_\varepsilon = 10^{-3}$)

state deviations in Figure 33e with those of the uncorrected trajectory in Figure 19, that the terminal constraints are met more accurately in the case where the control correction is performed.

f. Another observation is made at $t = 60$ days, and the resulting conditional mean deviations are presented in Figure 33f. From a comparison of the conditional mean deviations with the actual deviations in Figure 33e, it can be seen that the conditional mean deviations approximate the sample state deviations much more accurately, after two observations are made, than in the previous case.

g. The standard deviations of the state components are given in Figure 33g.

h. A control correction is made and the resulting mean state deviations are presented in Figure 33h.

i. The updated control deviation appears in Figure 33i.

j. The resulting state deviations of the sample trajectory are presented in Figure 33j. Note how the simulated state deviation components follow paths which are similar to the conditional mean state deviation values, shown in Figure 33h. It should also be noted that the conditional mean deviations are always controlled to meet the terminal constraints. For this reason, an indication as to how well the updated control program is performing, is how close the actual simulated trajectory state deviations are to the conditional mean deviations.

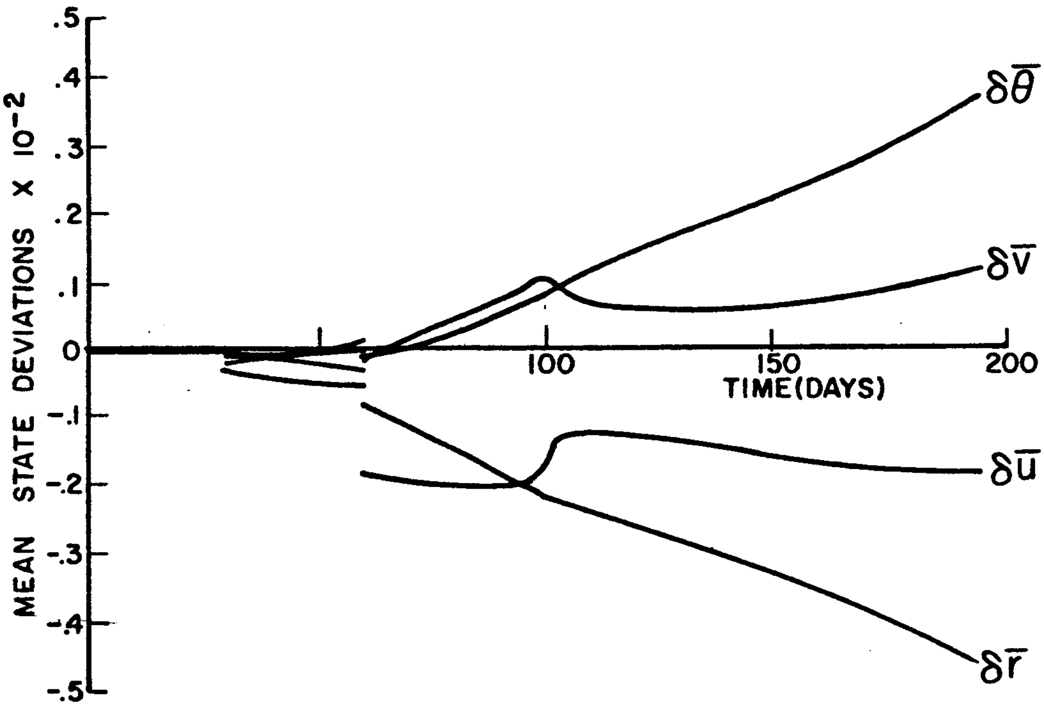

Figure 33f. Conditional Mean State Deviation Time Histories

(Observation at t = 30 Days and t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

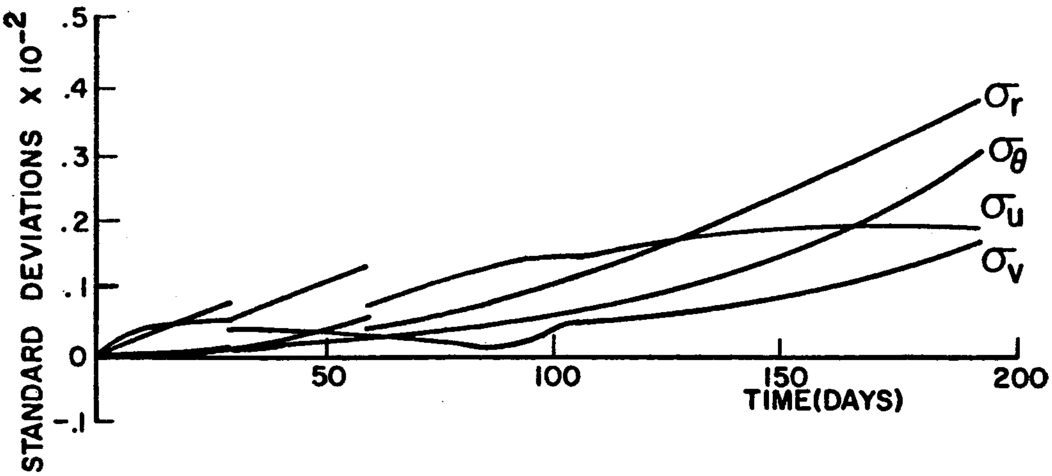

Figure 33g. Conditional Standard Deviation Time Histories

(Observations at t = 30 Days and t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

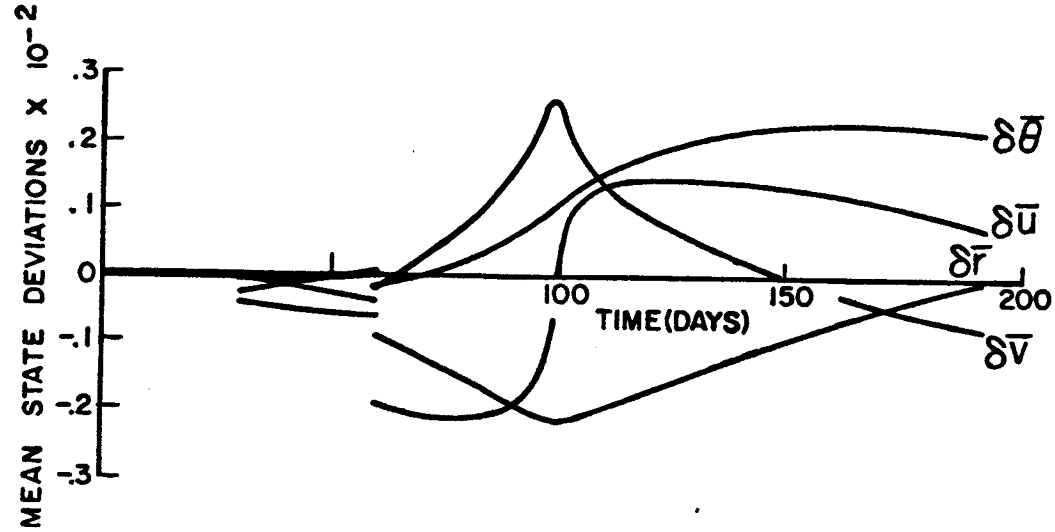

Figure 33h. Optimal Conditional Mean State Deviation Time Histories (Control Correction at t = 30 Days and t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

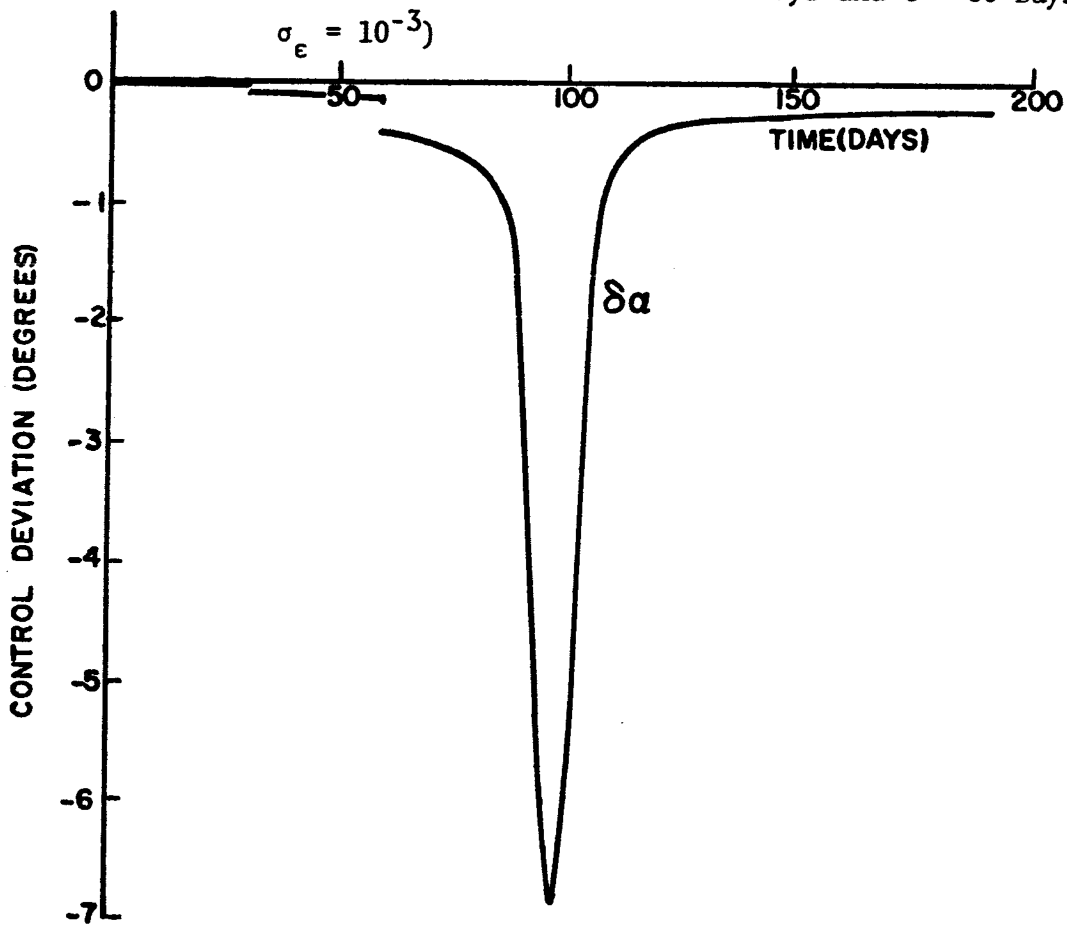

Figure 33i. Optimal Corrective Control Deviation Time History (Control Correction at t = 30 Days and t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

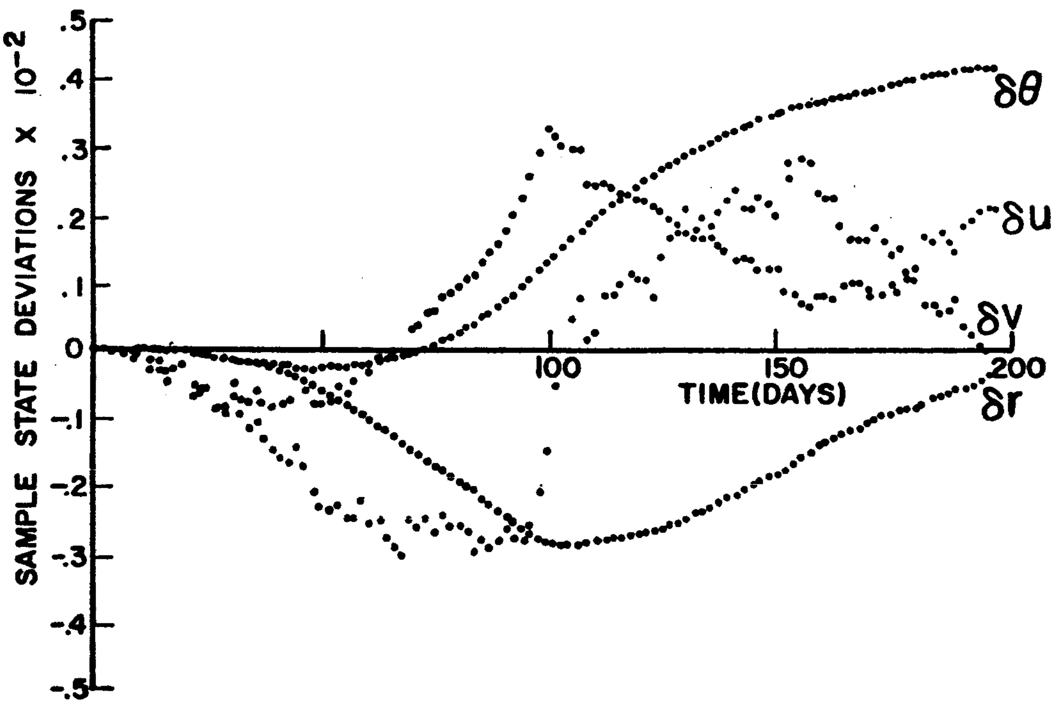

Figure 33j. Simulated State Deviation Time Histories (Control Corrections at t = 30 Days and t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

k. A third observation is made at $t = 90$ days, and the conditional means are given in Figure 33k. Note that the corrections in the conditional means at the third observation time are small in comparison to previous corrections. This is because the conditional mean deviations approximate the actual state deviations more closely than before observations were taken.

l. The updated standard deviations are presented in Figure 33l.

m. The mean state deviations for the third control correction appear in Figure 33m.

n. The third corrective control deviation appears in Figure 33n.

o. The resulting sample state deviations for the third control correction appear in Figure 33o.

p. In order to make a precise comparison of the conditional mean state deviations to the actual sample state deviations, for the case of three observations, Figure 33m is superimposed on Figure 33o. The superposition of the velocity deviations is presented in Figure 33p. Note that as each new observation is made available to the controller, the updated conditional mean deviations, which are computed on the basis of the most recent observation value, are closer to the sample state deviations than the previous conditional mean deviations.

q. The superposition of the position deviations is presented in Figure 33q.

r. In order to demonstrate the effectiveness of the preceding observation-correction scheme at increasing the terminal

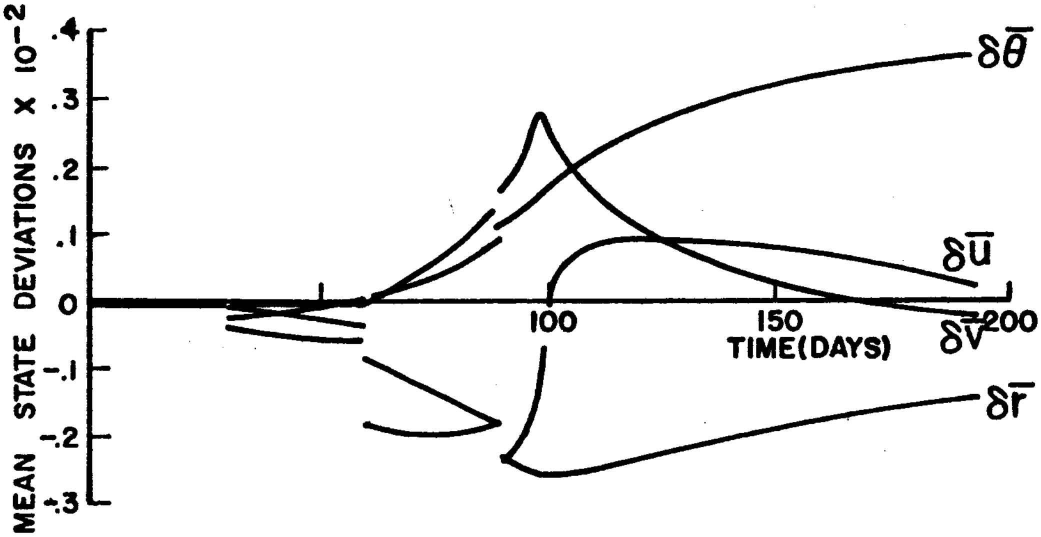

Figure 33k. Conditional Mean State Deviation Time Histories (Observation at t = 30 Days, t = 60 Days, and t = 90 Days, $\sigma_\varepsilon = 10^{-3}$)

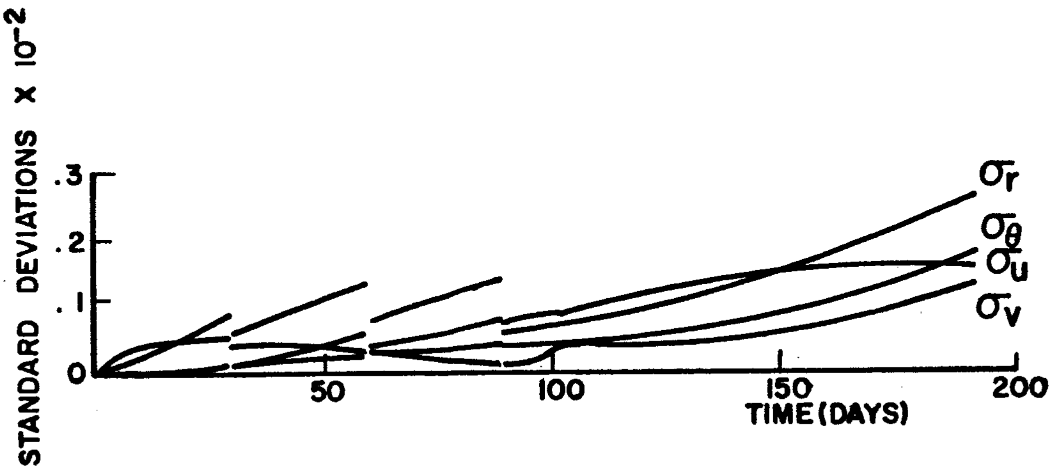

Figure 33l. Conditional Standard Deviation Time Histories (Observations at t = 30 Days, t = 60 Days, and t = 90 Days, $\sigma_\varepsilon = 10^{-3}$)

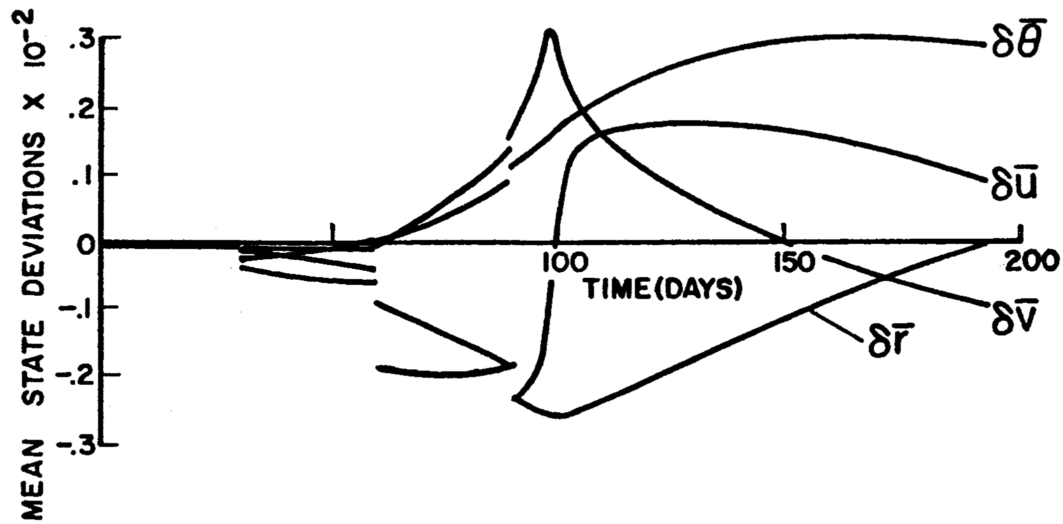

Figure 33m. Optimal Conditional Mean State Deviation Time Histories (Control Correction at t = 30 Days, t = 60 Days, t = 90 Days, $\sigma_\varepsilon = 10^{-3}$)

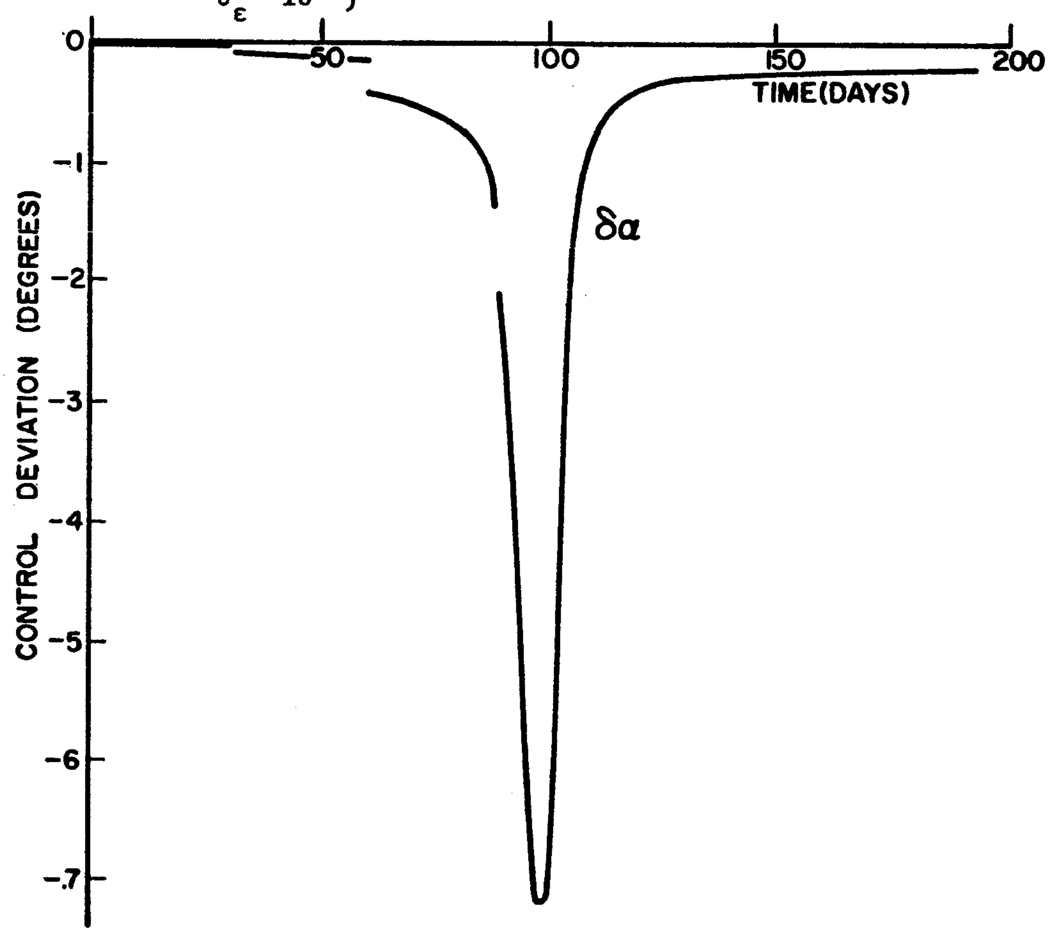

Figure 33n. Optimal Corrective Control Deviation Time History (Control Correction at t = 30 Days, t = 60 Days, and t = 90 Days, $\sigma_\varepsilon = 10^{-3}$)

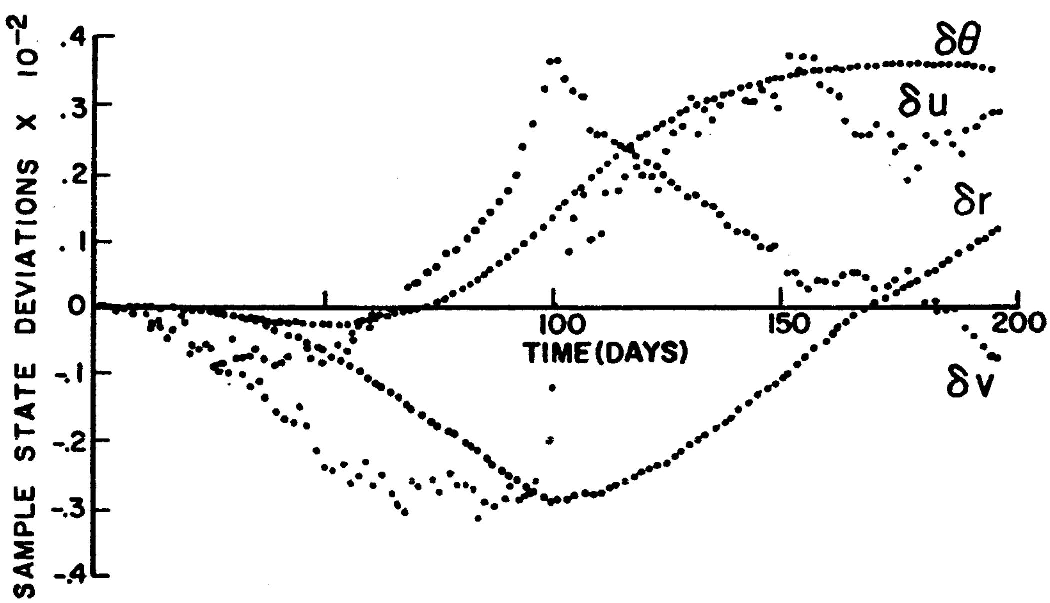

Figure 33o. Simulated State Deviation Time Histories (Control Correction at t = 30 Days, t = 60 Days, and t = 90 Days, $\sigma_\varepsilon = 10^{-3}$)

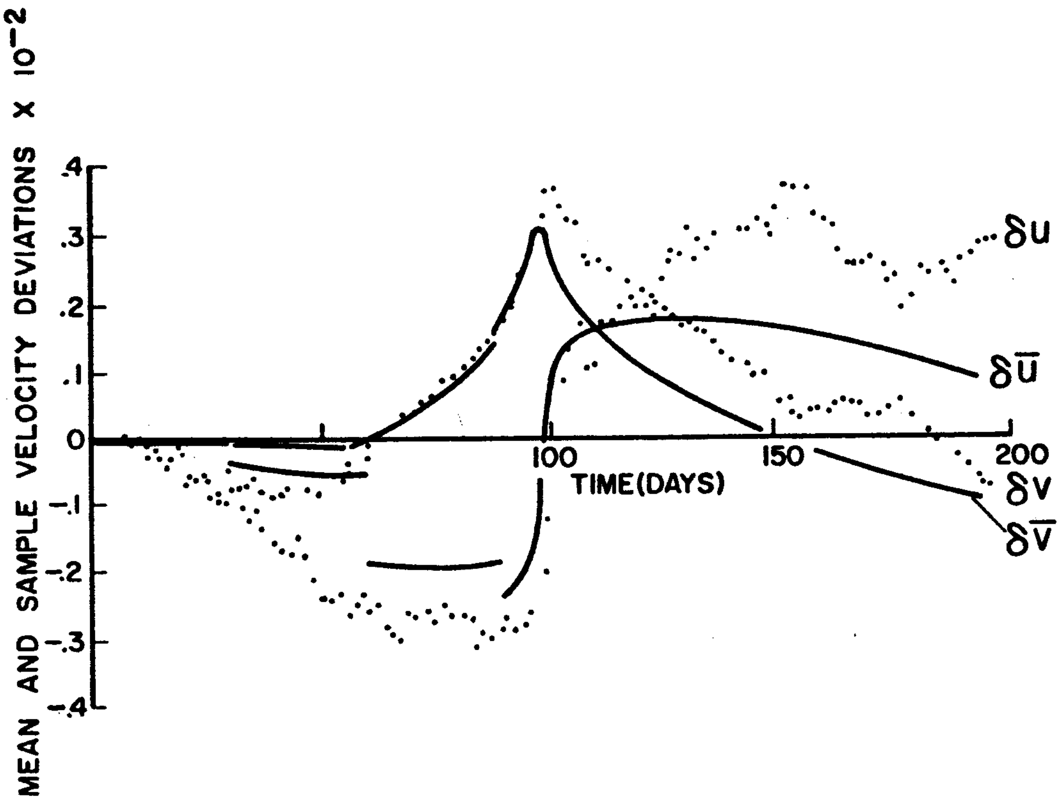

Figure 33p. Comparison of Conditional Mean State Deviations with Simulated State Deviations (Observations at t = 30, 60, 90 Days, $\sigma_\varepsilon = 10^{-3}$): Velocity Components

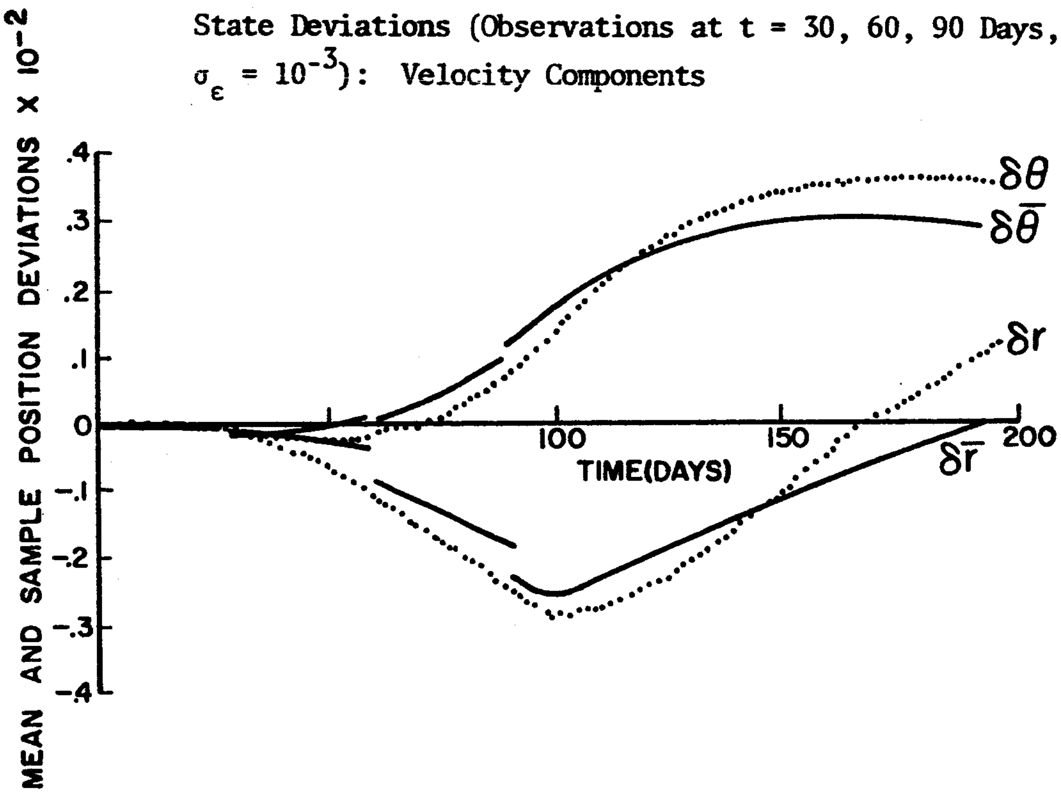

Figure 33q. Comparison of Conditional Mean State Deviations with Simulated State Deviations (Observations at t = 30, 60, 90 Days, $\sigma_\varepsilon = 10^{-3}$): Position Components

accuracy of the sample trajectory, the norm of the standard deviation components σ_u, σ_v, σ_r, and σ_θ, at the terminal time t_f, and the norm of the sample miss components $\delta u-\delta\bar{u}$, $\delta v-\delta\bar{v}$, $\delta r-\delta\bar{r}$, and $\delta\theta-\delta\bar{\theta}$, at the terminal time are plotted as a function of the number of observations. The results are shown in Figure 33r. The norm of the terminal standard deviations is defined in the following equation.

$$|\sigma| = \left|\sqrt{\sigma_u(t_f)^2 + \sigma_v(t_f)^2 + \sigma_r(t_f)^2 + \sigma_\theta(t_f)^2}\right| \qquad (5.76)$$

$|\sigma|$ could be called the "expected root square miss". The norm of the terminal sample miss components is defined in the following equation.

$$|\Delta x| = \left|\sqrt{[\delta u(t_f)-\delta\bar{u}(t_f)]^2 + [\delta v(t_f)-\delta\bar{v}(t_f)]^2 + [\delta r(t_f)-\delta\bar{r}(t_f)]^2 + [\delta\theta(t_f)-\delta\theta(t_f)]^2}\right| \qquad (5.77)$$

$|\Delta x|$ could be called the "sample root square miss". Both the expected root square miss and the sample root square miss are seen to decrease as a consequence of each additional observation. This indicates that the preceding sequence of observation-correction operations appears to be guiding the actual sample state closer to satisfying the terminal constraints as more observations and control corrections are made.

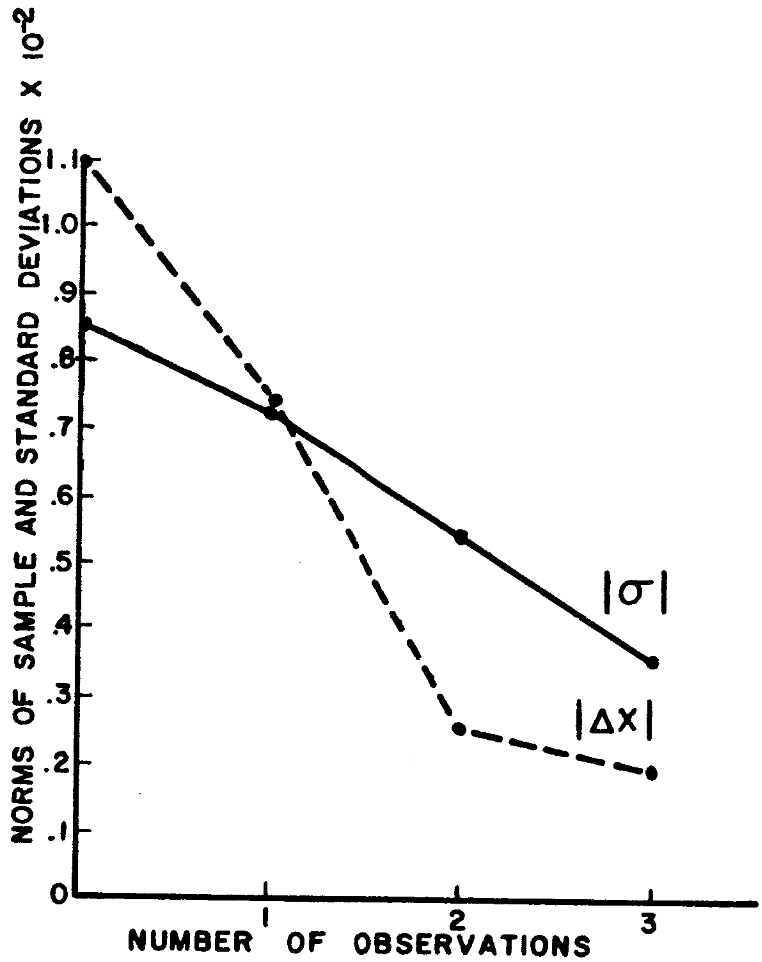

Figure 33r. Expected Root Square Miss and Sample Root Square Miss vs. the Number of Observations (Observations Taken Every 30 Days, $\sigma_\varepsilon = 10^{-3}$)

In order to illustrate the sensitivity of the preceding method of updating an optimal stochastic control program to the times at which the observations are taken, the same sample trajectory illustrated in Figure 19 is used for making single observation-correction operations at later times in the controlling interval. In particular, a single observation with an error standard deviation of $10^{-3}V_E$ is taken at $t = 60$ days, with the results given in Figure 34, and a single observation is taken at $t = 120$ days, with the results given in Figure 35. An outline of the results is given in the subsequent presentation.

For Figure 34,

a. The initial observation is made at $t = 60$ days. The conditional state mean deviations are given in Figure 34a.
b. The updated standard deviations are given in Figure 34b.
c. A control correction is made and the corrected mean state deviations are given in Figure 34c.
d. The corrected control deviation is given in Figure 34d.

For Figure 35,

a. The initial observation is made at $t = 120$ days. The conditional mean state deviations are given in Figure 35a.
b. The updated standard deviations are given in Figure 35b.
c. A control correction is made and the corrected mean state deviations are given in Figure 35c.
d. The corrected control deviation is given in Figure 35d.

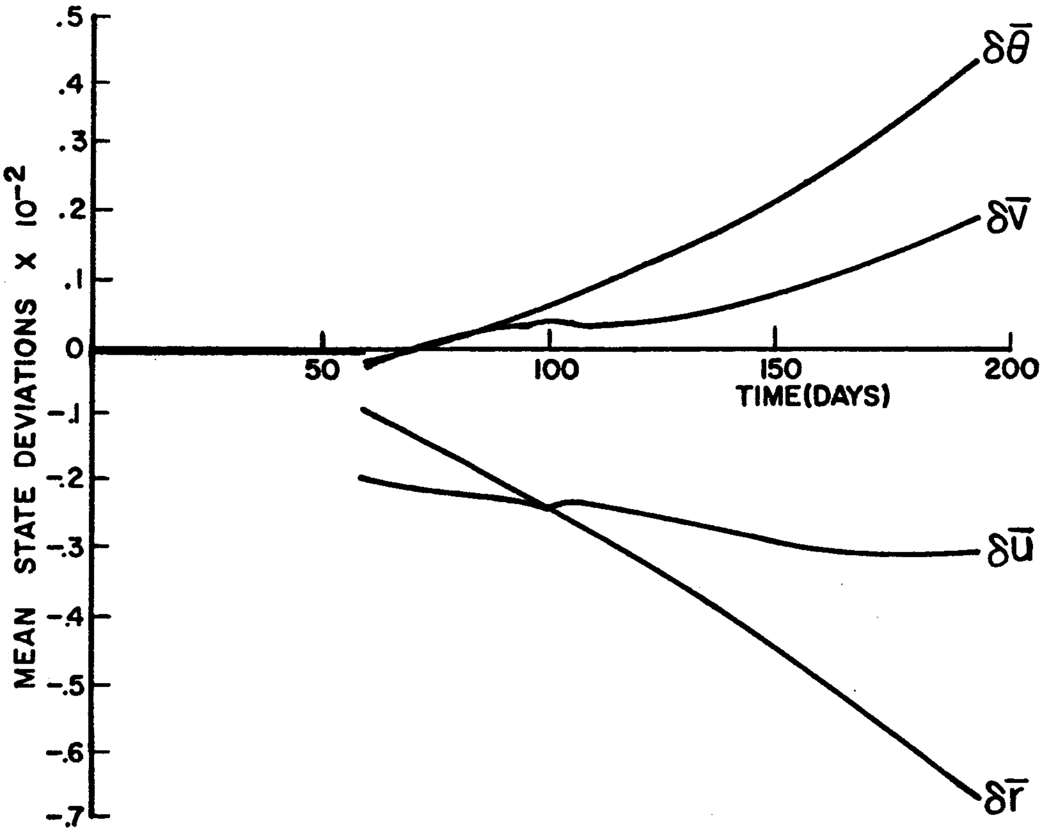

Figure 34a. Conditional Means State Deviation Time Histories

(Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

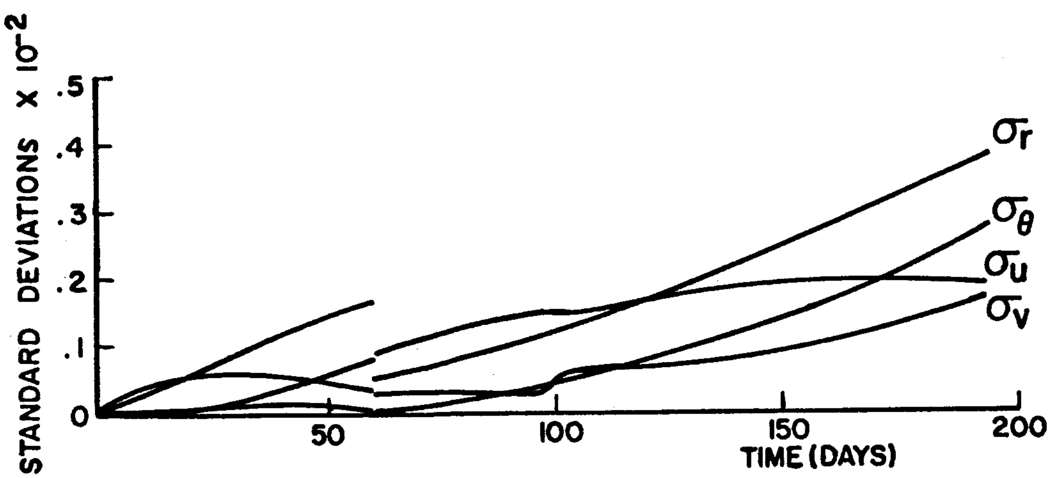

Figure 34b. Conditional Standard Deviation Time Histories

(Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

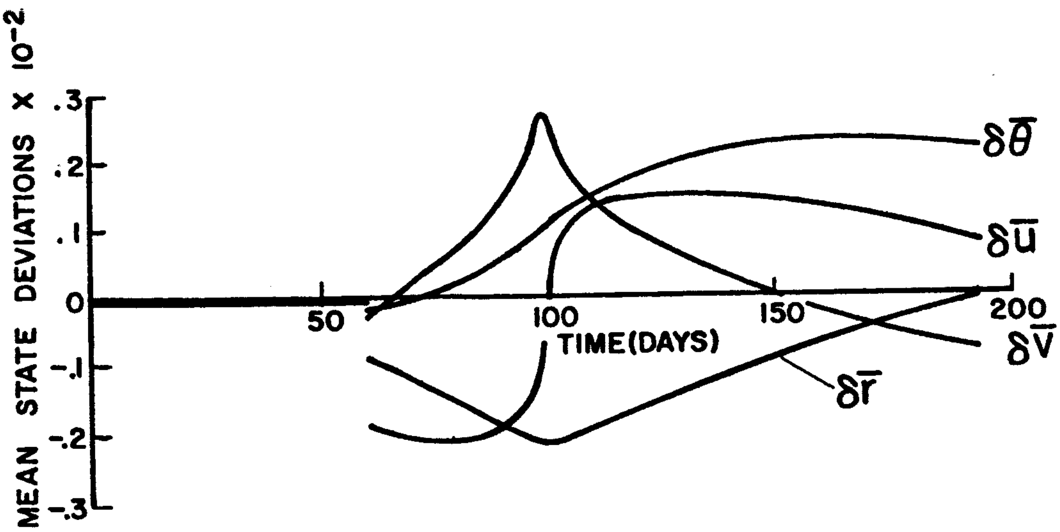

Figure 34c. Optimal Conditional Mean State Deviation Time Histories (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

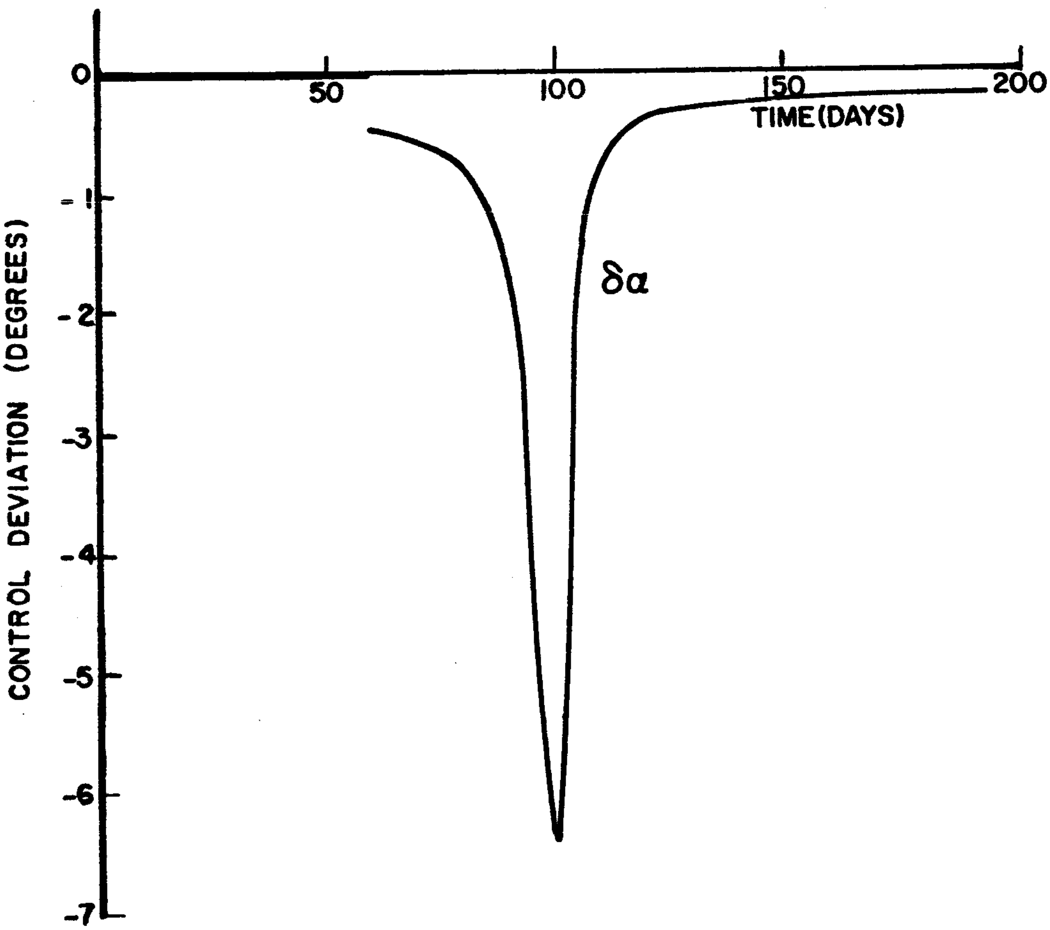

Figure 34d. Optimal Corrective Control Deviation Time History (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

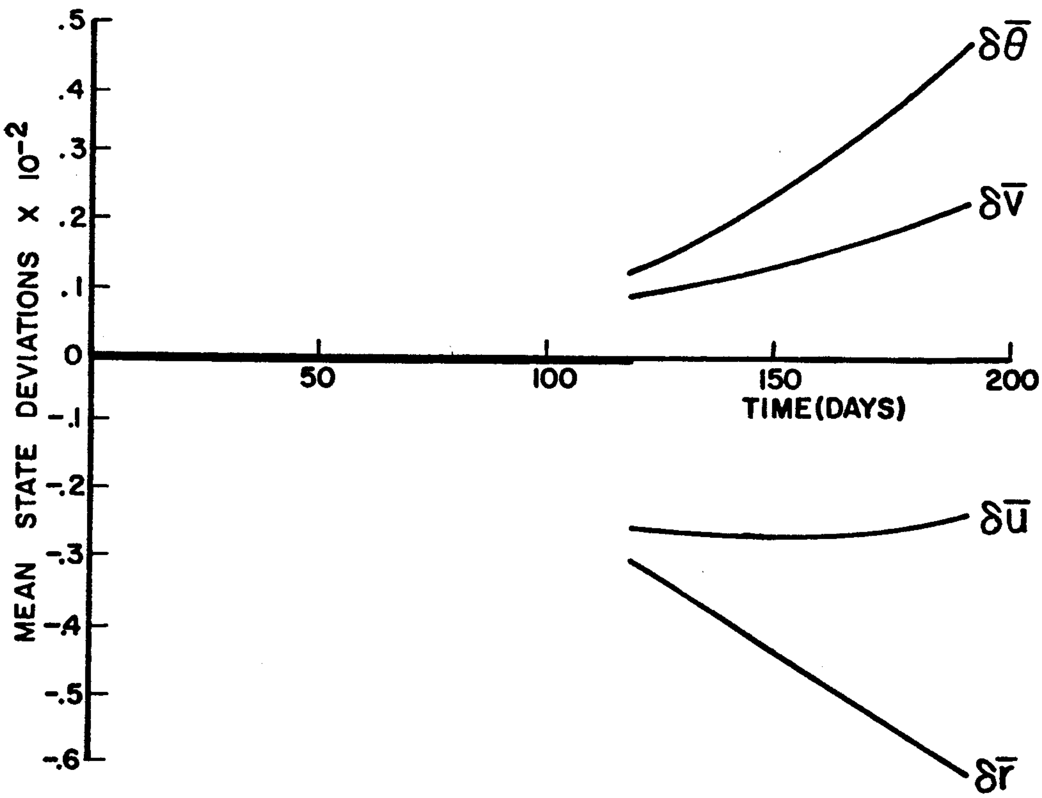

Figure 35a. Conditional Means State Deviation Time Histories

(Observation at t = 120 Days, $\sigma_{\varepsilon} = 10^{-3}$)

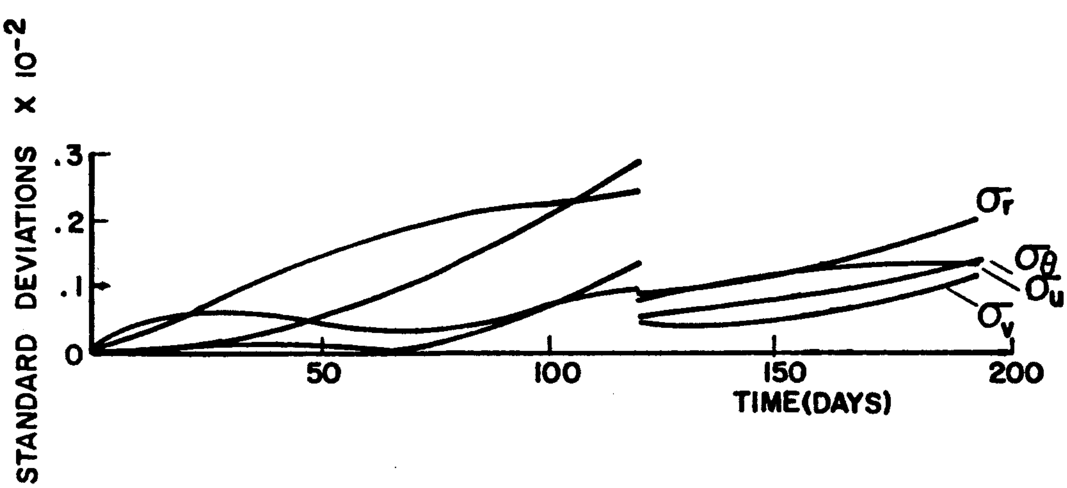

Figure 35b. Conditional Standard Deviation Time Histories

(Observation at t = 120 Days, $\sigma_{\varepsilon} = 10^{-3}$)

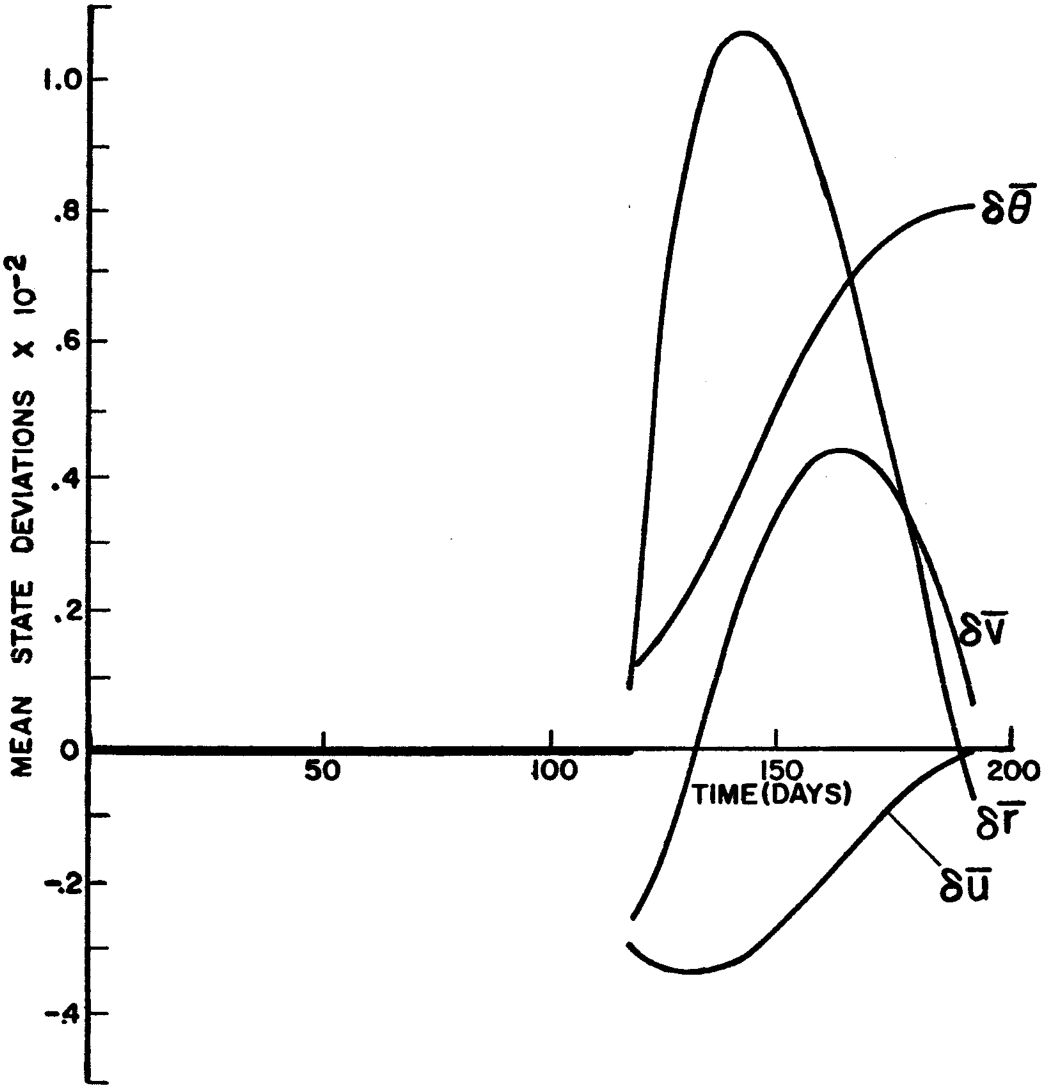

Figure 35c. Optimal Conditional Mean State Deviation Time Histories

(Control Correction at t = 120 Days, $\sigma_\varepsilon = 10^{-3}$)

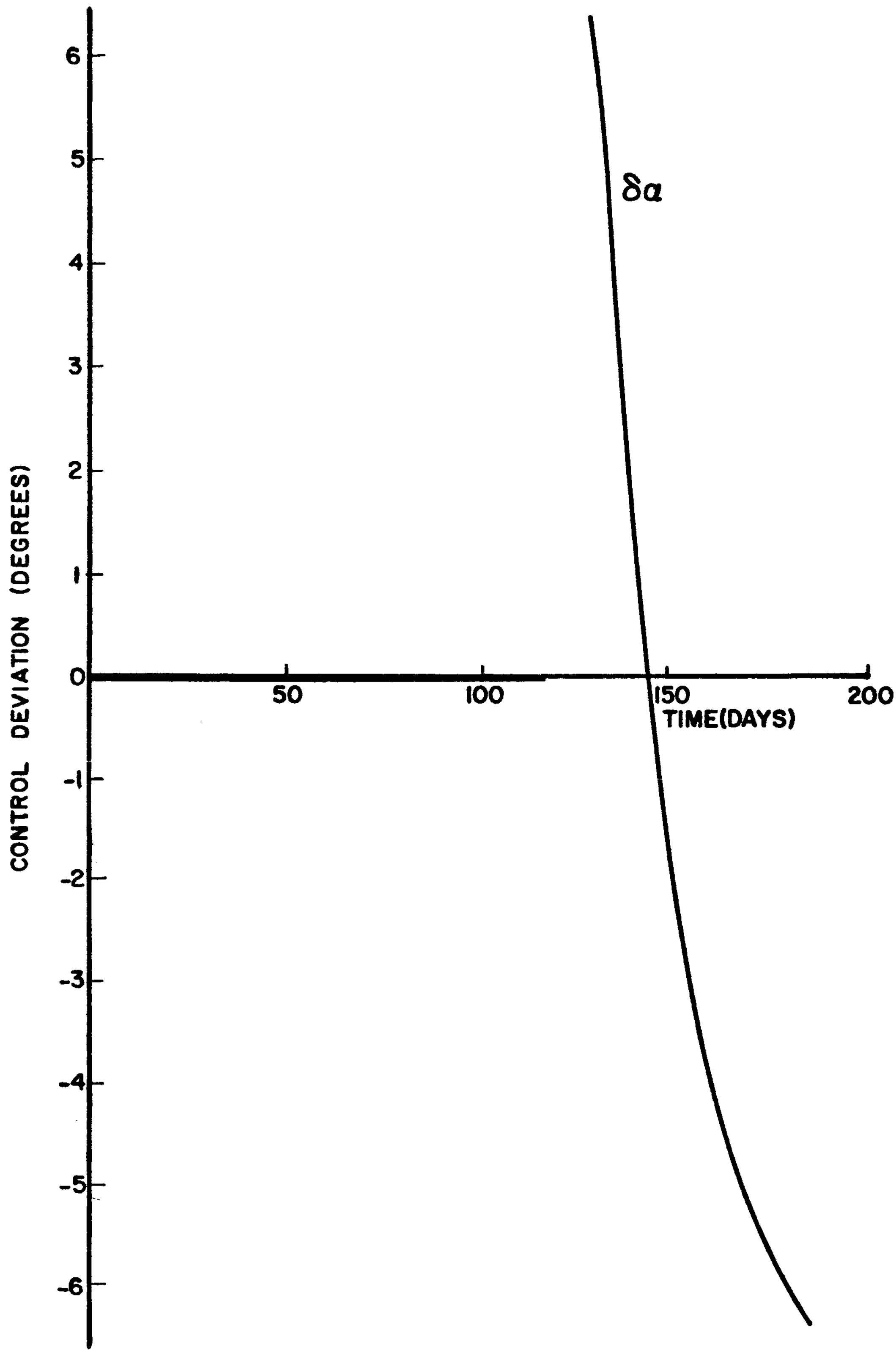

Figure 35d. Optimal Corrective Control Deviation Time History (Control Correction at t = 120 Days, $\sigma_\varepsilon = 10^{-3}$)

The results shown in Figure 34 and Figure 35 indicate first that the conditional mean state deviations which are derived from observations taken late in the controlling interval accurately predict the actual state deviations, but also that the control corrections, along with the resulting mean state deviations, are quite large. See Figures 35c and 35d.

In order to illustrate the performance of the preceding corrective control procedure on a trajectory which is perturbed by highly correlated noise, the corrective control scheme is applied to a second Monte Carlo simulated trajectory, and the results are presented in Figure 36. The parameters of interest for the trajectory are $\sigma_a = .05T$, $T_a = 1000$ days, $\sigma_\alpha = 0$, and $\sigma_0 = 0$. An outline of the results is presented in the subsequent presentation.

a. The sample state deviations which were simulated with the preceding set of noise parameters are presented in Figure 36a.

b. The highly correlated sample noise n_a, which occurs in the thrust/mass magnitude, is presented in Figure 36b.

c. The standard deviations associated with the preceding noise parameters are presented in Figure 35c.

d. An observation is made at $t = 60$ days, and the resulting conditional mean state deviations are given in Figure 36d. Note that $\sigma_\varepsilon = 10^{-3} V_E$.

e. The updated standard deviations of the state are presented in Figure 36e.

f. The resulting conditional mean of the noise is given in Figure 36f.

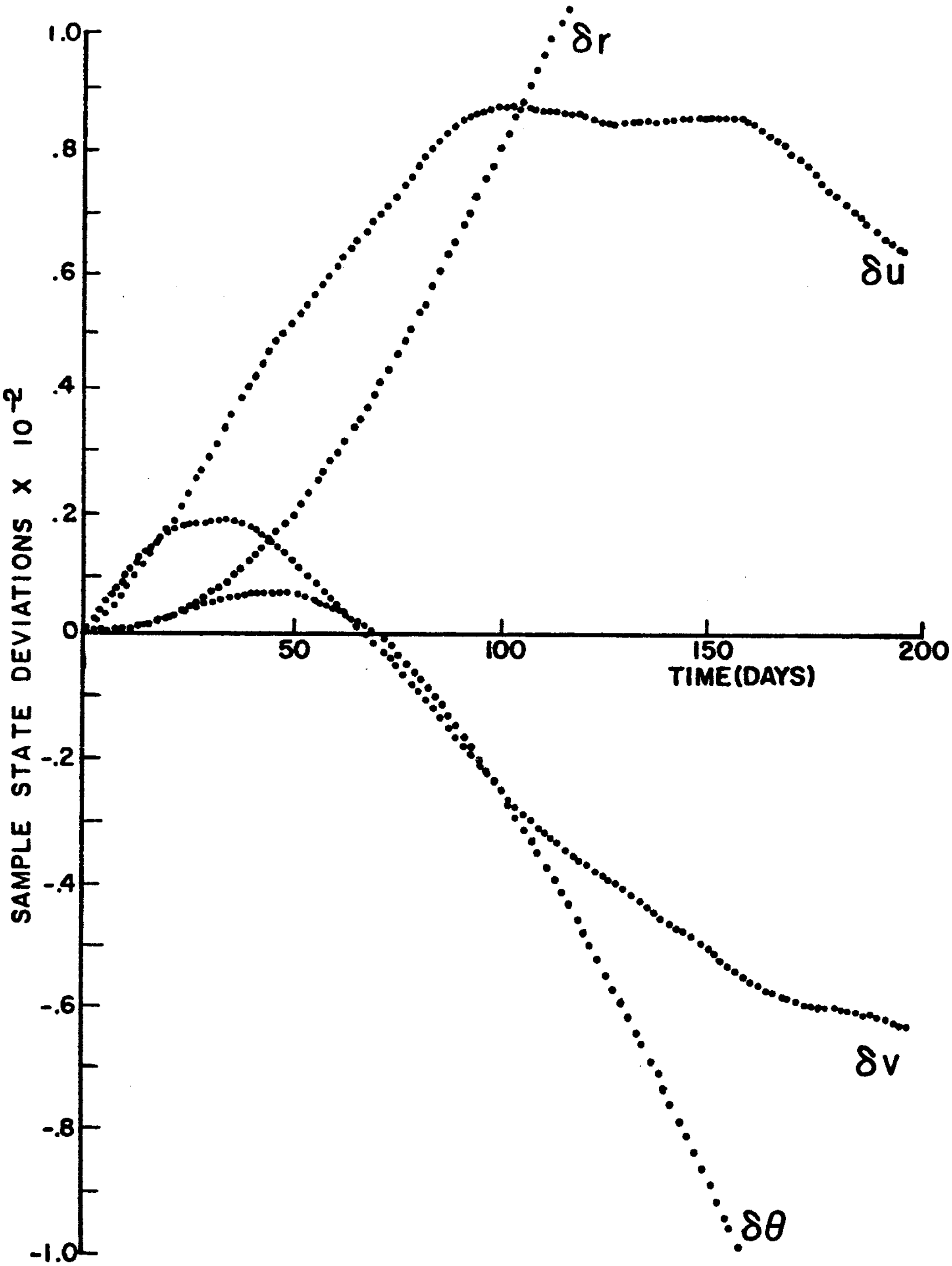

Figure 36a. Simulated State Deviation Time Histories

(σ_a = .05T, T_a = 1000 Days, σ_α = 0, σ_0 = 0)

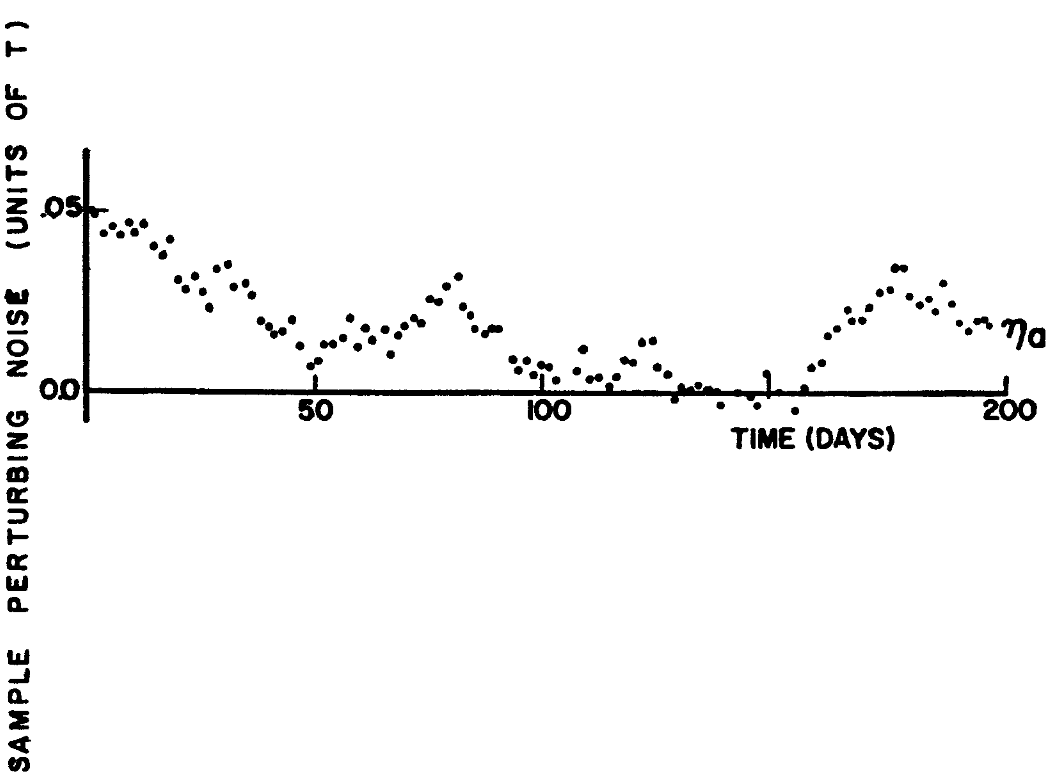

Figure 36b. Simulated Highly Correlated O.U. Process Noise Occurring in the Thrust/Mass Magnitude (σ_a = .05T, T_a = 1000 Days)

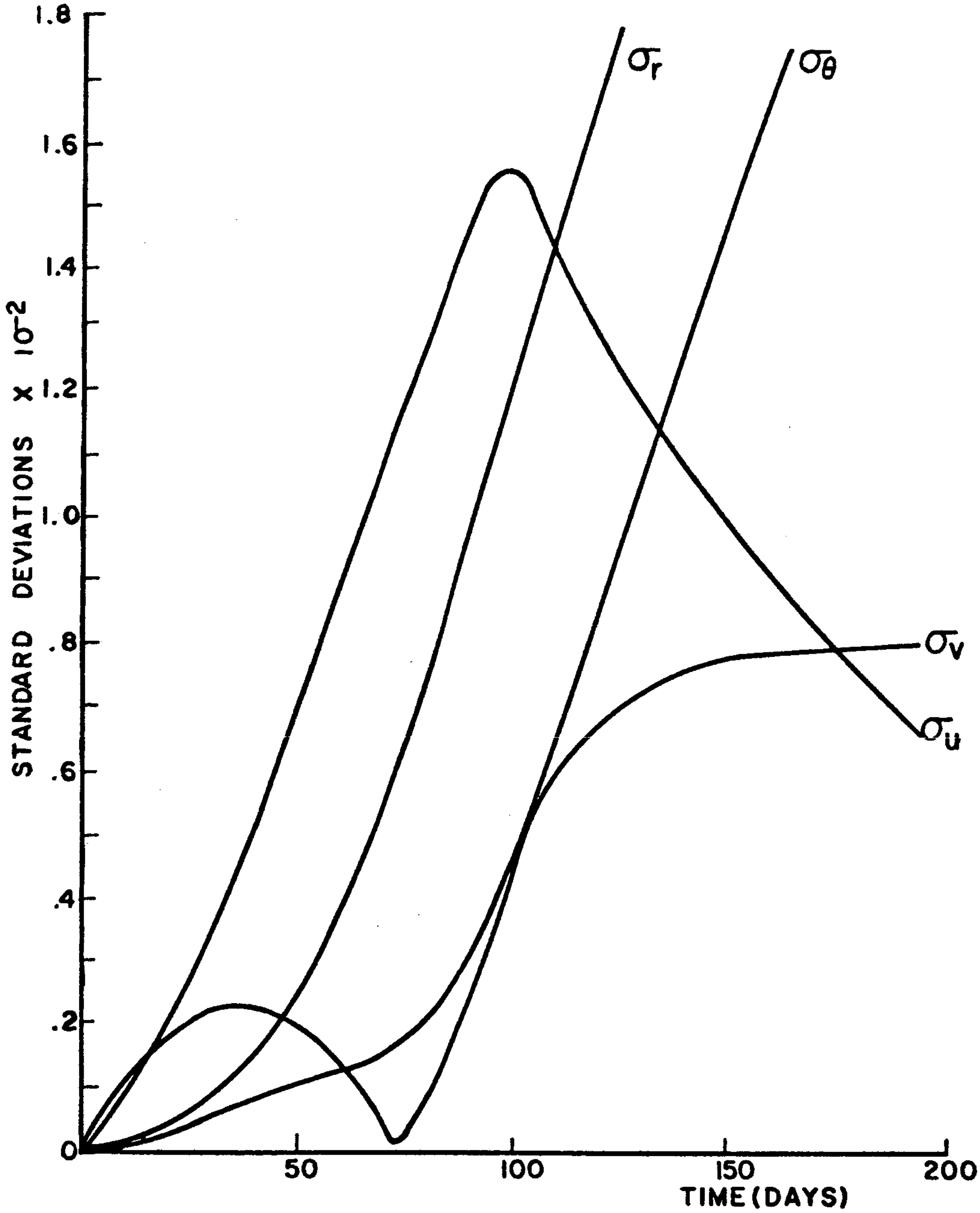

Figure 36c. Standard Deviation Time Histories

(σ_a = .05T, T_a = 1000 Days, σ_α = 0, σ_0 = 0)

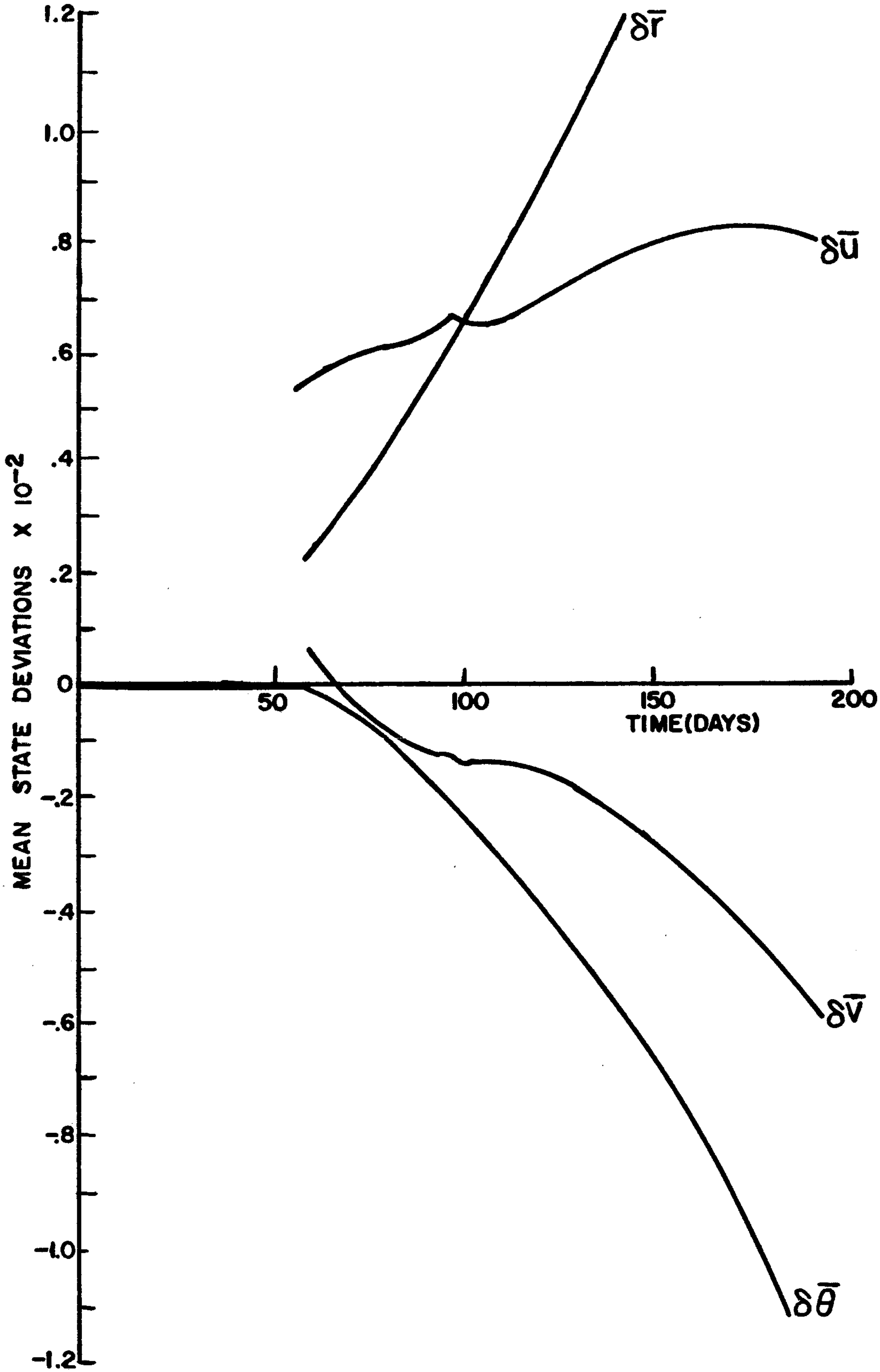

Figure 36d. Conditional Mean State Deviation Time Histories

(Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

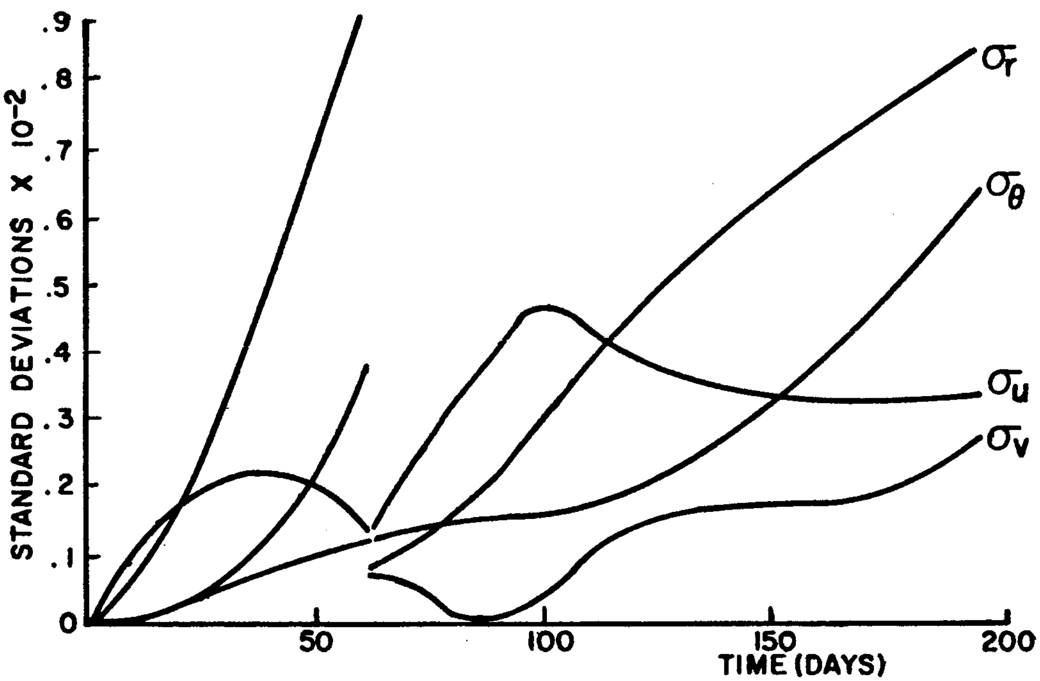

Figure 36e. Conditional Standard Deviation Time Histories

(Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

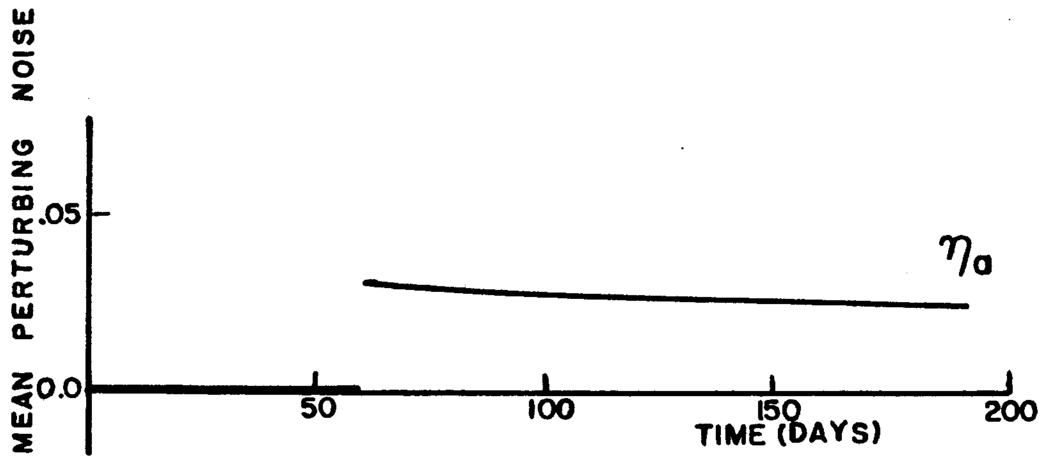

Figure 36f. Conditional Mean of the Noise (Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

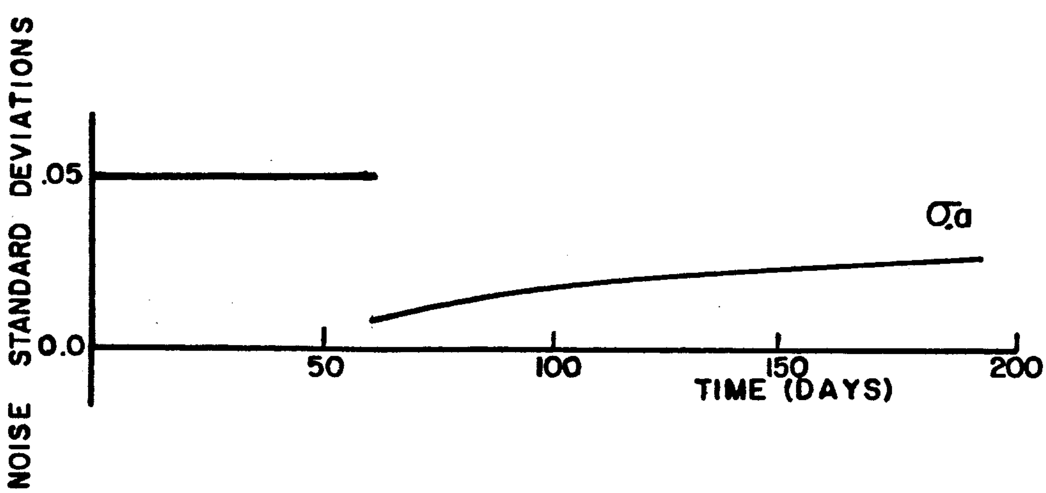

Figure 36g. Conditional Standard Deviation of the Noise (Observation at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

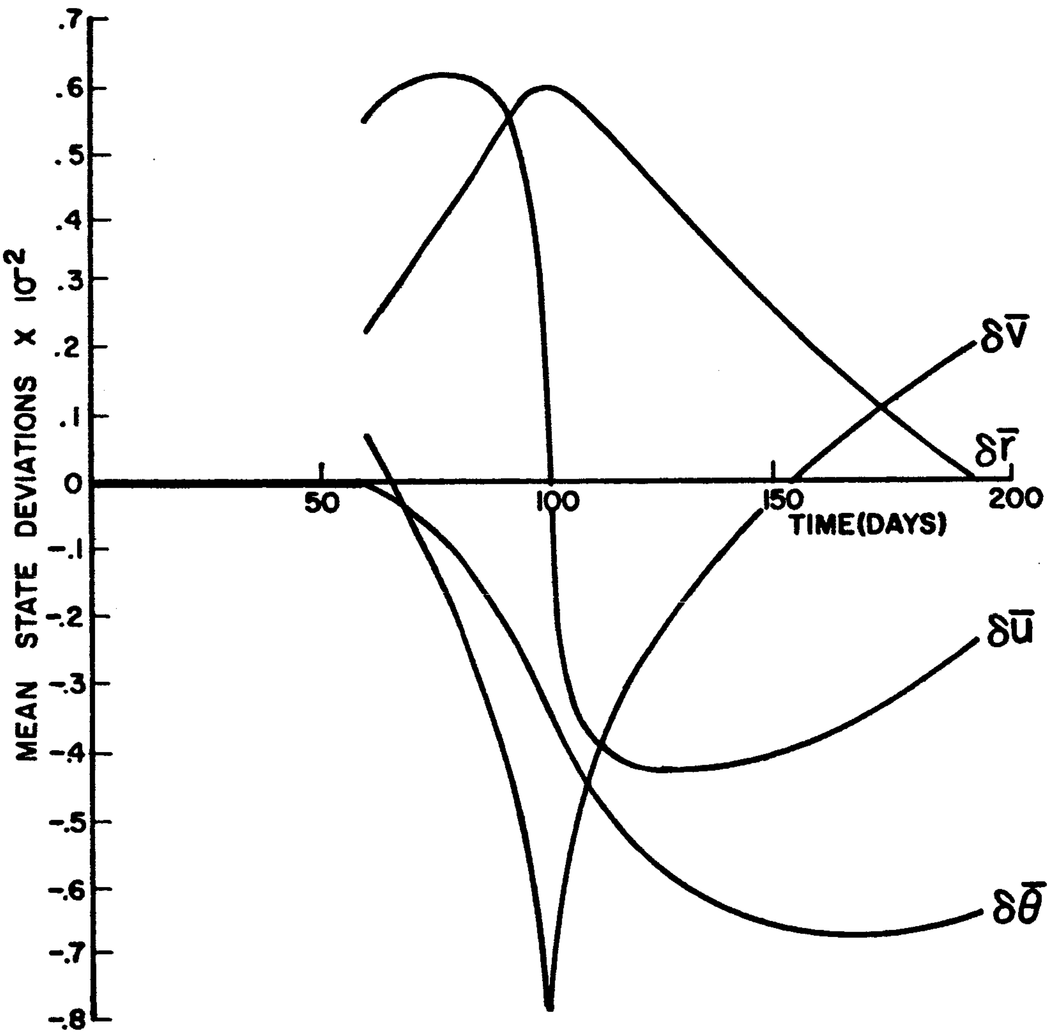

Figure 36h. Optimal Conditional Mean State Deviation Time Histories (Control Correction at t = 60 Days, $\sigma_{\varepsilon} = 10^{-3}$)

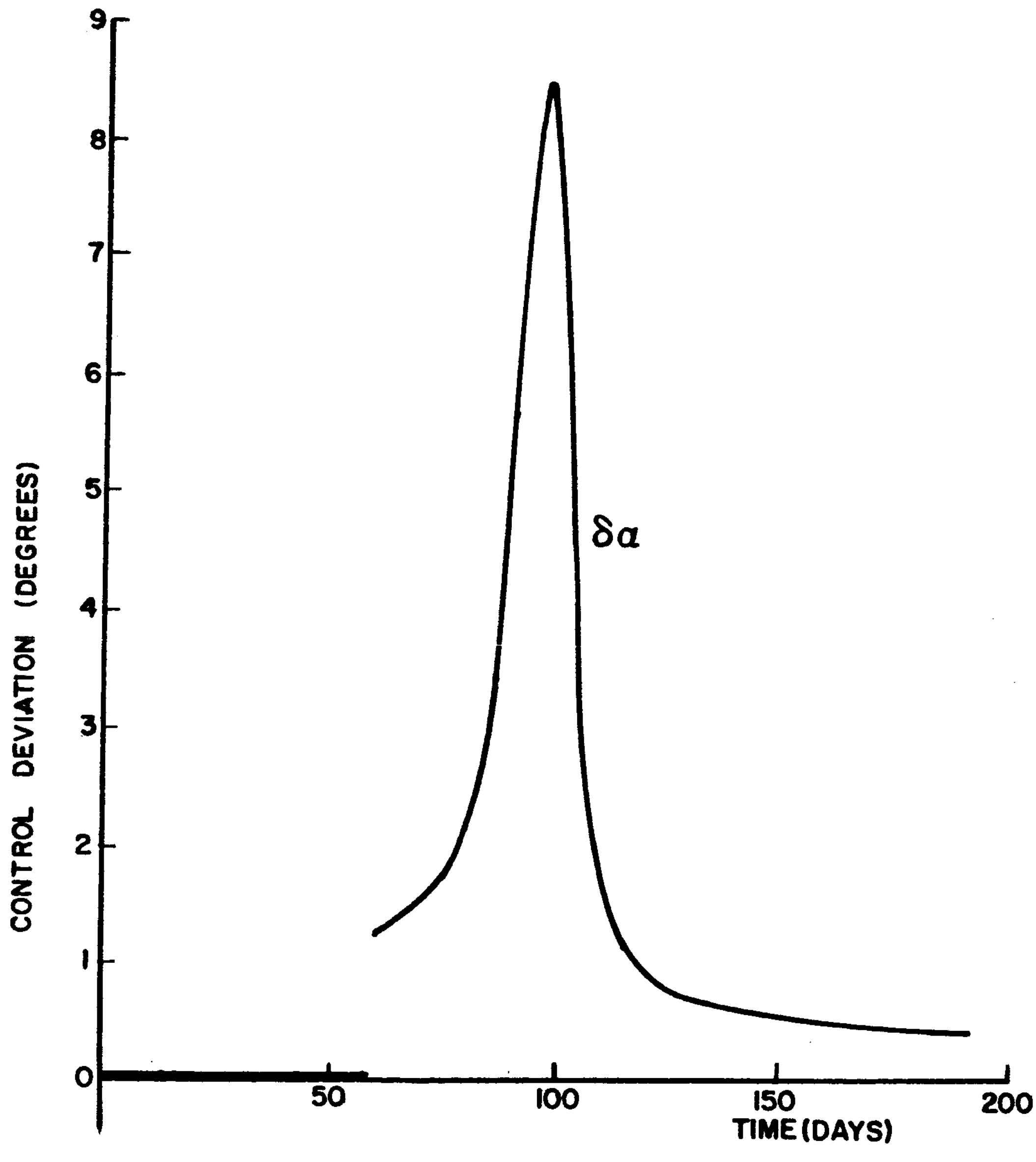

Figure 36i. Optimal Corrective Control Deviation Time History (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

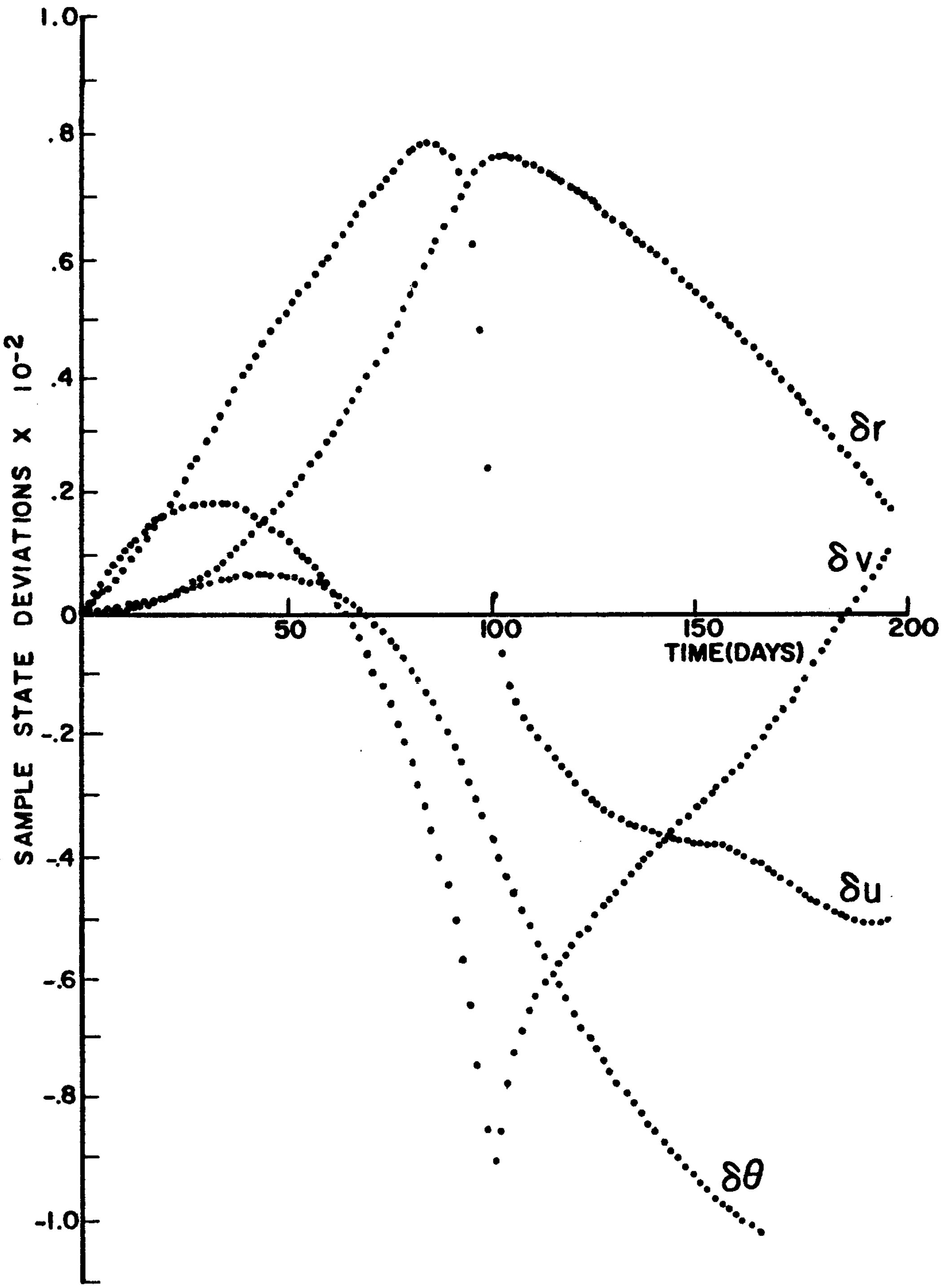

Figure 36j. Simulated Corrected State Deviation Time Histories (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$)

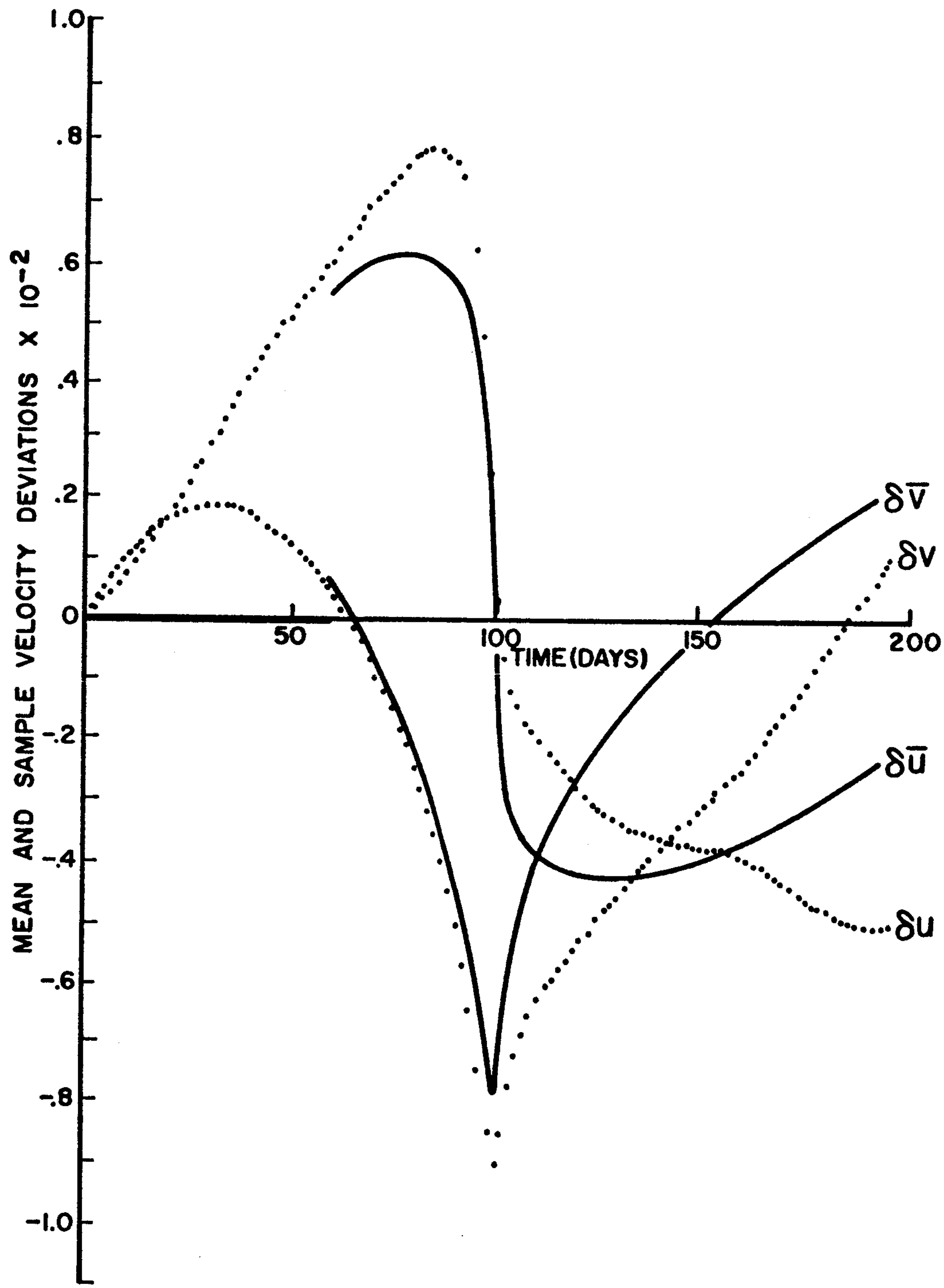

Figure 36k. Comparison of Conditional Mean State Deviations to Simulated Corrected State Deviations (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$): Velocity Components

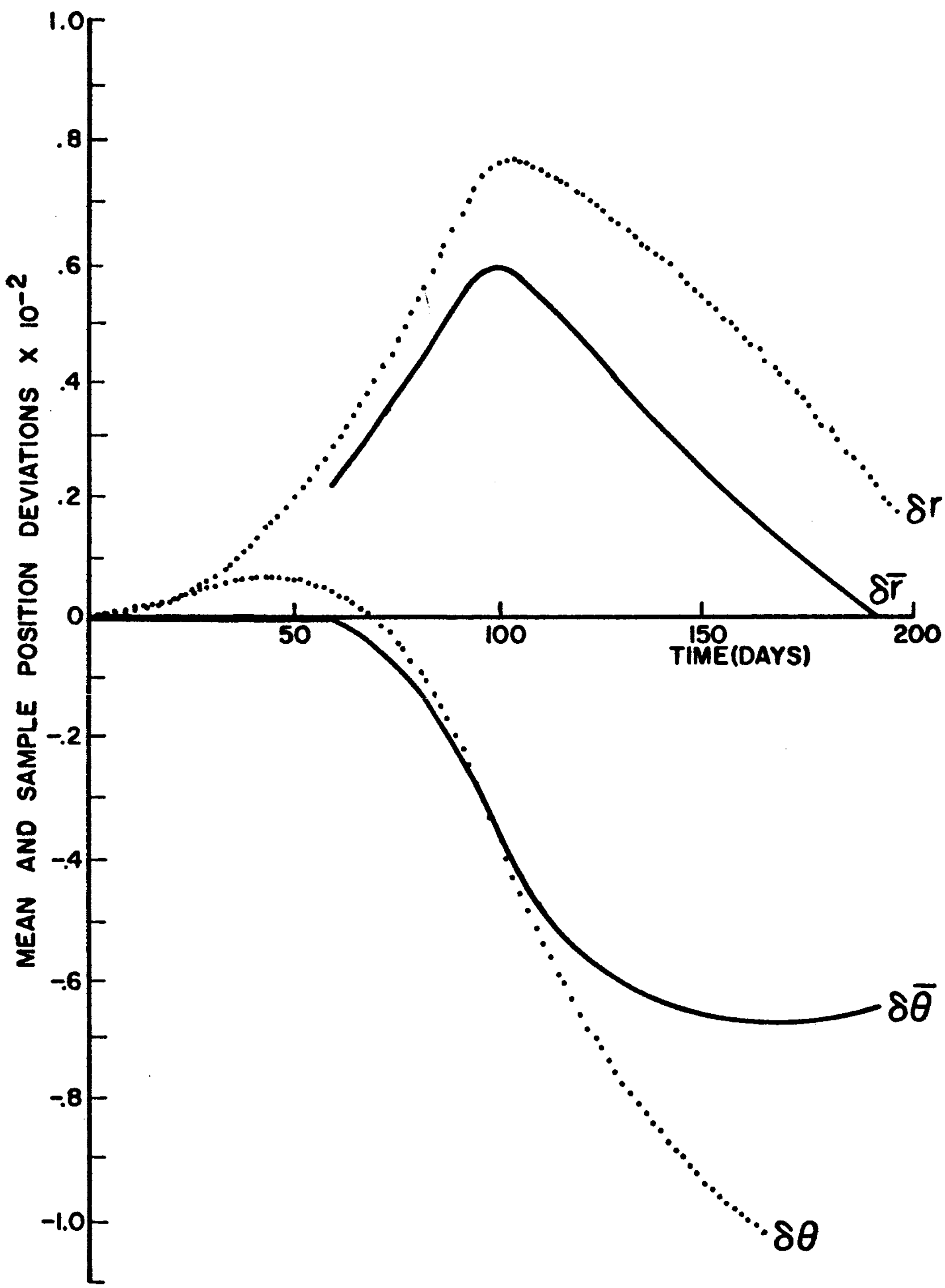

Figure 361. Comparison of Conditional Mean State Deviations to Simulated Corrected State Deviations (Control Correction at t = 60 Days, $\sigma_\varepsilon = 10^{-3}$): Position Components

g. The updated standard deviation of the noise is presented in Figure 36g.

h. A control correction is made, and the resulting mean state deviations are given in Figure 36h.

i. The corrected control deviation is presented in Figure 36i.

j. The sample corrected state deviations appear in Figure 36j.

k. A superposition of Figures 36i and 36j is made in order to illustrate the effectiveness of the corrected mean state deviations at approximating the corrected sample state deviations. The superposition of the velocity components of the state is given in Figure 36k.

l. The superposition of the position components of the state is given in Figure 36l.

The power of the observation process is illustrated in Figure 37. Observations are made every 30 days on a trajectory with the noise parameters $\sigma_a = .02T$, $T_a = 1$ day, $\sigma_\alpha = 0$, $\sigma_0 = 3x10^{-3}$. The sequence of Figures 37a through 37g illustrate how the series of recursive observations forces the standard deviations of the state to lesser and lesser values. Note that $\sigma_\varepsilon = 10^{-3}\ V_E$.

The sensitivity of the observation process to the observation accuracy is illustrated in Figure 38. The parameters of the trajectory are $\sigma_a = .05T$, $T_a = 1$ day, $\sigma_\alpha = 0$, $\sigma_0 = 0$. Observations are made every 60 days with the following standard deviations of the observation error.

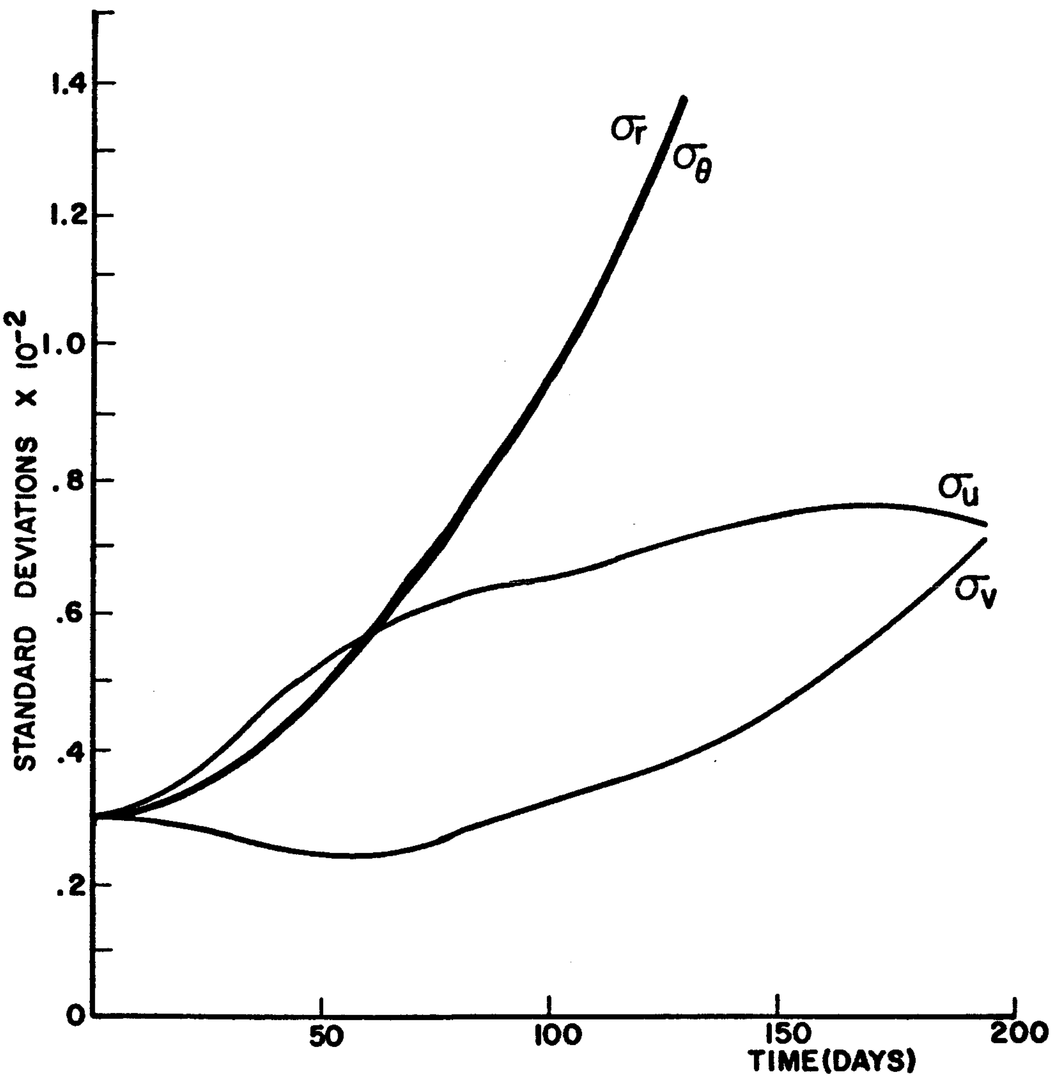

Figure 37a. Standard Deviation Time Histories

(σ_a = .02T, T_a = 1 Day, σ_α = 0, $\sigma_0 = 3 \times 10^{-3}$)

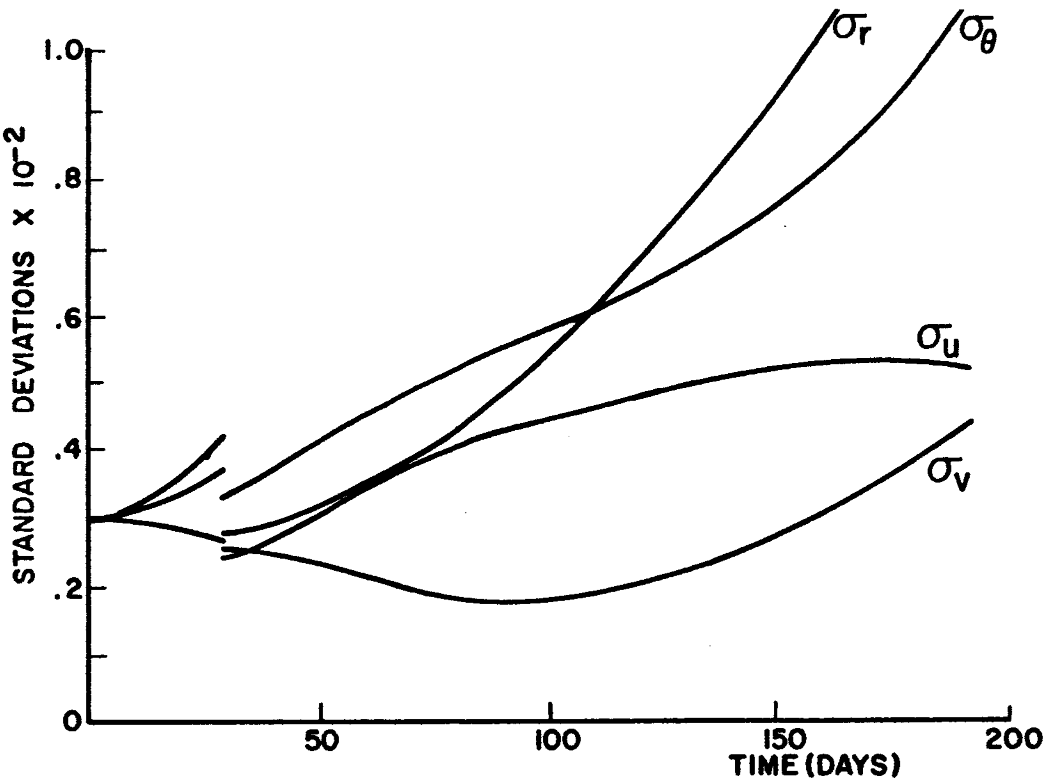

Figure 37b. Conditional Standard Deviation Time Histories (Observation at t = 30 Days, $\sigma_\varepsilon = 10^{-3}$)

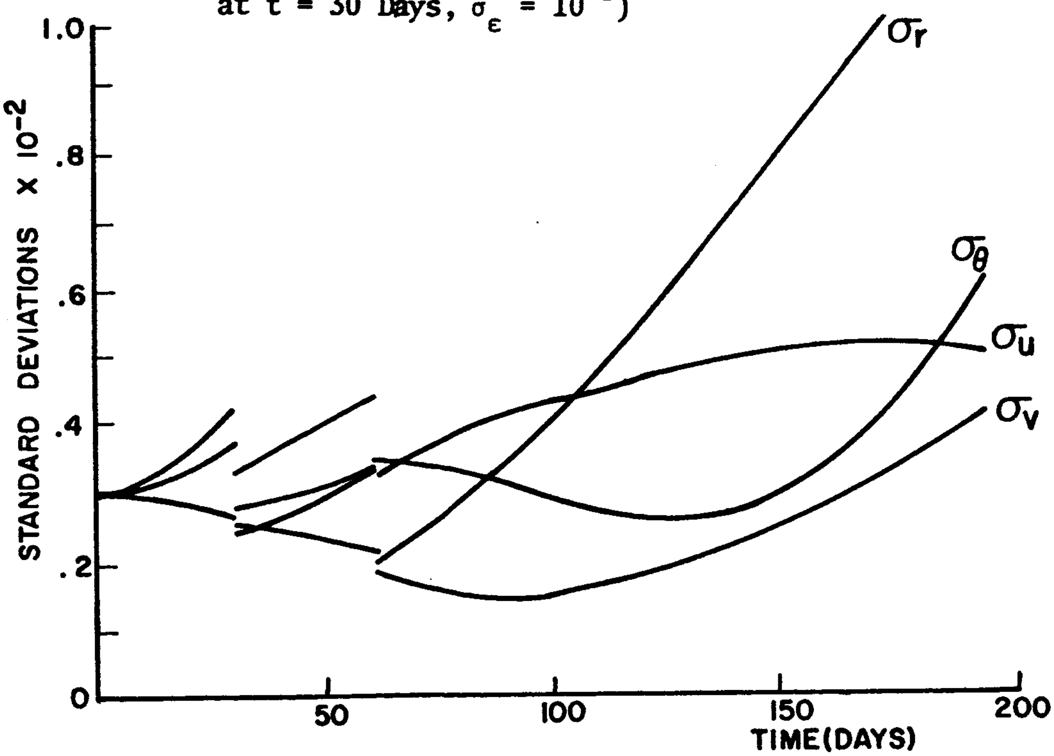

Figure 37c. Conditional Standard Deviation Time Histories (Observations at t = 30, 60 Days, $\sigma_\varepsilon = 10^{-3}$)

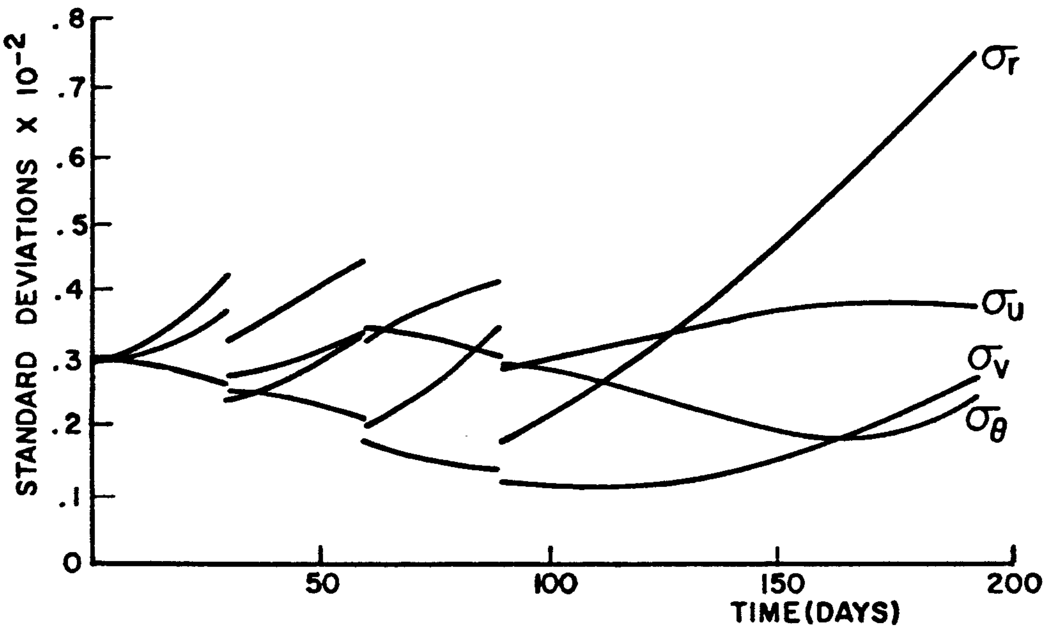

Figure 37d. Conditional Standard Deviation Time Histories (Observations at t = 30, 60, and 90 Days, $\sigma_\varepsilon = 10^{-3}$)

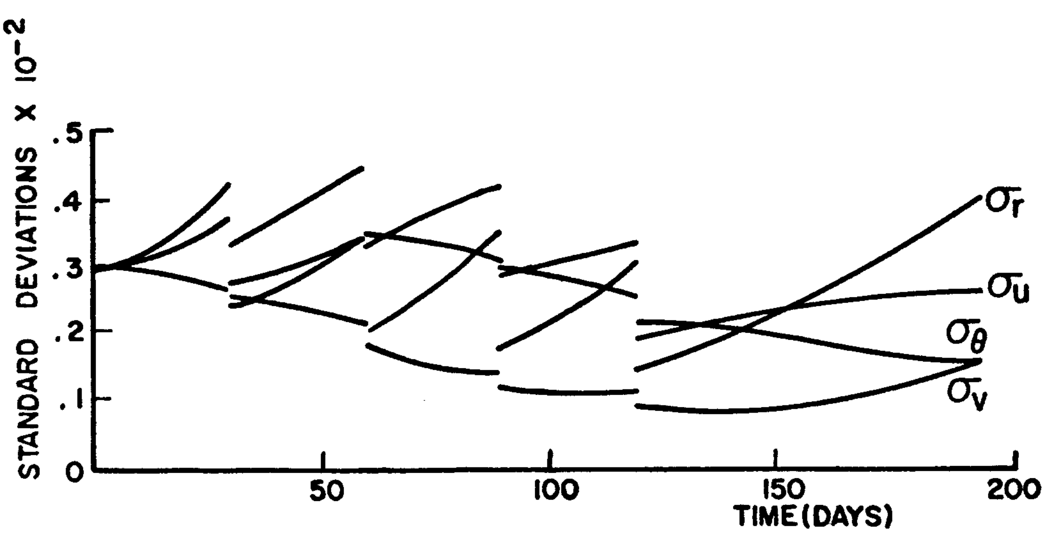

Figure 37e. Conditional Standard Deviation Time Histories (Observations at t = 30, 60, 90, and 120 Days, $\sigma_\varepsilon = 10^{-3}$)

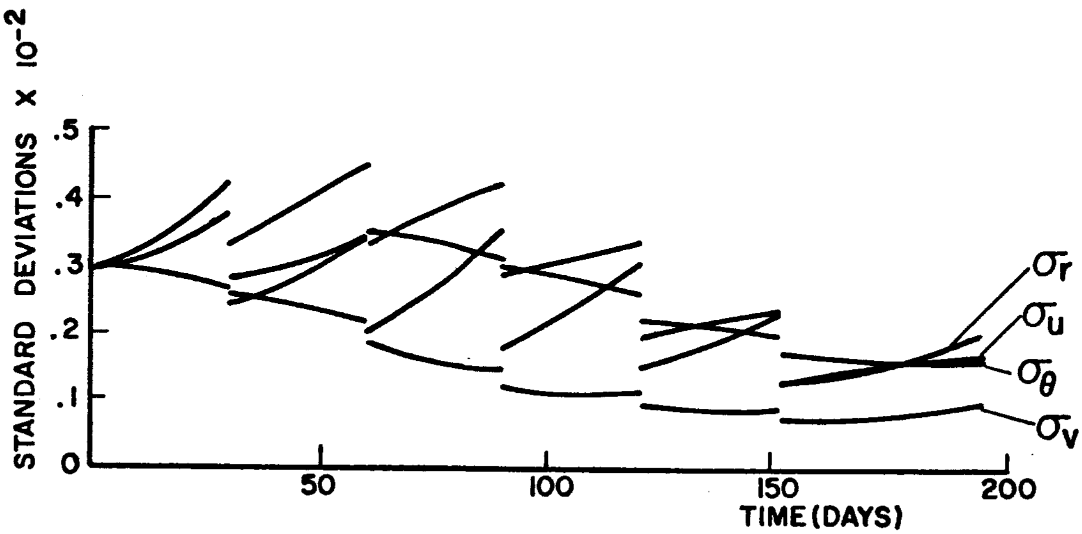

Figure 37f. Conditional Standard Deviation Time Histories (Observations at t = 30, 60, 90, 120, and 150 Days, $\sigma_\varepsilon = 10^{-3}$)

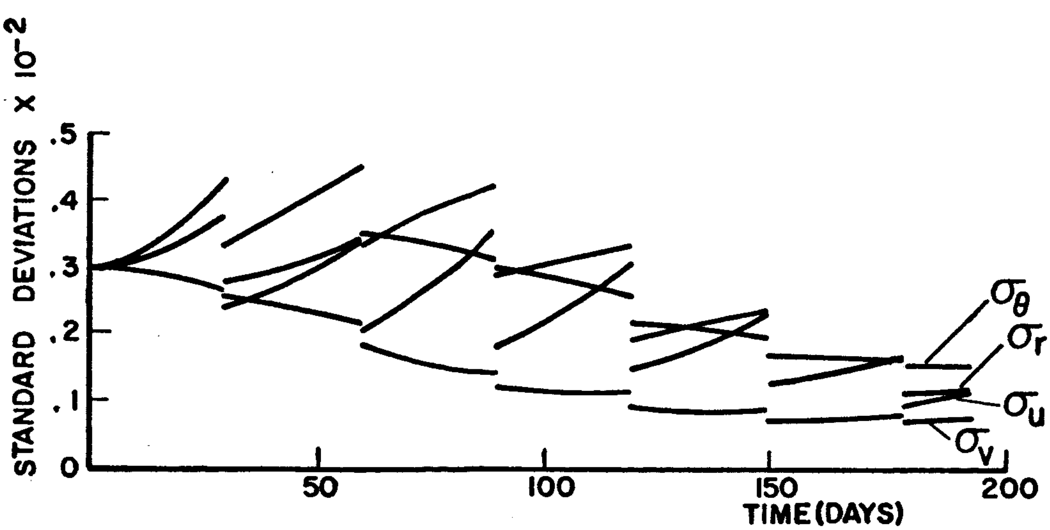

Figure 37g. Conditional Standard Deviation Time Histories (Observations at t = 30, 60, 90, 120, 150, and 180 Days, $\sigma_\varepsilon = 10^{-3}$)

For Figure 38a, $\sigma_\varepsilon = 0$

For Figure 38b, $\sigma_\varepsilon = 10^{-3}$

For Figure 38c, $\sigma_\varepsilon = 10^{-2}$

The standard deviations of the state components are presented in the figures. It is seen that the standard deviations of the state components vary directly with the standard deviation of the observation error σ_ε.

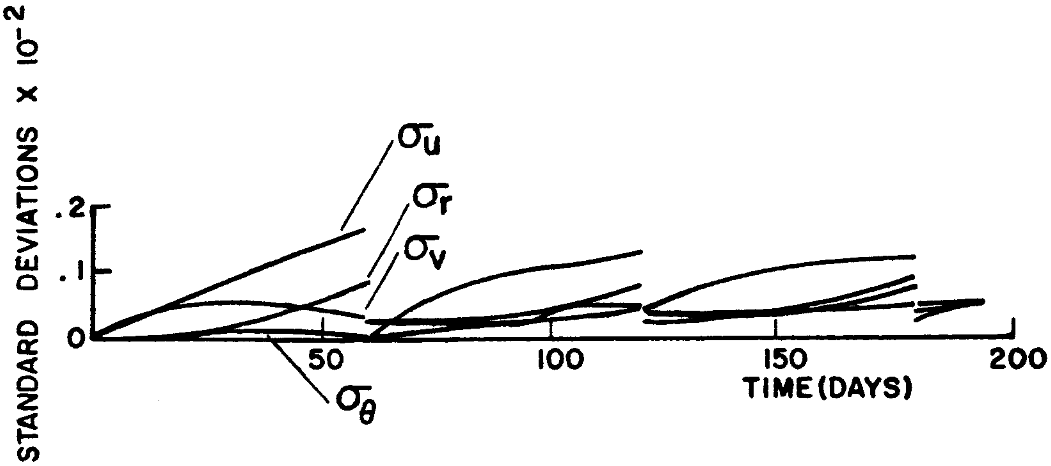

Figure 38a. Conditional Standard Deviation Time Histories (σ_a = .05T, T_a = 1 Day, σ_α = 0, σ_0 = 0, Observations at 60, 120, and 180 Days, σ_ε = 0)

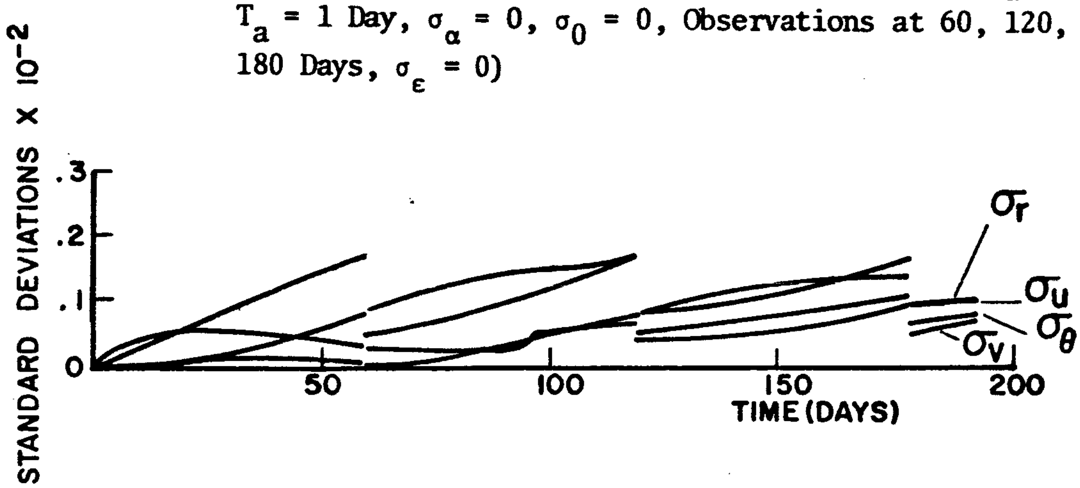

Figure 38b. Conditional Standard Deviation Time Histories (Observations at t = 60, 120, and 180 Days, σ_ε = 10^{-3})

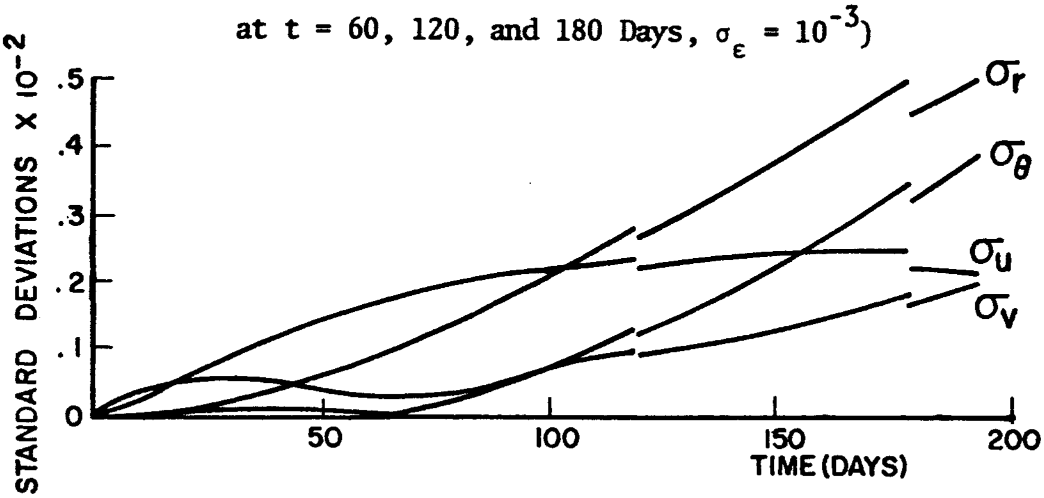

Figure 38c. Conditional Standard Deviation Time Histories (Observations at t = 60, 120, and 180 Days, σ_ε = 10^{-2})

CHAPTER 6

CONCLUSIONS AND EXTENSIONS TO THE WORK

Summary

In the investigation presented here, the problem of the optimal control of a nonlinear dynamic system in the presence of noise is studied. In particular, the investigation is concerned with continuous autocorrelated noise which perturbs the controls of the dynamic system.

A study is made of the effects of noise in the controls on an optimal deterministic trajectory. The effects are illustrated for a simulated study of a low-thrust spacecraft on a minimum time Earth-to-Mars transfer trajectory. The characteristics of the effects of the noise illustrated in the study indicate the necessity for developing an optimal stochastic control. The control procedure developed in the investigation is a nonrandom function of time, based on a priori knowledge of the statistical behavior of the noise process, and is designed to anticipate the expected effects of the noise on the dynamic system. A stochastic calculus of variations approach is employed to determine the control procedure for the stochastic system. The control essentially guides the expected value of the state to meet the terminal conditions, while extremizing the expected value of the original deterministic performance index functional. The behavior of the control procedure is studied for a simulated interplanetary transfer problem.

The results of the study indicate the necessity for presenting a scheme which will correct the control program, on the basis of information gained during the controlling interval, so that the actual state comes closer to satisfying the terminal constraints, while preserving the optimal nature of the control program. A method is presented for replacing the mean values of the state components and the Lagrange multipliers, with which the a priori control is computed, with conditional mean values of these quantities based on the values of state observations. The scheme is applied to the interplanetary transfer problem for the case where range-rate observations are taken at discrete instances of time.

Conclusions

From the study in Chapter 3 of the effects of noise on a nonlinear dynamic system, the following conclusions can be drawn:

1. Both the theory and the numerical studies of the interplanetary problem show that the occurrence of noise in a nonlinear dynamic system implies an ensemble of stochastic trajectories. The analysis shows that the mean of the ensemble differs from the deterministic trajectory.
2. In general the standard deviations of the state components increase with time indefinitely. However, the nonlinearity of the system and the optimal nature of the control strongly influence the values of the standard deviations.
3. The statistics of the ensemble of trajectories are highly dependent on both the variance and the correlation time associated with the perturbing noise. In general, the mean state deviations from the deterministic trajectory and the standard deviations both increase with increasing noise variance and/or increasing noise correlation time.
4. The results of the numerical studies on the interplanetary transfer problem show that the statistics of the ensemble of

trajectories for the case in which noise occurs in the thrust magnitude and for the case in which noise occurs in the thrust direction are quite different.

Study of the application of the optimal stochastic control to the interplanetary transfer problem in Chapter 4 has led to the following conclusions:

1. In the case of the interplanetary transfer problem the difference between the a priori optimal stochastic control and the optimal deterministic control is small in comparison with the perturbing noise. It should be noted that this may not be the case for highly nonlinear dynamic systems.
2. In the case of the interplanetary transfer problem, the implementation of an optimal stochastic control which is based only on an a priori knowledge of the statistics of the perturbing noise does not appreciably reduce the standard deviations of the state components at the final time. For this reason, it can be concluded that the control must be updated throughout the controlling interval if the terminal state is to satisfy approximately the terminal constraints.
3. The stochastic control deviation is highly dependent on the variance and correlation time of the noise, and whether it occurs in the thrust magnitude and/or the thrust direction angle.
4. In the case of the Earth-Mars transfer, the presence of noise in the thrust angle increases the expected value of the transfer time, while the presence of noise in the thrust magnitude slightly decreases the expected value of the transfer time.

The results, presented in Chapter 5, of updating the control program by the use of observation values made available to the controller during the controlling interval are summarized in the following statements:

1. The corrective control program based on the observed values of range-rate measurements appears to guide a simulated trajectory in such a manner that the terminal constraints are satisfied reasonably. However, if the control is not corrected early in the controlling interval, then the control corrections required late in the interval become very large.
2. The results indicate that, in the case of highly autocorrelated noise, there may be considerable advantage in computing the conditional mean of the perturbing noise as well as the conditional means of the state.

Unique Contributions of the Investigation

Some of the aspects of this study which are different from previous work in this area, are listed below.

1. The consideration of a nonlinear dynamic system perturbed by noise which is autocorrelated in time is an important step toward finding statistical models which accurately represent physical phenomena. Previous studies have mainly been concerned with systems perturbed by uncorrelated, or "white" noise. The white noise assumption may be an adequate approximation for representing the statistical properties of some external disturbances, but it is doubtful if it is

adequate for representing perturbing effects, such as electrical or mechanical malfunctions, in the controls of a dynamic system.

2. The expansion technique described in Chapter 3, which was used to derive differential equations for the mean state deviations from the deterministic trajectory when the dynamic system is subjected to perturbing noise, is, to the author's knowledge, original in this study. The inclusion of the nonlinear (quadratic) terms in the investigation which deals with continuously occurring noise has not been considered heretofore.
3. To the author's knowledge, this is the first work in which the stochastic calculus of variations has been applied to a variable final time problem, with constraints imposed on the statistics of the state at the initial and final times. Furthermore, the expansion technique described in Chapter 4, which was used for finding approximate equations for the necessary conditions of the variational problem, is an original development.
4. The procedure described in Chapter 5 for finding conditional means by an expansion about a deterministic value is new. The concept of computing the conditional mean of the perturbing noise as well as the conditional means of the state deviations, in order to compute an optimal corrective control, is original in this study.
5. The numerical results obtained in the simulated study of the interplanetary space guidance problem are the first to be

presented for continuously occurring noise.

Recommendations for Further Study

It is recommended that the study be extended to the following areas:

1. The effects of different types of noise on nonlinear dynamic systems should be investigated. In particular, the studies should include the analysis of random external effects and unknown model parameters.
2. The sensitivity of the control procedure to different types of observations should be investigated. In particular, the problem of implementing continuous control program corrections on the basis of information obtained from continuous observations should be considered. Internal measurements of the perturbing noise itself could also be considered.
3. Different approaches to the entire problem of optimal stochastic control should be studied. The dynamic programming method is an alternate approach which appears promising. If the joint probability density function of the system state could be easily computed, other statistical performance indices for optimality could be considered.

APPENDICES

APPENDIX A

THE CALCULUS OF VARIATIONS APPLIED TO OPTIMAL DETERMINISTIC CONTROL

The problem considered in the theory of optimal deterministic control is that of finding a set of admissible controls, $u_i(t)$, $i = 1, \ldots, m$, which govern a controllable dynamic system whose differential equations of motion are

$$\dot{x}_i = f_i(x,u,t) \qquad i = 1, \ldots n \tag{A.1}$$

in such a manner that

$$I[u] = \int_{t_0}^{t_f} f_{n+1}(x,u,t)\ dt \tag{A.2}$$

is an extremum. For a control to be admissible, it must lead to a state history $x_i(t)$ which satisfies the following initial and terminal constraints

$$x_i(t_0) = x_{i0}$$

$$x_i(t_f) = x_{if} \tag{A.3}$$

When the calculus of variations is applied to the problem of optimal control, the Equation (A.1) is adjoined to the functional given in Equation (A.2) with time dependent Lagrange multipliers $p_i(t)$, and the initial and terminal constraints given in Equation (A.3) are adjoined to the functional with constants μ_i and ν_i. Hence the extremal value of I[u] is found by extremizing the augmented integral J[u], where

$$J[u] = \nu_i(x(t_f)-x_{if}) + \mu_i(x_i(t_0)-x_{i0}) + \int_{t_0}^{t_f} f_{n+1}(x,u,t) + p_i(\dot{x}_i - f_i)\, dt \tag{A.4}$$

The generalized Hamiltonian associated with the variational problem is commonly defined as

$$H(x,u,p,t) = p_i f_i - f_{n+1} \tag{A.5}$$

and the functional to be extremized can be written

$$J[u] = \nu_i(x_i(t_f)-x_{if}) + \mu_i(x_i(t_0)-x_{i0}) + \int_{t_0}^{t_f} p_i\dot{x}_i - H\, dt \tag{A.6}$$

It is now assumed that the optimal control and the resulting optimal trajectory are denoted by

$$u_i^*(t),\ x_i^*(t),\ t_f^*,\ p_i^*(t),\ \mu_i^*,\ \nu_i^*$$

where

$$\dot{x}_i^* = f_i(x^*,u^*,t)$$

$$x_i^*(t_0) = x_{i0}$$

$$x_i^*(t_f^*) = x_{if} \qquad (A.7)$$

The introduction of the p_i's enables one to treat the x_i's and u_i's independently. Let the solution

$$u_i,\ x_i,\ p_i,\ t_f,\ \nu_i,\ \mu_i$$

be an arbitrary trajectory neighboring the optimal solution. This neighboring solution can be expressed in terms of the optimal solution by the following relations

$$\begin{aligned} x_i &= x_i^* + \varepsilon\,\delta x_i & t_f &= t_f^* + \varepsilon\,\delta t_f \\ u_i &= u_i^* + \varepsilon\,\delta u_i & \mu_i &= \mu_i^* + \varepsilon\,\delta\mu_i \\ p_i &= p_i^* + \varepsilon\,\delta p_i & \nu_i &= \nu_i^* + \varepsilon\,\delta\nu_i \end{aligned} \qquad (A.8)$$

where δx_i, δu_i, and δp_i, are arbitrary independent functions of time, and $\delta\nu_i$, $\delta\mu_i$, and δt_f, are arbitrary independent constants. ε is a small parameter. The functional $J[u]$ is now seen to be a function of the parameter ε for any set of functions δx_i, δu_i, δp_i, $\delta\mu_i$, $\delta\nu_i$ and δt_f. The necessary condition for optimality of the control $u_i^*(t)$ is

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{A.9}$$

By carrying out the differentiation with respect to Equation (A.6), the following expression is obtained

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \nu_i \left.\frac{dx_i(t_f)}{d\varepsilon}\right|_{\varepsilon=0} + \mu_i \left.\frac{dx_i(t_0)}{d\varepsilon}\right|_{\varepsilon=0} +$$

$$\left.\frac{d\nu_i}{d\varepsilon}\right|_{\varepsilon=0} (x_i(t_f)-x_{if}) + \left.\frac{d\mu_i}{d\varepsilon}\right|_{\varepsilon=0} (x_i(t_0)-x_{i0}) +$$

$$(p_i\dot{x}_i-H) \left.\frac{dt_f}{d\varepsilon}\right|_{\varepsilon=0} + \int_{t_0}^{t_f} \left\{ p_i \left.\frac{d\dot{x}_i}{d\varepsilon}\right|_{\varepsilon=0} + \right.$$

$$\left. \left.\frac{dp_i}{d\varepsilon}\right|_{\varepsilon=0} \dot{x}_i - H_{x_i} \left.\frac{dx_i}{d\varepsilon}\right|_{\varepsilon=0} - H_{u_i} \left.\frac{du_i}{d\varepsilon}\right|_{\varepsilon=o} - H_{p_i} \left.\frac{dp}{d\varepsilon}\right|_{\varepsilon=0} \right\} dt \tag{A.10}$$

It will be convenient to express Equation (A.10) in terms of the arbitrary functions defined in Equations (A.8). By taking the derivatives of Equations (A.8) with respect to ε, the following identities can be derived:

$$\left.\frac{dx_i}{d\varepsilon}\right|_{\varepsilon=0} = \delta x_i \qquad \left.\frac{d\mu_i}{d\varepsilon}\right|_{\varepsilon=0} = \delta\mu_i$$

$$\left.\frac{du_i}{d\varepsilon}\right|_{\varepsilon=0} = \delta u_i \qquad \left.\frac{d\nu_i}{d\varepsilon}\right|_{\varepsilon=0} = \delta\nu_i$$

$$\left.\frac{dp_i}{d\varepsilon}\right|_{\varepsilon=0} = \delta p_i \qquad \left.\frac{dt_f}{d\varepsilon}\right|_{\varepsilon=0} = \delta t_f \tag{A.11}$$

Now consider the first of Equations (A.8) evaluated at t_f^*

$$x_i(t_f^*) = x_i^*(t_f^*) + \varepsilon\, \delta x_i(t_f^*) \tag{A.12}$$

The quantity $x_i(t_f)$ can be approximated by the following expression

$$x_i(t_f) = x_i(t_f^*) + \dot{x}_i(t_f^*)(t_f - t_f^*) \tag{A.13}$$

If Equation (A.12) is substituted into Equation (A.13), the following expression for $x_i(t_f)$ is obtained

$$x_i(t_f) = x_i^*(t_f^*) + \varepsilon[\delta x_i(t_f^*) + \dot{x}_i(t_f^*)\ \delta t_f] \tag{A.14}$$

By taking the derivative of Equation (A.14) with respect to ε, the following relation is derived,

$$\left.\frac{dx_i(t_f)}{d\varepsilon}\right|_{\varepsilon=0} = \delta x_i(t_f^*) + \dot{x}_i(t_f^*)\ \delta t_f \tag{A.15}$$

If Equations (A.11) and (A.15) are substituted into the expression given in Equation (A.10), and the first term under the integral is integrated by parts, then the following expression is obtained

$$\delta J = [\nu_i + p_i(t_f)][\delta x_i(t_f) + \dot{x}_i \delta t_f] + [\mu_i - p_i(t_0)]\ \delta x_i(t_0) +$$

$$\delta\nu_i(x_i(t_f) - x_{if}) + \delta\mu_i(x_i(t_0) - x_{i0}) - H(t_f)\ \delta t_f -$$

$$\int_{t_0}^{t_f} (\dot{x} - H_{p_i})\delta p_i + (\dot{p}_i + H_{x_i})\delta x_i + H_{u_i}\ \delta u_i\ dt \tag{A.16}$$

where

$$\delta J = \left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} \tag{A.17}$$

By the fundamental lemma of the calculus of variations, the arbitrary nature of the terms δx_i, δu_i, δp_i, $\delta\mu_i$, $\delta\nu_i$, and δt_f, imply that their coefficients vanish identically. Thus the conditions necessary for the set x_i^*, u_i^*, p_i^*, μ_i^*, ν_i^*, t_f^*, to be an extremal solution are

$$\dot{x}_i - H_{p_i} = 0 \tag{A.18}$$

$$\dot{p}_i + H_{x_i} = 0 \tag{A.19}$$

$$H_{u_i} = 0 \tag{A.20}$$

at all points of time in the controlling interval $t_0 \leq t \leq t_f$,

$$x_i(t_0) = x_{i0} \tag{A.21}$$

$$p_i(t_0) = \mu_i \tag{A.22}$$

at the initial time t_0, and

$$x_i(t_f) = x_{if} \tag{A.23}$$

$$p_i(t_f) = -\nu_i \tag{A.24}$$

$$H(t_f) = 0 \tag{A.25}$$

at the terminal time t_f.

Equation (A.20) can be used to eliminate $u_i(t)$ from Equation (A.18) and (A.19). The $2n$ equations, i.e., Equations (A.18) and (A.19), then form a two point boundary value problem with $2n+1$ split end conditions, i.e., Equations (A.21) and (A.22), at t_0, and Equations (A.23), (A.24), and (A.25) at t_f. The problem can be solved for the values of the $2n$ unknown constants μ_i and ν_i, and for the final time t_f, by one of several existing numerical methods.

In applying the calculus of variations technique the Earth-Mars transfer problem, the transfer time, i.e.,

$$I[u] = \int_{t_0}^{t_f} 1 \, dt \tag{A.26}$$

is minimized subject to the differential equations of motion

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + a \sin\alpha$$

$$\dot{v} = -\frac{vu}{r} + a \cos\alpha$$

$$\dot{r} = u$$

$$\dot{\theta} = \frac{v}{r} \tag{A.27}$$

with $\quad a = \dfrac{T}{m_0 - \dot{m}(t-t_0)}$

The conditions at the final time are

$$u(t_f) = 0$$

$$v(t_f) = V_M \text{ (velocity of Mars)}$$

$$r(t_f) = R_M \text{ (radius of Martian Orbit)} \tag{A.28}$$

and the conditions at the initial time are

$$u(t_0) = 0$$

$$v(t_0) = V_E \text{ (velocity of Earth}$$

$$r(t_0) = R_E \text{ (radius of Earth's orbit)}$$

$$\theta(t_0) = 0 \tag{A.29}$$

θ is allowed to be unconstrained at the final time, hence the launch time will be selected, after the final solution is determined, in order to assure proper rendezvous configuration at Mars.

The integral to be extremized is

$$J[\alpha] = \nu_1(u(t_f)) + \nu_2(v(t_f)-V_M) + \nu_3(r(t_f)-R_M) +$$

$$\mu_1(u(t_0)) + \mu_2(v(t_0)-V_E) + \mu_3(r(t_0)-R_E) + \mu_4(\theta(t_0)) +$$

$$\int_{t_0}^{t_f} 1 + p_1(\dot{u} - \frac{v^2}{r} - \frac{\mu}{r^2} - a\sin\alpha) + p_2(\dot{v} + \frac{uv}{r} - a\cos\alpha)$$

$$+ p_3(\dot{r}-u) + p_4(\dot{\theta} - \frac{v}{r})\, dt \tag{A.30}$$

The resulting necessary conditions are

$$\dot{p}_1 = \frac{p_2 v}{r} - p_3$$

$$\dot{p}_2 = -\frac{2p_1 v}{r} + \frac{p_2 u}{r} - \frac{p_4}{r}$$

$$\dot{p}_3 = p_1\frac{v^2}{r^2} - \frac{2\mu p_1}{r^3} - \frac{p_2 uv}{r^2} + \frac{p_4 v}{r^2}$$

$$\dot{p}_4 = 0 \tag{A.31}$$

$$p_1\cos\alpha - p_2\sin\alpha = 0 \tag{A.32}$$

and Equations (A.27), in the controlling interval $t_0 \geq t \geq t_f$,

$$p_1(t_0) = \mu_1$$

$$p_2(t_0) = \mu_2$$

$$p_3(t_0) = \mu_3$$

$$p_4(t_0) = \mu_4 \tag{A.33}$$

and Equations (A.29), at the initial time t_0, and

$$p_1(t_f) = -\nu_1$$

$$p_2(t_f) = -\nu_2$$

$$p_3(t_f) = -\nu_3$$

$$p_4(t_f) = 0$$

$$p_1\left(\frac{v^2}{r} - \frac{\mu}{r^2} + a\ \sin\alpha\right)+p_2\left(-\frac{uv}{r} + a\ \cos\alpha\right)+p_3u+p_4\frac{v}{r}\Big|_{t_f} = 1 \qquad \text{(A.34)}$$

and Equations (A.28), at the terminal time t_f.

Equation (A.32) leads to the following relations

$$\sin\alpha = \frac{p_1}{+\sqrt{p_1^2+p_2^2}} \qquad \cos\alpha = \frac{p_2}{+\sqrt{p_1^2+p_2^2}} \qquad \text{(A.35)}$$

Note: An analysis of the second variation of the functional $J[u]$ leads to the selection of the plus signs for the radicals in Equations (A.35).

The necessary condition then reduce to the set of equations

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + a \frac{p_1}{\sqrt{p_1^2+p_2^2}}$$

$$\dot{v} = -\frac{uv}{r} + a \frac{p_2}{\sqrt{p_1^2+p_2^2}}$$

$$\dot{r} = u$$

$$\dot{\theta} = \frac{v}{r}$$

$$\dot{p}_1 = \frac{p_2 v}{r} - p_3$$

$$\dot{p}_2 = -\frac{2p_1 v}{r} + \frac{p_2 u}{r} - \frac{p_4}{r}$$

$$\dot{p}_3 = p_1 \frac{v^2}{r^2} - \frac{2\mu p_1}{r^3} - \frac{p_2 uv}{r^2} + \frac{p_4 v}{r^2}$$

$$\dot{p}_4 = 0 \tag{A.36}$$

$$\text{with } a = \frac{T}{m_0 - \dot{m}(t - t_0)}$$

at all points of time in the interval $t_0 \le t \le t_f$, Equations (A.29) and (A.33) at the initial time t_0, and Equations (A.28) and (A.34) at the terminal time t_f. Figure A.1 illustrates the time histories of the quantities u, v, r, θ, p_1, p_2, p_3 and p_4 for the converged optimal solution. Figure A.2 illustrates the time history of the optimal control angle $\alpha(t)$.

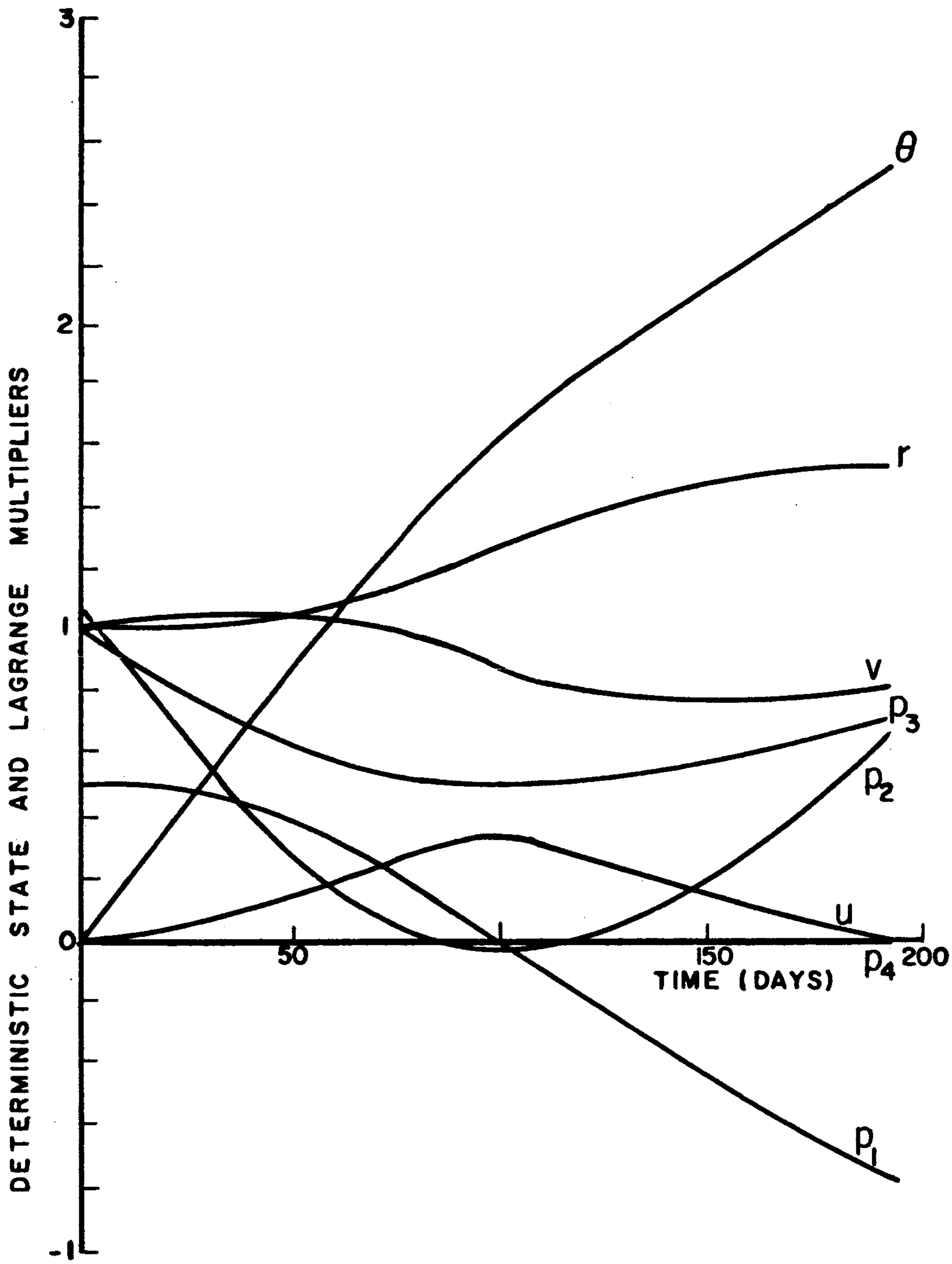

Figure A.1. Optimal Deterministic State and Lagrange Multiplier Time Histories for the Earth-Mars Transfer Problem

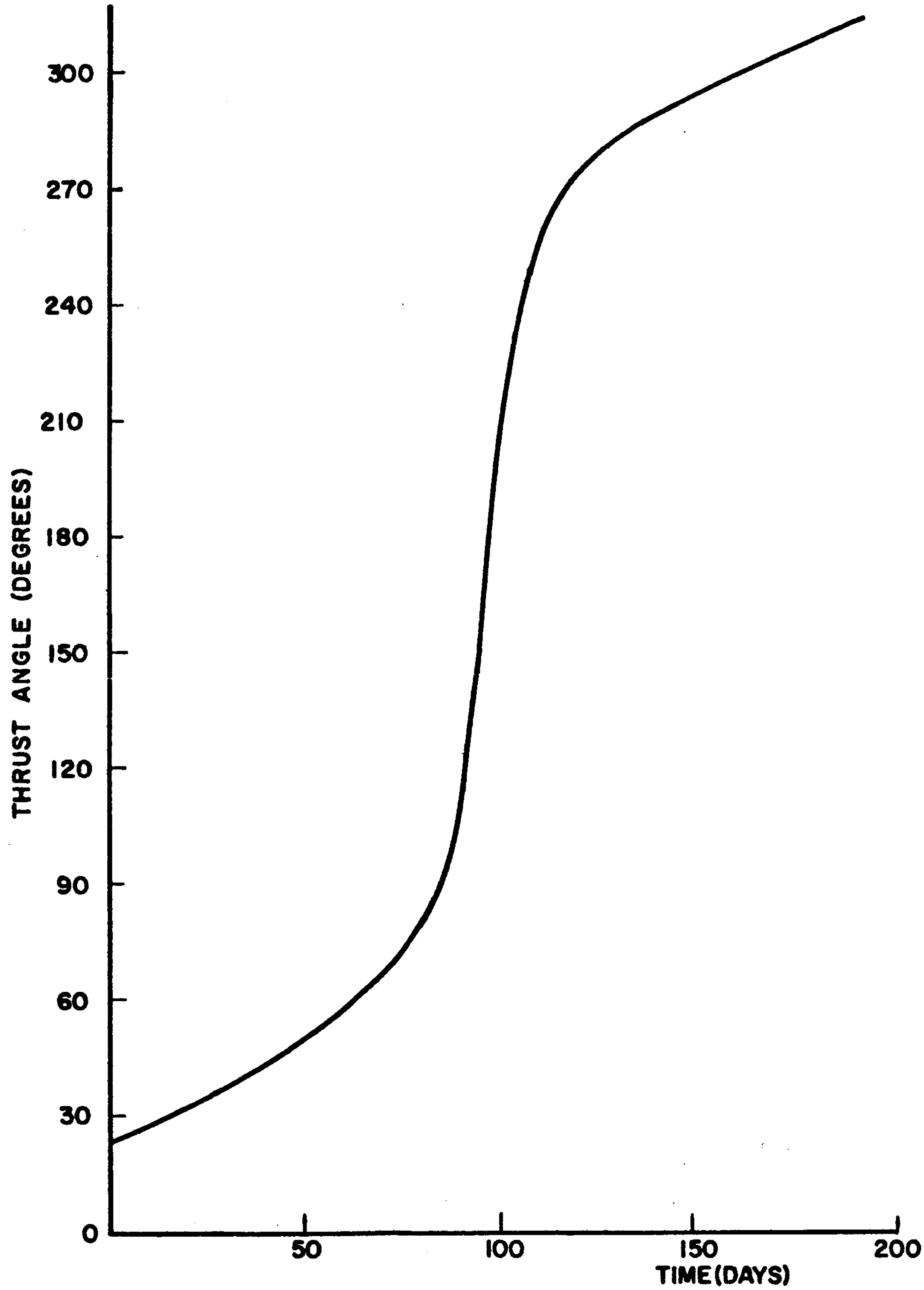

Figure A.2. Optimal Deterministic Control Variable Time History for the Earth-Mars Transfer Problem

APPENDIX B

MATRIX FORMULATION OF THE

EQUATIONS FOR THE EARTH-MARS

TRANSFER PROBLEM

The terms in the equations for the Earth-Mars transfer problem are listed below in matrix notation. The notation is defined by the following example.

If A_{ij} is an nm dimensional quantity, i.e., $i=1, \ldots, n$, $j=1, \ldots, m$, then the components of A_{ij} can be listed in the following manner

$$\left[A_{ij}\right] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & \\ \vdots & & & \vdots \\ A_{n1} & \cdots & & A_{nm} \end{bmatrix}$$

Chapter 3

$$\left[x_i\right] = \begin{bmatrix} u \\ v \\ r \\ \theta \end{bmatrix} \qquad \left[u_i\right] = \begin{bmatrix} a \\ \alpha \end{bmatrix}$$

$$\left[f_i\right] = \begin{bmatrix} \frac{v^2}{r} - \frac{\mu}{r^2} + a \sin\alpha \\ -\frac{uv}{r} + a \cos\alpha \\ u \\ \frac{v}{r} \end{bmatrix}$$

$$\left[\delta x_i\right] = \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \qquad \eta_i = \begin{bmatrix} \eta_a \\ \eta_\alpha \end{bmatrix}$$

$$\left[R_{ij}\right] = \begin{bmatrix} \sigma_a{}^2 & 0 \\ 0 & \sigma_\alpha{}^2 \end{bmatrix} \qquad \beta_{ij} = \begin{bmatrix} \beta_a & 0 \\ 0 & \beta_\alpha \end{bmatrix}$$

$$\left[f_{ix_j}\right] = \begin{bmatrix} 0 & \frac{2v}{r} & -\frac{v^2}{r^2} + \frac{2\mu}{r^3} & 0 \\ -\frac{v}{r} & -\frac{u}{r} & \frac{vu}{r} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r} & -\frac{v}{r^2} & 0 \end{bmatrix}$$

$$\left[f_{ux_ix_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{r} & -\frac{2v}{r^2} & 0 \\ 0 & -\frac{2v}{r^2} & \frac{2v^2}{r^3} - \frac{6\mu}{r^4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[f_{vx_ix_j}\right] = \begin{bmatrix} 0 & -\frac{1}{r} & \frac{v}{r^2} & 0 \\ -\frac{1}{r} & 0 & \frac{u}{r^2} & 0 \\ \frac{v}{r^2} & \frac{u}{r^2} & -\frac{2vu}{r^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[f_{rx_ix_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[f_{\theta x_i x_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & -\frac{1}{r^2} & \frac{2v}{r^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[f_{uu_i u_j}\right] = \begin{bmatrix} 0 & \cos\alpha \\ \cos\alpha & -a\sin\alpha \end{bmatrix} \quad \left[f_{vu_i u_j}\right] = \begin{bmatrix} 0 & -\sin\alpha \\ -\sin\alpha & -a\cos\alpha \end{bmatrix}$$

$$\left[f_{ru_i u_j}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \left[f_{\theta u_i u_j}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[M_{ij}\right] = E \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \begin{bmatrix} \delta u & \delta v & \delta r & \delta\theta \end{bmatrix} \quad \left[h_{ij}\right] = E \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \begin{bmatrix} \eta_a & \eta_\alpha \end{bmatrix}$$

Chapter 4

$$\left[\delta u_i\right] = \begin{bmatrix} 0 \\ \delta\alpha \end{bmatrix} \quad \left[p_i\right] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \quad \left[\delta p_i\right] = \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \delta p_3 \\ \delta p_4 \end{bmatrix}$$

$$H = p_1\left(\frac{v^2}{r} - \frac{\mu}{r^2} + a\sin\alpha\right) + p_2\left(-\frac{vu}{r} + a\cos\alpha\right) + p_3 u + p_4 \frac{v}{r}$$

$$\left[H_{x_i}\right] = \begin{bmatrix} -\dfrac{p_2 v}{r} + p_3 \\ \dfrac{2p_1 v}{r} - \dfrac{p_2 u}{r} + \dfrac{p_4}{r} \\ -\dfrac{p_1 v^2}{r^2} + \dfrac{2\mu p_1}{r^3} + \dfrac{p_2 ur}{r^2} - \dfrac{p_4 v}{r^2} \\ 0 \end{bmatrix}$$

$$H_{p_i} = f_i$$

$$H_\alpha = p_1\, a\cos\alpha - p_2\, a\sin\alpha$$

$$\left[H_{x_i x_j}\right] = \begin{bmatrix} 0 & -\dfrac{p_2}{r} & \dfrac{vp_2}{r^2} & 0 \\ -\dfrac{p_2}{r} & \dfrac{2p_1}{r} & \dfrac{up_2 - p_4 - 2vp_1}{r^2} & 0 \\ \dfrac{p_2 v}{r^2} & \dfrac{up_2 - p_4 - 2vp_1}{r^2} & \dfrac{2(p_1 v^2 + p_4 v - p_2 vu)}{r^3} - \dfrac{6\mu p_4}{r^4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_{x_i p_j} = f_{jx_i}$$

$$\left[H_{\alpha u_i}\right] = \left[p_1 \cos\alpha - p_2 \sin\alpha \;,\; - p_1 \, a \sin\alpha - p_2 \, a \cos\alpha\right]$$

$$\left[H_{\alpha p_i}\right] = \left[a \cos\alpha \;,\; - a \sin\alpha\right]$$

$$\left[H_{\alpha u_i u_j}\right] = \begin{bmatrix} 0 & -p_1\sin\alpha - p_2\cos\alpha \\ -p_1 \sin\alpha - p_2 \cos\alpha & 0 \end{bmatrix}$$

$$\left[H_{\alpha u_i p_j}\right] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ -a \sin\alpha & -a \cos\alpha & 0 & 0 \end{bmatrix}$$

$$\left[H_{u x_i u_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p_2}{r^2} & 0 \\ 0 & \frac{p_2}{r^2} & -\frac{2vp_2}{r^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{v x_i x_j}\right] = \begin{bmatrix} 0 & 0 & \frac{p_2}{r^2} & 0 \\ 0 & 0 & -\frac{2p_1}{r^2} & 0 \\ \frac{p_2}{r^2} & -\frac{2p_1}{r^2} & \frac{2(2vp_1+p_4)}{r^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{rx_ix_j}\right] = \begin{bmatrix} 0 & \frac{p_2}{r^2} & -\frac{2p_2v}{r^3} & 0 \\ \frac{p_2}{r^2} & -\frac{2p_1}{r^2} & \frac{2(2p_1v+p_4-p_2u)}{r^3} & 0 \\ -\frac{2p_2v}{r^3} & \frac{2(2p_1v+p_4-p_3u)}{r^3} & \frac{24\mu p_1}{r^5} - \frac{6(p_1v^2+p_4v-p_2uv)}{r^4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{\theta x_ix_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{ux_i p_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 & 0 \\ 0 & \frac{v}{r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{vx_i p_j}\right] = \begin{bmatrix} 0 & \frac{1}{r} & 0 & 0 \\ \frac{2}{r} & 0 & 0 & 0 \\ -\frac{2v}{r^2} & \frac{u}{r^2} & 0 & -\frac{1}{r^2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{rx_i x_j}\right] = \begin{bmatrix} 0 & \frac{v}{r^2} & 0 & 0 \\ -\frac{2v}{r^2} & \frac{u}{r^2} & 0 & -\frac{1}{r^2} \\ \frac{2v^2}{r^3} - \frac{6\mu}{r^4} & \frac{2uv}{r^3} & 0 & \frac{2v}{r^3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[H_{\theta x_i p_j}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[N_{ij}\right] = E \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \begin{bmatrix} \delta p_1 & \delta p_2 & \delta p_3 & \delta p_4 \end{bmatrix}$$

$$\left[f_{ij}\right] = E \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \delta p_3 \\ \delta p_4 \end{bmatrix} \begin{bmatrix} \eta_a & \eta_\alpha \end{bmatrix}$$

<u>Chapter 5</u>

$$y_i = \dot{\rho} \qquad g_i = \frac{u[r-R\cos(\theta-\omega(t-t_0))]+rR(\frac{V}{r}-\omega)\sin(\theta-\omega(t-t_0))}{\rho}$$

$$\delta y_i = \delta\dot{\rho}$$

$$\begin{bmatrix} g_{ix_j} \end{bmatrix} = \begin{bmatrix} g_u \\ g_v \\ g_r \\ g_\theta \end{bmatrix} \qquad \begin{aligned} g_u &= \frac{r - R\cos(\theta - \omega(t-t_0))}{\rho} \\ g_v &= \frac{R\sin(\theta-\omega(t-t_0))}{\rho} = \frac{R\sin(\gamma)}{\rho} \\ g_r &= \frac{u + R\sin(\gamma)(\frac{v}{r} - \omega) + \frac{Rv}{r}\sin(\gamma)}{\rho} - \frac{\dot{\rho} g_u}{\rho} \\ g_\theta &= \frac{uR\sin(\gamma) + R(v-\omega r)\cos(\gamma)}{\rho} - \frac{\dot{\rho} r g_r}{\rho} \end{aligned}$$

$$\begin{bmatrix} \delta x_i \end{bmatrix} = \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \\ \delta p_1 \\ \delta p_2 \\ \delta p_3 \\ \delta p_4 \\ \eta_a \\ \eta_\alpha \end{bmatrix} \qquad \begin{bmatrix} \mathcal{M}_{ij} \end{bmatrix} = \begin{bmatrix} [M_{ij}] & [N_{ij}] & [h_{ij}] \\ [N_{ji}] & 0 & [f_{ij}] \\ [h_{ji}] & [f_{ji}] & [R_{ij}] \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{ij} \end{bmatrix} \quad \begin{bmatrix} [f_{ix_j}] & 0 & [f_{iu_j}] \\ [-H_{x_i x_j}] & [-H_{x_i p_j}] & [-H_{x_i u_j}] \\ 0 & 0 & [-\beta_{ij}] \end{bmatrix}$$

APPENDIX C

NUMERICAL CONSTANTS FOR THE EARTH-MARS TRANSFER PROBLEM

<u>Numerical Constants in MKS System of Units</u>

Orbital Radius of Earth, R_E	=	1.4959870×10^{11} Meters
Velocity of Earth, V_E	=	2.9784901×10^{4} Meters/second
Orbital Radius of Mars, R_M	=	2.2794040×10^{11} Meters
Velocity of Mars, V_M	=	2.4129561×10^{4} Meters/second
Angular Velocity of Earth, ω	=	1.9909866×10^{-7} Radians/second
Gravitational Constant of Sun, μ	=	1.3271504×10^{20} Meters^3/second^2
Initial Spacecraft Mass, m_0	=	6.7978852×10^{3} Kilograms
Mass Flow Rate, $\dot{m}$	=	1.0123858×10^{-5} Kilograms/second
Thrust, T	=	4.0312370 Newtons

<u>Normalization Scheme</u>

Unit of Length	=	1 R_E, 1 AU
Unit of Velocity	=	1 $V_E = \sqrt{\frac{\mu}{R_E}}$
Unit of Mass	=	1 m_0

<u>Normalized Values of the Numerical Constants</u>

Initial Spacecraft Radius	=	1.0
Initial Spacecraft Velocity	=	1.0
Terminal Spacecraft Radius	=	1.5236790
Terminal Spacecraft Velocity	=	0.81012728
Angular Velocity of Earth	=	1.0

Normalized Values of the Numerical Constants - (continued)

Gravitational Constant of Sun	=	1.0
Initial Spacecraft Mass	=	1.0
Mass Flow Rate	=	0.074800391
Thrust	=	0.14012969

APPENDIX D

A THEOREM FROM PROBABILITY THEORY

Consider the set of random variables $x, y_1, y_2, y_3, \ldots, y_N$ that are distributed according to the joint probability density function $f(x,y_1, y_2, \ldots, y_N)$. The conditional expectation of the product $F(y_1)x$, where $F(y_1)$ is some function of y_1, given $y_2, y_3, \ldots, y_N$ is defined as follows

$$E\left[F(y_1)x \;\middle|\; y_2, \ldots, y_N\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(y_1)x\, f(x,y_1 \mid y_2,\ldots, y_N)\, dxdy_1 \tag{D.1}$$

where $f(x, y_1 \mid y_2, \ldots y_N)$ is the joint conditional probability density function of x, y_1 given $y_2, \ldots, y_N$. By the definition of the conditional density function (see Ref. 11), the following relation can be written

$$f(x,y_1 \mid y_2,\ldots,y_N) = \frac{f(x,y_1,y_2,\ldots,y_N)}{f(y_2,y_3,\ldots,y_N)} \tag{D.2}$$

Equation (D.2)can be written in the form

$$f(x,y_1 \mid y_2,\ldots,y_N) = \frac{f(x,y_1,y_2,\ldots,y_N)}{f(y_1,\ldots,y_N)} \cdot \frac{f(y_1,\ldots,y_N)}{f(y_2,\ldots,y_N)} \tag{D.3}$$

and from the definition of the conditional density function, Equation (D.3) reduces to the following relation

$$f(x,y_1 \mid y_2,\ldots,y_N) = f(x \mid y_1,\ldots,y_N)\, f(y_1 \mid y_2,\ldots,y_N) \qquad (D.4)$$

By substituting Equation (D.4) into the integral in Equation (D.1), the following expression is obtained

$$E\left[F(y_1)x \mid y_2,\ldots,y_N\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(y_1)x\, f(x \mid y_1,\ldots,y_N) f(y_1 \mid y_2,\ldots,y_N)\, dy_1 dx \qquad (D.5)$$

Rearranging Equation (D.5) leads to the following relation

$$E\left[F(y_1)x \mid y_2,\ldots,y_N\right] = \int_{-\infty}^{\infty} F(y_1)\left[\int_{-\infty}^{\infty} x\, f(x \mid y_1\ldots,y_N)\, dx\right] f(y_1 \mid y_2,\ldots,y_N)\, dy_1 \qquad (D.6)$$

By the definition of the expected value operation,Equation (D.6) reduces to the following equation

$$E\left[F(y_1)x \mid y_2,\ldots,y_N\right] = E\left[F(y_1)E(x \mid y_1,\ldots,y_N) \mid y_2,\ldots,y_N\right] \qquad (D.7)$$

In the notation of Chapter 5, Equation (D.7) can be generalized to the following form

$$E\left[F(\delta y(t_k))\delta x_i(t_k) \;\middle|\; y(t_{k-1}),\, y(t_{k-2}),\, \ldots,\, y(t_1)\right] =$$

$$E\left[F(\delta y(t_k))E(\delta x_i(t_k) \;|y(t_k),y(t_{k-1}),\ldots,y(t_1))\;\middle|\; y(t_{k-1}),y(t_{k-2}),\ldots,y(t_1)\right] \tag{D.8}$$

APPENDIX E

THE DIFFERENTIAL EQUATION FOR THE GENERALIZED COVARIANCE

Consider Equations (5.17) and (5.19) from Chapter 5. The equations can be written as follows

$$\delta\dot{\bar{x}}_i = f_{ix_j}\delta\bar{x}_j + f_{iu_j}(\delta u_j+\bar{\eta}_j) + \frac{1}{2} f_{ix_jx_k}M_{jk} +$$

$$\frac{1}{2} f_{iu_ju_k}(\delta u_j\delta u_k+\delta u_j\bar{\eta}_k+\bar{\eta}_j\delta u_k+R_{jk}) + f_{ix_ju_k}(\delta\bar{x}_j\delta u_k+h_{jk}) \quad \text{(E.1)}$$

$$\dot{M}_{ij} = f_{ix_k}M_{kj} + M_{ik} f_{jx_k} + f_{iu_k}(\delta u_k\delta\bar{x}_j+h_{jk})$$

$$+ (\delta\bar{x}_i\delta u_k+h_{ik}) f_{ju_k} \quad \text{(E.2)}$$

Now consider the covariance of the state P_{ij}, which is defined by the following expression

$$P_{ij} = M_{ij} - \delta\bar{x}_i\delta\bar{x}_j \quad \text{(E.3)}$$

In view of Equations (E.1) and (E.2), the derivative of P_{ij} with respect to time can be written in the following manner

$$\dot{P}_{ij} = \dot{M}_{ij}-\delta\bar{x}_i\delta\dot{\bar{x}}_j-\delta\dot{\bar{x}}_i\delta\bar{x}_j = f_{ix_k}M_{jk}$$

$$+ M_{ik} f_{jx_k} + f_{iu_k}(\delta u_k\delta\bar{x}_j+h_{jk}) \quad \text{(E.4)}$$

$$+ (\delta\bar{x}_i\delta u_k + h_{ik})\, f_{ju_k} - \delta\bar{x}_i [\, f_{jx_k}\delta\bar{x}_k + f_{ju_k}(\delta u_k + \bar{\eta}_k)]$$

$$+ [f_{ix_k}\delta\bar{x}_k + f_{iu_k}(\delta u_k + \bar{\eta}_k)]\, \delta\bar{x}_j$$

In view of Equation (E.3), Equation (E.4) reduces to the following expression

$$\dot{P}_{ij} = f_{ix_k}P_{jk} + P_{ik}\, f_{jx_k} + f_{iu_k}(h_{jk} - \delta\bar{x}_j\bar{\eta}_k) + (h_{ik} - \delta\bar{x}_i\bar{\eta}_k)\, f_{ju_k} \qquad \text{(E.5)}$$

In a manner similar to the manner in which Equation (E.5) was derived, Equations (5.18) and (5.20) may be used to derive the following expression

$$\frac{d}{dt}(N_{ij} - \delta x_i\delta p_j) = f_{ix_k}(N_{kj} - \delta\bar{x}_k\delta\bar{p}_j) - (N_{ik} - \delta\bar{x}_i\delta\bar{p}_k)\, f_{jx_k}$$

$$+ P_{ik}H_{x_k x_j} + f_{iu_k}(f_{jk} - \delta p_j\bar{\eta}_k) - (h_{ik} - \delta\bar{x}_i\bar{\eta}_k)\, H_{x_j u_k} \qquad \text{(E.6)}$$

In a similar manner the following equations can be derived from Equations (5.21) and (5.22).

$$\frac{d}{dt}(h_{ij}-\delta\bar{x}_i\bar{\eta}_j) = f_{ix_k}(h_{kj}-\delta x_k\bar{\eta}_j) + f_{iu_k}(R_{kj}-\bar{\eta}_k\bar{\eta}_j)$$

$$+ (h_{ik}-\delta\bar{x}_i\bar{\eta}_k)\,\beta_{kj} \tag{E.7}$$

$$\frac{d}{dt}(f_{ij}-\delta\bar{p}_i\bar{\eta}_j) = - H_{x_i x_k}(h_{kj}-\delta\bar{x}_k\bar{\eta}_j) - H_{x_i u_k}(R_{kj}-\bar{\eta}_k\bar{\eta}_j)$$

$$- H_{x_i p_j}(f_{kj}-\delta\bar{p}_k\bar{\eta}_j) -(f_{ik}-\delta p_i\bar{\eta}_k)\,\beta_{kj} \tag{E.8}$$

By recalling the definition of the generalized covariance P_{ij}, i.e., Equation (5.38), the set of Equations (E.4), (E.5), (E.6), (E.7), and (E.8) can be generalized to the following equation

$$\dot{P}_{ij} = \Gamma_{ik}P_{kj} + P_{ik}\Gamma_{jk} \tag{E.9}$$

where Γ_{ij} is a $(2n+m)^2$ dimensional quantity which is defined by the following relations

$$\Gamma_{ij} = f_{ix_j} \quad , \quad i = 1, \ldots, n; \quad j = 1, \ldots, n$$

$$\Gamma_{ij} = 0 \quad , \quad i = 1, \ldots, n; \quad j = n+1, \ldots, 2n$$

$$\Gamma_{ij} = f_{iu_{j-2n}} \quad , \quad i = 1, \ldots, n; \quad j = 2n+1, \ldots, 2n+m$$

$$
\begin{aligned}
\Gamma_{ij} &= -H_{x_{i-n}x_j}, && i = n+1, \ldots, 2n \\
&&& j = 1, \ldots, n \\
\Gamma_{ij} &= -H_{x_{i-n}p_{j-n}}, && i = n+1, \ldots, 2n \\
&&& j = n+1, \ldots, 2n \\
\Gamma_{ij} &= -H_{x_{i-n}u_{j-2n}}, && i = n+1, \ldots, 2n \\
&&& j = 2n+1, \ldots, 2n+m \\
\Gamma_{ij} &= 0, && i = 2n+1, \ldots, 2n+m \\
&&& j = 1, \ldots, n \\
\Gamma_{ij} &= 0, && i = 2n+1, \ldots, 2n+m \\
&&& j = n+1, \ldots, 2n \\
\Gamma_{ij} &= -\beta_{i-2n\ j-2n}, && i = 2n+1, \ldots, 2n+m \\
&&& j = 2n+1, \ldots, 2n+m
\end{aligned}
\tag{E.10}
$$

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