NASA TECHNICAL MEMORANDUM

NASA TM X-53478

June 23, 1966

NASA TM X-53478

GPO PRICE	$
CFSTI PRICE(S)	$
Hard copy (HC)	$3.00
Microfiche (MF)	1.00

ff 653 July 65

PROGRESS REPORT NO. 8
PROCEEDINGS OF THE TWENTY-FOURTH SEMINAR ON SPACE FLIGHT AND GUIDANCE THEORY

Sponsored by the Aero-Astrodynamics Laboratory

FACILITY FORM 602

N66 34926	N66 34933
(ACCESSION NUMBER)	(THRU)
146	1
(PAGES)	(CODE)
TMX-53478	30
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

TECHNICAL MEMORANDUM X-53478

PROGRESS REPORT NO. 8
PROCEEDINGS OF THE TWENTY-FOURTH SEMINAR
ON
SPACE FLIGHT AND GUIDANCE THEORY

Sponsored by the Aero-Astrodynamics Laboratory

George C. Marshall Space Flight Center
Huntsville, Alabama

ABSTRACT

Progress reports of NASA sponsored studies in space flight and guidance theory are presented. The studies are made by several universities and industrial firms under contract to MSFC. This progress report reflects work done on the contracts during the period from April 1, 1965 to December 31, 1965. The contracts are technically monitored by personnel of the Astrodynamics and Guidance Theory Division, Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center.

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

TECHNICAL MEMORANDUM X-53478

June 23, 1966

PROGRESS REPORT NO. 8

Proceedings of the Twenty-Fourth Seminar on Space Flight and Guidance Theory

Sponsored by the Aero-Astrodynamics Laboratory
George C. Marshall Space Flight Center

ASTRODYNAMICS AND GUIDANCE THEORY DIVISION
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

TABLE OF CONTENTS

TECHNICAL MEMORANDUM X-53478

PROGRESS REPORT NO. 8

Proceedings of the Twenty-Fourth Seminar
on
Space Flight and Guidance Theory

SUMMARY

Progress reports of NASA sponsored studies in space flight and guidance theory are presented. The studies are made by several universities and industrial firms under contract to MSFC. This progress report reflects work done on the contracts during the period from April 1, 1965 to December 31, 1965. The contracts are technically monitored by personnel of the Astrodynamics and Guidance Theory Division, Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center.

INTRODUCTION

This progress report contains eight papers whose subject matter lies within the areas of space flight and guidance theory. The papers have been written by investigators employed at universities and industrial firms under contract to MSFC.

This report is the eighth of the "Progress Reports" and covers the period from April 1, 1965 to December 31, 1965.

The contributing agencies and their fields of major interest are:

Stability of Dynamical Systems	Brown University General Precision Aerospace Grumman Aircraft
Trajectory Optimization	Southern Illinois University The Boeing Company Auburn University
Control Theory	Brown University

The objective of this introduction is to review briefly the contributions of each agency.

The first paper is concerned with finite time stability properties of periodic solutions of Hamiltonian systems. The contract for which it is a report of progress has for its principal objective the determination of that set of initial or injection conditions in which is included the initial conditions for a given (almost) periodic solution that will result in (almost) periodic solutions that lie within a prescribed "tube" of the given solution. A theoretical basis for this determination was given by Birkhoff. However, to obtain actual numerical estimates from the analytical theory an enormous amount of algebraic manipulation is required even in the simplest problems. For this reason, a digital computer and an appropriate non-numeric computer language to perform the required manipulations were employed.

As an initial application of the mechanization of the Birkhoff theory, the planar restricted three-body problem was chosen as a simplified dynamical model; the given solution in this model is a Lagrangian critical point.

The second paper is concerned with discontinuous vector fields which are encountered in problems of feedback control. It begins with the observation that if X is a discontinuous vector field then the study of stability under perturbations $\epsilon(t)$ is different if the perturbation enters the equation of motion as a summand in the argument of X, that is, as in the equation

$$\dot{x}(t) = X(x(t) + \epsilon(t))$$

from what it would be if the perturbation were not a part of the argument, as in the equation

$$\dot{x}(t) = X(x(t)) + \epsilon(t).$$

If X is continuous then this is not the case. Problems in feedback control lead to discontinuous vector fields in the form

$$X(x) = F(x, \mu(x))$$

where μ is a control function. The author discusses the distinction between a classical solution of the equations of motion and a Filippov solution, a generalization of the definition of solution. He then shows as his main result that if a vector field X is stable with respect to measurement, then every classical solution is a Filippov solution.

The third paper presents a survey of various approaches to the problem of estimating the domain of attraction of an equilibrium solution of a system of nonlinear autonomous differential equations. Based on observations resulting from this survey, the problem is reformulated as that of choosing optimally the Liapunov function from the space of positive definite quadratic forms. An estimate of the domain of attraction is then obtained as the solution of a minimization problem. This approach to the problem has the advantages of being suitable for machine computation, of yielding estimates that are easily visualized and of being relatively insensitive to system dimension. Some preliminary numerical results are presented for the Duffing equation with damping.

The fourth paper deals with the problem of obtaining a transformation technique which can be used to eliminate the control angles from the Euler-Lagrange equations to give a system of differential equations in the state variables and the Lagrange multipliers only. The problem arises in the study of trajectory optimization by classical calculus of variations techniques. In applying these techniques, certain Euler-Lagrange equations involving the control angles are encountered. In some cases these equations lead to a solution for the angles in terms of the Lagrange multipliers, and these solutions can be used to eliminate the control angles from the Euler-Lagrange equations resulting in a system of differential equations in generally desirable state variables and Lagrange multipliers only. The process, however, can be carried out more readily in some coordinate systems than in others. In this paper the technique for a general transformation of the state variables and their corresponding Lagrange multipliers from one coordinate system to another is discussed. The technique is then applied to a specific problem involving three-dimensional trajectory optimization.

The fifth paper is concerned with finding criteria for the stability of the zero solution of the differential equation

$$x^{(n)} + \rho_1(t)^{(n-1)} + \dots + \rho_{n-1}(t)\dot{x} + \rho_n(t)x = 0$$

which depend on the behavior of the real continuous functions $\rho_i(t)$, but not upon their derivatives.

Recently, Ghizzetti obtained simple stability criteria for this problem. The particularly attractive aspect of these criteria is that they depend only on n constants which locate a family of hyperellipsoids in the n-dimensional space of the $\rho_i(t)$. If the curve represented parametrically by the $\rho_i(t)$ is entirely contained within one of the hyperellipsoids, then the zero solution of the equation above is asymptotically stable.

In this paper the author uses the second method of Liapunov to obtain stability criteria for the above equation which depend on only n parameters which determine a family of elliptic paraboloids in the n-dimensional space of the $\rho_i(t)$. It can be shown that these elliptic paraboloids completely contain the hyperellipsoids of Ghizzetti. A practical technique for the application of the stability criteria obtained is discussed and is applied to two examples.

The objective of the sixth paper is to present a unified exposition of Liapunov's theory of stability that includes the classical Liapunov theorems on stability and instability as simple corollaries. The principal idea exploited in this paper was used by other investigators in the study of nonautonomous functional differential equations. Of considerable importance is the possibility of extending these concepts to more general classes of dynamical systems, especially to some types as defined by partial differential equations.

A noteworthy contribution is Theorem 1 and its corollary. The theorem, which is concerned with the nonautonomous system $\dot{x} = f(t,x)$, explains precisely the nature of the information given by a Liapunov function; it shows that a Liapunov function relative to a set G defines a set E which, under the conditions of the theorem, locates all positive limit sets of solutions $x(t)$ of $\dot{x} = f(t,x)$ that for positive time remain in G. However, in order to use the theorem, there must be some means of determining which solutions remain in G. A corollary, a consequence of the theorem, gives one way of doing this and also provides, for nonautonomous systems, a method for estimating regions of attraction (domains of stability).

A limit set of Ω is defined as the set approached by a solution $x(t)$ of a system of differential equations as $t \to \infty$. The points $p \epsilon \Omega$ are limit points. A limit set has an "invariance property" if all solutions $x(t)$ which start at $p \epsilon \Omega$ remain in Ω as $t \to \infty$. It is pointed out that there are special classes of differential equations where the limit sets of solutions have, additionally, an invariance property and that this property permits a refinement and sharpening of Theorem 1, mentioned above, for these special classes.

Because the paper is largely a survey of recent extensions of past investigations, formal proofs, except for corollary 6, are not given; but ample references and illustrative examples are provided for the reader who might wish to work out the proofs for himself.

In the seventh paper an analytical solution of the Euler-Lagrange equations for the Lagrange multipliers for optimum coast trajectories is obtained. Similar solutions have been obtained by other investigators, but all of these solutions had singularities for orbits with zero eccentricity. The solution presented in this paper does not have such a singularity, but there is a numerical difficulty due to a removable singularity at unit eccentricity. An approximate solution, accurate near unit eccentricity, is given. This solution reduces to the exact parabolic solution for unit eccentricity.

N66 34927

34927

AERОSPACE RESEARCH CENTER

Finite Time Stability of Periodic Solutions of Hamiltonian Systems

by

I. S. Bernstein
Mathematics Department

General Precision Aerospace
Little Falls, New Jersey

7

Abstract

This study deals with finite time stability properties of periodic solutions of Hamiltonian systems. It attempts to answer questions such as what tube of initial conditions about the periodic solution will keep solutions in some prescribed tube about the periodic solution over some prescribed time interval. A theoretical basis for answering questions such as the one formulated above was given by Birkhoff [1]. However, to obtain actual numerical estimates from the analytical theory requires a great deal of algebraic manipulation even in the simplest of problems. For this reason, it was decided to employ a computer and an appropriate computer language to perform the required manipulations.

As an initial application of the mechanization of the Birkhoff theory the system chosen was the planar restricted three-body problem and the solution chosen was a Lagrangian critical point.

Basic Theory

Let $H(x,y)$ be the Hamiltonian of a dynamical system so that the equations of motion can be written as

$$\begin{aligned} \dot{x}_v &= H_{y_v}(x,y) & x &= (x_1,\dots,x_n) \\ \dot{y}_v &= -H_{x_v}(x,y) & y &= (y_1,\dots,y_n) \end{aligned} \qquad v = 1,\dots,n, \qquad (1)$$

and let $x_v = \varphi_v(t)$, $y_v = \psi_v(t)$ be a periodic solution of (1) of period 2π. To study solutions in the neighborhood of this periodic motion we make the change of variables

$$\bar{x} = x - \varphi, \quad \bar{y} = y - \psi.$$

Then (1) takes the form

$$\dot{\bar{x}}_\nu = \tilde{H}_{\bar{y}_\nu}(\bar{x}, \bar{y}, t)$$

$$\dot{\bar{y}}_\nu = -\tilde{H}_{\bar{x}_\nu}(\bar{x}, \bar{y}, t) \qquad \nu = 1,\ldots,n, \qquad (2)$$

where $\tilde{H}$ has period 2π in t and

$$\tilde{H}_{\bar{y}_\nu}(0,0,t) = \tilde{H}_{\bar{x}_\nu}(0,0,t) = 0, \quad \nu = 1,\ldots,n.$$

Thus the origin is a critical point solution of (2). The problem is therefore reduced to the study of solutions in the neighborhood of an equilibrium solution of a Hamiltonian system with an explicit periodic time dependence. A theorem of Birkhoff, [1], is applicable to this problem.

<u>Theorem 1</u> Let the Hamiltonian $H(x,y,t)$ of a dynamical system with an equilibrium point at the origin, be analytic in x and y, periodic in t of period 2π, and thus representable in a convergent power series by

$$H(x,y,t) = \sum_{\nu_1+\nu_2+\ldots+\nu_{2n}=2}^{\infty} a_{\nu_1,\nu_2,\ldots,\nu_{2n}}(t)\, x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} y_1^{\nu_{n+1}} y_2^{\nu_{n+2}} \cdots y_n^{\nu_{2n}}$$

$$= H_2(x,y,t) + H_3(x,y,t) + \ldots + H_k(x,y,t) + \ldots$$

where $H_k(x,y,t)$ is a homogeneous polynomial in x,y of degree k with periodic coefficients of period 2π. Let the $2n$ characteristic exponents, [2], associated with H_2 be distinct and purely imaginary. As the system is Hamiltonian, they may be represented as, [3],

$$\lambda_1, \ldots, \lambda_n, \ -\lambda_1, \ldots -\lambda_n.$$

Furthermore let the exponents satisfy

<u>Assumption 1</u> $$m_1 \lambda_1 + m_2 \lambda_2 + \ldots + m_n \lambda_n + m_{n+1} \neq 0$$

for all integers m_i such that

$$0 < |m| = \sum_{i=1}^{n} |m_i| \leq N \geq 3 .$$

Then there exists a canonical change of variables

$$\begin{aligned} x_v &= f_v(\xi, \eta, t) \\ & \qquad\qquad v = 1, \ldots, n, \\ y_v &= g_v(\xi, \eta, t) \end{aligned} \tag{3}$$

where f_v and g_v are convergent power series without constant terms in the components of ξ and η with coefficients having period 2π in t, such that the Hamiltonian in the new variables has the form

$$\tilde{H}(\xi, \eta, t) = \tilde{H}_1(\xi_1 \eta_1, \xi_2 \eta_2, \ldots \xi_n \eta_n) + \tilde{H}_2(\xi, \eta, t),$$

where $\tilde{H}_1$ is a polynomial with constant coefficients of degree N if N is even and degree $N-1$ if N is odd in the variables $z_v = \xi_v \eta_v$, and where $\tilde{H}_2(\xi, \eta, t)$ is a power series in ξ_v, η_v, beginning with terms of degree $N+1$. With $\tilde{H}$ in this form the Hamiltonian is said to be normalized up to order N.

Theorem 2 (Special Case)

Let the Hamiltonian, $H(x,y)$, of a dynamical system with an equilibrium point at the origin, be analytic in x and y and thus representable in a convergent power series by

$$H(x,y) = \sum_{v_1+\ldots+v_{2n}=2}^{\infty} a_{v_1,\ldots,v_{2n}} x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} y_1^{v_{n+1}} y_2^{v_{n+2}} \cdots y_n^{v_{2n}} =$$

$$H_2(x,y) + H_3(x,y) + \ldots + H_n(x,y) + \ldots,$$

where $H_n(x,y)$ is a homogeneous polynomial in x, y of degree n with constant coefficients. Let the $2n$ eigenvalues associated with H_2 be distinct, purely imaginary and represented by

$$\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n.$$

Furthermore let the eigenvalues satisfy

Assumption 1' $\qquad m_1\lambda_1 + m_2\lambda_2 + \ldots, + m_n\lambda_n \neq 0$

for all such integers m_i such that

$$0 < |m| = \sum_{i=1}^{n} |m_i| \leq N \geq 3.$$

Then the conclusion of Theorem 1 holds. Moreover there is no explicit time dependence in the change of variables (3) and thus also in the new Hamiltonian $\tilde{H}(\xi, \eta)$.

The usefulness of this theorem is the following. If we are studying solutions near the equilibrium point $\xi = \eta = 0$, then $\tilde{H}_2$ is of higher order than $\tilde{H}_1$ and is discarded for the moment. The equations then have the form

$$\begin{aligned} \dot{\xi}_v &= (H_1)_{\eta_v}(z) = \xi_v (H_1)_{z_v} \\ & \qquad\qquad\qquad\qquad v = 1,\ldots,n. \\ \dot{\eta}_v &= -(H_1)_{\xi_v}(z) = -\eta_v (H_1)_{z_v} \end{aligned} \tag{4}$$

If we multiply the first equation by η_v, the second by ξ_v, and add, it follows that

$$\frac{d}{dt}(\xi_v \eta_v) = 0, \quad v = 1,\ldots,n.$$

Thus, $\xi_v \eta_v = c_v$ (constant) so that (4) becomes integrable yielding

$$\begin{aligned} \xi_v &= \xi_v(0)\, e^{i (H_1)_{z_v}(c)\, t} \\ & \qquad\qquad\qquad\qquad v = 1,\ldots,n. \\ \eta_v &= \eta_v(0)\, e^{-i (H_1)_{z_v}(c)\, t} \end{aligned} \tag{5}$$

If we restrict ourselves to a large finite time interval and a suitable region in phase space it can be shown that the higher order terms previously truncated can be made small so that (5) is a close representation to the actual solution in this region. By use of (3) approximate solutions to the original problem may be obtained. For precise statements along these lines see [1] and [3].

Rather than prove this general theorem we now illustrate how to carry out the normalization procedure for the particular Hamiltonian describing the planar restricted three-body problem and take for the periodic solution a Lagrange critical point. This problem falls under Theorem 2.

Application

The equations of motion for the planar restricted three-body problem in the rotating coordinate system are

$$\frac{d^2x}{dt^2} - 2\omega \frac{dy}{dt} = \omega^2 x - \frac{m_1 (x - x_1)}{r_1^3} - \frac{m_2 (x - x_2)}{r_2^3},$$

$$\frac{d^2y}{dt^2} + 2\omega \frac{dx}{dt} = \omega^2 y - \frac{m_1 y}{r_1^3} - \frac{m_2 y}{r_2^3},$$

where the gravitational constant has been set equal to one.

If we set

$$u = \frac{dx}{dt} - \omega y,$$

$$v = \frac{dy}{dt} + \omega x,$$

then the Hamiltonian takes the form

$$H = \frac{1}{2} (u^2 + v^2) + \omega (uy - vx) - \frac{m_1}{r_1} - \frac{m_2}{r_2}. \tag{6}$$

If we set $|x_1 - x_2| = d$ and

$$a = \frac{d}{2} \frac{m_1 - m_2}{m_1 + m_2}, \quad b = \frac{\sqrt{3}d}{2},$$

then the point

$$x = a, \quad y = b, \quad u = -\omega b, \quad v = \omega a,$$

is an equilibrium point solution for the system. We introduce dimensionless variables τ, q_1, q_2, p_1, p_2, in the neighborhood of this equilibrium point by

$$\tau = \omega t, \quad x = a + q_1 d, \ y = b + q_2 d, \ u = -\omega b + p_1 \omega d, v = \omega a + p_2 \omega d,$$

The Hamiltonian (3) is now defined in a neighborhood of the equilibrium point $q_1 = q_2 = p_1 = p_2 = 0$ and takes the form after expansion about this point

$$H = H_2 + H_3 + \dots + H_m + \dots$$

where

$$H_2 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + q_2 p_1 - q_1 p_2 + \frac{1}{8} q_1^2 - k q_1 q_2 - \frac{5}{8} q_2^2 \tag{7}$$

$$H_3 = -\frac{7\sqrt{3}k}{36} q_1^3 + \frac{3\sqrt{3}}{16} q_1^2 q_2 + \frac{11\sqrt{3}k}{12} q_1 q_2^2 + \frac{3\sqrt{3}}{16} q_2^3 \tag{8}$$

$$H_4 = \frac{37}{128} q_1^4 + \frac{25k}{24} q_1^3 q_2 - \frac{123}{64} q_1^2 q_2^2 - \frac{15k}{8} q_1 q_2^3 - \frac{3}{128} q_2^4 \tag{9}$$

with $k = \frac{3\sqrt{3}}{4} \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$.

We now carry out the normalization of the Hamiltonian up to order 3 ($N = 3$) for the Earth-Moon system. For this value of k the eigenvalues corresponding to H_2 are distinct and purely imaginary and Assumption 1' holds for $N = 3$.

From (7), H_2 can be written as

$$H_2 = \frac{1}{2} r^T \cdot E r$$

where $r^T = (q_1, q_2, p_1, p_2)$ and

$$E = \begin{bmatrix} 1/4 & -k & 0 & -1 \\ -k & -5/4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

The equations of motion then become

$$\dot{r} = (FE)\, r + \ldots$$

where

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

As the eigenvalues of FE, $\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$, are distinct there exists an A such that

$$A^{-1} FEA = D \qquad (10)$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix}$$

Moreover A can be chosen so that

$$A^T FA = F \tag{11}$$

which guarantees that

$$r = A\hat{r} \tag{12}$$

is a canonical mapping and thus preserves the Hamiltonian nature of the system. As the equations of motion of the system in the new variables become

$$\dot{\hat{r}} = D\hat{r} + \dots$$

the Hamiltonian in the new variable becomes

$$\hat{H}_2 (\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2) = \lambda_1 \hat{q}_1 \hat{p}_1 + \lambda_2 \hat{q}_2 \hat{p}_2 + \dots$$

Omitting the details, the set of all matrices that diagonalize FE and are canonical take the form

$$A = A_1 \Delta S,$$

where

$$A_1 = \begin{bmatrix} a_1 & a_2 & \bar{a}_1 & \bar{a}_2 \\ b_1 & b_2 & \bar{b}_1 & \bar{b}_2 \\ (\lambda_1 a_1 - b_1) & (\lambda_2 a_2 - b_2) & \overline{(\lambda_1 a_1 - b_1)} & \overline{(\lambda_2 a_2 - b_2)} \\ (a_1 + \lambda_1 b_1) & (a_2 + \lambda_2 b_2) & \overline{(a_1 + \lambda_1 b_1)} & \overline{(a_2 + \lambda_2 b_2)} \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\delta_i^{-1} = \lambda_i a_1 (11 - 4a_i), \qquad i = 1, 2,$$

A_1 being chosen such that (10) is satisfied and $A_1 \Delta$ so that both (10) and (11) are satisfied. The matrix S has the form

$$S = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1^{-1} & 0 \\ 0 & 0 & 0 & s_2^{-1} \end{bmatrix},$$

where s_1 and s_2 are free to vary. As r is real, from (12) it follows that we must guarantee that

$$A\tilde{r} = \overline{A\tilde{r}} \tag{13}$$

where the bar represents the complex conjugate. It can be shown that if we choose s_1 and s_2 by

$$s_1 = \frac{i}{\sqrt{|\delta_1|}},$$

$$s_2 = \frac{-i}{\sqrt{|\delta_2|}},$$

then a sufficient condition for (13) to hold is that

$$\begin{aligned} \hat{p}_1 &= i\,\bar{\hat{q}}_1 \\ \hat{p}_2 &= -i\,\bar{\hat{q}}_2 \;. \end{aligned} \tag{14}$$

Combining the above matrices, the matrix A has the form

$$\begin{bmatrix} a_1\sqrt{|\delta_1|} & a_2\sqrt{|\delta_2|} & -i\bar{a}_1\sqrt{|\delta_1|} & i\bar{a}_2\sqrt{|\delta_2|} \\ b_1\sqrt{|\delta_1|} & b_2\sqrt{|\delta_2|} & -i\bar{b}_1\sqrt{|\delta_1|} & +i\bar{b}_2\sqrt{|\delta_2|} \\ (\lambda_1 a_1 - b_1)\sqrt{|\delta_1|} & (\lambda_2 a_2 - b_2)\sqrt{|\delta_2|} & -i(\overline{\lambda_1 a_1 - b_1})\sqrt{|\delta_1|} & i(\overline{\lambda_2 a_2 - b_2})\sqrt{|\delta_2|} \\ (a_1 + \lambda_1 b_1)\sqrt{|\delta_1|} & (a_2 + \lambda_2 b_2)\sqrt{|\delta_2|} & -i(\overline{a_1 + \lambda_1 b_1})\sqrt{|\delta_1|} & i(\overline{a_2 + \lambda_2 b_2})\sqrt{|\delta_2|} \end{bmatrix}$$

and normalizes the terms of second degree of H.

The Hamiltonian under (12) takes the form

$$\tilde{H} = \lambda_1 \tilde{q}_1 \tilde{p}_1 + \lambda_2 \tilde{q}_2 \tilde{p}_2 + \sum_{v_1+\ldots+v_4=3} g_{v_1,v_2,v_3,v_4} \tilde{q}_1^{v_1} \tilde{q}_2^{v_2} \tilde{p}_1^{v_3} \tilde{p}_2^{v_4}$$

$$+ \sum_{v_1+v_2+v_3+v_4=4} h_{v_1,v_2,v_3,v_4} \tilde{q}_1^{v_1} \tilde{q}_2^{v_2} \tilde{p}_1^{v_3} \tilde{p}_2^{v_4} + \ldots \tag{15}$$

where (15) is obtained by substituting (12) into (7-9). This relatively simple symbolic operation, however, is quite cumbersome when attempted to be done by hand. It was done, though, for this particular model with the Earth-Moon constants and will be used for checking purposes.

We now normalize the third order terms of (15). As we shall see Assumption 1' for $N = 3$ is essential here. Let us introduce a canonical change of variables by the contact transformation, [3],

$$\xi_k = \tilde{q}_k + \frac{\partial V}{\partial \eta_k} \tag{16}$$

$$k = 1,2$$

$$\tilde{p}_k = \eta_k + \frac{\partial V}{\partial \tilde{q}_k} \tag{17}$$

where $V(\tilde{q}_k, \eta_k)$ has the form

$$V = \sum_{v_1+v_2+v_3+v_4=3} c_{v_1,v_2,v_3,v_4} \tilde{q}_1^{v_1} \tilde{q}_2^{v_2} \eta_1^{v_3} \eta_2^{v_4} \tag{18}$$

We attempt to choose c_{v_1, v_2, v_3, v_4} so as to eliminate as many third-order terms as possible in (15). Substituting (16,17) into (15) we obtain

$$\hat{H} = \lambda_1 \xi_1 \eta_1 + \lambda_2 \xi_2 \eta_2 + \lambda_1 \left(\xi_1 \frac{\partial V}{\partial \tilde{q}_1} - \eta_1 \frac{\partial V}{\partial \eta_1}\right) + \lambda_2 \left(\xi_2 \frac{\partial V}{\partial \tilde{q}_2} - \eta_2 \frac{\partial V}{\partial \eta_2}\right) \tag{19}$$

$$+ \sum_{v_1+v_2+v_3+v_4=3} g_{v_1,v_2,v_3,v_4} \xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4} + \text{terms of degree 4 \& higher.}$$

We note that V is a function of the old variables $\tilde{q}_k$. By a formal process we can solve for these variables from (16) and substitute for the $\tilde{q}_k$ in (19). Both the transformation from old to new variables and its inverse may be obtained by a formal procedure from (16), (17). Both lead to powers series representations which converge in a neighborhood of the origin. Eliminating all dependence on $\tilde{q}_k$ in (19) by this method, $\hat{H}$ takes the form

$$\hat{H} = \lambda_1 \xi_1 \eta_1 + \lambda_2 \xi_2 \eta_2 + \lambda_1 \left(\xi_1 \frac{\partial V(\xi,\eta)}{\partial \xi_1} - \eta_1 \frac{\partial V(\xi,\eta)}{\partial \eta_1}\right) \tag{20}$$

$$+ \lambda_2 \left(\xi_2 \frac{\partial V(\xi,\eta)}{\partial \xi_2} - \eta_2 \frac{\partial V(\xi,\eta)}{\partial \eta_2}\right) +$$

$$\sum_{v_1+v_2+v_3+v_4=3} g_{v_1,v_2,v_3,v_4} \xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4} + \text{terms of degree 4 \& higher.}$$

Collecting third order terms in $\xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4}$ in (20) and using (18) we obtain for a typical term

$$c_{v_1,v_2,v_3,v_4} [\lambda_1 (v_1 - v_3) + \lambda_2 (v_2 - v_4)] + g_{v_1,v_2,v_3,v_4}. \tag{21}$$

If the bracket in (21) doesn't vanish we can solve for c_{v_1,v_2,v_3,v_4} and eliminate the corresponding third order term from the Hamiltonian. But from

Assumption 1' the bracket can vanish only if $v_1 = v_3$, $v_2 = v_4$. However this would imply the order we are dealing with is even. Thus for $N = 3$ Assumption 1' guarantees all third order terms of the Hamiltonian can be eliminated by the change of variables (16,17). One must keep in mind that although all third order terms are eliminated many more fourth order terms arise from this process. These must be kept tract of for error bounds and also if higher order normalizations are to be carried out. This has been done by hand for the fourth-order terms of the Earth-Moon model and will be used for checking purposes.

Following the same procedure as above, normalization of the Hamiltonian up to degree s can be carried out if Assumption 1' holds for $N = s$. Let us assume that the normalization has been carried out up to degree $s - 1$. Then the change of variables defined implicitly by (16 - 18) with $v_1 + v_2 + v_3 + v_4 = s$ preserves the normal form up to degree $s - 1$. As before, collecting sth order terms in $\xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4}$ leads to an equation of the form (21). If $|v_1 - v_3| + |v_2 - v_4| \neq 0$ then c_{v_1, v_2, v_3, v_4} can be chosen to eliminate the corresponding sth order term from the new Hamiltonian. Reasoning as above, all sth order terms can be eliminated if s is odd. If s is even then all terms save those for which $v_1 = v_3$, $v_2 = v_4$, can be eliminated. We choose the corresponding c_{v_1, v_2, v_3, v_4} equal to zero in this case. However these terms are formed from products $\xi_k \ \eta_k$ and lead to an integrable Hamiltonian. It should be noted that the complexity of the operation of normalization increases with s. This manifests itself in keeping tract of all coefficients that combine to form a particular g_{v_1, v_2, v_3, v_4} in (21) which in turn is a function of all previous normalizations.

Returning to our original task, after third-order terms have been eliminated, the Hamiltonian takes the form

$$\hat{H}(\xi_1, \xi_2, \eta_1, \eta_2) = \lambda_1 \xi_1 \eta_1 + \lambda_2 \xi_2 \eta_2 +$$

$$\sum_{v_1+v_2+v_3+v_4=4} \hat{h}_{v_1,v_2,v_3,v_4} \xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4} + \text{higher order terms.}$$

Dropping the 4th and higher order terms the differential equations become

$$\dot{\xi}_k = \lambda_k \xi_k$$

$$\dot{\eta}_k = -\lambda_k \eta_k,$$

so that

$$\xi_k = \xi_k(0)\, e^{\lambda_k t}$$

$$\eta_k = \eta_k(0)\, e^{-\lambda_k t}.$$

It can be shown that if the initial conditions are chosen such that $\eta_k(0) = i\bar{\xi}_k(0)$ then solutions in the original variables will turn out to be real. Thus, if we invert all transformations, information about the original system may be obtained. The error in this case comes in because of the truncation of the 4th and higher order terms. It is intuitively obvious that this procedure will give better results than a linear analysis. For in such a linear analysis the error comes in by truncating cubic and higher terms in the Hamiltonian.

A computer program is being written to perform the algebraic manipulations described above. This program is utilizing the I.B.M. FORMAC language and is being written for the I.B.M. 7090/94 computer.

References

[1] Birkhoff, G.D., "Dynamical Systems", New York, 1927.

[2] Coddington, E.A., and Levinson, N., "Theory of Ordinary Differential Equations", McGraw Hill, New York, 1955.

[3] Siegel, C.L., "Vorlesungen über Himmelsmechanik", Springer, Berlin, 1956.

DISCONTINUOUS VECTOR FIELDS AND FEEDBACK CONTROL

H. Hermes[†]

[†]This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGR-40-002-015 and in part by the United States Air Force through the Air Force Office of Scientific Research under Grant No. AF-AFOSR-693-65.

DISCONTINUOUS VECTOR FIELDS AND FEEDBACK CONTROL

H. Hermes

Introduction. The study of "stability" under perturbations, $\varepsilon(t)$, for a C^1 vector field X is no different when the perturbation enters the equation as $\dot{x}(t) = X(x(t) + \varepsilon(t))$, $(\dot{x}(t) = \frac{dx(t)}{dt})$ or as $\dot{x}(t) = X(x(t)) + \varepsilon(t)$. This is no longer true if X is discontinuous. In particular, problems of feedback control naturally lead to discontinuous vector fields of the form $X(x) = F(x,u(x))$ where u is a control function. In practice, the value of u is determined after making a measurement on the state $x(t)$, at time t. If this measurement is in error, say $x(t) + \varepsilon(t)$ is measured rather than $x(t)$, the governing equation of motion will have the form

$$\dot{x}(t) = X(x(t) + \varepsilon(t)). \tag{1}$$

It is this concept which leads, in section 2, to the definition of stability with respect to measurement. Intuitively, X is stable with respect to measurement if any solutions of eq.(1) and $\dot{x}(t) = X(x(t))$, satisfying the same initial conditions, remain arbitrarily close over any finite positive time interval whenever the supremum of $|\varepsilon(t)|$ over this time interval is restricted to be sufficiently small.

In general, the initial value problem for a discontinuous vector field X need not have a solution. If, however, there is an absolutely continuous function φ of the real variable t which satisfies the initial condition and $\dot{\varphi}(t) = X(\varphi(t))$ almost everywhere; we will call φ a classical solution. There are many ways to generalize the difinition of solution, so that solutions will exist, even if X is merely measurable. A summary of the more standard notions, most of which replace the vector field X by an "averaged" or "smoothed" associated vector field, are given by Filippov in [3]. In [3] Filippov defines a new concept of a solution, which is motivated by control problems; we will discuss this notion in §1 and hereafter refer to such solutions as Filippov solutions.

It will be seen that control laws synthesized from "open loop" controls (hence classical solutions exist) may lead to vector fields which are not stable with respect to measurement. An example is given for which an optimal feedback control exists when solutions are taken in the classical sense, but does not exist if solutions are taken in the sense of Filippov.

The main result shows that if a vector field X is stable with respect to measurement (solutions taken in the classical sense in the definition of this stability) then every classical solution is a Filippov solution.

If X is stable with respect to measurement, solutions for $t \geqq 0$ of the initial value problem for the corresponding differential equation are unique, and such a solution when evaluated at a fixed positive time, varies continuously with the initial data. This means that, with increasing time, solutions may join but not branch. Thus it is felt that feedback controls which are meaningful from the viewpoint of applications should lead to vector fields which are stable with respect to measurement. To characterize such vector fields directly, however, is no easy task.

§1. A Reason for Discontinuous Fields; the Filippov Solution.

Consider a control system of the form

$$\dot{x} = g(x, u(x)) \quad , \quad x = (x_1, \ldots, x_n), \qquad u = (u_1, \ldots, u_r) \tag{2}$$

with values $u(x)$ to be chosen from a control set U. Let the terminal manifold (target) be a manifold S contained in $[0, \infty) \times E^n$, (E^n denotes Euclidean n space.) If g is bounded and Lipschitzian in both arguments and u is a given Lipschitzian control, then an initial value problem for (2) with data $x(0) = x^0$ has a unique solution, with value at time t denoted $\varphi(t,0,x^0)$. Suppose $\varphi(t_1, 0, x^0) \in S$. The question considered is the following: <u>If S has dimension less than n in E^{n+1}, is it possible that, for each x in some neighborhood $\mathcal{N}(x^0) \subset E^n$ of x^0, there exists a value $t(x)$, $0 \leq t(x) < \infty$, such that $\varphi(t(x), 0, x) \in S$?</u>

From a control system viewpoint, it would be desirable that this question have an affirmative answer (which is the case if u is allowed discontinuities). However for u Lipschitzian we will show, using a method related to a result of Bridgland [1, lemma 2], that the answer is negative. Indeed, for fixed t', $\varphi(t', 0, \cdot)$ is a homeomorphism therefore the image of an n neighborhood will have dimension n. To consider the case where the value of t may depend on the point $x \in \mathcal{N}(x^0)$ define the map $\psi : E^{n+1} \to E^{n+1}$ by $\psi(t,x) = (t,\varphi(t,0,x))$. Then ψ is a homeomorphism with inverse $\psi^{-1}(t,x) = (t,\varphi(0,t,x))$. Since S has dimension less than n, $\psi^{-1}(S)$ has dimension less than n. Let P be a projection defined by $P(t,x) = (0,x)$. Then $P(\psi^{-1}(S))$ has dimension less than n. But $P(\psi^{-1}(S))$ is precisely the set of initial points in E^n from which S is attainable. Indeed $x' \in P(\psi^{-1}(S))$ if and only if there exists $t'' \geq 0$ such that $(t', \varphi(t', 0, x')) \in S$. To see this, $x' \in P(\psi^{-1}(S)) \Longrightarrow$ for some t', $(t', x') \in \psi^{-1}(S) \Longrightarrow \psi(t', x') \in S$ or $(t', \varphi(t', 0, x')) \in S$. On the other hand $(t', \varphi(t', 0, x')) \in S \Longrightarrow (t', x') = (t', \varphi(0, t', \varphi(t', 0, x')) \in \psi^{-1}(S)$ and $x' \in P(\psi^{-1}(S))$. Thus the set of initial points from which S can be attained has dimension less than n.

It is useful, therefore, from a control viewpoint, to study differential equations with discontinuous right sides. For the sake of completeness we will briefly discuss the generalized concept of solutions for such equations as given by Filippov [3].

Let X be a measurable function defined almost everywhere in a domain $Q \subset E^n$ with values in a bounded set in E^n. With $X(x)$ associate the convex set

$$K\{X(x)\} = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{co}\,\{X(U(x,\delta) - N)\} \tag{3}$$

where $\overline{co}$ denotes closed convex hull, $U(x, \delta)$ is a closed δ neighborhood of x, N an arbitrary set in E^n and μ is n dimensional Lebesgue measure.

<u>An absolutely continuous vector valued function</u> φ, <u>defined on</u> $[0, T]$, <u>is called a solution in the sense of Filippov of</u> $\dot{x} = X(x)$ <u>if for almost all</u> t, $\dot{\varphi}(t) \in K\{X(\varphi(t))\}$. It is shown in [3] that such solutions will always exist, and many of their properties are discussed. In particular, if X is continuous, $K\{X(x)\} = X(x)$.

To illustrate this type of solution and its consequences we consider a very simple control problem.

<u>Example 1</u>. The problem will be that of minimum time transfer with terminal manifold $S = \{(t, x_1, x_2) : t \geqq 0, x_1 = 0, x_2 = 0\}$ and system equations

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

with control components subject to the constraint $0 \leqq |u_1| + |u_2| \leqq 1$. It is clear that the minimum time needed to attain S from the initial point (x_1^0, x_2^0)

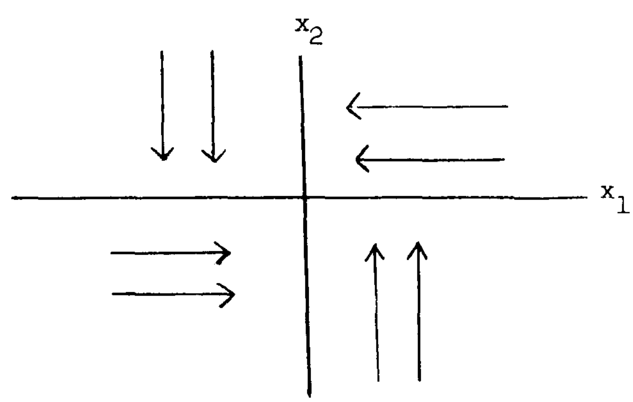
is $|x_1^o| + |x_2^o|$, and there are many ways in which this can be accomplished. We single out two such strategies; each will be given in closed loop (feedback) form as symthesized from obvious open loop strategies.

Strategy 1

$$u^1(x) = \begin{cases} (-1, \ 0) & , \quad x_1 > 0 \ , \ x_2 \geqq 0 \\ (0, \ -1) & , \quad x_1 \leqq 0 \ , \ x_2 > 0 \\ (1, \ \ 0) & , \quad x_1 < 0 \ , \ x_2 \leqq 0 \\ (0, \ \ 1) & , \quad x_1 \geqq 0 \ , \ x_2 < 0 \\ (0, \ \ 0) & , \quad x_1 = x_2 = 0 \end{cases}$$

Pictorially, the resulting vector field looks as follows:

Figure 1.



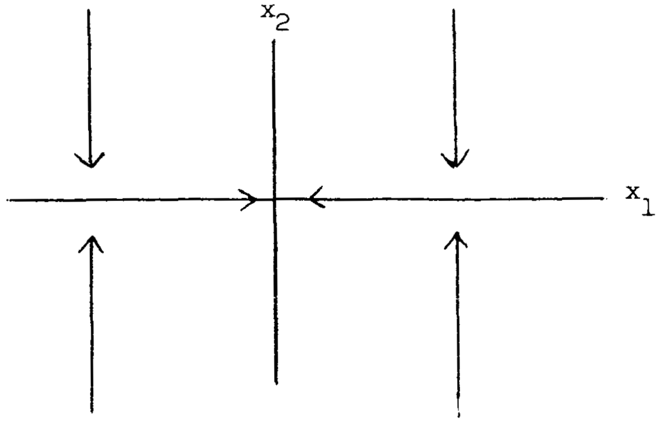
All vectors are unit vectors.

Strategy 2

$$u^2(x) = \begin{cases} (0, \ -1) & , \quad x_2 > 0 \\ (-1, \ 0) & , \quad x_2 = 0, \quad x_1 > 0 \\ (1, \ \ 0) & , \quad x_2 = 0, \quad x_1 > 0 \\ (0, \ \ 1) & , \quad x_2 < 0 \\ (0, \ \ 0) & , \quad x_1 = x_2 = 0 \ . \end{cases}$$

Pictorially:

Figure 2.



In each case, the classical solution of the equations of motion exists for arbitrary initial data, is uniquely defined for all $t \geqq 0$, depends continuously on the initial data and reaches the origin in minimum possible time. These same properties are true with strategy 1 when solutions are considered in the sense of Filippov, however in the case of strategy 2 the Filippov solutions become rest solutions when a state with $x_2 = 0$ is attained. Therefore, solutions in this sense, do not reach the target S. This occurs since the first component of the vector field $u^2(x)$ is zero except on a set of measure zero, i.e. the x_1 axis. From a practical viewpoint, since the control signal is determined by a state measurement, one should not expect sets of states having measure zero to influence the solution. From this viewpoint, the Filippov solution is the more realistic notion.

In the preceding example, with the proper choice of strategy, i.e. strategy 1, an optimal feedback control existed whether solutions are taken in the sense of Filippov or the classical sense. The following example will show that this need not always be the case; i.e. we will produce a feedback control synthesized from optimal open loop controls, which is an optimal feedback control if solutions are taken in the classical sense. However an optimal feedback control for solutions taken in the sense of Filippov will not exist.

Example 2. Let the equations of motion be:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u \quad , \quad 0 \leqq u \leqq 1 \quad , \quad |x^0| < 2 \tag{4}$$

with the optimization problem being to minimize the cost functional $\int_0^{t_f} u[(x_1-1)^2 + x_2^2 - 1]^2 dt$ where t_f is the smallest nonnegative time a solution reaches the origin.

The open loop strategy of $u = 0$ until the circle $(x_1-1)^2 + x_2^2 = 1$ is reached, at which time a switch to $u = 1$ allowing this circle to be traversed in a clockwise fashion, produces a trajectory which reaches the origin with zero cost. The corresponding synthesized feedback control leads to the following vector field for (4):

$$X(x) = \begin{cases} x_2 \\ -x_1 + u(x) \end{cases} \qquad \text{where} \quad u(x) = \begin{cases} 0 & \text{if} \quad (x_1-1)^2 + x_2^2 \neq 1 \\ 1 & \text{if} \quad (x_1-1)^2 + x_2^2 = 1 \end{cases}$$

On the other hand $K\{X(x_1,x_2)\} = \left\{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}\right\}$ since $u(x)$ is 1 only on a set of zero measure, and the corresponding Filippov solutions will not reach the origin. From the form of the cost functional, it is seen that for any function $u(x)$ for which the corresponding solutions in the sense of Filippov reach the origin, there will be a positive cost involved. Since this value can be made arbitrarily small, but not zero, an optimal feedback control for solutions in the sense of Filippov will not exist.

§2. Stability with Respect to Measurement.

An examination of the two vector fields of example 1 shows a type of stability present in the first which is not present in the second.

For notation ease, if f is a bounded function on a real interval $[0,T]$ with values in E^n, let $\|f\| = \text{ess. sup. } \{|f(t)|, t \in [0, T]\}$. We will use $U(x, \delta)$ to denote a compact, spherical neighborhood of radius δ, about the point $x \in E^n$, and coA to denote the convex hull of a set A.

Definition. A vector field X, for which a classical solution φ of $\dot{x} = X(x)$ with arbitrary initial data x^o exists, is said to be stable with respect to measurement if given $\epsilon > 0$ and finite $T > 0$, $\exists$ a $\delta > 0$ such that whenever $\mathcal{E}$ is a measurable function with values in E^n and norm less than δ for which a corresponding solution ψ (in classical sense) of $\dot{x}(t) = X(x(t) + \mathcal{E}(t))$, $x(o) = x^o$, exists on $[0, T]$, then $\|\varphi - \psi\| < \epsilon$.

For the remainder of this section we will assume X is a measurable function defined on a domain Q in E^n with values in a bounded set in E^n. Our concern will be with relating the concepts of stability with respect to measurement, Filippov solutions and classical solutions. In particular, lemma 3 will show that if ψ is a Filippov solution of $\dot{x} = X(x)$, $x(0) = x^o$ (such solutions do exist) then for any $\epsilon, \delta > 0$, there exists a measurable function $\mathcal{E}$ with $\|\mathcal{E}\| < \delta$ such that a classical solution φ of $\dot{x} = X(x + \mathcal{E}(t))$, $x(0) = x^o$ exists and satisfies $\|\varphi - \psi\| < \epsilon$. This essentially says that if one allows arbitrarily small perturbations of the argument, a response to any vector field X may be made to agree closely with a response to the associated Filippov generalized field $K\{X(\cdot)\}$. After this has been established one easily obtains:

Theorem 1. If X is stable with respect to measurement then every classical solution is a Filippov solution.

Lemma 1. Let ψ be an absolutely continuous function on $[0, T]$ with values in E^n, and z a measurable function with $z(t) \in K\{X(\psi(t))\}$, $t \in [0, T]$. Then given any $\delta > 0$, there exist a finite number of measurable functions $\omega^1, \ldots, \omega^k$ with $\omega^i(t) \in U(\psi(t), \delta)$ such that the function v^i defined by $v^i(t) = X(\omega^i(t))$ are measurable and for any $\epsilon > 0$, $z(t)$ is contained in an ϵ neighborhood of $co\{v^1(t), \ldots, v^k(t)\}$.

Proof Filippov [3] shows that the requirement $z(t) \in K\{X(\psi(t))\}$ is equivalent to the condition that for any vector η,

$$z(t) \cdot \eta \leqq \lim_{\gamma \to 0} (\text{ess. max } \{X(u) \cdot \eta : u \in U(\psi(t), \gamma)\}) \tag{5}$$

or equivalently $z(t) \cdot \eta \leqq \text{ess. max } \{X(u) \cdot \eta : u \in U(\psi(t), \gamma)\}$ for every $\gamma > 0$.

Let z be any measurable function with $z(t) \in K\{X(\psi(t))\}$. Suppose we are given $\delta, \epsilon > 0$. Pick an arbitrary vector $\eta \neq 0$; we will first show that one can construct a measurable function ω with $\omega(t) \in U(\psi(t), \delta)$ such that

$$z(t) \cdot \eta \leqq X(\omega(t)) \cdot \eta + \epsilon/2 \tag{6}$$

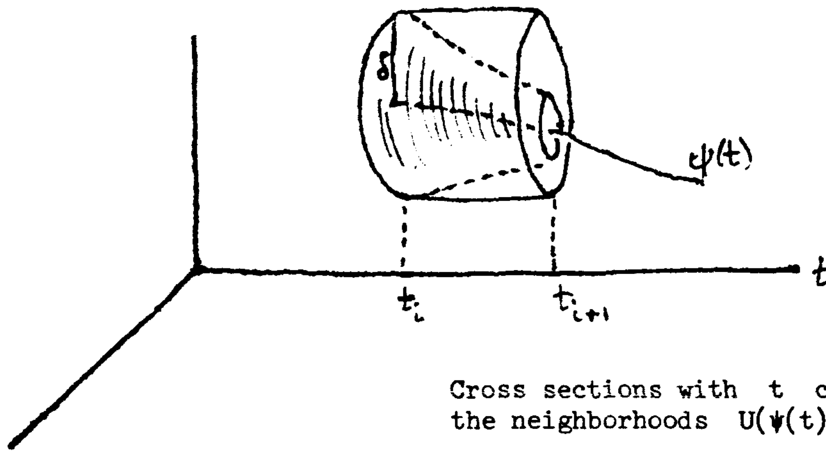
for $t \in [0, T]$.

Subdivide the interval $[0, T]$ into subintervals by a partition $0 = t_o < t_1 < \ldots < t_m = T$ and let $\delta(t-t_i)$ be a continuous real valued function defined on $[t_i, t_{i+1})$ with $\delta(0) = \delta$, $\delta \geqq \delta(t-t_i) \geqq \delta/2$ and such that

$$U(\psi(t), \delta(t-t_i)) \subset U(\psi(t'), \delta(t'-t_i)) \tag{7}$$

for $t_i \leqq t' \leqq t < t_{i+1}$. The existence of such a partition and function $\delta(t-t_i)$ is an immediate consequence of the uniform continunity of ψ on $[0, T]$.

Figure 4



Cross sections with t constant are the neighborhoods $U(\psi(t),\ \delta(t-t_i))$.

For any $\epsilon_1 > 0$, by Luzins' theorem we may cloose a compact subset $E_i(\epsilon_1)$ of $U(\psi(t_i),\ \delta)$ differing in measure from $U(\psi(t_i),\ \delta)$ by less than ϵ_1 and such that X is continuous on $E_i(\epsilon_1)$. Then for every $t \in [t_i,\ t_{i+1})$ the set $K_t = E_i(\epsilon_1) \cap U(\psi(t),\ \delta(t-t_i))$ is a nonempty compact subset of $E_i(\epsilon_1)$ on which $X(\cdot)\cdot\eta$ is continuous. Let $v_{\epsilon_1}(t) = \max.\{X(\omega)\cdot\eta : \omega \in K_t\}$. By (7), v_{ϵ_1} is a monotone decreasing function on $[t_i,\ t_{i+1})$ hence measurable. By theorem 1, [2] there exists a measurable function ω with $\omega(t) \in K_t$ such that $X(\omega(t))\cdot\eta = v_{\epsilon_1}(t)$, $t \in [t_i,\ t_{i+1})$. (Here we have replaced the condition of the sets K_t expanding, $K_t \subset K_{t'}$ for $t < t'$ in the cited theorem, by K_t contracting, $K_t \subset K_{t'}$ for $t' < t$; a condition which does not alter the proof since the direction of traversing the time axis is immaterial.)

This defines the function ω on the subinterval $[t_i,\ t_{i+1})$; since i was arbitrary we may assume ω to be defined on $[0,\ T]$ as that function whose restriction to $[t_i,\ t_{i+1})$ is defined as above.

For any $\epsilon_1 > 0$ either $v_{\epsilon_1}(t) \geq \text{ess. max.}\ \{X(u)\cdot\eta : u \in U(\psi(t),\ \delta/2)\} \geq z(t)\cdot\eta$, the latter inequality following from (5) , or $v_{\epsilon_1}(t) < \text{ess max.}\{X(u)\cdot\eta : u \in U(\psi(t),\ \delta/2)\}$. In the first case we deal with the situation where the maximum of $X(\cdot)\cdot\eta$ occurs in $U(\psi(t),\ \delta) - U(\psi(t),\ \delta/2)$ and inequality (6) holds even with $\epsilon = 0$.

In the second case, for fixed t, $v_{\epsilon_1}(t)$ is an increasing function of ϵ_1 since we may assume, without loss of generality, that $E_i(\epsilon_1') \supset E_i(\epsilon_1)$ if $\epsilon_1' < \epsilon_1$. Since, in this case $v_{\epsilon_1}(t) < \text{ess.max}\{X(u)\cdot\eta : u \in U(\psi(t),\delta/2)\}$, we may take a sequence of values ϵ_1 tending to zero. The corresponding bounded monotone sequence of real numbers $v_{\epsilon_1}(t)$ must converge to $\text{ess.max}\{X(u)\cdot\eta : u \in U(\psi(t), \delta/2)\}$; indeed if it converges to a number $\bar{m}$ less than this, by the definition of ess.max. there exists a set F of positive measure on which $X(\cdot)\cdot\eta > \bar{m}$. To obtain a contradiction we need only choose ϵ_1 less than the measure of F.

This establishes that for ϵ_1 sufficiently small, there exists a measurable function ω with values $\omega(t) \in U(\psi(t), \delta)$ such that $v(t) = X(\omega(t))$ is measurable and inequality (6) is satisfied. (Note: It is not true in general that a measurable function of a measurable function is measurable.)

Now let S^{n-1} be an n-1 sphere in E^n which contains $\cup_{t\in[0, T]} X(U(\psi(t), \delta)$; this exists by hypothesis. Since S^{n-1} is compact choose a finite number of vectors η^i , $i = 1, 2,\dots,k$ belonging to S^{n-1} and so that $\epsilon/2$ neighborhoods of the η^i cover S^{n-1}. For each η^i construct, as before, a function ω^i, measurable with values $\omega^i(t) \in X(U(\psi(t), \delta))$ satisfying (6). Let v^i be the corresponding measurable function; $v^i(t) = X(\omega^i(t))$. Then $z(t)$ is contained in an ϵ neighborhood of the convex hull of $\{v^1(t),\dots,v^k(t)\}$.

Lemma 2. *Let $v^1,\dots,v^k$ be bounded measurable functions defined on $[0, T]$ with $\mathcal{A}(t) = \{v^1(t),\dots,v^k(t)\}$. Let $\text{co}\mathcal{A}(t)$ denote the convex hull of $\mathcal{A}(t)$. Then if z is a measurable function with values $z(t)$ contained in an ϵ neighborhood of $\text{co}\mathcal{A}(t)$, there exists a measurable function v with values in $\mathcal{A}(t)$ such that $|\int_0^t[z(\tau) - v(\tau)]d\tau| < \epsilon(T+1)$ uniformly for $t \in [0, T]$.*

Proof. a) We will first show there is a measurable function y with values $y(t) \in \text{co}\mathcal{A}(t)$ such that $\|z-y\| \le \epsilon$.

Using the terminology of [4], if $z(\cdot)$ is measurable single valued function then $U(z(\cdot), \epsilon)$ is a measurable many valued function. Indeed, if $B(y^o, r)$ is a closed ball of radius r, center y^o, $\{t : U(z(t), \epsilon) \cap B(y^o, r) \neq \phi\} = \{t: |z(t) - y^o| \leq r + \epsilon\}$ which is measurable.

Next, since the functions v^i are measurable, we will show $\text{co}\mathcal{A}(\cdot)$ is a measurable set valued function. Obviously $\text{co}\mathcal{A}(t)$ is nonempty and closed for each t. Letting $B(y^o, r)$ be as above and S^{n-1} denote the unit $n-1$ sphere we note that the distance from $\text{co}\{(v^1(t)-y^o),\ldots,(v^k(t)-y^o)\}$ to the origin is $\max_{\eta\in S^{n-1}}(\min_{1\leq i\leq k} \eta\cdot(v^i(t)-y^o))$. Then

$$\{t : \text{co}\mathcal{A}(t) \cap B(y^o, r) \neq \phi\} = \{t : \max_{\eta\in S^{n-1}}(\min_{1\leq i\leq k} \eta\cdot(v^i(t)-y^o)) \leq r\}$$

which is measurable.

From [4], $U(z(\cdot), \epsilon) \cap \text{co}\mathcal{A}(\cdot)$ is again a measurable set valued function and there exists a measurable single valued function y with $y(t)\in U(z(t), \epsilon) \cap \text{co}\mathcal{A}(t)$.

b) We next show that if y is a measurable function on $[0, t']$ with $y(t)\in \text{co}\mathcal{A}(t)$ for each $t \in [0, t']$ then y admits the representation $y(t) = \sum_{i=1}^{k} \alpha_i(t)v^i(t)$ where the scalar valued functions α_i are measurable, $0 \leq \alpha_i(t) \leq 1$ and $\sum_{i=1}^{k}\alpha_i(t) = 1$ for all $t \in [0, t']$.

This result is closely related to lemma 1 [6]; which would yield the desired result if the functions v^i were continuous. To modify this to the present case where the v^i are measurable, let $f(t, \alpha) = \sum_{i=1}^{k}\alpha_i v^i(t)$, $Q = \{\alpha \in E^k : \sum_{i=1}^{k}\alpha_i = 1, 0 \leq \alpha_i \leq 1\}$, and $R(t) = f(t, Q)$. Then f if continuous in α for each fixed t. Referring now to the proof of the lemma of Filippov [5] and letting α_i, v^i play the role of the u_i, z^i, respectively, in that, proof, choose the set E to be so that $\alpha_1,\ldots,\alpha_{s-1}, v^1,\ldots,v^k$ and y are continuous in E. Because of the special form of f it follows that $f(t, \alpha)$ is continuous on $E \times Q$ and the Filippov argument may be applied to give the desired representation for y.

c) From theorem 1 [6], it now follows that for any interval $[0, t']$ there exists a measurable function v with values $v(t) \in \mathcal{A}(t)$ for each $t \in [0, t']$ such that

$$\int_0^{t'} y(\tau)d\tau = \int_0^{t'} v(\tau)d\tau \quad . \tag{8}$$

Since the functions v^i were bounded there is a constant M such that $\|y\| \leqq M$, $\|v\| \leqq M$. Subdivide the interval $[0, T]$ into m equal subintervals each of length T/m. Let I_j denote the interval $(jT/m, (j+1)T/m]$. Using (8) for each $j = 0,\ldots,(m-1)$, define v on I_j so that $\int_{I_j}[u(\tau)-v(\tau)]d\tau = 0$. Now if m is chosen so large that $m \geqq 2MT/\epsilon$ it follows that $|\int_0^t[y(\tau)-v(\tau)]d\tau| < \epsilon$ uniformly for $t \in [0, T]$.

d) To finish the proof we show the function v constructed in part c) satisfies the conclusions of lemma 2. Indeed $|\int_0^t [z(\tau)-v(\tau)]d\tau| =$ $|\int_0^t [z(\tau)-y(\tau) + y(\tau)-v(\tau)]d\tau| \leqq |\int_0^t [z(\tau)-y(\tau)]d\tau| + |\int_0^t[y(\tau)-v(\tau)]d\tau| \leqq$ $\epsilon T + \epsilon = \epsilon[T + 1]$; using the results of a) and c) , respectively.

<u>Lemma 3</u>. <u>Let</u> ψ <u>be a Filippov solution of</u> $\dot{x} = X(x)$, $x(0) = x^0$. <u>Then for any</u> ϵ, $\delta > 0$ <u>there exists a measurable function</u> $\mathcal{E} : [0, T] \to E^n$ <u>with</u> $\|\mathcal{E}\| < \delta$ <u>such that a classical solution</u> φ <u>exists, on the interval</u> $[0, T]$, <u>for the problem</u> $\dot{x} = X(x + \mathcal{E}(t))$, $x(0) = x^0$ <u>and satisfies</u> $\|\varphi-\psi\| < \epsilon$.

<u>Proof</u>. Let $\dot{\psi}(t) = z(t) \in K\{X(\psi(t))\}$. By lemma 1 there exist k measurable functions $\omega^1,\ldots,\omega^k$ with $\omega^i(t) \in U(\psi(t), \delta/2)$ such that the functions v^i

defined by $v^i(t) = X(\omega^i(t))$ are measurable and $z(t)$ is contained in an $\epsilon/(T+1)$ neighborhood of $co\{v^1(t),\ldots,v^k(t)\}$.

By lemma 2, there exists a measurable function v with $v(t)\epsilon\{v^1(t), \ldots,v^k(t)\}$ such that $|\int_0^t [v(\tau) - x(\tau)]d\tau| < \epsilon$ uniformly for $t \epsilon [0, T]$. Define the measurable function ω by $\omega(t) = \omega^i(t)$ if t is such that $v(t)=v^i(t)$. Then ω is measurable, $\omega(t) \epsilon U(\psi(t), \delta/2)$ and $X(\omega(t)) = v(t)$.

We next produce the absolutely continuous function φ and measurable function $\mathcal{E}$ in the statement of the lemma.

Define $\varphi(t) = x^o + \int_0^t v(\tau)d\tau$. Then $|\varphi(t)-\psi(t)| = |\int_0^t [v(\tau)-z(\tau)]d\tau| < \epsilon$ for $t \epsilon [0, T]$ hence $\varphi(t) \epsilon U(\psi(t), \epsilon)$ and $\|\varphi-\psi\| \leqq \epsilon$. Define $\mathcal{E}(t) = \omega(t) - \varphi(t)$. Then $\mathcal{E}$ is certainly measurable and $|\mathcal{E}(t)| = |\omega(t)-\psi(t)+\psi(t)-\varphi(t)| \leqq$ $\leqq \delta/2 + \epsilon$. There is no loss in generality if it is assumed $\epsilon < \delta/2$. Therefore $\|\mathcal{E}\| < \delta$.

Also, $\varphi(t) + \mathcal{E}(t) = \omega(t)$ hence $X(\varphi(t) + \mathcal{E}(t)) = v(t)$ and from the definition of φ, $\varphi(t) = x^o + \int_0^t X(\varphi(\tau) + \mathcal{E}(\tau))d\tau$ showing that φ is a classical solution of $\dot{x} = X(x + \mathcal{E}(t))$, $x(0) = x^o$.

Proof of Theorem 1: We shall prove the contrapositive; i.e. if some classical solution exists and is not a Filippov solution then the field X is not stable with respect to measurement.

The assumption that some classical solution is not a Filippov solution implies there exists x^o and classical solution φ through x^o existing on some interval $[0, t_1]$ such that there is a Filippov solution ψ through x^o with $\varphi(T) - \psi(T) \neq 0$ for some $T \epsilon (0, t_1]$. Let $|\varphi(T) - \psi(T)| = r > 0$, pick $\epsilon = r/2$. Then by lemma 3, for $\|\mathcal{E}\|$ arbitrarily small, we can find a classical solution ξ of $\dot{x} = X(x + \mathcal{E}(t))$, $x(0) = x^o$ such that $|\xi(T) - \psi(T)| < \epsilon$ hence $|\varphi(T) - \xi(T)| > \epsilon$, i.e. X is not stable with respect to measurement.

We shall end by briefly summarizing some additional properties of vector fields which are stable with respect to measurement.

If X is stable with respect to measurement, the solutions of $\dot{x} = X(x)$ for $t \geqq 0$, are unique. This follows as an immediate consequence of the definition.

If X is stable with respect to measurement and $\varphi(t_1, x^o)$ denotes the solution through initial data x^o evaluated at time $t_1 > 0$, then $\varphi(t_1, \cdot)$ is continuous.

Indeed suppose $x^k \to x^o$ but $\varphi(t_1, x^k) \not\to \varphi(t_1, x^o)$. Then there exists a $\delta > 0$ such that $|\varphi(t_1, x^k) - \varphi(t_1, x^o)| \geqq \delta$ for all k sufficiently large. Let $\varepsilon^k(t) \equiv x^o - x^k$; i.e. a constant measurement error. For k sufficiently large, $\|\varepsilon\|$ can be made arbitrarily small.

Since $\dot{\varphi}(t, x^k) = X(\varphi(t, x^k))$, if we define $\xi^k(t) = \varphi(t,x^k)-x^o + x^k$ then $\dot{\xi}^k(t) = \dot{\varphi}(t, x^k)$ hence $\dot{\xi}^k(t) = X(\xi^k(t) + \varepsilon^k(t))$ and $\xi^k(0) = x^o$. From the definition of ξ^k , for k sufficiently large $\|\xi^k - \varphi(\cdot, x^k)\|$ can be made arbitrarily small; it follows that $\|\xi^k - \varphi(\cdot, x^o)\| > \delta/2$ for k sufficiently large, hence X is not stable with respect to measurement.

References

[1] Bridgland, T. F. Jr., "On the Existence of Optimal Feedback Controls", J. SIAM Control, Ser. A, Vol. 1, No. 3, (1963), 261-274.

[2] Datko, R., "An Implicit Function Theorem With an Application to Control Theory", Mich. Math. J., 11, (1964), 345-351.

[3] Filippov, A. F., "Differential Equations with Discontinuous Right-Hand Sides", (English Translation), Trans. Amer. Math. Soc., Ser. 2, Vol. 42, (1964), 199-231.

[4] Filippov, A. F., "Differential Equations With Many Valued Discontinuous Right-Hand Side",(English Translation), Sov. Math., Vol. 4, No. 4, (1963), 941-945.

[5] Filippov, A. F., "On Certain Questions in the Theory of Optimal Control", (English Translation), J. SIAM Control, Ser. A, Vol. 1, No. 1, (1963), 76-84.

[6] Hermes, H., "A Note on the Range of a Vector Measure; Application to the Theory of Optimal Control", J. Math. Anal. and Appl., Vol. 8, No. 1, (1964), 78-83.

Grumman Research Department Memorandum RM-316J

34929

N66 34929

ESTIMATION OF THE DOMAIN OF ATTRACTION†

by

Gunther R. Geiss

Systems Research

March 1966

†Presented at the Twenty Fourth Guidance and Space Flight Theory Seminar at NASA George C. Marshall Space Flight Center, Huntsville, Alabama, February 16-17, 1966. To appear in "Progress Report No. 8 - Studies in the Fields of Space Flight and Guidance Theory."

Approved by: Charles E. Mack, Jr.

Charles E. Mack, Jr.
Director of Research

ACKNOWLEDGMENT

The author is indebted to Mr. R. McGill for furnishing information about the algorithm being used herein and to Mr. G. Taylor for programming and conducting the numerical experiments.

ABSTRACT

A survey of various approaches to the problem of estimating the domain of attraction of an equilibrium solution to a system of nonlinear autonomous differential equations is given. Based upon observations resulting from this survey the problem is reformulated as that of optimally choosing the Liapunov function from the space of positive definite quadratic forms. An estimate of the domain of attraction is then obtained as the solution of a minimization problem. This approach to the problem has the advantages of: 1) being designed specifically for machine computation; 2) yielding an estimate that is readily visualized; and 3) being relatively insensitive to system dimension. Some preliminary numerical results are presented for the Duffing equation with damping.

TABLE OF CONTENTS

Introduction

This presentation is concerned with the problem of computing the restrictions on the initial state errors in a dynamic system which will guarantee that these state errors will tend to zero as $t \to \infty$. Thus, we shall be concerned with developing an efficient numerical technique for estimating the domain of asymptotic stability or more succinctly the domain of attraction of the null solution. This problem has application in qualitatively predicting the attitude motions of a satellite and perhaps may provide a first step toward solving the problem of qualitatively evaluating the effects of disturbances upon various rocket guidance schemes.

The applicability of this analysis to rocket guidance problems is crucially dependant upon the assumption that motions of the vehicle off the nominal path can be described by an autonomous state differential equation, viz.,

$$\dot{x} = g\big(x,u(x)\big) = h(x), \quad h(0) = 0 \tag{1}$$

where $x(t)$ is the n-vector describing the deviation from the nominal state, $u(x)$ represents the control law designed to control this deviation, and the null solution is an equilibrium solution. The domain of attraction Ω is then defined as the set of all initial points that generate trajectories that tend toward the equilibrium solution, i.e.,

$$\Omega : \left(x^o \,\middle|\, \lim_{t\to\infty} x(t;x^o,t_o) = 0\right) \tag{2}$$

The only body of theory that has been applied to the general problem of estimating the domain of attraction is Liapunov's direct method. Within this theory there are two distinct approaches that have been taken to determine the domain of attraction.

The first of these approaches, due to V. I. Zubov [1], allows an exact solution to the problem, if an arbitrary function can be chosen such that a closed form solution is obtained for

Zubov's partial differential equation, or it allows an estimate of the domain of attraction via a truncated power series solution to the equation. That is, if a positive definite $\theta(x)$ can be found such that the Zubov linear partial differential equations

$$\nabla_x v(x) \cdot h(x) = -\theta(x) \Big(1 - v(x)\Big) \tag{3}$$

or

$$\nabla_x v(x) \cdot h(x) = -\theta(x) \Big(1 - v(x)\Big) \Big(1 + h(x) \cdot h(x)\Big) \tag{4}$$

can be solved exactly for $v(x)$, then Ω is given by

$$\Omega : \Big(x \Big| \; 0 \leq v(x) < 1\Big) \quad . \tag{5}$$

If a power series solution is obtained in the form

$$v^n(x) = \sum_{i=2}^{n} v_i(x) \quad , \quad v_i(\alpha x) = \alpha^i v_i(x) \quad , \tag{6}$$

a series of homogeneous forms, then an estimate Ω^n of Ω is obtained via

$$\Omega^n : \Big(x \Big| \; 0 \leq v^n(x) < 1\Big) \quad , \tag{7}$$

and

$$\Omega^n \subset \Omega \quad . \tag{8}$$

In 1962 Margolis and Vogt [2] reported on a procedure which employs a digital computer to develop the series solution to Zubov's equation for a class of differential equations of dimension two. The authors noted two principal problems: 1) computational problems arise for systems of higher dimension; and 2) the convergence of the series solution is far from uniform, i.e., the estimate obtained for the Van der Pol equation by using only second degree terms in the series was better than the estimate obtained by including terms up to sixth degree.

During 1962 and 1963 Szego [3] reported on methods for solving Zubov's equation in vector-matrix form (these methods are related to his earlier work [4] on generating Liapunov functions via a "quadratic form" whose coefficients are functions of the state variables) and in [5] on a generalization of Zubov's equations. The latter was pursued somewhat further by Szego and Geiss [6]. Although some results regarding identification of limit cycles and estimation of domains of attraction are reported, no results regarding the conversion of these processes to numerical algorithms are given.

Rodden [7] and [8] reported in 1964 on an algorithm he developed for both calculating the solution to Zubov's equation in series form and analyzing the resulting Liapunov function. His work was restricted to problems of dimension two and three, and his results indicated three principal problems: 1) lack of uniform convergence of the series solution to Zubov's equation; 2) strong dependence of the final result upon the choice of the arbitrary or "constraint" function $\theta(x)$ in Zubov's equations; and 3) visualization of the estimate of the domain of attraction, particularly for three dimensional systems. Rodden found, in some examples, that the second degree approximation was better than the 20th, and that the convergence of this series solution could be improved by solving a modified Zubov equation. However, this change still did not guarantee that higher order approximations would be better than lower order approximations.

The second principal approach to estimating the domain of attraction is to base the analysis upon La Salle's theorems on the extent of asymptotic stability [9] and use one of the many procedures for developing Liapunov functions that are available in the literature [10] and [11]. This tack was reported on in 1962 by Infante [12] and Infante and Clark [13]. Infante developed an ingenious and successful procedure for developing Liapunov functions for two dimensional systems based upon an approximation to the dynamic system. Although estimates are easily obtained from his Liapunov functions, the technique for generating the functions does not appear to be suited to machine computation. Infante's work was developed in 1964 by Walker [14] for systems of higher dimension but again the technique is not suited to machine computation. The present author [15] reported in 1964 some favorable results obtained from a cursory look at the value of using "optimal" quadratic form Liapunov functions for estimating the domain of attraction. (This paper reports on an extension of this concept.) In 1965, Weissenberger [16] and [17], using the analysis algorithm developed by Rodden [7], developed a numerical technique for estimating the domain of attraction of relay control systems via an "optimal" choice from the class of Lure-Liapunov functions.

Thus, upon reviewing the history of this problem the following remarks become apparent:

1. The majority of techniques for generating Liapunov functions are unsuitable as bases for machine computation of Liapunov functions because of the requirement of experience and ingenuity in their application. Of those which are acceptable, i.e., series solution of Zubov equations, Lure-Liapunov formulation, and quadratic forms, the Zubov approach suffers from erratic convergence and lack of knowledge of how to choose the "constraint" function.

2. The method of analyzing the Liapunov function to determine an estimate of the domain of attraction should be relatively insensitive to system dimension. Rodden's technique depends on geometric analysis to determine points of tangency of hypersurfaces and hence is directly dependent on system dimension.

3. The estimate of the domain should be easy to visualize if it is to be of engineering value. A glance at the figures constructed by Rodden for three dimensional problems, and recognition of the fact that rocket guidance systems are of at least dimension four gives strong motivation to this statement.

4. Little attention has been given to selecting the "optimal" Liapunov function from a given class of functions; rather, the emphasis has been on new methods of generating Liapunov functions.

Based upon these remarks, the Liapunov function to be used in this analysis will be restricted to be a member of the class of positive definite quadratic forms. This restriction guarantees that the estimate of the domain of attraction will always be an ellipsoid and thus easier to visualize than the results of higher order estimates. Secondly, based upon the results of Margolis and Vogt [2], and Rodden [7], there is reason to believe that this estimate may be better than those obtained by using functions of higher degree, particulary if the quadratic form parameters are optimally chosen. Finally, information may be gained that will aid in formulating a best choice of the "constraint" function $\theta(x)$ for Zubov's equations.

Problem Formulation

Consider the basis of this analysis, i.e.,

Theorem (La Salle [15]):

Let $V(x)$ be a scalar function with continuous first partial derivatives. Let Ω_ℓ designate the region where $V(x) < \ell$. Assume that Ω_ℓ is bounded and that within Ω_ℓ:

$$V(x) > 0 \quad \text{for} \quad x \neq 0$$
$$\dot{V}(x) < 0 \quad \text{for} \quad x \neq 0 \ .$$

Then the origin is asymptotically stable, and above all, every solution in Ω_ℓ tends to the origin as $t \to \infty$.

Thus, Ω_ℓ is an estimate of Ω and the problem is reduced to choosing $V(x)$ from the class of quadratic forms and then establishing that the required properties exist in some domain. That is, the following must be accomplished:

1. Prove that $V(x)$ is positive in some region that includes the origin.

2. Prove that $\dot{V}(x) = \nabla V \cdot h(x)$ is negative in some region including the origin.

3. Establish a region Ω_ℓ within which both 1 and 2 hold.

4. Prove that Ω_ℓ is bounded.

Now since $V(x)$ is restricted to be a positive definite quadratic form, viz.,

$$V(x) = x^T Px \quad , \quad P > 0 \tag{9}$$

it is positive everywhere. Further, restrict the system equation (1) to have a stable linear part, i.e., let

$$\dot{x} = h(x) = Ax + f(x) \tag{10}$$

where A is a stable matrix (its eigenvalues all have negative real parts) and $f(x)$ contains no terms of first order in x. This is not an inordinate assumption since our present technology only allows synthesis based upon essentially linear analysis and thus a system with stable linear approximation usually results.

Based upon the assumption of equation (10) the calculation of $\dot{V}(x)$ results in

$$\dot{V}(x) = x^T(A^T P + PA)x + 2x^T Pf(x) \tag{11}$$

and choosing

$$- Q = A^T P + PA \tag{12}$$

results in

$$\dot{V}(x) = -x^T Qx + 2x^T Pf(x) \quad . \tag{13}$$

Thus, if Q is positive definite $\dot{V}(x)$ will be negative in a region including the origin by virtue of the fact that $f(x)$ contains no terms of first order in x. Now, since A is assumed stable we know that every positive definite Q will produce a positive definite P (Kalman and Bertram [18]) and thus requirements 1 and 2 are satisfied.

The establishment of Ω_ℓ is next on the agenda. By virtue of our restriction on $V(x)$ the set of hypersurfaces $V = c_i$, $c_1 < c_2 < c_3 < \dots < c_i < \dots < c_n$ will be a set of ellipsoids of fixed orientation and increasing size. Thus Ω_ℓ should be chosen to be the interior of the largest such ellipsoid within which $\dot{V} < 0$, and that ellipsoid will be the smallest one which has a point of contact with the hypersurface given by $\dot{V} = 0$, $x \neq 0$ (see Fig. 1).

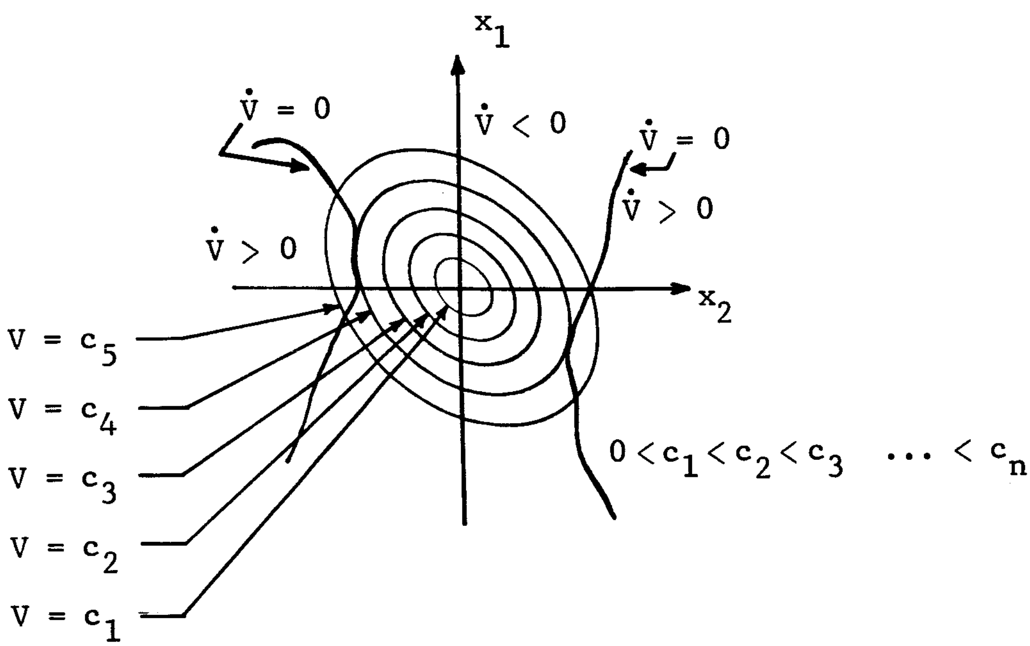


Fig. 1 Typical Relationship of the Loci $\dot{V} = 0$ and $V =$ constant

The problem is then one of calculating

$$\ell = \min_{x \neq 0} V(x) \quad \text{subject to} \quad \dot{V}(x) = 0 \quad . \tag{14}$$

It is exactly at this point that a computer is of most value. Also, note that since

$$\Omega_\ell : \left(x \mid V(x) < \ell\right) \tag{15}$$

is an ellipsoid it is bounded and requirement 4 is satisfied.

We have tacitly assumed above that there is a solution to the problem stated in (14), a sufficient condition for existence is obtained as follows. Consider that we must prove that

$$-x^T Qx + 2x^T P\, f(x) < 0 \tag{16}$$

in some domain D including the origin. Now note that

$$x^T Pf(x) \leq |x^T Pf(x)| \leq ||x^T P||\ ||f(x)|| \tag{17}$$

via the Schwartz inequality, and using the extremal properties of characteristic values of pencils of quadratic forms, Gantmacher [19],

$$||x^T P|| < \sqrt{\lambda^{max}(P^2)}\ ||x|| = \lambda^{max}(P)\ ||x|| \tag{18}$$

where $\lambda^{max}(P^2)$ is the maximal eigenvalue of P^2. Similarly,

$$x^T Qx \geq \lambda^{min}(Q)\ ||x||^2 \quad . \tag{19}$$

where $\lambda^{min}(Q)$ is the minimal eigenvalue of Q. Thus, (16) is satisfied if

$$||f(x)|| < \frac{\lambda^{\min}(Q)}{2\lambda^{\min}(P)} ||x|| \quad \text{in} \quad D \; . \tag{20}$$

Since Q and P are positive definite

$$\frac{\lambda^{\min}(Q)}{2\lambda^{\max}(P)} = K^2 > 0 \tag{21}$$

and thus (20) is a special case of a Lipschitz condition.

The results of this procedure are dependent upon the choice of Q and for each Q there will be a different Ω_ℓ , thus perhaps there is a best choice of Q for a particular criterion function. The most obvious criterion is the volume of Ω_ℓ and thus the last step in the analysis is to define

$$J(Q) = \ell^{n/2} \left(\prod_{i=1}^{n} \lambda_i(P) \right)^{-1/2} \tag{22}$$

and

$$J(Q^o) = \max_{Q>0} J(Q) \tag{23}$$

The computational procedure will then be as follows:

1. choose $Q > 0$
2. calculate P via (12)
3. compute ℓ via (14)
4. calculate $J(Q)$ via (22)
5. modify Q in direction of larger $J(Q)$
6. return to 2

Numerical Results

Consider the Duffing equation with damping, viz.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\epsilon_1 x_2 - x_1 + \epsilon_2 x_1^3 \end{aligned} \tag{24}$$

and the quadratic form Liapunov function

$$V = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \tag{25}$$

whose time derivative with respect to (24) is

$$\begin{aligned} \dot{V} = &-\left[2p_{12}x_1^2 + (2p_{22} + 2\epsilon_1 p_{12} - 2p_{11})x_1x_2 + (2\epsilon_1 p_{22} - 2p_{12})x_2^2\right] \\ &+\left[2\epsilon_2 p_{22}x_1^3x_2 + 2\epsilon_2 p_{12}x_1^4\right] \end{aligned} \tag{26}$$

Thus, the following relationships exist:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ -1 & -\epsilon_1 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ \epsilon_2 x_1^3 \end{pmatrix} \\ P &= \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \\ Q &= \begin{pmatrix} 2p_{12} & p_{22} + \epsilon_1 p_{12} - p_{11} \\ p_{22} + \epsilon_1 p_{12} - p_{11} & 2\epsilon_1 p_{22} - 2p_{12} \end{pmatrix} \end{aligned} \tag{27}$$

Application of the Sylvester criterion for positive definiteness of Q and P yields the parameter restrictions (for $\epsilon_1 = 1$):

$$\left(\frac{p_{11}}{p_{12}} - \frac{p_{22}}{p_{12}}\right)^2 < 2\left(\frac{p_{11}}{p_{12}} + \frac{p_{22}}{p_{12}}\right) - 5 \quad , \quad p_{12} > 0 \tag{28}$$

$$\begin{gathered} \frac{p_{11}}{p_{12}} \; \frac{p_{22}}{p_{12}} > 1 \\ \\ p_{11} > 0 \quad , \quad p_{22} > 0 \quad . \end{gathered} \tag{29}$$

This system has three equilibrium points, viz.,

$$(x_1, x_2) = \begin{cases} (0,0) \\ (\sqrt{\epsilon_2}, 0) \\ (-\sqrt{\epsilon_2}, 0) \end{cases} \tag{30}$$

the first being stable and the others unstable. Thus, one would not expect the domain of attraction to exceed $|x_1| = \epsilon_2^{-1/2}$ and it is reasonable to inquire whether (20) is satisfied in D, where

$$D : (x| \; ||x||^2 < \epsilon_2^{-1}) \tag{31}$$

Hence, the question is what is K^2 such that

$$\frac{||f(x)||}{||x||} = \frac{\epsilon_2 x_1^3}{\sqrt{x_1^2 + x_2^2}} < K^2 \quad \text{in} \quad D \tag{32}$$

and the solution is

$$K^2 > \sup_{x\epsilon D} \frac{\epsilon_2 x_1^3}{\sqrt{x_1^2 + x_2^2}} = 1 \tag{33}$$

Thus, the estimate of the domain of attraction will be larger than the circle $||x||^2 = \epsilon_2^{-1}$ if

$$K^2 = \frac{\lambda^{min}(Q)}{2\lambda^{max}(P)} > 1 \tag{34}$$

or if

$$\frac{\alpha + \beta + \sqrt{(\alpha + \beta)^2 + 4(1 - \alpha\beta)}}{\beta - \sqrt{(\beta - 2)^2 + (\beta - \alpha + 1)^2}} > 4 \tag{35}$$

where $\alpha = p_{11}(p_{12})^{-1}$ and $\beta = p_{22}(p_{12})^{-1}$. The parameter restrictions (28), (29) and (35) are illustrated in Fig. 2.

The allowable choice of parameters is as given in (28), (29) and (35) or Fig. 2 and the analysis in [15] has shown that the optimal choice, using the area of Ω_ℓ as criterion, to be

$$\frac{p_{11}}{p_{12}} = 2 \quad , \quad \frac{p_{22}}{p_{12}} = 1 \tag{36}$$

which is exactly on the boundary of the allowable region. Note that the resulting $\dot{V}$ is semidefinite, i.e., $\dot{V} = 0$ on $x_1 = 0$, and one must use another form of the stated theorem. See [15], or [9], p. 66. The corresponding Ω_ℓ^o (for $\epsilon_2 = 0.04$) is shown in Fig. 3 along with the estimate obtained by using the energy of the undamped system as the Liapunov function. The

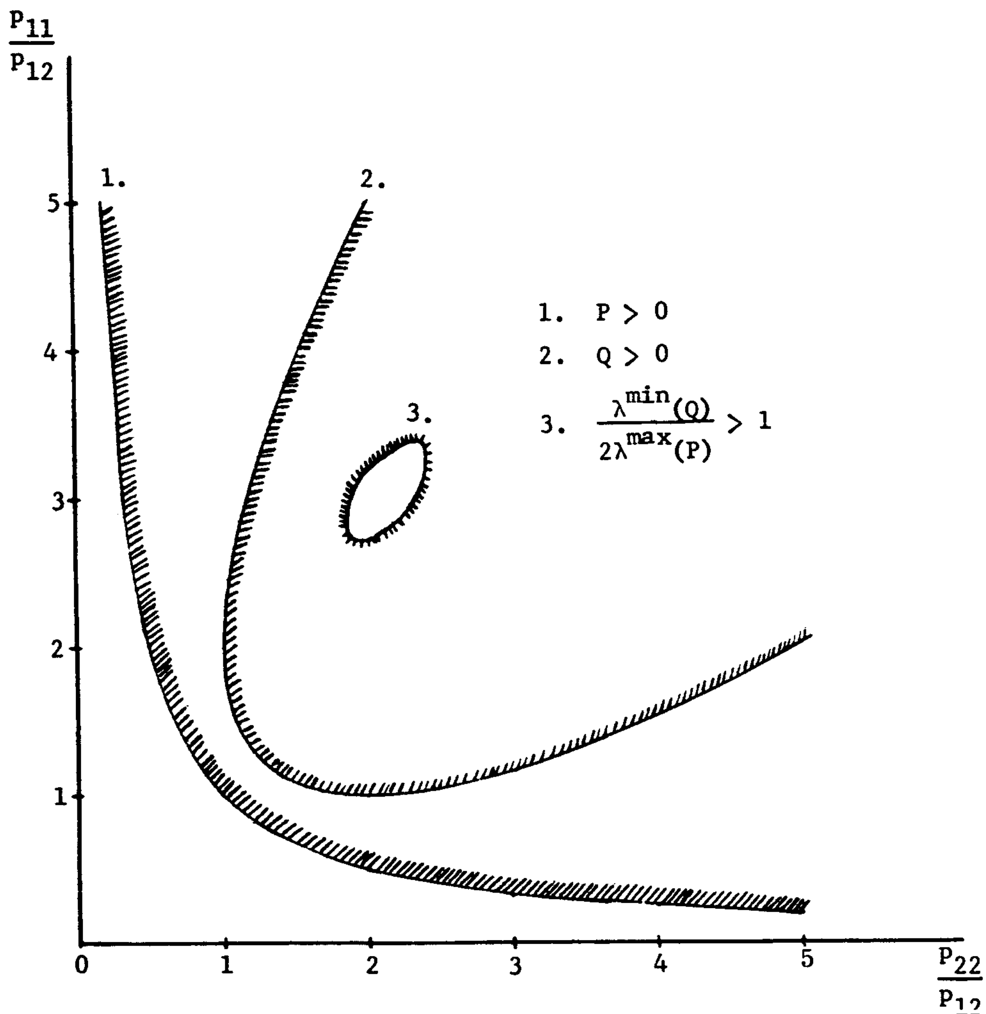


Fig. 2 Restrictions Upon Liapunov Function Parameters

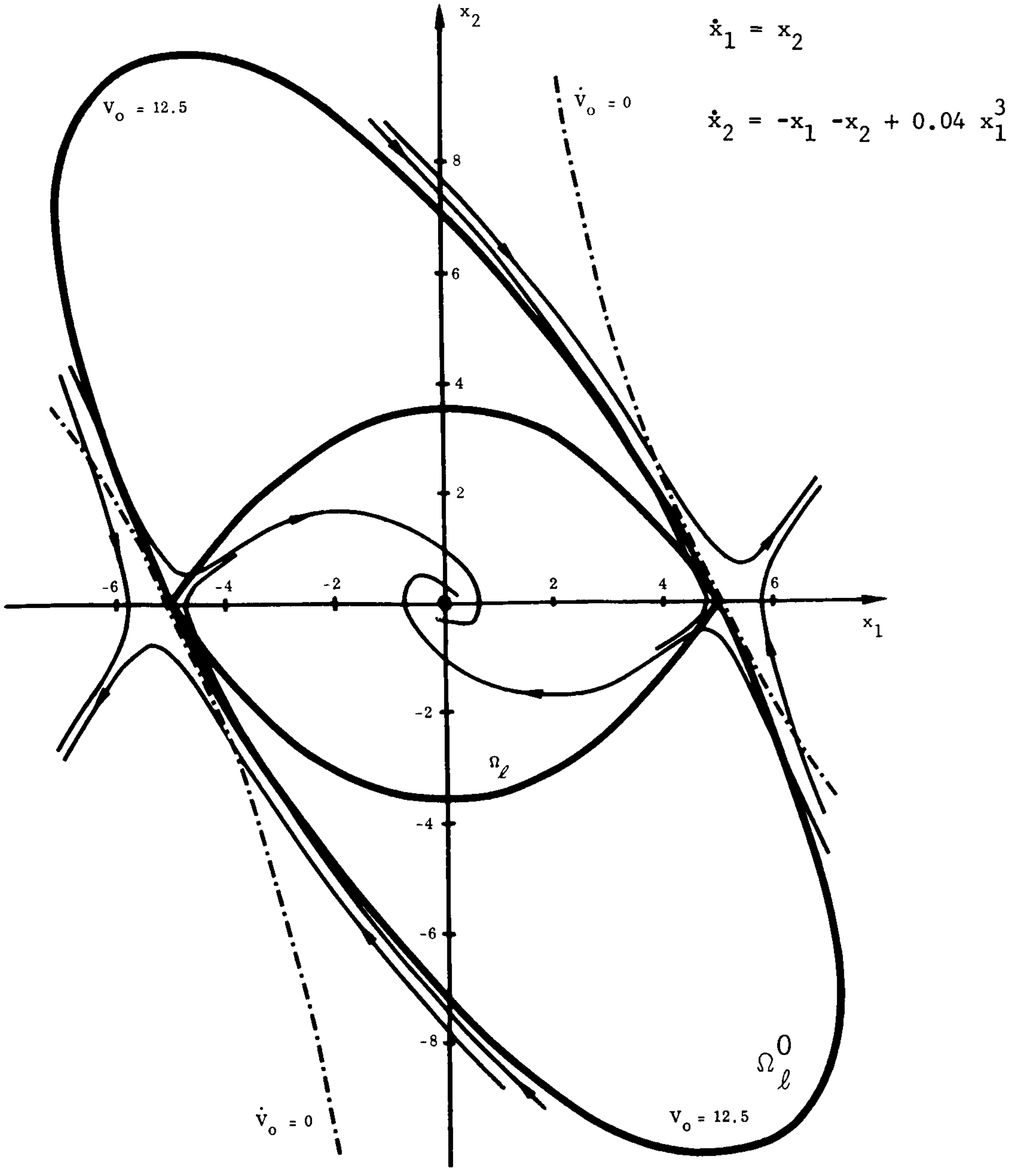


Fig. 3 Comparison of Best Estimate, Ω_ℓ^0 , with Energy Function Estimate, Ω_ℓ , and System Trajectories

value of ℓ is 12.5 when $p_{12} = 4$ and thus the estimate is

$$\Omega_{\ell}^{o} : \frac{1}{2} x_1^2 + \frac{1}{2} x_1 x_2 + \frac{1}{4} x_2^2 < 12.5 \tag{37}$$

The points of contact with $\dot{V} = 0$ are at the equilibrium points $(x_1, x_2) = (\pm 5, 0)$ and the estimate is seen to be close to the actual seperatrix and considerably larger than the estimate

$$\Omega_{\ell} : \begin{array}{c} \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - 0.01 x_1^4 < 6.25 \\ x_1^2 < 25 \end{array} \tag{38}$$

obtained by using the energy of the undamped system as the Liapunov function.

In Fig. 4 the optimal quadratic form estimate, Ω_3, is compared with the estimates obtained by Infante [12] for the system with $\epsilon_2 = 1$. Ingwerson's procedure [20] for generating Liapunov functions yields the following estimates:

$$\Omega_1 : x_1^2 - \frac{x_1^4}{4} + x_1 x_2 + \frac{x_2^2}{2} < \frac{1}{4} \tag{39}$$

while Infante's procedure yields

$$\Omega_2 : \begin{array}{c} x_1^2 \left[2 - \frac{x_1^2}{2} \right] + 2x_1 x_2 + x_2^2 < \frac{3}{2} \\ |x_1| < 1 \end{array} \tag{40}$$

Thus for this example it seems that the optimal quadratic form technique yields an improvement in accuracy, ease of solution, and ease of portrayal of the estimate of the domain of attraction.

The numerical solution of the constrained minimum problem (14) is obtained by solving an unconstrained problem, viz.,

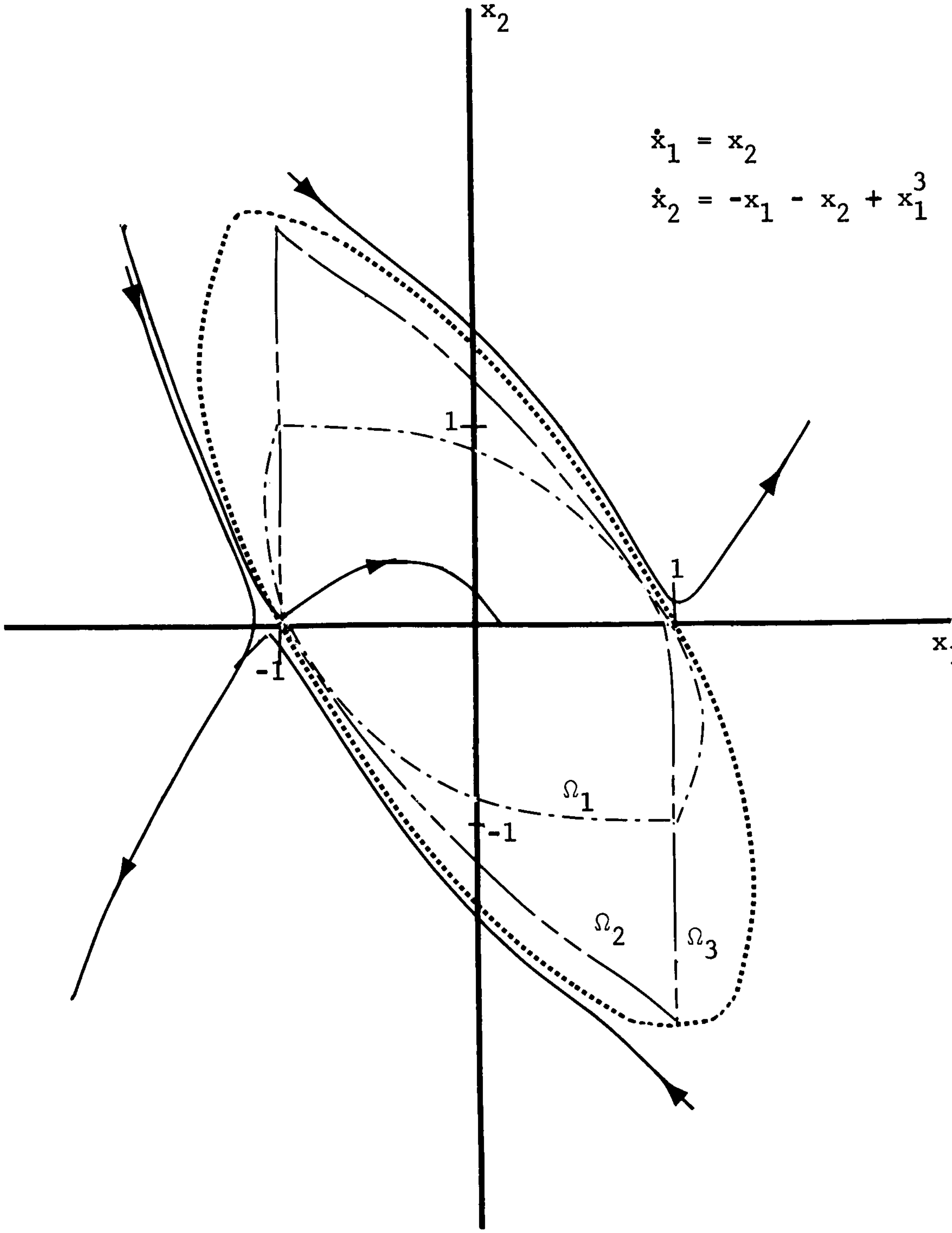


Fig. 4 Comparison of Best Estimate, Ω_3 , with Estimates Obtained by Infante, and System Trajectories

$$\ell = \min_{x} \left[V(x) + \frac{K_1^2\left(\dot{V}(x)\right)^2}{||x||^2} \right] , \tag{41}$$

where $||x||^2$ was introduced to avoid the trivial solution and K_1^2 is manipulated to assure satisfaction, to a prescribed accuracy, of the constraint $\dot{V}(x) = 0$.

At present, we are using a new algorithm for finding the minimum of a very general class of functionals to solve (41). (Eventually it will also be used to solve [23].) This algorithm is being developed at Grumman by Mr. R. McGill based on work by Davidon [21]. The algorithm being developed by McGill has the following salient characteristics:

1. It does not require numerical inversion of linear operators and thus is relatively free of dimensional limitations.

2. It is stable with respect to convergence, i.e., convergence to a local minimum is guaranteed.

3. It is efficient, i.e., convergence is quadratic in a neighborhood of a minimum.

4. It allows a tradeoff between precision and computing time.

5. It requires modest storage.

Typical computational results for $\epsilon_1 = 1$, $\epsilon_2 = 0.04$ are presented in Figs. 5 through 12. These figures show the boundary of the estimate Ω_ℓ and its relationship to the constraint $\dot{V} = 0$, and the area contained with Ω_ℓ. The estimate Ω_ℓ is the elliptical region surrounding the origin. The aberrations from ellipticity are due to the plotting machine routine being used and are not part of Ω_ℓ. The loci $\dot{V} = 0$ are those with the triangle and square markings. These markings are used to distinguish the branches corresponding to the positive and negative roots of the quadratic equation (in x_2) used to generate the loci $\dot{V} = 0$. These markings along the x_1 axis indicate that the roots are complex for those values of x_1 and are not part of the loci $\dot{V} = 0$. Note that when $p_{11} = 4.0$, $p_{12} = 0.5$, $p_{22} = 2.0$ $\dot{V} = 0$ appears to have a cusp at its point of contact with $V = \ell$. (This situation would be difficult to handle using a geometric approach such as Rodden's). The best computed result is about 10% off the optimal $J(Q^o) = 50\pi \doteq 157$. Some convergence difficulties have been observed as the boundary 2 of Fig. 2 is approached. This phenomenon has not yet been investigated.

Conclusions

Results obtained by investigators who pursued estimation of the domain of attraction via Zubov's technique, and the preliminary results presented here indicate that estimation of the domain of attraction for quasi linear dynamic systems via "optimal" quadratic form Liapunov functions, as formulated here, is feasible. This procedure, in conjunction with McGill's algorithm, offers the advantages of: 1) readily leading to an efficient algorithm for estimating the domain of attraction which is relatively insensitive to system dimension; and 2) providing an estimate which is easy to visualize, i.e., an ellipsoid.

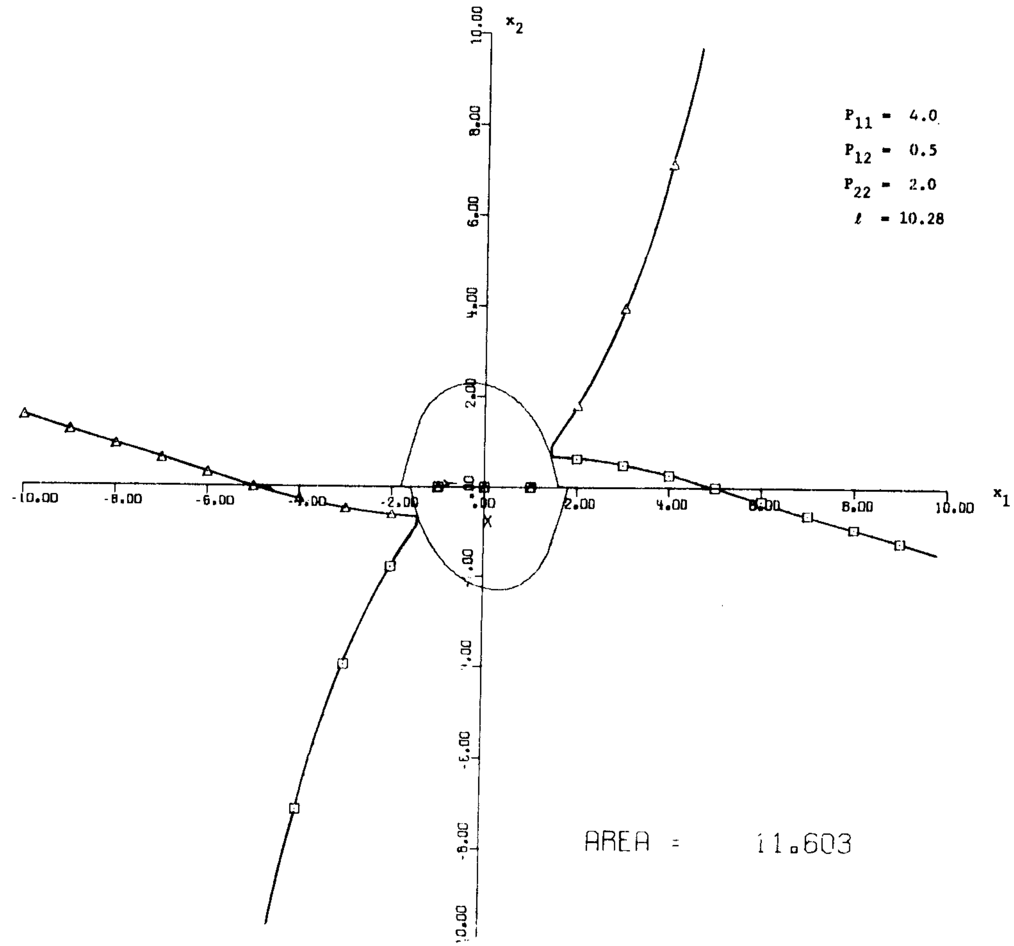


Fig. 5 Computer Solution to Unconstrained Minimum Problem

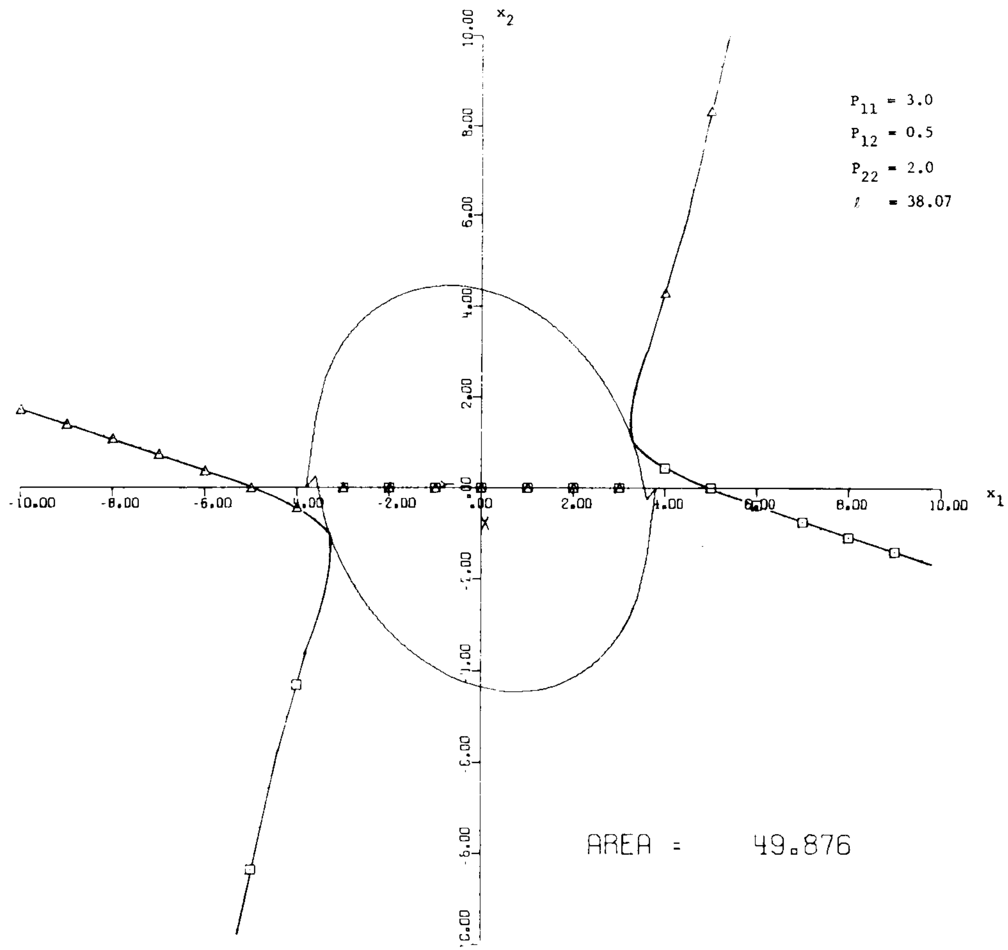


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

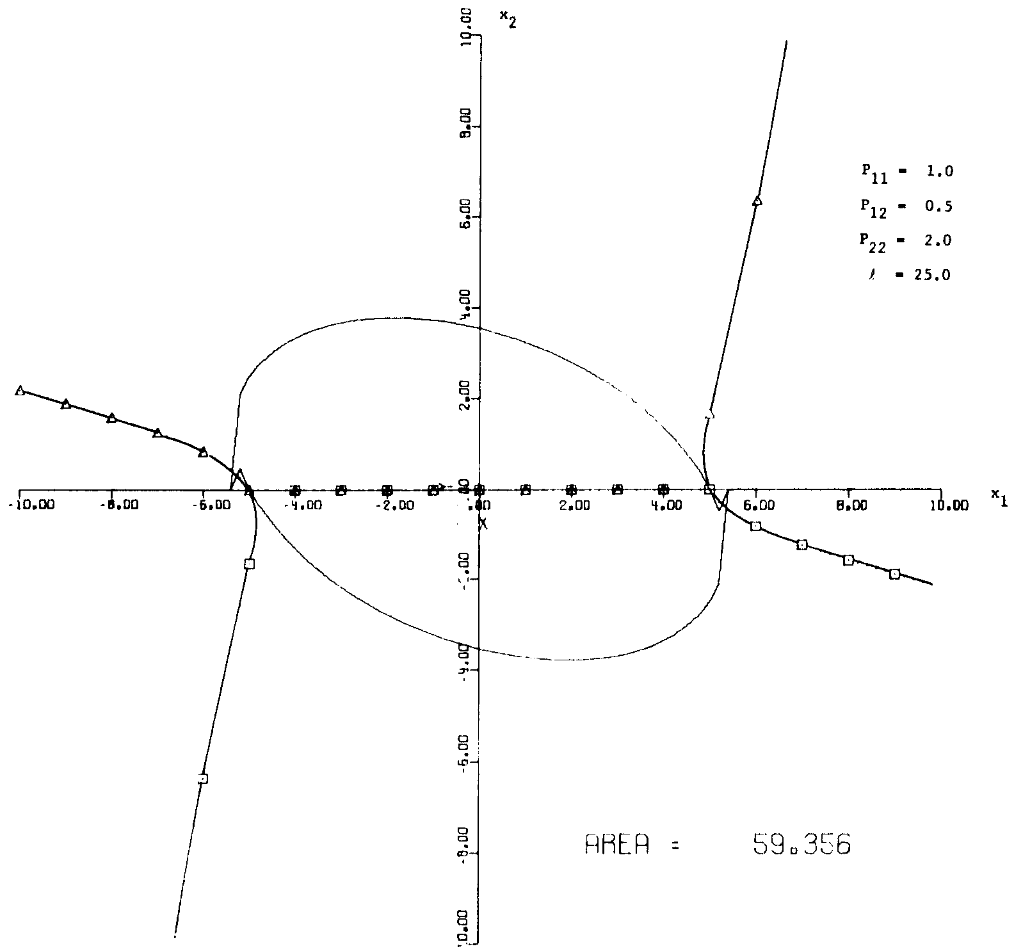


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

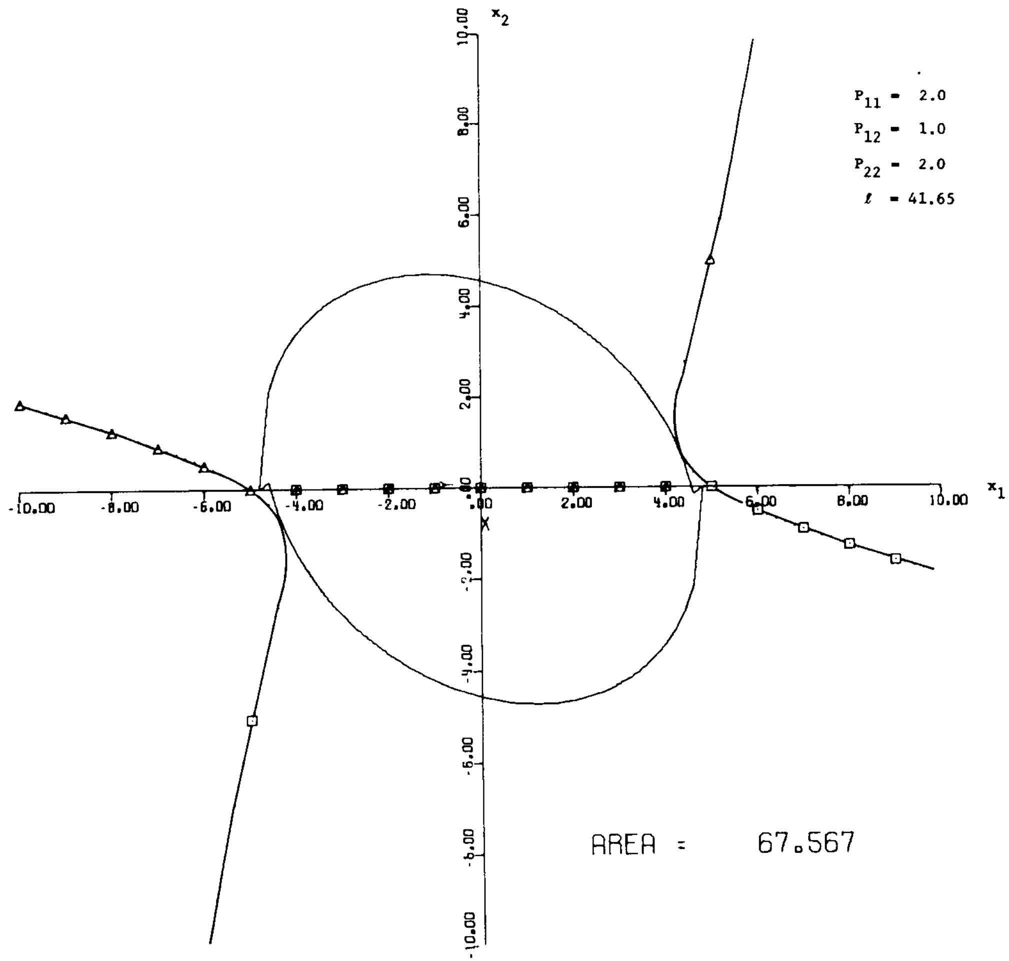


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

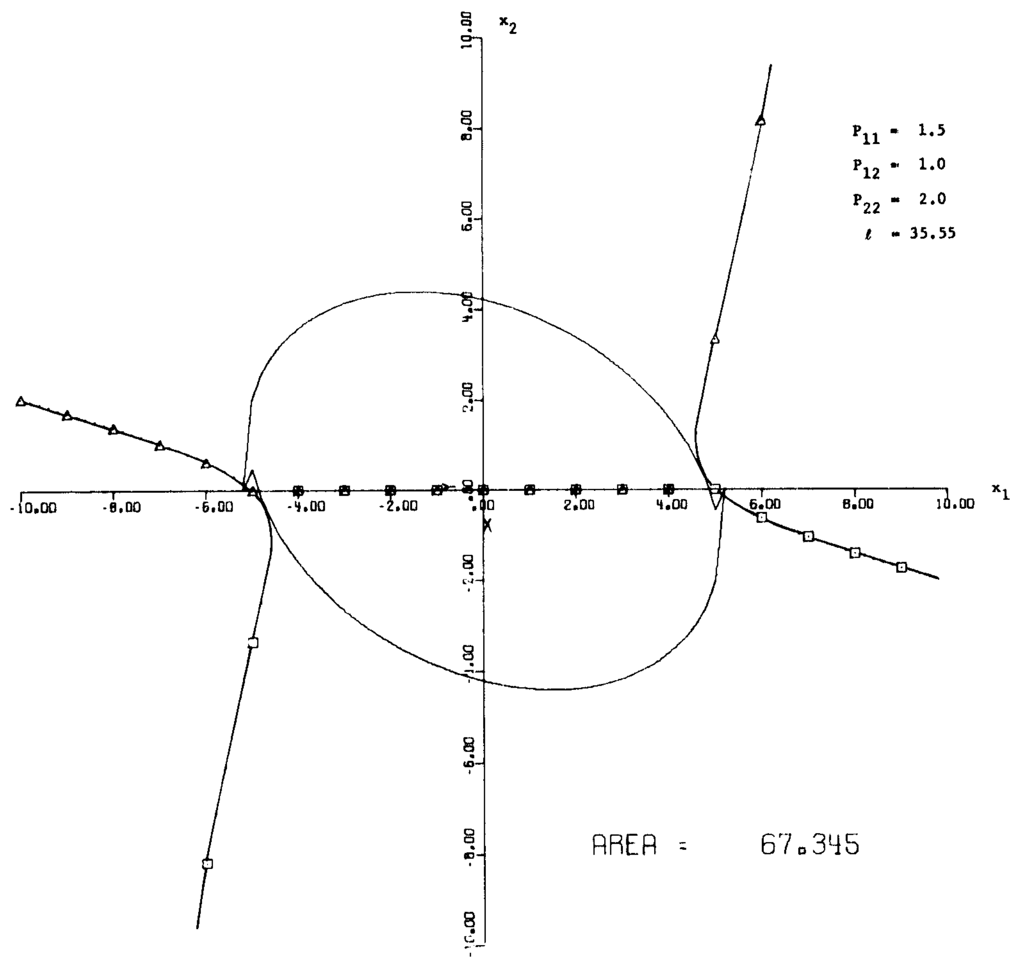


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

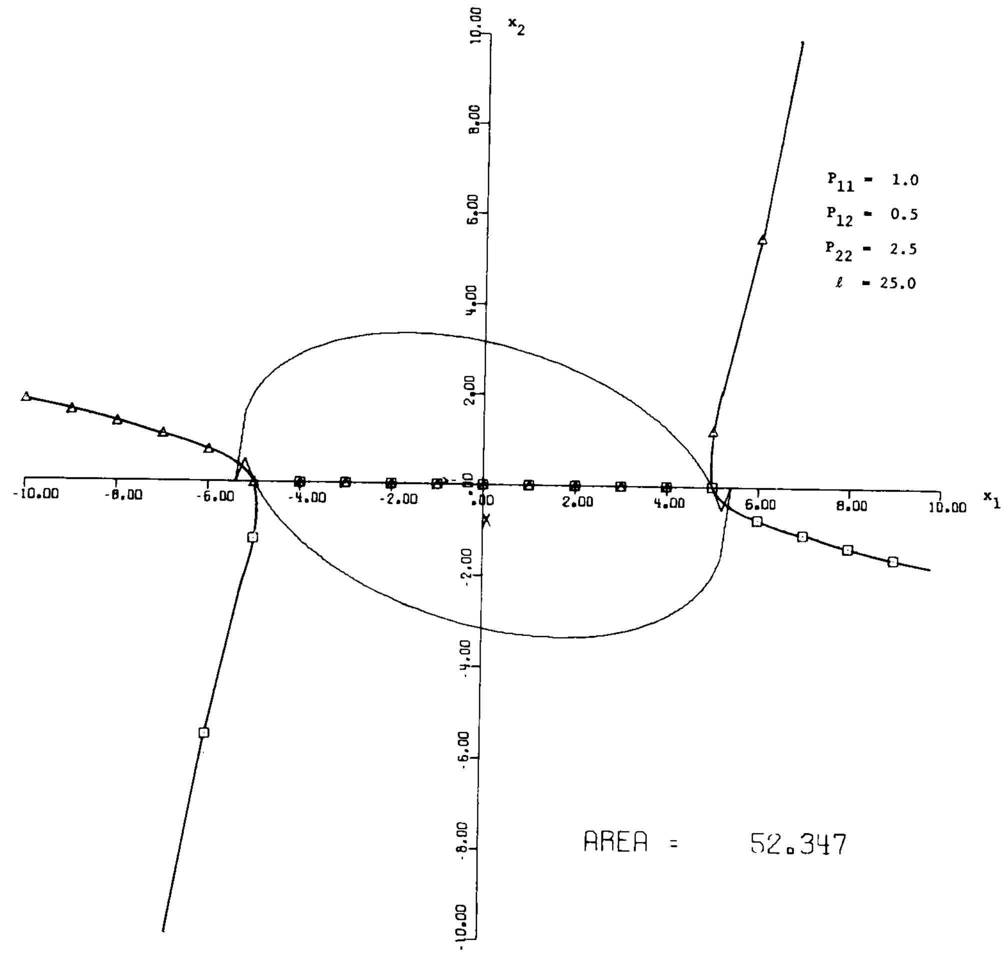


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

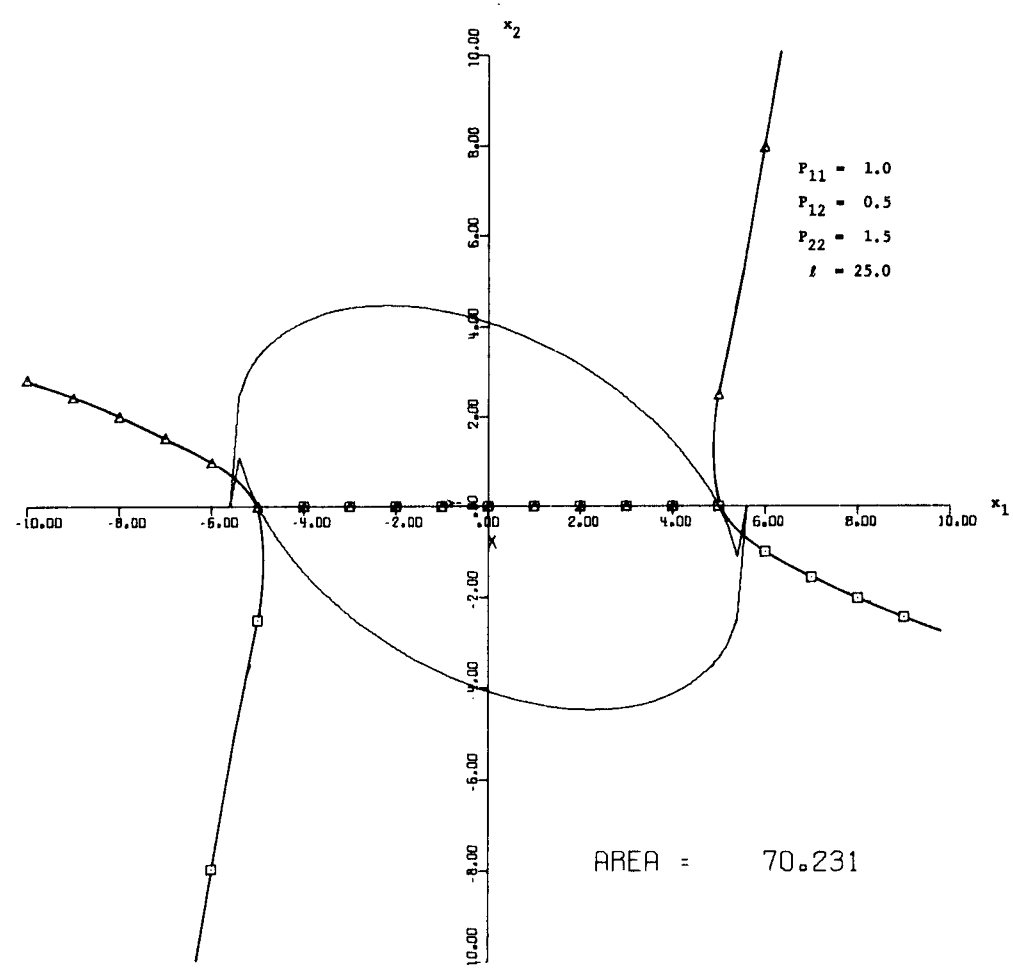


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

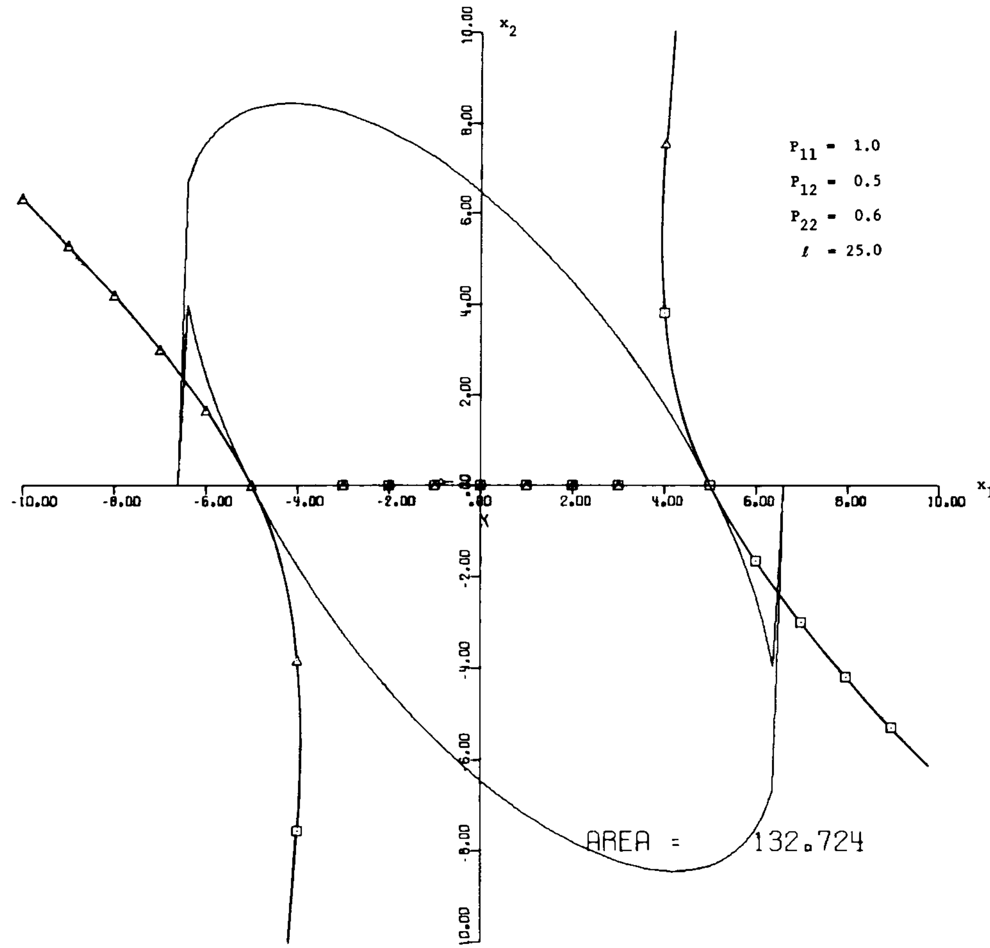


Fig. 5 (Cont.) Computer Solution to Unconstrained Minimum Problem

References

1. Zubov, V.I., "Questions in the Theory of Liapunov's Second Method; the Construction of the General Solution in the Domain of Asymptotic Stability," PMM, Vol. 19, 1955; Translation No. 389, Military Products Group Research Department, Minneapolis-Honeywell Regulator Company, August 1962.

2. Margolis, S.G. and Vogt, W.G., "Control Engineering Applications of V.I. Zubov's Construction Procedure for Liapunov Functions," IEEE Trans. on Automatic Control, Vol. AC-8, No. 2, April 1963, pp. 104-113; also Proc. of 1962 Joint Automatic Control Conference, New York Univ., N.Y.

3. Szego, G.P., "On the Application of Zubov's Method of Constructing Liapunov Functions for Nonlinear Control Systems," Trans. of the ASME, Journal of Basic Engineering, Vol. 85, Ser. D, No. 2, June 1963.

4. Szego, G.P., "A Contribution to Liapunov's Second Method: Nonlinear Autonomous Systems," Trans. of the ASME, Journal of Basic Engineering, Vol. 84, Ser. D, No. 4 , December 1962.

5. Szego, G.P., "On a New Partial Differential Equation for the Stability Analysis of Time Invariant Control Systems," Journal of SIAM on Control, Ser. A, Vol. 1, No. 1, 1962.

6. Szego, G.P. and Geiss, G.R., "A Remark on 'A New Partial Differential Equation for the Stability Analysis of Time Invariant Control Systems'," Journal of SIAM on Control, Ser. A, Vol. 1, No. 3, 1963.

7. Rodden, J.J., Applications of Liapunov Stability Theory, Ph.D. dissertation, Stanford Univ., 1964; also Lockheed Missile and Space Co., LMSC/A603476, April 1964.

8. Rodden, J.J., "Numerical Applications of Liapunov Stability Theory," Proc. of 1964 Joint Automatic Control Conference, Stanford Univ., Stanford, Calif.

9. La Salle, J.P., and Lefschetz, S., Stability by Liapunov's Direct Method with Applications, Academic Press, New York, 1961.

10. Geiss, G.R., The Analysis and Design of Nonlinear Control Systems Via Liapunov's Direct Method, RTD-TDR-63-4076, Air Force Flight Dynamics Laboratory, WPAFB, August 1964.

11. Drake, R.L., Methods for Systematic Generation of Liapunov Functions, Parts One and Two, Drexel Institute of Technology Project No. 243, NASA Marshall Space Flight Center, Contract NAS 8-11196.

12. Infante, E.F., A New Approach to the Determination of the Domain of Stability of Nonlinear Autonomous Second Order Systems, Ph.D. dissertation, University of Texas, 1962; University Microfilms 63-1669, Ann Arbor, Michigan.

13. Infante, E.F and Clark, L.G., "A Method for the Determination of the Domain of Stability of Second Order Nonlinear Autonomous Systems," Trans. of the ASME, Journal of Applied Mechanics, June 1964.

14. Walker, J.A., An Integral Method of Liapunov Function Generation for Nonlinear Autonomous Systems, Ph.D. dissertation, University of Texas, 1964; University Microfilms 64-11,853, Ann Arbor, Michigan.

15. Geiss, G.R., The Analysis and Design of Nonlinear Control Systems Via Liapunov's Direct Method, Ph.D. dissertation, Polytechnic Institute of Brooklyn, 1964.

16. Weissenberger, S., Stability Analysis of Relay Control Systems Via the Direct Method of Liapunov, Ph.D. dissertation, Stanford Univ., 1964, NASA Contractor Report CR-320, October 1965.

17. Weissenberger, S., "Stability-Boundary Approximation for Relay-Control Systems Via Steepest-Ascent Construction of Liapunov Functions," Proc. of 1965 Joint Automatic Control Conference, Rensselaer Polytechnic Institute, Troy, N.Y.

18. Kalman, R.E. and Bertram, J.E., "Control System Analysis and Design Via the 'Second Method' of Liapunov," ASME Journal of Basic Engineering, June 1960, pp. 371-400.

19. Gantmacher, F.R., The Theory of Matrices, Vol. 1, Chelsea Publishing Co., New York, 1959, p. 317ff.

20. Ingwerson, D.R., "A Modified Liapunov Method for Nonlinear Stability Analysis," IRE Trans. on Automatic Control, Vol. AC-6, No. 2, May 1961.

21. Davidon, W.C., Variable Metric Method for Minimization, A.E.C. Research and Development Report, ANL-5990 (Rev.), 1959.

A TRANSFORMATION TECHNIQUE TO OBTAIN CONTROL ANGLE SOLUTIONS IN CALCULUS OF VARIATIONS OPTIMIZATION PROBLEMS

by

Robert W. Hunt
Associate Professor of Mathematics
Southern Illinois University

In the study of trajectory optimization by classical calculus of variations techniques, one normally encounters certain Euler-Lagrange equations involving the control angles used in the formulation of the problem. In some cases, these equations lead to a solution for the angles (or, usually, the tangents of the angles) in terms of the Lagrange multipliers; and these solutions can be used to eliminate the control angles from the Euler-Lagrange equations and thus leave a system of differential equations in the state variables and the Lagrange multipliers. This is generally desirable. However, it seems that the process is more readily carried out in some coordinate systems than others and, in fact, virtually impossible in some systems. Thus, if one happens to be using a coordinate system in which the latter is true, it would be convenient to transform to another coordinate system in which the problem was not present, find the desired solutions, and then transform back to the original system. This involves transforming the state variables and their corresponding Lagrange multipliers from one system to another. In this discussion, the technique for a general transformation of this type is given and then applied to a specific problem involving three-dimensional trajectory optimization in a plumb-line coordinate system (with state variables x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, and control angles χ_{pitch} and χ_{yaw})[1] and in a spherical coordinate

[1] W. E. Miner, "Methods for Trajectory Computation", NASA-MSFC, Aeroballistics Internal Note No. 3-61, May 10, 1961

system (with state variables r, ϕ, θ, v, γ, δ and control angles α and β)[2]. In the former system, tan χ_{pitch} and tan χ_{yaw} are readily solved for whereas the same is not true in the latter system. However, the desired result is obtained for the latter system by application of the method just outlined.

Consider the equations of motion which simulate vehicle flight in three dimensions, through a vacuum, for a non-rotating spherical reference body. Thrust and weight flow are assumed constant and thrust and gravity are the only two forces acting on the vehicle. These equations, in flight path coordinates, are:

[2] D. H. Young, "Three Dimensional Vacuum Trajectory Optimization with End Points in Flight Path Coordinates", Douglas Aircraft Company Space and Missile Systems Division Memorandum

Symbols from the above are as follows:

v: total missile velocity, directed along the flight path

γ: vehicle elevation flight path angle (angle between the projection of the velocity vector on the local tangent plane and the velocity vector)

δ: vehicle azimuth flight path angle (angle between north and the projection of the velocity vector on the local tangent plane- positive, clockwise from north)

α: in-plane angle of attack (angle between the velocity vector and the projection of the thrust vector in the v-n plane- positive, counterclockwise from the velocity vector)

β: out-of-plane angle of attack (angle between the velocity vector and the projection of the thrust vector in the v-s plane- positive, clockwise from the velocity vector)

$$\dot{v} = \frac{T}{M(\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} - g \sin \gamma$$

$$\dot{\gamma} = \frac{T \tan \alpha}{Mv(\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} + \left(\frac{v}{r} - \frac{g}{v}\right) \cos \gamma$$

$$\dot{\delta} = \frac{T \tan \beta}{Mv \cos \gamma (\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} + \frac{v \cos \gamma \sin \delta}{r \cot \phi}$$

$$\dot{r} = v \sin \gamma$$

$$\dot{\phi} = (v \cos \gamma \cos \delta)/r$$

$$\dot{\theta} = (-v \cos \gamma \sin \delta)/r \cos \phi$$

where r, ϕ, θ are the usual spherical coordinates and v, γ, δ, α, and β are as previously defined.

In order to use classical calculus of variations techniques, form:

$$L = \lambda_v \left[\frac{T}{M(\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} - g \sin \gamma\right] +$$

$$\lambda_\gamma \left[\frac{T \tan \alpha}{Mv(\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} + \left(\frac{v}{r} - \frac{g}{v}\right) \cos \gamma\right] +$$

$$\lambda_\delta \left[\frac{T \tan \beta}{Mv \cos \gamma (\pm\sqrt{\tan^2 \alpha + \tan^2 \beta + 1})} + \frac{v \cos \gamma \sin \delta}{r \cot \phi}\right] +$$

λ_r [v sin γ] + λ_ϕ [(v cos γ cos δ)/r] +

λ_θ [(-v cos γ sin δ)/r cos ϕ]

It is then necessary that an extremizing trajectory satisfy

$$\dot{\lambda}_v = \partial L / \partial v$$

and similarly for γ , δ ,r, ϕ and θ ; and also,

$$\partial L/_{\partial \alpha} = 0 \ , \qquad \partial L/_{\partial \beta} = 0$$

along with certain end conditions and transversality conditions which are not pertinent to this discussion. The latter two equations yield

$$\tan \alpha = \frac{(\tan^2 \beta + 1)\,\lambda\gamma}{v\,\lambda v + \lambda\delta\ (\tan \beta / \cos \gamma)}$$

$$\tan \beta = \frac{(\tan^2 \alpha + 1)\lambda_\delta}{(\lambda\gamma \ \tan \alpha + v\,\lambda v) \cos \gamma}$$

It would now be desirable to solve these equations for α and β so as to eliminate α and β from the equations of motion and the Euler-Lagrange equations. However, this is precisely the situation mentioned earlier; namely, the desired solutions cannot be readily obtained.

Now, the same original problem can be considered in a plumbline coordinate system with state variables x, y, z, $\dot{x} = l$, $\dot{y} = m$, $\dot{z} = n$. The relations among the various state variables in this and the previous system are given by:

$$x = r \cos\phi \cos\theta$$

$$y = r \cos\phi \sin\theta$$

$$z = r \sin\phi$$

$$l = v[\sin\gamma \cos\phi \cos\theta - \sin\phi \cos\theta \cos\gamma \cos\delta + \sin\theta \cos\gamma \sin\delta]$$

$$m = v[\sin\gamma \cos\phi \sin\theta - \cos\theta \cos\gamma \sin\delta - \sin\phi \sin\theta \cos\gamma \cos\delta]$$

$$n = v[\sin\gamma \sin\phi + \cos\phi \cos\gamma \cos\delta]$$

and, inversely,

$$v = (l^2 + m^2 + n^2)^{1/2}$$

$$\gamma = \sin^{-1}\left[\frac{xl + ym + zn}{(x^2 + y^2 + z^2)^{1/2}(l^2 + m^2 + n^2)^{1/2}}\right]$$

$$\delta = \cos^{-1}\left[\frac{(x^2+y^2+z^2)n - z(xl+ym+zn)}{(x^2+y^2)^{1/2}((xm-yl)^2+(xn-zl)^2+(yn-zm)^2)^{1/2}}\right]$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$\phi = \sin^{-1}\left[\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$\theta = \tan^{-1}[y/x]$$

The equations of motion in the present coordinate system are:

$$\dot{l} = \frac{-F}{M \sin X_p \cos X_y} - \frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\dot{m} = \frac{F}{M \cos X_p \cos X_y} - \frac{\mu y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\dot{n} = \frac{F}{M \sin X_y} - \frac{\mu z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\dot{x} = l$$

$$\dot{y} = m$$

$$\dot{z} = n.$$

Then, for

$$\tilde{L} = \sigma_1 \left[\frac{-F}{M \sin X_p \cos X_y} - \frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

$$+ \sigma_m \left[\frac{F}{M \cos X_p \cos X_y} - \frac{\mu y}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

$$+ \sigma_n \left[\frac{F}{M \sin X_y} - \frac{\mu z}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

$$+ \sigma_x l + \sigma_y m + \sigma_z n$$

the Euler-Lagrange equations are:

$$\dot{\sigma}_l = -\sigma_x$$

$$\dot{\sigma}_m = -\sigma_y$$

$$\dot{\sigma}_n = -\sigma_z$$

$$\dot{\sigma}_x = \frac{\mu\sigma_l}{(x^2+y^2+z^2)^{3/2}} - \frac{3\mu x(x\sigma_l+y\sigma_m+z\sigma_n)}{(x^2+y^2+z^2)^{5/2}}$$

$$\dot{\sigma}_y = \frac{\mu\sigma_m}{(x^2+y^2+z^2)^{3/2}} - \frac{3\mu y(x\sigma_l+y\sigma_m+z\sigma_n)}{(x^2+y^2+z^2)^{5/2}}$$

$$\dot{\sigma}_z = \frac{\mu\sigma_n}{(x^2+y^2+z^2)^{3/2}} - \frac{3\mu z(x\sigma_l+y\sigma_m+z\sigma_n)}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial \tilde{L}}{\partial X_p} = \sigma_l \cos X_p \cos X_y + \sigma_m \sin X_p \cos X_y = 0$$

$$\frac{\partial \tilde{L}}{\partial X_y} = \sigma_l \sin X_p \sin X_y - \sigma_m \cos X_p \sin X_y + \sigma_n \cos X_y = 0$$

along with the equations of motion. The latter two equations above are analogous to those obtained for the other coordinate system from $\partial L/\partial\alpha = 0$ and $\partial L/\partial\beta = 0$. They give:

$$\tan X_p = -\sigma_l/\sigma_m \quad , \qquad \tan X_y = \sigma_n/(\sigma_l^2 + \sigma_m^2)^{1/2}$$

Substituting into the equations of motion gives:

$$\dot{l} = \frac{F\,\sigma_l}{M(\sigma_l^2 + \sigma_m^2 + \sigma_n^2)^{1/2}} - \frac{\mu x}{(x^2+y^2+z^2)^{3/2}}$$

$$\dot{m} = \frac{F\,\sigma_m}{M(\sigma_l^2 + \sigma_m^2 + \sigma_n^2)^{1/2}} - \frac{\mu y}{(x^2+y^2+z^2)^{3/2}}$$

$$\dot{n} = \frac{F\,\sigma_n}{M(\sigma_l^2 + \sigma_m^2 + \sigma_n^2)^{1/2}} - \frac{\mu z}{(x^2+y^2+z^2)^{3/2}}$$

$$\dot{x} = l \qquad \dot{y} = m \qquad \dot{z} = n$$

Thus, the latter coordinate system provides an example of a situation in which the desirable solution for, and elimination of, the control angles is readily obtained. If a transformation from the latter coordinate system to the former coordinate system can now be effected, a similar set of solutions for α and β should be immediately obtainable.

Suppose, then, that the immediate example is set aside momentarily in order that a transformation technique can be discussed. This technique can then, hopefully, be applied to the example. Suppose a function L is given in a certain coordinate system with variables x_i ($i = 1, 2, \ldots, n$) and Lagrange multipliers λ_i ($i = 1, 2, \ldots, n$) as:

$$L = \lambda_i f_i(t, x_1, x_2, \ldots, x_n)$$

Suppose, further, that a function $\tilde{L}$ is given in another coordinate system with variables h_i ($i = 1, 2, \ldots, n$) and Lagrange multipliers σ_i ($i = 1, 2, \ldots, n$), $\tilde{L} = \sigma_i g_i(t, h_1, h_2, \ldots, h_n)$.

L leads to the Euler equations:

$$\dot{\lambda}_j = - \lambda_i \, \partial f_i / \partial x_j \quad (j = 1, \dots , n)$$

and $\tilde{L}$ to:

$$\dot{\sigma}_j = - \sigma_i \, \partial g_i / \partial h_j \quad (j = 1, \dots , n)$$

(where in both cases, the summation convention applies to the repeated sub-script i).

Now, suppose that a transformation relationship is given for the variables in the two systems, say:

$$h_i = \tilde{h}_i(x_1, x_2, \dots , x_n)$$

or, inversely:

$$x_i = \tilde{x}_i(h_1, h_2, \dots , h_n)$$

Then, for

$$\lambda_i = \sigma_j \left(\frac{\partial \tilde{h}_j}{\partial x_i} \right) (\tilde{x}_1, \dots , \tilde{x}_n)$$

and

$$g_j = \left(\frac{\partial \tilde{h}_j}{\partial x_i} \right) (\tilde{x}_1, \dots , \tilde{x}_n) \; f_i(t, \tilde{x}_1, \dots , \tilde{x}_n),$$

the function $L = \lambda_i f_i$ transforms into

$$\left[\sigma_j \left(\frac{\partial \tilde{h}_j}{\partial x_i} \right) (\tilde{x}_1, \dots , \tilde{x}_n) \right] f_i(t, \tilde{x}_1, \dots , \tilde{x}_n) =$$

$$\sigma_j \left[\left(\frac{\partial \tilde{h}_j}{\partial x_i} \right) (\tilde{x}_1, \ldots, \tilde{x}_n) \; f_i(t, \tilde{x}_1, \ldots, \tilde{x}_n) \right]$$

$$= \sigma_j g_j = L.$$

The corresponding inverse multiplier transformation is:

$$\sigma_i = \lambda_j \left(\frac{\partial \tilde{x}_j}{\partial h_i} \right) (\tilde{h}_i, \ldots, \tilde{h}_n)$$

It can be demonstrated that, under these transformations, the Euler equations transform accordingly. Thus, if the functions $g_i(t, h_1, h_2, \ldots, h_n)$ are properly related to the functions $f_i(t, x_1, x_2, \ldots, x_n)$ by the above transformation, the Lagrange multipliers are transformed as shown.

In the example under consideration, all functions and transformation relationships are properly defined for this application. Thus,

$$\sigma_l = \lambda_v \, \partial v / \partial l + \lambda_\gamma \, \partial \gamma / \partial l + \lambda_\delta \, \partial \delta / \partial l$$
$$+ \lambda_r \, \partial r / \partial l + \lambda_\phi \, \partial \phi / \partial l + \lambda_\theta \, \partial \theta / \partial l$$

with corresponding equations for σ_m, σ_n, σ_x, σ_y and σ_z, where v, γ, δ, r, ϕ, θ are now expressed in terms of x, y, z, l, m, n by the transformation equations.

Now, the various partial derivatives involved in these preceding equations must be found in order to give the transformation equations explicitly. These partials can be found by differentiating the transformation equations with respect to the appropriate state variables and then solving the resulting algebraic system in the partial derivatives. Following this rather lengthy process will finally yield:

$$v_l = \sin\theta\cos\gamma\sin\delta + \cos\phi\cos\theta\sin\gamma - \sin\phi\cos\theta\cos\gamma\cos\delta$$

$$\gamma_l = 1/v[\cos\phi\cos\theta\cos\gamma + \sin\phi\cos\theta\sin\gamma\cos - \sin\theta\sin\gamma\sin\delta]$$

$$\delta_l = \frac{1}{v\cos\gamma}[\sin\theta\cos\delta + \sin\phi\cos\theta\sin\delta]$$

$$v_m = \cos\phi\sin\theta\sin\gamma - \cos\theta\cos\gamma\sin\delta - \sin\phi\sin\theta\cos\gamma\cos\delta$$

$$\gamma_m = 1/v[\cos\theta\sin\gamma\sin\delta + \cos\phi\sin\theta\cos\gamma + \sin\phi\sin\theta\sin\gamma\cos\delta]$$

$$\delta_m = \frac{1}{v\cos\gamma}[\sin\phi\sin\theta\sin\delta - \cos\theta\cos\delta]$$

$$v_n = \sin\phi\sin\gamma + \cos\phi\cos\gamma\cos\delta$$

$$\gamma_n = 1/v[\sin\phi\cos\gamma - \cos\phi\sin\gamma\cos\delta]$$

$$\delta_n = -\frac{1}{v\cos\gamma}[\cos\phi\sin\delta]$$

Further computation gives:

$$\sigma_l^2 + \sigma_m^2 + \sigma_n^2 = \lambda_v^2 + \lambda_\gamma^2/v^2 + \lambda_\delta^2/v^2\cos^2\gamma$$

Now, the Euler equations for the plumbline system for $\dot{l}$, $\dot{m}$, and $\dot{n}$ (in the form with tan X_p and tan X_y eliminated) may be transformed into the spherical system to yield an algebraic system of three equations in $\dot{v}$, $\dot{\gamma}$, and $\dot{\delta}$. This system may be solved algebraically for $\dot{v}$, $\dot{\gamma}$, and $\dot{\delta}$ to give:

$$\dot{v} = \frac{F\lambda_{v}}{M(\lambda_v^2 + \lambda_\gamma^2/v^2 + \lambda_\delta^2/v^2 \cos^2\gamma)^{1/2}} - g\sin\gamma$$

$$\dot{\gamma} = \frac{F\lambda_\gamma}{Mv(\lambda_v^2 + \lambda_\delta^2/v^2 + \lambda_\delta^2/v^2 \cos^2\gamma)^{1/2}} + \left(\frac{v}{r} - \frac{g}{v}\right)\cos\gamma$$

$$\dot{\delta} = \frac{F\lambda_\delta}{Mv\cos\gamma\,(\lambda_v^2 + \lambda_\delta^2/v^2 + \lambda_\delta^2/v^2 \cos^2\gamma)^{1/2}} + \frac{v\cos\gamma\sin\delta}{r\cot\phi}$$

These represent the Euler equations for the problem as expressed in the spherical coordinate system and with the control angles α and β eliminated. From them, or from direct transformation,

$$\tan\alpha = \lambda_\gamma/v\lambda_v$$

$$\tan\beta = \lambda_\delta/v\lambda_v\cos\gamma$$

This illustrates the transformation of Lagrange multipliers from one coordinate system to another and the accompanying transformation of the Euler equations. A particular value of such a transformation is that of obtaining expressions for the control angles in terms of the Lagrange multipliers in a coordinate system in which such expressions could not readily be obtained in a direct fashion, as noted in the above. Other motivations for carrying out such a transformation may also exist. For example, initial Lagrange multiplier values for one system may be used to give those for another. The simplicity and workability of such a transformation aid in making it a very valuable tool in trajectory optimization problems of the type illustrated.

Stability Criteria for n-th Order, Homogeneous

Linear Differential Equations†

E. F. Infante††

Center For Dynamical Systems, Brown University

1. Introduction

This note is concerned with the homogeneous differential equation

$$x^{(n)} + \rho_1(t)x^{(n-1)} + \ldots + \rho_{n-1}(t)\dot{x} + \rho_n(t)x = 0, \qquad (1.1)$$

where the $\rho_i(t)$ are real continuous functions. It is desired to determine appropriate criteria for the stability of the origin, criteria dependent on the behavior of the functions $\rho_i(t)$ but not of their derivatives.

This problem has been previously studied by Starzinski [1,2,3] for particular forms of this equation up to the fourth order, and by Razumichin [4] for the general matrix equation $\dot{x} = A(t)x$. The approach of these authors has been to use the direct method of Liapunov, using a constant quadratic Liapunov function $V(x) = x'Bx$ which is generated by determining the $n(n+1)/2$ constant elements of the symmetric matrix B. The determination of all these elements requires very heavy algebraic computations, computations which are completely unreasonable for $n > 2$. Recently, Ghizzetti [5,6] has obtained simple stability criteria for (1.1) by using some appropriate majoration formulae for all the integrals of this equation. The

† This research was supported by the National Aeronautics and Space Administration under Grant No. NAS8-11264.

†† On leave of absence from Department of Mechanical Engineering, University of Texas.

particularly attractive aspect of these criteria is that they depend on only n constant parameters which locate a family of hyperellipsoids in the n-dimensional space of the $\rho_i(t)$. If the curve parametrically represented by the $\rho_i(t)$ is entirely contained within one of these hyperellipsoids, then (1.1) is asymptotically stable.

In §2 of this note the second method of Liapunov is used to obtain stability criteria for (1.1) that depend on only n parameters which determine a family of elliptic paraboloids in the n-dimensional space $\rho_i(t)$. It can be shown that these elliptic paraboloids completely contain the hyperellipsoids of Ghizzetti. In §3 a practical technique for the application of the stability criteria obtained is discussed and is applied in the last section to two examples. The stability conditions presented in this note are not necessary. Indeed, they are probably not the best possible conditions obtainable from a quadratic Liapunov function. The technique presented in this note was devised with particular emphasis on ease of computability of some simple criteria.

2. Stability Criteria

Consider Eq. (1.1) rewritten in state-space coordinates as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\rho_n(t)x_1 - \dots -\rho_1(t)x_n. \end{aligned} \tag{2.1}$$

It is assumed that the $\rho_i(t)$, real continuous functions of time, satisfy the Routh-Hurwitz inequalities [7]. Let the n real numbers α_i, assumed to satisfy the Routh-Hurwitz inequalities, be associated to (2.1), which is rewritten as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_n &= -(\rho_n(t) - \alpha_n)x_1 - \dots -(\rho_1(t) - \alpha_1)x_n - \alpha_n x_1 - \dots - \alpha_1 x_n .
\end{aligned}
\tag{2.2}
$$

For economy of notation, (2.2) is rewritten as

$$\dot{x} = Ax - U(t)x, \tag{2.3}$$

where

$$
A = \begin{pmatrix}
0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 1 \\
-\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_2 & -\alpha_1
\end{pmatrix},
$$

$$
U(t) = \begin{pmatrix} 0 \\ \\ \\ u' \end{pmatrix}, \quad u = \begin{pmatrix} \eta_n \\ \eta_{n-1} \\ \vdots \\ \eta_1 \end{pmatrix} = \begin{pmatrix} u^u \\ \\ \eta_{n+1-i} \\ \\ u^\ell \end{pmatrix}, \tag{2.4}
$$

and where $\eta_i \equiv \rho_i(t) - \alpha_i$.

For the determination of the stability of the origin of (2.3), consider the Liapunov function $V(x) = x'Bx$, $B' = B = (\beta_{ij})$, β_{ij} = constant. Let b_n denote the n-th column of the matrix B, and

$$b_n = \begin{pmatrix} b_n^u \\ \beta_{ni} \\ b_n^\ell \end{pmatrix} . \tag{2.5}$$

The derivative $\dot{V}$ of the Liapunov function V in terms of (2.3) is given by

$$\dot{V} = x'(A'B + BA)x - x'(U'(t)B + BU(t))x, \tag{2.6}$$

or

$$-\dot{V} = x'Cx + x'(ub_n' + b_n u')x, \tag{2.7}$$

where $A'B + BA = -C$. If it were possible to determine a matrix B, positive definite, such that $-\dot{V}$ is positive definite for all $t \geq 0$, then asymptotic stability of the origin of (2.1) will have been determined by the well known theorem of Liapunov [8]. For this purpose, consider the following simple lemma:

Lemma 2.1: Given the constant matrix A, defined by (2.4), for any constant positive semidefinite diagonal matrix $C \not\equiv 0$ the equation $A'B + BA = -C$ has a unique solution B, and B is positive definite.

Proof. The matrix B, obviously symmetric, is unique since all the eigenvalues of A have negative real parts. Now, let $V(x_o) = x_0'Bx_0 < 0$ for some $x_o \neq 0$, and define δ_o as the trajectory of $\dot{x} = Ax$ issuing from x_o at $t = 0$. Along δ we then have $V(x) \leq V(x_o) < 0$. But δ approaches the origin and $V(0) = 0$. Hence $V(x) \geq 0$. Similarly, let $V(x_1) = 0$, $x_1 \neq 0$, and δ_1 the trajectory emanating from x_1 at t_o. Since this trajectory approaches the origin, it must lie on the manifold $x'Cx = 0$. But this is clearly impossible with C diagonal and A in the form (2.4). Hence B is positive definite.

Hence, let the matrix B be generated by the diagonal matrix

$$C = \begin{pmatrix} C^u & & \\ & 0 & \\ & & C^\ell \end{pmatrix} , \tag{2.8}$$

where C^u and C^ℓ are constant nonsingular positive definite diagonal square matrices, and where the zero element in the diagonal is located in the i,i position. On the basis of the above lemma $V(x) = x'Bx$ will be positive definite. In this case, Eq. (2.7) then becomes

$$-\dot{V} = x'\begin{pmatrix} C^u & & \\ & 0 & \\ & & C^\ell \end{pmatrix}x + x'\begin{pmatrix} u^u b_n^{u'} + b_n^u u^{u'} & u^u \beta_{ni} + \eta_{n+1-i} b_n^u & u^u b_n^{\ell'} + b_n^{u'} u^{\ell'} \\ \eta_{n+1-i} b_n^{u'} + \beta_{ni} u^{u'} & 2\beta_{ni}\eta_{n+1-i} & \eta_{n+1-i} b_n^{\ell'} + \beta_{ni} u^{\ell'} \\ u^\ell b_n^{u'} + b_n^\ell u^{u'} & u^\ell \beta_{ni} + \eta_{n+1-i} b_n^\ell & u^\ell b_n^{\ell'} + b_n^\ell u^{\ell'} \end{pmatrix} x$$

(2.9)

Assume $\beta_{ni} > 0$ (it is always possible to find a $\beta_{ni} > 0$, namely β_{nn}) and consider the regular transformation $x = Sy$,

$$S = \begin{pmatrix} I & 0 & 0 \\ -\dfrac{b_n^{u'}}{\beta_{ni}} & 1 & -\dfrac{b_n^{\ell'}}{\beta_{ni}} \\ 0' & 0 & I \end{pmatrix}, \tag{2.10}$$

where the unit element is in the i,i position and the I are unit matrices of appropriate dimensions. If this transformation is applied to Eq. (2.9), one obtains

$$-\dot{V} = y' \begin{pmatrix} C^u & & \\ & 0 & \\ & & C^\ell \end{pmatrix} y + y' \begin{pmatrix} 0 & \beta_{ni}u^u - \eta_{n+1-i}b_n^u & 0 \\ \beta_{ni}u^{u'} - \eta_{n+1-i}b_n^{u'} & 2\beta_{ni}\eta_{n+1-i} & \beta_{ni}u^{\ell'} - \eta_{n+1-i}b_n^{\ell'} \\ 0' & \beta_{ni}u^\ell - \eta_{n+1-i}b_n^\ell & 0 \end{pmatrix} y \tag{2.11}$$

or

$$-\dot{V} = y' \begin{pmatrix} C^u & \beta_{ni}u^u - \eta_{n+1-i}b_n^u & 0 \\ \beta_{ni}u^{u'} - \eta_{n+1-i}b_n^{u'} & 2\beta_{ni}\eta_{n+1-i} & \beta_{ni}u^{\ell'} - \eta_{n+1-i}b_n^{\ell'} \\ 0' & \beta_{ni}u^\ell - \eta_{n+1-i}b_n^\ell & C^\ell \end{pmatrix} y \tag{2.12}$$

It now becomes necessary to determine under what conditions (2.12) is positive definite. For this purpose, consider the second transformation. $y = Tz$,

$$T = \begin{pmatrix} I & -C^{u^{-1}} v^u & 0 \\ 0' & 1 & 0' \\ 0' & -C^{\ell^{-1}} v^{\ell} & I \end{pmatrix} , \tag{2.13}$$

where the unit element is in the i,i position, the I are unit matrices of appropriate dimensions and $v^u = \beta_{ni} u^u - \eta_{n+1-i} b_n^u$, $v^{\ell} = \beta_{ni} u^{\ell} - \eta_{n+1-i} b_n^{\ell}$. This transformation is obviously regular and when applied to Eq. (2.12) yields

$$-\dot{V} = z' \begin{pmatrix} C^u & 0 & 0 \\ 0' & \omega & 0' \\ 0' & 0 & C^{\ell} \end{pmatrix} z \tag{2.14}$$

where

$$\begin{aligned} \omega = 2\beta_{ni}\eta_{n+1-i} &- (\beta_{ni} u^u - \eta_{n+1-i} b_n^u)' C^{u^{-1}} (\beta_{ni} u^u - \eta_{n+1-i} b_n^u) + \\ &- (\beta_{ni} u^{\ell} - \eta_{n+1-i} b_n^{\ell})' C^{\ell^{-1}} (\beta_{ni} u^{\ell} - \eta_{n+1-i} b_n^{\ell}) \end{aligned} \tag{2.15}$$

Since (2.14) is diagonal, it can then be concluded that $\dot{V}$ will be negative definite if $\omega \geq \delta > 0$.

On the basis of what has been said above, it is then possible to state:

Theorem 2.1: Given the homogeneous differential equation

$$x^{(n)} + \rho_1(t)x^{(n-1)} + \ldots + \rho_{n-1}(t)\dot{x} + \rho_n(t)x = 0 \,, \qquad (2.16)$$

with $\rho_i(t)$ real continuous functions for $t \geq 0$, associate with this equation the n real constants $\alpha_1,\ldots,\alpha_n$ satisfying the Routh-Hurwitz inequalities, and define $\eta_i = \rho_i(t) - \alpha_i$. Let the matrix $B = (\beta_{ij})$ be the solution of the matrix equation

$$A'B + BA = -\begin{pmatrix} C^u & & \\ & 0 & \\ & & C^\ell \end{pmatrix} , \qquad (2.17)$$

where C^u, C^ℓ are constant, positive definite diagonal matrices, and the zero element in the diagonal appears in the i,i position; and where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdot\ \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot\ \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdot\ \cdot & -\alpha_2 & -\alpha_1 \end{pmatrix} . \qquad (2.18)$$

Let b_n denote the n-th column of B and define

$$b_n = \begin{pmatrix} b_n^u \\ \beta_{ni} \\ b_n^\ell \end{pmatrix} , \quad u = \begin{pmatrix} \eta_{n+1-1} \\ \vdots \\ \eta_{n+1-i} \\ \vdots \\ \eta_1 \end{pmatrix} = \begin{pmatrix} u_u \\ \eta_{n+1-i} \\ u^\ell \end{pmatrix} . \qquad (2.19)$$

Then, if for any $\delta > 0$ *and any* $i = 1, \ldots, n$

$$2\beta_{ni}\eta_{n+1-i} - (\beta_{ni}u^u - \eta_{n+1-i}b_n^u)' C^{u^{-1}} (\beta_{ni}u^u - \eta_{n+1-i}b_n^u) + \\ - (\beta_{ni}u^\ell - \eta_{n+1-i}b_n^\ell)' C^{\ell^{-1}} (\beta_{ni}u^\ell - \eta_{n+1-i}b_n^\ell) \geq \delta \tag{2.20}$$

for all $t \geq 0$, *the null solution of* (2.16) *is asymptotically stable*.

This theorem is not as general as it would have been possible to state, yet it is still too general for practical applications because of the generality of the matrices C^u and C^ℓ. Before restricting the theorem, it is desirable to make some remarks concerning the results so far obtained.

First of all we wish to point out that Eq. (2.20) represents, in the parameter space of the η's, an elliptic paraboloid. This can be easily seen by introducing the transformation of coordinates for the parameter space given by

$$v = \begin{pmatrix} \gamma_{n+1-1} \\ \vdots \\ \gamma_{n+1-i} \\ \vdots \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} v^u \\ \\ \gamma_{n+1-i} \\ \\ v^\ell \end{pmatrix} = \begin{pmatrix} \beta_{ni}I & -b_n^u & 0 \\ \\ 0' & 1 & 0' \\ \\ 0' & -b_n^\ell & \beta_{ni}I \end{pmatrix} \begin{pmatrix} u^u \\ \\ \eta_{n+1-i} \\ \\ u^\ell \end{pmatrix} \tag{2.21}$$

This transformation is obviously regular if $\beta_{ni} > 0$, which as was previously pointed out, is no restriction. In the new coordinates, Eq. (2.20) becomes

$$2\beta_{ni}\gamma_{n+1-i} - v' \begin{pmatrix} C^{u^{-1}} & & \\ & 0 & \\ & & C^{\ell^{-1}} \end{pmatrix} v \geq \varepsilon . \qquad (2.22)$$

This is evidently the equation of an elliptic paraboloid. If $\beta_{ni} > 0$, as assumed, the domain defined in the parameter space by (2.22), hance by (2.20). is nonempty.

Secondly, it is evident that, for any C^u and C^ℓ satisfying the conditions of Theorem 2.1, the domain of the η parameter space defined by any of the (2.20) is strictly contained within the domain where the $\rho_i(t)$ satisfy the Routh-Hurwitz inequalities. On the other hand, it is easily shown that every point of the domain of the parameter space where the $\rho_i(t)$ satisfy the Routh-Hurwitz inequalities is contained in at least one of the domains defined by (2.20). To prove this, let $\rho_i(t) \equiv \overline{\rho_i}$ = constants. Since the $\overline{\rho_i}$ satisfy the Routh-Hurwitz inequalities, it is possible to select the n numbers α_i, themselves satisfying these inequalities, and such that $\eta_{n+1-j} = \overline{\rho}_{n+1-j} - \alpha_{n+1-j} \equiv \varepsilon > 0$ for some j and $\overline{\rho}_{n+1-i} - \alpha_{n+1-i} = 0$ for all $i \neq j$. Under these conditions Eq. (2.20) reduces to

$$2\beta_{nj}\eta_{n+1-j} - \eta_{n+1-j} b_n^{u'} C^{u^{-1}} b_n^u \eta_{n+1-j} - \eta_{n+1-j} b_n^{\ell'} C^{\ell^{-1}} b_n^\ell \eta_{n+1-j} \geq \delta . \qquad (2.23)$$

But for any $\varepsilon > 0$ sufficiently small, a $\delta > 0$ can be found such that (2.23) is satisfied. Hence the remark.

Finally, it is noted that the continuity condition imposed by Theorem 2.1 on the $\rho_i(t)$ imply that Eq. (2.16) does not have a finite escape time. It is therefore possible on the basis of this remark and the two previous ones to state:

Corollary 2.1: Given the differential equation (2.16) with $\rho_i(t)$ *real continuous functions for* $t \geq 0$, *if there exist a* $\tau > 0$ *such that for all* $t \geq \tau$ *(2.20) is satisfied for some* $\delta > 0$ *and some* $i = 1,\ldots,n$, *then the null solution of (2.16) is asymptotically stable.*

Corollary 2.2: If, in Eq. (2.16), the $\rho_i(t)$ *are real continuous functions for* $t \geq 0$ *and* $\lim_{t \to \infty} \rho_i(t) = \overline{\rho_i}$, *where the* $\overline{\rho_i}$ *satisfy the Routh-Hurwitz inequalities, then the null solution of (2.16) is asymptotically stable.*

This last corollary is very well known [7], and can be traced directly to Liapunov.

3. Application of Stability Criteria

The positive definite diagonal matrices C^u and C^ℓ have not been so far specified. The first step in the application of the stability criteria obtained to a specific example is the selection of these two matrices, from which the matrix B is obtained as the solution of the equation $A'B + BA = -C$. Algorithms for the solution of this matrix equation are available. A particularly simple one has been recently given by Smith [9] in the case matrix A has the form (2.18).

It is particularly convenient, to obtain algebraically simple forms for B, to select the matrices C^u and C^ℓ to be composed of linear combinations of matrices of the form

$$C_1 = 2 \operatorname{diag} (\mu, 0,\ldots,0) \qquad (3.1)$$

and

$$C_k = 2 \text{ diag } (0,\ldots,\frac{\mu}{\alpha_n},0,\ldots,0), \quad k \neq 1, \tag{3.2}$$

where μ is the Hurwitz determinant [7] of the α:

$$\mu = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \cdot & \cdot & \cdot & \alpha_{2n-1} \\ 1 & \alpha_2 & \alpha_4 & \cdot & \cdot & \cdot & \alpha_{2n-2} \\ 0 & \alpha_1 & \alpha_3 & \cdot & \cdot & \cdot & \alpha_{2n-3} \\ 0 & 1 & \alpha_2 & \cdot & \cdot & \cdot & \alpha_{2n-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \alpha_n \end{vmatrix} \tag{3.3}$$

The matrix equation $A'B_k + B_kA = -C_k$, where A is given by (2.18) can be rapidly solved for B_k when C_k is of the suggested form. The matrices obtained in this manner for $n = 2,3$ are shown in Table 1. Ingwerson [10] previously published these matrices for $n = 2,3,4$. If C^u and C^ℓ are obtained, as suggested, by linear combinations of the C_k, then the matrix B will be the corresponding linear combination of the B_k.

Table I

<u>n = 2</u>

$$B_1 = \begin{pmatrix} \alpha_1^2 + \alpha_2 & \alpha_1 \\ \alpha_1 & 1 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 2\alpha_1\alpha_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2\alpha_1 \end{pmatrix}$$

<u>n = 3</u>

$$B_1 = \begin{pmatrix} \alpha_2(\alpha_1\alpha_2-\alpha_3)+\alpha_1^2\alpha_3 & \alpha_1^2\alpha_2 & \alpha_1\alpha_2-\alpha_3 \\ \alpha_1^2\alpha_2 & \alpha_1^3+\alpha_3 & \alpha_1^2 \\ \alpha_1\alpha_2-\alpha_3 & \alpha_1^2 & \alpha_1 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 2\alpha_3(\alpha_1\alpha_2-\alpha_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} \alpha_1\alpha_3 & \alpha_3 & 0 \\ \alpha_3 & \alpha_1^2+\alpha & \alpha_1 \\ 0 & \alpha_1 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(\alpha_1\alpha_2-\alpha_3) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} \alpha_3^2 & \alpha_2\alpha_3 & 0 \\ \alpha_2\alpha_3 & \alpha_1\alpha_3+\alpha_2^2 & \alpha_3 \\ 0 & \alpha_3 & \alpha_2 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(\alpha_1\alpha_2-\alpha_3) \end{pmatrix}$$

4. Two Examples

In this section, the stability criteria obtained is applied to two simple but illustrative example problems.

As a first example, consider the second order equation

$$\ddot{x} + p\dot{x} + q(t)x = 0 \tag{4.1}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -q(t)x_1 - px_2 \end{aligned} \tag{4.2}$$

where, $p > 0$ is a constant and $0 < q_1 + \xi \le q(t) \le q_2 - \xi$, for $\xi > 0$. It is desired to determine conditions on q_1, q_2 and p that guarantee the asymptotic stability of the null solution of (4.2). This same problem has been treated by Ghizzetti [5], with whom we wish to compare our results.

In the case of a second order equation, inspection of the matrices B_1 and B_2 of table one indicates that, for $\beta_{ni} > 0$ one must select $i = 2$. With this choice one immediately obtains

$$C = \begin{pmatrix} 2\alpha_1\alpha_2 & 0 \\ 0 & 0 \end{pmatrix} , \qquad C^{u^{-1}} = \frac{1}{2\alpha_1\alpha_2} ,$$

$$b_n = \begin{pmatrix} \beta_{21} \\ \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix} ; \quad u = \begin{pmatrix} \eta_2 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} q - \alpha_2 \\ p - \alpha_1 \end{pmatrix} , \tag{4.3}$$

upon which the stability equation given by (2.20) becomes

$$2(p - \alpha_1) - [(q - \alpha_2) - \alpha_1(p - \alpha_1)] \frac{1}{2\alpha_1\alpha_2} [(q - \alpha_2) - \alpha_1(p - \alpha_1)] \geq \delta > 0 \tag{4.4}$$

or, letting $\nu_1 = \frac{\alpha_1}{p}$, $\nu_2 = \frac{\alpha_2}{p^2}$ and $z(t) = \frac{q(t)}{p^2}$,

$$4\nu_1\nu_2(1 - \nu_1) - [z(t) - \nu_2 - \nu_1(1 - \nu_1)]^2 \geq \varepsilon > 0 \tag{4.5}$$

To determine the appropriate values of ν_1 and ν_2 for this expression, let

$$\begin{aligned} z_1 &= \frac{q_1}{p^2} = \nu_2 + \nu_1(1 - \nu_1) - 2\sqrt{\nu_1\nu_2(1 - \nu_1)} \quad , \\ z_2 &= \frac{q_2}{p^2} = \nu_2 + \nu_1(1 - \nu_1) + 2\sqrt{\nu_1\nu_2(1 - \nu_1)} \quad , \end{aligned} \tag{4.6}$$

and to maximize the difference between z_2 and z_1 let $\nu_1 = 1/2$. Then

$$\begin{aligned} z_1 &= \frac{1}{4} + \nu_2 - \sqrt{\nu_2} \quad , \\ z_2 &= \frac{1}{4} + \nu_2 + \sqrt{\nu_2} \quad . \end{aligned} \tag{4.7}$$

Solving now for ν_2 from the first of these equations

$$\nu_2 = \frac{1}{4} + z_1 + \sqrt{z_1} \tag{4.8}$$

is obtained. With these two particular values of ν_1 and ν_2 (4.7) yields

$$z_2 = z_1 + 2\sqrt{\frac{1}{4} + z_1 + \sqrt{z_1}} \quad .$$

Hence, if $0 < z_1 + \xi < z(t) < \xi$ for some $\xi > 0$, an $\varepsilon > 0$ can be found such that Eq. (4.5) is satisfied. Therefore, Eq. (4.1) is asymptotically stable if, for some $\xi > 0$,

$$0 < q_1 + \xi \leq q(t) \leq q_2 - \xi \tag{4.10}$$

and

$$\frac{q_2}{p^2} = \frac{q_1}{p^2} + 2\sqrt{\frac{1}{4} + \frac{q_1}{p^2} + \sqrt{\frac{q_1}{p^2}}} \quad . \tag{4.11}$$

This result is represented in graphical form in Figure 1: if $\frac{q(t)}{p^2}$ is strictly internal to the domain Δ of the parameter space q_1/p^2 vs. q_2/p^2, then Eq. (4.1) is asymptotically stable. The domain Δ obtained by Ghizzetti [5] is shown also.

As a second example, consider the differential equation

$$\dddot{x} + p\ddot{x} + \dot{x} + r(t)x = 0 \ , \qquad (4.11)$$

where $p > 0$ is a constant and $0 < \xi \leq r(t) \leq r_2 - \xi$ for some $\xi > 0$. It is desired to determine conditions on r_2 to guarantee the asymptotic stability of the null solution of this equation. This equation has been studied by Starzinski [3], who generated a constant Liapunov function by determining, through a very laborious process, appropriate values for all six elements of the 3×3 B matrix.

Inspection of the third order matrices of Table 1 indicates that, for $\beta_{ni} > 0$ one must select either $i = 2$ or $i = 3$. Let $i = 3$ upon which the stability equation (2.20) becomes

$$2\beta_{33}(p - \alpha_1) - [\beta_{33}(r(t) - \alpha_3) - \beta_{31}(p - \alpha_1)]^2 C^{u^{-1}} + \\ - [\beta_{33}(1 - \alpha_2) - \beta_{32}(p - \alpha_1)]^2 C^{\ell^{-1}} \geq \delta > 0. \tag{4.12}$$

Since $i = 3$, let $C = C_1 + \lambda C_2$ where C_1 and C_2 are the two matrices shown in Table 1, and $\lambda > 0$. From Table 1, then

$$\beta_{31} = \alpha_1\alpha_2 - \alpha_3, \quad \beta_{32} = \alpha_1^2 + \lambda\alpha_1, \quad \beta_{33} = \lambda + \alpha_1 \\ C^{u^{-1}} = \frac{1}{2\alpha_3(\alpha_1\alpha_2-\alpha_3)}, \quad C^{\ell^{-1}} = \frac{1}{2\lambda(\alpha_1\alpha_2-\alpha_3)} \tag{4.13}$$

are immediately obtained. Equation (4.12) can be therefore rewritten as

$$4\alpha_3(\lambda + \alpha_1)(\alpha_1\alpha_2-\alpha_3)(p - \alpha_1) - [(\lambda + \alpha_1)(r(t) - \alpha_3) - (\alpha_1\alpha_2-\alpha_3)(p - \alpha_1)]^2 + \\ - \frac{\alpha_3}{\lambda}[(\lambda + \alpha_1)(1 - \alpha_2) - (\alpha_1^2 + \lambda\alpha_1)(p - \alpha_1)]^2 \geq \varepsilon > 0. \tag{4.14}$$

The second quadratic term vanishes if

$$(1 - \alpha_2) = \alpha_1(p - \alpha_1). \tag{4.15}$$

Furthermore, (4.14) can be satisfied as $r(t)$ becomes very small only if

$$p - \alpha_1 = \alpha_3 \frac{\lambda + \alpha_1}{\alpha_1\alpha_2 - \alpha_3} \quad . \tag{4.16}$$

Assuming these two conditions, Eq. (4.14) yields

$$0 < \xi \le r(t) \le 4\alpha_3 - \xi \ , \tag{4.17}$$

where $\xi \to 0$ as $\varepsilon \to 0$. Equations (4.15) and (4.16) yield

$$\alpha_3 = \frac{\alpha_2 - \alpha_2^2}{\lambda + p} \ , \quad \alpha_1 = \frac{p + \sqrt{p^2 - 4 + 4\alpha_2}}{2} \ ; \tag{4.18}$$

therefore, let

$$\begin{aligned} \alpha_2 &= 1 - (\tfrac{p}{2})^2 \qquad \text{if} \qquad 0 < p \le \sqrt{2} \\ \alpha_2 &= \tfrac{1}{2} \qquad\qquad\quad \text{if} \qquad \sqrt{2} \le p \end{aligned} \tag{4.19}$$

upon which one obtains that Eq. (4.11) is asymptotically stable if

$$\begin{aligned} 0 < \xi \le r(t) \le \frac{1}{\lambda + p} (p^2 + \frac{p^4}{4}) - \xi \qquad &\text{if} \quad 0 < p \le \sqrt{2} \\ 0 < \xi \le r(t) \le \frac{1}{\lambda + p} - \xi \qquad &\text{if} \quad p \ge \sqrt{2} \end{aligned} \tag{4.20}$$

for some $\xi > 0$ and $\lambda > 0$, since the α's obtained from Eq. (4.18) and

(4.19) satisfy the Routh-Hurwitz inequalities.

This same result would have been obtained if the stability Eq. (2.20) for i = 2 had been used. The stability conditions (4.20) are identical to those obtained by Starzinski [3].

References

[1] Starzinski, V. M., "Sufficient Conditions for Stability of a Mechanical System with One Degree of Freedom", PMM, v. 16, 1952, p. 369, (Russian).

[2] Starzinski, V. M., "On the Stability of Unsteady Motion in One Case", PMM, v. 16, 1952, p. 500, (Russian).

[3] Starzinski, V. M., "On Stability of Unsteady Motions in a Special Case", PMM, v. 19, 1955, p. 471, (Russian).

[4] Razumichin, B. S., "On the Application of Liapunov's Method to Stability Problems", PMM, v. 22, 1958, p. 338.

[5] Ghizzetti, A., "Some Sufficient Stability Criteria for the Integrals of a Linear Homogeneous Differential Equation of Second Order", Rendiconti, Acc. Naz. dei Lincei, Series VIII, v. 33, 1962, p. 219, (Italian).

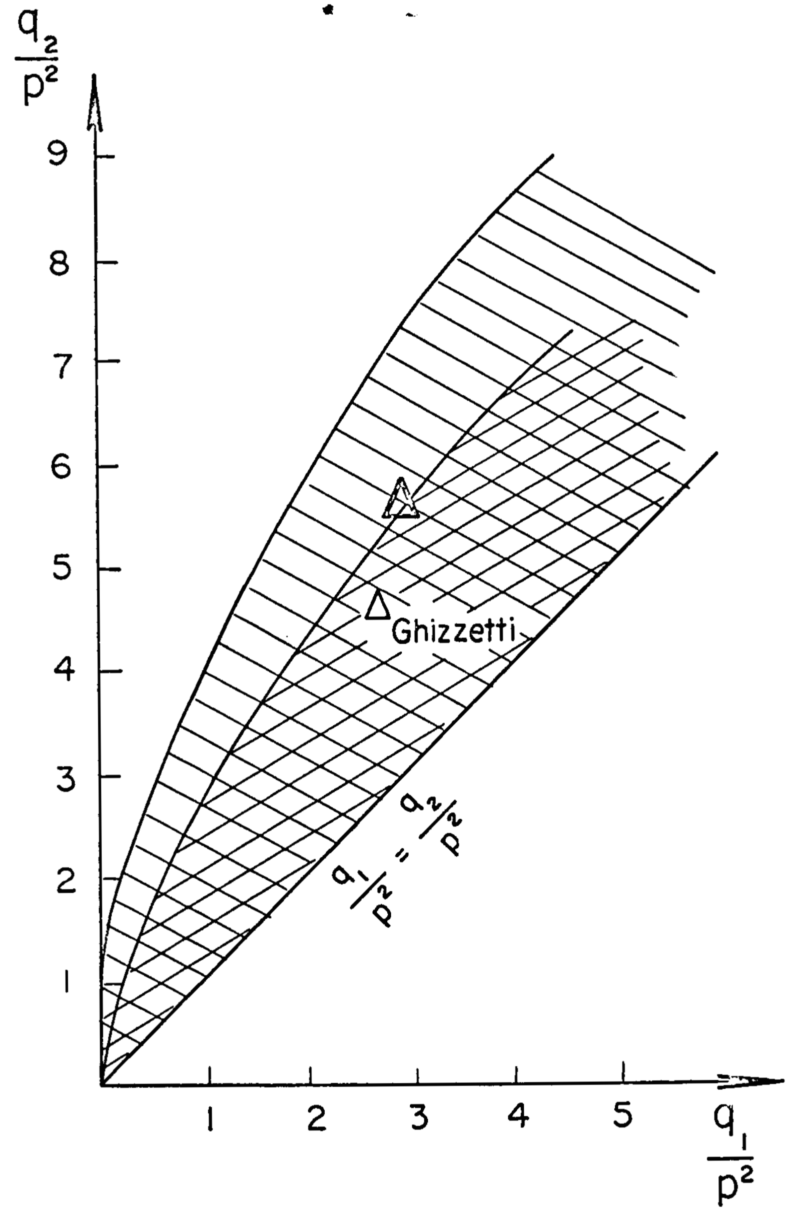
[6] Ghizzetti, A., "Majoration Formulae and Sufficient Criteria for the Stability of the Integrals of a Linear Homogeneous Differential Equation of n-th Order", Memorie, Acc. Naz. dei Lincei, Series VIII, v. 7, 1963, p. 17, (Italian).

[7] Cesari, L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer-Verlag, 1953, p. 21.

[8] LaSalle, J. P. and S. Lefschetz, Stability by Liapunov's Direct Method, Academic Press, 1961, p. 40.

[9] Smith, R. A., "Matrix Calculations for Liapunov Quadratic Forms", to appear, Journal of Differential Equations.

[10] Ingwerson, D. R., "A Modified Lyapunov Method for Nonlinear Stability Analysis", IRE Trans., v. AC-6, 1961, p. 199.



$\frac{q_2}{p^2}$
9
8
7
6
5
4
3
2
1
1
2
3
4
5
$\frac{q_1}{p^2}$
Ghizzetti
$\frac{q_1}{p^2} = \frac{q_2}{p^2}$

FIGURE 1

AN INVARIANCE PRINCIPLE IN THE THEORY OF STABILITY

by

J. P. LaSalle

Center for Dynamical Systems
Brown University

1. Introduction.

The purpose of this paper is to give a unified presentation of Liapunov's theory of stability that includes the classical Liapunov theorems on stability and instability as well as their more recent extensions. The idea being exploited here had its beginnings some time ago. It was, however, the use made of this idea by Yoshizawa in [1] in his study of nonautonomous differential equations and by Hale in [2] in his study of autonomous functional differential equations that caused the author to return to this subject and to adopt the general approach and point of view of this paper. This produces some new results for dynamical systems defined by ordinary differential equations which demonstrate the essential nature of a Liapunov function and which may be useful in applications. Of greater importance, however, is the possibility, as already indicated by Hale's results for functional differential equations,

*This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGR-40-002-015 and under Contract No. NAS8-11264, in part by the United States Air Force through the Air Force Office of Scientific Research under Grant No. AF-AFOSR-693-65 and in part by the United States Army Research Office, Durham, under Contract No. DA-31-124-ARO-D-270.

that these ideas can be extended to more general classes of dynamical systems. It is hoped, for instance, that it may be possible to do this for some special types of dynamical systems defined by partial differential equations.

In section 2 we present some basic results for ordinary differential equations. Theorem 1 is a fundamental stability theorem for nonautonomous systems and is a modified version of Yoshizawa's Theorem 6 in [1]. A simple example shows that the conclusion of this theorem is the best possible. However, whenever the limit sets of solutions are known to have an invariance property then sharper results can be obtained. This "invariance principle" explains the title of this paper. It had its origin for autonomous and periodic systems in [3] - [5], although we present here improved versions of those results. Miller in [6] has established an invariance property for almost periodic systems and obtains thereby a similar stability theorem for almost periodic systems. Since little attention has been paid to theorems which make possible estimates of regions of attraction (regions of asymptotic stability) for nonautonomous systems results of this type are included. Section 3 is devoted to a brief discussion of some of Hale's recent results [2] for autonomous functional differential equations.

2. Ordinary differential equations.

Consider the system

$$\dot{x} = f(t,x) \tag{1}$$

where x is an n-vector, f is a continuous function on R^{n+1} to R^n and satisfies any one of the conditions guaranteeing uniqueness of solutions. For each x in R^n we define $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$, and for E a closed set in R^n we define $d(x,E) = \text{Min}\,\{|x-y|: y \text{ in } E\}$. Since we do not wish to confine ourselves to bounded solutions, we introduce the point at ∞ and define $d(x,\infty) = |x|^{-1}$. Thus when we write $E^* = E \cup \{\infty\}$, we shall mean $d(x,E^*) = \text{Min}\{d(x,E), d(x,\infty)\}$. If $x(t)$ is a solution of (1), we say that $x(t)$ approaches E as $t \to \infty$ if $d(x(t),E) \to 0$ as $t \to \infty$. If we can find such a set E, we have obtained information about the asymptotic behavior of $x(t)$ as $t \to \infty$. The best that we could hope to do is to find the smallest closed set Ω that $x(t)$ approaches as $t \to \infty$. This set Ω is called the <u>positive limit set</u> of $x(t)$ and the points p in Ω are called the <u>positive limit points</u> of $x(t)$. In exactly the same way one defines $x(t) \to E$ as $t \to -\infty$, negative limit sets, and negative limit points. This is exactly G. D. Birkhoff's concept of limit sets. A point p is a positive limit point of $x(t)$ if and only if there is a sequence of times t_n approaching ∞ as $n \to \infty$ and such that $x(t_n) \to p$ as $n \to \infty$. In the above it may be that the maximal interval of definition of $x(t)$ is $[0,\tau)$. This causes no difficulty since in the results to be presented here we need only with respect to time t replace ∞ by τ. We usually ignore this possibility and speak as though our solutions are defined on $[0,\infty)$ or $(-\infty,\infty)$.

Let $V(t,x)$ be a C^1 function on $[0,\infty) \times R^n$ to R, and let G be any set in R^n . We shall say that V is a <u>Liapunov function</u> on G for equation (1) if $V(t,x) \geqq 0$ and $\dot{V}(t,x) \leqq -W(x) \leqq 0$ for all $t > 0$ and all x in G where W is continuous on R^n to R and

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i \quad .$$

We define ($\overline{G}$ is the closure of G)

$$E = \{x;\ W(x) = 0,\quad x \text{ in } \overline{G}\}.$$

The following result is then a modified but closely related version of Yoshizawa's Theorem 6 in [1].

THEOREM 1. If V is a Liapunov function on G for equation (1), then each solution $x(t)$ of (1) that remains in G for all $t > t_o \geqq 0$ approaches $E^* = E \cup \{\infty\}$ as $t \to \infty$, provided one of the following conditions is satisfied:

(i) For each p in $\overline{G}$ there is a neighborhood N of p such that $|f(t,x)|$ is bounded for all $t > 0$ and all x in N.

(ii) W is C^1 and $\dot{W}$ is bounded from above or below along each solution which remains in G for all $t > t_o \geqq 0$.

If E is bounded, then each solution of (1) that remains in G for $t > t_0 \geqq 0$ either approaches E or ∞ as $t \to \infty$.

Thus this theorem explains precisely the nature of the information given by a Liapunov function. A Liapunov function relative to a set G defines a set E which under the conditions of the theorem contains (locates) all the positive limit sets of solutions which for positive time remain in G. The problem in applying the result is to find "good" Liapunov functions. For instance, the zero function $V = 0$ is a Liapunov function for the whole space R^n and condition (ii) is satisfied but gives no in-information since $E = R^n$. It is trivial but useful for applications to note that if V_1 and V_2 are Liapunov functions on G, then $V = V_1 + V_2$ is also a Liapunov function and $E = E_1 \cap E_2$. If E is smaller than either E_1 or E_2, then V is a "better" Liapunov function than either V_1 or V_2 and is always at least as "good" as either of the two.

Condition (i) of Theorem 1 is essentially the one used by Yoshizawa. We now look at a simple example where condition (ii) is satisfied and condition (i) is not. The example also shows that the conclusion of the theorem is the best possible. Consider $\ddot{x} + p(t)\dot{x} + x = 0$ where $p(t) \geqq \delta > 0$. Define $2V = x^2 + y^2$, where $y = \dot{x}$. Then $\dot{V} = -p(t)y^2 \leqq -\delta y^2$ and V is a Liapunov function on R^2. Now $W = \delta y^2$ and $\dot{W} = 2\delta y\dot{y} = -2\delta(xy + p(t)y^2) \leqq -2\delta xy$. Since all solutions are evidently bounded for all $t > 0$, condition (ii) is satisfied. Here E is the x-axis $(y = 0)$ and for each solution $x(t)$, $y(t) = \dot{x}(t) \to 0$ as $t \to \infty$. Noting that the equation $\ddot{x} + (2 + e^t)\dot{x} + x = 0$ has a solution $x(t) = 1 + e^{-t}$, we see that this is the best possible result without further restrictions on p.

In order to use Theorem 1 there must be some means of determining which solutions remain in G. The following corollary, which is an obvious consequence of Theorem 1, gives one way of doing this and also provides for nonautonomous systems a method for estimating regions of attraction.

Corollary 1. Assume that there exist continuous functions $u(x)$ and $v(x)$ on R^n to R such that $u(x) \leqq V(t,x) \leqq v(x)$ for all $t \geqq 0$. Define $Q_\eta^+ = \{x ; u(x) < \eta\}$ and let G^+ be a component of Q_η^+. Let G denote the component of $Q_\eta = \{x ; v(x) < \eta\}$ containing G^+. If V is a Liapunov function on G for (1) and the conditions of Theorem 1 are satisfied, then each solution of (1) starting in G^+ at any time $t_o \geqq 0$ remains in G for all $t > t_o$ and approaches E^* as $t \to \infty$. If G is bounded and $E^o = \overline{E \cap G} \subset G^+$, then E^o is an attractor and G^+ is in its region of attraction.

In general we know that if $x(t)$ is a solution of (1)--in fact, if $x(t)$ is any continuous function on R to R^n--then its positive limit set is closed and connected. If $x(t)$ is bounded, then its positive limit set is compact. There are, however, special classes of differential equations where the limit sets of solutions have an additional invariance property which makes possible a refinement of Theorem 1. The first of these are the autonomous systems

$$\dot{x} = f(x) \tag{3}$$

The limit sets of solutions of (3) are invariant sets. If $x(t)$ is defined on $[0,\infty)$ and if p is a positive limit point of $x(t)$, then the points on the solution through p on its maximal interval of definition are positive limit points of $x(t)$. If $x(t)$ is bounded for $t > 0$, then it is defined on $[0,\infty)$, its positive limit set Ω is compact, nonempty and solutions through points p of Ω are defined on $(-\infty,\infty)$ (i.e., Ω is invariant). If the maximal domain of definition of $x(t)$ for $t > 0$ is finite, then $x(t)$ has no finite positive limit points: that is, if the maximal interval of definition of $x(t)$ for $t > 0$ is $[0,\beta)$, then $x(t) \to \infty$ as $t \to \beta$. As we have said before, we will always speak as though our solutions are defined on $(-\infty,\infty)$ and it should be remembered that finite escape time is always a possibility unless there is, as for example in Corollary 2 below, some condition that rules it out. In Corollary 3 below, the solutions might well go to infinity in finite time.

The invariance property of the limit sets of solutions of autonomous systems (3) now enables us to refine Theorem 1. Let V be a C^1 function on R^n to R. If G is any arbitrary set in R^n, we say that V is a <u>Liapunov function on</u> G for equation (3) if $\dot{V} = (\text{grad } V) \cdot f$ does not change sign on G. Define $E = \{ x ; \dot{V}(x) = 0, x \text{ in } \bar{G} \}$, where $\bar{G}$ is the closure of G. Let M be the largest invariant set in E. M will be a closed set. The fundamental stability theorem for autonomous systems is then the following:

THEOREM 2. If V is a Liapunov function on G for (3), then each solution $x(t)$ of (3) that remains in G for all $t > 0$ $(t < 0)$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$ $(t \to -\infty)$. If M is bounded, then either $x(t) \to M$ or $x(t) \to \infty$ as $t \to \infty$ $(t \to -\infty)$.

This one theorem contains all of the usual Liapunov like theorems on stability and instability of autonomous systems. Here however, there are no conditions of definiteness for V or $\dot{V}$, and it is often possible to obtain stability information about a system with these more general types of Liapunov functions. The first corollary below is a stability result which for applications has been quite useful and the second illustrates how one obtains information on instability. Četaev's instability theorem is similarly an immediate consequence of Theorem 2 (see section 3).

COROLLARY 2. Let G be a component of $Q_\eta = \{ x ; V(x) < \eta \}$. Assume that G is bounded, $\dot{V} \leq 0$ on G, and $M^o = \overline{M \cap G} \subset G$. Then M^o is an attractor and G is in its region of attraction. If, in addition, V is constant on the boundary of M^o, then

M^o is a stable attractor.

Note that if M^o consists of a single point p, then p is asymptotically stable and G provides an estimate of its region of asymptotic stability.

COROLLARY 3. Assume that relative to (3) that $V\dot{V} > 0$ on G and on the boundary of G that $V = 0$. Then each solution of (3) starting in G approaches ∞ as $t \to \infty$ (or possibly in finite time).

There are also some special classes of nonautonomous systems where the limit sets of solutions have an invariance property. The simplest of these are periodic systems (see [3]).

$$\dot{x} = f(t,x) \quad , \; f(t + T,x) = f(t) \text{ for all } t \text{ and } x \, . \tag{4}$$

Here in order to avoid introducing the concept of a periodic approach of a solution of (4) to a set and the concept of a periodic limit point let us confine ourselves to solutions $x(t)$ of (4) which are bounded for $t > 0$. Let Ω be the positive limit set of such a solution $x(t)$, and let p be a point in Ω. Then there is a solution of (4) starting at p which remains in Ω for all t in $(-\infty,\infty)$; that is, if one starts at p at the proper time the solution remains in Ω for all time. This is the sense now in which Ω is an invariant set. Let $V(t,x)$ be C^1 on $R \times R^n$ and periodic in t of period T. For an arbitrary set G of R^n we say that V is a Liapunov function on G for

for the periodic system (4) if $\dot{V}$ does not change sign for all t and all x in G. Define $E = \{ (t,x);\ \dot{V}(t,x) = 0,\ x \text{ in } \overline{G} \}$ and let M be the union of all solutions $x(t)$ of (4) with the property that $(t,x(t))$ is in E for all t. M could be called "the largest invariant set relative to E". One then obtains the following version of Theorem 2 for periodic systems:

THEOREM 3. If V is a Liapunov function on G for the periodic system (4), then each solution of (4) that is bounded and remains in G for all $t > 0$ $(t < 0)$ approaches M as $t \to \infty$ $(t \to -\infty)$.

In [6] Miller showed that the limit sets of solutions of almost periodic systems have a similar invariance property and from this he obtains a result quite like Theorem 3 for almost periodic systems. This then yields for periodic and almost periodic systems a whole chain of theorems on stability and instability quite similar to that for autonomous systems. For example, one has

COROLLARY 4. Let $Q_\eta^+ = \{ x;\ V(t,x) < \eta, \text{ all } t \text{ in } [0,T] \}$, and let G^+ be a component of Q_η^+. Let G be the component of $Q_\eta = \{ x;\ V(t,x) < \eta \text{ for some } t \text{ in } [0,T] \}$ containing G^+. If G is bounded, $\dot{V} \leq 0$ for all t and all x in G, and if $M^0 = \overline{M \cap G} \subset G^+$, then M^0 is an attractor and G^+ is in its region of attraction. If $V(t,x) = \varphi(t)$ for all t and all x on the boundary of M^0, then M^0 is a stable attractor.

Our last example of an invariance principle for ordinary differential equations is that due to Yoshizawa in [1] for "asymptotically autonomous" systems. It is a consequence of Theorem 1 and results by Markus and Opial (see [1] for references) on the limit sets of such systems. A system of the form

$$\dot{x} = F(x) + g(t,x) + h(t,x) \tag{5}$$

is said to be <u>asymptotically</u> <u>autonomous</u> if (i) $g(t,x) \to 0$ as $t \to \infty$ uniformly for x in an arbitrary compact set of R^n, (ii) $\int_0^\infty |h(t,\varphi(t))|\, dt < \infty$ for all φ bounded and continuous on $[0,\infty)$ to R^n. The combined results of Markus and Opial then state that the positive limit sets of solutions of (5) are invariant sets of $\dot{x} = F(x)$. Using this, Yoshizawa then improved Theorem 1 for asymptotically autonomous systems.

It turns out to be useful, as we shall illustrate in a moment on the simplest possible example, in studying systems (1) which are not necessarily asymptotically autonomous to state the theorem in the following manner:

THEOREM 4. If, in addition to the conditions of Theorem 1, it is known that a solution $x(t)$ of (1) remains in G for $t > 0$ and is also a solution of an asymptotically autonomous system (5), then $x(t)$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$, where M is the largest invariant set of $\dot{x} = F(x)$ in E.

It can happen that the system (1) is itself asymptotically autonomous in which case the above theorem can be applied. However, as the following example illustrates, the original system may not itself be asymptotically autonomous but it still may be possible to construct for each solution of (1) an asymptotically autonomous system (5) which it also satisfies.

Consider again the example

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - p(t)y \qquad , \qquad 0 < \delta \leq p(t) \leq m \\ & \qquad\qquad\qquad\qquad \text{for all } \ t > 0 \end{aligned} \tag{6}$$

Now we have the additional assumption that $p(t)$ is bounded from above. Let $(\bar{x}(t), \bar{y}(t))$ be any solution of (6). As was argued previously below Theorem 1, all solutions are bounded and $\bar{y}(t) \to 0$ as $t \to \infty$. Now $(\bar{x}(t), \bar{y}(t))$ satisfies $\dot{x} = y$, $y =$ $-x - p(t)\bar{y}(t)$, and this system is asymptotically autonomous to (*) $\dot{x} = y$, $\dot{y} = -x$. With the same Liapunov function as before, E is the x-axis and the largest invariant set of (*) in E is the origin. Thus for (6) the origin is asymptotically stable in the large.

3. Autonomous functional differential equation.

Difference differential equations of the form

$$\dot{x}(t) = f(t,x(t),x(t-r)) \qquad , \qquad r > 0 \tag{7}$$

have been studied almost as long as ordinary differential equations and these as well as other types of systems are of the general form

$$\dot{x}(t) = f(t,x_t) \tag{8}$$

where x is in R^n and x_t is the function defined on $[-r,0]$ by $x_t(\tau) = x(t+\tau)$, $-r \leqq \tau \leqq 0$. Thus x_t is the function that describes the past history of the system on the interval $[t-r,t]$ and in order to consider it as an element in the space C of continuous functions all defined on the same interval $[-r,0]$, x_t is taken to be the function whose graph is the translation of the graph of x on the interval $[t-r,t]$ to the interval $[-r,0]$. Since such equations have had a long history it seems surprising that it is only within the last 10 years or so that the geometric theory of ordinary differential equations has been successfully carried over to functional differential equations. Krasovskii [8] has demonstrated the effectiveness of a geometric approach in extending the classical Liapunov theory, including the converse theorems, to functional differential equations. An account of other aspects of their theory which have yielded to this geometric approach can be found in the paper [9] by Hale. What we wish to do here is to present Hale's extension in [2] of the results of Section 2 of this paper to autonomous functional differential equations

$$\dot{x} = f(x_t) . \tag{9}$$

It is this extension that has had so far the greatest success in studying stability properties of the solutions of systems (9), and it is possible that this may lead to a similar theory for special classes of systems defined by partial differential equations.

With $r \geqq 0$ the space C is the space of continuous functions φ on $[-r,0]$ to R^n with $\|\varphi\| = \max\{|\varphi(\tau)|;\ -r \leqq \tau \leqq 0\}$. Convergence in C is uniform convergence on $[-r,0]$. A function x defined on $[-r,\infty)$ to R^n is said to be a <u>solution</u> of (9) <u>satisfying the initial condition</u> φ at time $t = 0$ if there is an $a > 0$ such that $\dot{x}(t) = f(x_t)$ for all t in $[0,a)$ and $x_0 = \varphi$. Remember $x_0 = \varphi$ means $x(\tau) = \varphi(\tau)$, $-r \leqq \tau \leqq 0$. At $t = 0$, $\dot{x}$ is the right hand derivative. The existence uniqueness theorems are quite similar to those for ordinary differential equations. If f is locally Lipschitzian on C, then for each φ in C there is one and only one solution of (9) and the solution depends continuously on φ. The solution can also be extended in C for $t > 0$ as long as it remains bounded. As in Section 2, we will always speak as though solutions are defined on $[-r,\infty)$. The space C is now the state space of (9) and through each point φ of C there is the motion or flow x_t starting at φ defined by the solution $x(t)$ of (9) satisfying at time $t = 0$ the initial condition φ; x_t, $0 \leqq t < \infty$, is a curve in C which starts at time $t = 0$ at φ. In analogy to Section 2 with C replacing R^n, x_t replacing $x(t)$, and $\|x_t\|$ replacing $|x(t)|$, we define the <u>distance</u> $d(x_t,E)$ of x_t from a closed set E of C to be $d(x_t,E) = \min\{\|x_t-\psi\|;\ \psi \in E\}$. The positive limit set of x_t is then defined in a manner completely analogous to Section 2. Because there are some important differences we shall be satisfied here with restricting ourselves to motions

x_t bounded for $t > 0$. One of the differences here is that in C closed and bounded sets are not always compact. Another is that although we have uniqueness of solutions in the future two motions starting from different initial conditions can come together in finite time $t_0 > 0$; after this they coincide for $t \geqq t_0$. (The motions define semi-groups and not necessarily groups.)

Hale in [2] has, however, shown that the positive limit sets Ω of bounded motions x_t are nonempty, compact, connected, invariant sets in C. Invariance here is in the sense that, if x_t is a motion starting at a point of Ω, then there is an extension onto $(-\infty,-r]$ such that $x(t)$ is a solution of (9) for all t in $(-\infty,\infty)$ and x_t remains in Ω for all t. With this result he is then able to obtain a result which is similar to Corollary 1 of Section 2.

For $\varphi \in C$ let $x_t(\varphi)$ denote the motion defined by (9) starting at φ. For V a continuous function on C to R define $\dot{V}$ and Q_ℓ by

$$\dot{V}(\varphi) = \overline{\lim_{\tau \to 0+}} \frac{1}{\tau}[V(x_\tau(\varphi))-V(\varphi)]. \tag{10}$$

and

$$Q_\ell = \{\varphi \ ; \ V(\varphi) < \ell\} .$$

THEOREM 5. If V is a Liapunov function* on G for (9) and x_t is a trajectory of (9) which remains in G and is bounded for $t > 0$, then $x_t \to M$ as $t \to \infty$.

* As before, V is a Liapunov function on G, if $\dot{V}$ does not change sign on G.

Hale has also given the following more useful version of this result.

COROLLARY 5. Define $Q_\eta = \{\varphi;\ V(\varphi) < \eta\}$ and let G be Q_η or a component of Q_η. Assume that V is a Liapunov function on G for (9) and that either (i) G is bounded or (iii) $|\varphi(0)|$ is bounded for φ in G. Then each trajectory starting in G approaches M as $t \to \infty$.

The following is an extension of Četaev's instability theorem. This is a somewhat simplified version of Hale's Theorem 4 in [2], which should have stated "$V(\varphi) > 0$ on U when $\varphi \neq 0$ and $V(0) = 0$" and at the end "... intersect the boundary of C_γ ...". This is clear from his proof and is necessary since he wanted to generalize the usual statment of Četaev's theorem to include the possibility that the equilibrium point be inside U as well as on its boundary.

COROLLARY 6. Let $p \in C$ be an equilibrium point of (9) contained in the closure of an open set U and let N be a neighborhood of p. Assume that (i) V is a Liapunov function on $G = U \cap N$, (ii) $M \cap G$ is either the empty set or p, (iii) $V(\varphi) < \eta$ on G when $\varphi \neq p$, and (iv) $V(p) = \eta$ and $V(\varphi) = \eta$ on that part of the boundary of G inside N. Then p is unstable. In fact, if N_0 is a bounded neighborhood of p properly contained in N then each trajectory starting at a point of $G_0 = G \cap N_0$ other than p leaves N_0 in finite time.

Proof. By the conditions of the corollary and Theorem 6 each trajectory starting inside G_o at a point other than p must either leave G_o, approach its boundary or approach p. Conditions (i) and (iv) imply that it cannot reach or approach that part of the boundary of G_o inside N_o nor can it approach p as $t \to \infty$. Now (ii) states that there are no points of M on that part of the boundary of N_o inside G. Hence each such trajectory must leave N_o in finite time. Since p is either in the interior or on the boundary of G, each neighborhood of p contains such trajectories, and p is therefore unstable.

In [2] it was shown that the equilibrium point $\varphi = 0$ of

$$\dot{x}(t) = ax^3(t) + bx^3(t-r)$$

was unstable if $a > 0$ and $|b| < |a|$. Using the same Liapunov function and Theorem 5 we can show a bit more. With

$$V(\varphi) = -\frac{\varphi^4(0)}{4a} + \tfrac{1}{2}\int_{-r}^{0} \varphi^6(\theta)d\theta \ ,$$

$$V(x_t) = -\frac{x^4(t)}{4a} + \tfrac{1}{2}\int_{t-r}^{t} x^6(\theta)d\theta$$

and

$$\dot{V}(\varphi) = -\tfrac{1}{2}(\varphi^6(0) + 2\,\frac{b}{a}\,\varphi^3(0)\varphi^3(-r) + \varphi^6(-r))$$

which is nonpositive when $|b| < |a|$ (negative definite with respect to $\varphi(0)$ and $\varphi(-r)$); that is, V is a Liapunov function on C and $E = \{\varphi;\ \varphi(0) = \varphi(-r) = 0\}$. Therefore M is simply the null function $\varphi = 0$. If $a > 0$, the region $G = \{\varphi;\ V(\varphi) < 0\}$

is nonempty, and no trajectory starting in G can have $\varphi = 0$ as a positive limit point nor can it leave G. Hence by Theorem 5 each trajectory starting in G must be unbounded. Since $\varphi = 0$ is a boundary point of G, it is unstable. It is also easily seen [2] that if $a < 0$ and $|b| < |a|$, then $\varphi = 0$ is asymptotically stable in the large.

In [2] Hale has also extended this theory for systems with infinite lag $(r = \infty)$, and in that same paper gives a number of significant examples of the applications of this theory.

References

[1] Yoshizawa, T., Asymptotic behavior of solutions of a system of differential equations, Contrib. to Diff. Eq., 1(1963), 371-387.

[2] Hale, J., Sufficient conditions for stability and instability of autonomous functional differential equations, J. of Diff. Eq., 1(1965), 452-482.

[3] LaSalle, J., Some extensions of Liapunov's second method, IRE Trans. on Circuit Theory, CT-7(1960), 520-527.

[4] LaSalle, J., Asymptotic stability criteria, Proc. of Symposia in Applied Mathematics, Vol. 13, Hydrodynamic Instability, Amer. Math. Soc., Providence, R.I., 1962, 299-307.

[5] LaSalle, J., and Lefschetz, S., Stability by Liapunov's Direct Method with Applications, Academic Press, New York, 1961.

[6] Miller, R., On almost periodic differential equations, Bul. Amer. Math. Soc., 70(1964), 792-795.

[7] Miller, R., Asymptotic behavior of nonlinear delay-differential equations, J. of Diff. Eq., 3(1965), 293-305.

[8] Krasovskii, N.N., Stability of Motion, Stanford University Press, 1963(translation of 1959 Russian Edition).

[9] Hale, J., "Geometric theory of functional differential equations", Proc. of An International Symposium on Differential Equations and Dynamical Systems, University of Puerto Rico, Mayaguez, P. R., Dec. 1965, Academic Press, New York (to appear).

ANALYTICAL SOLUTION OF EULER-LAGRANGE EQUATIONS FOR OPTIMUM COAST TRAJECTORIES†

Chiu H. Ng* and
Peter J. Palmadesso*

The Boeing Company
Huntsville, Alabama

An analytical solution of the Euler-Lagrange equations for the Lagrange multipliers for optimum coast trajectories (minimum fuel consumption) is obtained. Previous solutions have a singularity at zero eccentricity. The present solution does not have this singularity, but there is a numerical difficulty due to a removable singularity at unit eccentricity. An approximate solution, accurate near unit eccentricity, is given. This solution reduces to the exact parabolic solution for unit eccentricity.

---

† This study was part of the work done under Contract NAS8-11244, "Advanced Flight Mechanics and Guidance Studies", carried out for the Aero-Astrodynamics Laboratory of the Marshall Space Flight Center, Huntsville, Alabama.

* Research Engineers, Flight Mechanics and Guidance Research Group

1. INTRODUCTION

Optimization of the flight trajectory of a rocket powered space vehicle with the indirect method of calculus of variations requires the simultaneous integration of the equations of motion (constraint equations) and the Euler-Lagrange equations for the Lagrange multipliers. Because of the difficulty in obtaining an analytical solution to this problem during powered portions of the trajectory, the integration must be performed numerically. For coasting portions (zero thrust) of the trajectory an analytical solution for the motion of the vehicle can be produced if it is assumed that the vehicle moves in a vacuum under the action of the gravitational field of a single central body (spherical earth). The motion is, of course, governed by the classical Kepler solution of the two body problem. However, the Lagrange multipliers are not constant during coasting portions of the trajectory and it is necessary to solve the Euler-Lagrange equations for the multipliers to determine the optimal direction of thrust at the end of each coasting arc. It is the purpose of this paper to present a new analytical solution for the multipliers. This solution used with the appropriate form of Kepler's solution gives a particularly convenient analytical form of solution for optimal coasting arcs. When implemented in a numerical routine for trajectory computation, the analytical solution not only reduces computational time, but also eliminates the errors due to numerical integration for long coasting arcs. In studies made of the earth orbit rendezvous problem[1], it was found that long coasting arcs were often a necessity to avoid severe payload penalties. In this case, the analytical solution for coast is of especial advantage.

Analytical solutions for coast were presented by W. E. Miner[2] in 1963 and by S. A. Jurovics[3]. In the June, 1965, issue of the AIAA Journal, M. W. Eckenwiler[4] presented a solution very similar to Miner's. These solutions have the disadvantage of a singularity at zero eccentricity. Since circular and near circular orbits are of major interest, this singularity is removed in the present solution. A numerical difficulty appears here for near unit eccentricity. However, an exact solution for parabolic orbits and, in addition, an approximate solution for near parabolic orbits that eliminates the numerical difficulty are given.

2. MINIMUM FUEL TRAJECTORY

The equations of motion for a rocket in the gravity field of a spherical, homogeneous earth are:

$$\begin{aligned} \dot{\overline{R}} &= \overline{V} \\ \dot{\overline{V}} &= -\frac{\mu}{r^3}\overline{R} + \frac{\overline{T}}{m} \end{aligned} \qquad (1)$$

2. (Continued)

The state variables are the position vector $\bar{R}$, the velocity $\bar{V}$, and the mass m. The control variable is the thrust vector $\bar{T}$. The gravitational constant of the central boyd and the magnitude of $\bar{R}$ are denoted by μ and r, respectively. The mass flow rate of the rocket is assumed to be constant, and given by

$$\dot{m} = -\dot{M} \tag{2}$$

For minimum fuel consumption, it is required that

$$\int_{t_o}^{t_f} - \dot{m}\, dt = \text{MINIMUM} \tag{3}$$

Appending the equations of motion, Eqs. (1) and the mass flow requirement Eq. (2) as constraints with the Lagrange multiplier techniques, the following integral must be a minimum

$$\Phi = \int_{t_o}^{t_f} [-\dot{m} + \bar{\lambda} \cdot (\dot{\bar{V}} + \frac{\mu}{r^3} \bar{R} - \frac{\bar{T}}{m}) + \bar{\gamma} \cdot (\dot{\bar{R}} - \bar{V}) + \sigma(\dot{m} + \dot{M})]dt \tag{4}$$

The components of the vectors $\bar{\lambda}$ and $\bar{\gamma}$, and σ are the Lagrange multipliers. The Euler-Lagrange equations for these Lagrange multipliers are:

$$\dot{\bar{\lambda}} + \bar{\gamma} = 0$$

$$\dot{\bar{\gamma}} - \frac{\mu}{r^3} \bar{\lambda} + \frac{3\mu}{r^5} (\bar{\lambda} \cdot \bar{R})\bar{R} = 0$$

$$\frac{\bar{T}}{T} - \frac{\bar{\lambda}}{\lambda} = 0$$

$$\sigma - \frac{1}{m^2} \bar{\lambda} \cdot \bar{T} = 0 \tag{5}$$

where T and λ are the magnitudes of $\bar{T}$ and $\bar{\lambda}$, respectively. The third of these equations shows that $\bar{\lambda}$ is the same direction as the thrust. When the thrust is zero, the last equation implies that σ is a constant. The following differential equation for $\bar{\lambda}$ comes from the first two of Eqs. (5).

$$\ddot{\bar{\lambda}} = - \frac{\mu}{r^3} \bar{\lambda} + \frac{3\mu}{r^5} (\bar{\lambda} \cdot \bar{R})\bar{R} \tag{6}$$

This equation and the constraint equations have to be satisfied along the optimum trajectory.

3. ANALYTICAL SOLUTION FOR LAGRANGE MULTIPLIERS DURING COAST PERIOD

During coast the plane of motion is fixed, and from Eq. (6) the differential equation for the component of $\bar{\lambda}$ normal to the plane of motion becomes

$$\ddot{\lambda}_N = -\frac{\mu}{r^3} N \tag{7}$$

Let x and y represent the components of the Kepler solution in a cartesian coordinate system with the x axis directed toward perigee and the y axis in the plane of motion. By comparing Eq. (7) with the equation of motion during coast, we see that the solution of Eq. (7) is

$$\lambda_N = K_1 x + K_2 y \tag{8}$$

where K_1 and K_2 are the integration constants. It is particularly convenient to assume the solution of Eq. (6) for the projection of $\bar{\lambda}$ on the plane of motion to have the form

$$\bar{\lambda}_P = F\bar{R} + G\dot{\bar{R}} \tag{9}$$

The F and G are, in general, functions of time and uniquely define $\bar{\lambda}_P$ when the position and the velocity vectors do not coincide.

Note that the form of Eq. (9) is similar to that assumed for the position vector in terms of the classical f and g series[5]. Here, F and G are not these series, but, as shall be found, finite expressions involving quantities from the Kepler solution.

Substituting Eq. (9) into Eq. (6), we have after manipulation

$$\ddot{\bar{\lambda}}_P = \frac{\mu}{r^3}\left(2F + 3\frac{\dot{r}}{r} G\right)\bar{R} - \frac{\mu}{r^3} G\dot{\bar{R}} \tag{10}$$

Eq. (9) is differentiated twice with respect to time. This yields:

$$\ddot{\bar{\lambda}}_P = \left(\ddot{F} + \frac{3\mu}{r^4}\dot{r}G - \frac{2\mu}{r^3}\dot{G} - \frac{\mu}{r^3}F\right)\bar{R} + \left(2\dot{F} - \frac{\mu}{r^3}G + \ddot{G}\right)\dot{\bar{R}} \tag{11}$$

By comparing Eqs. (10) and (11), we see that:

$$\ddot{G} = -2\dot{F} \tag{12}$$

$$\ddot{F} = \frac{\mu}{r^3}\left(3F + 2\dot{G}\right) \tag{13}$$

3. (Continued)

From Eq. (12), we have

$$\dot{G} = K_3 - 2F \tag{14}$$

By substituting Eq. (14) into Eq. (13), a differential equation for F alone is obtained

$$\ddot{F} + \frac{\mu}{r^3}(F - 2K_3) = 0 \tag{15}$$

This equation has the solution

$$F = K_4 X + K_5 Y + 2K_3 \tag{16}$$

where, again, x and y are the components of the Kepler solution and K_3, K_4, K_5 are integration constants. The function G is obtained by integrating Eq. (14).

$$G = \int (K_3 - 2F)dt = - \int (3K_3 + 2K_4 X + 2K_5 Y)dt$$

In order to carry out this integration, the eccentric anomaly of the Keplerian motion is used as the integration variable for elliptical orbits.

$$G = -3K_3 t - 2K_4 \int \frac{a}{n}(\cos u - e)(1 - e \cos u)du - 2K_5 \int \frac{a}{n}(1 - e^2)^{\frac{1}{2}}$$
$$\sin u(1 - e \cos u)du$$

The integration gives

$$G = -3K_3 t + \frac{1}{2}K_4 \frac{\mu}{E}[\frac{Y}{L}(r + P) - 3et] + K_5 \frac{X}{L} P(r + P) + K_6 \tag{17}$$

where
- t = time
- a = semi-major axis
- n = mean motion
- E = total energy
- L = angular momentum
- P = semi-latus rectum
- e = eccentricity
- K_6 = integration constant

The G function has exactly the same form for hyperbolic orbits when the integration is performed by using hyperbolic anomaly as the integration variable. Similarly, for parabolic orbits, the G function becomes

$$G = -3K_3 t + \frac{2}{5}K_4 \frac{Y}{L}[r^2 - P(r + P)] + K_5 \frac{X}{L} P(r + P) + K_6 \tag{18}$$

3. (Continued)

Finally, $\overline{Y}$ can be obtained from

$$\overline{Y} = -\dot{\overline{\lambda}} = -G\ddot{\overline{R}} + L\overline{D} + K_3\dot{\overline{R}} - (K_1\dot{X} + K_2\dot{Y})\overline{k} \tag{19}$$

with

$$\overline{D} = -K_5\ \overline{i} + K_4\ \overline{j}$$

where $\overline{i}$, $\overline{j}$ and $\overline{k}$ are unit vectors of the perigee oriented coordinates.

4. APPROXIMATE SOLUTION OF COAST FOR ORBITS WITH ECCENTRICITY VERY CLOSE TO UNITY

The elliptical solution as found in the previous section is

$$\lambda_N = K_1X + K_2Y \tag{8}$$

$$\overline{\lambda}_P = F\overline{R} + G\dot{\overline{R}} \tag{9}$$

$$F = K_4X + K_5Y + 2K_3 \tag{16}$$

$$G = -3K_3t + \frac{1}{2}K_4\frac{\mu}{E}\ [\frac{Y}{L}(r + P) - 3et] + K_5\frac{X}{L}P(r + P) + K_6 \tag{17}$$

All terms are well behaved numerically as eccentricity approaches unity except the term with coefficient K_4 in the G-function. The energy E and the squared bracket term vanish separately at unit eccentricity but their ratio is finite. To eliminate this numerical difficulty, let eccentricity $e = 1 - \varepsilon$ with $\varepsilon << 1$, then the eccentric anomaly u is also much less than 1 and can be expanded in a series of sin u, which is:

$$u = \sin u + \frac{1}{2} \cdot \frac{1}{3} \sin^3 u + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \sin^5 u + \ldots. \tag{20}$$

This series is truncated after the third term and substituted into Kepler's equation:

$$t = \frac{1}{n}(u - e \sin u) \tag{21}$$

This yields

$$t = \frac{1}{n}[\sin u + \frac{1}{6}\sin^3 u + \frac{3}{40}\sin^5 u - (1 - \varepsilon)\sin u]$$

$$= \frac{1}{n}[\varepsilon \sin u + \frac{1}{6}\sin^3 u + \frac{3}{40}\sin^5 u]$$

4. (Continued)

Since

$$\sin u = \frac{Y}{P}(1 - e^2)^{\frac{1}{2}} = \frac{Y}{P}\,\varepsilon^{\frac{1}{2}}(2 - \varepsilon)^{\frac{1}{2}}$$

and

$$n = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} = \frac{L}{P^2}(1 - e^2)^{\frac{3}{2}} = \frac{L}{P^2}\,\varepsilon^{\frac{3}{2}}(2 - \varepsilon)^{\frac{3}{2}}$$

Kepler's equation becomes

$$t = \frac{Y}{3L}\left[\frac{3}{2}P\left(1 + \frac{\varepsilon}{2}\right) + \frac{1}{2}\frac{Y^2}{P} + \varepsilon\frac{9}{20}\,\frac{Y^4}{P^3}\right] \tag{22}$$

The second and higher order terms in ε have been dropped in the last expression for time. These terms will also be neglected in the derivation hereafter.

Now

$$Y^2 = a^2(1 - e^2)\sin^2 u = aP(1 + \cos u)(1 - \cos u)$$

$$\frac{Y^2}{P} = (1 + \cos u)a[1 - (e + \varepsilon)\cos u]$$

$$= (1 + \cos u)(r - a\varepsilon\cos u)$$

Since

$$\cos u = \frac{1}{e}\left(1 - \frac{r}{a}\right) = (1 + \varepsilon)\left(1 - \frac{r}{a}\right) = 1 + \varepsilon\left(1 - 2\frac{r}{P}\right)$$

with

$$a\varepsilon = \frac{P}{2}\left(1 + \frac{\varepsilon}{2}\right)$$

thus

$$\frac{Y^2}{P} = (2r - P) + 2\varepsilon\left[(2r - P) - \frac{r^2}{P}\right] \tag{23}$$

Substituting Eq. (23) into Eq. (22), yields

$$t = \frac{Y}{3L}\ (r + P) + \frac{\varepsilon}{5}\left[(r + P) + 4\frac{r^2}{P}\right] \tag{24}$$

4. (Continued)

With this expression for time, the term with coefficient K_4 in the G-function is

$$\frac{1}{2}K_4 \frac{\mu}{E} \left\{\frac{Y}{L}(r + P) - (1 - \varepsilon) \frac{Y}{L} [(r + P) + \frac{\varepsilon}{5}(r + P + 4\frac{r^2}{P})]\right\}$$

$$= \frac{2}{5}K_4 \frac{Y}{L} \left\{r^2 - P(r + P) - \frac{\varepsilon}{4}[P(r + P) + 4r^2]\right\}$$

Finally we have

$$G = -3K_3t + \frac{2}{5}K_4 \frac{Y}{L} \left\{r^2 - P(r + P) - \frac{\varepsilon}{4}[P(r + P) + 4r^2]\right\} + \qquad (25)$$

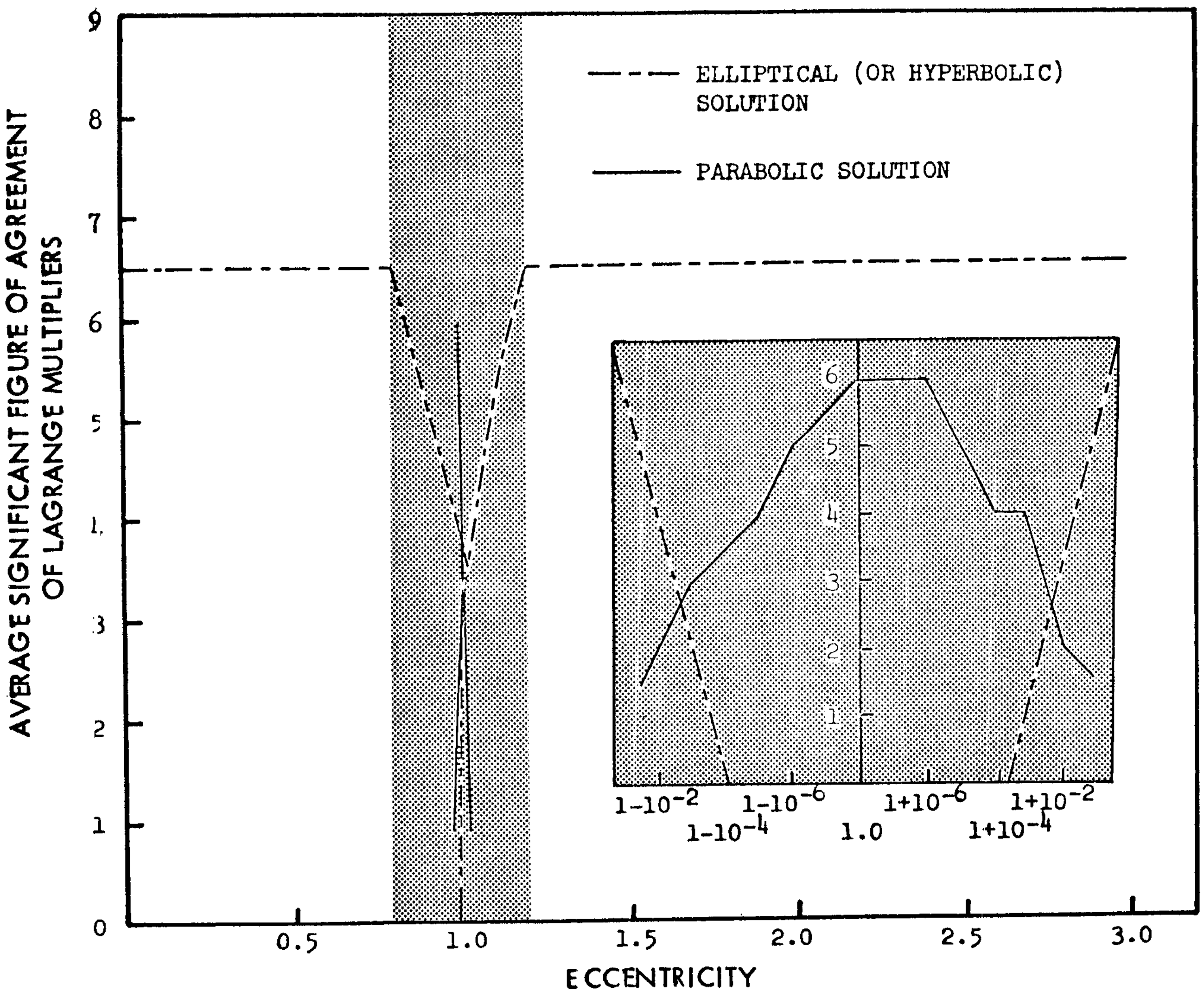
$$K_5\frac{X}{L}P(r + P) + K_6$$

When $\varepsilon = 0$, this becomes the parabolic solution given by Eq. (18).

5. DISCUSSION

The singularity at zero eccentricity that exists in previous analytical coast solutions does not appear in the solution presented here. The form of solution taken for the projection of $\bar{\lambda}$ onto the plane of motion eliminated this singularity and also gave a particularly simple form for the resulting solution. The absence of the singularity is of assistance for study of optimal trajectories for boost to circular and near circular orbits. With the improved accuracy due to elimination of integration and round off errors, the analytical solution presented is also useful for optimizing parking orbits.

The numerical difficulty for very nearly parabolic orbits can be handled by employing the approximate solution presented in Section 4. It is of interest to note, however, that in actual numerical work carried out, the exact parabolic solution gives satisfactory results in that region. The average of the significant figures of agreement of the Lagrange multipliers obtained from the analytical solution with the multipliers calculated by direct numerical integration is shown in the figure below. The comparison was made 100 seconds after the beginning of coast. Coast was initiated 1000 seconds after perigee. The values were calculated carrying eight digits. In the ranges of eccentricity away from unity, the difference in analytical and numerical values is due to error in the numerical integration. In these regions, there is no numerical difficulty with the analytical solution and it gives precise values. The results in the neighborhood of eccentricity $e = 1 \pm 10^{-3}$ can be improved by using the approximate solution.



ELLIPTICAL (OR HYPERBOLIC) SOLUTION
PARABOLIC SOLUTION
AVERAGE SIGNIFICANT FIGURE OF AGREEMENT OF LAGRANGE MULTIPLIERS
ECCENTRICITY
0.5
1.0
1.5
2.0
2.5
3.0
1-10^{-2}
1-10^{-4}
1-10^{-6}
1.0
1+10^{-6}
1+10^{-4}
1+10^{-2}

5. (Continued)

The analytical coast solution can be employed for three dimensional problems as well as planar cases. It is only necessary to rotate into the plane of motion at the start of coast and rotate back out of the plane to the original reference at the end of coast. The axes used must coincide with the perigee oriented axes used in developing the coast solution.

REFERENCES

1. "Trajectory Optimization and Guidance for the Earth Orbit Rendezvous Mission", Vol. 1, Boeing Document No. D5-13136 (1965)

2. W. E. Miner, "Transformation of the λ-Vector and Closed Form Solution of the λ-Vector on Coast Arcs", Aeroballistics Internal Note #20-63, George C. Marshall Space Flight Center, Huntsville, Alabama.

3. S. A. Jurovics, "Orbital Transfer by Optimum Thrust Direction and Duration", NASA TM X-53150, October 16, 1964.

4. M. W. Eckenwiler, "Closed Form Lagrangian Multipliers for Coast Periods of Optimum Trajectories", AIAA Journal, June, 1145-1151 (1965).

5. D. Brouwer and G. M. Clemence, "Methods of Celestial Mechanics", (Academic Press, 1961), Chapter I, P. 52.

NASA TM X-53478 APPROVAL June 23, 1966

PROGRESS REPORT NO. 8

Proceedings of the Twenty-Fourth Seminar
on
Space Flight and Guidance Theory

- - - - -

Sponsored by the Aero-Astrodynamics Laboratory
George C. Marshall Space Flight Center

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

E. D. Geissler, Director
Aero-Astrodynamics Laboratory

DISTRIBUTION

DIR

I-I/IB-MGR, Col. James

R-ASTR, Mr. Swanson
Mr. Moore
Mr. Richard
Mr. Gassaway
Mr. Taylor
Mr. Brooks
Mr. Hosenthien
Mr. Scofield
Mr. Woods
Mr. Digesu
Dr. George
Dr. R. Decher
Mrs. Neighbors

R-COMP, Dr. Arenstorf
Mr. Davidson
Mr. Horton
Mr. Schollard
Mr. Seely
Mr. Calhoun
Dr. Fehlberg
Mr. French

R-AERO, Dr. Geissler
Mr. Jean
Mr. Baker (25)
Mr. Dearman
Mr. Lovingood
Mr. Schwaniger
Mr. Ingram
Mr. Lisle
Mr. Hart
Mr. Thomae
Mr. Lindberg
Dr. H.G.L. Krause
Mr. R. Hill
Mr. Goldsby
Mr. Horn
Mr. LeMay
Mr. Hurst
Mr. T. Deaton

R-AERO, Mr. Dickey
Mr. Blair
Mrs. Chandler
Mr. Stone
Dr. Sperling
Dr. Heybey
Mr. Cummings
Mr. Vaughan

MS-T, Mr. Bland (6)

AST-S, Dr. Lange

R-DIR, Mr. Weidner
Dr. McCall

MS-IP

MS-IL (8)

I-RM-M

MS-H

CC-P

NASA Technical & Scientific
Information Facility (25)
ATTN: NASA Representative
P. O. Box 33
College Park, Maryland 20740

Mr. Richard Auelmann
Astrodynamics Section
Astrosciences Department
AERONUTRONIC Division of Philco Corporation
Ford Road
Newport Beach, California 92660

Dr. W. A. Shaw
Mechanical Engineering Department
AUBURN University
Auburn, Alabama 36830

Mr. Richard Hardy
Mail Stop AG-05
BOEING Company
P. O. Box 1680
Huntsville, Alabama 35807

Dr. J. P. La Salle
Center for Dynamical Systems
BROWN University
Providence, Rhode Island 02912

Dr. P. B. Krishnaswamey
Department of Electrical Engineering
Drexel Institute of Technology
32nd and Chestnut Streets
Philadelphia, Pennsylvania 19104

Dr. Edward Norman
Department of Mathematics
DREXEL Institute of Technology
32nd and Chestnut Streets
Philadelphia, Pennsylvania 19104

Dr. T. J. Pignani
Department of Mathematics
East Carolina College
Greenville, North Carolina 27834

Dr. Paul B. Richards
Aerospace Group
GENERAL PRECISION, INC.
1150 McBride Avenue
Little Falls, New Jersey 07424

Mr. Otto E. Kirchner, Jr.
Box 85-90, Space Division
The Boeing Company
P. O. Box 3707
Seattle, Washington 98124

Dr. Bernard Friedland
Staff Scientist-Control
General Precision, Inc.
Research Center
Little Falls, New Jersey 07424

Dr. Gunther Geiss
Control Theory Research Group
Grumman Aircraft Engineering Corporation
Bethpage, Long Island, New York 11714

Mr. Harry Passmore
Hayes International Corporation
P. O. Box 2287
Birmingham, Alabama 35201

Mr. Alan L. Friedlander
Research Engineer
Guidance & Control Section
IIT Research Center
10 W. 35th Street
Chicago, Illinois 60616

Dr. David F. Bender
Space Sciences Laboratory
Space and Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Dr. John Morrison
Applied Mathematics Subdivision
Research Division
Engineering Department
Republic Aviation Corporation
Farmingdale, Long Island, New York 11735

Dr. Robert W. Hunt
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62903

Mr. J. L. Kooley
Research Laboratories
United Aircraft Corporation
East Hartford, Connecticut 06108

Mr. J. W. Hanson
Computation Center
University of North Carolina
Chapel Hill, North Carolina 27515

Dr. M. G. Boyce
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37203

Dr. C. D. Johnson
University of Alabama
Research Institute
301 Sparkman Drive, N. W.
Huntsville, Alabama 35805

Prof. Y. H. Ku
Moore School of Electrical Engineering
University of Pennsylvania
Philadelphia, Pa. 19104

Prof. L. Markus
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455

Prof. L. G. Clark
Department of Engineering Mechanics
University of Texas
Austin, Texas 78712

Prof. S. Lefschetz
11 Lake Lane
Princeton, New Jersey

Dr. R. P. O'Shea
Department of Electrical Engineering
University of Pittsburg
Pittsburg, Pa. 15213

Dr. Charles C. Conley
Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53705

Dr. Bob Plunket
University of Alabama
University, Alabama 35486

Dr. K. J. Davis
Department of Mathematics
East Carolina College
Greenville, North Carolina 27834

Dr. Seymour Sherman
General Precision, Inc.
Aerospace Research Center
Little Falls, New Jersey 07424

Mr. Samuel Pines
Analytical Mechanics Associates, Inc.
57 Old Country Road
Westbury, New York 11590

Mr. Dahlard Lukes
Military Products Group
Systems and Research Division
2345 Walnut Street
Honeywell, Inc.
St. Paul, Minnesota 55113

Dr. I. E. Perlin, Head
Mathematical Analysis Branch
Rich Computer Center
Georgia Institute of Technology
Atlanta, Georgia 30332

Dr. William Nesline
Raytheon Company
Sudbury, Massachusetts 01776

Mr. R. K. Squires
Code 640
Goddard Space Flight Center
Greenbelt, Maryland 20771

Mr. Charles F. Pontious
Office of Advanced Research & Technology
National Aeronautics & Space Administration
Washington, D. C. 20025

Mr. Jules Kanter
Office of Advanced Research & Technology
National Aeronautics & Space Administration
Washington, D. C. 20025

Mr. Arthur G. Bennett
Mail Stop AK-39
Boeing Company
Huntsville, Alabama

Mr. David Thomas
NASA-Langley Research Center
Space Mechanics Division
Langley Station
Hampton, Virginia 23362

Mr. R. E. Holmen
Guidance & Control Section, A2-263
Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California 90405

Dr. William Byrd
Department of Physics
East Carolina College
Greenville, North Carolina 27834

Dr. Robert Novosad
Mail #0522
Martin Company
P. O. Box 179
Denver, Colorado 80201

Mr. Donald Jezewski
Guidance Analysis Branch
Spacecraft Technology Division
Manned Spacecraft Center
Houston, Texas 77001

Mr. Ernest S. Armstrong
NASA-Langley Research Center
Hampton, Virginia 23365

Mr. J. D. Byrd
Langley Research Center
Langley Station
Hampton, Virginia 23365

Mr. J. S. White
Ames Research Center
Moffett Field, California 94035

Dr. Ted Guinn
Solid & Fluid Physics Department
Research and Development
Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California 90405

Mr. Walter L. Portugal
Manager, Systems Sales
General Precision, Inc.
Little Falls, New Jersey 07424

Dr. S. H. Lehnigk
Physical Sciences Laboratory
Army Missile Command - Bldg. 5429
Redstone Arsenal, Alabama

Dr. Rudolf Hermann, Director
University of Alabama Research Institute
4701 University Avenue, N. W.
Huntsville, Alabama

Dr. M. L. Anthony
Space Flight Technology
Mail No. A-153
The Martin Company
Denver, Colorado 80201

Mr. Wes Morgan
Mail Stop AF-74
Boeing Company
P. O. Box 1680
Huntsville, Alabama

Mr. Robert A. Lerman
Senior Analytical Engineer
Technical Planning
Hamilton Standard Division
United Aircraft Corporation
Windsor Locks, Connecticut 06096

Mr. Robert G. Chilton
NASA-Manned Spacecraft Center
P. O. Box 1537
Houston, Texas 77001

Dr. W. G. Melbourne
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

Dr. J. S. Rosen
General Electric Company
Holiday Office Center
Suite 22
Huntsville, Alabama

Dr. E. R. Rang
Military Products Group
Research Laboratory
Honeywell, Inc.
2345 Walnut Street
St. Paul, Minnesota 55113

Dr. C. A. Harvey
Military Products Group
Research Laboratory
Honeywell, Inc.
2345 Walnut Street
St. Paul, Minnesota 55113

Dr. E. G. Rynaski
Flight Research Department
Cornell Aeronautical Laboratory
Box 235
Buffalo, New York 14221

Mr. J. M. Johnson, Jr.
AUTONETICS
A Division of North American Aviation
Data Systems Division
Department 345-4
P. O. Box 4171
Anaheim, California 92803

Dr. N. N. Puri
General Electric Company
Reentry Systems Division
32nd and Chestnut Streets
Philadelphia, Pennsylvania 19104

Dr. Steve Hu
Northrop Space Laboratories
Technology Drive
Huntsville, Alabama

Mr. Frank J. Sullivan
Code RE
National Aeronautics & Space Administration
Washington, D. C. 20025

Dr. Joseph F. Shea
Apollo Program Manager
Apollo Spacecraft Program Office
Manned Spacecraft Center
Houston, Texas 77058

Mr. R. J. Hayes
Space Guidance Laboratory
NASA-Electronics Research Center
575 Technology Square
Cambridge, Massachusetts 02139

Mr. T. Perkins
Chrysler Corporation
HIC Building
Huntsville, Alabama

Mr. Gary P. Herring
Chrysler Corporation
HIC Building
Huntsville, Alabama

Mr. Y. L. Luke
Mathematics & Physics Division
Midwest Research Institute
425 Volker Boulevard
Kansas City, Missouri 64110

Mr. J. W. Scheuch
North American Aviation, Inc.
P. O. Box 557
Huntsville, Alabama

Mr. R. W. Reck
Martin Company
3312 S. Memorial Parkway
Huntsville, Alabama

Dr. Stanley E. Ross
Chief, Advanced Program Analysis
Advanced Manned Missions
Systems Engineering
National Aeronautics & Space Administration
Washington, D. C. 20003

Dr. Joseph W. Siry
Theory & Analysis Office
Data Systems Division
Goddard Space Flight Center
Greenbelt, Maryland 20771

Dr. Henry Hermes
Assistant Professor
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

Dr. J. C. Eaves
Department of Mathematics & Astronomy
University of Kentucky
Lexington, Kentucky

Dr. Byron D. Tapley
Department of Aerospace Engineering
University of Texas
Austin, Texas

Mr. J. S. Farrior
Lockheed
P. O. Box 1103 West Station
Huntsville, Alabama

Mr. G. E. Christopher
Manager, Dynamics & Guidance
Lockheed Missiles & Space Company
4800 Bradford Drive
Huntsville, Alabama

Dr. David A. Ford
Department of Mathematics
Emory University
Atlanta, Georgia

Mr. D. K. Sharmack
Military Products Group
Systems & Research Division
2345 Walnut Street
Honeywell, Inc.
St. Paul, Minnesota 55113

Dr. E. B. Lee
Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota

Dr. J. Andrus
General Electric Company
Holiday Office Center
Suite 22
Huntsville, Alabama

Mr. Daniel B. Killeen
Computer Laboratory
Norman Mayer Bldg.
Tulane University
New Orleans, Louisiana 70118

Mr. Myron Schall
Supervisor, Powered Trajectory
Space & Information Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Mr. J. Mayne Lewallen
NASA, Manned Spacecraft Center, ED13
Houston, Texas 77058

Mr. Yoshihide Kozai
Smithsonian Institution
Astrophysical Observatory
60 Garden Street
Cambridge, Massachusetts 02138

Dr. Rudolph Kalman
Research Institute for Advanced Study
7212 Bellona Avenue
Baltimore, Maryland 21212

Mr. Ken Kissel
Aeronautical Systems Division
Applied Mathematics Research Branch
Wright-Patterson Air Force Base
Dayton, Ohio 45433

Mr. Jack Funk
Manned Spacecraft Center
Flight Dynamics Branch
Houston, Texas 77058

Dr. J. B. Rosser
Department of Mathematics
Cornell University
Ithaca, New York 14850

Mr. Bryan F. Dorlin
Theoretical Guidance & Control Branch
NASA, Ames Research Center
Moffett Field, California 94035

Lt. Col. R. A. Newman
Air Force Space Systems Division
M-SSVH
Building 5250
Redstone Arsenal, Alabama

Dr. Paul Degrarabedian
Astro Science Laboratory
Building G
Space Technology Laboratory, Inc.
One Space Park
Redondo Beach, California 90277

Dr. Ray Wilson
OART - CODE RRA
National Aeronautics & Space Administration
Washington, D. C. 20025

Mr. Joe Mason
Space & Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Mr. Joe Santa
A2-863
Douglas Aircraft Company
3000 Ocean Park Blvd.
Santa Monica California 90405

Mr. J. B. Cruz, Jr.
Research Associate Professor
Coordinated Science Laboratory
Urbana, Illinois 61801

Dr. Imre Izsak
Smithsonian Institution
Astrophysical Observatory
60 Garden Street
Cambridge, Massachusetts 02138

Mr. Howard Haglund
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

Mr. Hewitt Phillips
Langley Research Center
Langley Station
Hampton, Virginia 23365

Mr. H. A. McCarty
Space & Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Mr. M. L. Bell
Chrysler Corporation
HIC Building
Huntsville, Alabama

Mr. George Cherry
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Mr. C. R. Coates
Astrosciences Department
Aeronutronic Division of Ford Motor Co.
Ford Road
Newport Beach, California

Mr. M. D. Anderson, Jr.
General Dynamics Corporation
Suite 42, Holiday Office Center
South Memorial Parkway
Huntsville, Alabama 35801

Dr. B. B. Gregg
Dept. 574, Bldg. 74
Autonetics Division of Philco Corp.
Ford Road
Newport Beach, California

Dr. Stephen J. Kahne
Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota 55455

Mr. Dale B. Ruhmel
Staff Associate
Aeronautical Research Associates of Princeton, Inc.
50 Washington Road
Princeton, New Jersey 08541

Dr. Raymond Rishel
Mathematical Analysis Staff
Organization 2-5330
Mail Stop 89-75
Boeing Company
P. O. Box 3707
Seattle, Washington 98124

Mr. Harold Chestnut
1 River Road
Schenectady, New York 12305

Dr. William A. Mersman, Chief
Electronic Machine Computing Branch
Ames Research Center
Moffett Field, California 94035

Dr. Siegfried J. Gerathewohl
Bioscience Program Manager
Manned Space Science Program
Code SM
National Aeronautics & Space Administration
600 Independence Avenue
Washington, D. C. 20003

Dr. Herman M. Dusek
A. C. Research and Development
A. C. Spark Plug
950 N. Sepulveda Blvd.
El Segundo, California 90245

Dr. D. M. Schrello, Director
Flight Sciences
Space & Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Mr. Howard S. London
Bellcomm, Inc.
1100 17th Street, N. W.
Washington, D. C. 20036

Dr. J. P. deVries
Astrodynamics Operation
Space Sciences Laboratory
Missile & Space Vehicle Department
General Electric Company
Valley Forge Space Technology Center
P. O. Box 8555
Philadelphia, Pennsylvania 19104

Dr. Lembit Pottsepp
Dept. A-260
Douglas Aircraft Company
3000 Ocean Park 'lvd.
Santa Monica, California 90405

Mr. William R. Wells
Mail Stop #304
Langley Research Center
Space Mechanics Division
Langley Station
Hampton, Virginia 23365

Dr. R. M. L. Baker
4444 So. Via Marina, Apt. P-88
Marina del Rey, California 90292

Dr. John W. Drebinger
Manager, Systems Analysis
Advanced Projects Laboratories
Space Systems Division
Aerospace Group
Hughes Aircraft Company
El Segundo, California 90246

Mr. Carl B. Cox
Organization 2-5700
Mail Stop 22-85
The Boeing Company
P. O. Box 3707
Seattle, Washington 98124

Mr. Dave Engels
Space and Information Systems Division
Dept 595/720
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California 90241

Mr. Bill Putnam
Lockheed Missiles & Space Company
P. O. Box 1103, West Station
Huntsville, Alabama 35807

Mr. William F. Powers
3501 Owens Circle
Austin, Texas 78705

Dr. Daniel E. Dupree
Department of Mathematics
Northeast Louisiana State College
Monroe, Louisiana 71201

Mr. T. L. Campbell
Dept. 7162, Applied Mathematics
Chrysler Corporation Missile Division
Sixteen Mile Road and Van Dyke
P. O. Box 2628
Detroit, Michigan 48231

Mr. Robert Glasson
Bendix Systems Division
The Bendix Corporation
3322 Memorial Parkway South
Huntsville, Alabama 35801

Mr. C. H. Tross, President
Council for Technological and
Industrial Development
P. O. Box 931
Rancho Santa Fe, California 92067

SSD (Space Guidance Program Division/SSTDG)
ATTN: Maj. C. M. Waespy
AF Unit Post Office
Los Angeles, California 90045

Mr. E. J. Luedtke, Jr.
Autonetics
Division of North American Aviation
3322 Memorial Parkway
Huntsville, Alabama 35801

Dr. Peter Musen
Goddard Space Flight Center
Greenbelt, Maryland 20771

Dr. George Leitmann
Associate Professor, Engineering Science
University of California
Berkeley, California 94720

Dr. R. P. Agnew
Department of Mathematics
Cornell University
Ithaca, New York 14850

Dr. Jurgen Moser
Professor of Mathematics
Graduate School of Arts & Sciences
New York University
New York, New York 10019

Dr. Lu Ting
Professor of Aeronautics & Astronautics
New York University
University Heights
Bronx, New York 10453

Mr. Clint Pine
Department of Mathematics
Northwestern State College
Natchitoches, Louisiana 71457

Dr. John Gates
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California 91103

Auburn Research Foundation (2)
Auburn University
Auburn, Alabama 36830

Grumman Aircraft Engineering Corporation
Grumman Library
Bethpage, L. I., New York 11714

Jet Propulsion Laboratory Library
4800 Oak Grove Drive
Pasadena, California 91103

NASA Ames Research Center (2)
Mountain View, California 94035
ATTN: Librarian

NASA Flight Research Center (2)
Edwards Air Force Base, California 93523
ATTN: Librarian

NASA Goddard Space Flight Center (2)
Greenbelt, Maryland 20771
ATTN: Librarian

NASA Langley Research Center (2)
Hampton, Virginia 23365
ATTN: Librarian

John F. Kennedy Space Center (2)
Kennedy Space Center, Florida 32899
ATTN: Librarian

NASA Lewis Research Center (2)
Cleveland, Ohio 44135
ATTN: Librarian

NASA Manned Spacecraft Center (2)
Houston, Texas 77058
ATTN: Librarian

NASA Electronics Research Center (2)
575 Technology Square
Cambridge, Massachusetts 02139
ATTN: Librarian

NASA Wallops Space Flight Station (2)
Wallops Island, Virginia 23337
ATTN: Librarian

Space Flight Library (2)
University of Kentucky
Lexington, Kentucky 40506

University of Kentucky Library (5)
University of Kentucky
Lexington, Kentucky 40506