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ATTITUDE STABILITY OF A SPINNING PASSIVE SATELLITE IN A CIRCULAR ORBIT

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Principal Investigator: Leonard Meirovitch

ABSTRACT

The spinning satellite dynamics problems defined by differential equations with constant coefficients, called autonomous systems, have received considerable attention. This report concentrates on the far more complicated problem of nonautonomous systems and in particular on the case in which the coefficients are periodic. This type of problems has been treated by means of Floquet's Theory but the treatment is somewhat unsatisfactory since stability can be checked only for given sets of parameters. The approach adopted here is broader in scope. It can be regarded as consisting of three complementing aspects: a Liapounov-type analysis, an infinite determinant approach and an asymptotic expansion approach. The first one is expected to yield the location of the instability regions whereas the latter two should furnish the width of these regions as a function of a small parameter ϵ. The parameter ϵ is related to the difference in the moments of inertia of a spinning satellite about transverse axes and in many practical satellites this difference is made small so as to minimize periodic torques resulting from gravitational forces, etc. For the Liapounov-type analysis a new theorem, more suitable for the class of problems under investigation, has been proposed. The infinite determinant approach as well as the asymptotic expansion approach are believed to be used for the first time for the treatment of problems involving gyroscopic terms.

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I. Introduction.

This report covers the investigation of the stability of motion of spinning, passive satellites conducted during the period December 1, 1965 - May 31, 1966. In contrast with the first six-month period, the research done during this period concentrates heavily on nonautonomous systems and in particular on systems with periodic coefficients.

During the present period some attention has been given to the problem of stability of motion of a spinning, rigid, symmetric satellite under the influence of aerodynamic and gravitational torques. The aerodynamic torques may play an important role in the stability of motion of relatively low-altitude satellites and it was felt that an investigation of this particular aspect of the problem should be beneficial. A study was conducted on the stability of motion of such a satellite moving in a circular orbit. The governing differential equations of motion for this problem have constant coefficients, hence they are autonomous. It follows that the stability of motion could be studied using the Liapounov direct method as was employed on a different problem during the previous report period.(1)* The work discussing the combined aerodynamic and gravitational effects has been submitted for publication in a national journal and will not be presented in this progress report. It should be mentioned, however, that in this particular problem the aerodynamic torques could be described by a potential function, a factor which made that analysis possible.

The major effort of this report period was devoted to the development of mathematical techniques for the investigation of the stability of motion of a satellite for the case in which the motion can be described by ordinary differential equations with periodic coefficients. A large number of important satellite dynamics problems are of this type, including the following

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* A paper describing this problem has been accepted for publication by the Journal of the Astronautical Sciences.

a. Spinning, rigid, unsymmetrical satellite in a circular orbit.
b. Spinning, rigid, symmetrical satellite in an elliptic orbit.
c. Rigid, symmetrical satellite with elastically connected moving parts moving in a circular orbit and possessing an angular motion relative to an orbiting frame of reference.
d. Spinning, rigid, symmetrical satellite in a circular orbit under the influence of periodic torques due to solar pressure.
e. Spinning, rigid, symmetrical satellite in a circular orbit under the influence of torques due to passage through an atmosphere with nonconstant density.

The first two of the problems listed above have been studied by Kane et.al.(2)(3) using an analysis based on Floquet's Theory.(15) This analysis involves numerical integration of the linearized equations of motion for specific values of the parameters of the problem. Since that is an infinitesimal analysis, conclusive statements can be made only about the instability of motion and, furthermore, it is restricted to individual points in the parameter space and it does not present a continuous picture.

In the present study attempts are being made to obtain analytical relationships with which regions (rather than individual points) in the parameter space may be checked for stability. For the purpose of developing the required analytical techniques, a two-degree-of-freedom system consisting of a rigid, spinning body with two moments of inertia almost equal is being studied. The body moves in a circular orbit. The problem formulation is shown in Section II. Three methods of analysis are being explored and their feasibility evaluated.

Sections III, IV and V describe the techniques being applied in the current study of periodic systems. Section III presents a stability theorem similar to the one of Liapounov, which has promise of more satisfactory adaptation to nonautonomous systems and in particular to systems with periodic coefficients. A discussion is presented in which it is shown that the difference between the Hamiltonian and the Hamiltonian at an equillibrium position is a reasonable testing function and the corresponding theorem is applied to the two-degree-of-freedom system. Section IV describes analysis based upon the infinite determinant idea. The analysis allows for the estimation of the resonance instability regions for a linear system. Section V presents a similar approach for determining the regions of instability, but is based upon an asymptotic expansion of the solution in terms of a small parameter.

During the next report period the present work with nonautonomous systems will be continued.

II. Spinning, Unsymmetrical Satellite in a Circular Orbit. Problem Formulation.

We shall be concerned with the problem of stability of motion of a spinning, unsymmetrical, rigid satellite in a circular orbit. When the satellite possesses rotational motion relative to an orbiting frame of reference the problem formulation involves periodic coefficients. Hence, we shall be interested in developing techniques for the treatment of systems with periodic coefficients. The same techniques, with slight modifications, should be applicable to an entire breed of problems such as the ones listed in the preceding section.

Previous work[(1)] has shown that for rigid bodies there is no coupling between the orbital motion of the center of mass of the body and the attitude motion of the body about the center of mass. This assumption, referred to as orbital constraints, will be used in the present study.

Particular emphasis will be placed upon the case in which the body is nearly, but not exactly, symmetrical with respect to the spin axis. It is felt that this is a case of great interest since, for spin-stabilized satellites, a practical satellite system would be made nearly symmetrical with respect to the spin axis to minimize the periodic excitations caused by gravitational torques.

1. Coordinate systems.

An orbital frame of reference with its origin at the satellite center of mass and its orientation, as shown in Figure 1a, is chosen. Axis a is along a radial line from the center of force (center of the earth) to the center of the satellite, axis b along the orbit path, and axis c perpendicular to axes a and b. The orbit angular velocity, denoted by Ω_0, is related to the constant K, which is the product of the universal gravitational constant times the earth's mass, and the orbit radius R_c by $\Omega_0{}^2 = K/R_c{}^3$. Hence a, b, c forms an orbiting frame of reference. The orientation of the satellite relative to the a, b, c reference system is obtained by three successive rotations θ_2, θ_1, and φ as shown on Figure 1b.

The z axis is taken as the spin axis and the mass moments of inertia about the axes x, y and z are denoted by A, B and C, respectively.

The direction cosines between the x, y, z axes and the a, b, c axes may be written in terms of the following matrix equation:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \ell_{xa} & \ell_{xb} & \ell_{xc} \\ \ell_{ya} & \ell_{yb} & \ell_{yc} \\ \ell_{za} & \ell_{zb} & \ell_{zc} \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$= \begin{bmatrix} c\theta_2 c\varphi - s\theta_1 s\theta_2 s\varphi & c\theta_1 s\varphi & -s\theta_2 c\varphi - c\theta_2 s\theta_1 s\varphi \\ -c\theta_2 s\varphi - s\theta_1 s\theta_2 c\varphi & c\theta_1 c\varphi & s\theta_2 s\varphi - c\theta_2 s\theta_1 c\varphi \\ s\theta_2 c\theta_1 & s\theta_1 & c\theta_2 c\theta_1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{1}$$

where $c\theta_2 = \cos\theta_2$, $s\theta_1 = \sin\theta_1$, etc. The angular velocities about the x, y, and z axes may be written

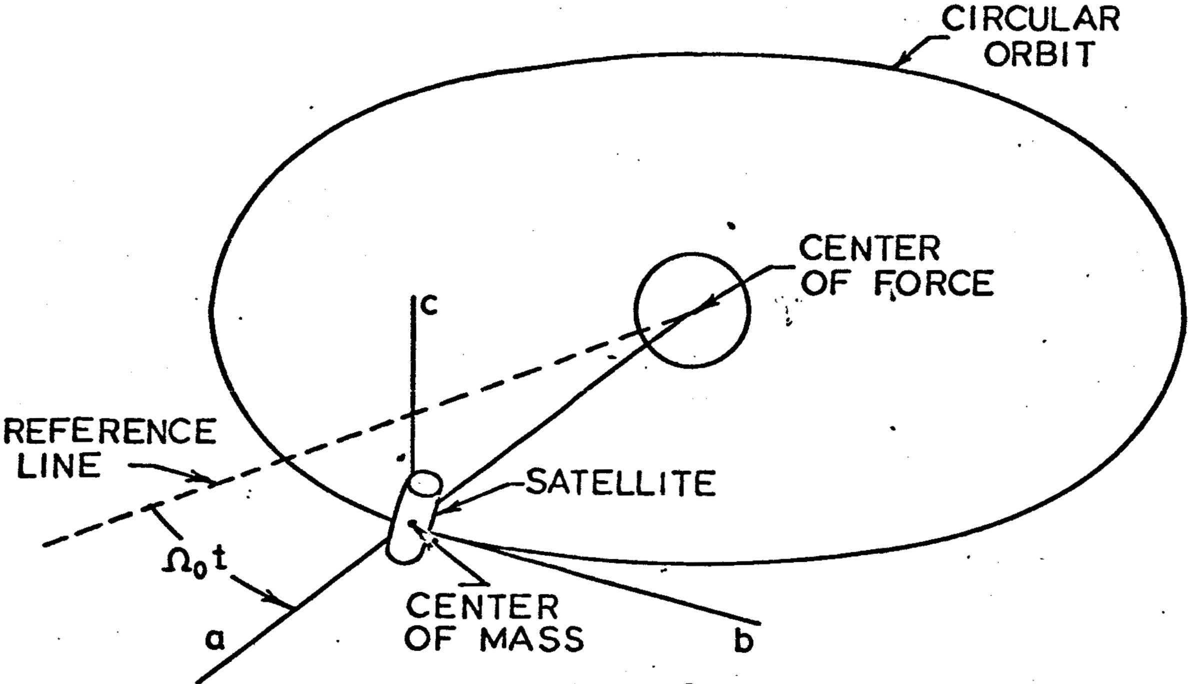


FIG. 1a
THE SATELLITE AND THE ORBITAL AXES

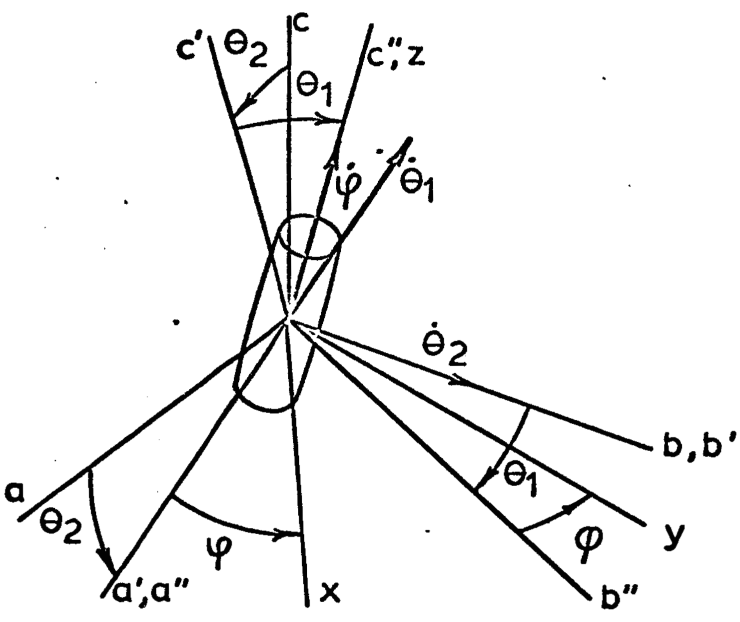


FIG. 1b
COORDINATE SYSTEMS AND ANGULAR VELOCITIES

$$\Omega_x = \Omega_0\,(-s\theta_2 c\varphi - c\theta_2 s\theta_1 s\varphi) + \dot{\theta}_2 c\theta_1 s\varphi - \dot{\theta}_1 c\varphi$$

$$\Omega_y = \Omega_0\,(s\theta_2 s\varphi - c\theta_2 s\theta_1 c\varphi) + \dot{\theta}_2 c\theta_1 c\varphi + \dot{\theta}_1 s\varphi \tag{2}$$

$$\Omega_z = \Omega_0\,(c\theta_2 c\theta_1) + \dot{\theta}_2 s\theta_1 + \dot{\varphi}$$

We will introduce the following dimensionless quantities:

$$r = C/A, \qquad \epsilon = (B - A)/A,$$
$$\alpha = \dot{\varphi}/\Omega_0, \qquad \alpha_1 = \omega_0/\Omega_0 \tag{3}$$

where $\dot{\varphi}$ is the instantaneous spin rate and ω_0 is the average spin rate. Note that ϵ is a measure of the asymmetry of the body with respect to the spin axis and will generally be a small number as mentioned previously.

2. Energy expressions.

The kinetic and potential energy expressions may be written as follows

$$\text{K.E.} = \frac{1}{2} A\,\Omega_x^2 + \frac{1}{2} B\,\Omega_y^2 + \frac{1}{2} C\,\Omega_z^2$$
$$\text{P.E.} = -\frac{3}{4}\frac{K}{R_c^3}\Big[(C + B - A)\ell_{xa}^2 + (C + A - B)\ell_{ya}^2 + (A + B - C)\ell_{za}^2\Big] \tag{4}$$

3. The equations of motion.

The Lagrangian function, L = K.E. - P.E., may be used in conjunction with the Lagrangian formulation to derive the differential equations of motion. For a conservative system, Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad (i = 1,2,---,n) \tag{5}$$

where n is the degree of freedom of the system. Equations (5) lead to

$$
\begin{aligned}
&\ddot\theta_1 + \dot\theta_2\Omega_0 c\theta_2 + \dot\theta_2^2\, s\theta_1 c\theta_1 - \Omega_0(\dot\theta_2 c\theta_2 s^2\theta_1 - \dot\theta_2 c\theta_2 c^2\theta_1)\\
&\quad - \Omega_0^2 c^2\theta_2 s\theta_1 c\theta_1 - r[\dot\varphi\dot\theta_2 c\theta_1 + \dot\theta_1^2\, s\theta_1 c\theta_1 + \Omega_0\dot\theta_2 c\theta_2(c^2\theta_1 - s^2\theta_1)\\
&\quad - \Omega_0\dot\varphi c\theta_2 s\theta_1 - \Omega_0^2 c^2\theta_2 s\theta_1 c\theta_1] + 3(1-r)\Omega_0^2 s^2\theta_2 s\theta_1 c\theta_1 + \epsilon\,[\ddot\theta_1 s^2\varphi\\
&\quad + 2\dot\theta_1\dot\varphi s\varphi c\varphi + \ddot\theta_2 c\theta_1 s\varphi c\varphi - \dot\theta_1\dot\theta_2 s\theta_1 s\varphi c\varphi + \dot\theta_2\dot\varphi c\theta_1(c^2\varphi - s^2\varphi)\\
&\quad + \Omega_0\dot\theta_2(c\theta_2 s^2\varphi + s\theta_2 s\theta_1 s\varphi c\varphi) + 2\,\Omega_0\dot\varphi s\theta_2 s\varphi c\varphi - \Omega_0\dot\theta_1 c\theta_2 c\theta_1 s\varphi c\varphi\\
&\quad - \Omega_0\dot\varphi c\theta_2 s\theta_1(c^2\varphi - s^2\varphi) + \dot\theta_1\dot\theta_2 s\theta_1 s\varphi c\varphi + \dot\theta_2^2\, s\theta_1 c\theta_1 c^2\varphi\\
&\quad + \Omega_0\dot\theta_2 s\theta_2 s\theta_1 s\varphi c\varphi - \Omega_0\dot\theta_2 c\theta_2 s^2\theta_1 c^2\varphi + \Omega_0\dot\theta_1 c\theta_2 c\theta_1 s\varphi c\varphi + \Omega_0\dot\theta_2 c\theta_2 c^2\theta_1 c^2\varphi\\
&\quad + \Omega_0^2\,(4s\theta_2 c\theta_2 c\theta_1 s\varphi c\varphi - c^2\theta_2 s\theta_1 c\theta_1 c^2\varphi + 3s^2\theta_2 s\theta_1 c\theta_1 c^2\varphi)] = 0
\end{aligned}
$$

$$
\begin{aligned}
&\ddot\theta_2 c^2\theta_1 - 2\dot\theta_2\dot\theta_1 s\theta_1 c\theta_1 - \Omega_0(-\dot\theta_2 s\theta_2 s\theta_1 c\theta_1 + \dot\theta_1 c\theta_2 c^2\theta_1 - \dot\theta_1 c\theta_2 s^2\theta_1 + \dot\theta_1 c\theta_2\\
&\quad + \dot\theta_2 s\theta_2 s\theta_1 c\theta_1) - \Omega_0^2(s\theta_2 c\theta_2 - s\theta_2 c\theta_2 s^2\theta_1) + r\,[\ddot\varphi s\theta_1 + \dot\varphi\dot\theta_1 c\theta_1\\
&\quad + \ddot\theta_2 s^2\theta_1 + 2\dot\theta_2\dot\theta_1 s\theta_1 c\theta_1 + \Omega_0(-\dot\theta_2 s\theta_2 s\theta_1 c\theta_1 + \dot\theta_1 c\theta_2 c^2\theta_1 - \dot\theta_1 c\theta_2 s^2\theta_1\\
&\quad + \dot\varphi s\theta_2 c\theta_1 + \dot\theta_2 s\theta_2 s\theta_1 c\theta_1 + \Omega_0 s\theta_2 c\theta_2 c^2\theta_1)] - 3(1-r)\,\Omega_0^2 s\theta_2 c\theta_2 c^2\theta_1\\
&\quad + \epsilon[\ddot\theta_1 c\theta_1 s\varphi c\varphi - \dot\theta_1^2 s\theta_1 s\varphi c\varphi + \dot\theta_1\dot\varphi c\theta_1(c^2\varphi - s^2\varphi) + \ddot\theta_2 c^2\theta_1 c^2\varphi \qquad (6)\\
&\quad - 2\dot\theta_2\dot\theta_1 s\theta_1 c\theta_1 c^2\varphi - 2\dot\theta_2\dot\varphi c^2\theta_1 s\varphi c\varphi + \Omega_0(-\dot\theta_1 s\theta_1 s\theta_2 s\varphi c\varphi + \dot\theta_2 c\theta_1 c\theta_2 s\varphi c\varphi\\
&\quad + \dot\varphi c\theta_1 s\theta_2 c^2\varphi - \dot\varphi c\theta_1 s\theta_2 s^2\varphi + \dot\theta_2^{1} s\theta_2 s\theta_1 c\theta_1 c^2\varphi - \dot\theta_1 c\theta_2 c^2\theta_1 c^2\varphi
\end{aligned}
$$

$$+ \dot\theta_1 c\theta_2 s^2\theta_1 c^2\varphi + 2\dot\varphi c\theta_2 s\theta_1 c\theta_1 s\varphi c\varphi - \dot\theta_1 c\theta_2 s^2\varphi - \dot\theta_2 c\theta_2 c\theta_1 s\varphi c\varphi$$

$$- \dot\theta_1 s\theta_2 s\theta_1 s\varphi c\varphi - \dot\theta_2 s\theta_2 s\theta_1 c\theta_1 c^2\varphi) - 4\Omega_0^2(s\theta_2 c\theta_2 s^2\varphi + s^2\theta_2 s\theta_1 s\varphi c\varphi$$

$$- c^2\theta_2 s\theta_1 s\varphi c\varphi - s\theta_2 c\theta_2 s^2\theta_1 c^2\varphi)] = 0$$

$$r[\ddot\varphi + \ddot\theta_2 s\theta_1 + \dot\theta_2\dot\theta_1 c\theta_1 + \Omega_0(-\dot\theta_2 s\theta_2 c\theta_1 - \dot\theta_1 c\theta_2 s\theta_1)] - \epsilon[\dot\theta_1^2 s\varphi c\varphi$$

$$+ \dot\theta_2\dot\theta_1 c\theta_1(c^2\varphi - s^2\varphi) - \dot\theta_2^2 c^2\theta_1 s\varphi c\varphi + \Omega_0(\dot\theta_1 s\theta_2 s\varphi c\varphi - \dot\theta_1 c\theta_2 s\theta_1 c^2\varphi$$

$$- \dot\theta_2 s\theta_2 c\theta_1 s^2\varphi + \dot\theta_2 c\theta_2 s\theta_1 c\theta_1 s\varphi c\varphi + \dot\theta_1 s\theta_2 s\varphi c\varphi + \dot\theta_1 c\theta_2 s\theta_1 s^2\varphi$$

$$+ \dot\theta_2 s\theta_2 c\theta_1 c^2\varphi + \dot\theta_2 c\theta_2 s\theta_1 c\theta_1 s\varphi c\varphi) + \Omega_0^2(s^2\theta_2 s\varphi c\varphi + s\theta_2 c\theta_2 s\theta_1 s^2\varphi$$

$$- s\theta_2 c\theta_2 s\theta_1 c^2\varphi - c^2\theta_2 s^2\theta_1 s\varphi c\varphi - 3c^2\theta_2 s\varphi c\varphi - 3s\theta_2 c\theta_2 s\theta_1 c^2\varphi$$

$$+ 3s\theta_2 c\theta_2 s\theta_1 s^2\varphi + 3s^2\theta_2 s^2\theta_1 s\varphi c\varphi)] = 0$$

4. Equilibrium positions. The linearized equations of motion.

Inspection of the differential equations of motion, Eqs. (6), shows that an equilibrium position exists when $\theta_1 = \dot\theta_1 = \theta_2 = \dot\theta_2 = \dot\varphi = 0$ and $\varphi = \frac{n\pi}{2}$. This is the equilibrium position studied by De Bra and Delp[4] in which the unsymmetrical rigid body has a position that is fixed with respect to an orbiting frame of reference. The corresponding equilibrium position was studied by Meirovitch[1] for the case of a rigid satellite with elastically connected moving parts. We are now interested in the case in which the satellite has a spinning motion relative to the orbiting frame of reference so that φ and $\dot\varphi$ are not constant. However, we notice that by linearizing Eqs. (6) the equation for the coordinate φ becomes uncoupled, hence one can solve for φ independently.

In view of the above conclusions, we wish to define an equilibrium position $\theta_1 = \theta_2 = 0$. We note that θ_1 and θ_2 define the attitude of the spin axis relative to the orbiting frame of reference and $\theta_1 = \theta_2 = 0$ corresponds to the position in which the spin axis is perpendicular to the orbit plane.

The equations of motion, Eqs. (6), can be linearized about the position $\theta_1 = \theta_2 = 0$. To this end it will prove convenient to change the time scale to the nondimensional one defined by

$$\tau = \Omega_0 t, \quad \frac{d}{dt} = \Omega_0 \frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = \Omega_0^2 \frac{d^2}{d\tau^2} \tag{7}$$

so that Eqs. (6) can be written as

$$\begin{aligned}
&\theta_1'' + \theta_2'[2 - r(1+\alpha)] - \theta_1[1 - r(1+\alpha)] \\
&\quad + \epsilon\Big\{\theta_1'\left(1 - \tfrac{1}{2} r\right)(1+\alpha)s2\varphi + \theta_2'\Big[\tfrac{1}{2} r(1+\alpha) \\
&\quad + \left(1 - \tfrac{1}{2} r\right)(1+\alpha)\, c2\varphi\Big] - \theta_1\Big[\tfrac{1}{2} r\,(1+\alpha) \\
&\quad + \left(1 - \tfrac{1}{2} r\right)(1+\alpha)\, c2\varphi\Big] + \theta_2\left(1 - \tfrac{1}{2} r\right)(4+\alpha)s2\varphi\Big\} = 0
\end{aligned}$$

$$\begin{aligned}
&\theta_2'' - \theta_1'\,[2 - r(1+\alpha)] - \theta_2[4 - r(4+\alpha)] \\
&\quad + \epsilon\Big\{\theta_1'\Big[-\tfrac{1}{2} r(1+\alpha) + \left(1 - \tfrac{1}{2} r\right)(1+\alpha)\, c2\varphi\Big] \\
&\quad - \theta_2'\left(1 - \tfrac{1}{2} r\right)(1+\alpha)\, s2\varphi + \theta_1\left(1 - \tfrac{1}{2} r\right)(1+\alpha)\, s2\varphi \\
&\quad - \theta_2\Big[\tfrac{1}{2} r\,(4+\alpha) - \left(1 - \tfrac{1}{2} r\right)(4+\alpha)\, c2\varphi\Big]\Big\} = 0
\end{aligned} \tag{8}$$

$$\varphi'' + \frac{3}{2}\frac{\epsilon}{r}\, s2\varphi = 0$$

where primes indicate differentiations with respect to τ.

5. The spinning motion.

We notice from the third of Eqs. (8) that, for motion in the neighborhood of $\theta_1 = \theta_2 = 0$, the spinning motion is independent of the coordinates θ_1 and θ_2. One can attempt a solution for φ in terms of a power series in $\epsilon = (B - A)/A$ as follows

$$\varphi = \omega_0 t + \epsilon\varphi_1(t) + \epsilon^2\varphi_2(t) + --- \tag{9}$$

from which it follows immediately that

$$\alpha = \alpha_1 + \epsilon\,\varphi_1'(\tau) + \epsilon^2\varphi_2'(\tau) + --- \tag{10}$$

Substituting the above into the third of Eqs. (8), and equating terms of equal powers of ϵ to zero, we obtain a sequence of ordinary differential equations. The sequential solution of these differential equations allows one to write

$$\alpha = \alpha_1 + \epsilon\left(\frac{3}{4r\alpha_1}\, c\, 2\alpha_1\tau\right) + \epsilon^2\left(\frac{9}{32r^2\alpha_1^3}\, c\, 4\alpha_1\tau\right)$$

$$+ \epsilon^3\left[\frac{9}{2048r^3\alpha_1{}^5}(c\, 6\alpha_1\tau + 15\, c\, 2\alpha_1\tau)\right] + ---$$

$$c2\varphi = c\, 2\alpha_1\tau + \epsilon\left[\frac{3}{8r\alpha_1^2}(c\, 4\alpha_1\tau - 1)\right] \tag{11}$$

$$+ \epsilon^2\left[\frac{9}{64r^4\alpha_1^4}(c\, 6\alpha_1\tau - c\, 2\alpha_1\tau)\right] + ---$$

$$s2\varphi = s\, 2\alpha_1\tau + \epsilon\left(\frac{3}{8r\alpha_1^2}\, s\, 4\alpha_1\tau\right)$$

$$+ \epsilon^2\left[\frac{9}{64r^4\alpha_1^4}(s\, 6\alpha_1\tau - s\, 2\alpha_1\tau)\right] + ---$$

An alternate approach that will also prove useful results from noting that the third of Eqs. (8) can be integrated once to yield

$$\dot{\varphi} = \omega_0 \left[1 + \frac{3}{2} \frac{\epsilon}{r\alpha_1^2} c\, 2\varphi \right]^{1/2} \tag{12}$$

6. <u>The Hamiltonian function</u>.

Our interest is in the motion about the point $\theta_1 = \theta_2 = 0$. In this neighborhood, as can be seen from the third of Eqs. (8), the coordinate φ can be considered as an explicit function of time. One can conceive of a constrained system as a system identical with the system under consideration but with the φ coordinate a known function of time, namely the solution of the third of Eqs. (8). When θ_1 and θ_2 are not small, the motion of this system is not in accordance with the complete, nonlinear equations which indicates that constraint forces must be added. However, as θ_1 and θ_2 approach zero, the terms coupling the φ motion with the θ_1 and θ_2 motions approach zero and the constrain forces approach zero. As a result of this assumption, one can devise a Hamiltonian function containing θ_1 and θ_2 as variables and φ as an explicit function of time. This Hamiltonian is consistent only with the linearized equations and <u>must be used only when θ_1 and θ_2 are small</u>. The Hamiltonian in question can be written as

$$H = \frac{\partial L}{\partial \dot{\theta}_1} \dot{\theta}_i + \frac{\partial L}{\partial \dot{\theta}_2} \dot{\theta} - L = \frac{1}{2} A \Big\{ \dot{\theta}_1^2 (1 + \epsilon\, s^2\varphi) + \dot{\theta}_2^2 (c^2\theta_1 + r s^2\theta_1$$

$$+ \epsilon\, c^2\theta_1 c^2\varphi) + \epsilon\, \dot{\theta}_1 \dot{\theta}_2\, c\theta_1 s2\varphi + \Omega_0^2 \,[- r(\alpha + c\theta_1 c\theta_2)^2 - s^2\theta_2$$

$$- c^2\theta_2 s^2\theta_1 + 3(r - 1) s^2\theta_2 c^2\theta_1] + \epsilon\, \Omega_0^2 [3(c\theta_2 s\varphi + s\theta_1 s\theta_2 c\varphi)^2$$

$$- (s\theta_2\, s\varphi - c\theta_2 s\theta_1 c\varphi)^2] \Big\} \tag{13}$$

III. Liapounov-type Stability Analysis.

A stability theorem which is applicable to nonautonomous mechanical systems (systems characterized by differential equations with time-dependent coefficients) is presented. While similar to the theorem of Liapounov, the proposed theorem broadens the scope in a manner which is found to be useful in dealing with undamped systems with periodic coefficients. The difference between the Hamiltonian and the Hamiltonian function evaluated at the equilibrium position is shown to be a suitable function for testing the stability of motion of the nonautonomous systems under consideration. The stability theorem is applied to the problem developed in the preceding section.

The conclusive stability statement that can be made by means of the Liapounov type of analysis on autonomous systems (see Reference 1) would be of great value if it could be made in connection with multi-degree-of-freedom conservative* systems with periodic coefficients. Such an approach may also be useful in locating regions of instability in linearized systems with periodic coefficients. The periodic terms may enter the equations by virtue of a periodically varying potential function or by means of assumed periodic behavior of one, or more, coordinate that is not subject to the stability investigation but appears in the form of known time-dependent coefficients.

Given a system whose essential features are described by n/2 generalized coordinates and n/2 generalized velocities (of course, n must be an even number which is equal to twice the degree of freedom of the system), $x_i (i = 1,2,---,n)$, an equilibrium position is said to exist at $x_i = c_i$, $(i = 1,2,---,n)$, where c_i are constants, if these values satisfy the differential equations of motion. By a suitable coordinate transformation, any equilibrium position of a mechanical system can be translated to the origin, $x_1 = x_2 = ---- = x_n = 0$, and this will be assumed to be the case in further discussion. An equilibrium position will be defined as stable in the sense

*Conservative is used here in the sense that the external forces are derivable from a potential function that is independent of the velocities even though the potential function and the resulting forces may be time-dependent.

of Liapounov[5] if there exist positive numbers ε and η and time t_0 such that

$$\sum_{i=1}^{n} x_i^2 \leq \varepsilon \qquad \text{for } t \geq t_0 \tag{14}$$

for all motion subsequent to an initial perturbation from the equilibrium position, where the initial perturbation satisfies

$$\sum_{i=1}^{n} x_{i0}^2 \leq \eta \qquad \text{at } t = t_0 \tag{15}$$

It should be noted that the equilibrium position may be defined in terms of a restricted number of coordinates (for example the attitude stability problem of a spinning satellite may be defined in terms of the position of the spin axis as in the case of the constrained system of Section II.6) and other coordinates such as the orbit parameters or spin angle may appear as time-dependent coefficients.

The Hamiltonian function has been widely used as a testing function, in conjunction with the Liapounov stability analyses, in the case of autonomous mechanical systems. However, it has not been used in the case of systems with periodic coefficients. To discover why this had been the case, let us consider an unconstrained conservative system with nonrotating coordinates. The usual forms of the Liapounov theorem state[6] that the motion in the neighborhood of an equilibrium position will be stable if a function of the coordinates and time can be found which is positive definite in the neighborhood of the equilibrium and has a negative or zero time derivative. The total energy is a suitable Liapounov function in this nonrotating coordinate problem, so the Liapounov theorem can be seen to require that the total energy have a relative minimum at the equilibrium position and that energy is either dissipated or unchanged during <u>any</u> small motions near the equilibrium. This latter requirement appears to be too stringent in the case of systems with periodic coefficients, since intuition tells us that a stable equilibrium could exist such that energy could flow in and out of the system during small motions near the equilibrium, as long as the net energy addition after a period of time is not a cumulative effect. Consequently, the following stability theorem is suggested.

1. Stability theorem.

Given a system of equations described by the differential equations

$$\dot{x}_s = X_s(x_1, x_2, ----, x_n, t), \quad (s = 1,2,---,n) \tag{16}$$

for which an equilibrium position, E, exists at $x_1 = x_2 = --- = x_n = 0$, then the perturbed motion about this equilibrium position is said to be stable if a continuous function V can be found such that

a. $V(x_1, x_2, ---, n_n, t)$ is positive definite in the neighborhood of E, zero at E, and

b. $\int_{t_0}^{t} \frac{dV}{dt}\, dt \le M\,(x_{10}^2 + x_{20}^2 + --- + x_{n0}^2)$

for motion subsequent to $t = t_0$, in which M is any finite, positive constant and $x_{10}, x_{20}, ---, x_{n0}$ are initial small displacementsat $t = t_0$.

Proof of the preceding theorem is not given here because of lack of space. Similar theorems giving the conditions for asymptotic stability or instability may be developed in the same manner. It should be noted that the preceding stability theorem gives conditions that are sufficient for concluding that a given motion is stable but does not give the necessary conditions. Consequently, stable motions may exist which would not meet the requirements of this theorem.

To make use of the stability theorem it is necessary to select a testing function which can have a relative minimum at an equilibrium position and for which a meaningful value of the integral, Eq. (17), can be obtained. The Hamiltonian function will be investigated for this purpose and shown to be a reasonable choice based on physical reasoning. This physical reasoning will be applied to a nonrotating coordinate system before applying it to rotating systems.

2. Testing function. Nonrotating coordinates.

First let us introduce the rotation:

K.E. = kinetic energy

P.E. = potential energy

L = K.E. - P.E. = Lagrangian function

q_i = generalized coordinate

$\dot{q}_i$ = generalized velocity

p_i = generalized momentum

Q_i = generalized conservative force

By definition, the generalized momentum and generalized force are related to the Lagrangian by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad Q_i = \frac{\partial L}{\partial q_i} \tag{18}$$

Furthermore, we have the Lagrange's equations of motion for a conservative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad \therefore \qquad \dot{p}_i = Q_i \tag{19}$$

which states that the time rate of change of the generalized momentum is equal to the generalized conservative force. The definition of the Hamiltonian is

$$H(q,p,t) = \sum_i p_i\dot{q}_i - L\,(q,\dot{q},t) \tag{20}$$

from which we can derive the canonical equations of motion

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -Q_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \tag{21}$$

and also that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{22}$$

When a nonrotating coordinate system is used to describe a mechanical system subjected to conservative external forces, the above general relations take on special special forms such that the kinetic energy is quadratic in the velocities, the potential energy is a function of only the spatial coordinates, the total energy (K.E. + P.E.) is equal to the Hamiltonian, and the generalized force and momentum are equal to the linear force and momentum.

In the absence of explicit time dependence in the Hamiltonian, the total energy is constant. With time dependence the total energy changes in accordance with Eq. (22).

Now we can consider the mechanisms through which an instability exists in the neighborhood of an equilibrium position E (the origin). If we consider motion near E in which the energy level is larger by an amount ΔH than the energy level at E, we see that an instability could exist due to exchange of energy between K.E. and P.E. in one or more coordinates. But for a mechanical system, K.E. is positive definite (i.e. K.E. increases as the p_i depart from zero). So, if the motion is not to diverge from the neighborhood of the equilibrium it is only necessary that the potential energy increases as q_i increase. More specifically, along any path diverging from E, such as the path

$$\bar{r} = a_1 q_1 \bar{e}_1 + a_2 q_2 \bar{e}_2 + \dots + a_n q_n \bar{e}_n \tag{23}$$

where a_1, $a_2, \dots,$ a_n are arbitrary positive constants, we require that

$$\nabla P.E. \cdot \bar{r} = \nabla H \cdot \bar{r} = -\bar{Q} \cdot \bar{r} > 0 \tag{24}$$

This essentially states that the generalized force Q_i must act towards the equilibrium. This is equivalent to the requirement that P.E. (and H) have a relative minimum in the neighborhood of the origin. A key element in the above discussion is that if H is time dependent, it is reasonable to require that H be positive definite <u>for all time</u> and, thus, would fulfill the first stability requirement of the proposed theorem.

Another possible mechanism for instability exists, even if H is positive definite in the neighborhood of E. It is possible that the energy of the system will build up over a period of time such that the integral of the time derivative of the difference between the energy of the motion and the energy at the equilibrium position increases without bound

$$I = \int_{t_0}^{t} \frac{d}{dt} (H - H_E)\, dt \tag{25}$$

where H_E is the Hamiltonian evaluated at the equilibrium position and it is a function which depends on t only. If this integral increases without bound, then unbounded values of one or more of the q_i or $\dot{q}_i$ would be expected. Conversely, if this integral can be shown to be bounded in accordance with the second requirement of the stability theorem, the motion will be bounded and of arbitrarily small magnitude, depending upon the initial disturbance that is assumed.

No criterion is known to exist for the selection of an optimum testing function for the purpose of a Liapounov type of stability analysis. However, the above arguments give a physical interpretation of the requirements on the Hamiltonian for a stable equilibrium to exist in a nonautonomous system and shows the relationship between these requirements and those of the proposed stability theorem. Consequently, $H - H_E$ appears as a likely choice for use as a testing function in conjunction with the proposed theorem.

3. Testing function. Rotating coordinates.

In this case the Hamiltonian can be shown to be

$$H = \sum_i \dot{q}_i p_i - L = K.E.^* + U \tag{26}$$

where $K.E.^*$ is the portion of the kinetic energy expression that is quadratic in the velocities and U is the dynamic potential given by

$$U = P.E. - \gamma \tag{27}$$

in which γ is the portion of the kinetic energy expression that does not depend on the velocity. The discussion of the preceding section is equally valid for the rotating coordinate system, except that we must use K.E.* and U instead of K.E. and P.E., respectively. It should be noted that the Hamiltonian is no longer equal to the total energy of the system and the generalized force and momentum are not, in general, equal to the linear force and momentum. Once again the function $V = H - H_E$ appears to be a reasonable testing function for use with the proposed stability theorem.

4. General application of the stability theorem.

In many cases, the parameters of the problem can be specified so as to satisfy the first condition of the stability theorem, namely that the testing function V be positive definite in the neighborhood of the equilibrium position, at all times. This may be done in a direct way by proving that the Hessian matrix associated with V is positive definite for all times or, alternately, by means of a comparison testing function.(3) The latter consists of assuming that it is possible to find a positive definite function $W(x_1, x_2, ---, x_n)$ which does not depend explicitly on t and such that

$$V(x_1, x_2, ---, x_n, t) \geq W(x_1, x_2, ---, x_n) \tag{28}$$

When the time dependence of V is periodic one may regard the function $V = c$ as representing a pulsating n-dimensional surface. In order to check the second stability requirement we must obtain information about the integral

$$I = \int_{t_0}^{t} \frac{dV}{dt}\, dt = \int_{t_0}^{t} \frac{\partial V}{\partial t}\, dt \tag{29}$$

In general, V will have the form of a series of terms consisting of a periodic function multiplied by second, or higher, power functions of the generalized coordinates q_i and generalized velocities $\dot{q}_i$. Consequently, we must at least know the form of the solution, q_i and $\dot{q}_i$, in order to determine the behavior of the integral, Eq. (29).

For some types of problems it is possible to state that the solution is of the form

$$q_i = \sum_j f_j^{(i)}(t)\, e^{i\gamma_j t} \tag{30}$$

where $f_j^{(i)}(t)$ are nonperiodic functions of time (or constants) and γ_j are real numbers. Equation (29) can be re-expressed in the form of terms such as

$$\int_{to}^{t} g_\ell(t)\, e^{i\sigma_\ell t} \cos \omega t\, dt \tag{31}$$

where ω is the frequency of one of the periodic terms in the Hamiltonian. In the case of undamped systems in which the periodic terms in the Hamiltonian are small (which is certainly the case in many satellite dynamics problems) the $g_\ell(t)$ will be constant or slowly varying functions of time and the integral, Eq. (31), will behave like the product of trigonometric functions and will diverge only if one of the σ_ℓ becomes equal to one of the frequencies in the periodic forcing function. We will call this a resonance oscillation.

Estimation of the frequencies of oscillation of the system will depend on the particular system under consideration. In some cases it could be accomplished by means of Floquet's theorem or an asymptotic expansion in terms of a small parameter.

The conclusions can be stated as a corollary to the stability theorem:

<u>Corollary</u> - An undamped system subjected to conservative forces and characterized by differential equations and a Hamiltonian function with periodic coefficients will admit stable motion in the neighborhood of an equilibrium position if

a. the Hamiltonian function has a relative minimum at the equilibrium position and

b. no resonance occurs between the motion in the neighborhood of the equilibrium and the periodic coefficients in the Hamiltonian.

In some cases, such as the following example, it may be desirable to apply the stability theorem to a linearized system, in which case the statement concerning the stability of motion should be regarded as pertinent to the small motion only. Hence, in this case, stability will occur in the indicated regions of the parameter space only if the linear terms dominate the motion.

5. Application of the stability theorem to the spinning, unsymmetrical satellite in a circular orbit.

Application of the proposed theorem to the problem of the spinning, unsymmetrical, rigid satellite moving in a circular orbit will now be undertaken. To this end we make the assumption that φ is an explicit function of t so that we can use the Hamiltonian function as given by Eq. (13). The Hamiltonian function evaluated at the equilibrium position, $\theta_1 = \theta_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$, is

$$H_E = \frac{1}{2} A \left[- r (1 + \alpha)^2 \Omega_0^2 + 3\epsilon\Omega_0^2 \, s^2\varphi\right] \tag{32}$$

so that the testing function can be written as

$$\begin{aligned} V = H - H_E = \frac{1}{2} A \Big\{ & \dot{\theta}_1^2(1 + \epsilon \, s^2\varphi) + \dot{\theta}_2^2(c^2\theta_1 + r \, s^2\theta_1 + \epsilon c^2\theta_1 c^2\varphi) \\ & + \epsilon \, \dot{\theta}_1\dot{\theta}_2 \, c\theta_1 \, s2\varphi + \Omega_0^2 \,[r(1 - c^2\theta_1 c^2\theta_2) + 2r\alpha(1 - c\theta_1 c\theta_2) \\ & - s^2\theta_2 - c^2\theta_2 \, s^2\theta_1 + 3(r - 1) \, s^2\theta_2 \, c^2\theta_1] + \epsilon \, \Omega_0^2 \, [3(c\theta_2 s\varphi + \\ & s\theta_1 \, s\theta_2 \, c\varphi)^2 - 3s^2\varphi - (s\theta_2 \, s\varphi - c\theta_2 s\theta_1 c\varphi)^2] \Big\} \end{aligned} \tag{33}$$

Recalling that $V = K.E.^* + U$, where $K.E.^*$ includes the terms that are quadratic in the velocities and is a positive definite function in the neighborhood of the equilibrium position E, we can write the dynamic potential, U, in the form

$$U = \frac{1}{2} A \Omega_0^2 [r(1 - c^2\theta_1 c^2\theta_2) + 2r\alpha (1 - c\theta_1 c\theta_2) - s^2\theta_2 - c^2\theta_2 s^2\theta_1 + 3(r - 1) s^2\theta_2 c^2\theta_1 + 3\epsilon(c\theta_2 s\varphi + s\theta_1 s\theta_2 c\varphi)^2 - 3\epsilon s^2\varphi - \epsilon(s\theta_2 s\varphi - c\theta_2 s\theta_1 c\varphi)^2] \tag{34}$$

As pointed out in Section 3 it is sufficient to check U for positive definiteness rather that V.

It is easy to check that

$$\left.\frac{\partial U}{\partial \theta_1}\right|_{\theta_1=\theta_2=0} = \left.\frac{\partial U}{\partial \theta_2}\right|_{\theta_1=\theta_2=0} = 0 \tag{35}$$

which confirms the existence of the equilibrium at $\theta_1 = \theta_2 = 0$. To determine the conditions for positive definiteness of U we apply Sylvester's criterion.(6) According to this criterion we conclude that the system is stable if the following conditions are satisfied

$$\left.\frac{\partial^2 U}{\partial \theta_1^2}\right|_E = A \Omega_0^2 [r(1 + \alpha) - 1 - \epsilon c^2\varphi] > 0$$

$$\left[\frac{\partial^2 U}{\partial \theta_1^2} \frac{\partial^2 U}{\partial \theta_2^2} - \left(\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}\right)^2\right]_E = A^2\Omega_o^4 [(r - 1 + r\alpha)(4r - 4 + r\alpha) + \epsilon(-4r + 4 - 4r\alpha + 3r\alpha c^2\varphi) - 12\epsilon^2 s^2\varphi c^2\varphi] > 0 \tag{36}$$

It should be noted that if we let $\epsilon = 0$ we obtain the same stability criteria as obtained by Pringle(7) and Likins(8) for the symmetrical body. When ϵ is not zero we obtain time dependent terms through α

and φ. For small values of ϵ we can determine the boundary of the region within which U is positive definite by neglecting terms in ϵ^2. Writing the binomial expansion of Eq. (12) and retaining the first two terms only, we obtain

$$\alpha \cong \alpha_1 + \epsilon \frac{3}{4r\alpha_1} c2\varphi = \alpha_1 + \epsilon \frac{3}{4r\alpha_1} (2c^2\varphi - 1) \tag{37}$$

so that Eqs. (36) become

$$\begin{aligned} &r(1 + \alpha_1) - 1 - \epsilon \left[\frac{3}{4\alpha_1} + \left(1 - \frac{3}{2\alpha_1}\right) c^2\varphi \right] > 0 \\ &(r - 1 + r\alpha_1)(4r - 4 + r\alpha_1) + \epsilon \left[\left(4 - 4r - 4r\alpha_1 + \frac{15}{4\alpha_1} - \frac{15r}{4\alpha_1}\right)\right. \\ &\qquad \left. + \left(3r\alpha_1 - \frac{15}{2\alpha_1} + \frac{15r}{2\alpha_1}\right) c^2\varphi \right] > 0 \end{aligned} \tag{38}$$

The expansion used in Eq. (37) is valid only when $\frac{3\epsilon}{2r\alpha_1} < 1$ and the retention of the first two terms only is justified when $\frac{3\epsilon}{2r\alpha_1} \ll 1$. Consequently, the above expression will not be used to investigate the region in which $r\alpha_1$ is small. In addition, we will consider only the region in which α is positive and ϵ is sufficiently small to neglect terms in ϵ^2. Under these restrictions Eqs. (37) and (38) may be extremized by selecting $c^2\varphi$ to be zero or one. It is then found that Eq. (38) is the most stringent requirement and that the requirement for stable motion is satisfied if the following two conditions are fulfilled:

$$\begin{aligned} r &> \frac{4}{\alpha_1 + 4} + \epsilon \frac{16\alpha_1^2 + 11\alpha_1 - 10}{4\alpha_1(\alpha_1^2 + 5\alpha_1 + 4)} \\ r &> \frac{4}{\alpha_1 + 4} + \epsilon \frac{3}{4\alpha_1(\alpha_1 + 4)} \end{aligned} \tag{39}$$

Figure 2 shows the resulting boundary of the stability region for $\epsilon = 0$ and $\epsilon = 0.1$, where the boundary for $\epsilon = 0$ is identical with that shown by Pringle[7] and Likins[8] for the symmetrical body.

The second part of the stability theorem can now be applied. If we assume that the linearized equations describe the motion near E, we may use Floquet's theorem[9] to state that the solutions are of the form

$$\begin{aligned} \theta_1 &= \sum_{j=1}^{4} f_j(t)\, e^{(u_j + iv_j)t} \\ \theta_2 &= \sum_{j=1}^{4} g_j(t)\, e^{(u_j + iv_j)\, t} \end{aligned} \tag{40}$$

in which $f_j(t)$ and $g_j(t)$ are periodic functions with the same period as the periodic coefficients appearing in the differential equations of motion, which is $2\pi/2\omega_0$ in this case. Consequently, $f_j(t)$ and $g_j(t)$ may be expanded in terms of Fourier series so that

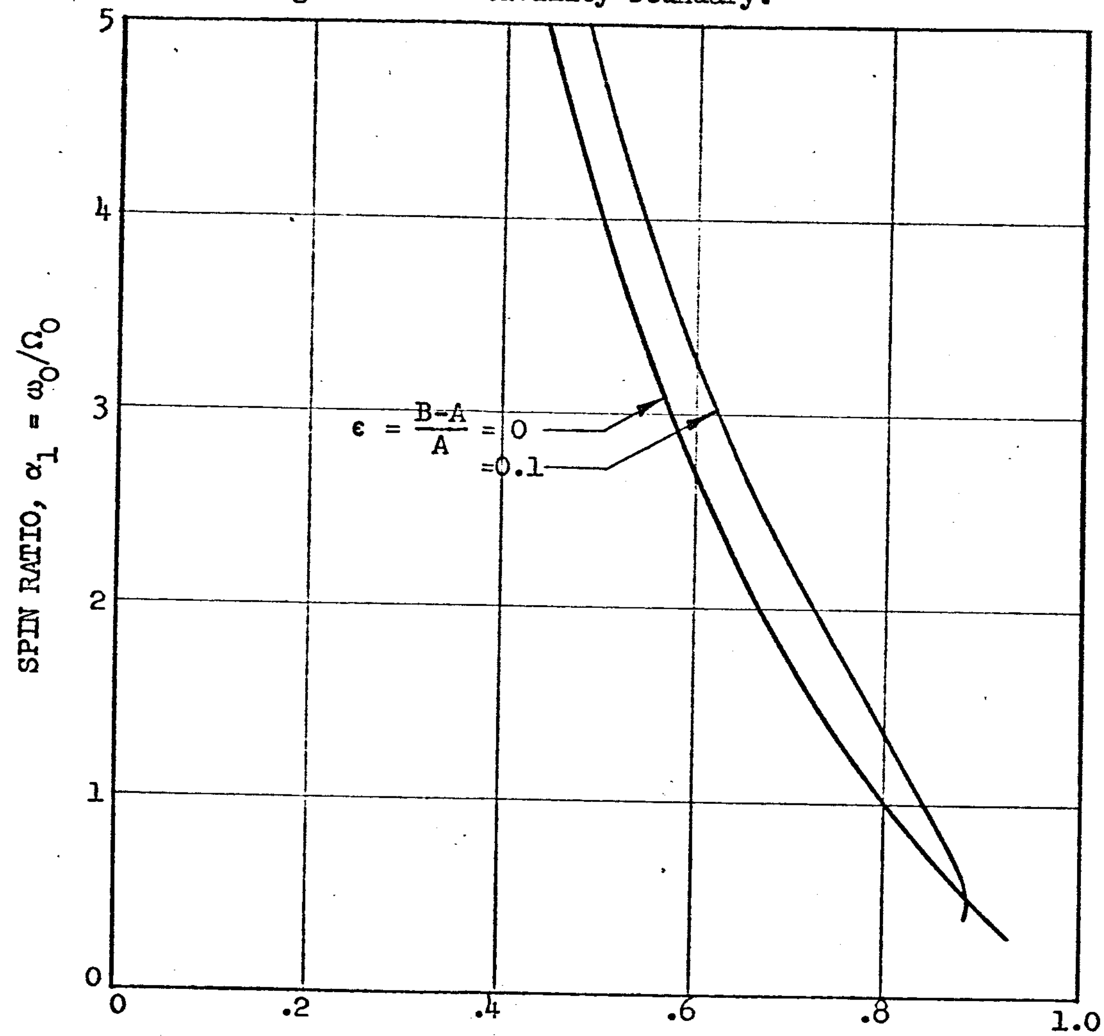
$$\theta_1 = \sum_{j=1}^{4} \sum_{k=-\infty}^{\infty} f_{jk}\, e^{u_j t}\, e^{i(v_j + 2k\omega_0)t}$$

$$\theta_2 = \sum_{j=1}^{4} \sum_{k=-\infty}^{\infty} g_{jk}\, e^{u_j t}\, e^{i(v_j + 2k\omega_0)t}$$

In the neighborhood of the equilibrium position, we will neglect terms in the third and higher powers of the coordinates and velocities as small compared with second power terms, so that the partial derivative of the testing function V with respect to time may be written as

$$\begin{aligned} \frac{\partial V}{\partial t} = \frac{\partial (H - H_E)}{\partial t} &= \frac{1}{2}\, \epsilon\, A\Omega_0 \Big\{ \alpha_1 (\dot{\theta}_1^2 - \dot{\theta}_2^2)\ s\ 2\omega_0 t + 2\alpha_1 \dot{\theta}_1 \dot{\theta}_2\ c\ 2\omega_0 t \\ &- \Omega_0^2 \left[\left(\frac{3}{2} - \alpha_1 \right) \theta_1^2 + \left(\frac{3}{2} + 4\alpha_1 \right) \theta_2^2 \right] s 2\omega_0 t \\ &+ 8\alpha_1\, \Omega_0^2\, \theta_1 \theta_2\ c\ 2\omega_0 t \Big\} + O(\epsilon^2) \end{aligned} \tag{42}$$

NOTE: Nonresonance oscillations are stable for configurations to the right of the stability boundary.



RATIO OF MOMENTS OF INERTIA, $r = C/A$

FIGURE 2

STABILITY BOUNDARY DICTATED BY POSITIVE DEFINITE HAMILTONIAN (LINEARIZED SYSTEM) SPINNING, UNSYMMETRICAL SATELLITE

Substituting Eqs. (41) into Eq. (42) and integrating from t_0 to t we obtain an expression of the term

$$I = \sum_{j=1}^{4} \sum_{\ell=1}^{4} \sum_{k=-\infty}^{\infty} h_{j\ell k} \int_{t_0}^{t} e^{(u_j+u_\ell)t} e^{i(v_j+v_\ell + 2k\omega_0)t} \sin(2\omega_0 t + \delta)\, dt \tag{43}$$

We must investigate four cases

$$\begin{aligned}
&(a)\quad u_j + u_\ell \le 0, \quad v_j + v_\ell + 2k\omega_0 \neq 2\omega_0 \\
&(b)\quad u_j + u_\ell > 0, \quad v_j + v_\ell + 2k\omega_0 \neq 2\omega_0 \\
&(c)\quad u_j + u_\ell \le 0, \quad v_j + v_\ell + 2k\omega_0 = 2\omega_0 \\
&(d)\quad u_j + u_\ell > 0, \quad v_j + v_\ell + 2k\omega_0 = 2\omega_0
\end{aligned} \tag{44}$$

The first two cases are nonresonant cases, so that the value of the integral varies periodically with time. Case (a) represents stable motion whereas case (b) is clearly impossible since, for large t, the Hamiltonian would oscillate with increasing amplitude, which is not possible for a nonresonant case. Cases (c) and (d) correspond to resonant motion. Case (c) represents bounded resonant motion and case (d) represents divergent motion in which the Hamiltonian tends to increase without bound with time. It is possible to say, however, that unstable motion does not occur in the nonresonant case for which the Hamiltonian is positive definite.

As the value of $\epsilon = (B - A)/A$ approaches zero, in the limit, the motion must reduce to

$$\begin{aligned}
\theta_1 &= a_{11}\, e^{i\omega_1 t} + a_{12}\, e^{-i\omega_1 t} + a_{13}\, e^{i\omega_2 t} + a_{14}\, e^{-i\omega_2 t} \\
\theta_2 &= a_{21}\, e^{i\omega_1 t} + a_{22}\, e^{-i\omega_1 t} + a_{23}\, e^{i\omega_2 t} + a_{24}\, e^{-i\omega_2 t}
\end{aligned} \tag{45}$$

Where $\pm\omega_1$ and $\pm\omega_2$ are characteristic numbers associated with the equations

$$\begin{aligned} \theta_1'' + \theta_2' \, [2-r(1+\alpha_1)] - \theta_1^2 \, [1-r(1+\alpha_1)] &= 0 \\ \theta_2'' - \theta_1' \, [2-r(1+\alpha_1)] - \theta_2^2 \, [4-r(4+\alpha_1)] &= 0 \end{aligned} \tag{46}$$

The characteristic number have the values

$$\omega_1 = \sqrt{b + \sqrt{b^2 - c}}, \quad \omega_2 = \sqrt{b - \sqrt{b^2 - c}}$$

in which

$$\begin{aligned} b &= -\frac{1}{2}\Omega_0^2 \, [1 - r(1 - 2\alpha_1) - r^2(1+\alpha_1)^2] \\ c &= -\Omega_0^4 \, [4 - r(5+\alpha_1) + r^2(1+\alpha_1)(4+\alpha_1)] \end{aligned} \tag{48}$$

A comparison of Eqs. (41) and (45) shows that

$$\lim_{\epsilon \to 0} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \omega_1 \\ -\omega_1 \\ \omega_2 \\ -\omega_2 \end{Bmatrix} \tag{49}$$

Consequently when ϵ is small, Eqs. (44) lead us to the conclusion that resonance must occur near the values

$$\left.\begin{aligned} \omega_1 &= m\omega_0 \quad , \quad \omega_2 = m\omega_0 \\ \omega_1 + \omega_2 &= 2m\omega_0 \quad , \quad \omega_1 - \omega_2 = 2m\omega_0 \end{aligned}\right\} (m = \pm 1, \pm 2, ---) \tag{50}$$

Figure 3 shows in the plane α_1 vs r the location near which resonance will occur in the linearized system and for small ϵ. It should be noted that the present method gives us the location of instability regions. To describe the width of these regions we must choose different methods, as will be discussed in the next section. It is interesting to note, however, that the existence of resonance conditions is in contradiction with results obtained by Kane and Shippy[(2)], which showed stability to occur throughout these regions. The stability conclusions reached in Reference (2), however, were based upon spot-checking of the stability at a few fixed values of spin rate and moments of inertia ratio. The results showing stability at a number of points apparently led the above investigators to believe that large regions were stable.

IV. Analysis Based on an Infinite Determinant.

A method similar to that employed by Bolotin[(10)] has been adapted for use in defining the regions of resonance of the linearized system. The present study represents a substantial advance in the use of this type of analysis, including its application to a system of second order equations which contain the velocity terms that are typical of gyroscopic motions. This method has been applied to the problem of the spinning, unsymmetrical body.

1. Discussion of the motion of linear, periodic systems.

According to Floquet's theorem, a system of n/2 second order linear differential equations with periodic coefficients possess n linearly independent solutions* of the form

$$\left\{ x^{(j)} \right\} + \left\{ f^{(j)} \right\} e^{\frac{t}{T} \ln \rho_j} \quad (j = 1, 2, \ldots, n) \tag{51}$$

*Discrete characteristic multipliers are assumed.

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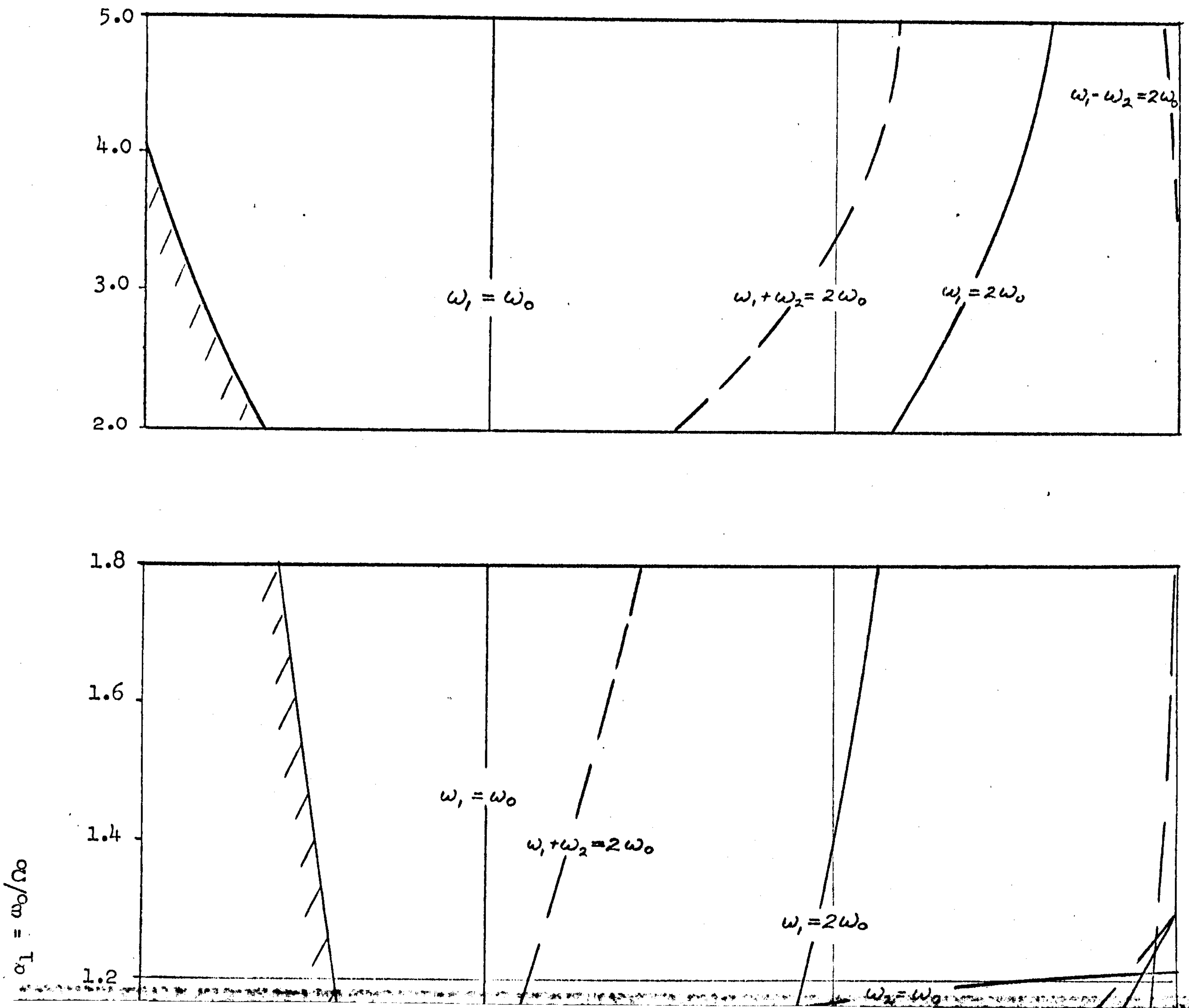
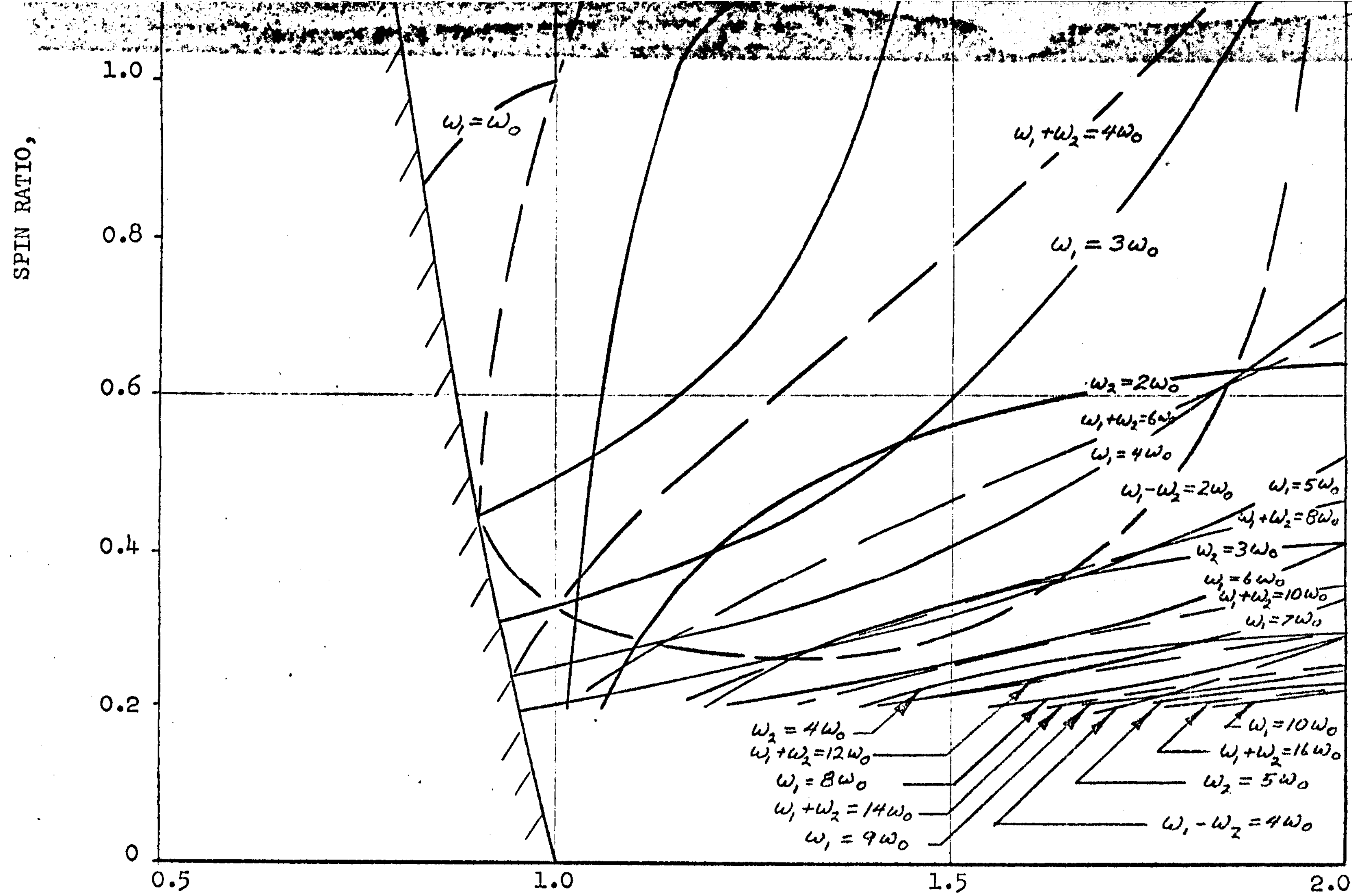


5.0
4.0
3.0
2.0
$\omega_1 = \omega_0$
$\omega_1 + \omega_2 = 2\omega_0$
$\omega_1 = 2\omega_0$
$\omega_1 - \omega_2 = 2\omega_0$
1.8
1.6
1.4
1.2
$\omega_1 = \omega_0$
$\omega_1 + \omega_2 = 2\omega_0$
$\omega_1 = 2\omega_0$
$\omega_2 = \omega_0$



RATIO OF MOMENTS OF INERTIA, $r = C/A$

FIGURE 3

UNSYMMETRICAL SPINNING SATELLITE

APPROXIMATE LOCATIONS OF RESONANCE REGIONS FOR $\epsilon \ll 1$

where the $f_k^{(j)}$ are periodic functions with period T and the ρ_j are called the characteristic multipliers. Note that

$$\ln \rho = \ln |\rho| + i \arg \rho \tag{52}$$

The motion about the identically zero solution will be stable if no characteristic multiplier has an absolute value that is greater than one. The motion will be asymptotically stable if all characteristic multipliers have absolute values that are less than one and will be unstable if any characteristic multiplier has an absolute value that is greater than one.

The system under consideration is linear with periodic coefficients and has a Hamiltonian function from which the equations of motion may be written in canonical form, i.e.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{53}$$

A theorem due to Liapounov[6] states that for such a case the characteristic multipliers occur in reciprocal pairs. Consequently, if ρ_j is a characteristic multiplier, then $1/\rho_j$ is also a characteristic multiplier. Also the characteristic multipliers for the system under consideration occur in complex conjugate pairs.*

A pair of particular solutions corresponding to reciprocal roots may be written

$$\begin{aligned} \left\{x^{(j)}\right\} &= \left\{f^{(j)}\right\} e^{\frac{t}{T} \ln \rho_j} \\ \left\{x^{(k)}\right\} &= \left\{f^{(k)}\right\} e^{-\frac{t}{T} \ln \rho_j} \end{aligned} \tag{54}$$

*This is not known to have been proven for the system under study.

The region of parameter space in which ρ_j is real and different from ± 1 is clearly the region of unstable motion, since one of the reciprocal characteristic multipliers must be greater than one. Upon further variation of the parameters of the problem, the roots will become complex conjugate pairs

$$\rho_j = a + ib, \quad \rho_k = \rho_j^{-1} = a - ib \tag{55}$$

and since $\rho_j \rho_k = 1$ they will have an absolute value equal to one. This then indicates that the region of complex ρ_j is the region of bounded motion. Since the characteristic multipliers are continuous functions of the parameters of the problem, the boundaries of the regions of stability will be given by the cases when pairs of roots, $\rho = 1$ or $\rho = -1$, occur. But we can use Floquet's theorem to show that

$$\begin{aligned}\left\{x^{(j)}\ (t + T)\right\} = \left\{f^{(j)}\ (t + T)\right\} e^{\frac{t + T}{T} \ln \rho_j} &= \rho_j \left\{x^{(j)}(t)\right\} \\ &= \rho_j \left\{ f^{(j)}(t) \right\} e^{\frac{t}{T} \ln \rho_j}\end{aligned} \tag{56}$$

which, on the boundaries of the regions of instability, gives us

$$\begin{aligned}\rho_j &= 1, \qquad x^{(j)}(t + T) = x^{(j)}(t) \\ \rho_j &= 1, \qquad x^{j}(t + T) = - x^{(j)}(t)\end{aligned} \tag{57}$$

The first of Eqs. (57) tells us that a motion which is periodic with period T will be admitted on a boundary where $\rho = 1$. The second of Eqs. (57) indicates that a motion that is periodic with period 2T will be admitted on a boundary where $\rho = -1$. In addition, any distinct instability region (region of real characteristic multipliers) must be bounded by a single value of ρ (i.e. $\rho = 1$ or $\rho = -1$). This is seen to be true because, in order for the values $\rho = 1$ and $\rho = -1$ to occur on different boundaries of a given instability region, it would be necessary that $\rho_j = 0$ and $1/\rho_j = \infty$ at some location within the instability region, since ρ is a continuous function of the parameters of the system. But this is not possible. This leads to the formulation of a stability theorem.

2. Stability theorem for linear systems.

Theorem

Periodic solutions with period T or 2T are admitted on the boundaries between regions of stability and regions of instability in canonical systems which are described by systems of linear differential equations with periodic coefficients. Solutions of the same period bound each distinct region of instability.

3. General application of the stability theorem.

Application of the preceding stability theorem consists of investigations to define the locations in parameter space along which solutions with period T or 2T can exist. Separate Fourier expansions with period T and 2T may be made such that for period T we have

$$\begin{aligned}\{x\} = \sum_{n=-\infty}^{\infty} \{a_n\} e^{i\frac{2n\pi t}{T}} &= \sum_{n=1}^{\infty} \{b_n\} \sin \frac{2n\pi t}{T} \\ &+ \sum_{n=0}^{\infty} \{c_n\} \cos \frac{2n\pi t}{T}\end{aligned} \tag{58}$$

and for period 2T we can write

$$\begin{aligned}\{x\} = \sum_{n=-\infty}^{\infty} \{a_n\} e^{i\frac{n\pi t}{T}} &= \sum_{n=1}^{\infty} \{b_n\} \sin \frac{n\pi t}{T} \\ &+ \sum_{n=0}^{\infty} \{c_n\} \cos \frac{n\pi t}{T}\end{aligned} \tag{59}$$

Either the exponential or trigonometric form of one of the above solutions may be substituted into the differential equations and the resulting coefficients of equal harmonics may be equated, giving an n times infinite system of linear equations in terms of n times infinite coefficients. For a nontrivial solution to exist, the determinant of the coefficients must be zero. Evaluation of the infinite determinant has been possible in the case of Hill's equation(11), and this process has been used further by Mettler(12). In both cases, the equations being studied are of the second order, and do not include velocity terms. The same techniques do not appear applicable to systems with gyroscopic terms.

In some cases, a reasonable approximation may be achieved by taking only the first few terms of the periodic expansion, Eq. (58) or (59). This approach will be taken in the following example.

4. Determination of the regions of instability of a spinning, unsymmetrical satellite in a circular orbit.

The linearized equations of motion of a spinning, unsymmetrical satellite in a circular orbit may be shown to be:

$$\theta_1'' + \theta_2'\left[-\beta + \epsilon\left(r_1 - \frac{3}{4\alpha_1}\right)\cos 2\alpha_1\tau\right] + \theta_1'\,\epsilon r_1 \sin 2\alpha_1\tau$$

$$+ \theta_2\,\epsilon r_2 \sin 2\alpha_1\tau + \theta_1\left[\beta+1 - \epsilon\left(r_1 - \frac{3}{4\alpha_1}\right)\cos 2\alpha_1\tau\right] + \theta(\epsilon^2) = 0$$

$$\theta_2'' + \theta_1'\left[\beta + \epsilon\left(r_1 + \frac{3}{4\alpha_1}\right)\cos 2\alpha_1\tau\right] - \theta_2'\,\epsilon r_1 \sin 2\alpha_1\tau \tag{60}$$

$$+ \theta_1\,\epsilon r_1 \sin 2\alpha_1\tau + \theta_2\left[\beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2\right.$$

$$\left. + \epsilon\left(r_2 + \frac{3}{4\alpha_1}\right)\cos 2\alpha_1\tau\right] + O(\epsilon^2) = 0$$

in which

$$\beta = r(\alpha_1 + 1)\left(1 - \frac{\epsilon}{2}\right) - 2$$

$$r_1 = \left(1 - \frac{r}{2}\right)(1 + \alpha_1) \tag{61}$$

$$r_2 = \left(1 - \frac{r}{2}\right)(4 + \alpha_1)$$

and terms in the second and higher powers of ϵ are grouped in the terms $O(\epsilon^2)$ and will be neglected.

The regions of instability are bounded by periodic solutions of period 2T and T which can be written in the form

Period 2T

$$\theta_1 = \sum_{n=1,3,5,\ldots}^{\infty} (a_{1n} \sin n\alpha_1\tau + b_{1n} \cos n\alpha_1\tau)$$

$$\theta_2 = \sum_{n=1,3,5,\ldots}^{\infty} (a_{2n} \sin n\alpha_1\tau + b_{2n} \cos n\alpha_1\tau) \tag{62}$$

Period T

$$\theta_1 = \sum_{n=0,2,4,\ldots}^{\infty} (a_{1n} \sin n\alpha_1\tau + b_{1n} \cos n\alpha_1\tau)$$

$$\theta_2 = \sum_{n=0,2,4,\ldots}^{\infty} (a_{2n} \sin n\alpha_1\tau + b_{2n} \cos n\alpha_1\tau) \tag{63}$$

where advantage has been taken of the specific form of Eqs. (60) in eliminating the even n terms in Eqs. (62) and the odd n terms in Eqs. (63).

The general procedure at this point consists of substituting a finite number of terms of either Eqs. (62) or (63) into the equations of motion, equating the coefficients of equal harmonics, setting up the determinant of the coefficients of the assumed periodic series, expanding this determinant, and solving the resulting equation for the instability boundaries.

Experience with this procedure tells us that for the problem under consideration the use of only the $n = 1$ terms of Eqs. (62) will define a first approximation for the regions near $\omega_1 = \omega_0$ and $\omega_2 = \omega_0$, where ω_1 and ω_2 are the natural frequencies of the unperturbed system. If we include the $n = 3$ terms also, we will obtain a better approximation of the regions near $\omega_1 = \omega_0$ and $\omega_2 = \omega_0$, and will obtain a first approximation of the regions near $\omega_1 = 3\omega_0$ and $\omega_2 = 3\omega_0$. As we include more terms, we improve the approximation of the lower order parametric resonances, and define regions at which the natural frequencies are approximately equal to successively higher odd multiples of the average spin frequency, ω_0. When we use Eqs. (63) rather than Eqs. (62), we define the even numbered regions in a similar manner.

Inspection of Figure 2 of the previous section shows us that regions near $\omega_1 = 2\omega_0$ and $\omega_2 = 2\omega_0$ are to be expected for a satellite configurations with $1 < r < 1.6$. We can define these regions, in the first approximation, by assuming

$$\begin{aligned} \theta_1 &= a_1 + b_1 \sin 2\alpha_1\tau + c_1 \cos 2\alpha_1\tau \\ \theta_2 &= a_2 + b_2 \sin 2\alpha_1\tau = c_2 \cos 2\alpha_1\tau \end{aligned} \tag{64}$$

Substitution into the equations of motion, Eqs. (60), and equating the constant terms, the coefficients of $\sin 2\alpha_1\tau$ and the coefficients of $\cos 2\alpha_1\tau$ to zero, we obtain the following six algebraic equations:

$$a_1(\beta + 1) + b_2\epsilon \left[\alpha_1\left(r_1 - \frac{3}{4\alpha_1}\right) + \frac{1}{2} r_2\right] + c_1\epsilon\left[-\alpha_1 r_1 - \frac{1}{2}\left(r_1 - \frac{3}{4\alpha_1}\right)\right] = 0$$

$$a_1\epsilon\left(r_1 - \frac{3}{4\alpha_1}\right) - b_2\, 2\alpha_1\beta + c_1\,(-4\alpha_1^2 + \beta + 1) = 0$$

$$a_1\epsilon r_1 + b_2\left[-4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2\right] - c_1 2\alpha_1\beta = 0$$

$$a_2\left[\beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2\right] + b_1\epsilon\left[\alpha_1\left(r_1 + \frac{3}{4\alpha_1}\right) + \frac{1}{2} r_1\right]$$

$$+ c_2\epsilon\left[\alpha_1 r_1 + \frac{1}{2}\left(r_2 + \frac{3}{4\alpha_1}\right)\right] = 0$$

$$a_2\epsilon r_2 + b_1(-4\alpha_1^2 + \beta + 1) + c_2\epsilon\left[\alpha_1 r_1 + \frac{1}{2}\left(r_2 + \frac{3}{4\alpha_1}\right)\right] = 0$$

$$a_2\epsilon\left(r_2 + \frac{3}{4\alpha_1}\right) + b_1\, 2\alpha_1\beta + c_2\left[-4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2\right] = 0$$

For nontrimial solutions to exist, the determinant of the coefficients must be zero, which in this case can be expressed as two 3 x 3 determinants.

$$\begin{vmatrix} \beta + 1 & \epsilon\left[\alpha_1\left(r_1 - \frac{3}{4\alpha_1}\right) + \frac{1}{2} r_2\right] & \epsilon\left[-\alpha_1 r_1 - \frac{1}{2}\left(r_1 - \frac{3}{4\alpha_1}\right)\right] \\ \epsilon\left(r_1 - \frac{3}{4\alpha_1}\right) & -2\alpha_1\beta & -4\alpha_1^2 + \beta + 1 \\ \epsilon r_1 & -4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 & -2\alpha_1\beta \end{vmatrix} = 0$$

$$\begin{vmatrix} \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 & \epsilon\left[\alpha_1\left(r_1 + \frac{3}{4\alpha_1}\right) + \frac{1}{2} r_1\right] & \epsilon\left[\alpha_1 + \frac{1}{2}\left(r_2 + \frac{3}{4\alpha_1}\right)\right] \\ \epsilon r_2 & -4\alpha_1^2 + \beta + 1 & 2\alpha_1\beta \\ \epsilon\left(r_2 + \frac{3}{4\alpha_1}\right) & 2\alpha_1\beta & -4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 \end{vmatrix} = 0 \tag{66}$$

In the process of determining the boundaries of the regions of instability (which are surfaces in α_1, r, ϵ space) it is convenient to express the value of α_1 in the following terms:

$$2\alpha_1 = \frac{\omega_i}{\Omega_0} + \Delta_1\epsilon + \Delta_2\epsilon^2 + \dots \qquad (i = 1,2) \tag{67}$$

where ω_i is the natural frequency of the system in the absence of the periodic terms in the equations of motion and is given by

$$\frac{\omega_i}{\Omega_0} = \left[\left[\frac{\beta^2 + 2\beta + 3r(1-\frac{\epsilon}{2}) + 1}{2}\right] \pm \left\{\left[\frac{\beta^2 + 2\beta + 3r(1-\frac{\epsilon}{2}) - 1}{2}\right]^2 - (\beta+1)\left[\beta + 3r\left(1-\frac{\epsilon}{2}\right) - 2\right]\right\}^{1/2}\right]^{1/2} \tag{68}$$

and we will associate ω_1 with the positive sign under the radical and ω_2 with the negative sign. The boundaries of the regions of instability can then be expressed in terms of Eq. (67). We find that $\Delta_1 = 0$ and Δ_2 takes on either of the following values

$$\Delta_2 = \frac{1}{(\beta+1)[16\alpha_1^3 + 2\alpha_1(-2\beta-3r+1-\beta^2)]}\left\{-\left(r - \frac{3}{4\alpha_1}\right)^2\left(2\alpha_1^2\beta + 2\alpha_1^2 - \frac{1}{2}\beta - \frac{3}{2}r + 1\right) + \left(r - \frac{3}{4\alpha_1}\right)\left[-r_2\alpha_1\beta + r_1\alpha_1\left(-8\alpha_1^2 + \beta + 3r - 1\right)\right] - \frac{r_1 r_2}{3}\left(-4\alpha_1^2 + \beta + 1\right) + 2\alpha_1^2 r_1^2\beta\right\}$$

or

$$\Delta_2 = \frac{1}{(\beta+3r-2)[16\alpha_1^3 + 2\alpha_1(-2\beta-3r+1-\beta^2)]}\left\{\frac{1}{2}\left(r_2 + \frac{3}{4\alpha_1}\right)^2\left(-4\alpha_1^2 + \beta + 1\right) - \left(r_2 + \frac{3}{4\alpha_1}\right)\left(4\alpha_1^3 r_1 - \alpha_1 r_1 + \alpha_1 r_2\beta + 2\alpha_1^2 r_1\beta + \frac{3}{2}\alpha_1\beta\right) + \alpha_1 r_2\left(r_1 + \frac{3}{4\alpha_1}\right)\left(-4\alpha_1^2 + \beta + 3r - 2\right) + \frac{r_1 r_2}{1}\left(-4\alpha_1^2\beta - 4\alpha_1^2 + \beta + 3r - 2\right)\right\} \tag{69}$$

Figure 4 shows the resulting stability boundaries for r = 1.5.

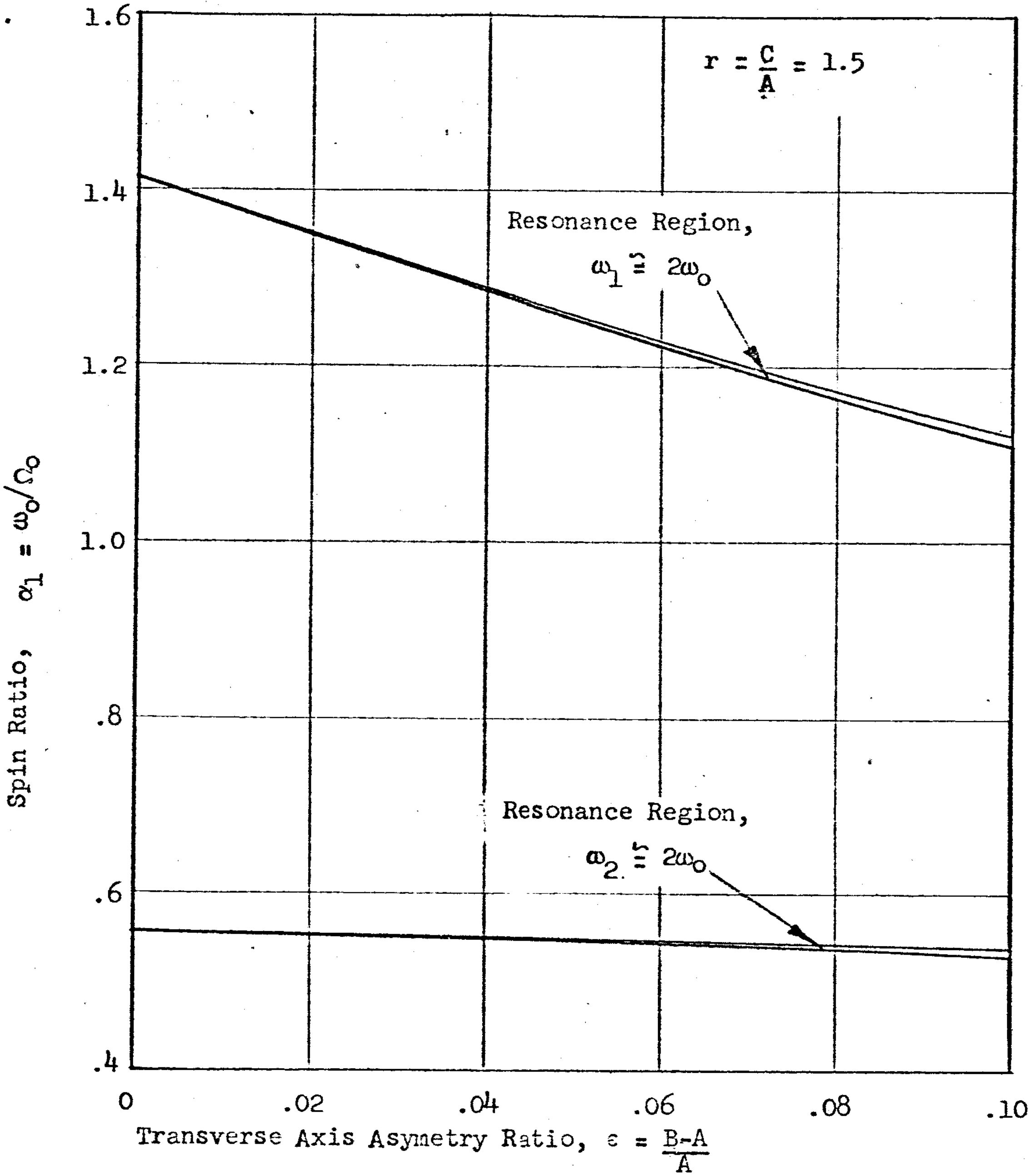


FIGURE 4

UNSYMMETRICAL SPINNING SATELLITE.
RESONANCE REGIONS FOR OSCILLATIONS
WITH FREQUENCY $2\omega_o$ (LINEARIZED SYSTEM)

V. Analysis Based on Asymptotic Expansion in Terms of a Small Parameter.

A method of analysis which is similar to that originated by Kryloff and Bogoliuboff(14) and further developed by Bogoliuboff and Mitropolski(15) has been employed. In contrast with these authors, however, the method used in this case involves an expansion of the equations of motion using assumed simultaneous resonance and nonresonance solutions. As a result, one can define unstable regions of motion of a multi-degree-of-freedom system including those regions in which resonance occurs between periodic terms (i.e. $2n\omega_0$) and the sum or difference of natural frequencies (i.e. $\omega_1 \pm \omega_2$). Systems involving gyroscopic terms have not been discussed in References (14) or (15) and, to the principal investigator's knowledge, this is the first time that this type of asymptotic expansions has been employed for the treatment of satellite dynamics problems. This method has been applied to the problem of defining some of the instability regions of the unsymmetrical, spinning satellite, but will not be reported until a subsequent report, when additional results are available.

The methods employed by Kryloff, Bogoliuboff, and Mitropolski are generally applied to systems with equations of motion of the form

$$\ddot{x} + \omega^2 x = \epsilon f(x,\dot{x},t) \qquad (70)$$

in which ϵ is a small parameter and $f(x,\dot{x},t)$ is an arbitrary, periodic function of time and may be either linear or nonlinear in x and $\dot{x}$. In the limit (as $\epsilon \to 0$) the motion is periodic and of the form

$$x = a \cos (\omega t + \delta) \qquad (71)$$

where a, ω, and δ are constant. We may look upon the left side of Eq. (70) as the unperturbed equation and the terms on the right side will be regarded as a perturbation. The assumption is then made that, for small ϵ, the amplitude and phase angle are no longer constant but functions of ϵ and time and may be expressed as

$$\frac{da}{dt} = \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \dots$$
$$\frac{d\delta}{dt} = \epsilon \delta_1 + \epsilon^2 \delta_2 + \epsilon^3 \delta_3 + \dots \tag{72}$$

The assumed solution is substituted into the equations of motion, the coefficients of every power of ϵ is set equal to zero, and the resulting algebraic equations are solved in succession.

Some problems of satellite dynamics may be expressed in the form

$$\theta_1'' + \beta_1 \theta_2' + \beta_2 \theta_1 = \epsilon\, f_1(\theta_1, \theta_2, \theta_1', \theta_2', t) + O(\epsilon^2)$$
$$\theta_2'' + \beta_3 \theta_1' + \beta_4 \theta_2 = \epsilon\, f_2(\theta_1, \theta_2, \theta_1', \theta_2', t) + O(\epsilon^2) \tag{73}$$

where in the limit, as ϵ approaches zero, the solution becomes of the form

$$\theta_1 = a \cos(\omega_1 t + \delta) + b \cos(\omega_2 t + \delta_2)$$
$$\theta_2 = a\lambda_1 \sin(\omega_1 t + \delta_1) + b\lambda_2 \sin(\omega_2 t + \delta_2) \tag{74}$$

in which ω_1 and ω_2 are the natural frequencies of the unperturbed system, a and b are arbitrary amplitudes, δ_1 and δ_2 are arbitrary phase angles, and λ_1 and λ_2 are constants that are obtained in the process of solving the unperturbed equations of motion. We may assume in general that the functions f_1 and f_2 appearing on the right side of Eqs. (73) can be expressed in terms of products of periodic functions (i.e. with frequency $2\omega_0$) and powers of the coordinates and velocities, θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$.

One can distinguish two significantly different types of motion: resonant and nonresonant. Resonant motion will be shown to take place when ω_1 or ω_2 is sufficiently close to

$$\ell\omega_1 + m\omega_2 + n\omega_0 \qquad (\ell, m, n = 0, \pm 1, \pm 2, \dots) \tag{75}$$

Use of the asymptotic method shows that the nonresonant motion will be stable and bounded, and that resonant motion can be divergent (unstable). Further, following the approach described in the previous section, resonant motion that is bounded and periodic may take place on the boundaries of the regions of instability. The asymptotic method has been adapted to allow calculation of the location of these boundaries.

Apparent advantages of the asymptotic method include its relative ease of calculation of the instability boundaries, its ability to define stability boundaries where the sum or difference of the natural frequencies is resonance with a parametric excitation, and its possible applicability to nonlinear systems.

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