RESEARCH REPORT 248

# ON THE EQUILIBRIUM OF ELECTRON CLOUDS IN TOROIDAL MAGNETIC FIELDS*

by

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## ABSTRACT

The self-consistent equilibrium configurations available to low density ($q = \omega_p^2/\omega_c^2 \ll 1$) toroidal electron clouds under the influence of an external magnetic field and the electric fields due to space and image charges are discussed. The equilibria are dynamic rather than static, but steady. $\nabla B$, centrifugal and other drifts are shown to be unimportant, in contrast to the situation in toroidal neutral plasmas. Typical equilibria are shown in detail, to zero order in $q$. The construction of equilibria accurate to order $q$ is shown to be possible but is not carried out in detail.

## 1. INTRODUCTION

Two new applications[1,2] for magnetically contained clouds of electrons require toroidal vacuum vessels and magnetic fields. It appears that self-consistent azimuthally symmetric toroidal equilibria are possible for such clouds without the necessity for a rotational transform.[3] Roughly, the existence of such equilibria is due to the favorable effect of the strong electric fields which arise both from the presence of unneutralized space charge and from the corresponding surface charges on the wall of the containing vessel. These electric fields constrain the electrons to move in closed orbits even when account is taken of the first order particle drifts usually found in a torus ($\nabla B$, etc.). This situation is in striking contrast to the neutral plasma case where, in the absence of important electric fields, special techniques (such as the rotational transform) are required to provide containment against the first order drifts.

In the applications cited above,[1,2] the electric field is due entirely to a combination of space and image charges, there being no applied potential. Further, the ratio $q = \omega_p^2/\omega_c^2$ is small, on the order of $10^{-3}$. It follows that the electron motion is essentially a balance between the electric and magnetic forces, inertial forces being less than either by the factor $q$. Alternatively, if we suppose that a typical "thermal" velocity is not greater than the electric drift velocity $E/B$, we find that the typical electron gyro radius is not greater than $mE/eB^2$. But from Gauss' law,

$E \sim nea/\epsilon_0$ (see Fig. 1), and it follows that q can be interpreted equivalently as the ratio of the gyro radius to the "scale size" of the electric field. The "scale size" of the magnetic field, due to the toroidal geometry, is $b(> a)$. Thus for moderate radius ratios, q is a rough measure of the "adiabaticity parameter," the parameter which measures the appropriateness of the guiding-center approximation. For the present case, this approximation should be very accurate indeed. Under the same assumption on the electron thermal velocities, q has still another interpretation, namely the ratio of the electron kinetic pressure to the electric pressure.

These considerations suggest the following treatment of the problem of equilibrium: first (Section 2) we neglect all effects of order q, that is, all those effects proportional to the electron mass. This yields for the electron motion the simple expression

$$\underset{\sim}{v}_E = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} \tag{1.1}$$

which must be interpreted as the zero-order solution to the electronic equation of motion. In this approximation the electrons are supposed cold, and no distinction need be made between the particles and their guiding centers. Explicit solutions to the equilibrium problem are given in this limit.

Section 3 is devoted to considerations of the effects of order q taking the particle drift point of view. The principal result of Section 3 is the observation that since the electron energy, magnetic moment, and

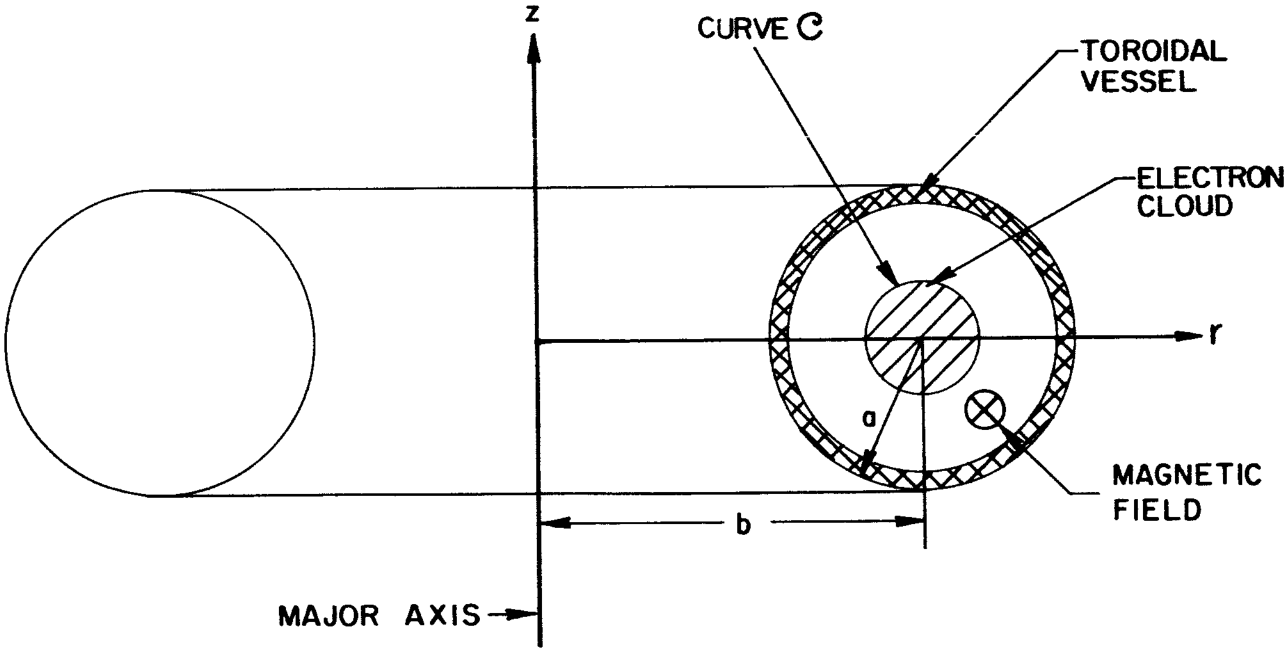


Fig. 1 This figure shows the general configuration discussed in the text. The electrons are subject to the space charge field and the imposed toroidal magnetic field; equilibrium configurations are sought. Azimuthal symmetry is assumed. The curve $\mathcal{C}$ marks the boundary of the electron cloud.



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angular momentum are conserved in the adiabatic approximation, the allowable positions of the particle are restricted to the immediate vicinity of a definite equipotential. The conservation of the electron energy is exhibited explicitly in an adiabatic (guiding center) approximation to the electron motion. Use of the guiding center drift velocity in the equation of continuity yields the necessary functional form of the guiding center distribution for a steady state. It is argued that this form, together with Poisson's equation for the electrostatic potential $\phi$ may be solved self-consistently to yield steady equilibria. This result implies that the "cold" equilibria discussed in Section 2 are in fact very close to true self-consistent equilibria of the "warm" electron cloud. More generally, containment in the presence of $\nabla B$ drifts is seen to be accomplished by finding an equilibrium position for the plasma in which the $\nabla B$ drifts which are, say, upward, are just cancelled by a mean electric drift downwards.

The remaining sections deal with the effect of finite $v_E/c$, and the connection between the derived equilibria and Earnshaw's theorem.

## 2. BASIC CASE, $q = 0$

We consider the geometry illustrated in Fig. 1. Azimuthal symmetry is assumed. We neglect "thermal" motion, and the electron inertia. Since only steady equilibria are being sought, we can derive the electric field from an electrostatic potential $\phi(r, z)$. The electron motion is then solely in the meridian plane and is given by

$$\underset{\sim}{v}_E = \frac{-\nabla\phi \times \underset{\sim}{B}}{B^2} \quad . \tag{2.1}$$

This motion is along an equipotential, so that the potential energy of each electron is conserved. The steady state continuity equation is

$$\text{div } n\underset{\sim}{v}_E = 0 \tag{2.2}$$

and, on substitution and expansion we find:

$$\left(\nabla\phi \times \nabla\frac{n}{B^2}\right) \bullet \underset{\sim}{B} = 0 \quad . \tag{2.3}$$

To reach this result it is necessary to observe that

$$\nabla\phi \bullet \text{curl } \underset{\sim}{B} = \mu_0 \nabla\phi \bullet \underset{\sim}{j} = -\mu_0 e\, n \left\{\nabla\phi \bullet \underset{\sim}{v}_E\right\} = 0 \tag{2.4}$$

where we have identified $\underset{\sim}{j}$ as the conduction current $-ne\underset{\sim}{v}_E$. Thus, the result depends on the fact that, from (2.1), $\underset{\sim}{v}_E$ and $\nabla\phi$ are perpendicular; it does not depend on the vanishing of curl $\underset{\sim}{B}$ .

The interpretation of (2.3) is as follows: by symmetry $\nabla\phi$ and $\nabla(n/B^2)$ both lie in the meridional plane. Their cross product is therefore parallel to $\underset{\sim}{B}$ , and (2.3) can be satisfied only if $\nabla\phi$ and $\nabla n/B^2$ are parallel. This in turn requires that

$$\frac{n}{B^2} = f(\phi) \tag{2.5}$$

where f is an arbitrary function. Now in two dimensional motion perpendicular to the magnetic field the quantity $n/B$ is preserved by each parcel of electrons, this being the number of electrons per unit length in a flux tube. In a torus, however, the flux tubes change their length. Since the length of the tube is proportional to $B^{-1}$ , the quantity $n/B^2$ now has the significance of the number of electrons per flux tube. Note that the ratio q is proportional to $n/B^2$ . Now we have seen that the electrons move at constant $\phi$ . The significance of (2.5) is therefore that steady motion is possible when the number of electrons per flux tube is the same at all points on the same equipotential. This is the most general condition for steady equilibrium under the approximations of this section.

It is a rather simple matter to use the condition (2.5) to derive actual equilibria in detail. To see this, we use Poisson's equation:

$$\nabla^2\phi = ne/\epsilon_0 \quad . \tag{2.6}$$

Also,

$$\text{curl } B = \mu_0 \underset{\sim}{j} \qquad (2.7)$$

implies (in view of the azimuthal symmetry)

$$B \propto \frac{1}{r} + O(v_E{}^2/c^2) \qquad (2.8)$$

Neglecting the correction of order $v_E{}^2/c^2$ in this equation, we can combine (2.5), (2.6) and (2.8) to yield

$$\nabla^2\phi = r^{-2}f(\phi) \qquad (2.9)$$

where some constants have been absorbed into f. (In Section 4 we shall see that this equation holds true even if $v_E/c$ is not negligibly small.) Consider, for example, the situation in which $n/B^2$ (or q) is constant not only along each equipotential, but has the same value for each equipotential. In this case $f(\phi)$ is constant and solutions of (2.9) are readily obtained in the form

$$\phi = (\ell n r)^2 + \text{harmonic functions} \quad . \qquad (2.10)$$

An example of a solution of this type is given in Fig. 2. Such solutions can easily be found to fit any equipotential boundary, or, in the converse procedure, any equipotential arising in this way can be taken as a boundary of the electron cloud. It is a simple matter also to solve for the potential in any vacuum space between the boundary of the cloud and the containing vessel if this is not too distant. For other choices of the function $f(\phi)$ the computation becomes more difficult in practice, but not in principle.

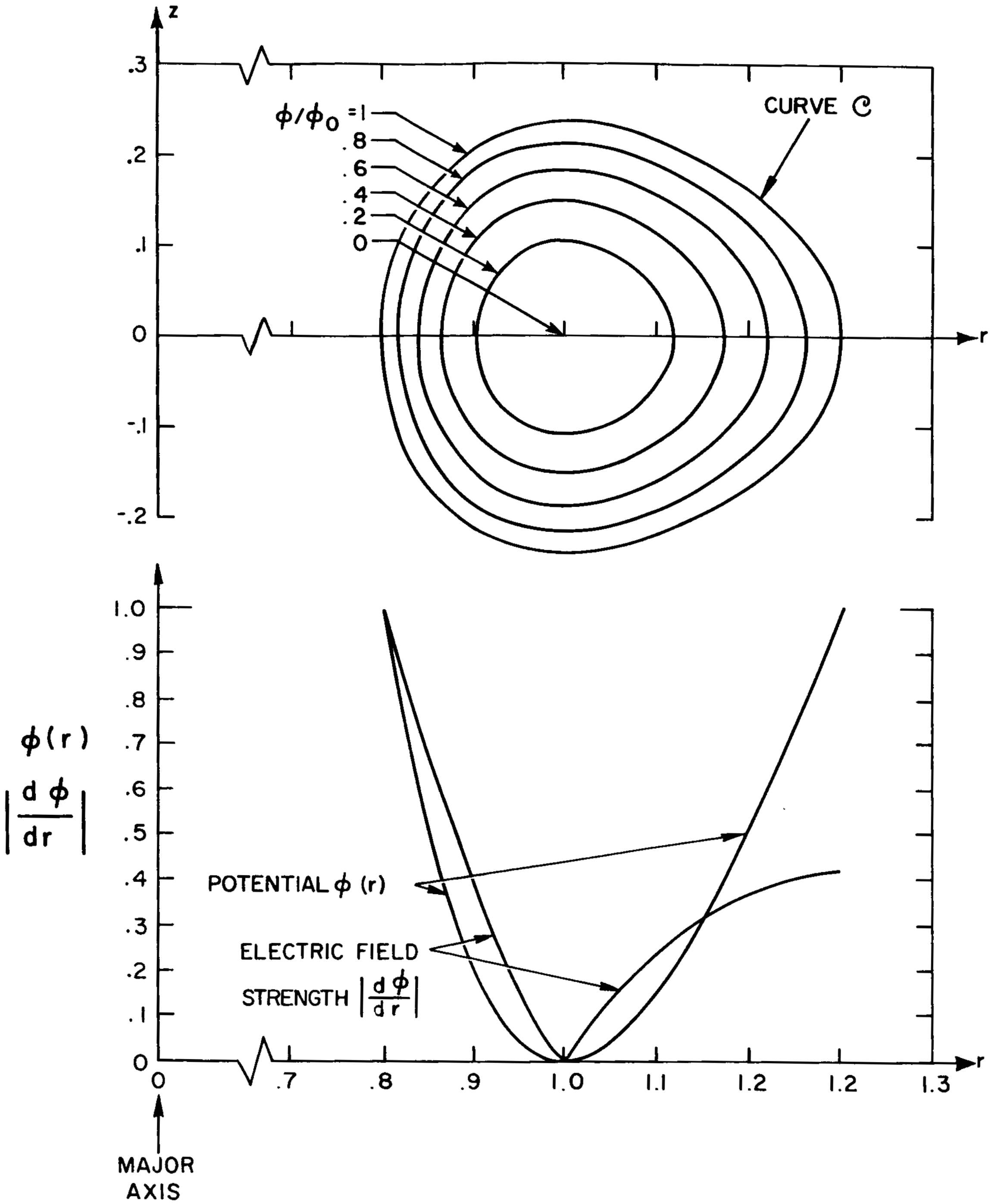


Fig. 2 Equipotentials associated with the potential function $\phi \propto 2(\ell nr)^2 + \ell nr - 1/2\,(r^2 - 1) + z^2$ for which $\nabla^2\phi \propto 4r^{-2}$. The constant of proportionality is adjusted to make $\phi = \phi_0$ on the edge $\mathcal{C}$ of the electron cloud. In the lower part of the figure the potential and absolute value of the electric field in the plane of symmetry $z = 0$ are shown on arbitrary scales. The equilibrium shown here is subject to various small corrections as indicated in Sections 3 and 4.



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We conclude this section by observing that the arbitrary function $f(\phi)$ appearing in (2.5) or (2.9) will be determined in any given case by the way the electron cloud is set up. The details of the way in which the setting-up process determines $f(\phi)$ are outside the scope of this paper.

## 3. PARTICLE DRIFT DESCRIPTION: FINITE q .

In steady fields, the total energy, H , of each electron is necessarily conserved. In addition, for our assumed azimuthally symmetric system, each electron's angular momentum, J , about the $\hat{z}$ axis is also conserved. Because of the smallness of the characteristic ratio q , the adiabatic description[4] of the electrons' motion in terms of a drifting guiding center (velocity = $\underset{\sim}{v}_D$) and a constant adiabatic magnetic moment

$$\mu = \frac{w_\perp}{B} \tag{3.1}$$

is applicable. This additional constraint together with H and J yields the trajectory of the guiding center to within the accuracy of the predictions of the adiabatic theory. Evaluating all quantities at the position of the guiding center one may define a total energy

$$\begin{aligned} H_g &= -e\phi + w_\perp + w_{||} + \frac{m}{2} v_E^2 \\ &= -e\phi + \mu B + \frac{J^2}{2mr^2} + \frac{m}{2} \frac{(\nabla\phi)^2}{B^2} \end{aligned} \tag{3.2}$$

which differs from the true electron energy by terms of order $q^2 e\phi \sim 10^{-6} e\phi$ . It is this energy that is conserved in the guiding center approximation. Equation (3.2) defines systems of surfaces of constant $H_g$ which constrain the positions of the electron guiding centers and hence the position of each individual electron. For reasonable radius ratios, we can assume that the terms $w_\perp$ and $w_{||}$ are comparable in magnitude to the term $\frac{1}{2}m(\nabla\phi/B)^2$. The basis for this assumption is the observation that in the applications considered, many of the electrons will be formed essentially at rest. In these conditions $w_\perp$ will be close to $\frac{1}{2}m(\nabla\phi/B)^2$ initially. It may then be expected that $w_{||}$ is of the same order of magnitude. The last three terms in (3.2) are smaller by the factor q than the first term, and it follows that the surfaces of constant $H_g$ depart from the equipotentials by distances on the order of the gyro-radius, i.e., $\sim qa$. In particular, when the equipotentials are closed, so are the surfaces of constant $H_g$, which therefore ensures containment.

Several deductions follow from equation (3.2). First, if there is no electric field (neutral plasma), the first and last terms drop out. It can then be seen that the guiding center motion is constrained to a surface of constant r, that is, to a cylinder; such a cylinder inevitably intersects the boundary of the containing vessel. As we know, the $\nabla B$ and centrifugal drifts do in fact confine the particles to such a cylinder; study of the kinematics yields the drift velocities and directions along the cylinder.

In a general discussion of particle containment, Budker[5] obtained a closely related result. Using an assumed electric field in a torus, Budker obtained a closed circular orbit for an electron which was displaced laterally (towards $r = 0$ in Fig. 1) a distance of the order of $(w_\perp + 2w_{||})a/eEb$ from his assumed circular equipotentials. For a given field, this shift is readily reconciled with equation (3.2) if one takes $\Delta H_g = 0$ over a circular orbit, assuming circular equipotentials.

However, the interesting physical point is that Budker's lateral shift may be interpreted to yield an average effective electric field which in turn yields an average $\underset{\sim}{E}_{eff} \times \underset{\sim}{B}$ drift. This drift just cancels the drifts due to $\nabla B$ and centrifugal forces, giving us a complementary picture of the manner in which containment is achieved.

Another observation of Budker's is of interest in the present context. He considers a secondary magnetic field $\underset{\sim}{B}'$ such as would arise from a current in the plasma parallel to the primary magnetic field, or from a current in external windings of the rotational transform type. The magnetic field lines associated with $\underset{\sim}{B}'$ are nested as are the equipotentials of, say, Fig. 2. It is then clear that a particle having a large $v_{||}$ (to the primary magnetic field) will experience a force in the same direction as the electric force of Fig. 2 or Fig. 3, and the containment picture is essentially the same, with E replaced by $v_{||}$ B'. In certain respects, then, the toroidal containment of charged plasmas by electric fields is comparable to the toroidal containment of neutral plasmas using the rotational transform. However, from the single particle point of view, containment by rotational transforms must fail for all those particles having sufficiently small $|v_{||}|$. No such difficulty arises for the charged plasma case.

Thus far we have argued that simple conservation principles are sufficient to ensure that each electron's guiding center moves in a closed orbit in the presence of steady fields. It remains in this section to show how closed orbits of this kind can be combined into a fully self-consistent equilibrium. These equilibria may be expected to differ from those

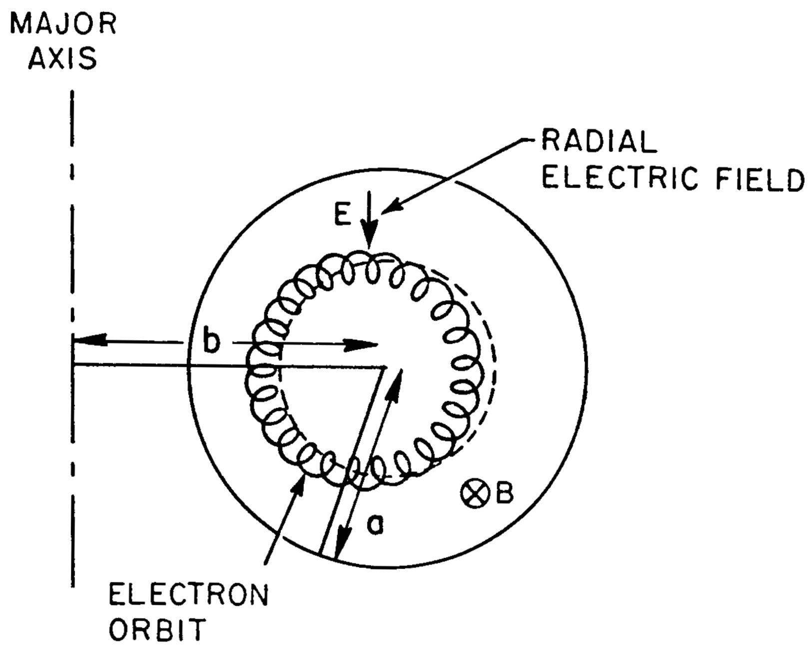


Fig. 3 Periodic drift orbit of an electron in a toroidal magnetic field, with a radial electric field. Note the displacement of the large orbit from the center of the cross-section. This displacement is roughly $(w/e\phi) \bullet (a^2/b)$ $(\ll a)$, where $w$ is the total kinetic energy in the sense of (3.2).



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discussed in Section 2 by an amount of order $q$. In view of the smallness of $q$, it follows that more interest attaches to a demonstration of the existence of such self-consistent equilibria, than to detailed calculations concerning their nature.

Assuming the adiabatic approximation, we may classify each particle according to its "constants" $\mu$ and $J$. Writing $n'(\mu, J, r, z)$ as the density of guiding centers of the indicated class of electrons and $\underset{\sim}{v}_D(\mu, J, r, z)$ as its (Eulerian) velocity in the transverse $(r, z)$ plane, then for azimuthal symmetry, the continuity equation requires

$$\text{div}\,(n' \underset{\sim}{v}_D) = 0 \tag{3.3}$$

for each $\mu$ and $J$. Following the treatment of Northrop,[4] one obtains the result that for our toroidal system, the perpendicular component of the guiding center velocity is given by

$$\underset{\sim}{v}_D = -\nabla\left(\phi - \frac{\mu}{e} B - \frac{m}{2e} v_D^{\,2} - \frac{J^2}{2mer^2}\right) \times \frac{\underset{\sim}{B}}{B^2}$$

$$+ \frac{m}{e}\left(\frac{\underset{\sim}{B}}{B^2} \bullet \nabla \times \underset{\sim}{v}_D\right) \underset{\sim}{v}_D \tag{3.4}$$

We have used the fact that $\underset{\sim}{\ddot{r}} = (\underset{\sim}{\dot{r}} \bullet \nabla)\underset{\sim}{\dot{r}} = \frac{1}{2}\nabla \dot{r}^2 - \underset{\sim}{\dot{r}} \times \nabla \times \underset{\sim}{\dot{r}} = \frac{1}{2}\nabla\left(v_D^{\,2} + \frac{J^2}{m^2 r^2}\right) - \underset{\sim}{v}_D \times \nabla \times \underset{\sim}{v}_D$ in obtaining (3.4) from Northrop's result (his equation 1.13).

Equation (3.4) contains all of the first order drifts applicable to a non-relativistic treatment — (1) the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, (2) the $\nabla B$ drift, and

(3) the acceleration drifts due to the curvature of the guiding center trajectory.

It is immediately clear from (3.4) that the combined influence of the first order drifts is to cause the guiding center to move perpendicular to the gradient of the quantity $\left(-e\phi + \mu B + \frac{m}{2} v_D{}^2 + \frac{J^2}{2mr^2}\right)$. Retaining quantities of O(1) and O(q), this quantity is simply $H_g$ as given by (3.2). Hence, to the accuracy of the usual drift approximation, $H_g$ is conserved as previously asserted. To $O(q^2)$ in (3.4) the quantity $\nabla \times \underset{\sim}{v}_D$ may be taken as $\nabla \times \underset{\sim}{v}_E$ and so we find that

$$\underset{\sim}{v}_D = \frac{\nabla H_g \times \underset{\sim}{B}}{eB^2}\left(1 - \frac{m}{e}\frac{\underset{\sim}{B}}{B^2} \bullet \nabla \times \underset{\sim}{v}_E\right)^{-1} . \qquad (3.5)$$

The derivation of (3.5) has followed from a non-relativistic treatment of the electron motion and so is correct through terms of order $q v_E{}^2/c^2$ in a two-parameter expansion of the equation of motion. Observing then that

$$\frac{m}{e}\frac{\underset{\sim}{B}}{B^2} \bullet \nabla \times \underset{\sim}{v}_E = \frac{nm}{\epsilon_0 B^2} + O\left(q \frac{v_E{}^2}{c^2}\right)$$

where n is the total particle number density we may finally write

$$\underset{\sim}{v}_D = \left(1 - \frac{nm}{\epsilon_0 B^2}\right)^{-1} \frac{\nabla H_g \times \underset{\sim}{B}}{eB^2} = (1-q)^{-1} \frac{\nabla H_g \times \underset{\sim}{B}}{eB^2} \qquad (3.6)$$

to a consistent accuracy of $O(q \frac{v_E^2}{c^2})$ .

Now we find that substitution of (3.6) in the continuity equation (3.3) yields

$$\left[ \nabla H_g \times \nabla \left\{ (1 - \frac{nm}{\epsilon_0 B^2})^{-1} \frac{n'}{B^2} \right\} \right] \bullet \underset{\approx}{B} = 0 \qquad (3.7)$$

where we have neglected terms of $O(\frac{v_E^2}{c^2})$ in the equation for curl B for simplicity in our present treatment (see Section 4). The neglect of these terms implies $B = B_0$ and so equation (3.7) has the solution

$$n'(\mu, J, r, z) = B_0^2 (1 - \frac{mn}{\epsilon_0 B_0^2}) G(\mu, J, H_g(\mu, J, r, z)) \qquad (3.8)$$

where G is any arbitrary function of the constants of the "adiabatic" motion $\mu$, J, and $H_g$ .

At this point, we have shown the following:

(a) Taking account of all first-order drifts, the individual guiding centers move on closed trajectories; these trajectories depart from the equipotentials by distances on the order of the gyro-radius.

(b) The class of electrons having given adiabatic invariants will be in steady equilibrium if equation (3.8) for the density of this class is satisfied.

It therefore remains to show that we can sum over all values of $\mu$ and J in such a way as to be consistent with the macroscopic number density and electric field. Our procedure is to exhibit this consistency in a simple case, emphasizing the physical approximations involved, rather than to attempt a more general existence theorem.

We observe first that with error $\sim q^2$ the number densities of the particles and of the guiding centers are the same. Next, we assume $\partial G/\partial H_g \approx G/(H_g)_{max}$ and so may expand the function $G$ as

$$G(\mu, J, H_g) = G(\mu, J, -e\phi) + (H_g + e\phi) \left.\frac{\partial G}{\partial H_g}\right|_{H_g=-e\phi} + O(q^2) . \quad (3.9)$$

From (3.2), the quantity $H_g + e\phi$ is just the particle kinetic energy, and is a factor $\sim q$ smaller than either $H_g$ or $e\phi$ . Neglecting terms of order $q^2$ , then (3.8) may be written:

$$\frac{n'(\mu, J, r, z)}{B_0^2} = G(\mu, J, -e\phi) + \left[ (H_g + e\phi) \left.\frac{\partial G}{\partial H_g}\right|_{H_g=-e\phi} - \frac{m}{\epsilon_0} \frac{n}{B_0^2} \cdot G(\mu, J, -e\phi) \right] . \quad (3.10)$$

We now consider the following special form for $G$ :

$$G(\mu, J, H_g) = \begin{cases} f(\mu, J) & (H_g > \phi_0) \\ 0 & (H_g < \phi_0) . \end{cases} \quad (3.11)$$

$\phi_0$ is the potential of the edge of the electron cloud.

The effect of this choice for G is, to zero order, to let $n'(\mu, J, r, z)/B_0^2$

be uniform for $\phi < \phi_0$ , and zero for $\phi > \phi_0$ . Now let $\phi'(\mu, J, r, z)$ satisfy

$$\nabla^2 \phi' = \frac{n'e}{\epsilon_0} \qquad (3.12)$$

so that $\phi'$ is the potential arising from the particles having invariants $\mu$ and $J$ . For $G$ defined by (3.11), the solution of (3.10) and (3.12) is then essentially the same problem as was studied at the end of Section 2. The general solution is given in equation (2.10); a particular case is illustrated in Fig. 2. Clearly $\phi'$ is constant on the curve $\mathcal{C}$ (i.e., the edge of the electron cloud). Now, by superposition,

$$\phi(r, z) = \sum_{\mu, J} \phi'(\mu, J, r, z) \qquad (3.13)$$

Evaluating this on the curve $\mathcal{C}$ defines the constant $\phi_0$

At this stage we have reconstructed the solution of Fig. 2 as the zero order solution to (3.10). We now consider the two correction terms of order $q$ . Dealing first with the second term we see that it represents a volume effect. Provided we redefine $f(\mu, J)$ so that

$$\sum_{\mu, J} f(\mu, J) = \frac{n(r, z)/B_0^{\,2}}{1 - \frac{m}{\epsilon_0} \frac{n(r, z)}{B_0^{\,2}}} \qquad (3.14)$$

the density will be correct with errors of $O(q^2)$ . Satisfaction of (3.14) is possible, since the right-hand side is an absolute constant (i.e., independent of $r$ and $z$ ) for $\phi < \phi_0$ .

We now consider the other correction term in (3.10). From the definition (3.11) of G , $\partial G/\partial H_g \big|_{H_g=-e\phi}$ vanishes everywhere, except that it has a delta-function behavior at the edge of the electron cloud. This term therefore represents a correction to the density $n'$ on the edge of the electron cloud. As such it is (in first order) susceptible of two equivalent interpretations. We can either think of it as a blurring of the edge, over a distance like a gyro-radius. The alternative (and more useful) interpretation is as a surface charge distributed on the zeroth order cloud edge and of magnitude

$$\frac{\sigma(r_1, z_1)}{B_0^2(r_1)} = \sum_{\mu, J} \frac{\sigma'(\mu, J, r_1, z_1)}{B_0^2}$$

$$= -\sum_{\mu, J} \frac{\left[\mu B_0 + \frac{J^2}{2mr_1^2} + \frac{m}{2}\left(\frac{E_n}{B_0}\right)^2\right] f(\mu, J)}{E_n(r_1, z_1)} \tag{3.15}$$

In evaluating the right-hand side of (3.15), all quantities are evaluated at the edge of the electron cloud, that is on the curve $\mathcal{C}$; $E_n$ is the electric field on $\mathcal{C}$, pointing into the cloud.

Equation (3.15) represents a surface charge of order $q$ , distributed around the curve $\mathcal{C}$ . The last step in our demonstration is to consider the electric field due to this charge. But this electric field can be supposed to vanish inside $\mathcal{C}$ (i.e., in the electron cloud) provided that an appropriate normal electric field exists outside it. Thus, seen from outside the cloud, the potential on $\mathcal{C}$ is zero, and the normal electric field is $E_n$ plus a small correction. By integrating Laplace's equation forward along the electric field lines, the equipotentials can be located. Any of these is a consistent location for the conducting wall of the containing vessel. Even though, for computational reasons, this procedure will be unsatisfactory if the wall is very far from the edge of the electron cloud, it nevertheless guarantees that when the cloud is fairly near the wall, simple self-consistent solutions for the fields exist, correct to order $q$. There seems to be no reason why such solutions should not also exist for choices of the function $G$ other than that of (3.11).

## 4. FINITE VALUE OF $v_E^2/c^2$

For the applications cited[1,2] in Section 1, it is expected that the parameter $v_E^2/c^2$ will have a characteristic value of about 0.05; we must therefore consider the small change in the magnetic field due to the current of electrons flowing at a speed $\sim .25c$, and dynamical effects resulting from the use of the correct relativistic equation of electron motion. Of these two effects we shall ignore the second entirely, for the following reason: all effects involving the electron mass are of order $q(\sim 10^{-3})$. The relativistic effect may be thought of as an effective correction to the electron mass of order $v_E^2/c^2$. The combined effect yields a correction of order $qv_E^2/c^2 \sim 5 \times 10^{-5}$, and there seems to be no reason to suppose that this numerically small effect should have an importance beyond its apparent magnitude.

With respect to the modification to the magnetic field caused by the electron current, we consider the changes of order $v_E^2/c^2$ produced in the "zero-order" solutions exemplified by Fig. 2. For the cited numbers these changes are somewhat larger than the corrections of order $q$.

We observe first that the result (2.5), that $n/B^2 = f(\phi)$, was shown in Section 2 to be valid even when curl $\underset{\sim}{B}$ was not negligible. This result followed from the fact that

$$\text{curl}\,\underset{\sim}{B} = \mu_0 \underset{\sim}{j} = \mu_0 \frac{ne}{B^2} (\nabla\phi \times \underset{\sim}{B}) \tag{4.1}$$

so that curl $\underset{\sim}{B}$ is perpendicular to $\nabla\phi$ . But we can make use of our result (2.5) to rewrite (4.1) in the form

$$\text{curl}\ \underset{\sim}{B} = \mu_0 e\, \nabla \left\{ \int f(\phi)\, d\phi \right\} \times \underset{\sim}{B} \ , \tag{4.2}$$

and this equation can be solved explicitly for $B$ , as:

$$B = B_0 \exp \left\{ \mu_0 e \int_{\phi_0}^{\phi} f(\phi)\, d\phi \right\} \tag{4.3}$$

where $B_0 \propto r^{-1}$ is the vacuum field due to the external windings, and $\phi_0$ is the potential of the edge of the electron cloud. Clearly, (2.5) and (4.3) together imply that $n/B_0{}^2$ , which is proportional to $nr^2$ , is (as we asserted in Section 2) in all cases a function of $\phi$ .

By way of example, the potential illustrated in Fig. 2 can be seen to satisfy $n/B_0{}^2$ = constant. This means that the $f(\phi)$ appearing in (4.3) is just $B_0{}^2/B^2$ . On solving we find simply

$$B^2 = B_0{}^2 \left[ 1 + \frac{ne(\phi - \phi_0)}{B_0{}^2/2\mu_0} \right] \ . \tag{4.4}$$

At the bottom of the potential well, $\phi = 0$ and $B_0{}^2$ is reduced by the amount $2\mu_0 e n \phi_0$ . As expected, the ratio of this quantity to $B_0{}^2$ is of order $v_E{}^2/c^2$ . For simplicity, in Section 3 we neglected terms of order $v_E{}^2/c^2$ in the calculation of equation (3.6). With a generalization from $\phi$ to $H_g$ and from $n$ to a summation over all particles, one finds for the calculations of Section 3 a correction to B and $n/B^2$ analogous to that of equation (4.3). To a consistent order of accuracy (i.e. up to terms of $O(q\frac{v_E^2}{c^2})$ one finds no qualitative differences from the corrections discussed above.

In conclusion, the effect of small but finite $v_E^2/c^2$ is to modify slightly the equilibria that would otherwise have been expected, but, and this is the important point, it does not substantially change their character from that discussed in the previous sections.

## 5. EARNSHAW'S THEOREM

It is sometimes thought that Earnshaw's theorem (as described, say, by Coulson[6]) makes equilibria of unneutralized electron clouds impossible. This is not so. Earnshaw's theorem states that stable static equilibrium is impossible for a collection of free charges, and it follows from the observation that solutions of Laplace's equation cannot have maxima or minima except at boundaries. The electron clouds described in this paper are indeed not in static equilibrium, but move about, for the most part at the velocity $\underset{\sim}{E} \times \underset{\sim}{B}/B^2$. However, if the orbits described by the particles are periodic, the configuration can be described as being in dynamic equilibrium, and this type of equilibrium is beyond the scope of Earnshaw's theorem. A well-known example of the distinction we are making here is the solar system which also cannot be in static equilibrium due to the gravitational analog of Earnshaw's theorem.

## 6. CONCLUSIONS

In conclusion, we have brought out the factors contributing to the possibility of finding self-consistent steady equilibria for azimuthally symmetric toroidal magnetically confined electron clouds. While strict proofs are lacking, the indications are that a variety of equilibria do exist, and these are close to those that result from a simple approximate calculation. Rotational transforms are not necessary. Theoretical discussions of stability problems affecting electron clouds of the type discussed have been given by Levy,[7] Buneman et al,[8] and others. Experimental evidence in general support of these ideas will be found in a forthcoming paper.[9]


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13. ABSTRACT


The self-consistent equilibrium configurations available to low density $(q = \omega_p^2/\omega_c^2 \ll 1)$ toroidal electron clouds under the influence of an external magnetic field and the electric fields due to space and image charges are discussed. The equilibria are dynamic rather than static, but steady. $\nabla B$, centrifugal and other drifts are shown to be unimportant, in contrast to the situation in toroidal neutral plasmas. Typical equilibria are shown in detail, to zero order in $q$. The construction of equilibria accurate to order $q$ is shown to be possible but is not carried out in detail.

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