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APPLICATION OF THE METHODS OF CELESTIAL MECHANICS TO THE RIGID BODY PROBLEM

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ABSTRACT

This report contains the results of a preliminary study aimed at the application of the perturbation methods of celestial mechanics to the rigid body problem, with particular emphasis on the problem of the motion of an artificial earth satellite about its center of mass.

Detailed considerations of various representations of the equations of motion and their solution, necessary to a complete understanding of the problem, are given. The pertinent coordinate transformations are described.

The method of Hamilton-Jacobi is used to obtain the perturbed equations of motion for a uniaxial body, and groundwork is laid toward the development of a complete first-order theory for the case of gravity-gradient perturbations for such a body.

Preliminary results are given for a simplified study of the effects of damping, and a computer integration routine developed for this work is described.

CONTENTS

CHAPTER VI

I. INTRODUCTION

In NASA Contract NAS8-20175, the Contractor was responsible for a preliminary investigation of the problem of describing the motion of a rigid artificial earth satellite about its center of mass. The perturbation methods of celestial mechanics were to have been employed to study the motion of such a body under the influence of certain small perturbing torques acting upon it.

During the period covered by the contract the following tasks were performed:

1) A literature search was made to determine the present state of the art in this area.

2) Studies of a general nature relating to the problem area under investigation were conducted. These studies were concerned, for the most part, with the examination of various representations of the pertinent equations of motion with the view in mind of trying to decide which representations lend themselves most readily to perturbation work.

3) The perturbations of uniaxial bodies by both gravitational torques and special cases of magnetic torques were examined. The method of Hamilton-Jacobi was used in the gravitational case and the development of a first-order theory for this problem appears imminent.

4) The qualitative nature of the effects of various perturbing torques was compared with observed motions.

5) In anticipation of a need for computer support and verification of analytical results, a general integration routine was developed.

Detailed results of these investigations follow.

II. BASIC CONSIDERATIONS

A. Equations of Motion.

1. Euler Equations.

The basic vector equation of motion for a rigid body rotating about its center of mass is

$$L = \dot{H} , \tag{1}$$

where L is the torque and H is the angular momentum, both referenced to the center of mass.

Equation (1) holds in any non-rotating reference frame. In a frame rotating in inertial space with angular velocity ω, equation (1) becomes

$$L = \dot{H} + \Omega H, \tag{2}$$

where

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{3}$$

and the non-zero elements of the matrix (3) are the components of ω along the cartesian axes indicated by the subscripts. A similar designation of components of other vectors is made in what follows. The tensor Ω is equivalent to the vector operator (ω x).

If the rotating coordinate system is fixed in the rigid body, then

$$H = I\omega , \tag{4}$$

where I is the moment of inertia tensor. Then equation (2) becomes

$$L' = \dot{H}' + \Omega' I' \omega' . \tag{5}$$

(The prime notation is used to indicate the body-fixed reference frame.)

Finally, if the body-fixed reference frame is aligned with the principal axes, I' is diagonal.

Specifically

$$I' = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \tag{6}$$

where A, B, and C are the moments of inertia referenced to the principal axes.

Equation (5) may be written in the scalar form

$$\left.\begin{aligned} L'_x &= A\dot{\omega}'_x + (C-B)\omega'_y\omega'_z \\ L'_y &= B\dot{\omega}'_y + (A-C)\omega'_x\omega'_z \\ L'_z &= C\dot{\omega}'_z + (B-A)\omega'_x\omega'_y \end{aligned}\right\} \tag{7}$$

and, in this form, the three equations in (7) are known as the Euler equations. They constitute a set of differential equations for the components of ω'. Note that they do not give the orientation of the body. Further, since L' in general depends on the orientation, they cannot be solved without additional relations. One such relation gives the transformation matrix.

2. Poisson Equations.

If a vector transforms from the body-fixed frame to the space-fixed one by

$$\rho = T\rho', \tag{8}$$

where T is the transformation matrix,

then

$$\dot{T} = \Omega T. \tag{9}$$

Also, since Ω is a tensor and transforms according to

$$\Omega = T\Omega' T^I , \tag{10}$$

then

$$\dot{T} = T\Omega' . \tag{11}$$

In scalar form,

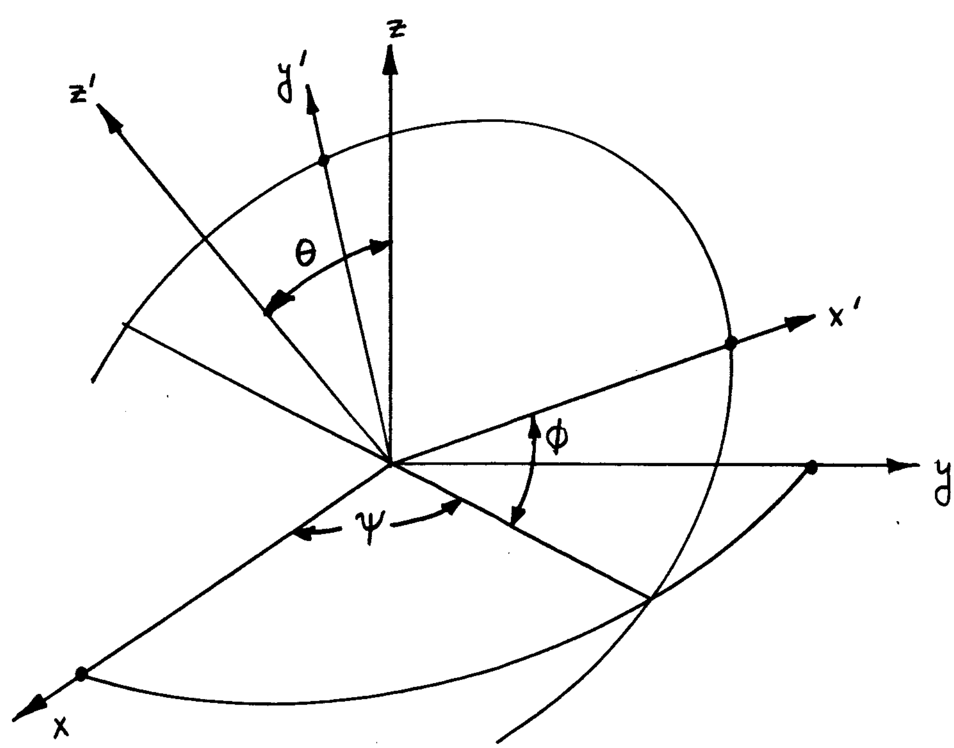
$$\left.\begin{aligned}
\dot{T}_{11} &= T_{12}\omega'_z - T_{13}\omega'_y \\
\dot{T}_{12} &= -T_{11}\omega'_z + T_{13}\omega'_x \\
\dot{T}_{13} &= T_{11}\omega'_y - T_{12}\omega'_x \\
\dot{T}_{21} &= T_{22}\omega'_z - T_{23}\omega'_y \\
\dot{T}_{22} &= -T_{21}\omega'_z + T_{23}\omega'_x \\
\dot{T}_{23} &= T_{21}\omega'_y - T_{22}\omega'_x \\
\dot{T}_{31} &= T_{32}\omega'_z - T_{33}\omega'_y \\
\dot{T}_{32} &= -T_{31}\omega'_z + T_{33}\omega'_x \\
\dot{T}_{33} &= T_{31}\omega'_y - T_{32}\omega'_x \quad .
\end{aligned}\right\} \tag{12}$$

Equations (12) are the Poisson equations. Combined with the Euler equations, they comprise a set of simultaneous differential equations for T and ω'.

3. Euler Angles.

Typically the Euler angles are defined as shown in figure 1.

Figure 1.



The corresponding transformation matrix between the body-fixed and space-fixed frames is

$$T = \begin{pmatrix} \cos\psi\cos\phi-\sin\psi\sin\phi\cos\theta, & -\cos\psi\sin\phi-\sin\psi\cos\phi\cos\theta, & \sin\psi\sin\theta \\ \sin\psi\cos\phi-\cos\psi\sin\phi\cos\theta, & -\sin\psi\sin\phi+\cos\psi\cos\phi\cos\theta, & -\cos\psi\sin\theta \\ \sin\phi\sin\theta & , \quad \cos\phi\sin\theta & , \quad \cos\theta \end{pmatrix}. \tag{13}$$

The following relations may be shown to hold:

$$\omega = M\xi, \tag{14}$$

$$\omega' = N\xi, \tag{15}$$

$$M = TN, \tag{16}$$

where

$$\xi = \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}, \tag{17}$$

$$M = \begin{pmatrix} 0, & \cos\psi, & \sin\psi\ \sin\theta \\ 0, & \sin\psi, & -\cos\psi\ \sin\theta \\ 1, & 0\ , & \cos\theta \end{pmatrix}, \tag{18}$$

and

$$N = \begin{pmatrix} \sin\phi\ \sin\theta, & \cos\phi, & 0 \\ \cos\phi\ \sin\theta, & -\sin\phi, & 0 \\ \cos\theta\ , & 0\ , & 1 \end{pmatrix}. \tag{19}$$

The inverse relations for M and N are

$$M^{I} = \begin{pmatrix} -\sin\psi\ \cot\theta, & \cos\psi\ \cot\theta, & 1 \\ \cos\psi\ , & \sin\psi\ , & 0 \\ \sin\psi\ \csc\theta, & -\cos\psi\ \csc\theta, & 0 \end{pmatrix}, \tag{20}$$

$$N^{I} = \begin{pmatrix} \sin\phi\ \csc\theta, & \cos\phi\ \csc\theta, & 0 \\ \cos\phi\ , & -\sin\phi\ , & 0 \\ -\sin\phi\ \cot\theta, & -\cos\phi\ \cot\theta, & 1 \end{pmatrix}. \tag{21}$$

An important relation involves the kinetic energy, $\mathcal{T}$. Specifically

$$\mathcal{T} = {}^1/_2\ \omega^T I \omega = {}^1/_2\ \omega'^T I' \omega' \quad . \tag{22}$$

Using relation (15), one may write

$$\mathcal{T} = {}^1/_2\ \xi^T X \xi \ , \tag{23}$$

where

$$X = N^T I' N \quad . \tag{24}$$

Explicitly, X and its inverse are:

$$X = \begin{pmatrix} (A\sin^2\phi + B\cos^2\phi)\sin^2\theta + C\cos^2\theta \ , & (A-B)\sin\phi\ \cos\phi\ \sin\theta \ , & C\cos\theta \\ (A-B)\sin\phi\ \cos\phi\ \sin\theta & , \quad (A\cos^2\phi + B\sin^2\phi) & , \quad 0 \\ C\cos\theta & , \quad 0 & , \quad C \end{pmatrix} \tag{25}$$

$$X^I = \begin{pmatrix} \left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\frac{1}{\sin^2\theta} \ , & \left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\phi\cos\phi}{\sin\theta} \ , & -\left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\frac{\cot\theta}{\sin\theta} \\ \left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\phi\cos\phi}{\sin\theta} \ , & \left(\frac{\cos^2\phi}{A} + \frac{\sin^2\phi}{B}\right) \ , & \left(\frac{1}{B} - \frac{1}{A}\right)\sin\phi\ \cos\phi\ \cot\theta \\ -\left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\frac{\cot\theta}{\sin\theta} \ , & \left(\frac{1}{B} - \frac{1}{A}\right)\sin\phi\cos\phi\cot\theta & \left[\left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\cot^2\theta + \frac{1}{C}\right] \end{pmatrix} \tag{26}$$

Of particular importance are the equations obtained by reverting (15). In scalar form,

$$\left.\begin{aligned} \dot\psi &= \sin\phi\ \csc\theta\ \omega'_x + \cos\phi\ \csc\theta\ \omega'_y \\ \dot\theta &= \cos\phi\ \omega'_x - \sin\phi\ \omega'_y \\ \dot\phi &= -\sin\phi\ \cot\theta\ \omega'_x - \cos\phi\ \cot\theta\ \omega'_y + \omega'_z \quad . \end{aligned}\right\} \tag{27}$$

These equations express the derivatives of the Euler angles in terms of the Euler angles and components of ω'. Together with the Euler equations, they comprise a set of differential equations in ω' and the Euler angles.

B. Standardized Symbols.

In the problem of the motion of a satellite there are four coordinate systems which are of particular interest, and six sets of Euler angles relating them. A standard notation has been adopted here for these parameters. They are as follows:

1. Cartesian Coordinate Systems.

(x^*,y^*,z^*) The reference inertial coordinate system. Unless specified otherwise, this is an equatorial system with the z^*-axis directed along the earth's spin vector and the x^*-axis toward the vernal equinox.

(x_o,y_o,z_o) A coordinate system associated with the satellite orbit plane, with the z_o-axis directed along the normal to the orbit in the sense of the orbital angular momentum associated with the satellite, and the x_o-axis toward the ascending node.

(x,y,z) A coordinate system associated with the angular momentum vector of the satellite as it rotates about its center of mass. The z-axis is directed along the positive angular momentum vector. The position of the x-axis will be left undefined at this time.

(x',y',z') The body-fixed axis system, oriented along the principal axes of the body. The choice of the six possible orientations will not be completely specified, but in general it will be made so that the angular momentum vector lies "close" to the positive z'-axis.

2. Euler Angles.

$$\left.\begin{array}{llll}
(\Omega,i,-) & \text{rotates} & (x^*,y^*,z^*) & \text{into} \quad (x_o,y_o,z_o) \\
(\psi^*,\theta^*,\phi^*) & \text{rotates} & (x^*,y^*,z^*) & \text{into} \quad (x,y,z) \\
(\psi,\theta,\phi) & \text{rotates} & (x^*,y^*,z^*) & \text{into} \quad (x',y',z') \\
(\psi_L,\theta_L,\phi) & \text{rotates} & (x_o,y_o,z_o) & \text{into} \quad (x,y,z) \\
(\psi_o,\theta_o,\phi_o) & \text{rotates} & (x_o,y_o,z_o) & \text{into} \quad (x',y',z') \\
(\psi',\theta',\phi') & \text{rotates} & (x,y,z) & \text{into} \quad (x',y',z')
\end{array}\right\} \quad (28)$$

C. Torque-Free Motion.

The classical solutions for the torque-free motion are obtained from the Euler equations, (7), and equations (27) for the Euler angles. If L = 0, equations (7) become

$$\left.\begin{aligned} A\dot{\omega}_x' + (C-B)\omega_y'\omega_z' &= 0 \\ B\dot{\omega}_y' + (A-C)\omega_x'\omega_z' &= 0 \\ C\dot{\omega}_z' + (B-A)\omega_x'\omega_y' &= 0 \ . \end{aligned}\right\} \tag{29}$$

Since these do not contain the Euler angles, equations (29) may be solved independently of (27).

1. Uniaxial Bodies.

A uniaxial body is one that has two of its principal moments of inertia equal. If A = B in (29), we get

$$\left.\begin{aligned} A\dot{\omega}_x' + (C-A)\omega_y'\omega_z' &= 0 \\ A\dot{\omega}_y' + (A-C)\omega_x'\omega_z' &= 0 \\ C\dot{\omega}_z' &= 0 \ . \end{aligned}\right\} \tag{30}$$

The solutions have the form

$$\left.\begin{aligned} \omega_x' &= \rho \cos(nt+\delta) \\ \omega_y' &= \rho \sin(nt+\delta) \\ \omega_z' &= \omega_{zo}' \end{aligned}\right\} \tag{31}$$

where

$$n = \left(\frac{C-A}{A}\right) \omega_{zo}' \ .$$

and ρ, ω_{zo}', and δ are determined by the initial conditions.

The Euler angles are obtained by combining equations (31) with the differential equations (27). The general solution, however, is complicated. To simplify the problem, a particular orientation of the space-fixed coordinate system is assumed; the z-axis is assumed to lie along the angular momentum vector. This implies a priori knowledge that the vector H is, itself, space-fixed.

We require

$$H = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}. \tag{32}$$

Then the relation

$$H' = I'\omega' = T^{I} H \tag{33}$$

gives the following conditions

$$\left.\begin{aligned} \tan\phi' &= \omega'_x/\omega'_y \\ \cos\theta' &= C\omega'_z/h \end{aligned}\right\} \tag{34}$$

(In keeping with the convention of Section II.B.2, primes are used to indicate the angles are referred to the angular momentum coordinate system.)

We also have, from (27),

$$\dot{\psi}' = \sin\phi' \csc\theta' \, \omega'_x + \cos\phi' \csc\theta' \, \omega'_y. \tag{35}$$

Solution of (34) and (35) gives

$$\left.\begin{aligned} \psi' &= \psi'_o + ht/A \\ \theta' &= \theta'_o \\ \phi' &= \phi'_o - nt = (\pi/2-\delta)-nt \quad . \end{aligned}\right\} \tag{36}$$

The z'-axis or axis of symmetry is thus seen to describe a cone of half-angle θ'_o about the angular momentum vector, at a rate $\dot{\psi}' = h/A$. This motion is commonly called Eulerian motion.

2. Triaxial Bodies .

If the three moments of inertia are unequal, equations (29) must be solved. They have solutions in terms of the Jacobian elliptic functions. Using (29), and the energy and momentum relations

$$\mathcal{T} = {}^1/_2 \, (A\omega'^2_x + B\omega'^2_y + C\omega'^2_z) = \text{constant}, \tag{37}$$

$$H'^2 = A^2\omega'^2_x + B^2\omega'^2_y + C^2\omega'^2_z = h^2 = \text{constant}, \tag{38}$$

we obtain

$$\left.\begin{aligned} \omega'^2_x &= a_1{}^2 - b_1{}^2\omega'^2_y, \\ \omega'^2_z &= a_3{}^2 - b_3{}^2\omega'^2_y, \end{aligned}\right\} \tag{39}$$

where

$$\left.\begin{aligned} a_1{}^2 &= \frac{h^2-2C\mathfrak{J}}{A(A-C)}, \\ a_3{}^2 &= \frac{2A\mathfrak{J}-h^2}{C(A-C)}, \\ b_1{}^2 &= \frac{B(B-C)}{A(A-C)}, \\ \text{and} \qquad b_3{}^2 &= \frac{B(A-B)}{C(A-C)}. \end{aligned}\right\} \tag{40}$$

If (39) is solved for $\omega'_x\omega'_z$ and the results are substituted into (29) one obtains

$$B\dot{\omega}'_y = (C-A)\sqrt{(a_1{}^2-b_1{}^2\omega'^2_y)(a_3{}^2-b_3{}^2\omega'^2_y)}\ . \tag{41}$$

The integral of (41) is an elliptic integral which may be expressed in terms of the Jacobian elliptic functions.

If

$$h^2 < 2B\mathfrak{J}, \tag{42}$$

then

$$\omega'_y = (a_1/b_1)\mathrm{sn}(pt+c)\ , \tag{43}$$

where

$$p = \left(\frac{C-A}{B}\right) b_1a_3 \tag{44}$$

and k, the modulus of the elliptic function, is given by

$$k = b_3a_1/a_3b_1. \tag{45}$$

The remaining components of ω' are obtained in a similar way and, hence, we obtain the complete solution

$$\left.\begin{aligned} \omega'_x &= a_1\ \mathrm{cn}(pt+c) \\ \omega'_y &= (a_1/b_1)\mathrm{sn}(pt+c) \\ \omega'_z &= a_3\ \mathrm{dn}(pt+c). \end{aligned}\right\} \tag{46}$$

The above solution is valid only when condition (42) is satisfied. However, we may note that interchange of the x'- and z'-axes reverses the inequality in (42). Thus condition (42) may be assured (except when $h^2 = 2B\mathcal{T}$) by properly designating the principal axes. Note, however, that this designation depends on the initial conditions. The case $h^2 = 2B\mathcal{T}$ is covered by the limiting case $k^2 \to 1$, for which

$$\left.\begin{aligned} \omega'_x &= a_1 \operatorname{sech}(pt+c) \\ \omega'_y &= (a_1/b_1)\tanh(pt+c) \\ \omega'_z &= a_3 \operatorname{sech}(pt+c). \end{aligned}\right\} \tag{47}$$

The Euler angles as functions of time are found, in the same way as before, from equations (34) and (35). Equations (34) give

$$\left.\begin{aligned} \cot\phi' &= \frac{B}{Ab_1} \operatorname{tn}(pt+c) \\ \cos\theta' &= \frac{Ca_3}{h} \operatorname{dn}(pt+c) . \end{aligned}\right\} \tag{48}$$

Equation (35) gives

$$\psi' = \psi'_o + h \int_o^t \left(\frac{2\mathcal{T} - C\omega'^2_z}{h^2 - C^2\omega'^2_z}\right) dt$$

or

$$\psi' = \psi'_o + \frac{h}{A} \int_o^t \left[\frac{1 + \alpha\ \operatorname{sn}^2(pt+c)}{1 + \beta\ \operatorname{sn}^2(pt+c)}\right] dt \tag{49}$$

where

$$\alpha = \frac{A-B}{B-C}$$

and

$$\beta = C\alpha/A. \tag{50}$$

Equation (49) may be solved in terms of an elliptic integral of the third kind:

$$\Pi(n,k,u) = \int \frac{du}{1 + n\ \operatorname{sn}^2 u}.$$

However, the formulation so obtained does not readily reduce to the uniaxial case when A = B. Instead, we introduce the function

$$F(n,k,u) = \int \frac{sn^2u \; du}{1 + n \; sn^2u} . \tag{51}$$

The expression for ψ' then takes the form

$$\psi' = \psi_o' + ht/A + \alpha \left[F(\beta,k,nt+c) - F(\beta,k,c) \right] \tag{52}$$

Since $\alpha = 0$ when A = B, this immediately gives the uniaxial case.

III. DAMPED UNIAXIAL BODIES

As shown previously, a free uniaxial body executes Eulerian motion, with the axis of symmetry maintaining a constant angle, θ, with the angular momentum vector. The observations of the Pegasus satellites, however, indicate that θ increases with time for those bodies, eventually reaching 90^0. Such a motion indicates a loss in energy, so that some damping torque appears to be acting on the satellite. A preliminary study was performed to determine the effects of certain types of damping.

The Euler equations for a uniaxial body are

$$\left.\begin{aligned} A\dot{\omega}_x' + (C-A)\omega_y'\omega_z' &= L_x' \\ A\dot{\omega}_y' + (A-C)\omega_x'\omega_z' &= L_y' \\ C\dot{\omega}_z' &= L_z' . \end{aligned}\right\} \tag{53}$$

To consider the simplest form of damping, let

$$L' = - k\omega' , \tag{54}$$

where k is a real constant.

The last of equations (53) becomes

$$C\dot{\omega}_z' = - k\omega_z' ,$$

which gives

$$\omega_z' = \omega_{zo}' \, e^{- t/\tau_1} \tag{55}$$

where

$$\tau_1 = C/k . \tag{56}$$

For the other two components, we require the equations to have the same form as in the torque-free case

$$\omega'_x = \rho \cos(nt+\delta)$$
$$\omega'_y = \rho \sin(nt+\delta) \qquad \left[\text{from (31)}\right]$$

where now ρ and n are allowed to vary with time. Solving for these two variables one obtains

$$\rho = \rho_o e^{-t/\tau_2} \tag{57}$$

where

$$\tau_2 = A/k \tag{58}$$

and

$$nt = \beta\left(1-e^{-t/\tau_1}\right) \tag{59}$$

where

$$\beta = \tau_1 n_o = \tau_1\left(\frac{C-A}{A}\right)\omega'_{zo}. \tag{60}$$

The angle γ between the symmetry axis and the angular momentum vector is given by

$$\tan\gamma = \frac{\sqrt{H_x'^2 + H_y'^2}}{H_z'} = \frac{A\rho}{C\omega_z'} . \tag{61}$$

Substituting for ω'_z from (55), we obtain

$$\tan\gamma = Ge^{\lambda t} , \tag{62}$$

where

$$G = \tan\gamma_o = A\rho_o/C\omega'_{zo} , \tag{63}$$

and

$$\lambda = k\left(\frac{A-C}{AC}\right) . \tag{64}$$

Note that λ depends in sign on the relative magnitudes of A and C. If $A > C$, $\lambda > 0$ and γ increases with time. If $A < C$, λ is negative and γ approaches zero.

It should be noted that γ is <u>not</u> the angle between the symmetry axis and any space-fixed axis. Since a torque is present, H is not a fixed vector. However, if k is small, H varies very little from its original direction. In this case γ is essentially the Euler coning angle θ'.

In general, the damping constant k must be regarded as a tensor, rather than a simple scalar constant. The form of the tensor will depend on the distribution of the physical properties (mass, conductivity, etc.) of the body. It is reasonable to suppose that the transformation to principal axes, which diagonalizes the inertia tensor, will also diagonalize the damping tensor. Hence we may suppose that

$$K' = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}. \tag{65}$$

If the body is a body of rotation then $k_1 = k_2$. The motion for this case is immediately given by letting

$$\left.\begin{aligned} \tau_1 &= C/k_3 \\ \tau_2 &= A/k_1 \\ \text{and} \qquad \lambda &= \frac{Ak_3 - Ck_1}{AC} \end{aligned}\right\} \tag{66}$$

If $k_1 \neq k_2$, the solutions are different, and contain periodic terms. The essential features, however, are unchanged.

The variation of γ with time given by the above theory has been found to agree well, qualitatively, with the observed motions of the Pegasus satellites. However, the damping constant required to fit the data is considerably higher than would be expected if damping were only magnetic in nature. It appears that other types of damping, perhaps aerodynamic in nature, are acting. Furthermore, they appear to have different magnitudes for the different Pegasus satellites.

It is no doubt a gross over-simplification to assume that the damping constant is independent of time. In further work attempts should be made to consider the variations of the damping torques with attitude, altitude, etc.

IV. HAMILTONIAN MECHANICS

The great bulk of effort in this contract has been devoted to the application of the methods of Hamiltonian Mechanics to the rigid body problem. This is because these methods are well suited to perturbation problems, whereas other methods, such as used in section III, appear to be more difficult to apply except in special cases. While some work has included consideration of triaxial bodies, most of the attention has been given to the case of uniaxial bodies. Since, for bodies which have two nearly equal principal moments of inertia, such as the Pegasus satellites, motion is very nearly Eulerian, this concentration on uniaxial bodies is felt to be justified.

A. Equations of Motion.

Let $q_1 = \psi$, $q_2 = \theta$, $q_3 = \phi$, so that

$$Q = \begin{pmatrix} \psi \\ \theta \\ \phi \end{pmatrix} . \tag{67}$$

The momenta canonical to the q_i are given by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} , \tag{68}$$

where $\mathcal{L}$ is the Lagrangian function. If the potential is independent of the $\dot{q}_i$'s,

$$p_i = \frac{\partial \mathfrak{T}}{\partial \dot{q}_i} , \tag{69}$$

where

$$\mathfrak{T} = {}^1/_2\, \xi^T X \xi = {}^1/_2\, \dot{Q}^T X \dot{Q} \qquad \text{[from (23).]}$$

Performing the differentiation, we obtain

$$P = \begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = X\xi \quad . \tag{70}$$

Relations can also be derived between P and H. We find

$$P = N^T H' = M^T H. \tag{71}$$

In terms of P, the kinetic energy is

$$\mathcal{T}(q_i,p_i) = {}^1/_2 \; P^T X^I P \; . \tag{72}$$

The Hamiltonian function is

$$\mathcal{H}(q_i,p_i) = {}^1/_2 \; P^T X^I P + \mathcal{V}(q_i). \tag{73}$$

The Hamiltonian equations of motion are

$$\left.\begin{aligned} \dot{q}_k &= \frac{\partial \mathcal{H}(q_i,p_i)}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial \mathcal{H}(q_i,p_i)}{\partial q_k} \quad . \end{aligned}\right\} \tag{74}$$

The first of these equations repeats equations (69). The second becomes

$$\dot{p}_k = -\,{}^1/_2 \; P^T \frac{\partial X^I}{\partial q_k} P - \frac{\partial \mathcal{V}}{\partial q_i} \quad . \tag{75}$$

The derivatives of X^I have been evaluated. They are:

$$\left.\begin{aligned} \frac{\partial X^I}{\partial \psi} &= 0 \\ \frac{\partial X^I}{\partial \theta} &= \left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\csc^3\theta \begin{pmatrix} -2\cos\theta & 0 & (1+\cos^2\theta) \\ 0 & 0 & 0 \\ (1+\cos^2\theta) & 0 & -2\cos\theta \end{pmatrix} \\ &+ \left(\frac{1}{A} - \frac{1}{B}\right)\sin\phi\cos\phi\csc^2\theta \begin{pmatrix} 0 & -\cos\theta & 0 \\ -\cos\theta & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \frac{\partial X^I}{\partial \phi} &= 2\left(\frac{1}{A} - \frac{1}{B}\right)\sin\phi\cos\phi\csc^2\theta \begin{pmatrix} 1 & 0 & -\cos\theta \\ 0 & -\sin^2\theta & 0 \\ -\cos\theta & 0 & \cos^2\theta \end{pmatrix} \\ &+ \left(\frac{1}{A} - \frac{1}{B}\right)(\cos^2\phi - \sin^2\phi)\csc\theta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\cos\theta \\ 0 & -\cos\theta & 0 \end{pmatrix} \end{aligned}\right\} \tag{76}$$

The following equations of motion are obtained.

$$\left.\begin{aligned}
\dot{p}_\psi &= -\frac{\partial \mathcal{V}}{\partial \psi} \\
\dot{p}_\theta &= \left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)\csc^3\theta \left[(p_\psi{}^2+p_\phi{}^2)\cos\theta - (1+\cos^2\theta)p_\psi p_\phi\right] \\
&+ \left(\frac{1}{A} - \frac{1}{B}\right) \sin\phi \cos\phi \csc^2\theta \; p_\theta(p_\psi\cos\theta - p_\phi) - \frac{\partial \mathcal{V}}{\partial \theta} \\
\dot{p}_\phi &= \left(\frac{1}{A} - \frac{1}{B}\right) \left\{\sin\phi \cos\phi \left[p_\theta{}^2 - \csc^2\theta(p_\psi - p_\phi\cos\theta)^2\right]\right. \\
&\left. - (\cos^2\phi - \sin^2\phi)\csc\theta \; p_\theta(p_\psi - p_\phi\cos\theta)\right\} - \frac{\partial \mathcal{V}}{\partial \phi} \quad .
\end{aligned}\right\} \quad (77)$$

For the torque-free motion of a uniaxial body, equations (70), (71), and (77) become

$$\mathcal{H} = \frac{1}{2A}\left[\csc^2\theta(p_\psi - p_\phi\cos\theta)^2 + p_\theta{}^2\right] + \frac{p_\phi{}^2}{2C} \quad (78)$$

$$\left.\begin{aligned}
\dot{\psi} &= \frac{1}{A} \csc^2\theta(p_\psi - p_\phi\cos\theta) \\
\dot{\theta} &= p_\theta/A \\
\dot{\phi} &= -\frac{1}{A} \csc\theta \cot\theta(p_\psi - p_\phi\cos\theta) + p_\phi/C \\
\dot{p}_\psi &= 0 \\
\dot{p}_\theta &= \frac{1}{A} \csc^3\theta \left[(p_\psi{}^2+p_\phi{}^2)\cos\theta - (1+\cos^2\theta)p_\psi p_\phi\right] \\
\dot{p}_\phi &= 0 \quad .
\end{aligned}\right\} \quad (79)$$

B. Hamilton-Jacobi Method.

In the method of Hamilton-Jacobi, we seek a transformation to new variables which are, in the torque-free case, canonical constants of the motion. The perturbed motion is then to be determined by studying the perturbing Hamiltonian function.

The variables (q_i, p_i) are to be transformed to new variables (α_i, β_i). The new variables will be canonical if (q_i, p_i) are canonical and

$$\left.\begin{aligned} p_j &= \frac{\partial S(q_i,\alpha_i)}{\partial q_j} \\ \beta_j &= -\frac{\partial S(q_i,\alpha_i)}{\partial \alpha_j} \end{aligned}\right\} \tag{80}$$

where $S(q_i,\alpha_i)$ is an arbitrary generating function. The Hamilton-Jacobi equation is

$$\mathcal{H} + \partial S/\partial t = 0. \tag{81}$$

If $\mathcal{H}$ is independent, explicitly, of the time it is a constant of the motion. Let it be α_1. Equation (78) becomes, using (80),

$$\alpha_1 = \frac{1}{2A}\left[\csc^2\theta\left(\frac{\partial S}{\partial \psi} - \cos\theta\,\frac{\partial S}{\partial \phi}\right)^2 + \left(\frac{\partial S}{\partial \theta}\right)^2\right] + \frac{1}{2C}\left(\frac{\partial S}{\partial \phi}\right)^2 \tag{82}$$

Assuming

$$S = -\alpha_1 t + S_\psi(\psi) + S_\theta(\theta) + S_\phi(\phi) , \tag{83}$$

we get

$$2A\alpha_1 = \csc^2\theta\,(S'_\psi - \cos\theta\; S'_\phi)^2 + S'^2_\theta + \frac{A}{C}\,S'^2_\phi , \tag{84}$$

where the primes denote differentiation. Solving for S'_ψ,

$$S'_\psi = \cos\theta\; S'_\phi + \sin\theta\,\sqrt{2A\alpha_1 - S'^2_\theta - \frac{A}{C}\,S'^2_\phi} .$$

Since the right-hand side of this equation is independent of ψ, it must be a constant. Thus

$$\left.\begin{aligned} S'_\psi &= \alpha_3 \\ S_\psi &= \alpha_3\psi . \end{aligned}\right\} \tag{85}$$

Similarly, we obtain

$$S_\phi = \alpha_2\phi , \tag{86}$$

and

$$S'^2_\theta = 2A\alpha_1 - \csc^2\theta\,(\alpha_3 - \alpha_2\cos\theta)^2 - \frac{A}{C}\,\alpha_2^2 ,$$

or

$$S'_\theta = \pm\sqrt{F(\theta)} , \tag{87}$$

where

$$F(\theta) = 2A\alpha_1 - \csc^2\theta(\alpha_3-\alpha_2\cos\theta)^2 - \frac{A}{C}\alpha_2{}^2 \, . \tag{88}$$

In equation (87), either sign may be adopted. It turns out that, to agree with conventional notation, the negative one should be used. From (87), then,

$$S_\theta = - \int \sqrt{F(\theta)} \; d\theta \, . \tag{89}$$

Let

$$I_1 = - S_\theta = \int \sqrt{F(\theta)} \; d\theta \, . \tag{90}$$

This may be written

$$I_1 = \int \csc\theta \sqrt{a+b\sin^2\theta+c\cos\theta} \; d\theta, \tag{91}$$

where

$$\left.\begin{aligned} a &= -(\alpha_2^2+\alpha_3^2) \\ b &= 2A\alpha_1 + \left(\frac{C-A}{C}\right)\alpha_2^2 \\ c &= 2\alpha_2\alpha_3 \, . \end{aligned}\right\} \tag{92}$$

To perform the integration, let

$$\cos\theta = \alpha - \beta\cos\gamma \quad , \tag{93}$$

where α and β are to be chosen to simplify the resulting integral. We find

$$I_1 = -\beta \int \frac{\sqrt{\left[a + b(1-\alpha^2) + c\alpha\right] + (2\alpha\beta b - c\beta)\cos\gamma - b\beta^2\cos^2\gamma} \; \sin\gamma d\gamma}{1 - (\alpha-\beta\cos\gamma)^2} \tag{94}$$

If

$$2\alpha b - c = 0 \, , \tag{95}$$

the second term vanishes.

If, further,

$$\left[a + b(1-\alpha^2) + c\alpha\right] = b\beta^2, \tag{96}$$

then the term under the radical sign becomes

$$b\beta^2(1-\cos^2\gamma) = b\beta^2\sin^2\gamma \, .$$

Then (94) becomes

$$I_1 = - b^{1/2} \beta^2 \int \frac{\sin^2\gamma \; d\gamma}{1 - (\alpha-\beta\cos\gamma)^2}. \tag{97}$$

From (92), we note that $b = h^2$, where h is the angular momentum. Conditions (95) and (96) require

$$\alpha = \alpha_2\alpha_3/h^2$$
$$\beta = \sqrt{1 - \alpha_2{}^2/h^2}\ \sqrt{1 - \alpha_3{}^2/h^2}\ .$$

Let

$$\left.\begin{aligned} \cos\theta' &= \alpha_2/h \\ \cos\theta^* &= \alpha_3/h \end{aligned}\right\} \tag{98}$$

Then

$$\left.\begin{aligned} \alpha &= \cos\theta'\ \cos\theta^* \\ \beta &= \sin\theta'\ \sin\theta^*\ . \end{aligned}\right\} \tag{99}$$

Equation (93) becomes

$$\cos\theta = \cos\theta'\ \cos\theta^* - \sin\theta'\ \sin\theta^*\ \cos\gamma\ , \tag{100}$$

which has the form of the law of cosines for spherical triangles.

The integral in (97) may be evaluated by making the transformation

$$z = \tan\frac{\gamma}{2}\ , \tag{101}$$

and expanding in partial fractions. We find

$$I_1 = -\frac{8h\beta^2}{d}\int\frac{z^2dz}{(1+z^2)(1+e^2z^2)(1+f^2z^2)}$$

or

$$I_1 = -\frac{8h\beta^2}{d(e^2-f^2)}\left[\frac{e^2-f^2}{(1-e^2)(1-f^2)}I_2 - \frac{e^4}{(1-e^2)}I_3 + \frac{f^4}{(1-f^2)}I_4\right] \tag{102}$$

where

$$\left.\begin{aligned} e^2 &= \left(\frac{1+\alpha+\beta}{1+\alpha-\beta}\right) \\ f^2 &= \left(\frac{1-\alpha-\beta}{1-\alpha+\beta}\right) \\ d &= (1-\alpha+\beta)(1+\alpha-\beta) \\ I_2 &= \int\frac{z^2dz}{1+z^2} \\ I_3 &= \int\frac{z^2dz}{1+e^2z^2} \\ I_4 &= \int\frac{z^2dz}{1+f^2z^2} \end{aligned}\right\} \tag{103}$$

Let

$$\left.\begin{aligned} \tan\lambda &= ez \\ \tan\mu &= fz \,. \end{aligned}\right\} \tag{104}$$

Then the integrals (103) become

$$\left.\begin{aligned} I_2 &= z - \gamma/2 \\ I_3 &= z/e^2 - \lambda/e^3 \\ I_4 &= z/f^2 - \mu/f^3. \end{aligned}\right\} \tag{105}$$

After substituting (105) into (102) and simplifying, we obtain

$$I_1 = h\left[e(1+\alpha-\beta)\lambda + f(1-\alpha+\beta)\mu - \gamma\right] . \tag{106}$$

From (99), we can show

$$\left.\begin{aligned} 1+\alpha+\beta &= 2\cos^2 {}^1/_2\,(\theta'-\theta^*) \\ 1+\alpha-\beta &= 2\cos^2 {}^1/_2\,(\theta'+\theta^*) \\ 1-\alpha+\beta &= 2\sin^2 {}^1/_2\,(\theta'+\theta^*) \\ 1-\alpha-\beta &= 2\sin^2 {}^1/_2\,(\theta'-\theta^*) \,. \end{aligned}\right\} \tag{107}$$

Then

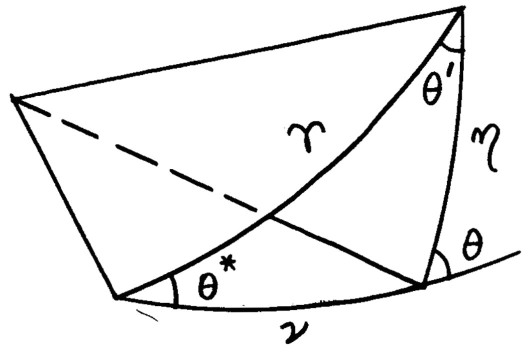
$$\left.\begin{aligned} e^2 &= \frac{\cos^2 {}^1/_2\,(\theta'-\theta^*)}{\cos^2 {}^1/_2\,(\theta'+\theta^*)} \\ f^2 &= \frac{\sin^2 {}^1/_2\,(\theta'-\theta^*)}{\sin^2 {}^1/_2\,(\theta'+\theta^*)} \,. \end{aligned}\right\} \tag{108}$$

Equation (106) becomes

$$I_1 = h\left[\cos\theta'\,(\lambda-\mu) + \cos\theta^*\,(\lambda+\mu) - \gamma\right]. \tag{109}$$

Now, consider the spherical triangle in figure 2:

Figure 2.



One may write the following identities from spherical trigonometry:

$$\left.\begin{aligned} \tan {}^1/_2\ (\nu+\eta) &= \frac{\cos {}^1/_2\ (\theta'-\theta^*)}{\cos {}^1/_2\ (\theta'+\theta^*)} \tan\frac{\gamma}{2} = ez \\ \tan {}^1/_2\ (\nu-\eta) &= \frac{\sin {}^1/_2\ (\theta'-\theta^*)}{\sin {}^1/_2\ (\theta'+\theta^*)} \tan\frac{\gamma}{2} = fz\ . \end{aligned}\right\} \tag{110}$$

Comparison of these relations with (104) gives

$$\nu + \eta = 2\lambda$$

$$\nu - \eta = 2\mu\ ,$$

or

$$\left.\begin{aligned} \nu &= \lambda + \mu \\ \eta &= \lambda - \mu\ . \end{aligned}\right\} \tag{111}$$

Equation (109) then gives

$$I_1 = h\Big[\eta\cos\theta' + \nu\cos\theta^* - \gamma\Big]. \tag{112}$$

But

$$h\cos\theta' = \alpha_2\ ,$$

$$h\cos\theta^* = \alpha_3\ , \quad \text{so}$$

$$I_1 = \alpha_2\eta + \alpha_3\nu - h\gamma\ . \tag{113}$$

The generating function, from (85), (86), and (87) is

$$S = -\alpha_1 t + \alpha_3\psi + \alpha_2\phi - I_1(\theta)\ . \tag{114}$$

From equations (80),

$$\left.\begin{aligned} p_\psi &= \alpha_3 \\ p_\theta &= -\frac{\partial I_1}{\partial\theta} \\ p_\phi &= \alpha_2 \\ \beta_1 &= t + \frac{\partial I_1}{\partial\alpha_1} \\ \beta_2 &= -\phi + \frac{\partial I_1}{\partial\alpha_2} \\ \beta_3 &= -\psi + \frac{\partial I_1}{\partial\alpha_3} \end{aligned}\right\} \tag{115}$$

To obtain the derivatives of I_1, use is made of relations (98) and

(100), the expression for h,

$$h^2 = 2A\alpha_1 + \left(\frac{C-A}{C}\right)\alpha_2^2 \,, \qquad \text{[from (92)]}$$

and the spherical trigonometry identities

$$\left.\begin{aligned} \cos\theta' &= \cos\theta^* \cos\theta + \sin\theta^* \sin\theta \cos\nu \\ \cos\theta^* &= \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos\eta \\ \frac{\sin\gamma}{\sin\theta} &= \frac{\sin\nu}{\sin\theta'} = \frac{\sin\eta}{\sin\theta^*} \,. \end{aligned}\right\} \qquad (116)$$

After differentiating and much simplification we find

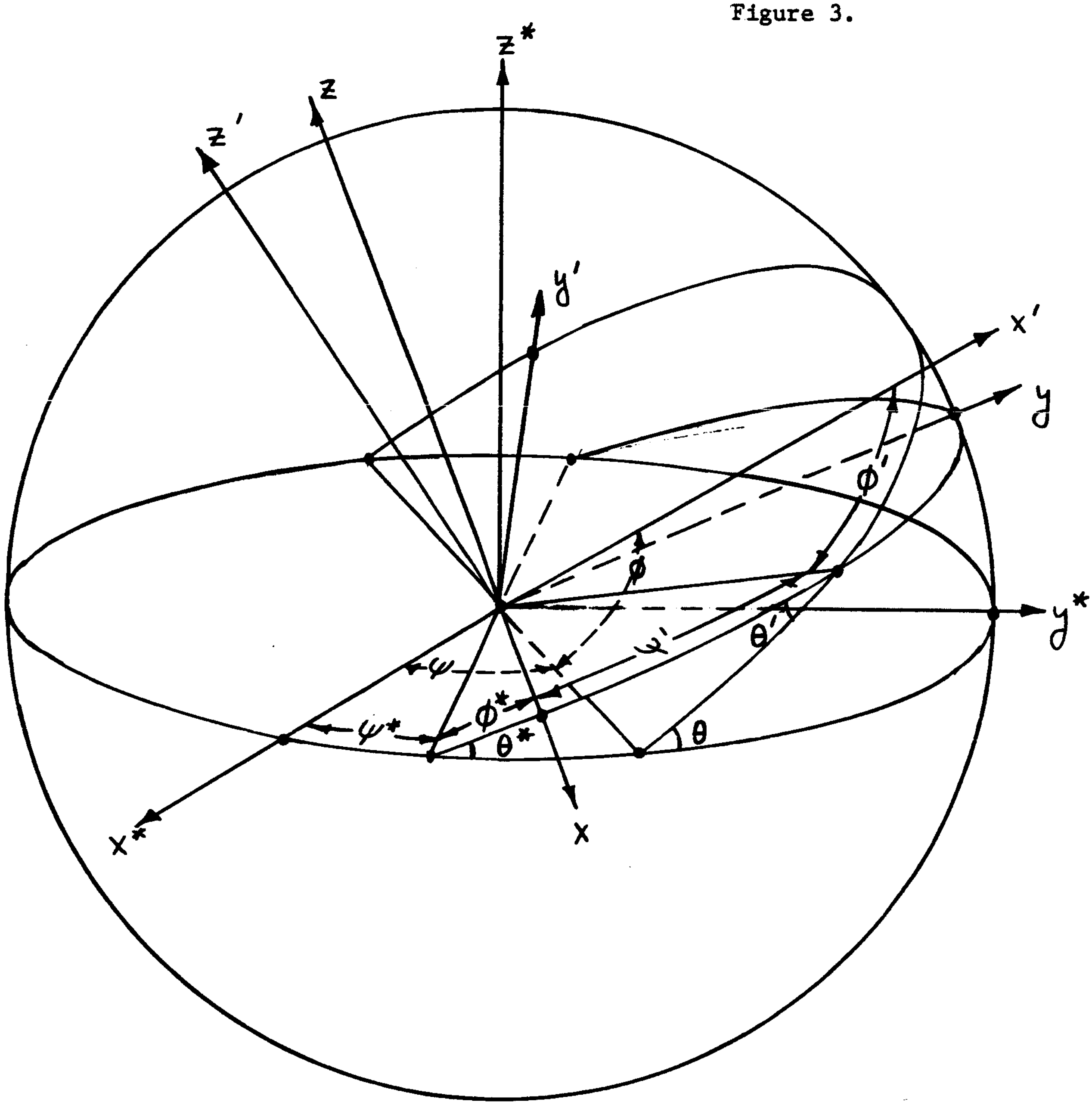
$$\left.\begin{aligned} \frac{\partial I_1}{\partial\theta} &= h \sin\theta^* \sin\nu \\ \frac{\partial I_1}{\partial\alpha_1} &= -A\gamma/h \\ \frac{\partial I_1}{\partial\alpha_2} &= \eta - \left(\frac{C-A}{C}\right)\gamma\cos\theta' \\ \frac{\partial I_1}{\partial\alpha_3} &= \nu. \end{aligned}\right\} \qquad (117)$$

The equations of transformation then become

$$\left.\begin{aligned} p_\psi &= \alpha_3 \\ p_\theta &= -h \sin\theta^* \sin\nu \\ p_\phi &= \alpha_2 \\ \beta_1 &= t - A\gamma/h \\ \beta_2 &= -\phi + \eta - \left(\frac{C-A}{C}\right)\gamma\cos\theta' \\ \beta_3 &= -\psi + \nu \,. \end{aligned}\right\} \qquad (118)$$

Up to now we have been using angles θ' and θ^*, although no attempt has been made to identify them with the Euler angles defined in Section II.B.2. It is now asserted that these are the same angles. θ^* is found to be the angle between the z*-axis and the angular momentum vector, and θ' that between H and the symmetry axis. Furthermore, consider figure 3:

Figure 3.



This figure shows the relation between the Euler angle sets (ψ,θ,ϕ), (ψ^*,θ^*,ϕ^*), and (ψ',θ',ϕ'). Comparison of this figure with that of figure 2 suggests the equivalences

$$\left.\begin{aligned} \nu &= \psi - \psi^* \\ \eta &= \phi - \phi' \\ \gamma &= \phi^* + \psi' \end{aligned}\right\} \qquad (119)$$

Equations (98) and (100) are also consistent with this identification.

Equations (118) then become

$$\left.\begin{aligned} p_\psi &= \alpha_3 \\ p_\theta &= -\ h \sin\theta \ \ \sin(\psi - \psi^*) \\ p_\phi &= \alpha_2 \end{aligned}\right\} \qquad (120)$$

$$\left.\begin{aligned} \beta_1 &= t - (A/h)(\phi^* + \psi') \\ \beta_2 &= -\ \phi' - \left(\frac{C-A}{C}\right)(\phi^* + \psi')\cos\theta' \\ \beta_3 &= -\ \psi^* \end{aligned}\right\} \qquad (121)$$

Equations (121) may be written

$$\left.\begin{aligned} \phi^* + \psi' &= \frac{h}{A}\,(t - \beta_1) \\ \phi' &= -\ \beta_2 - \frac{\alpha_2}{A}\left(\frac{C-A}{C}\right)(t - \beta_1) \\ \beta_3 &= -\ \psi^* \end{aligned}\right\} \qquad (122)$$

Now consider the solution obtained previously for this motion as given by

$$\left.\begin{aligned} \psi' &= \psi'_o + ht/A \\ \theta' &= \theta'_o \\ \phi' &= \phi'_o - nt \end{aligned}\right\} \qquad (36)$$

Note that

$$n = \left(\frac{C-A}{A}\right)\omega'_z = \frac{\alpha_2}{A}\left(\frac{C-A}{C}\right) .$$

By comparing (122) with (36), we may identify the β 's.

Let $\mathcal{T}$ = the time at which

$$\gamma = \phi^* + \psi' = 0 .$$

Then

$$\left.\begin{aligned} \beta_1 &= \mathcal{T} \\ \beta_2 &= -\ \phi'_o \qquad\qquad (-\phi' \text{ at } t = \mathcal{T}) \\ \beta_3 &= -\ \psi^* \ . \end{aligned}\right\} \qquad (123)$$

The complete set of equations for the angles is

$$\left.\begin{aligned} \psi^* &= \psi^*_o \\ \theta^* &= \theta^*_o = \cos^{-1}(\alpha_3/h) \\ \gamma &= \frac{h}{A}(t-\mathcal{T}) \\ \theta' &= \theta'_o = \cos^{-1}(\alpha_2/h) \\ \phi' &= \phi'_o - n(t-\mathcal{T}) \end{aligned}\right\} \tag{124}$$

Since ϕ^* and ψ' enter only through $\gamma = \phi^* + \psi'$, it is not necessary to consider both angles. One may be set at any convenient value.

One of the properties of the Hamilton-Jacobi method is that the equations of motion in the perturbed case are of a relatively simple form. Suppose the Hamiltonian function may be split into two parts:

$$\mathcal{H} = \mathcal{H}_o + \mathcal{H}_1(\alpha_i, \beta_i) \tag{125}$$

where $\mathcal{H}_o$ is the Hamiltonian for the unperturbed problem. The unperturbed problem has a solution given by algebraic transformations to the canonical constants α_i, β_i. The motion for the perturbed problem is given by considering the α_i, β_i to be variables, with the equations of motion

$$\left.\begin{aligned} \dot{\alpha}_i &= \frac{\partial \mathcal{H}_1(\alpha_i, \beta_i)}{\partial \beta_i} \\ \dot{\beta}_i &= -\frac{\partial \mathcal{H}_1(\alpha_i, \beta_i)}{\partial \alpha_i} \,. \end{aligned}\right\} \tag{126}$$

Note, however, that the task of expressing $\mathcal{H}_1$ in terms of the canonical constants may not be an easy one.

C. <u>Poisson Brackets</u>.

Consider any set $\{y_i\}$ of variables. These variables may be written as functions of the canonical constants.

$$y_i = y_i(\alpha_1, \alpha_2, \ldots, \alpha_N, \beta_1, \beta_2, \ldots, \beta_N, t), \tag{127}$$

where N is the number of degrees of freedom. (Note that the number of variables y_i is not restricted to be equal to N.) Differentiating (127), one obtains

$$\dot{y}_i = \sum_{j=1}^{N} \left(\frac{\partial y_i}{\partial \alpha_j} \dot{\alpha}_j + \frac{\partial y_i}{\partial \beta_j} \dot{\beta}_j \right) + \frac{\partial y_i}{\partial t} . \tag{128}$$

But, from (126),

$$\dot{\alpha}_j = \frac{\partial \mathcal{H}_1}{\partial \beta_j} , \quad \dot{\beta}_j = - \frac{\partial \mathcal{H}_1}{\partial \alpha_j} ,$$

so

$$\dot{y}_i = \sum_{j=1}^{N} \left(\frac{\partial y_i}{\partial \alpha_j} \frac{\partial \mathcal{H}_1}{\partial \beta_j} - \frac{\partial y_i}{\partial \beta_j} \frac{\partial \mathcal{H}_1}{\partial \alpha_j} \right) + \frac{\partial y_i}{\partial t} . \tag{129}$$

Now, the perturbing potential $\mathcal{H}_1$ may be expressed as a function of the y_i 's, i.e.,

$$\mathcal{H}_1 = \mathcal{H}_1(y_1, y_2, \ldots, y_M, t) . \tag{130}$$

Differentiating (130), we obtain

$$\left.\begin{aligned} \frac{\partial \mathcal{H}_1}{\partial \alpha_j} &= \sum_{k=1}^{M} \frac{\partial \mathcal{H}_1}{\partial y_k} \frac{\partial y_k}{\partial \alpha_j} \\ \frac{\partial \mathcal{H}_1}{\partial \beta_j} &= \sum_{k=1}^{M} \frac{\partial \mathcal{H}_1}{\partial y_k} \frac{\partial y_k}{\partial \beta_j} . \end{aligned}\right\} \tag{131}$$

Substituting into (129) we find

$$\dot{y}_i = \frac{\partial y_i}{\partial t} + \sum_{k=1}^{M} \left\{ y_i, y_k \right\} \frac{\partial \mathcal{H}_1}{\partial y_k} , \tag{132}$$

where $\left\{ y_i, y_k \right\}$, the Poisson bracket, is defined as

$$\left\{ y_i, y_k \right\} = \sum_{j=1}^{N} \left(\frac{\partial y_i}{\partial \alpha_j} \frac{\partial y_k}{\partial \beta_j} - \frac{\partial y_i}{\partial \beta_j} \frac{\partial y_k}{\partial \alpha_j} \right) . \tag{133}$$

In equation (132), note that if $\mathcal{H}_1 = 0$, corresponding to the unperturbed case, then

$$\dot{y}_i = \frac{\partial y_i}{\partial t} .$$

Thus $\partial y_i/\partial t$ is just the unperturbed rate of change of y_i, call it $(\dot{y}_i)_o$. Also, noting that the sum in (132) is of the form of a matrix product, we define matrices Y, B, and $\mathcal{H}_y$ by

$$\begin{aligned} Y_i &= y_i \\ B_{ij} &= \{y_i, y_j\} \\ \mathcal{H}_{yj} &= \frac{\partial \mathcal{H}_1}{\partial y_j} . \end{aligned} \tag{134}$$

Then we have

$$\dot{Y} = (\dot{Y})_o + B\mathcal{H}_y . \tag{135}$$

Equation (135) gives, explicitly, the perturbed rate of change of the variables.

At this point, we have two possible approaches which may be taken. The equations of motion as expressed in equations (126) may be used by expressing $\mathcal{H}_1$ in terms of the canonical constants α_i, β_i, and differentiating to obtain terms of the form $\partial \mathcal{H}_1/\partial \alpha_i$. After integrating to obtain the time variations in α_i, β_i, the corresponding variations in the angular variables may be obtained through the transformation equations (120) and (121).

Alternatively, we may elect to use the equations of motion in the form of equations (135), by expressing $\mathcal{H}_1$ in terms of the variables of the array Y and differentiating with respects to these variables. Note that equations (135) bear a close analogy to the Lagrange planetary equations of Celestial Mechanics.

The latter course is felt at this time to be the more promising one, and future work should be directed along this line.

V. GRAVITY-GRADIENT TORQUE

The gravitational potential energy of a rigid body in the vicinity of a central point mass is, to a good approximation,

$$\mathcal{V} = -\frac{GMm}{R_o} + \frac{GM}{2R_o^3}\left[3\rho^T I\rho - t_I\right], \tag{136}$$

where

G = universal gravitational constant

M = mass of central body

m = mass of rigid body

R_o = distance between the centers of mass of M and m

ρ = unit vector directed from the c.m. of m to that of M

t_I = trace of I

In (136), the first term is the potential energy of two point masses.

Suppose that the coordinate system to which I and ρ are referenced is the body-fixed principal-axis system. Then

$$I \rightarrow I' = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

and $t_I = A + B + C$. If this system is used, then, of the terms in (136), only the term containing ρ' will depend on the orientation. Thus we may let the perturbing potential be

$$\left.\begin{aligned} \mathcal{H}_1 &= K\,\rho'^T I'\rho' \\ \text{or} \qquad \mathcal{H}_1 &= K(Al^2+Bm^2+Cn^2) \end{aligned}\right\} \tag{137}$$

where

$$K = \frac{3GM}{2R_o^3} \tag{138}$$

where l, m, n are the components of ρ'.

Now ρ' transforms according to

$$\rho = T\rho' \ ,$$

$$\text{or} \qquad \rho' = T^I \rho \ . \tag{139}$$

Then (137) becomes

$$\mathcal{H}_1 = K\rho^T T I' T^I \rho \ . \tag{140}$$

If the components of ρ are (λ,μ,ν), we obtain, in scalar form,

$$\mathcal{H}_1 = K\Big\{(a\cos^2\psi + d\sin^2\psi)\lambda^2 + (a\sin^2\psi + b\cos^2\psi)\mu^2 + e\nu^2 + 2c(\lambda\cos\psi + \mu\sin\psi)\Big[\nu\sin\theta - (\lambda\sin\psi - \mu\cos\psi)\cos\theta\Big] - 2f(\lambda\sin\psi - \mu\cos\psi)\nu\sin\theta\cos\theta\Big\} \tag{141}$$

where

$$\left.\begin{aligned} a &= (A\cos^2\phi + B\sin^2\phi) \\ b &= (A\sin^2\phi + B\cos^2\phi) \\ c &= (A-B)\sin\phi\cos\phi \\ d &= b\cos^2\theta + C\sin^2\theta \\ e &= b\sin^2\theta + C\cos^2\theta \\ f &= b - C. \end{aligned}\right\} \tag{142}$$

In the uniaxial case, $A = B$, and $\mathcal{H}_1$ simplifies considerably. Thus

$$\mathcal{H}_1 = K\Big\{\Big[A\cos^2\psi + (A\cos^2\theta + C\sin^2\theta)\sin^2\psi\Big]\lambda^2 + \Big[A\sin^2\psi + (A\cos^2\theta + C\sin^2\theta)\cos^2\psi\Big]\mu^2 + (A\sin^2\theta + C\cos^2\theta)\nu^2 - 2(A-C)(\lambda\sin\psi - \mu\cos\psi)\nu\sin\theta\cos\theta\Big\}. \tag{143}$$

(A=B)

Now let us return to equation (138). We have

$$\mathcal{H}_1 = K(Al^2 + Bm^2 + Cn^2),$$

which, for $A = B$, becomes

$$\mathcal{H}_1 = K\Big[A(l^2+m^2) + Cn^2\Big]. \tag{144}$$

Let δ be the angle between ρ' and the z'-axis. Then

$$\left.\begin{aligned} \cos\delta &= \rho'\cdot\hat{k}' = n \\ \sin^2\delta &= 1 - n^2 = l^2 + m^2. \end{aligned}\right\} \tag{145}$$

Equation (144) becomes

$$\mathcal{H}_1 = K\Big[A\sin^2\delta + C\cos^2\delta\Big].$$

This may be written in two equivalent forms:

$$\mathcal{H}_1 = K\Big[(A-C)\sin^2\delta + C\Big] \tag{146}$$

or

$$\mathcal{H}_1 = K\Big[(C-A)\cos^2\delta + A\Big]. \tag{147}$$

VI. NUMERICAL INTEGRATION ROUTINE

It has been anticipated that, at some time during the performance of this work, a need would arise for the computer integration of differential equations such as those of equations (126) or (135). To this end, a computer subroutine has been written which will numerically integrate simultaneous differential equations.

A. Description of the Routine.

The subroutine, called QUAD1, effects the solution of a system of first order differential equations of the form

$$y_i' = f(x, y_1, y_2, \ldots, y_N) \ , \ i = 1, 2, \ldots, N \tag{148}$$

where N may be any number from 1 to 12, or larger, with slight modifications.

The integration method used is a modified Runge-Kutta method known as Merson's method. The chief advantage to this method is that it provides a very good estimate of the fifth-order truncation error. From the limited tests conducted, it also appears to be about five times as accurate as Runge-Kutta using Runge's coefficients. The equations for Merson's method for one dependent variable are:

$$\left.\begin{aligned}
k_1 &= hf(x,y) \\
k_2 &= hf(x + h/3, y + k_1/3) \\
k_3 &= hf(x + h/3, y + k_1/6 + k_2/6) \\
k_4 &= hf(x + h/2, y + k_1/8 + 3k_3/8) \\
k_5 &= hf(x + h, y + k_1/2 - 3k_3/2 + 2k_4) \\
y_{n+1} &= y_n + 1/6 \ (k_1 + 4k_4 + k_5) \\
\varepsilon &= 1/30 \ (2k_1 - 9k_3 + 8k_4 - k_5) \quad .
\end{aligned}\right\} \tag{149}$$

QUAD1 is a control routine. Once it is called it remains in control until integration is completed, calling subroutines to evaluate the functions f_i, to integrate, to adjust the integration interval and to print at specified intervals. The incrementing of the independent and dependent variables is performed in double precision. Integration proceeds until the independent variable exceeds a specified value. A detailed description and listing of the program is available to interested persons.