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THE CLASSICAL "SPHERE-OF-INFLUENCE"

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RESEARCH AND DEVELOPMENT OPERATIONS

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SUMMARY

The classical definition of the "sphere-of-influence" is given and the equation implicitly defining its radius is derived. This equation was solved using a little known iteration technique attributed to Wegstein. Two analytic approximations are derived and the Fourier coefficients through 12th order for the exact result are displayed for the moon and each planet in the solar system. Several "spheres" about the moon computed according to these results are displayed graphically for the earth-moon system.

I. INTRODUCTION

It has been an increasingly popular and useful tool in the preliminary design phases of interplanetary trajectories and even lunar probes to approximate the actual motion of a rocket vehicle by segments of trajectories found from the classical two-body equations. This means, for example, that the motion of a Mars probe might be approximated to a high degree of accuracy by three segments: an earth-centered conic segment, a sun-centered conic segment and a Mars-centered conic segment. The point at which the transition from one segment to the next occurs is somewhat arbitrary, but it has been customary to use an approximation to the "sphere-of-influence" as the criteria. This modern term, "sphere-of-influence," is a carry-over from the classical work of Laplace and Tisserand who used the term "activity-sphere." The surface to which both these terms refer is not truly spherical, but generically the terms are descriptive. It is not the intent here to discuss the effect of the "sphere-of-influence" on the accuracy of patched-conic techniques but merely to make available a reference on a much seen but little discussed term.

II. DISCUSSION

Considering n point masses, the equations of motion for the kth body are

$$\ddot{\vec{r}}_k = - \sum_{j=1}^{n}{}' K m_j \frac{(\vec{r}_k - \vec{r}_j)}{|\vec{r}_k - \vec{r}_j|^3}$$

where K is the gravitational proportionality constant, ' indicates that the dummy variable j cannot equal k, and the origin of the coordinate system is arbitrary except that it is not coincident with one of the masses. If the origin is translated to one of the masses, say the ith mass, the equations of motion can be written

$$\ddot{\vec{r}}_k^* = -K(m_i + m_k)\frac{\vec{r}_k^*}{|\vec{r}_k^*|^3} - \sum_{j=1}^{n}{}' Km_j\left(\frac{\vec{r}_k^* - \vec{r}_j^*}{|\vec{r}_k^* - \vec{r}_j^*|^3} + \frac{\vec{r}_j^*}{|\vec{r}_j^*|^3}\right) \tag{1}$$

where $k \neq i$, $j \neq k$, $j \neq i$ and * indicates that the origin for the radius vectors is coincident with the ith mass. Notice for the classical two-body problem, equation (1) reduces to the well-known result

$$\ddot{\vec{r}}_2^* = -K(m_1 + m_2)\frac{\vec{r}_2^*}{|\vec{r}_2^*|^3}, \tag{2}$$

where m_1 is at the origin of the coordinate system. The right-hand side (R.H.S.) of (2) is the force per unit mass acting on m_2 and its magnitude will be called the two-body force.

Now consider three bodies where the coordinate system in Figure 1 is referenced.

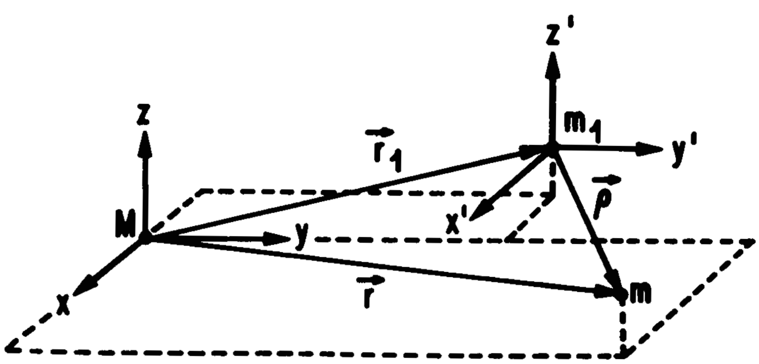

Figure 1

With the aid of (1), dropping the * notation, the equation of motion of m with respect to M becomes

$$\ddot{\vec{r}} = -K(m + M)\frac{\vec{r}}{|\vec{r}|^3} - Km_1\left(\frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + \frac{\vec{r}}{|\vec{r}_1|^3}\right).$$

From Figure 1, it is seen that $\vec{\rho} = \vec{r} - \vec{r}_1$ and making the assumption that $m \ll M$, the last equation becomes

$$\ddot{\vec{r}} + Km\frac{\vec{r}}{|\vec{r}|^3} = -Km_1\left(\frac{\vec{\rho}}{|\vec{\rho}|^3} + \frac{\vec{r}_1}{|\vec{r}_1|^3}\right). \tag{3}$$

The R.H.S. of (3) may be considered a perturbing force due to m_1 which disturbs the motion of m about M. Calling this force $\vec{F}_1$, its magnitude is

$$|\vec{F}_1| = +\sqrt{\vec{F}_1 \cdot \vec{F}_1} = Km_1\left(\frac{1}{\rho^4} + \frac{1}{r_1^4} + \frac{2\vec{\rho} \cdot \vec{r}_1}{\rho^3\, r_1^3}\right)^{1/2},$$

where $\rho = |\vec{\rho}|$ and $r_1 = |\vec{r}_1|$. The ratio of the magnitude of the perturbing force to the magnitude of the two-body force for this case is

$$R_1 = r^2\frac{m_1}{M}\left(\frac{1}{\rho^4} + \frac{1}{r_1^4} + \frac{2\vec{\rho} \cdot \vec{r}_1}{\rho^3\, r_1^3}\right)^{1/2}. \tag{4}$$

Using (1) again, the equation of motion for m with respect to m_1 is

$$\ddot{\vec{\rho}} = -K(m + m_1)\frac{\vec{\rho}}{|\vec{\rho}|^3} - KM\left(\frac{\vec{\rho} - (-\vec{r}_1)}{|\vec{\rho} - (-\vec{r}_1)|^3} + \frac{(-\vec{r}_1)}{|-\vec{r}_1|^3}\right)$$

$$= -K(m + m_1)\frac{\vec{\rho}}{|\vec{\rho}|^3} - KM\left(\frac{\vec{\rho} + \vec{r}_1}{|\vec{\rho} + \vec{r}_1|^3} - \frac{\vec{r}_1}{|\vec{r}_1|^3}\right).$$

Assuming $m \ll m_1$ and noticing that $\vec{r} = \vec{\rho} + \vec{r}_1$, this becomes

$$\ddot{\vec{\rho}} + Km_1 \frac{\vec{\rho}}{|\vec{\rho}|^3} = - KM \left(\frac{\vec{r}}{|\vec{r}|^3} - \frac{\vec{r}_1}{|\vec{r}_1|^3} \right). \tag{5}$$

Again, the R.H.S. of (5) may be considered a perturbing force due to M which disturbs the motion of m about m_1. Calling this force $\vec{F}_2$, its magnitude is

$$|\vec{F}_2| = + \sqrt{\vec{F}_2 \cdot \vec{F}_2} = KM \left(\frac{1}{r^4} + \frac{1}{r_1^4} - \frac{2\vec{r} \cdot \vec{r}_1}{r^3\, r_1^3} \right)^{1/2} ,$$

where $r = |\vec{r}|$ and $r_1 = |\vec{r}_1|$. The ratio of the magnitude of the perturbing force to the magnitude of the two-body force for this case is

$$R_2 = \rho^2 \frac{M}{m_1} \left(\frac{1}{r^4} + \frac{1}{r_1^4} - \frac{2\vec{r} \cdot \vec{r}_1}{r^3\, r_1^3} \right)^{1/2} . \tag{6}$$

The "sphere-of-influence" is now defined as the locus of points for which these two ratios are equal. Equating (4) and (6), evaluating the scalar products and rearranging yields

$$\rho^4 = (m_1/M)^2\, r^4 \left(\frac{\rho^4 + r_1^4 + 2\rho^2\, r_1^2 \cos\theta}{r^4 + r_1^4 - 2r\, r_1^2\, \rho \cos\theta - 2r\, r_1^3} \right)^{1/2} , \tag{7}$$

where θ is the angle between $\vec{r}_1$ and $\vec{\rho}$. This is the exact defining equation for ρ, the radius of the "sphere-of-influence." ρ is the only unknown in (7) since $r = (r_1^2 + \rho^2 + 2r_1\, \rho \cos\theta)^{1/2}$, r_1 is supposed known, and θ is a parameter. For $m_1 < M$, equation (7) represents a surface enclosing m_1 and $\rho/r_1 < 1$. Equation (7) must be solved for ρ numerically using an iterative procedure. Wegstein's iteration, discussed in Appendix I is applicable and extremely rapid. However, numerical experience with a normalized form of (7) indicated that its denominator was not being evaluated accurately. The fundamental reason for this is due to the R.H.S. of (5) which requires differencing two nearly equal quantities; always an inaccurate process unless special procedures are adopted. This computational problem is neatly circumvented by applying an idea of Encke's which he employed in computing planetary orbits. The specific details of the application are discussed in Appendix II with the following result

$$R^4 = (m_1/M)^2 \frac{(1 + R^2 + 2R \cos \theta)\ (1 + R^4 + 2R^2 \cos \theta)^{1/2}}{[F^2(1 + R^2 + 2R \cos \theta) + 2F(R^2 + R \cos \theta) + R^2]^{1/2}}$$

$$= (m_1/M)^2 \frac{(1 + Q_1)\ (1 + R^4 + 2R^2 \cos \theta)^{1/2}}{[F^2(1 + Q_1) + 2F(Q_1 + R^2) + R^2]^{1/2}}, \tag{8a}$$

where $R = \rho/r_1$, $Q_1 = R^2 + 2R \cos \theta$ and $F = F(Q_i)$ is defined as

$$F = F(Q_1) = -\frac{3}{2} Q_1 + \frac{15}{8} Q_1^2 + \ldots + (-1)^n \frac{(2n + 1)!}{n!\ n!\ 2^{2n}} Q_1^n + \ldots$$

$$n = 1,2,\ \ldots,\ \text{if } 0 \leqq Q_1 < 1 \tag{8b}$$

$$F = F(Q_2) = \frac{3}{2}\left(Q_2 + \frac{Q_2^2}{4} - \frac{1}{24} Q_2^3 + \frac{1}{64} Q_2^4 + \ldots\right.$$

$$\left. + (-1)^n \frac{(2n - 5)!}{n!(n - 3)!} \frac{1}{2^{2n-4}} Q_2^n + \ldots \right) \tag{8c}$$

$$n = 3,4,5,\ \ldots,\ \text{if } -1 < Q_1 < 0,$$

where

$$Q_2 = -\frac{Q_1}{1 + Q_1}.$$

The Wegstein iterative technique was able to solve equations (8) to 12 significant digits in at most 6 trials and in the majority instances 3 or 4 trials for the moon and planets. This precision is not justified by the limited accuracy of the mass ratios involved, but assuming them exact, the speed of convergence of the iteration scheme is as stated. Because the complexity of equations (8) requires an electronic computer or desk calculator for efficient evaluation, two simplified approximations were developed in Appendices III and IV. These are given as (9) and (10) which explicitly define ρ.

$$\rho = \frac{r_1}{(M/m_1)^{2/5}\ (1 + 3\cos^2\theta)^{1/10} - \frac{2}{5}\cos\theta\left(\frac{1 + 6\cos^2\theta}{1 + 3\cos^2\theta}\right)} \tag{9}$$

$$\rho = \left(\frac{2}{5}\right)^{1/10}\left(\frac{m_1}{M}\right)^{2/5} r_1\left[1 + \left(\frac{4}{5}\right)\left(\frac{2}{5}\right)^{1/10}\left(\frac{m_1}{M}\right)^{2/5}\cos\theta - \frac{3}{50}\cos 2\theta + \ldots\right]$$

$$= .91244\left(\frac{m_1}{M}\right)^{2/5} r_1\left[1 + .72995\left(\frac{m_1}{M}\right)^{2/5}\cos\theta - .06\cos 2\theta + \ldots\right]. \tag{10}$$

They are essentially variations of Tisserand's development. The maximum relative error incurred with (9) is less than 1.5 percent for the moon, negligible for the planets out to Mars, and less than .2 percent (usually much less) for the remainder. The maximum relative error using (10) is less than 5 1/2 percent for the moon and less than 3 1/2 percent for the planets. These two equations should be compared with the frequently seen expression for ρ

$$\rho = (m_1/M)^{2/5}\, r_1. \tag{11}$$

Equation (10) suggests a Fourier expansion for ρ. Since ρ is an even function of ϑ, the expansion is of the form

$$\rho(\vartheta) = a_o + a_1\cos\vartheta + a_2\cos 2\theta + \ldots\, a_n\cos n\theta + \ldots,$$

where

$$a_o = \frac{1}{\pi} \int_0^{\pi} \rho(\vartheta)\, d\vartheta \quad \text{and} \quad a_n = \frac{2}{\pi} \int_0^{\pi} \rho(\vartheta) \cos n\theta\, d\vartheta,$$

$$\text{for } n = 1, 2, \ldots .$$

The coefficients through 12th order were computed for the moon and planets and are displayed in Table I along with the largest relative error incurred in truncating the series at that coefficient. Trapezoidal integration was used for the calculation using the planetary data in Table II. A comparison of the actual coefficients with the first three coefficients yielded by (10) is made in Table III.

For the earth-moon system, equations (8), (9), (10) and (11) were used to plot typical "spheres-of-influence" about the moon. These results are shown in Figure 2. Notice that these are cross-sectional shapes, and the three-dimensional surface is obtained by a revolution about $\vec{r}_1$ or, in this case, the earth-moon line. The maximum radius occurs at about 80.8° and is not on the earth-moon line as implied by Tisserand. Also shown for comparison is the gravity sphere defined as the locus of points for which the gravitational attraction of the moon equals the gravitational attraction of the earth. Its equation is

$$\rho^2 - 2 \frac{1}{\left(\frac{M}{m_1} - 1\right)} r_1 \rho \cos \theta - \frac{1}{\left(\frac{M}{m_1} - 1\right)} r_1^2 = 0, \tag{12}$$

which is the equation of a circle with radius

$$\sqrt{M/m_1} \frac{r_1}{\left(\frac{M}{m_1} - 1\right)}$$

and center displaced away from the moon's center by an amount

$$\frac{r_1}{\left(\frac{M}{m_1} - 1\right)} .$$

III. CONCLUSIONS

Considering three bodies, two massy and one a particle of negligible mass, the "sphere-of-influence" has been defined as the locus of points for which the ratio found by taking the perturbing force of one massy body on the particle and dividing it by the two-body force between the particle and the other massy body is set equal to a similar ratio with the roles of the two massy bodies reversed. The resulting equation for the "sphere-of-influence" was conveniently and rapidly solved by using Wegstein's iteration. Formulas for two analytic approximations and coefficients for Fourier expansions through 12 terms have been displayed. Which of these should be used depends on the accuracy required and the computational aids available. Finally, graphical display of some of these results indicates that, while the departure from sphericity of the "sphere-of-influence" is relatively small, it is nonetheless significant.

TABLE I

"Sphere-of-Influence" Fourier Coefficients for the Solar System and the Moon

	ORDER											
	1	2	3	4	5	6	7	8	9	10	11	12
SUN-MERCURY Coeff. (km)	103,235.	114.303	- 6918.37	1.74079	1270.62	- 2.09308	- 296.734	.921785	76.7033	- .362768	- 20.9659	.125578
SUN-VENUS Coeff. (km)	568,684.	1861.44	-38110.8	28.2095	6999.38	- 33.9591	-1634.50	15.0440	422.454	- 5.84190	-115.452	2.09871
SUN-EARTH Coeff. (km)	856,430.	3040.03	-57394.3	46.0335	10541.0	- 55.4427	-2461.50	24.5680	636.186	- 9.53524	-173.857	3.43056
SUN-MARS Coeff. (km)	535,962.	779.091	-35917.8	11.8606	6596.62	- 14.2531	-1540.53	6.28983	398.210	- 2.46343	-108.844	.864290
SUN-JUPITER Coeff. (100 km)	444,203.	16022.3	-29766.0	217.481	5468.40	-283.608	-1266.67	126.465	322.275	-48.6675	- 85.9805	17.6671
SUN-SATURN Coeff. (100 km)	504,860.	11170.8	-33835.5	159.173	6214.49	-200.181	-1446.77	89.1459	371.738	-34.3955	-100.692	12.5276
SUN-URANUS Coeff. (100 km)	478,662.	4978.53	-32078.8	73.7720	5891.54	- 90.1770	-1374.92	40.0756	354.934	-15.4907	- 96.8235	5.63513
SUN-NEPTUNE Coeff. (100 km)	798,960	8848.94	-53544.6	130.835	9833.92	-160.181	-2294.75	71.1975	592.279	-27.5167	-161.526	10.0128
SUN-PLUTO Coeff. (100 km)	329,438	1145.52	-22077.5	17.3497	4054.73	- 20.8933	- 946.855	9.25768	244.721	- 3.59352	- 66.8779	1.29240
Maximum Relative Error for the Above (%)		7.7	1.5	1.5	.368	.361	.0993	.0949	.0297	.0263	.00896	.00752
EARTH-MOON Coeff. (km)	61,649.2	6355.15	- 4093.37	66.6156	758.304	-106.255	- 166.426	47.1779	37.6895	-17.5965	8.16862	6.06283
% Error		7.65	1.53	1.51	.412	.348	.128	.094	.0409	.0238	.0135	.00592

TABLE II

Solar System Parameters

Planet	r_1(km)*	$(m_1/M)^{2/5}$	$(M/m_1)^{2/5}$	M/m_1*
Mercury	57.9×10^6	$.19313 \times 10^{-2}$	517.78	6.1×10^6
Venus	108×10^6	$.57037 \times 10^{-2}$	175.32	4.07×10^5
Earth	150×10^6	$.61846 \times 10^{-2}$	161.69	3.3244×10^5
Mars	229×10^6	$.25352 \times 10^{-2}$	394.44	3.09×10^6
Jupiter	776×10^6	$.61937 \times 10^{-1}$	16.145	1.0474×10^3
Saturn	143×10^7	$.38226 \times 10^{-1}$	26.160	3.5×10^3
Uranus	287×10^7	$.18064 \times 10^{-1}$	55.358	2.28×10^4
Neptune	450×10^7	$.19230 \times 10^{-1}$	52.002	1.95×10^4
Pluto	589×10^7	$.60585 \times 10^{-2}$	165.05	3.5×10^5
Moon**	384.402×10^3	.17212	5.8099	81.357

*These values taken from "Lunar Flight Handbook," NAS8-5031, Martin Co., 1963.

**These values refer to earth-moon system.

TABLE III

Comparison Between Analytic Fourier Coefficients from Equation (10) and Their Numerically Determined Values

	ORDER 1	2	3
SUN-MERCURY			
Approximate	102036.	143.852	-6122.15
Exact	103235.	114.303	-6918.37
SUN-VENUS			
Approximate	562072.	2340.19	-33724.3
Exact	568684.	1861.44	-33111.0
SUN-EARTH			
Approximate	846471.	3821.40	-50788.2
Exact	856430.	3040.03	-57394.3
SUN-MARS			
Approximate	529734.	980.326	-31784.1
Exact	535962.	779.091	-35917.8
SUN-JUPITER			
Approximate	43855400.	1982780.	-2631320.
Exact	44420300.	1602230.	-2976600.
SUN-SATURN			
Approximate	49878700.	1391820.	-2992720.
Exact	50486000.	1117080.	-3383550.
SUN-URANUS			
Approximate	47305800.	623787.	-2838350.
Exact	47866200.	497853.	-3207880.
SUN-NEPTUNE			
Approximate	78959600.	1108380.	-4737570.
Exact	79896000.	884894.	-5354460.
SUN-PLUTO			
Approximate	32560700.	144000.	-1953640.
Exact	32943800.	114552.	-2207750.
EARTH-MOON			
Approximate	60371.7	7585.28	-3622.30
Exact	61649.2	6355.15	-4093.37

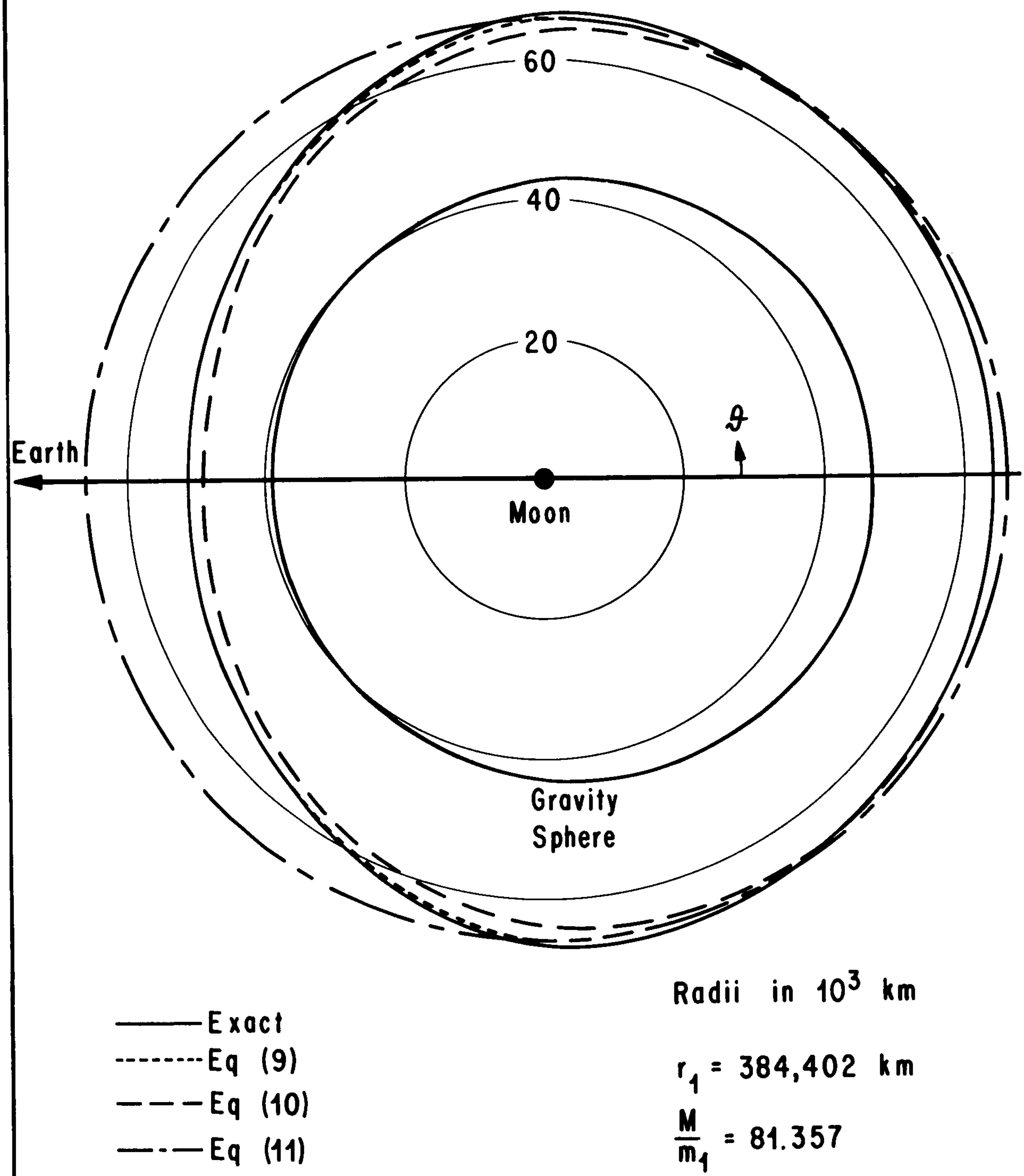

FIG. 2. SPHERE OF INFLUENCE ABOUT MOON

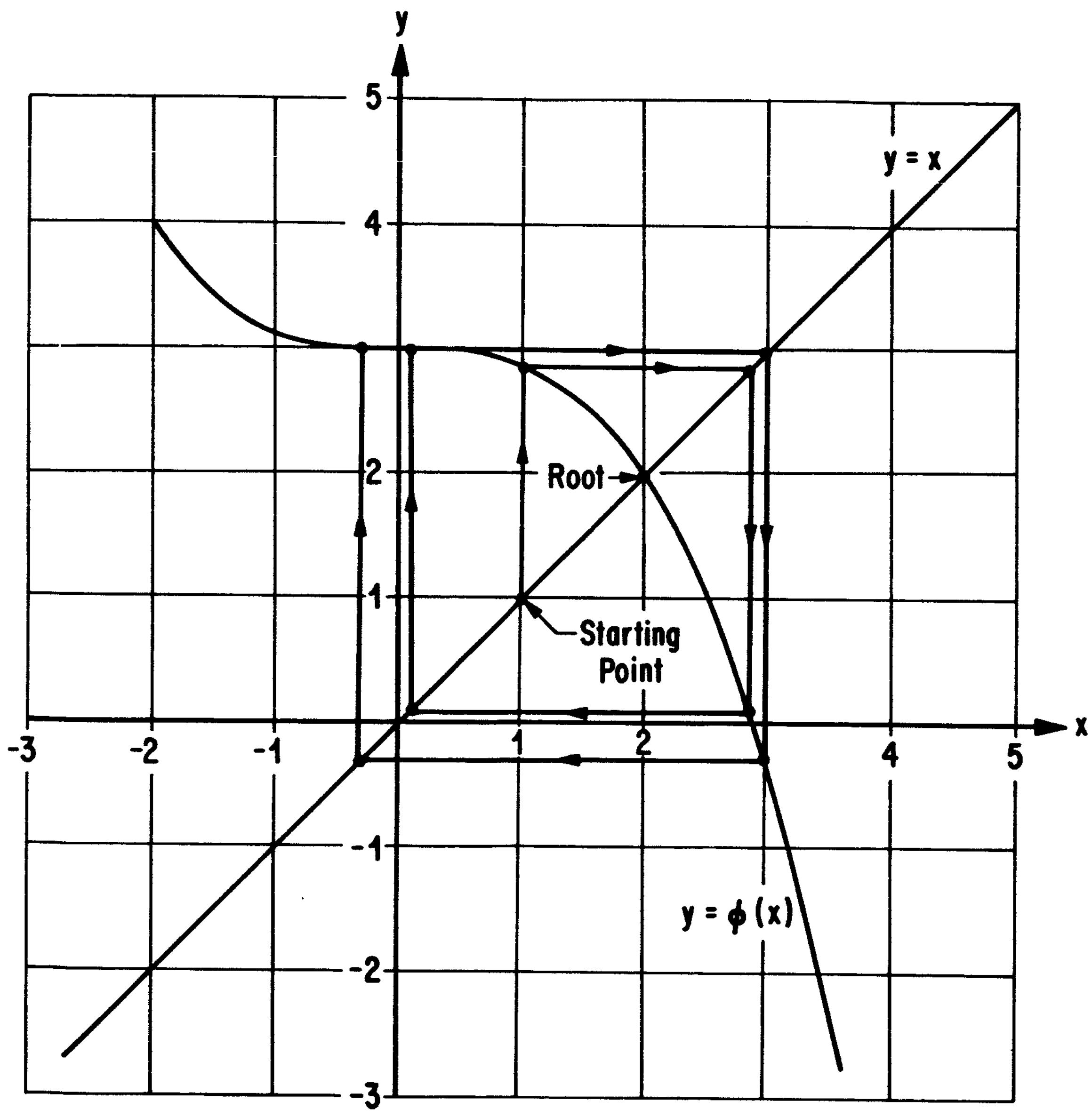

FIG. 3. THE CLASSICAL "METHOD OF ITERATION"

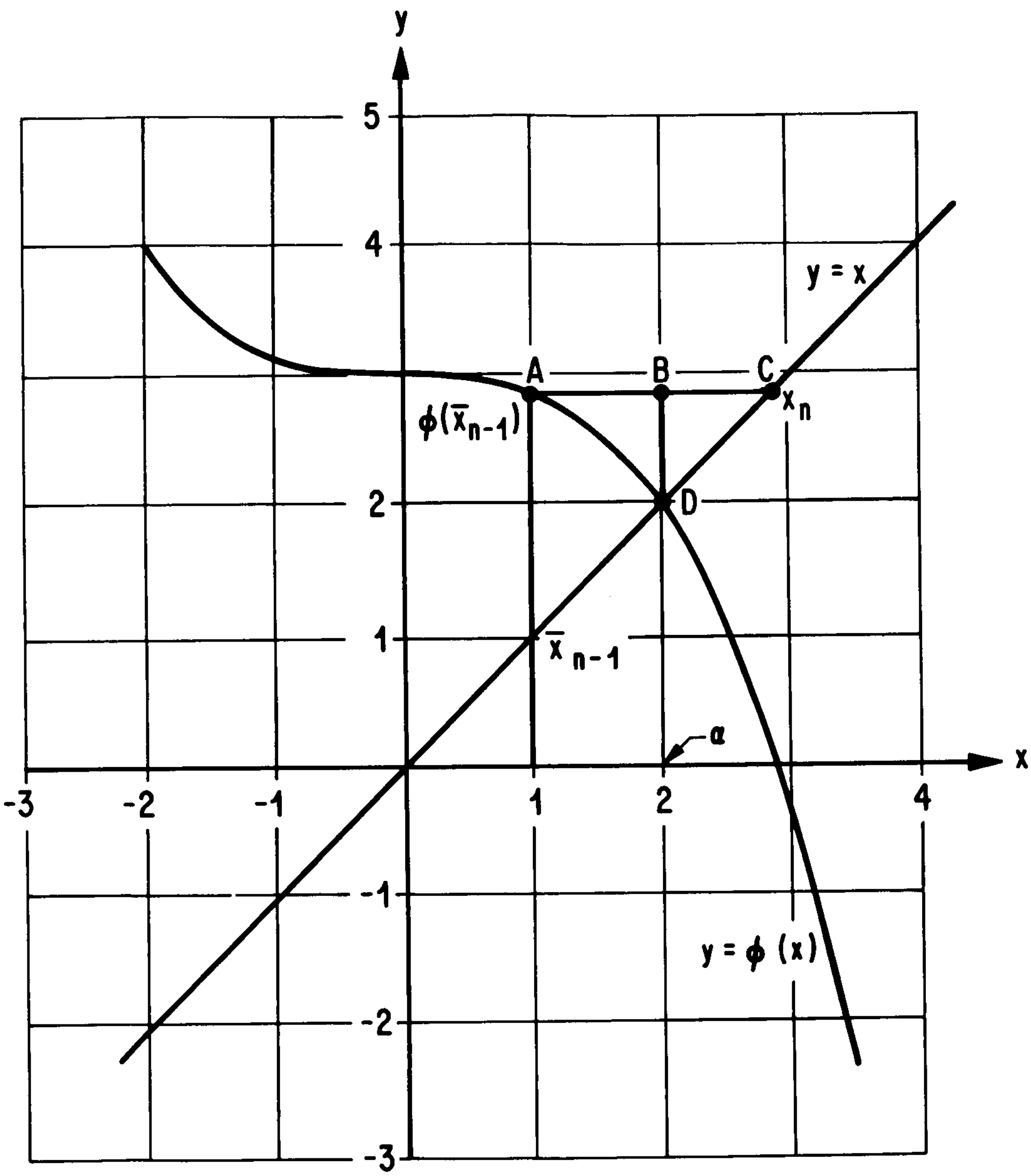

FIG. 4. THE DEVELOPMENT OF WEGSTEIN ITERATION

APPENDIX I

A classical method for finding the root of a nonlinear equation is known as the method of iteration. If a root of $F(x) = 0$ is desired and $F(x)$ can be solved for x in some form $x = \emptyset(x)$, then the algorithm is

$$x_n = \emptyset(x_{n-1}). \tag{I-1}$$

A sufficient condition for convergence to a root is

$$|\emptyset'(x_{n-1})| < 1 \tag{I-2}$$

for each of the x_i in some neighborhood of the root. The sequence defined by (I-1) is demonstrated graphically in Figure 3. The desired root, α, is obviously the intersection of the straight line and the curve $\emptyset(x)$. In this particular example, the sequence (I-1) is oscillatory, nonconvergent, principally because $|\emptyset'(\alpha)| > 1$. Wegstein devised a modification of (I-1) that can force convergence even when (I-2) is violated. The modification consists of altering the successive values of x_n. Denoting the altered values by $\bar{x}_i$, then

$$\bar{x}_n = x_n + K\,(x_n - \bar{x}_{n-1}), \tag{I-3}$$

where now $x_n = \emptyset(\bar{x}_{n-1})$. An estimate for K can be derived by referring to Figure 4. Ideally, $K(x_n - \bar{x}_{n-1}) = -BC$, and since $x_n - \bar{x}_{n-1} = AC$,

$$K = -\frac{BC}{AC}.$$

Now

$$1 + K = \frac{AC - BC}{AC} = \frac{AB}{AC}$$

so that

$$\frac{K}{1 + K} = -\frac{BC}{AC}\,\frac{AC}{AB} = -\frac{BC}{AB}.$$

From Figure 4, BC = BD, and by the mean value theorem, there exists a $\bar{x}_{n-1} < \xi < \alpha$ such that

$$\phi'(\xi) = \frac{\phi(\alpha) - \phi(\bar{x}_{n-1})}{\alpha - \bar{x}_{n-1}} = -\frac{BD}{AB} .$$

Therefore,

$$\frac{K}{1 + K} = -\frac{BC}{AB} = -\frac{BD}{AB} = \phi'(\xi)$$

or

$$K = \frac{\phi'(\xi)}{1 - \phi'(\xi)} . \qquad \text{(I-4)}$$

$\phi'(\xi)$ can be estimated as

$$\phi'(\xi) = \frac{\phi(\bar{x}_{n-1}) - \phi(\bar{x}_{n-2})}{\bar{x}_{n-1} - \bar{x}_{n-2}}$$

$$= \frac{x_n - x_{n-1}}{\bar{x}_{n-1} - \bar{x}_{n-2}} \qquad \text{(I-5)}$$

Substituting (I-5) into (I-4) into (I-3) yields

$$\bar{x}_n = x_n + \frac{(x_n - x_{n-1})\ (x_n - \bar{x}_{n-1})}{\bar{x}_{n-1} - \bar{x}_{n-2} + x_{n-1} - x_n} , \qquad \text{(I-6)}$$

which is Wegstein's iteration formula.

As an example of the rapid convergence properties of (I-6), consider the equation $x^3 + 8x - 24$ with roots 2, $-1 \pm j\sqrt{11}$. Equation (I-1) becomes

$$x = \phi(x) = \frac{24 - x^3}{8} . \qquad \text{(I-7)}$$

Notice that $\phi'(2) = -3/2$. Actually, Figures 3 and 4 are plots of (I-7) and show that (I-7) unmodified diverges as expected. The classical iteration and the Wegstein procedure are compared in the following table:

n	$\bar{x}_n$	x_n
1	1.00000000000	1.00000000000
2	2.87500000000	2.87500000000
3	.029541015625	.029541015625
4	1.74476338246	2.99999677755
5	1.93922535110	-.374989124219
6	2.00484760174	3.00659122337
7	1.99991096179	-.397294289403
8	1.99999987059	3.00783875302
9	2.00000000000	-.401524978457
10		3.00809184799
11		-.402383715948
12		3.00814387697
13		-.402560265342
14		3.00815460127
15		-.402596656742
16		3.00815681300
17		-.402604161995
18		3.00815726919
19		-.402605709998
20		3.00815736328
21		-.402606029282
22		3.00815738269
23		-.402606095173
24		3.00815738669
25		-.402606108735
26		3.00815738752
27		-.402606111558
28		3.00815738768
29		-.402606112111
30		3.00815738773
31		-.402606112257
32		3.00815738773
33		-.402606112257

APPENDIX II

The numerical problem mentioned in the text comes about as a result of attempting to evaluate the magnitude of

$$\frac{\vec{r}}{r^3} - \frac{\vec{r}_1}{r_1^3}, \tag{II-1}$$

where $r = |\vec{r}|$ and $r_1 = |\vec{r}_1|$. When $\vec{r}$ and $\vec{r}_1$ are approximately equal in magnitude, the subtraction in (II-1) causes a loss of significant digits. This means that, if a certain number of significant digits are desired in the result, $\vec{r}$ and $\vec{r}_1$ must be known with a greater number of significant digits (assuming $\vec{r}$ and $\vec{r}_1$ are sufficiently close in magnitude). The requirement for extra precision can be by-passed using a modification of Encke's well known astronomical method. Substituting $\vec{r} = \vec{r}_1 + \vec{\rho}$ into (II-1) gives

$$\frac{\vec{r}_1 + \vec{\rho}}{r^3} - \frac{\vec{r}_1}{r_1^3}$$

or

$$\left(\frac{1}{r^3} - \frac{1}{r_1^3}\right)\vec{r}_1 + \frac{\vec{\rho}}{r^3} = \frac{1}{r_1^3}\left\{\left(\frac{r_1^3}{r^3} - 1\right)\vec{r}_1 + \frac{r_1^3}{r^3}\vec{\rho}\right\}. \tag{II-2}$$

The R.H.S. of (II-2) can be evaluated accurately if $(r_1/r)^3 - 1$ can be found accurately. There are essentially two cases to consider: (1) $r \geqq r_1$ and (2) $r < r_1$.

<u>Case 1</u>: Define Q_1 by

$$1 + Q_1 = (r/r_1)^2. \tag{II-3}$$

Then

$$(r_1/r)^3 = (1 + Q_1)^{-3/2}$$

$$= 1 - \frac{3}{2} Q_1 + \frac{3 \cdot 5}{2 \cdot 2 \cdot 2} Q_1^2 + \ldots + (-1)^n \frac{(2n + 1)!}{n!\ n!\ 2^{2n}} Q_1^n + \ldots$$

by the binomial expansion, valid for $|Q_1| < 1$. Consequently,

$$(r_1/r)^3 - 1 = - \frac{3}{2} Q_1 + \frac{3 \cdot 5}{2 \cdot 2 \cdot 2} Q_1^2 + \ldots + (-1)^n \frac{(2n + 1)!}{n!\ n!\ 2^{2n}} Q_1^n + \ldots \qquad \text{(II-4)}$$

$$n = 1, 2, \ldots$$

$$= F(Q_1).$$

Substituting (II-4) into (II-2) yields

$$\frac{1}{r_1^3} \left\{ F(Q_1)\ \vec{r}_1 + \Big(1 + F(Q_1)\Big) \vec{\rho} \right\}, \qquad \text{(II-5)}$$

where from (II-3)

$$Q_1 = (r/r_1)^2 - 1 = \frac{r^2 - r_1^2}{r_1^2} = \frac{(\vec{r}_1 + \vec{\rho}) \cdot (\vec{r}_1 + \vec{\rho}) - r_1^2}{r_1^2}$$

$$= \frac{2\vec{\rho} \cdot \vec{r}_1 + \vec{\rho} \cdot \vec{\rho}}{r_1^2} = \frac{\rho^2 + 2\rho\, r_1 \cos\theta}{r_1^2}$$

$$= R^2 + 2R \cos\theta \qquad \text{(II-6)}$$

where

$$R = \rho/r_1.$$

Case 2: Define Q_2 by

$$1 + Q_2 = (r_1/r)^2. \tag{II-7}$$

Then

$$(r_1/r)^3 = (1 + Q_2)^{3/2}$$

$$= 1 + \frac{3}{2} Q_2 + \frac{3}{2} \cdot \frac{1}{2} \frac{Q_2^2}{2} - \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \frac{Q_2^3}{3 \cdot 2} + \ldots$$

by the binomial expansion, valid for $|Q_2| < 1$. Consequently,

$$(r_1/r)^3 - 1 = \frac{3}{2}\Bigg(Q_2 + \frac{Q_2^2}{2^2} - \frac{Q_2^3}{3 \cdot 2^3} + \frac{Q_2^4}{4 \cdot 2^4} - \frac{Q_2^5}{4 \cdot 2^5} + \ldots$$

$$+ (-1)^n \frac{(2n - 5)!}{n!(n - 3)!} \frac{1}{2^{2n-4}} Q_2^n + \ldots, \qquad n = 3, 4, 5, \ldots \tag{II-8}$$

$$= F(Q_2).$$

Substitution of (II-8) into (II-2) yields a form similar to (II-5)

$$\frac{1}{r_1^3} \left\{F(Q_2)\ \vec{r}_1 + \Big(1 + F(Q_2)\Big)\ \vec{\rho}\right\}, \tag{II-9}$$

where from (II-7)

$$Q_2 = (r_1/r)^2 - 1 = \frac{r_1^2 - r^2}{r^2} = \frac{r_1^2 - (\vec{r}_1 + \vec{\rho}) \cdot (\vec{r}_1 + \vec{\rho})}{(\vec{r}_1 + \vec{\rho}) \cdot (\vec{r}_1 + \vec{\rho})}$$

$$= -\frac{(2\vec{\rho} \cdot \vec{r}_1 + \vec{\rho} \cdot \vec{\rho})}{\vec{r}_1 \cdot \vec{r}_1 + 2\vec{\rho} \cdot \vec{r}_1 + \vec{\rho} \cdot \vec{\rho}} = -\frac{(\rho^2 + 2\rho\ r_1 \cos\ \theta)}{r_1^2 + \rho^2 + 2\rho\ r_1 \cos\ \theta}$$

$$Q_2 = - \frac{R^2 + 2R \cos \theta}{1 + R^2 + 2R \cos \theta}$$

$$= - \frac{Q_1}{1 + Q_1} . \qquad \text{(II-10)}$$

Referencing (II-5) and (II-9) in the form

$$\frac{1}{r_1^3} \left\{ F \vec{r}_1 + (1 + F) \vec{\rho} \right\} ,$$

the magnitude of the perturbing force of M disturbing the motion of m about m_1 is

$$|\vec{F}_2| = + \sqrt{\vec{F}_2 \cdot \vec{F}_2} = \frac{KM}{r_1^3} \left| F \vec{r}_1 + (1 + F) \vec{\rho} \right| = \frac{KM}{r_1^3} \left| F(\vec{r}_1 + \vec{\rho}) + \vec{\rho} \right|$$

$$= \frac{KM}{r_1^3} \left[F^2 (\vec{r}_1 + \vec{\rho}) \cdot (\vec{r}_1 + \vec{\rho}) + 2F(\vec{r}_1 + \vec{\rho}) \cdot \vec{\rho} + \vec{\rho} \cdot \vec{\rho} \right]^{1/2}$$

$$= \frac{KM}{r_1^3} \left[F^2 (r_1^2 + 2\rho\, r_1 \cos \theta + \rho^2) + 2F(\rho^2 + \rho\, r_1 \cos \theta) + \rho^2 \right]^{1/2}$$

$$= \frac{KM}{r_1^2} \left[F^2 (1 + 2R \cos \theta + R^2) + 2F(R^2 + R \cos \theta) + R^2 \right]^{1/2}$$

$$= \frac{KM}{r_1^2} \left[F^2 (1 + Q_1) + F(Q_1 + R^2) + R^2 \right]^{1/2} ,$$

so that R_2 becomes

$$R_2 = \frac{M}{m_1} R^2 \left[F^2 (1 + Q_1) + F(Q_1 + R^2) + R^2 \right]^{1/2} .$$

Setting $R_1 = R_2$ gives

$$\frac{m_1}{M} r^2 \left(\frac{1}{\rho^4} + \frac{1}{r^4} + \frac{2\vec{\rho} \cdot \vec{r}_1}{\rho^3\, r_1^3}\right)^{1/2} = \frac{M}{m_1} R^2 \left[F^2(1 + Q_1) + F(Q_1 + R^2) + R^2\right]^{1/2}$$

$$(m_1/M)(r_1^2 + \rho^2 + 2\rho\, r_1 \cos\theta) \left(\frac{r_1^4 + \rho^4 + 2\rho^2\, r_1^2 \cos\theta}{\rho^4\, r_1^4}\right)^{1/2} =$$

$$\frac{M}{m_1} R^2 \left[F^2(1 + Q_1) + F(Q_1 + R^2) + R^2\right]^{1/2},$$

or

$$R^4 = (m_1/M)^2 \frac{(1 + R^2 + 2R\cos\theta)(1 + R^4 + 2R^2 \cos\theta)^{1/2}}{[F^2(1 + Q_1) + F(Q_1 + R^2) + R^2]^{1/2}}$$

$$= (m_1/M)^2 \frac{(1 + Q_1)(1 + R^4 + 2R^2 \cos\theta)^{1/2}}{[F^2(1 + Q_1) + 2F(Q_1 + R^2) + R^2]^{1/2}},$$

which is equation (8a) of the text. To apply the Wegstein iteration, this is rewritten as

$$R = \sqrt{m_1/M}\; \frac{(1 + Q_1)^{1/4}(1 + R^4 + 2R^2 \cos\theta)^{1/8}}{[F^2(1 + Q_1) + 2F(Q_1 + R^2) + R^2]^{1/8}}.$$

The $F(Q_1)$ and $F(Q_2)$ series are both alternating series; thus, the truncation errors are less than the first term of the neglected remainder of the series. Simple calculations show that 22 terms of the $F(Q_1)$ series and 12 terms of the $F(Q_2)$ series are sufficient to yield an error less than 10^{-8} if Q_1 and Q_2 are less than or equal to .4. Actually, for the planets, the maximum Q_1 and Q_2 are an order of magnitude smaller than this; therefore retaining the quoted number of terms yields a precision much greater than required. In using these series, Q_1 is always calculated, and $F(Q_1)$ is computed if $Q_1 \geqq 0$ and $F(Q_2)$ is computed if $Q_1 < 0$.

APPENDIX III

Equations (9) and (10) are derived by making use of the fact that $\rho/r_1 < 1$. First, rewrite (7) as

$$\left(\frac{m_1}{M}\right)^2 = \left(\frac{\rho}{r_1}\right)^4 \frac{1}{\left(\frac{r}{r_1}\right)^4} \frac{\left\{1 + \left(\frac{r}{r_1}\right)^4 - 2\frac{r}{r_1}\left[1 + \left(\frac{\rho}{r_1}\right)\cos\theta\right]\right\}^{1/2}}{\left[1 + \left(\frac{\rho}{r_1}\right)^4 + 2\left(\frac{\rho}{r_1}\right)^2 \cos\theta\right]^{1/2}} .$$

Substituting $\frac{r}{r_1} = \left[1 + \left(\frac{\rho}{r_1}\right)^2 + 2\frac{\rho}{r_1}\cos\theta\right]^{1/2}$ and $R = \frac{\rho}{r_1}$ yields

$$\left(\frac{m_1}{M}\right)^2 = \frac{R^4\sqrt{1 + (1 + R^2 + 2R\cos\theta)^2 - 2(1 + R\cos\theta)(1 + R^2 + 2R\cos\theta)^{1/2}}}{(1 + R^2 + 2R\cos\theta)^2\sqrt{1 + R^4 + 2R^2\cos\theta}} .$$

Expanding the radical and applying the binomial theorem several times gives

$$\left(\frac{m_1}{M}\right)^2 = R^4 (1 - 4R\cos\theta + \ldots)\sqrt{R^2(1 + 3\cos^2\theta) + 4R^3\cos\theta + \ldots}$$

$$= R^5 (1 - 4R\cos\theta + \ldots)\sqrt{(1 + 3\cos^2\theta)\left(1 + \frac{4R\cos\theta}{1 + 3\cos^2\theta} + \ldots\right)}$$

$$= R^5 (1 - 4R\cos\theta + \ldots)\sqrt{1 + 3\cos^2\theta}\sqrt{1 + \frac{4R\cos\theta}{1 + 3\cos^2\theta} + \ldots}$$

$$= R^5 \, (1 - 4R \cos\theta + \ldots) \sqrt{1 + 3\cos^2\theta} \left(1 + \frac{2R\cos\theta}{1 + 3\cos^2\theta} + \ldots\right)$$

$$= R^5 \sqrt{1 + 3\cos^2\theta} \left(1 + \frac{2R\cos\theta}{1 + 3\cos^2\theta} - 4R\cos\theta + \ldots\right)$$

$$= R^5 \sqrt{1 + 3\cos^2\theta} \left(1 + \frac{2R\cos\theta - 4R\cos\theta - 12R\cos^3\theta}{1 + 3\cos^2\theta} + \ldots\right)$$

$$= R^5 \sqrt{1 + 3\cos^2\theta} \left(1 - 2R\cos\theta \frac{(1 + 6\cos^2\theta)}{1 + 3\cos^2\theta} + \ldots\right).$$

Thus,

$$R^5 = \left(\frac{m_1}{M}\right)^2 \frac{1}{\sqrt{1 + 3\cos^2\theta}} \left(1 - 2R\cos\theta \frac{(1 + 6\cos^2\theta)}{1 + 3\cos^2\theta} + \ldots\right)^{-1}$$

or

$$R = \left(\frac{m_1}{M}\right)^{2/5} \frac{1}{(1 + 3\cos^2\theta)^{1/10}} \left(1 - 2R\cos\theta \frac{(1 + 6\cos^2\theta)}{1 + 3\cos^2\theta} + \ldots\right)^{-1/5}.$$

Therefore,

$$R = \left(\frac{m_1}{M}\right)^{2/5} \frac{1}{(1 + 3\cos^2\theta)^{1/10}} \left(1 + \frac{2}{5} R\cos\theta \frac{(1 + 6\cos^2\theta)}{(1 + 3\cos^2\theta)} + \ldots\right)$$

which, solved for R, gives

$$R = \frac{\left(\frac{m_1}{M}\right)^{2/5} (1 + 3\cos^2\theta)^{-1/10}}{1 - \left(\frac{m_1}{M}\right)^{2/5} \frac{1}{(1 + 3\cos^2\theta)^{1/10}} \left(\frac{2}{5}\right) (\cos\theta) \frac{(1 + 6\cos^2\theta)}{(1 + 3\cos^2\theta)}} + \ldots$$

so that

$$\rho = \frac{r_1}{\left(\frac{M}{m_1}\right)^{2/5} (1 + 3\cos^2\theta)^{1/10} - \left(\frac{2}{5}\right) (\cos\theta) \frac{(1 + 6\cos^2\theta)}{1 + 3\cos^2\theta} + \ldots}$$

which yields equation (9) and differs from Tisserand's result only in that he expands the denominator in the next to last equation.

APPENDIX IV

Equation (10) is derived as follows. In equation (8), the denominator through second order terms in R becomes $R^2 + 3R^2 \cos^2 \theta$. Neglecting terms higher than first order elsewhere yields

$$R^4 = \left(\frac{m_1}{M}\right)^2 (1 + 4R \cos \theta + \ldots) \frac{1}{\sqrt{R^2(1 + 3 \cos^2 \theta) + \ldots}}$$

or

$$R^5 = \left(\frac{m_1}{M}\right)^2 (1 + 4R \cos \theta + \ldots) (1 + 3 \cos^2 \theta + \ldots)^{-1/2}$$

so that

$$R = \left(\frac{m_1}{M}\right)^{2/5} (1 + 4R \cos \theta + \ldots)^{1/5} (1 + 3 \cos^2 \theta + \ldots)^{-1/10}$$

$$R = \left(\frac{m_1}{M}\right)^{2/5} (1 + \frac{4}{5} R \cos \theta + \ldots) (1 + 3 \cos^2 \theta + \ldots)^{-1/10}.$$

Substituting $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$ gives

$$R = \left(\frac{m_1}{M}\right)^{2/5} (1 + \frac{4}{5} R \cos \theta + \ldots) \left((5/2) (1 + \frac{3}{5} \cos 2\theta) + \ldots\right)^{-1/10}.$$

solving for R yields

$$R = \left(\frac{m_1}{M}\right)^{2/5} \left(\frac{2}{5}\right)^{1/10} \left[(1 + \frac{3}{5} \cos 2\theta)^{1/10} - \left(\frac{2}{5}\right)^{1/10} \left(\frac{4}{5}\right) \left(\frac{m_1}{M}\right)^{2/5} \cos \theta + \ldots\right]^{-1}$$

$$R = \left(\frac{m_1}{M}\right)^{2/5} \left(\frac{2}{5}\right)^{1/10} \left[1 + \frac{3}{50} \cos 2\theta + \ldots - \left(\frac{4}{5}\right)\left(\frac{2}{5}\right)^{1/10} \left(\frac{m_1}{M}\right)^{2/5} \cos \theta + \ldots\right]^{-1} .$$

$$= \left(\frac{2}{5}\right)^{1/10} \left(\frac{m_1}{M}\right)^{2/5} \left[1 + \left(\frac{4}{5}\right)\left(\frac{2}{5}\right)^{1/10} \left(\frac{m_1}{M}\right)^{2/5} \cos \theta - \frac{3}{50} \cos 2\theta + \ldots\right].$$

Substituting $\left(\frac{2}{5}\right)^{1/10} = 0.91244$, $\frac{4}{5} = 0.8$, and $\frac{3}{50} = .06$ yields equation (10).

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By Roger R. Burrows

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