A RIGOROUS DERIVATION OF SECOND-APPROXIMATION THEORY OF ELASTIC SHELLS

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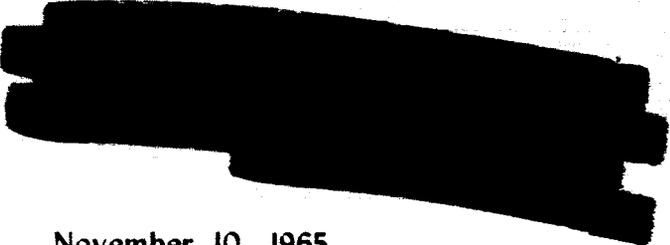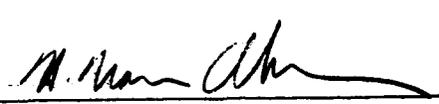Contract NASr-94(06)
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ABSTRACT

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A rigorous derivation and new analytical viewpoint of linear shell theory are presented which aim at resolving some fundamental difficulties in elastic shell theory. The approach is based on the concept that the stress and displacement components in three-dimensional elasticity can be expanded into infinite series of the Legendre polynomials of a dimensionless thickness variable, which converge uniformly and rapidly in the thickness interval. The shell equations are derived through integration of the linear elasticity equations. The orthogonality property of the Legendre polynomials uncouples most higher order terms during the integration process. A minimum number of assumptions are then introduced after the integration and only when necessary. The a priori Kirchhoff-Love hypothesis is replaced by a more rigorous accuracy criteria.

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NOMENCLATURE

A, B		Metric or Lamé coefficients of the midsurface of the shell
C	=	$Eh/(1 - v^2)$, extensional rigidity
D	=	$Eh^3/12(1 - v^2)$, flexural rigidity
E		elastic modulus
e_{ij}		strain tensor
H_i		Metric or Lamé coefficients or orthogonal curvilinear coordinates
h		thickness
i, j	=	1, 2, 3, indices corresponding to x, y, z or ξ, η, ζ respectively
M_1, M_{12}, etc.		couple-resultants
N_1, N_{12}, etc.		stress-resultants
n		nonnegative integers
P_n		Legendre polynomials of degree n
Q_1, Q_2		transverse shearing stress-resultants
R_1, R_2		principal radii of curvature of the midsurface
R		minimum radius of curvature of the midsurface
t		time
U, V, W		displacement components
u, v, w		plate or shell displacement components
u_n, v_n, w_n		nth coefficients of the Legendre series of U, V, W, respectively
x, y, z		Cartesian coordinates
β_1, β_2		average rotation of a normal

ϵ_1, ϵ_2		expressions of u, v, w
ζ	$=$	$z/(h/2)$ dimensionless thickness variable
ξ, η		curvilinear coordinates along lines of principal curvature
κ		shear constant
$\kappa_1, \kappa_2, \kappa_1^*, \kappa_2^*$		expressions of β_1, β_2 and w_1
ν		Poisson's ratio
ρ		mass density
σ_{ij}		stress tensor
$\tilde{\sigma}_{ij}$		weighted stress components
$\sigma_{ij}^{(n)}, \tilde{\sigma}_{ij}^{(n)}$		n^{th} coefficient of the Legendre series of σ_{ij} and $\tilde{\sigma}_{ij}$, respectively
$\sigma_{3j}^+, \sigma_{3j}^-$		loading functions at $z = h/2$ and $z = -h/2$, respectively
τ_1, τ_2		expressions of β_1 and β_2
Φ_1, Φ_2		expressions of U, V, W
Ψ_1, Ψ_2		expressions of U, V
ω_1, ω_2		expressions of u and v

INTRODUCTION

As a branch of the well-established theory of elasticity, the theory of thin elastic shells has been ironically and persistently defying a satisfactory derivation, free from unnecessary assumptions, approximations and inconsistency. Despite the considerable amount of reexamination and rederivation of the shell equations made by numerous authors, some basic difficulties and questions[1*] remain unsolved in the foundations of the subject. Research workers in many fields dealing with shell-type structures may find themselves confronting scores of linear shell theories which differ more or less from each other. The choice among these theories has been left mainly to personal preference because of the lack of a universally accepted criterion for the numerous, explicit and implicit assumptions involved.

The fact that elastic shell theory is a natural generalization of the membrane theory has unfortunately led most authors to treat the thin shell as an elastic "surface" with added bending rigidity, and to give little emphasis to the fact that the shell thickness, though small compared to other dimensions, is a _finite_ quantity; in other words, the shell space has been considered as a small "neighborhood" of its midsurface; thereof, a direct connection with the three-dimensional elasticity has never been

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*Superscripts refer to references cited at the end of this paper.

established. Under this mathematical model, the elegance of the surface differential geometry, and more recently, of the tensor calculus, has diverted most theoretical attention to the abstract construction of the equations themselves, while leaving the gap between the shell theory and the three-dimensional elasticity in obscurity. Some recent development in attempting to derive shell equations through the general theory of Cosserat surfaces[2,3] is an evidence of this trend.

While numerous papers have been devoted to the development of the linear shell theory[1,4,5,6], the basic approaches employed in these derivations can be generally summarized into the following four categories.

(1) Direct approach, as used by Love[7], Flügge[8], and other pioneer investigators, who worked directly on a differential element of the midsurface in deriving equilibrium and kinematic equations.

(2) Variational methods, as elaborated by E. Reissner[9,10], Naghdi[1,11,12], Koiter[13], Saunders[14], and many other authors who favor the energy methods in which, once the basic assumptions and strain energy expression are set forth, no further approximations or inconsistencies may creep into the derivation. In addition, all the field equations and boundary conditions are obtained in a single variational process, which provides some confidence of analytical unity. However, the question, "How satisfactory are the set of initial assumptions and the strain energy expression for the shell?" remains to be examined by more basic formulations. Literature surveys[1,4,5,6] indicate that the variational methods have been by far the most fruitful approach in the rederivation of shell theories.

(3)   Method of parametric expansion and asymptotic integration, as used by Johnson and E. Reissner[15], Reiss[16], Green[17], E. Reissner[18], Gol'denveizer[19], etc. In this method, appropriate quantities in three-dimensional elasticity are expanded into power series of some small thickness-parameter, then the shell equations are derived by asymptotic integration.   The results from this approach are admittedly not encouraging both from analytical and from applicational viewpoints.

(4)   Method of Taylor series expansion, as employed by Vlasov[20], Kennard[21], and very recently by Kil'chevskiy[28]. This method can be evidently traced back to the plate theory due to Cauchy and Poisson*, which is based on the expansion of the displacements and stresses in power series of the thickness coordinate z.   As pointed out by St. Venant,* the corresponding series will, as a rule, diverge eventually. From an analytical viewpoint, the region and nature of the convergence of the Taylor series in these derivations are indeed uncertain. Furthermore, as remarked by Naghdi[5], this process is essentially "regarding the system of three-dimensional equations as defining an initial-value problem (the middle surface being the initial manifold);'' while in the present author's opinion, they really define a boundary-value problem in the interval $-h/2 \leqslant z \leqslant h/2$.

Bearing in mind the intrinsic relations between the derivation of a shell theory and the solution of a boundary-value problem, one naturally

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*Refer to the historic remarks by Love[7] and by Novozhilov[22]. Cauchy and and Poisson's plate theory was later superseded by Kirchhoff's plate theory.

turns to the various methods of solving a boundary-value problem, among which the method using series expansion in orthogonal functions outstands. It can be easily seen that by using orthogonal functions, the two major difficulties in the Taylor series method are completely removed: First, while the Taylor series are "point expansions" and converge only within some small neighborhood, the series of orthogonal functions are "interval expansions," which converge uniformly in the entire interval of interest. Second, while the terms of a power series are not orthogonal to each other (thus a priori assumptions, i.e., series truncations have to be introduced before the integration of the three-dimensional equations), the orthogonality property in the new approach uncouples most high-order terms during the integration process, and assumptions are then introduced after the integration and only when necessary.

In the following sections, the new derivation and analytical view-point of linear shell theory are presented, based on the concept that the stresses and displacements in three-dimensional elasticity can be expanded into series of a selected set of orthogonal functions, which converge rapidly in the thickness interval. The a priori Kirchhoff-Love hypothesis is replaced by more rigorous accuracy criteria. Although the choice of the coordinate functions can be arbitrary as long as they form a complete, orthogonal set, the Legendre polynomials evidently are the most natural and convenient ones, since, as will be seen, they preserve the definitions of the stress- and couple-resultants.

For the purpose of clarity, we will first illustrate the new approach by the simpler case of the flat plate, and show that both Mindlin's plate theory[23] and classical plate theory can be reproduced without using Kirchhoff hypothesis on displacements (Eq. 10, Ref. 23). In the subsequent sections, a second-approximation shell theory is derived and discussed; then, a first-approximation theory will be proposed in the spirit of Love's shell theory[7].

PLATE THEORY AS TRUNCATED LEGENDRE SERIES SOLUTION

Consider a flat plate referring to Cartesian coordinates x, y, z. The faces of the plate are the planes $z = \pm h/2$, where h denotes the constant thickness. Under arbitrary dynamic load on the faces, the exact elasticity solutions of the problem are completely described by nine quantities, namely, six components of the <u>symmetric</u> stress tensor σ_{ij}, where i, j = 1, 2, 3 (corresponding to the three directions x, y, z, respectively), and three displacement components U, V, W; all of the nine quantities are, in general, functions of four independent variables (x, y, z, t). If we use the superscripts "+" to denote the values of the corresponding functions at z = h/2, and "-" to denote those at z = -h/2, then the boundary conditions at the faces $z = \pm h/2$ are given by the prescribed loads:

$$\text{at } z = h/2, \quad \sigma_{3j} = \sigma_{3j}^{+} \quad , \quad j = 1, 2, 3 \tag{1}$$

$$\text{at } z = -h/2, \quad \sigma_{3j} = \sigma_{3j}^{-} \quad , \quad j = 1, 2, 3$$

In addition to (1) and the boundary conditions at the plate edges, the nine quantities must satisfy nine field equations*, namely, the three equations of motion:

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*Since the six strain components can be very easily eliminated, they are excluded to avoid complication. Also, the body forces are not included in the equations of motion, but their inclusion does not change the derivation essentially.

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{21}}{\partial y} + \frac{\partial \sigma_{31}}{\partial z} = \rho \ddot{U} \tag{2a}$$

$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{32}}{\partial z} = \rho \ddot{V} \tag{2b}$$

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} = \rho \ddot{W} \tag{2c}$$

where dot denotes time differentiation, and the six stress-displacement relations,

$$\frac{\partial U}{\partial x} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) \tag{3a}$$

$$\frac{\partial V}{\partial y} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11} - \nu\sigma_{33}) \tag{3b}$$

$$\frac{\partial W}{\partial z} = \frac{1}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) \tag{3c}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{2(1+\nu)}{E}\sigma_{12} \tag{4a}$$

$$\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = \frac{2(1+\nu)}{E}\sigma_{23} \tag{4b}$$

$$\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = \frac{2(1+\nu)}{E}\sigma_{13} \tag{4c}$$

Now if we define the dimensionless variable $\zeta = z/(h/2)$, then $-1 \leqslant \zeta \leqslant 1$ is the thickness interval; the nine quantities can be expanded[24] into infinite series of Legendre polynomials of ζ

$$\sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)} P_n(\zeta) \quad , \qquad i, j = 1, 2, 3 \tag{5}$$

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} P_n(\zeta) \tag{6}$$

where the coefficients are functions of (x, y, t), and are defined by[24]

$$\sigma_{ij}^{(n)} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} \sigma_{ij} P_n(\zeta) d\zeta \quad , \qquad i, j = 1, 2, 3 \tag{7}$$

$$\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} P_n(\zeta) d\zeta \tag{8}$$

Note that $P_0(\zeta) = 1$, $P_1(\zeta) = \zeta$, $P_2(\zeta) = \frac{1}{2}(3\zeta^2 - 1)$, etc. It is easily seen that, within a constant factor, the plate stress- and couple-resultants are nothing but the coefficients $\sigma_{ij}^{(0)}$ and $\sigma_{ij}^{(1)}$

$$N_1 = \int_{-h/2}^{h/2} \sigma_{11} dz = h\sigma_{11}^{(0)}$$

$$N_{12} = \int_{-h/2}^{h/2} \sigma_{12} dz = h\sigma_{12}^{(0)} = N_{21}$$

$$Q_1 = \int_{-h/2}^{h/2} \sigma_{13} dz = h\sigma_{13}^{(0)} \tag{9}$$

$$M_1 = \int_{-h/2}^{h/2} \sigma_{11} z \, dz = \frac{h^2}{6} \sigma_{11}^{(1)}$$

$$M_{12} = \int_{-h/2}^{h/2} \sigma_{12} z \, dz = \frac{h^2}{6} \sigma_{12}^{(1)} = M_{21} \quad , \quad \text{etc.}$$

We now define the "plate displacements" by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \equiv \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \frac{1}{h} \int_{-h/2}^{h/2} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} dz \tag{10}$$

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \equiv \begin{Bmatrix} \dfrac{u_1}{h/2} \\ \dfrac{v_1}{h/2} \end{Bmatrix} = \frac{12}{h^3} \int_{-h/2}^{h/2} \begin{Bmatrix} U \\ V \end{Bmatrix} z \, dz \tag{11}$$

It is seen that β_1 and β_2 are identical to E. Reissner's[25] "average change of slope of the normal," but w is somewhat different from the "weighted average" of W used in his variational derivation[25].

Theoretically, we can multiply Eqs. (2), (3), and (4) by $P_n(\zeta)$, $n = 0 \sim \infty$, then integrate through the thickness, to obtain an infinite number of equations governing all the coefficients in (5) and (6). The purpose of deriving a plate theory is to obtain a determinate set of equations governing the plate stresses and plate displacements defined in (9), (10), and (11).

Exact Equations of Motion for Plates

Multiplying (2) by $P_0(\zeta)$ and integrating through thickness, using the definitions (9) and (10), we get

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_{12}}{\partial y} + (\sigma_{31}^+ - \sigma_{31}^-) = \rho h \ddot{u} \tag{12a}$$

$$\frac{\partial N_{12}}{\partial x} + \frac{\partial N_2}{\partial y} + (\sigma_{32}^+ - \sigma_{32}^-) = \rho h \ddot{v} \tag{12b}$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + (\sigma_{33}^+ - \sigma_{33}^-) = \rho h \ddot{w}$$

Multiplying (2a, b) by $P_1(\zeta)$ then integrating through the thickness (with the help of integration by parts), using the definitions (9) and (11), we get:

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_{12}}{\partial y} - Q_1 + \frac{h}{2}(\sigma_{31}^+ + \sigma_{31}^-) = \frac{\rho h^3}{12}\ddot{\beta}_1 \tag{13a}$$

$$\frac{\partial M_{12}}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 + \frac{h}{2}(\sigma_{32}^+ + \sigma_{32}^-) = \frac{\rho h^3}{12}\ddot{\beta}_2 \tag{13b}$$

These five equations of motion are exact, which govern the _five_ degrees
of freedom of the _gross_ motion of a "needle-like" differential element.
All higher equations of motion, which can be obtained similarly, describe
more complicated _elastic_ motion of the normal fibre, for example, the
sixth equation of motion governs the thickness stretch mode. Some
authors (e. g. , Ref. 20) have attempted to include an equation of motion
governing the rotation of the differential element about the z-azis; that
this is a trivial identity is evident if we recall that the differential element
has zero moment of inertia about the z-axis.

Constitutive Equations for Plates

Multiplying (3) by $P_0(\zeta)$ and (3a, b) by $P_1(\zeta)$, and then integrating
through the thickness, we get

$$\frac{\partial u}{\partial x} = \frac{1}{Eh}\left[N_1 - \nu N_2 - \nu h\sigma_{33}^{(0)}\right]$$

$$\frac{\partial v}{\partial y} = \frac{1}{Eh}\left[N_2 - \nu N_1 - \nu h\sigma_{33}^{(0)}\right]$$

$$W^+ - W^- = \frac{1}{E}\left[-\nu(N_1 + N_2) + h\sigma_{33}^{(0)}\right] \tag{14}$$

$$\frac{\partial \beta_1}{\partial x} = \frac{12}{Eh^3}\left[M_1 - \nu M_2 - \frac{\nu h^2}{6}\sigma_{33}^{(1)}\right]$$

$$\frac{\partial \beta_2}{\partial y} = \frac{12}{Eh^3}\left[M_2 - \nu M_1 - \frac{\nu h^2}{6}\sigma_{33}^{(1)}\right]$$

Since $\sigma_{33}^{(0)}$ and $\sigma_{33}^{(1)}$, which represent the effects of thickness normal stress, are not included in the category "plate stresses", Eq. (9), we shall introduce the "generalized plane-stress assumption".

Plate Assumption 1: For a thin plate, the normal stress σ_{33} has the following properties*

$$\sigma_{33}^{(0)} << \sigma_{11}^{(0)} \text{ or } \sigma_{22}^{(0)}$$

$$\sigma_{33}^{(1)} << \sigma_{11}^{(1)} \text{ or } \sigma_{22}^{(1)}$$
(15)

With the help of Assumption 1, (14) can be rearranged as follows:

$$N_1 = C\left(\frac{\partial u}{\partial x} + \nu\,\frac{\partial v}{\partial y}\right)$$
(16a)

$$N_2 = C\left(\frac{\partial v}{\partial y} + \nu\,\frac{\partial u}{\partial x}\right)$$
(16b)

$$M_1 = D\left(\frac{\partial \beta_1}{\partial x} + \nu\,\frac{\partial \beta_2}{\partial y}\right)$$
(17a)

$$M_2 = D\left(\frac{\partial \beta_2}{\partial y} + \nu\,\frac{\partial \beta_1}{\partial x}\right)$$
(17b)

$$W^+ - W^- = -\frac{\nu h}{1 - \nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
(18)

where $C = Eh/(1 - \nu^2)$, $D = Eh^3/12(1 - \nu^2)$. From (18), it is seen that Assumption 1 does not imply that the thickness change is negligible (for $\nu = 0.3$, it is nearly -43% of the sum of membrane strains; however, it

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*Note, a much better approximation can be obtained by assuming $\sigma_{33}^{(0)} = (\sigma_{33}^+ + \sigma_{33}^-)/2$ and $\sigma_{33}^{(1)} = (\sigma_{33}^+ - \sigma_{33}^-)/2$, which may be called Assumption 1 for improved plate theory The same concept applies to shell theory also.

will be found that (18) is uncoupled from the set of plate equations governing flexural motion. This uncoupling is unfortunately not true for shell theory.

Now multiplying (14a) by $P_0(\zeta)$ and $P_1(\zeta)$, then integrating through the thickness, we get

$$N_{12} = N_{21} = \frac{1-\nu}{2} C \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{19}$$

$$M_{12} = M_{21} = \frac{1-\nu}{2} D \left(\frac{\partial \beta_1}{\partial y} + \frac{\partial \beta_2}{\partial x} \right) \tag{20}$$

To determine the equations for the transverse shear stress-resultants Q_1 and Q_2, we simply integrate (4b, c) through the thickness,

$$Q_1 = \frac{1-\nu}{2} C \left[\frac{\partial w}{\partial x} + \frac{1}{h} (U^+ - U^-) \right] \tag{21a}$$

$$Q_2 = \frac{1-\nu}{2} C \left[\frac{\partial w}{\partial y} + \frac{1}{h} (V^+ - V^-) \right] \tag{21b}$$

Equations (21a, b) contain the undesirable quantities $(U^+ - U^-)$ and $(V^+ - V^-)$, which must be removed by some means. Two possible ways will be used, one leads to the "shear-constant plate theory" similar to those derived by E. Reissner[25] and by Mindlin[23]; the other leads to the classical plate theory due to Kirchhoff.

(A) Shear-Constant Plate Theory

Since $P_n(1) = 1$, and $P_n(-1) = (-1)^n$, we have, from (6),

$$U^+ - U^- = 2(u_1 + u_3 + u_5 + \ldots)$$

$$V^+ - V^- = 2(v_1 + v_3 + v_5 + \ldots)$$

Substituting these into (21a, b), using (11), we get

$$Q_1 = \frac{1-\nu}{2} C \left[\frac{\partial w}{\partial x} + \beta_1 + \frac{2}{h} (u_3 + u_5 + \dots) \right] \tag{22a}$$

$$Q_2 = \frac{1-\nu}{2} C \left[\frac{\partial w}{\partial y} + \beta_2 + \frac{2}{h} (v_3 + v_5 + \dots) \right] \tag{22b}$$

Now we introduce the "shear-constant assumption."

Plate Assumption 2A. The normals of a thin plate remain <u>nearly</u> <u>straight</u> after deformation, and the <u>antisymmetric</u> part of the slight deviation from a straight line can be approximated by

$$u_3 + u_5 + \dots = \frac{\kappa - 1}{2} \left(\frac{\partial w}{\partial x} + \beta_1 \right) \tag{23a}$$

$$v_3 + v_5 + \dots = \frac{\kappa - 1}{2} \left(\frac{\partial w}{\partial y} + \beta_2 \right) \tag{23b}$$

where κ is a shear constant to be determined by some physical argument[23]. Substituting (23) into (22), we get

$$Q_1 = \frac{1-\nu}{2} C\kappa \left(\frac{\partial w}{\partial x} + \beta_1 \right) \tag{24a}$$

$$Q_2 = \frac{1-\nu}{2} C\kappa \left(\frac{\partial w}{\partial y} + \beta_2 \right) \tag{24b}$$

Altogether, (12), (13), (16), (17), (19), (20), and (24), consist of a determinate set of thirteen plate equations for thirteen unknowns.

From the above analysis, it becomes clear that the shear constant κ, which has been introduced by Timoshenko, E. Reissner, Mindlin, etc., through physical intuition, does not arise from some mysterious source, rather it is a remedy to account for the fact that normals do <u>not</u> remain straight. Strictly speaking, κ should be a different function of (x, y, t) in (23a) and in (23b), and also depend on loading, but its variation has been

found to be very small. According to the variational derivation due to E. Reissner, $\kappa = 5/6$, while according to a limiting process concerning flexural waves, Mindlin[23] found

$$\kappa \approx 0.76 + 0.30\,\nu$$

For $\nu = 0.3$, this gives $\kappa \approx 0.85$, which agrees favorably with Reissner's value 0.833.

(B) Classical Plate Theory

If we neglect the "rotatory inertia" terms in (13) to get

$$Q_1 = \frac{\partial M_1}{\partial x} + \frac{\partial M_{12}}{\partial y} + \frac{h}{2}(\sigma_{31}^+ + \sigma_{31}^-) \tag{25a}$$

$$Q_2 = \frac{\partial M_{12}}{\partial x} + \frac{\partial M_2}{\partial y} + \frac{h}{2}(\sigma_{32}^+ + \sigma_{32}^-) \tag{25b}$$

and make an alternative assumption to express β_1 and β_2 in terms of w, we obtain the classical plate theory, from which, the two undesirable equations (21a, b) are <u>uncoupled</u>.

<u>Plate Assumption 2B</u>. The plate displacements obey the following approximate relations

$$\beta_1 = -\frac{\partial w}{\partial x}, \qquad \beta_2 = -\frac{\partial w}{\partial y} \tag{26}$$

By this assumption, we can eliminate β_1 and β_2 from (17) and (20).

$$M_1 = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{27a}$$

$$M_2 = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{27b}$$

$$M_{12} = -(1-\nu)D \cdot \frac{\partial^2 w}{\partial x\partial y} \tag{28}$$

Altogether, (12), (25), (16), (27), (19), and (28) consist of a determinate set of eleven equations for eleven unknowns.

It should be noted that, in both plate theories, the in-plane motions (determined by N_1, N_2, N_{12}, and u, v) are always uncoupled from the flexural motions. This uncoupling again does not exist in the shell theory.

SECOND-APPROXIMATION SHELL THEORY

In the previous section, the derivation procedure is illustrated by deriving the plate theory in Cartesian coordinates (x, y, z). Now we proceed to derive a consistent linear shell theory in orthogonal, curvilinear coordinates (ξ, η, ζ), which are chosen to be the lines of principal curvature of the shell midsurface and its normals. For convenience, we shall use the same notation as before, since the plate is merely a special case of the curved shell. The thickness variable z (distance from midsurface) will be used freely and interchangeably with ζ, whichever is more convenient, since they differ only by a constant factor.

The shell space is defined as the space bounded by the two curved surface $\zeta = \pm 1$, or equivalently $z = \pm h/2$, where h is the constant thickness, and by the edges of the shell, if any. Under general dynamic loads on the two faces, $\sigma_{3j}^+(\xi, \eta, t)$ on the face $z = h/2$, and $\sigma_{3j}^-(\xi, \eta, t)$ on the face $z = -h/2$, the exact elasticity solution, as before, are completely determined by nine scalar quantities, the symmetric stress tensor σ_{ij}, i, j = 1, 2, 3 (referring to the curvilinear coordinates ξ, η, ζ, respectively), and U, V, W, which are, in general, functions of four independent variables (ξ, η, ζ, t).

The first quadratic form of the orthogonal coordinates can be written:

$$ds^2 = H_1^2 d\xi^2 + H_2^2 d\eta^2 + H_3^2 dz^2$$

$$\equiv A^2 \left(1 + \frac{z}{R_1}\right)^2 d\xi^2 + B^2 \left(1 + \frac{z}{R_2}\right)^2 d\eta^2 + dz^2 \qquad (29)$$

where H_1, H_2 and H_3 are the metric or Lamé coefficients, A and B the metric coefficients of the midsurface, R_1 and R_2 the principal radii of curvature. For future reference we note the formulas

$$\frac{\partial H_1}{\partial \eta} = \left(1 + \frac{z}{R_2}\right) \frac{\partial A}{\partial \eta}$$

$$\frac{\partial H_2}{\partial \xi} = \left(1 + \frac{z}{R_1}\right) \frac{\partial B}{\partial \xi} \qquad (30)$$

which may be deduced from the Mainardi-Codazzi relations[11]. The nine field equations.* corresponding to (2), (3), and (4), in the curvilinear coordinates may be found from Ref. 7 or 26, namely, the three equations of motion

$$\frac{\partial(H_2 \sigma_{11})}{\partial \xi} + \frac{\partial(H_1 \sigma_{21})}{\partial \eta} + \frac{\partial(H_1 H_2 \sigma_{31})}{\partial z} +$$

$$+ \sigma_{12} \frac{\partial H_1}{\partial \eta} + H_2 \sigma_{13} \frac{\partial H_1}{\partial z} - \sigma_{22} \frac{\partial H_2}{\partial \xi} = \rho H_1 H_2 \ddot{U} \qquad (31a)$$

$$\frac{\partial(H_2 \sigma_{12})}{\partial \xi} + \frac{\partial(H_1 \sigma_{22})}{\partial \eta} + \frac{\partial(H_1 H_2 \sigma_{32})}{\partial z} +$$

$$+ \sigma_{21} \frac{\partial H_2}{\partial \xi} + H_1 \sigma_{23} \frac{\partial H_2}{\partial z} - \sigma_{11} \frac{\partial H_1}{\partial \eta} = \rho H_1 H_2 \ddot{V} \qquad (31b)$$

*As before, the six strain components e_{ij} are eliminated to avoid complication, and the body forces are not included.

$$\frac{\partial(H_2\sigma_{13})}{\partial\xi} + \frac{\partial(H_1\sigma_{23})}{\partial\eta} + \frac{\partial(H_1H_2\sigma_{33})}{\partial z}$$

$$- H_2\sigma_{11}\frac{\partial H_1}{\partial z} - H_1\sigma_{22}\frac{\partial H_2}{\partial z} = \rho H_1 H_2 \ddot{W} \qquad (31c)$$

and the six stress-displacement relations

$$(e_{11} =)\frac{1}{1+z/R_1}\left[\frac{1}{A}\frac{\partial U}{\partial\xi} + \frac{1}{AB}\frac{\partial A}{\partial\eta}V + \frac{W}{R_1}\right] = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) \qquad (32a)$$

$$(e_{22} =)\frac{1}{1+z/R_2}\left[\frac{1}{B}\frac{\partial V}{\partial\eta} + \frac{1}{AB}\frac{\partial B}{\partial\xi}U + \frac{W}{R_2}\right] = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11} - \nu\sigma_{33}) \qquad (32b)$$

$$(e_{33} =)\frac{\partial W}{\partial z} = \frac{1}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) \qquad (32c)$$

$$(2e_{12} =) \frac{1}{1+z/R_1}\left[\frac{1}{A}\frac{\partial V}{\partial\xi} - \frac{1}{AB}\frac{\partial A}{\partial\eta}U\right] +$$

$$+ \frac{1}{1+z/R_2}\left[\frac{1}{B}\frac{\partial U}{\partial\eta} - \frac{1}{AB}\frac{\partial B}{\partial\xi}V\right] = \frac{2(1+\nu)}{E}\sigma_{12} \qquad (33a)$$

$$(2e_{13} =)\frac{\partial U}{\partial z} + \frac{1}{1+z/R_1}\left[\frac{1}{A}\frac{\partial W}{\partial\xi} - \frac{U}{R_1}\right] = \frac{2(1+\nu)}{E}\sigma_{13} \qquad (33b)$$

$$(2e_{23} =)\frac{\partial V}{\partial z} + \frac{1}{1+z/R_2}\left[\frac{1}{B}\frac{\partial W}{\partial\eta} - \frac{V}{R_2}\right] = \frac{2(1+\nu)}{E}\sigma_{23} \qquad (33c)$$

If we let

$$\Phi_1 = \frac{1}{A}\frac{\partial U}{\partial\xi} + \frac{1}{AB}\frac{\partial A}{\partial\eta}V + \frac{W}{R_1}$$

$$\Phi_2 = \frac{1}{B}\frac{\partial V}{\partial\eta} + \frac{1}{AB}\frac{\partial B}{\partial\xi}U + \frac{W}{R_2}$$

$$\Psi_1 = \frac{1}{A} \frac{\partial V}{\partial \xi} - \frac{1}{AB} \frac{\partial A}{\partial \eta} U$$

$$\Psi_2 = \frac{1}{B} \frac{\partial U}{\partial \eta} - \frac{1}{AB} \frac{\partial B}{\partial \xi} V$$

then (32a, b) and (33a) can be written as

$$\sigma_{11} = \frac{E}{1 - \nu^2} \left[\frac{1}{1 + z/R_1} \Phi_1 + \frac{\nu}{1 + z/R_2} \Phi_2 \right] + \frac{\nu}{1 - \nu} \sigma_{33} \tag{32a'}$$

$$\sigma_{22} = \frac{E}{1 - \nu^2} \left[\frac{1}{1 + z/R_2} \Phi_2 + \frac{\nu}{1 + z/R_1} \Phi_1 \right] + \frac{\nu}{1 - \nu} \sigma_{33} \tag{32b'}$$

$$\sigma_{12} = \frac{E}{2(1 + \nu)} \left[\frac{1}{1 + z/R_1} \Psi_1 + \frac{1}{1 + z/R_2} \Psi_2 \right] \tag{33a'}$$

Guided by the form of the equations of motion, and by the definitions of the shell stress- and couple-resultants, we define the "weighted stress components" $\tilde{\sigma}_{ij}$ by

$$\tilde{\sigma}_{1j} = \sigma_{1j}(1 + z/R_2),$$

$$\tilde{\sigma}_{2j} = \tilde{\sigma}_{2j}(1 + z/R_1), \qquad\qquad j = 1, 2, 3 \tag{34}$$

$$\tilde{\sigma}_{3j} = \tilde{\sigma}_{3j}(1 + z/R_1)(1 + z/R_2),$$

Note that $\tilde{\sigma}_{ij}$ is no longer symmetric. With the help of (30), introduction of (34) into (31) yields the equations of motion in the weighted stresses.

$$\frac{\partial(B\tilde{\sigma}_{11})}{\partial \xi} + \frac{\partial(A\tilde{\sigma}_{21})}{\partial \eta} + \frac{\partial(AB\tilde{\sigma}_{31})}{\partial z} + \tilde{\sigma}_{12} \frac{\partial A}{\partial \eta} +$$

$$+ \frac{AB}{R_1} \tilde{\sigma}_{13} - \tilde{\sigma}_{22} \frac{\partial B}{\partial \xi} = \rho AB \left(1 + \frac{z}{R_1}\right) \left(1 + \frac{z}{R_2}\right) \ddot{U} \tag{35a}$$

$$\frac{\partial(B\tilde{\sigma}_{12})}{\partial\xi} + \frac{\partial(A\tilde{\sigma}_{22})}{\partial\eta} + \frac{\partial(AB\tilde{\sigma}_{32})}{\partial z} + \tilde{\sigma}_{21} \frac{\partial B}{\partial\xi} +$$

$$+ \frac{AB}{R_2} \tilde{\sigma}_{23} - \tilde{\sigma}_{11} \frac{\partial A}{\partial\eta} = \rho AB \left(1 + \frac{z}{R_1}\right) \left(1 + \frac{z}{R_2}\right) \ddot{V} \qquad (35b)$$

$$\frac{\partial(B\tilde{\sigma}_{13})}{\partial\xi} + \frac{\partial(A\tilde{\sigma}_{23})}{\partial\eta} + \frac{\partial(AB\tilde{\sigma}_{33})}{\partial z}$$

$$- \frac{AB}{R_1} \tilde{\sigma}_{11} - \frac{AB}{R_2} \tilde{\sigma}_{22} = \rho AB \left(1 + \frac{z}{R_1}\right) \left(1 + \frac{z}{R_2}\right) \ddot{W} \qquad (35c)$$

Similarly, the stress-displacement relations can be written

$$\tilde{\sigma}_{11} = \frac{E}{1 - \nu^2}\left[\frac{1 + z/R_2}{1 + z/R_1} \Phi_1 + \nu\Phi_2\right] + \frac{\nu}{1 - \nu} \frac{1}{1 + z/R_1} \tilde{\sigma}_{33} \qquad (36a)$$

$$\tilde{\sigma}_{22} = \frac{E}{1 - \nu^2}\left[\frac{1 + z/R_1}{1 + z/R_2} \Phi_2 + \nu\Phi_1\right] + \frac{\nu}{1 - \nu} \frac{1}{1 + z/R_2} \tilde{\sigma}_{33} \qquad (36b)$$

$$\frac{\partial W}{\partial z} = \frac{1}{E}\left[\frac{1}{(1 + z/R_1)(1 + z/R_2)} \tilde{\sigma}_{33} - \frac{\nu}{1 + z/R_2} \tilde{\sigma}_{11} - \frac{\nu}{1 + z/R_1} \tilde{\sigma}_{22}\right]$$
$$(36c)$$

$$\tilde{\sigma}_{12} = \frac{E}{2(1 + \nu)}\left[\frac{1 + z/R_2}{1 + z/R_1} \Psi_1 + \Psi_2\right] \qquad (37a)$$

$$\tilde{\sigma}_{21} = \frac{E}{2(1 + \nu)}\left[\Psi_1 + \frac{1 + z/R_1}{1 + z/R_2} \Psi_2\right] \qquad (37b)$$

$$\tilde{\sigma}_{13} = \frac{E}{2(1 + \nu)}\left[\frac{1 + z/R_2}{1 + z/R_1}\left(\frac{1}{A} \frac{\partial W}{\partial\xi} - \frac{U}{R_1}\right) - \frac{U}{R_2} + \frac{\partial(1 + z/R_2)U}{\partial z}\right] \qquad (38a)$$

$$\tilde{\sigma}_{23} = \frac{E}{2(1 + \nu)}\left[\frac{1 + z/R_1}{1 + z/R_2}\left(\frac{1}{B} \frac{\partial W}{\partial\eta} - \frac{V}{R_2}\right) - \frac{V}{R_1} + \frac{\partial(1 + z/R_1)V}{\partial z}\right] \qquad (38b)$$

The other two equations for $\tilde{\sigma}_{31}$ and $\tilde{\sigma}_{32}$ are not needed.

Now, analogously to the derivation of plate theory, we expand the weighted stresses into infinite series of Legendre polynomials of ζ.

$$\tilde{\sigma}_{ij} = \sum_{n=0}^{\infty} \tilde{\sigma}_{ij}^{(n)} P_n(\zeta), \quad i, j = 1, 2, 3 \tag{39}$$

where

$$\tilde{\sigma}_{ij}^{(n)} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} \tilde{\sigma}_{ij} P_n(\zeta) \, d\zeta \tag{40}$$

It is seen that, within a constant factor, the shell stress- and couple-resultants are nothing but the ten coefficients $\tilde{\sigma}_{ij}^{(0)}$ (i = 1, 2; j = 1, 2, 3) and $\tilde{\sigma}_{ij}^{(1)}$ (i, j = 1, 2):

$$\begin{Bmatrix} N_1 \\ N_{12} \\ Q_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \left(1 + \frac{z}{R_2}\right) dz = \begin{Bmatrix} h\tilde{\sigma}_{11}^{(0)} \\ h\tilde{\sigma}_{12}^{(0)} \\ h\tilde{\sigma}_{13}^{(0)} \end{Bmatrix} \tag{41}$$

$$\begin{Bmatrix} M_1 \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \end{Bmatrix} \left(1 + \frac{z}{R_2}\right) z \, dz \begin{Bmatrix} \frac{h^2}{6}\tilde{\sigma}_{11}^{(1)} \\ \frac{h^2}{6}\tilde{\sigma}_{12}^{(1)} \end{Bmatrix} \tag{42}$$

and others obtained by interchanging the subscripts 1 and 2. The expansions of the displacements U, V, W and the definition of the five shell displacements u, v, w, β_1, β_2 are the same as (6), (8), (10), and (11).

From the form of the Eqs. (35) through (38), it is evident that the integration process will be much more lengthier than in the case of the flat plate.

However, the basic principle is the same, namely, to uncouple all the higher coefficients and to obtain a set of determinate equations which govern the ten shell stresses and five (or three for the classical theory) shell displacements, under minimum number of necessary and consistent asumptions.

Equations of Motion for Shells

Due to the factors $(1 + z/R_1)$ and $(1 + z/R_2)$ in the equations of motion, the integration process will not uncouple* the coefficients u_2, v_2, w_2, and u_3, v_3 which do not belong to the category of shell displacements. To overcome this difficulty, we now introduce two assumptions.

Assumption 1. The thickness h of a thin shell is small compared to the minimum radius of curvature R so that the quadratic terms of h/R are negligible compared to unity

$$(h/R)^2 << 1 \tag{43}$$

From this assumption, all the quadratic or higher terms of (z/R) are also negligible compared to unity. This assumption should be used both before and after the integration process to avoid inconsistent retention of small terms. If we restrict ourselves to use (43) only after integration, detailed study indicates that same results will be obtained if all the higher coefficients, $\{u_n, v_n, w_n\}$, $n \geq 2$, are at most of the same order as the leading terms $\{u_0, v_0, w_0\}$, which is evidently true for physical reasons.

*We can derive an alternative shell theory by using the "weighted displacements," $\{U*, V*, W*\} = \{U, V, W\}(1 + z/R_1)(1 + z/R_2)$ in place of $\{U, V, W\}$. This will make the equations of motion slightly simpler, but will make the stress-displacement relations much more complex.

As typical examples,

$$\int_{-h/2}^{h/2} \left(1 + \frac{z}{R} + \frac{z^2}{R^2} + \frac{z^3}{R^3} + \ldots\right) W\,dz$$

$$= h\left[w_0 + \frac{h}{R}\frac{w_1}{6} + \frac{h^2}{R^2}\left(\frac{w_0}{12} + \frac{w_2}{30}\right) + \frac{h^3}{R^3}\left(\frac{w_1}{126} + \frac{w_3}{140}\right) + \ldots\right]$$

$$\approx h\left[w_0 + \frac{h}{R}\frac{w_1}{6}\right]$$

and

$$\int_{-h/2}^{h/2} \left(1 + \frac{z}{R} + \frac{z^2}{R^2} + \ldots\right) zW\,dz$$

$$= h^2\left[\frac{w_1}{6} + \frac{h}{R}\left(\frac{w_0}{12} + \frac{w_2}{30}\right) + \frac{h^2}{R^2}\left(\frac{w_1}{126} + \frac{w_3}{140}\right) + \ldots\right]$$

$$\approx h^2\left[\frac{w_1}{6} + \frac{h}{R}\left(\frac{w_0}{12} + \frac{w_2}{30}\right)\right]$$

The results do not depend on whether we apply Assumption 1 before or after integration. Therefore, Assumption 1 uncouples all $\{u_n, v_n, w_n\}$, $n \geqslant 3$ from the shell equations except the two stress-displacement relations for Q_1 and Q_2, where $U^{\pm}$ and $V^{\pm}$ appear.

Assumption 2. The normals of a thin shell remain nearly straight after deformation, and the symmetric part of the slight deviation from a straight line can be neglected

$$\{u_2, v_2\} << \{u_0, v_0\} \tag{44}$$

Now, multiplying (35) by $P_0(\zeta)$ and (35a, b) by $P_1(\zeta)$, then integrating through the thickness, we get:

$$\frac{\partial(BN_1)}{\partial\xi} + \frac{\partial(AN_{21})}{\partial\eta} + N_{12}\frac{\partial A}{\partial\eta} - N_2\frac{\partial B}{\partial\xi} + \frac{AB}{R_1}Q_1 +$$

$$+ AB(\tilde{\sigma}_{31}^+ - \tilde{\sigma}_{31}^-) = \rho hAB\left[\ddot{u} + \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{\beta}_1\right] \qquad (45a)$$

$$\frac{\partial(BN_{12})}{\partial\xi} + \frac{\partial(AN_2)}{\partial\eta} + N_{21}\frac{\partial B}{\partial\xi} - N_1\frac{\partial A}{\partial\eta} + \frac{AB}{R_2}Q_2 +$$

$$+ AB(\tilde{\sigma}_{32}^+ - \tilde{\sigma}_{32}^-) = \rho hAB\left[\ddot{v} + \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{\beta}_2\right] \qquad (45b)$$

$$\frac{\partial(BQ_1)}{\partial\xi} + \frac{\partial(AQ_2)}{\partial\eta} - AB\left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) +$$

$$+ AB(\tilde{\sigma}_{33}^+ - \tilde{\sigma}_{33}^-) = \rho hAB\left[\ddot{w} + \frac{h}{6}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{w}_1\right] \qquad (45c)$$

$$\frac{\partial(BM_1)}{\partial\xi} + \frac{\partial(AM_{21})}{\partial\eta} + M_{12}\frac{\partial A}{\partial\eta} - M_2\frac{\partial B}{\partial\xi}$$

$$- ABQ_1 + AB\cdot\frac{h}{2}(\tilde{\sigma}_{31}^+ + \tilde{\sigma}_{31}^-) = \rho AB\frac{h^3}{12}\left[\ddot{\beta}_1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{u}\right] \qquad (46a)$$

$$\frac{\partial(BM_{12})}{\partial\xi} + \frac{\partial(AM_2)}{\partial\eta} + M_{21}\frac{\partial B}{\partial\xi} - M_1\frac{\partial A}{\partial\eta}$$

$$- ABQ_2 + AB\cdot\frac{h}{2}(\tilde{\sigma}_{32}^+ + \tilde{\sigma}_{32}^-) = \rho AB\frac{h^3}{12}\left[\ddot{\beta}_2 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{v}\right] \qquad (46b)$$

It can be seen that these equations of motion are the same as those derived by Naghdi in Ref. 11.

It should be remarked that, though the inertia terms of the equations of motion are not exact, the left-hand sides of these equations (45) and (46), are an exact reduction from the elasticity equation. Therefore, in static cases, the equations of equilibrium are exact.

Constitutive Equations for Shells

Since (36), (37) and (38) also contain the factors $(1 + z/R_1)$ and $(1 + z/R_2)$, we shall apply Assumption 1 to eliminate all quadratic and higher terms of (z/R_1) and (z/R_2). In addition, we introduce the "generalized plane stress assumption" to eliminate the effects of $\tilde{\sigma}_{33}$.

Assumption 3. The thickness normal stress of a thin shell has negligible effects on the constitutive equations according to the following criterion:

$$\frac{h}{R}\tilde{\sigma}_{33}^{(1)} \quad , \quad \tilde{\sigma}_{33}^{(0)} << \tilde{\sigma}_{11}^{(0)} \quad , \quad \tilde{\sigma}_{22}^{(0)}$$

$$\frac{h}{R}\tilde{\sigma}_{33}^{(0)} \quad , \quad \tilde{\sigma}_{33}^{(1)} << \tilde{\sigma}_{11}^{(1)} \quad , \quad \tilde{\sigma}_{22}^{(1)}$$

(47)

Note that when $R \rightarrow \infty$, (47) reduces to (15).

To write the resulting stress-displacement relations in a legible form, we define the following quantities, which are similar to the conventional midsurface strains and curvatures, though we do not intend to interpret their physical meaning in the conventional manner.

$$\epsilon_1 = \frac{1}{h}\int_{-h/2}^{h/2} \Phi_1 dz = \frac{1}{A}\frac{\partial u}{\partial \xi} + \frac{1}{AB}\frac{\partial A}{\partial \eta}v + \frac{w}{R_1}$$

(48a)

$$\epsilon_2 = \frac{1}{h} \int_{-h/2}^{h/2} \Phi_2 dz = \frac{1}{B} \frac{\partial v}{\partial \eta} + \frac{1}{AB} \frac{\partial B}{\partial \xi} u + \frac{w}{R_2} \tag{48b}$$

$$\omega_1 = \frac{1}{h} \int_{-h/2}^{h/2} \Psi_1 dz = \frac{1}{A} \frac{\partial v}{\partial \xi} - \frac{1}{AB} \frac{\partial A}{\partial \eta} u \tag{48c}$$

$$\omega_2 = \frac{1}{h} \int_{-h/2}^{h/2} \Psi_2 dz = \frac{1}{B} \frac{\partial u}{\partial \eta} - \frac{1}{AB} \frac{\partial B}{\partial \xi} v \tag{48d}$$

$$\kappa_1^* = \frac{12}{h^3} \int_{-h/2}^{h/2} \Phi_1 z dz = \frac{1}{A} \frac{\partial \beta_1}{\partial \xi} + \frac{1}{AB} \frac{\partial A}{\partial \eta} \beta_2 + \frac{2w_1}{hR_1} \tag{49a}$$

$$\kappa_2^* = \frac{12}{h^3} \int_{-h/2}^{h/2} \Phi_2 z dz = \frac{1}{B} \frac{\partial \beta_2}{\partial \eta} + \frac{1}{AB} \frac{\partial B}{\partial \xi} \beta_1 + \frac{2w_1}{hR_2} \tag{49b}$$

$$\tau_1 = \frac{12}{h^3} \int_{-h/2}^{h/2} \Psi_1 z dz = \frac{1}{A} \frac{\partial \beta_2}{\partial \xi} - \frac{1}{AB} \frac{\partial A}{\partial \eta} \beta_1 \tag{49c}$$

$$\tau_2 = \frac{12}{h^3} \int_{-h/2}^{h/2} \Psi_2 z dz = \frac{1}{B} \frac{\partial \beta_1}{\partial \eta} - \frac{1}{AB} \frac{\partial B}{\partial \xi} \beta_2 \tag{49d}$$

Since $z^2 = (h^2/12)[P_0(\zeta) + 2P_2(\zeta)]$, we have, by virtue of Assumption 2

$$\frac{12}{h^3} \int_{-h/2}^{h/2} \Phi_1 z^2 dz = \epsilon_1 + \frac{2}{5} \frac{w_2}{R_1} \tag{50a}$$

$$\frac{12}{h^3} \int_{-h/2}^{h/2} \Phi_2 z^2 dz = \epsilon_2 + \frac{2}{5} \frac{w_2}{R_2} \qquad (50b)$$

$$\frac{12}{h^3} \int_{-h/2}^{h/2} \Psi_1 z^2 dz = \omega_1 \qquad (50c)$$

$$\frac{12}{h^3} \int_{-h/2}^{h/2} \Psi_2 z^2 dz = \omega_2 \qquad (50d)$$

It is seen that the undesirable unknowns w_1 and w_2 are involved in (49a, b) and (50a, b), which will be eliminated later with the help of equation (36c).

Now, multiplying (36a, b) and (37a, b) by $P_0(\zeta)$ and by $P_1(\zeta)$, then integrating through the thickness, we obtain

$$N_1 = C \left[\epsilon_1 + \nu\epsilon_2 + \frac{h^2}{12} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \kappa_1^* \right] \qquad (51a)$$

$$N_2 = C \left[\epsilon_1 + \nu\epsilon_2 + \frac{h^2}{12} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \kappa_2^* \right] \qquad (51b)$$

$$N_{12} = \frac{1-\nu}{2} C \left[\omega_1 + \omega_2 + \frac{h^2}{12} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \tau_1 \right] \qquad (51c)$$

$$N_{21} = \frac{1-\nu}{2} C \left[\omega_1 + \omega_2 + \frac{h^2}{12} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tau_2 \right] \qquad (51d)$$

$$M_1 = D \left[\kappa_1^* + \nu\kappa_2^* + \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \left(\epsilon_1 + \frac{2}{5} \frac{w_2}{R_1} \right) \right] \qquad (52a)$$

$$M_2 = D \left[\kappa_2^* + \nu\kappa_1^* + \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \left(\epsilon_2 + \frac{2}{5} \frac{w_2}{R_2} \right) \right] \qquad (52b)$$

$$M_{12} = \frac{1-\nu}{2} D \left[\tau_1 + \tau_2 + \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \omega_1 \right] \tag{52c}$$

$$M_{21} = \frac{1-\nu}{2} D \left[\tau_1 + \tau_2 + \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \omega_2 \right] \tag{52d}$$

To eliminate w_1 and w_2, we shall find two additional relations from (36c). Integrating (36c) through the thickness, we get

$$W^+ - W^- = - \frac{\nu}{E} \left[N_1 + N_2 - \frac{M_1}{R_2} - \frac{M_2}{R_1} \right] \tag{53a}$$

Multiplying (36c) by z, then integrating through the thickness (with the help of integration by parts), we get

$$\frac{h}{2} (W^+ + W^-) - hw_0 = - \frac{\nu}{E} \left[M_1 + M_2 - \frac{h^2}{12} \left(\frac{N_1}{R_2} + \frac{N_2}{R_1} \right) \right] \tag{53b}$$

It is now necessary to introduce the following assumption.

Assumption 4.   The even terms and odd terms of the Legendre series of W both converge rapidly so that

$$W^+ - W^- = 2(w_1 + w_3 + w_5 + \dots) \approx 2w_1$$

$$\frac{1}{2} (W^+ + W^-) - w_0 = (w_0 + w_2 + w_4 + \dots) - w_0 \approx w_2 \tag{54}$$

Therefore, (53) can be rewritten as

$$w_1 = - \frac{\nu}{2E} \left[N_1 + N_2 - \frac{M_1}{R_2} - \frac{M_2}{R_1} \right] \tag{55a}$$

$$w_2 = - \frac{\nu}{hE} \left[M_1 + M_2 - \frac{h^2}{12} \left(\frac{N_1}{R_2} + \frac{N_2}{R_1} \right) \right] \tag{55b}$$

Substitution of (51a, b) and (52a, b) into (55b) yields

$$w_2 = O(h^2 \kappa^*_{1,2}) + O\left(\frac{h^2}{R} \epsilon_{1,2}\right) \tag{55b'}$$

Substituting this into (52a, b), we find that the effects of w_2 is negligible by virtue of Assumption 1. Substituting (51a, b) and (52a, b) into (55a) and using Assumption 1, we have

$$\frac{2w_1}{h} = -\frac{\nu}{1-\nu}\left[\epsilon_1 + \epsilon_2 - \frac{h^2}{12}\left(\frac{\kappa^*_1}{R_1} + \frac{\kappa^*_2}{R_2}\right)\right] \tag{55a'}$$

Within the accuracy of Assumption 1, the three equations (55a') and (49a, b) can be rewritten

$$\frac{2w_1}{h} = -\frac{\nu}{1-\nu}\left[\epsilon_1 + \epsilon_2 - \frac{h^2}{12}\left(\frac{\kappa_1}{R_1} + \frac{\kappa_2}{R_2}\right)\right] \tag{56}$$

$$\kappa^*_1 = \kappa_1 - \frac{\nu}{1-\nu}\frac{\epsilon_1 + \epsilon_2}{R_1} \tag{57a}$$

$$\kappa^*_2 = \kappa_2 - \frac{\nu}{1-\nu}\frac{\epsilon_1 + \epsilon_2}{R_2} \tag{57b}$$

where

$$\kappa_1 = \frac{1}{A}\frac{\partial \beta_1}{\partial \xi} + \frac{1}{AB}\frac{\partial A}{\partial \eta}\beta_2 \tag{58a}$$

$$\kappa_2 = \frac{1}{B}\frac{\partial \beta_2}{\partial \eta} + \frac{1}{AB}\frac{\partial B}{\partial \xi}\beta_1 \tag{58b}$$

Substituting (57a, b) into (51a, b) and (52a, b), (with w_2 neglected), and using Assumption 1, we finally get

$$N_1 = C\left[\epsilon_1 + \nu\epsilon_2 + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\kappa_1\right] \tag{51a'}$$

$$N_2 = C\left[\epsilon_2 + \nu\epsilon_1 + \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\kappa_2\right] \tag{51b'}$$

$$M_1 = D\left[\kappa_1 + \nu\kappa_2 + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\epsilon_1 - \frac{\nu}{1-\nu}\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)(\epsilon_1 + \epsilon_2)\right] \tag{52a'}$$

$$M_2 = D\left[\kappa_2 + \nu\kappa_1 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\epsilon_2 - \frac{\nu}{1-\nu}\left(\frac{1}{R_2} + \frac{\nu}{R_1}\right)(\epsilon_1 + \epsilon_2)\right] \tag{52b'}$$

It now remains to derive stress-displacement relations for Q_1 and Q_2 by integrating (38a, b). From (38a)

$$Q_1 = \frac{1-\nu}{2}C\left[\frac{1}{A}\frac{\partial w}{\partial \xi} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)u + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{1}{A}\frac{\partial w_1}{\partial \xi}\frac{2}{h} - \frac{\beta_1}{R_1}\right) + \right.$$
$$\left. + \frac{1}{h}\left(1 + \frac{h}{2R_2}\right)U^+ - \frac{1}{h}\left(1 - \frac{h}{2R_2}\right)U^-\right] \tag{59}$$

Note that

$$\frac{1}{h}\left(1 + \frac{h}{2R_2}\right)U^+ - \frac{1}{h}\left(1 - \frac{h}{2R_2}\right)U^- = \frac{2}{h}(u_1 + u_3 + u_5 + \dots)$$
$$+ \frac{1}{R_2}(u_0 + u_2 + u_4 + \dots) \tag{60}$$

To eliminate the undesirable quantities $U^\pm$ and $V^\pm$, we introduce the "shear-constant assumption" similar to the plate theory.

<u>Assumption 5</u>. By introducing a shear-constant κ, we can neglect the higher coefficients u_n and v_n, $n \geqslant 2$ in the constitutive equations for Q_1 and Q_2. With this assumption and the help of (56) and Assumption 1, (59) can be written:

$$Q_1 = \frac{1-\nu}{2} C\kappa \left[\frac{1}{A} \frac{\partial w}{\partial \xi} + \beta_1 - \frac{u}{R_1} \right.$$

$$\left. - \frac{\nu}{1-\nu} \frac{h^2}{12} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \frac{1}{A} \frac{\partial}{\partial \xi} \left\{ \epsilon_1 + \epsilon_2 - \frac{h^2}{12} \left(\frac{\kappa_1}{R_1} + \frac{\kappa_2}{R_2} \right) \right\} \right] \quad (61a)$$

A similar equation for Q_2 can be easily written

$$Q_2 = \frac{1-\nu}{2} C\kappa \left[\frac{1}{B} \frac{\partial w}{\partial \eta} + \beta_2 - \frac{v}{R_2} \right.$$

$$\left. - \frac{\nu}{1-\nu} \frac{h^2}{12} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{B} \frac{\partial}{\partial \eta} \left\{ \epsilon_1 + \epsilon_2 - \frac{h^2}{12} \left(\frac{\kappa_1}{R_1} + \frac{\kappa_2}{R_2} \right) \right\} \right] \quad (61b)$$

The fifteen equations (45a, b, c), (46a, b); (51a', b', c, d), (52a', b', c, d) and (61a, b) form a determinate set of shell equations which are a direct reduction from elasticity equations under Assumption 1-5.

Classical Shell Theory

We use the name "classical shell theory" to designate a shell theory not involving β_1 and β_2. The elimination of β_1 and β_2 can be achieved by neglecting $\ddot{\beta}_1$ and $\ddot{\beta}_2$ in the equations of motion, and replacing Assumption 6 by the following assumption.

Assumption 5 (Classical). The shell displacements β_1 and β_2 in (53a', b', c, d) and (54a', b', c, d) can be approximated by

$$\beta_1 = -\frac{1}{A} \frac{\partial w}{\partial \xi} + \frac{u}{R_1}$$

$$(62)$$

$$\beta_2 = -\frac{1}{B} \frac{\partial w}{\partial \eta} + \frac{u}{R_2}$$

Substituting (62) into (49a', b', c, d), we find

$$\kappa_1 = \frac{1}{A} \frac{\partial}{\partial \xi} \left(-\frac{1}{A} \frac{\partial w}{\partial \xi} + \frac{u}{R_1} \right) + \frac{1}{AB} \frac{\partial A}{\partial \eta} \left(-\frac{1}{B} \frac{\partial w}{\partial \eta} + \frac{v}{R_2} \right) \tag{63a}$$

$$\kappa_2 = \frac{1}{B} \frac{\partial}{\partial \eta} \left(-\frac{1}{B} \frac{\partial w}{\partial \eta} + \frac{v}{R_2} \right) + \frac{1}{AB} \frac{\partial B}{\partial \xi} \left(-\frac{1}{A} \frac{\partial w}{\partial \xi} + \frac{u}{R_1} \right) \tag{63b}$$

$$\tau_1 = \frac{1}{A} \frac{\partial}{\partial \xi} \left(-\frac{1}{B} \frac{\partial w}{\partial \eta} + \frac{v}{R_2} \right) - \frac{1}{AB} \frac{\partial A}{\partial \eta} \left(-\frac{1}{A} \frac{\partial w}{\partial \xi} + \frac{u}{R_1} \right) \tag{63c}$$

$$\tau_2 = \frac{1}{B} \frac{\partial}{\partial \eta} \left(-\frac{1}{A} \frac{\partial w}{\partial \xi} + \frac{u}{R_1} \right) - \frac{1}{AB} \frac{\partial B}{\partial \xi} \left(-\frac{1}{B} \frac{\partial w}{\partial \eta} + \frac{v}{R_2} \right) \tag{63d}$$

Comparing (63) to the equivalent expressions derived by Love (Ref. 7, p. 524), we find Love's expression for τ is incorrect and must be replaced by

$$\tau = \frac{1}{2} (\tau_1 + \tau_2) \tag{64}$$

This correction has been pointed out by many previous investigators.

The approximation (62) should not be used in the two constitutive equations for Q_1 and Q_2, i.e., (59) or (61), since the purpose of introduction of this approximation is to <u>uncouple</u> two equations and two unknowns from the set of shell equations. Therefore, in a classical shell theory, (59) is not needed because of the absence of β_1 and β_2 from the rest of the shell equations. In the following discussion, we shall consider a classical shell theory as a variation of its equivalent "shear-constant" shell theory.

First-Approximation Shell Theory

If the shell deformations do not involve severe bending or flexural

motion with short wave-length, * or more precisely, if

$$\kappa_{1,2} \approx O(\epsilon_{1,2}/R) \quad , \quad \tau_{1,2} \approx O(\omega_{1,2}/R) \tag{65}$$

then the last terms in (53a', b', c, d) and (61a, b) can be neglected by virtue of

Assumption 1.   The resulting constitutive equations will be called first-

approximation shell theory, which, if used in compliance with condition

(65), will give the same accuracy as the second-approximation shell theory.

One more remark seems to be appropriate concerning the order

of magnitude of the different terms in the shell equations.   All the terms

do not necessarily have the same order of magnitude in all

problems.   Some terms may be found quite small in one problem

while becoming important in other cases.

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*cf., Ref. 7, p. 532.

COMPARISON AND DISCUSSION

Since the equations of motion are well-understood and present no fundamental questions in the existing literature, the present discussion will be concentrated on the constitutive equations.

In spite of the fact that the midsurface displacements have been abandoned and new definitions for shell displacements introduced, the constitutive equations appear to possess a striking formal resemblance to those of the conventional shell theories. In fact, except the last terms of (52a', b'), the constitutive equations are formally identical to the Flügge-Lure-Byrne equations as discussed by Naghdi in Ref. 1, where some terms of the order of (h^2/R^2) have also been retained to satisfy _exactly_ some invariant requirements and the identity

$$N_{12} + \frac{M_{12}}{R_1} = N_{21} + \frac{M_{21}}{R_2} \tag{66}$$

From the viewpoint of the present approach, the shell equations are merely truncated Legendre series solution to the elasticity equations; therefore, we shall be satisfied if these invariant requirements, energy requirements and identities are met _within the accuracy of Assumption 1._

On the other hand, the correction terms in (52a', b') obtained through the rigorous derivation, represent a significant improvement of the accuracy and consistency of the shell equations. It is clear that these terms represent the effects of the thickness change, which cannot be obtained in all the existing derivations based on Kirchhoff-Love hypothesis.

If we compare the present results with the constitutive equations derived by Naghdi[11] in 1957, through a variational method, we note that the correction terms in (52a', b') do exist in his equations for M_1 and M_2; however, the constitutive equations in Ref. 11, known as Reissner-Naghdi's theory, also include many other additional terms which are not obtainable through the present derivation. The details of the comparison are listed in Table 1. It might be remarked that in a recent numerical investigation, Klosner and Levine[27] compared the elasticity and shell theory solutions to a specific boundary-value problem of a cylindrical shell, and found that the solutions of Reissner-Naghdi's theory have not shown improvement over those of Flügge's theory and simpler theories. This may be attributed to the fact that additional and missing terms introduce equivalent numerical errors.

For a discussion of the constitutive equations due to Novozhilov, Koiter, Sanders, etc., the reader is referred to Ref. 1, where it may be seen that these theories deviate further from the rigorous second-approximation than Flügge-Lure-Byrne's theory or Reissner-Naghdi's theory.

TABLE 1. COMPARISON OF SHELL THEORIES*

Constitutive equations for	$\dfrac{1-\nu^2}{Eh} N_1$	$\dfrac{2(1+\nu)}{Eh} N_{12}$	$\dfrac{12(1-\nu^2)}{Eh^3} M_1$	$\dfrac{24(1+\nu)}{Eh^3} M_{12}$	$\dfrac{4(1+\nu)}{Eh} Q_1$
Love's first approx.** Ref. 7, pp. 521-531	$\epsilon_1 + \nu\epsilon_2$	$\omega_1 + \omega_2$	$\kappa_1 + \nu\kappa_2 + \dfrac{1}{R_2}(\epsilon_1 + \nu\epsilon_2)$	$\tau_1 + \tau_2 + \dfrac{1}{R_2}(\omega_1 + \omega_2)$	
Love's first approx. (1888) (Ref. 1)	$\epsilon_1 + \nu\epsilon_2$	$\omega_1 + \omega_2$	$\kappa_1 + \nu\kappa_2$	$\tau_1 + \tau_2 + \dfrac{\omega_1}{R_2} + \dfrac{\omega_2}{R_1}$	
Flügge-Luré-Byrne's second approx. (Ref. 1)	$\epsilon_1 + \nu\epsilon_2 + \dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\kappa_1$	$\omega_1 + \omega_2 + \dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\tau_1$	$\kappa_1 + \nu\kappa_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\epsilon_1$	$\tau_1 + \tau_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\omega_1$	$\dfrac{5}{6}\left(\dfrac{1}{A}\dfrac{\partial w}{\partial\xi} + \beta_1 - \dfrac{u}{R_1}\right)$
Reissner-Naghdi's second approx.† (Ref. 11)	$\epsilon_1 + \nu\epsilon_2 + \dfrac{h^2}{12}\dfrac{\nu}{1-\nu}\left(\dfrac{1}{R_1} + \dfrac{\nu}{R_2}\right)(\kappa_1 + \kappa_2)$ $-\dfrac{h^2}{24}\dfrac{\nu}{1-\nu}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\kappa_1$ $-\dfrac{h^2}{12}\dfrac{\nu}{1-\nu}\left(\dfrac{\kappa_1+\nu\kappa_2}{R_1} + \dfrac{\kappa_2+\nu\kappa_1}{R_2}\right)$	$\omega_1 + \omega_2 + \dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\tau_1$	$\kappa_1 + \nu\kappa_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\epsilon_1$ $-\dfrac{\nu}{1-\nu}\left(\dfrac{1}{R_1} + \dfrac{\nu}{R_2}\right)(\epsilon_1 + \epsilon_2)$ $-\dfrac{1}{5}\dfrac{\nu}{1-\nu}\left(\dfrac{\epsilon_1+\nu\epsilon_2}{R_1} + \dfrac{\epsilon_2+\nu\epsilon_1}{R_2}\right)$	$\tau_1 + \tau_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\omega_1$	$\dfrac{5}{6}\left(\dfrac{1}{A}\dfrac{\partial w}{\partial\xi} + \beta_1 - \dfrac{u}{R_1}\right)$ $-\dfrac{\nu}{1-\nu}\dfrac{h^2}{40}\dfrac{\partial}{\partial\xi}\left[(1+\nu)(\kappa_1 + \kappa_2)\right]$ $-\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)(\epsilon_1 - \epsilon_2)$ $-\dfrac{\nu}{1-\nu}(1+\nu)\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)(\epsilon_1 + \epsilon_2)$ $-\dfrac{2}{5}\dfrac{\nu}{1-\nu}\left(\dfrac{\epsilon_1+\nu\epsilon_2}{R_1} + \dfrac{\epsilon_2+\nu\epsilon_1}{R_2}\right)$
Rigorous second approx.	$\epsilon_1 + \nu\epsilon_2 + \dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\kappa_1$	$\omega_1 + \omega_2 + \dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\tau_1$	$\kappa_1 + \nu\kappa_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\epsilon_1$ $-\dfrac{\nu}{1-\nu}\left(\dfrac{1}{R_1} + \dfrac{\nu}{R_2}\right)(\epsilon_1 + \epsilon_2)$	$\tau_1 + \tau_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\omega_1$	$\kappa\left[\dfrac{1}{A}\dfrac{\partial w}{\partial\xi} + \beta_1 - \dfrac{u}{R_1}\right]$ $-\dfrac{\nu}{1-\nu}\dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\dfrac{1}{A}\dfrac{\partial}{\partial\xi}\left\{\epsilon_1 + \epsilon_2\right\}$ $-\dfrac{h^2}{12}\left(\dfrac{\kappa_1}{R_1} + \dfrac{\kappa_2}{R_2}\right)$
Rigorous first approx.	$\epsilon_1 + \nu\epsilon_2$	$\omega_1 + \omega_2$	$\kappa_1 + \nu\kappa_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\epsilon_1$ $-\dfrac{\nu}{1-\nu}\left(\dfrac{1}{R_1} + \dfrac{\nu}{R_2}\right)(\epsilon_1 + \epsilon_2)$	$\tau_1 + \tau_2 + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\omega_1$	$\kappa\left[\dfrac{1}{A}\dfrac{\partial w}{\partial\xi} + \beta_1 - \dfrac{u}{R_1}\right]$ $-\dfrac{\nu}{1-\nu}\dfrac{h^2}{12}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\dfrac{1}{A}\dfrac{\partial}{\partial\xi}(\epsilon_1 + \epsilon_2)$

*The other five constitutive equations may be obtained by interchanging subscripts 1 and 2. All terms of the order (h^2/R^2) are not included in the comparison.

**The expression corresponding to $(\tau_1 + \tau_2)$ is erroneous in Ref. 7, (p. 524) and must be corrected as in the text (eq. 64). For other versions of Love's first approximation, refer to discussion by Naghdi (Ref. 1, pp. 66-68).

†The terms associated with loading are not included.

APPENDIX

CONSTITUTIVE EQUATIONS INCORPORATING THE EFFECTS OF THICKNESS NORMAL STRESS

In the preceding derivation, the effects of the thickness normal stress σ_{33} (or $\tilde{\sigma}_{33}$) have been neglected (Assumption 3) for the sake of clarity. It can be easily shown, for example, by considering the simple case of a spherical shell subjected to internal pressure, that the thickness normal stress is of the order of (h/R) compared to the membrane stress $\sigma_{11}^{(0)}$ and $\sigma_{22}^{(0)}$ if there are distributed loads on the faces $z = \pm h/2$. This makes the accuracy of Assumption 3 considerably poorer than the other four assumptions. To improve this, we may replace it as follows.

Assumption 3 (Improved). The thickness normal stress σ_{33} is nearly linear in the z-direction and

$$\sigma_{33}^{(0)} = \frac{1}{2}(\sigma_{33}^{+} + \sigma_{33}^{-}), \qquad \sigma_{33}^{(1)} = \frac{1}{2}(\sigma_{33}^{+} - \sigma_{33}^{-})$$

From this we write the constitutive equations as follows, without giving the detailed derivation.

$$N_1 = C\left[\epsilon_1 + \epsilon_2 + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\kappa\right] + \frac{\nu}{1-\nu}\frac{h}{2}(\sigma_{33}^{+} + \sigma_{33}^{-})$$

$$N_{12} = \frac{1-\nu}{2}C\left[\omega_1 + \omega_2 + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\tau_1\right]$$

$$M_1 = D\left[\kappa_1 + \nu\kappa_2 + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\epsilon_1 - \frac{\nu}{1-\nu}\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)(\epsilon_1 + \epsilon_2)\right] +$$

$$+ \frac{\nu}{1-\nu}\frac{h^2}{12}(\sigma_{33}^{+} - \sigma_{33}^{-})$$

$$M_{12} = \frac{1-\nu}{2} D \left[\tau_1 + \tau_2 + \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \omega_i \right]$$

$$Q_1 = \frac{1-\nu}{2} C\kappa \left[\frac{1}{A} \frac{\partial w}{\partial \xi} + \beta_1 - \frac{u}{R_1} - \frac{\nu}{1-\nu} \frac{h^2}{12} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \frac{1}{A} \frac{\partial}{\partial \xi} \left\{ \epsilon_1 + \epsilon_2 \right.\right.$$

$$\left.\left. - \frac{h^2}{12} \left(\frac{\kappa_1}{R_1} + \frac{\kappa_2}{R_2} \right) + \frac{(1-\nu)}{2E} (\sigma_{33}^+ + \sigma_{33}^-) \right\} \right]$$

The accuracy of this shell theory (or the shell theory derived in the text when no distributed normal loads $\sigma_{33}^{\pm}$ are present) is of the order of $(h/R)^2$. It should be remarked, however, that when we speak of accuracy of the shell theory compared to the theory of elasticity, we should compare the shell stresses and displacements to the corresponding terms of the corresponding Legendre series. For example, we should compare the transverse shearing stress-resultant Q_1 only to the elasticity solution $\sigma_{13}^{(0)}$ rather than to σ_{13}, since all the shell theory solutions are insufficient to describe the detailed variation through the thickness due to their two-dimensional nature.

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