NASA CONTRACTOR REPORT

NASA CR-572

NASA CR-572

THE AXISYMMETRIC RESPONSE OF CYLINDRICAL AND HEMISPHERICAL SHELLS TO TIME-DEPENDENT LOADING

by James Ting-Shun Wang, Wolfram Stadler, and Chi-wen Lin

Prepared by
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Ga.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1966

NASA CR-572

THE AXISYMMETRIC RESPONSE OF CYLINDRICAL AND HEMISPHERICAL SHELLS TO TIME-DEPENDENT LOADING

By James Ting-Shun Wang, Wolfram Stadler, and Chi-wen Lin

Prepared under Grant No. NsG-571 by
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Ga.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $2.00

Contents

Part I

Cylindrical Shell

By

Wolfram Stadler[1]
James T. S. Wang[2]

Summary

The bending theory of thin shells is used to obtain the dynamic response of a complete cylindrical shell of finite length. The shell is subjected to axisymmetric dynamic loads, normal to the shell surface. The effects of the inertial load in the longitudinal direction are assumed to be negligible. The general solution is exact within the theory, and it is presented in a concise form, which simplifies its application to specific boundary-value problems. The natural frequencies are calculated from transcendental equations obtained by the application of the respective boundary conditions to the general solution. An illustrative example of a clamped cylindrical shell subjected to a uniform shock with linear decay is presented, including numerical results.

Symbols

d	Duration of loading
D	Plate rigidity
E	Modulus of Elasticity
h	Shell thickness
L	Shell length
q	Normal load
R	Radius of cylinder
t	Time
w	Normal displacement
x,y	Longitudinal and tangential coordinates

[1]Graduate Research Assistant, School of Engineering Mechanics.

[2]Associate Professor of Engineering Mechanics. Georgia Institute of Technology. Atlanta, Georgia 30332.

Introduction

The analogy between the governing equation of a beam on an elastic foundation, and the dynamic response of a cylindrical shell, with axisymmetric loading and negligible longitudinal inertia is obvious. Full use was made of this fact in obtaining the present solution.

The case of simply-supported cylindrical shells subjected to dynamic loads has been considered by several authors, for example by P. G. Bhuta [1], who used the same differential equation employed here. F. I. Baratta [2], made extensive use of the solutions obtained for the motion of a beam on an elastic foundation subject to various boundary conditions and loaded by constant velocity shock-waves. He solved individually for each type of loading and presented the respective solutions in tabular form. C. N. DeSilva and G. E. Tersteeg [3] obtained expressions for the lowest frequency of vibration for various types of shells of revolution, based on membrane theory, with the inclusion of bending effects at the boundary. Their expression for the natural frequencies of a clamped circular cylinder agrees with the one presented in this paper to within a small factor. Little work has been done considering boundary conditions other than simply supported edges.

There was a definite need to consolidate some of this work and to generalize the solution subject to the assumptions indicated above. The presented general solution is exact within the theory and is applicable to any problem involving finite cylindrical shells with axisymmetric normal, dynamic loads, and arbitrary geometric boundary conditions.

The governing differential equations are based on the bending theory of thin shells. The effects of longitudinal inertia are neglected and axisymmetric loading is assumed. With these assumptions it is possible to reduce the system of two equations to a single, linear, fourth-order partial differential equation, analogous to that of a beam on an elastic foundation. First, the homogeneous equation is solved for the general eigenfunctions of free vibration, in a form, which may easily be modified to suit the particular boundary conditions. The dynamic response is then obtained by expanding both, the loading function and the response in terms of the general, normalized eigenfunctions. The final, general solution also appears in a form, which is easy to apply to a particular problem.

To illustrate the use of the general solution, the problem of a clamped, cylindrical shell, subjected to a uniform shock, with a decay-function, varying linearly with time, is solved, including numerical results.

Equation of Motion

In Fig. 1 the general terminology used in this paper is illustrated. The differential equation of motion

$$\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{DR^2} w + \frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} = \frac{q}{D} \tag{1}$$

is based on the bending theory of thin shells with the additional assumptions of negligible inertial loading in the x-direction, and subjection only to dynamic loading, which is axisymmetric, and normal to the surface of the shell. In equation (1), w denotes the radial deflection, x the longitudinal variable, E Young's modulus, h the thickness of the shell, ρ the mass per unit volume, t the time variable, R the radius measured to the middle surface, and D is the rigidity of the shell. q represents the loading as a function of x and t.

Equation (1) is non-dimensionalized by introducing the following non-dimensional variables:

$$\overline{W} = \frac{w}{L} ; \ \xi = \frac{x}{L} ; \ \tau = \frac{t}{d} ; \ Q = \frac{q}{E} ,$$

where d represents some characteristic time interval, like the duration time of the shock-load, for example. The result of this non-dimensionalization is

$$\frac{\partial^4 \overline{W}}{\partial \xi^4} + a_1 \overline{W} + a_2 \frac{\partial^2 \overline{W}}{\partial \tau^2} = a_3 Q , \tag{2}$$

where

$$a_1 = \frac{EhL^4}{DR^2} ; \ a_2 = \frac{\rho h L^4}{Dd^2} ; \ a_3 = \frac{EL^3}{D} .$$

In order to solve equation (2) the eigenfunctions corresponding to the free vibration are obtained first by assuming

$$\overline{W}(\xi,\tau) = W(\xi) e^{i\omega\tau}$$

Substitution in

$$\frac{\partial^4 \overline{W}}{\partial \xi^4} + a_1 \overline{W} + a_2 \frac{\partial^2 \overline{W}}{\partial \tau^2} = 0$$

results in

$$\frac{d^4 W}{d\xi^4} - \lambda^4 W = 0 , \tag{3}$$

where $\lambda^4 = a_2\omega^2 - a_1$. The solution to equation (3) is well known. However, instead of writing this solution in the usual form, it will here be presented as it was first obtained by W. Nowacki [4] by employing Laplace transforms.

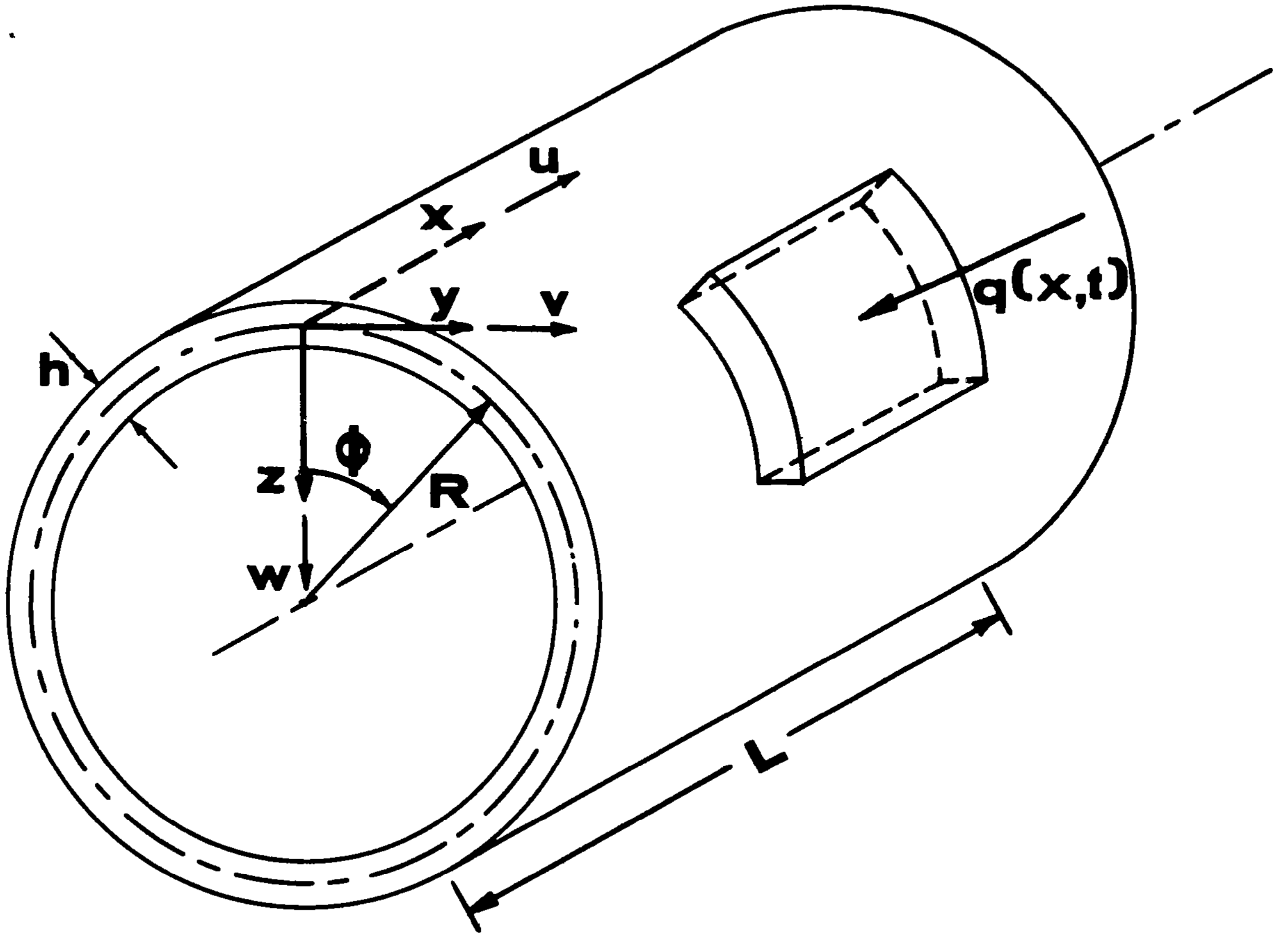

FIGURE 1.

SCHEMATIC REPRESENTATION OF CYLINDRICAL SHELL

In this form the solution is

$$W(\xi) = W(0)T(\lambda\xi) + \frac{1}{\lambda} W'(0)S(\lambda\xi) + \frac{1}{\lambda^2} W''(0)N(\lambda\xi) + \frac{1}{\lambda^3} W'''(0)M(\lambda\xi), \quad (4)$$

where the primes denote the derivatives with respect to ξ, and where

$$\begin{aligned} M(\lambda\xi) &= 1/2\ (\sinh\lambda\xi - \sin\lambda\xi) \\ N(\lambda\xi) &= 1/2\ (\cosh\lambda\xi - \cos\lambda\xi) \\ S(\lambda\xi) &= 1/2\ (\sinh\lambda\xi + \sin\lambda\xi) \\ T(\lambda\xi) &= 1/2\ (\cosh\lambda\xi + \cos\lambda\xi) \end{aligned} \quad (5)$$

with $M(0) = N(0) = S(0) = 0$ and $T(0) = 1$. The advantage of writing the solution in this form lies in the fact that the boundary conditions appear explicitly and hence simplify the application of the general solution to a specific eigenvalue problem. The frequency equations are obtained in the usual manner by substituting the boundary conditions into equation (4), as illustrated in the included example. The resulting eigenvalues λ_n are then substituted into (4) to obtain the eigenfunctions $W_n(\xi)$. The arbitrary constant contained in the eigenfunctions is used to normalize the $W_n(\xi)$ to facilitate further calculations.

It might be noted at this point that the eigenvalues λ_n in this particular case form a denumerable sequence of positive values due to the positive definiteness of the operator $L = \frac{d^4}{d\xi^4}$. The eigenfunctions $W_n(\xi)$ constitute a complete, orthonormal system, and any function f, which satisfies the boundary conditions, and for which L[f] is piecewise continuous may be expanded in terms of these eigenfunctions in an absolutely and uniformly convergent series [5].

It may now be assumed that the forcing function Q satisfies these conditions. If the forcing function does not satisfy the boundary conditions, only the values of the response near the boundary are affected; convergence is still uniform and absolute on any closed subinterval strictly interior to the basic interval, as long as the remaining hypotheses are satisfied.

With these facts in mind, it is then possible to write

$$Q(\xi\tau) = \sum_{n=1}^{\infty} q_n(\tau)W_n(\xi) \quad (6)$$

$$\overline{W}(\xi,\tau) = \sum_{n=1}^{\infty} \psi_n(\tau)W_n(\xi) ,$$

where $\overline{W}(\xi,\tau)$ now is the actual response, and where, in the usual manner

$$q_j(\tau) = \int_0^1 Q(\xi,\tau)W_j(\xi)d\xi$$

is a known quantity. Equations (6) are then substituted in

$$\frac{\partial^4\overline{W}}{\partial\xi^4} + a_1\overline{W} + a_2\frac{\partial^2\overline{W}}{\partial\tau^2} = a_3Q .$$

Multiplication of the result by $W_j(\xi)$ and integration w.r.t. ξ over the basic interval $[0,1]$, results in

$$\frac{d^2\Psi_j}{d\tau^2} + \omega_j{}^2\Psi_j = \frac{a_3}{a_2} q_j \tag{7}$$

where $\omega_j{}^2 = \frac{1}{a_2}(\lambda_j{}^4 + a_1)$. In obtaining (7), use was made of the facts that integral and summation are interchangeable due to the uniform convergence of (6), and the orthonormality of the $W_n(\xi)$, i.e.,

$$\int_0^1 W_i(\xi)W_j(\xi)d\xi = \delta_i{}^j ,$$

where $\delta_i{}^j$ is the Kronecker delta. For a system initially at rest in the equilibrium position the application of Laplace transforms to equation (7) results in

$$\Psi_j(\tau) = \frac{a_3}{a_2\omega_j}\int_0^\tau q_j(\zeta) \sin \omega_j(\tau - \zeta)\, d\zeta$$

and hence, the dynamic response of a cylindrical shell subject to an axisymmetric, normal dynamic loading Q, where L[Q] is piecewise continuous, is given by

$$\overline{W}(\xi,\tau) = \frac{Ed^2}{\rho hL}\sum_{i=1}^{\infty}\frac{W_i(\xi)}{\omega_i}\int_0^\tau q_i(\zeta) \sin \omega_i(\tau - \zeta)d\zeta. \tag{8}$$

Here, the $W_i(\xi)$ are the orthonormal eigenfunctions corresponding to a particular boundary value problem.

Illustrative Example

Clamped cylinder subjected to a uniform shock subsiding linearly with time.

a) The dynamic response.

Consider a cylindrical shell clamped at both ends and loaded axisymmetrically by a uniform shock decaying linearly w.r.t. time. The nondimensionalized geometric boundary conditions to be used in equation (4), and its derivative w.r.t. ξ, are:

$$W(0) = W(1) = W'(0) = W'(1) = 0 , \tag{9}$$

with the system at rest initially. The resulting eigenfunctions for this particular boundary value problem become

$$W_n(\xi) = C_n\left[N(\lambda_n\xi) - \frac{N(\lambda_n)}{M(\lambda_n)} M(\lambda_n\xi)\right] , \tag{10}$$

where the C_n are written as

$$C_n = \left\{ \int_0^1 \left[N(\lambda_n \xi) - \frac{N(\lambda_n)}{M(\lambda_n)} M(\lambda_n \xi) \right]^2 d\xi \right\}^{-1/2}$$

in order to normalize the W_n. The eigenvalues λ_n are obtained from the transcendental equation

$$\cosh \lambda_n \cos \lambda_n = 1 , \tag{11}$$

resulting in the non-dimensional eigenfrequencies of free vibration of the cylinder,

$$\omega_n^2 = \frac{\lambda_n^4 + a_1}{a_2} = \frac{Dd^2}{\rho h L^4} (\lambda_n^4 + \frac{EhL^4}{DR^2}). \tag{12}$$

The numerical values for the λ_n are given by

$\lambda_1 = 4.730; \lambda_2 = 7.853; \lambda_3 = 10.996;$

$\lambda_4 = 14.137; \lambda_5 = 17.279; \lambda_n = \pi/2(2n + 1)$ for $n > 5$.

It is obvious that $\lambda = 0$ also is a solution of (11) but the eigenfunction corresponding to $\lambda = 0$ is $W_o \equiv 0$, and hence is of no interest. It must be noted that expression (12) agrees with that obtained in [3] to within a factor of $1/(1- \nu^2)$ multiplying the second term.

The loading q is taken to be internal and of the form

$$q(x,t) = -p_o(1 - \frac{t}{d}) , \tag{13}$$

where p_o is the peak overpressure, and d is the equivalent decay time, equivalent in the sense that the magnitude of the impulse due to the linear decay is the same as that due to the actual decay. The non-dimensional form of (13) is

$$Q(\xi,\tau) = - \frac{p_o}{E}(1 -\tau).$$

It is clear that the final solution of the problem must consist of two parts, one, the response during the duration of the shock, two, the subsequent free vibration of the shell with corresponding initial conditions. The concern here shall be the former only.

As indicated in equation (6), the loading Q is expressed in terms of the $W_n(\xi)$ with

$$q_j(\tau) = \int_0^1 Q(\xi,\tau) \, W_j(\xi) d\xi = - \frac{p_o}{E} K(\lambda_j)(1-\tau) ,$$

where

$$K(\lambda_j) = \frac{C_j}{\lambda_j} M(\lambda_j) - \frac{N(\lambda_j)}{M(\lambda_j)} [T(\lambda_j) - 1] \quad .$$

The integral

$$\int_0^1 q_j(\zeta) \sin \omega_j(\tau-\zeta) \, d\zeta$$

then is

$$-\frac{p_o}{E\omega_j} K(\lambda_j) \left[1 - \tau - \cos \omega_j \tau + \frac{1}{\omega_j} \sin \omega_j \tau\right]. \tag{14}$$

The substitution of expression (14) into equation (8) results in the non-dimensional dynamic response

$$\overline{W}(\xi,\tau) = -\frac{d^2 p_o}{\rho h L} \sum_{j=1}^{\infty} \frac{K(\lambda_j) W_j(\xi)}{\omega_j{}^2} \left[1 - \tau - \cos \omega_j \tau + \frac{1}{\omega_j} \sin \omega_j \tau\right] . \tag{15}$$

This solution reduces to the static solution corresponding to a uniform load of unit magnitude, when the expression in brackets is omitted.

b) The numerical results

In the numerical evaluation of equation (15) the shell is assumed to be of steel, with the following material parameters and dimensions:

$L = 10.0$ in;	$R = 5.0$ in ;	$E = 30 \times 10^6$ psi;
$h = 0.05$ in;	$d = 20$ m sec;	$\nu = 0.3$;

$$\rho = 0.0007298 \text{ lb sec}^2/\text{in}^4.$$

For convenience in the calculations the peak overpressure of the loading is taken to be unity.

The series are evaluated, approximately, at $\xi = 1/2$ for the presumed maximum deflection. Convergence is excellent, with an accuracy to three significant figures obtained by taking only the first five terms of the series.

Table I lists the non-dimensional natural frequencies. Since there is very little spread in the natural frequencies of these lower modes, it is feasible to speak of a "quasi-period" of the motion. If the "mean" frequency

Table I
Non-dimensional Natural Frequencies

n = 1	811.11
n = 2	811.87
n = 3	814.38
n = 4	820.21
n = 5	831.42

is taken to be 812.5, the non-dimensional "quasi-period" becomes approximately 0.0077, or 0.154 m sec, which is born out also by the representation of the exact non-dimensional response in Fig. 2. A schematic representation of the response throughout the duration of the load may be found in Fig. 3, where, as indicated, the maximum dynamic deflection occurs at $\tau = .74$ and is about 2.4 times the static deflection due to a uniform load of unit magnitude.

To check the accuracy of the solution the length L is varied, and, as is to be expected, this variation has little effect on the magnitude of the deflection, when L is kept in the long shell range. No other parameters or dimensions were varied.

Conclusion

The dynamic response of a cylindrical shell, subjected to axisymmetric loads is obtained. The solution is exact within the assumptions of the theory, and it is presented in an easily applied form. The numerical results are in accord with those obtained in [1], in that the dynamic loading factor as defined therein, is about 2.4 for the illustrative example considered. The solution presented here unifies and supplements previous work done by other authors almost exclusively on simply supported cylindrical shells.

Acknowledgements

The authors gratefully acknowledge the fruitage of discussions with Dr. Wilton W. King. The indulgence of the personnel of the Rich Electronic Computer Center at Georgia Tech in running the computer program is also appreciated.

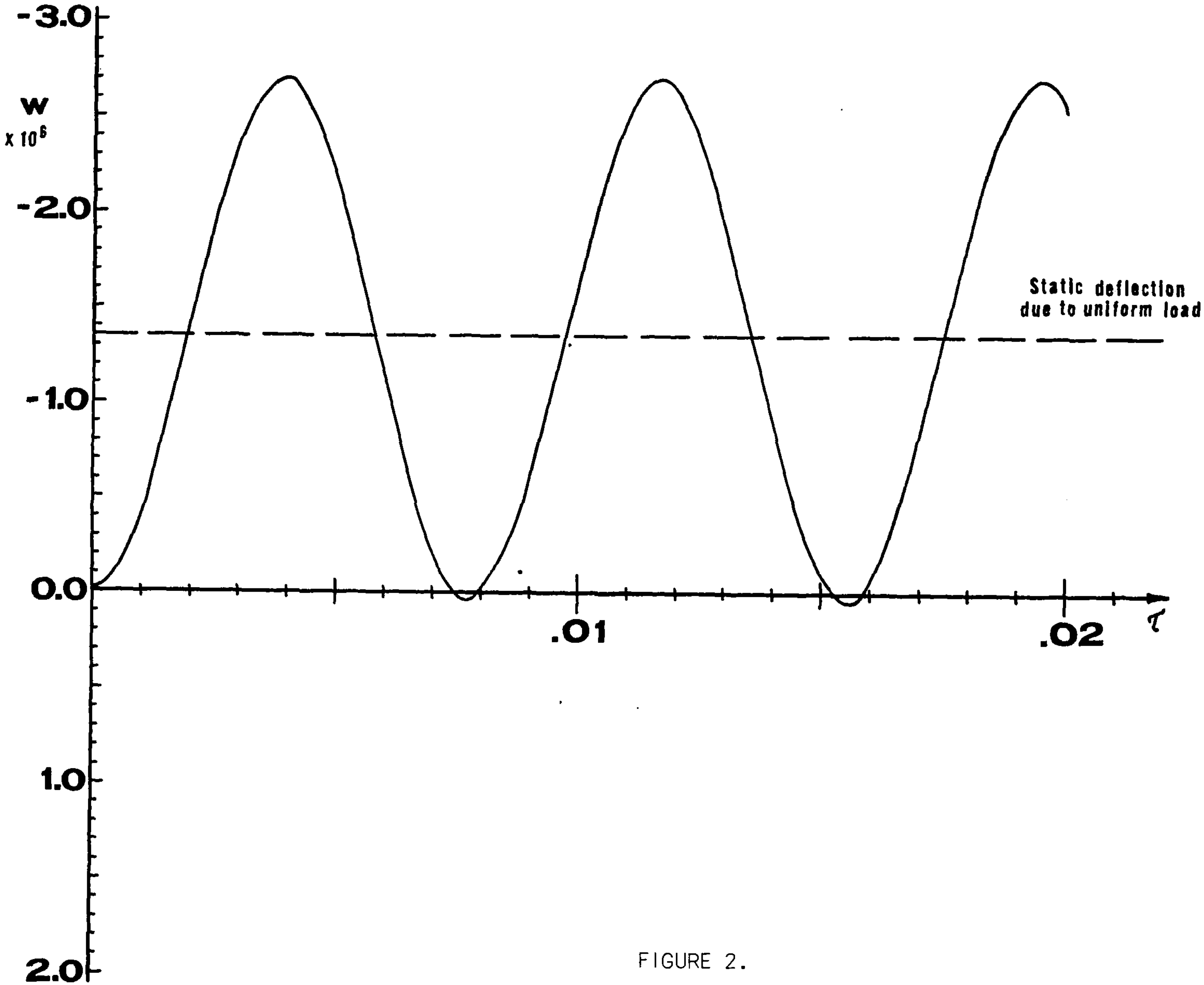

FIGURE 2.

NON-DIMENSIONAL DYNAMIC RESPONSE OF CYLINDRICAL SHELL

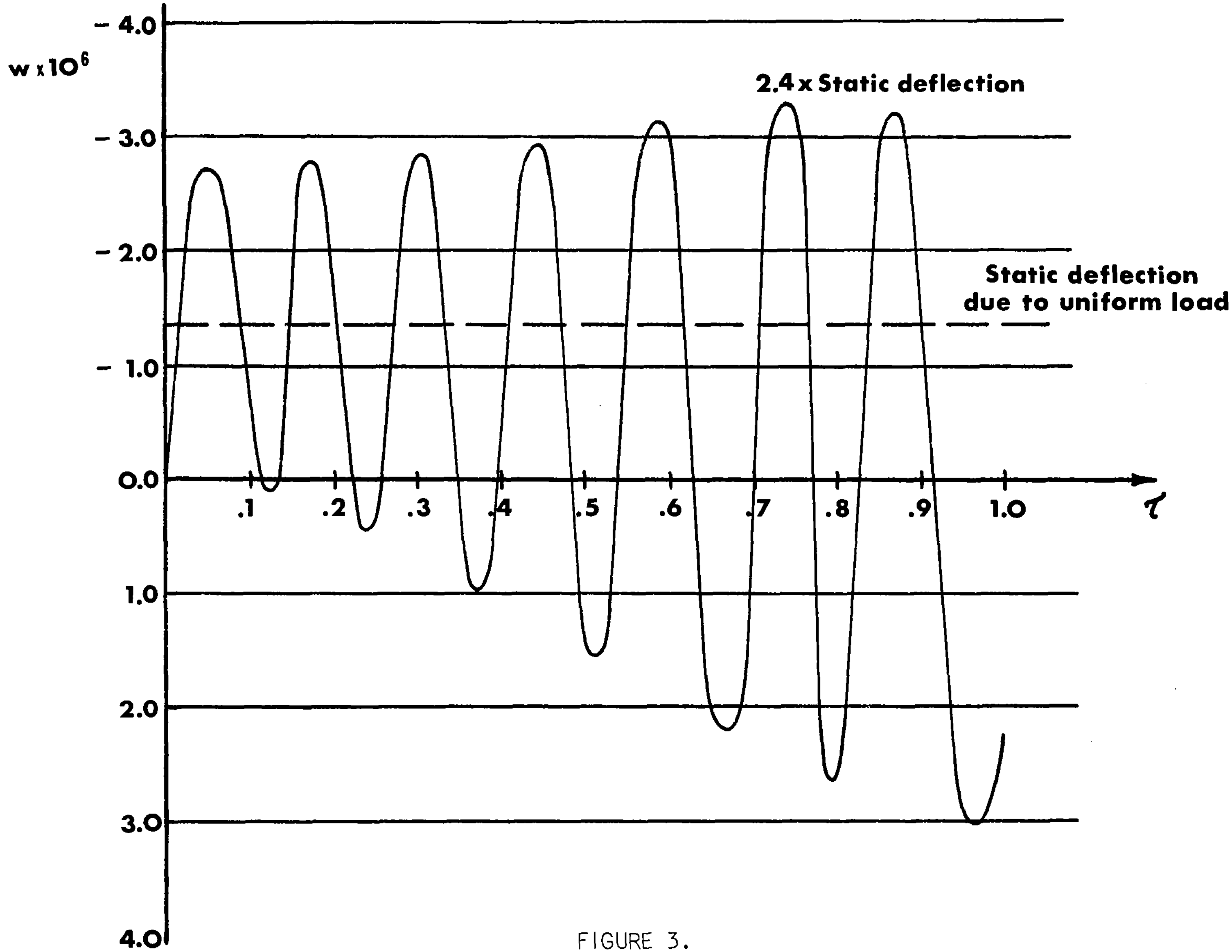

FIGURE 3.

SCHEMATIC REPRESENTATION OF NON-DIMENSIONAL
DYNAMIC RESPONSE

References

1. Bhuta, Pravin G., "Transient Response of a Thin Elastic Cylindrical Shell to a Moving Shock Wave", J. Acous. Soc. Amer., 35, 1, (January 1963), pp. 25-30.

2. Baratta, F. I., "The Rotationally Symmetric Motion of a Cylindrical Shell with Various End-Conditions, due to Various Constant Velocity Loads", Watertown Arsenal Labs, WAL-TR 850/13, (November 1962).

3. De Silva, C. N. and Tersteeg, G. E., "Axisymmetric Vibrations of Thin Elastic Shells", J. Acous. Soc. Amer., 36, 4, (April 1964), pp. 666-672.

4. Nowacki, Witold, Dynamics of Elastic Systems, John Wiley & Sons, Inc., 1963, p. 119.

5. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Interscience Publishers, Inc., New York, 1963, p. 363.

Part II

Hemispherical Shell

By

James Ting-Shun Wang[1]
Chi-wen Lin[2]

Summary

The axisymmetric dynamic response of hemispherical shells subjected to arbitrary loading is formulated according to elastic bending theory of shells. Generalized Green's functions are constructed and closed form solutions are obtained for hemispherical shells having roller hinged and roller-clamped edges. The results can be reduced to the solutions obtained by other authors according to membrane shell theory. Comparison is made on the frequencies obtained according to the present bending theory to the frequencies based on membrane theory. When harmonic loadings are applied along the edge of roller-hinged and along the edge of roller-clamped shells, the frequency equations for a shell with hinged edge and a shell completely clamped along its edge are obtained. The lowest three frequencies for a fixed edge hemispherical shell are calculated for three different thickness-radius rations.

[1]Associate Professor of Engineering Mechanics
[2]Graduate Research Assistant of Engineering Mechanics

Symbols

a	Radius of the hemispherical shell
E	Modulus of elasticity
G, G^*	Generalized Green's functions
h	Shell thickness
M_ϕ, M_θ	Components of stress couple
N_ϕ, N_θ	Components of stress resultant
q_v, q_w	Tangential and normal components of surface loading
Q	Transverse shear along meridianal line
s	Laplacian parameter
t	Time
v, w	Components of displacements
α	$\frac{1}{12} \left(\frac{h}{a}\right)^2$
λ	$\rho a^2(1 - \mu^2)/E$
ϕ,θ	Spherical coordinates
$\varepsilon_\phi,\varepsilon_\theta$	Components of strain
χ_ϕ,χ_θ	Components of change of curvature
μ	Poisson's ratio
ρ	Density
ω	Frequency

Introduction

The free vibration of hemispherical shells has been studied recently by a number of authors. Naghdi and Kalnins [1] obtained the natural frequencies for free-edged shells with thickness-radius ratio larger than 0.01. The frequency equations corresponding to spherical shells and for hemispherical shells are discussed by Kalnins in [2]. Huang [3] obtained the natural frequencies for a hemispherical shell using a method, similar to that used by Baker [4] to obtain the frequencies for a complete spherical shell, neglecting bending effects. In [5] Kalnins presents a numerical method for the calculation of the natural frequencies and normal modes of arbitrary rotationally symmetric shells.

The present study is concerned with the following aspects:

(A) The general solution of the dynamic response of roller-hinged and roller-clamped hemispherical shells subjected to arbitrary loads.

(B) The free vibration of hemispherical shells with roller-hinged and roller-clamped edges.

(C) The free vibration of hemispherical shells with hinged and completely clamped edge by using the results obtained in (A) and (B).

The solutions are obtained in closed form, based on the bending theory of shells.

Basic Elasto-Kinetic Equations

The equations of motion for the axisymmetric deformation of a spherical shell are

$$\frac{\partial N}{\partial \phi} + (N_\phi - N_\theta) \operatorname{Cot} \phi - Q = \rho h a \frac{\partial^2 v}{\partial t^2} - q_v a \tag{1a}$$

$$\frac{\partial Q}{\partial \phi} + Q \operatorname{Cot} \phi + N_\theta + N_\phi = \rho h a \frac{\partial^2 w}{\partial t^2} - q_w a \tag{1b}$$

$$\frac{\partial M_\phi}{\partial \phi} + (M_\phi - M_\theta) \operatorname{Cot} \phi - Q a = 0 \tag{1c}$$

Figure (1) shows the geometry of a hemispherical shell. The stress-strain-displacement relations are

$$N_\phi = \frac{Eh}{1-\mu^2} (\varepsilon_\phi + \mu\varepsilon_\theta) = \frac{Eh}{a(1-\mu^2)} \left[\left(\frac{\partial v}{\partial \phi} - w\right) + \mu(v \operatorname{Cot} \phi - w) \right] \tag{2a}$$

$$N_\theta = \frac{Eh}{1-\mu^2} (\varepsilon_\phi + \mu\varepsilon_\theta) = \frac{Eh}{a(1-\mu^2)} \left[(v \operatorname{Cot} \phi - w) + \mu\left(\frac{\partial v}{\partial \phi} - w\right) \right] \tag{2b}$$

$$M_\phi = -D(\chi_\phi + \mu\chi_\theta) = -\frac{D}{a^2}\left[\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2} + \mu\left(v + \frac{\partial w}{\partial \phi}\right) \operatorname{Cot} \phi \right] \tag{2c}$$

$$M_\theta = -D(\chi_\theta + \mu\chi_\phi) = -\frac{D}{a^2}\left[\left(v + \frac{\partial w}{\partial \phi}\right) \operatorname{Cot} \phi + \mu\left(\frac{\partial w}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) \right] \tag{2d}$$

By eliminating Q in Eqs. (1a) - (1c) and substituting the stress-displacement relations in Eq. (1) results in the following governing differential equations:

$$\frac{\partial}{\partial \phi}\Big[\alpha\, L(w) + (1+\alpha)L(\Psi) - (1+\mu)(1+\alpha)(w+\Psi) + 2\alpha w + 2(1+\alpha)\Psi - \lambda \frac{\partial^2 \Psi}{\partial t^2}\Big] = -q_v \frac{a^2(1-\mu^2)}{Eh} \tag{3a}$$

$$\alpha[LL(w+\Psi) - (1+\mu)L(w)] - [(1+\mu)(1+\alpha)]\, L(\Psi) + 2(1+\mu)\, w + \lambda \frac{\partial^2 w}{\partial t^2} = -q_w \frac{a^2(1-\mu^2)}{Eh} \tag{3b}$$

where

$$\alpha = \frac{1}{12}\left(\frac{h}{a}\right)^2, \quad \lambda = \rho \frac{a^2(1-\mu^2)}{E}, \quad v = \frac{\partial x}{\partial \phi} \tag{4a}$$

and

$$L(\cdot) = \frac{\partial^2(\cdot)}{\partial \phi^2} + \operatorname{Cot}\phi \frac{\partial(\cdot)}{\partial \phi} \tag{4b}$$

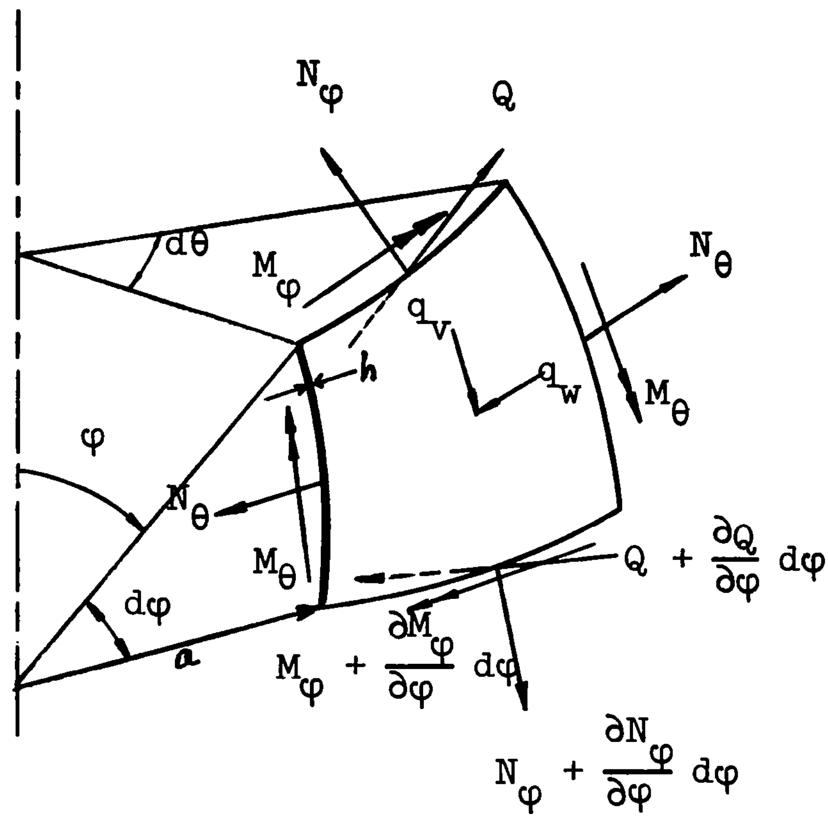

Fig. (1) Geometry of a Hemispherical Shell

With the usual assumption of $(\frac{h}{a})^2 << 1$ for thin shells, Eqs. (3a) and (3b) can be simplified by neglecting α when compared to 1. Eqs. (3a) and (3b) thence have the form

$$L(\Psi) = -\alpha\, L(w) + (1+\mu)\, w - (1-\mu)\Psi + \lambda \cdot \frac{\partial^2 \Psi}{\partial t^2} - \frac{a^2(1-\mu^2)}{Eh} \int_o^\phi q_v \, d\phi \qquad (5a)$$

$$LL(w) = -LL(\Psi) + (1+\mu) L(w) + \frac{(1+\mu)}{\alpha} L(\Psi)\, \frac{2(1+\mu)}{\alpha} w - \lambda \frac{\partial^2 w}{\partial t^2} - q_w \frac{a^2(1-\mu^2)}{Eh} \qquad (5b)$$

Let $x = \text{Cos}\ \phi$, the operator L becomes (6)

$$L(\cdot) = (1 - x^2) \frac{\partial^2(\cdot)}{\partial x^2} - 2x \frac{\partial(\cdot)}{\partial x} \qquad (7)$$

Shells, subject to the following boundary conditions, will be considered first:

(a) Roller-hinged edge: $w = \frac{\partial v}{\partial \phi} = \frac{\partial^2 w}{\partial \phi^2} = 0$ along $\phi = \pi/2$ (8a)

(b) Roller-clamped edge: $v + \frac{\partial w}{\partial \phi} = \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^3 w}{\partial \phi^3} = 0$ along $\phi = \pi/2$ (8b)

Construction of the Generalized Green's Functions

The generalized Green's Functions associated with the linear operator L and LL shall be denoted by G (x,ξ) and G* (x, ξ) respectively. They satisfy the equations

$$L(G) = \delta\ (x - \xi) \tag{9a}$$

$$LL(G^*) = \delta\ (x - \xi) \tag{9b}$$

and the conditions for

(a) Roller-hinged edge:

$$G^* = G = \frac{\partial^2 G^*}{\partial \xi^2} = 0 \qquad \text{at } \xi = 0 \tag{10a}$$

(b) Roller-clamped edge:

$$\frac{\partial G}{\partial \xi} = \frac{\partial G^*}{\partial \xi} = \frac{\partial^3 G^*}{\partial \xi^3} = 0 \qquad \text{at } \xi = 0 \tag{10b}$$

where $\delta(x-\xi)$ is the singularity function.

The general solution of Legendre's equation

$$(1 - \xi^2)\frac{\partial^2 G}{\partial \xi^2} - 2\xi\frac{\partial G}{\partial \xi} + n(n + 1)\ G = 0 \tag{11}$$

has the form

$$G = \sum_{n=0}^{\infty} A_n P_n(\xi) + \sum_{n=0}^{\infty} B_n Q_n(\xi) \tag{12}$$

where $P_n(\xi)$ and $Q_n(\xi)$ are the Legendre polynomials of the 1st and the 2nd kind respectively.

Since $G(x, \xi)$ must be finite for $\xi = 1$, hence

$$B_n = 0 \tag{13}$$

and Eq. (12) becomes

$$G = \sum_{n=0}^{\infty} A_n P_n(\xi) \tag{14}$$

Substitution of Eq. (14) into Eq. (9a) in conjunction with (11) leads to

$$-\sum_{n=0}^{\infty} n(n + 1)\ A_n P_n(\xi) = \delta\ (x - \xi) \tag{15}$$

where

$$A_n = -\frac{2n+1}{n(n+1)} \int_1^0 P_n(\xi)\ \delta(x - \xi)\ d\xi = \frac{2n+1}{n(n+1)} P_n(x) \tag{16}$$

Therefore

$$G(x,\ \xi) = \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} P_n(x)\ P_n(\xi) \tag{17}$$

In a similar manner, the generalized Green's function associated with the operator LL is found to be

$$G^*(x,\ \xi) = -\sum_{n=0}^{\infty} \frac{2n+1}{n^2(n+1)^2}\ P_n(x)\ P_n(\xi) \tag{18}$$

The multiplication of Eqs. (5a) and (5b) by $G(x,\ \xi)$ and $G^*(x,\ \xi)$ respectively, and their integration over the interval [0, 1] in conjunction with Eqs. (10a) and (10b) results in

$$\Psi = -\int_0^1 L(\xi)\ G(x,\xi)\ d\xi \tag{19}$$

$$w = -\int_0^1 LL(w)\ G^*(w,\ \xi)\ d\xi \tag{20}$$

in which

$$G(x,\ \xi) = \sum_{\substack{n=1,3,5---\\ \text{or}\\ n=2,4,6---}}^{\infty} -\frac{2n+1}{n(n+1)}\ P_n(x)\ P_n(\xi) \tag{21a}$$

$$G^*(x,\ ) = \sum_{\substack{n=1,3,5---\\ \text{or}\\ n=2,4,6---}}^{\infty} \frac{2n+1}{n^2(n+1)^2} P_n(x)\ P_n(\xi) \tag{21b}$$

where n=1,3,5,--- corresponds to the roller-hinged case, and n=2,4,6--- corresponds to the roller-clamped case.

Using Eqs. (5) and (21) in the Eq. (20) it is found that Ψ and w are in the following general form:

$$\Psi(x,\ t) = \sum_{\substack{n=1,3,5--\\ \text{or}\\ n=2,4,6--}}^{\infty} \Psi_n(t)\ P_n(x) \tag{22a}$$

$$w(x,\ t) = \sum_{\substack{n=1,3,5---\\ \text{or}\\ n=2,4,6}}^{\infty} w_n(t)\ P_n(x) \tag{22b}$$

where Ψ (t) and w (t) satisfy the following differential equations:

$$\lambda \frac{\partial^2 \Psi_n}{\partial t^2} - [(1-\mu) - n(n+1)]\Psi_n + [\alpha n(n+1) + (1+\mu)]w_n$$
$$= \frac{a^2(1-\mu^2)}{Eh}\quad (2n+1)\int_0^1 P\ (\xi)\int_0^{(\xi)}\ q_v\ d\ \phi\ d\xi \tag{23a}$$

and

$$\lambda \frac{\partial^2 w_n}{\partial t^2} + \{2(1 + \mu) + \alpha n(n+1) \, [n(n + 1) + (1 + \mu)] \, w_n$$

$$+ n(n +1) \, [(1 + \mu) + \alpha n(n + 1)]\Psi_n \tag{23b}$$

$$= -(2n + 1) \, \frac{a^2(1 - \mu^2)}{Eh} \int_0^1 q \quad P \, (\xi)d\xi$$

By eliminating w_n and Ψ_n between Eqs. (23a) and (23b), two uncoupled linear differential equations are obtained.

$$\lambda^2 \frac{\partial^4 \Psi_n}{\partial t^4} + A_{1n} \frac{\partial^2 \Psi_n}{\partial t^2} + A_{2n} \Psi_n$$

$$= \frac{a^2(1-\mu^2)}{Eh} (2n + 1) \left\{\left[A_{3n} + \lambda \frac{\partial^2}{\partial t^2}\right] \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi d\xi\right.$$

$$\left. + [n(n + 1) + (1+\mu)] \int_0^1 q_n \, P_n(\xi) \, d\xi \right\} \tag{24a}$$

$$\lambda^2 \frac{\partial^4 {}_n}{\partial t^4} + A_{1n} \frac{\partial^2 {}_n}{\partial t^2} + A_{2n} w_n$$

$$= - \frac{a^2(1-\mu^2)}{Eh} (2n + 1) \left\{ \left[n(n + 1) - (1 - \mu) + \frac{\partial^2}{\partial t^2} \right] \cdot \right.$$

$$\left. \int_0^1 q_w \, P_n(\xi) \, d\xi + n(n + 1) \, [(1 + \mu) + \alpha n(n + 1)] \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi d\xi \right\} \tag{24b}$$

where

$$A_{1n} = \left[\alpha n(n + 1) \, (n^2 + n + 1 + \mu)\right] + \left[n(n + 1) + (1 + 3\mu)\right] \tag{25a}$$

$$A_{2n} = \{\alpha \, [n(n + 1) - (1 - \mu)][n(n + 1)(n^2 + n + 1 + \mu)]$$

$$- \alpha^2 n^3 (n + 1 \;^3 - 2\alpha n^2 (n + 1)^2 (1 + \mu)\}$$

$$+ (1 - \mu^2) \, [n(n + 1) - 2] \tag{25b}$$

$$A_{3n} = \left[\alpha n(n +1)(n^2 + n + 1 + \mu)\right] + \left[2(1 + \mu)\right] \tag{25c}$$

The general solutions of Eqs. (24a) and (24b) may be obtained by use of Laplace transforms, they are

$$\Psi_n = B_{1n} \text{ Cos } \omega_{1n}t + B_{2n} \text{ Cos } \omega_{2n}t + C_{1n} \text{ Sin } \omega_{1n}t + C_{2n} \text{ Sin } \omega_{2n}t +$$

$$+ \frac{a^2(1-\mu^2)}{\lambda^2 Eh} \quad \frac{2n+1}{(\omega_{1n}^2-\omega_{2n}^2)} \left\{ \frac{[A_{3n}-\lambda\omega_{1n}^2(\omega_{1n}^2-\omega_{2n}^2)]}{\omega_{2n}} \int_0^t [\text{Sin } \omega_{1n}(t-\tau) \cdot \right.$$

$$\int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi d\xi \,] \, d\tau - \frac{A_{3n}}{\omega_{1n}} \int_0^t [\text{Sin } \omega_{1n}(t-\tau) \cdot$$

$$\int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi d\xi \,] \, d\tau + \frac{[\alpha n(n+1) + (1+\mu)]}{\omega_{1n} \, \omega_{2n}} \cdot$$

$$\int_0^1 [\omega_{1n} \text{ Sin } \omega_{2n} (t-\tau)] - \omega_{2n} \text{ Sin } \omega_{1n} (t-\tau) \int_0^1 P_n(\xi) \, q_w \, d\xi d\tau -$$

$$\left. \lambda D_1 (\text{Cos } \omega_{2n}t - \text{Cos } \omega_{1n}t) + \lambda \, D_2 \left(\frac{1}{\omega_{2n}} \text{ Sin } \omega_{2n}t - \frac{1}{\omega_{1n}} \text{ Sin } \omega_{1n}t\right) \right\} \qquad (26a)$$

$$w_n = B_{1n}^* \text{ Cos } \omega_{1n}t + B_{2n}^* \text{Cos } \omega_{2n}t + C_{1n}^* \text{ Sin } \omega_{1n}t + C_{2n}^* \text{ Sin } \omega_{2n}t$$

$$- \frac{a^2(1-\mu^2)}{\lambda^2 Eh} \cdot \frac{2n+1}{(\omega_{1n}^2-\omega_{2n}^2)} \quad \frac{[n(n+1)-(1-\mu)-\lambda\omega_{1n}^2(\omega_{1n}^2-\omega_{2n}^2)]}{\omega_{2n}} \cdot$$

$$\int_0^t [\text{Sin } \omega_{2n}(t-\tau) \int_0^1 P_n(\xi) \, q_w \, d\xi] \, d\tau -$$

$$\frac{n(n+1)-(1-\mu)}{\omega_{1n}} \int_0^t \left[\text{Sin } \omega_{1n}(t-\tau) \int_0^1 P_n(\xi) \, q_w \, d\xi d\tau + \right.$$

$$n(n+1) \quad \frac{[(1+\mu) + \alpha n(n+1)]}{\omega_{1n} \, \omega_{2n}} \int_0^t [\, \omega_{1n} \text{ Sin } \omega_{2n}(t-\tau) - \omega_{2n} \text{ Sin } \omega_{1n}(t-\tau)] \cdot$$

$$\left. \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi \, d\xi \right] d\tau - \lambda \, D_1^* \, (\text{Cos}\omega_{2n}t - \text{Cos } \omega_{1n}t)$$

$$-\lambda \, D_2^* \left(\frac{1}{\omega_{2n}} \text{ Sin}\omega_{2n}t - \frac{1}{\omega_{1n}} \text{ Sin } \omega_{1n}t\right) \qquad (26b)$$

where

$$\omega_{1n}^{\ 2} = \frac{1}{2\lambda} \; A_{1n} + [(A_{1n}^{\ 2} - 4 \; A_{2n})^{1/2}] \qquad (27a)$$

$$\omega_{2n}^{\ 2} = \frac{1}{2\lambda} \; A_{1n}^{\ 2} - [(A_{1n}^{\ 2} - 4 \; A_{2n})^{1/2}] \qquad (27b)$$

represent respectively the upper and lower branches of the natural frequencies for the free vibration of the shells, B_{1n}, B_{2n}, C_{1n}, C_{2n}, B_{1n}^*, B_{2n}^*, C_{1n}^*, C_{2n}^* are arbitrary constants which can be determined by the initial conditions,

$$D_1 \left[\int_0^1 P_n(\xi)\int_0^{\phi(\xi)} q_v \, d\phi \, d\xi\right]_{t=0} \tag{28a}$$

$$D_2 \left[\frac{\partial}{\partial t}\int_0^1 P_n(\xi) \int_0^{\phi(\xi)} q_v \, d\phi \, d\xi\right]_{t=0} \tag{28b}$$

$$D_1^* \left[\int_0^1 P_n(\xi) \, q_w \, d\xi\right]_{t=0} \tag{28c}$$

$$D_2^* \left[\frac{\partial}{\partial t}\int_0^1 P_n(\xi)\right] q_w \, d\xi_{\,t=0} \tag{28d}$$

$\lambda\omega_{1n}^2$ and $\lambda\omega_{2n}^2$ are plotted for $\frac{h}{a}$ = 0.01, 0.02, and 0.05 and are shown in Figure (2). It is seen that the variation of thickness has very little effect to the upper branch frequencies. However, the lower branch frequencies change significantly as the thickness of shell increases, particularly at the higher modes. In addition, the lower branch frequencies are not bounded which contradicts the results obtained according to membrane theory by Baker [4].

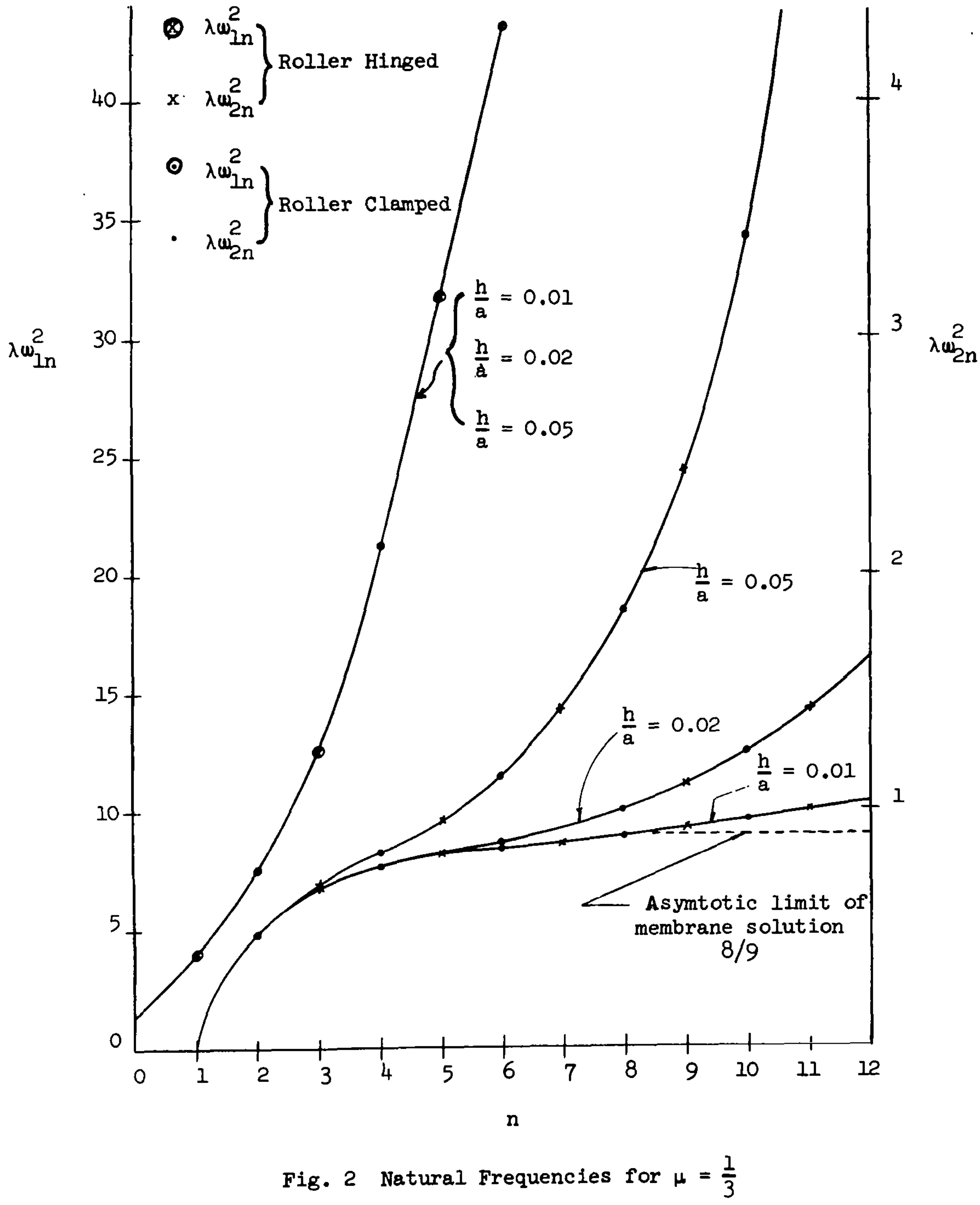

Fig. 2 Natural Frequencies for $\mu = \frac{1}{3}$

Free Vibrations of Hinged and Completely Clamped Shell

If the loading function appearing in equations (26a) and (26b) are chosen to be harmonic and assumed to be acting along the edge of the shell, the shell can be forced to satisfy the boundary conditions for the hinged and the completely clamped edge.

(a) Hinged boundary: The forcing functions are taken in the following form

$$q_v = q_o\ \delta(x - 0)e^{i\omega t} \tag{29a}$$

$$q_w = 0 \tag{29b}$$

where q_o is the amplitude of the load with frequency ω equal to the natural frequency of the hinged shell.

Eqs. (29a) and (29b) are substituted in (26a) first and consider only the steady state solution. The resulting equation is then substituted into Eq. (22a) for the case where n is taken on odd integers. In order that the final solution satisfies the following condition

$$u = 0 \quad \text{or} \quad \frac{\partial \Psi}{\partial x} = 0 \qquad \text{at} \qquad x = 0$$

for hinged shell, the frequency equation is found to be

$$S = \sum_{n=1,3,5---}^{\infty} \frac{n(2n+1)\ (A_{3n} - \lambda\omega^2)\ \Gamma\ (\frac{1}{2}n)\Gamma(\frac{1}{2}n)}{\lambda^2(\omega^2 - \omega^2{}_{1n})(\omega^2 - \omega^2{}_{2n})\ \ (n+1)\Gamma(\frac{1}{2} + \frac{n}{2})\Gamma(\frac{1}{2} + \frac{n}{2})} = 0 \tag{30}$$

(b) Completely clamped shell:

The forcing functions are taken as

$$q_v = 0 \tag{31a}$$

$$q_w = q_o\ \delta(x - 0)e^{i\omega t} \tag{31b}$$

Substitution of Eqs. (31) into Eq. (26b) and then Eq. (22b) for the case where n is taken on even integers and satisfies

$$w\Big|_{x=0} = 0 \tag{32}$$

for completely clamped shell, the frequency equation is obtained as follows:

$$S = \sum_{n=2,4,6---}^{\infty} \frac{(2n+1)\ [\lambda\omega^2 - n(n+1) + (1-\mu)]\ \ P_n^2(0)}{\lambda^2(\omega^2 - \omega_{1n}{}^2)(\omega^2 - \omega_{2n}{}^2)} = 0 \tag{33}$$

The natural frequencies, ω, for both hinged and clamped shells may be obtained by trial and error procedure on an electronic computer. A value of n is first selected. The roots of Equations (30) and (33) are searched between every two consecutive $\lambda\omega^2_{in}$ (i = 1 and 2 correspond to upper and lower branches respectively). A number of trial values of $\lambda\omega^2$ with constant increment are fed in the Eqs. (30) or (33). The curves S vs $\lambda\omega^2$ are plotted. The frequencies may be obtained by interpolating between the two consecutive $\lambda\omega^2$ where the corresponding values of S change signs. The value of n for the series solutions is then increased until the variation of frequencies obtained is acceptable. After the approximate values of ω are estimated, a more sophisticated numerical device similar to the method of false positions is programmed which may be used to search for more accurate ω in the vicinities of the approximate ones.

The first three natural frequencies for a hemispherical shell completely clamped along its edge with $\frac{h}{a}$ = 0.01, 0.02, and 0.05 and n = 60, obtained by interpretation, are listed in Table 1. A plot of S vs frequencies are shown in Figure (3). From the figure, it is seen that ω approach to ω_{2n} as the shell becomes thinner and increase as the shell thickness increases. For the case $\frac{h}{a}$ = 0.01, the largest difference between the three lowest frequencies for the roller-clamped and completely clamped cases is 8%. It may be concluded that for a sufficiently thin hemispherical shell, the results obtained for roller-clamped edge may be used for completely clamped case.

$\frac{h}{a}$ = 0.01	$\frac{h}{a}$ = 0.02	$\frac{h}{a}$ = 0.05
0.51	0.53	0.57
0.78	0.81	0.98
0.85	0.95	1.56

Table 1. $\lambda\omega^2$ For Completely Clamped Shells for $\mu = \frac{1}{3}$

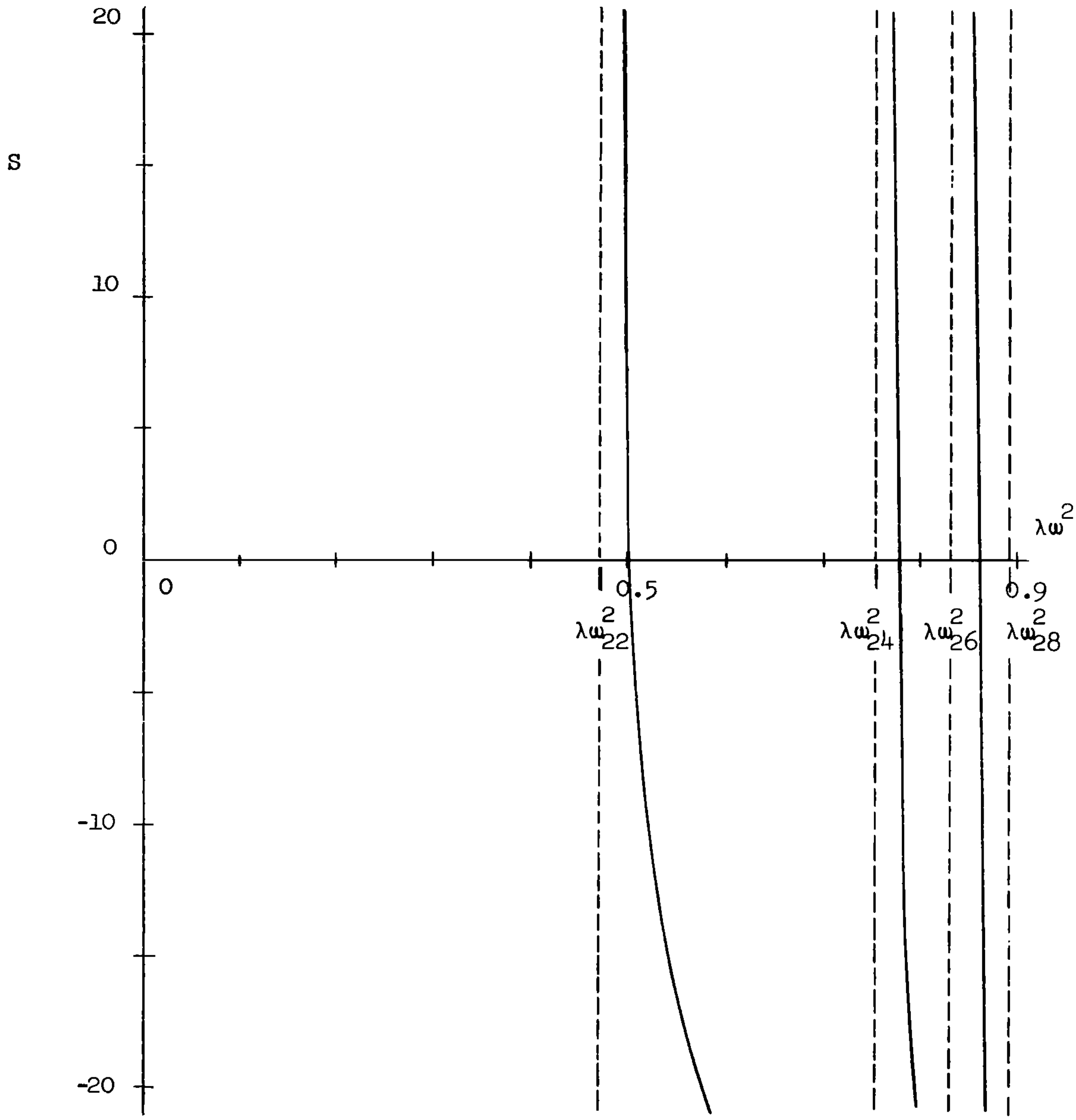

Fig. (3-a) S vs. $\lambda\omega^2$ for $\frac{h}{a} = 0.01$ and $\mu = \frac{1}{3}$

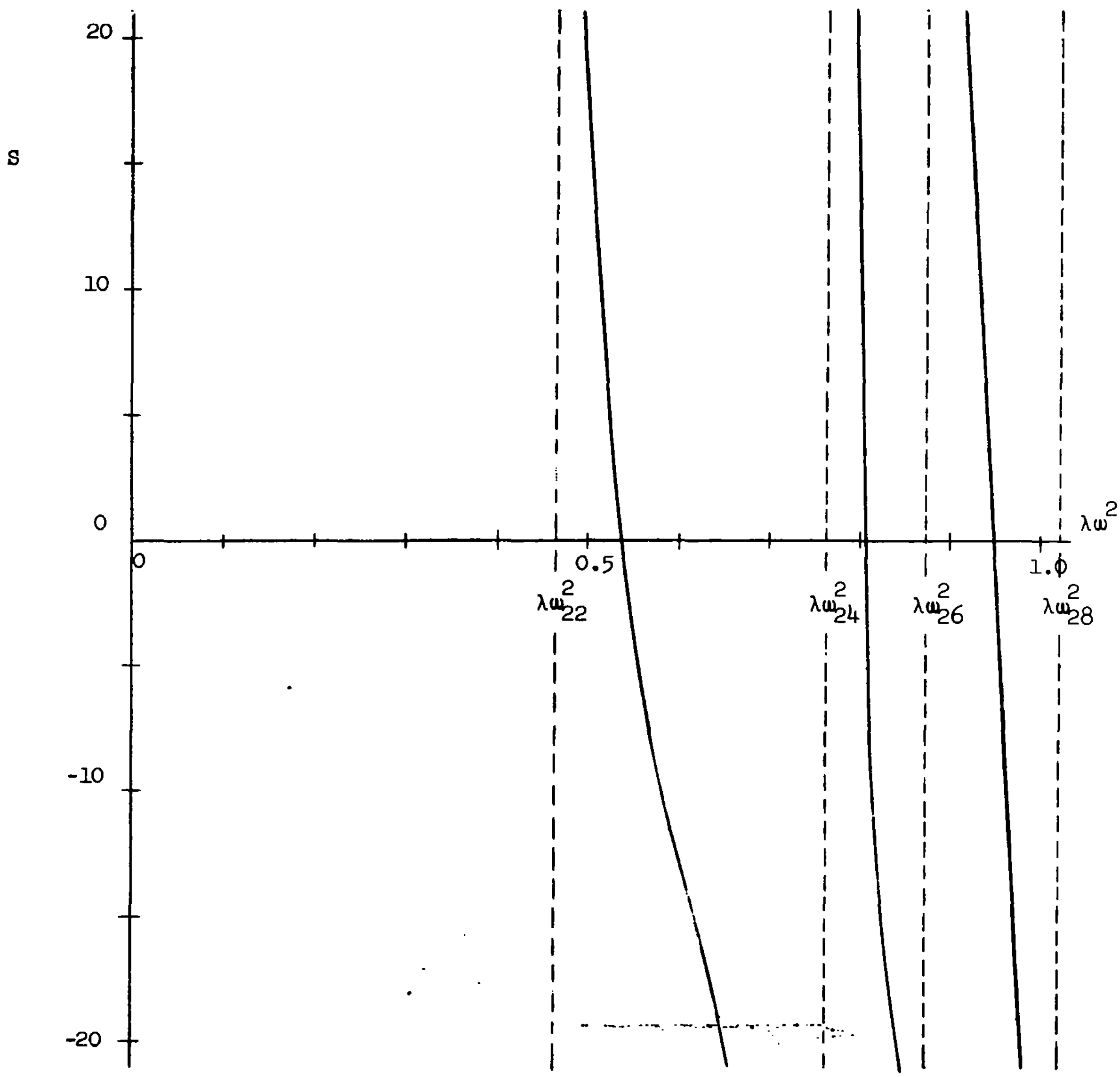

Fig. (3-b) S vs. $\lambda\omega^2$ for $\frac{h}{a} = 0.02$ and $\mu = \frac{1}{3}$

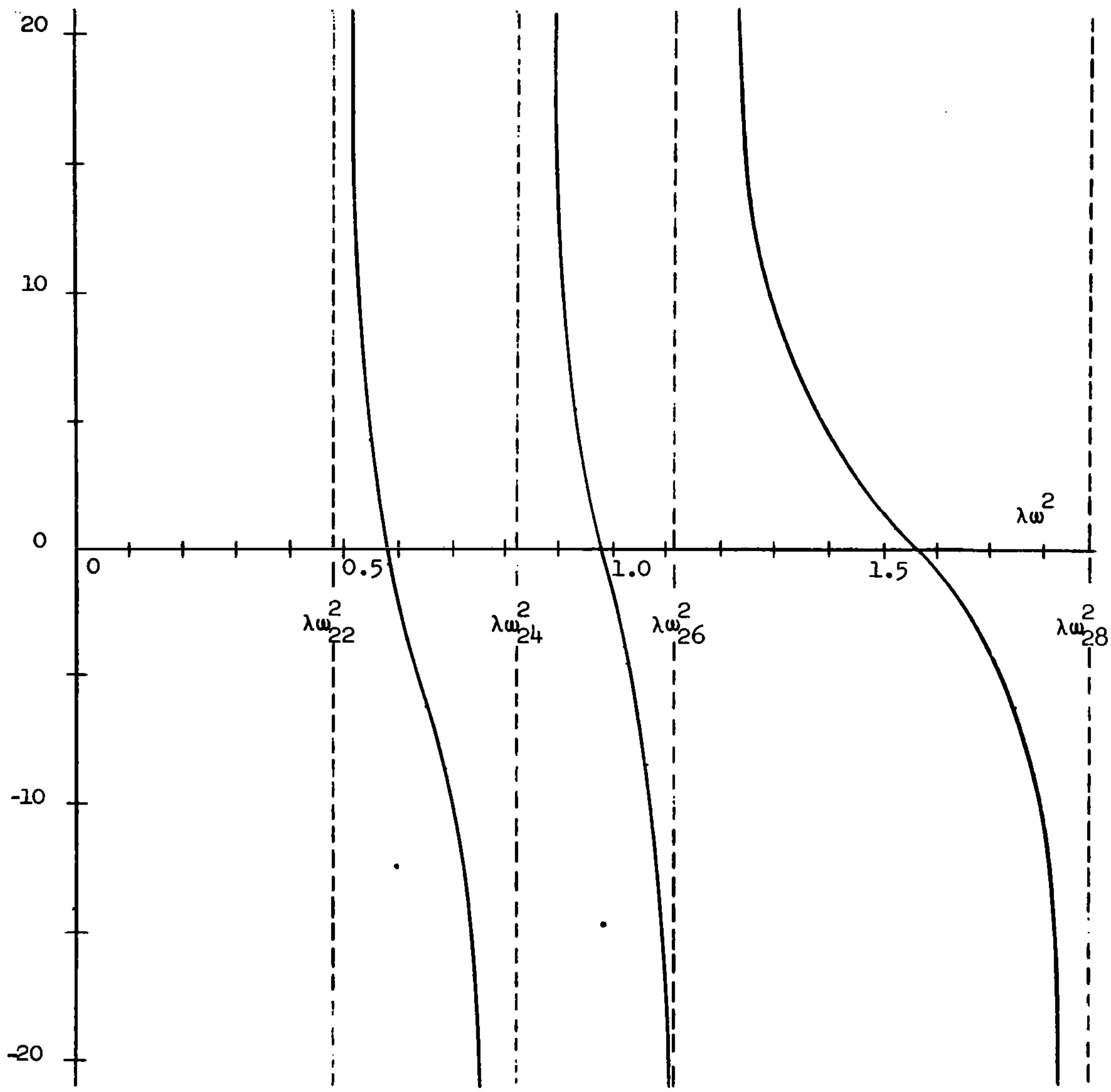

Fig. (3-c) S vs. $\lambda\omega^2$ for $\frac{h}{a}$ = 0.05 and $\mu = \frac{1}{3}$

Conclusions

1. The variation of shell thickness has very little effect on the upper branch frequencies for hemispherical shells having roller-hinged edge and roller-clamped edge. However, lower branch frequencies are affected significantly as the shell thickness increases.

2. The lower branch frequencies for roller-hinged and roller-clamped shells are not bounded which contradicts the results obtained according to membrane theory given in [4].

3. The frequencies of a shell completely clamped along its edge approach to the frequencies of a roller-clamped shell when the thickness of the shell decreases. It may be concluded that for a sufficiently thin shell, the dynamic response for a roller-clamped shell may be used to represent a completely clamped case for the regions away from the boundary.

Bibliography

[1] Naghdi, P. M., Kalnins, A., "On Vibrations of Elastic Spherical Shells", J. Appl. Mech., March 1962, p. 65.

[2] Kalnins, A., "Effect of Bending on Vibrations of Spherical Shells", J. Acous. Soc. Amer., 36, No. 1, Jan. 1964, pp. 74-81.

[3] Hwang, C., "Extensional Vibration of Axisymmetric Shells", AIAA Journal, Jan. 1965.

[4] Baker, Wilered E., "Axisymmetric Modes of Vibration of Thin Spherical Shell", The J. Acous. Soc. Amer., 33, 12, Dec. 1961, pp. 1749-1758.

[5] Kalnins, A., "Free Vibration of Rotationally Symmetric Shells", J. Acous. Soc. Amer., 36, 7, July 1964, pp. 1355-1365.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546