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A CLASS OF RELATED SPACE-TIMES

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1. Introduction.

In a recent paper Kerr and Schild [1] have considered the solution of the vacuum field equations $R_{ab} = 0$, for a space-time[1)] with metric tensor g_{ab} of the form

$$g_{ab} = \eta_{ab} + k_a k_b .$$

Here η_{ab} is the metric of Minkowski space in coordinates which are cartesian, but not necessarily rectangular, and k_a is a null vector field; $g^{ab}k_a k_b = 0$. The contravariant components of the metric tensor take the simple form

$$g^{ab} = \eta^{ab} - k^a k^b ,$$

where $k^a = g^{ab}k_b$; consequently k_a is also a null vector of the flat space-time. An important result for the above space-times is that they are algebraically special in the sense of the Pirani-Petrov classification.

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1) By space-time we will mean a four-dimensional Riemannian space of signature +2. Quantities in space-time will be defined as in: Riemannian Geometry by L. P. Eisenhart (Princeton University Press).

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More generally, given an arbitrary space-time $\tilde{V}_4$ with metric tensor $\tilde{g}_{ab}$, and a null vector field k_a, then we can consider the tensor

$$g_{ab} = \tilde{g}_{ab} + k_a k_b \,, \tag{1.1}$$

as the metric tensor of a related space-time V_4. In this note the relationship of V_4 to $\tilde{V}_4$ is considered and a generalisation of the above result concerning the Petrov type of V_4, when $\tilde{V}_4$ is a flat space-time, is given.

2. <u>Preliminary Results</u>. Let g^{ab} and $\tilde{g}^{ab}$ denote the contravariant metric tensors of V_4 and $\tilde{V}_4$ respectively. The tensor S^{ab} is defined by:

$$S^{ab} = g^{ab} - \tilde{g}^{ab} \,.$$

Hence

$$g^{ab} g_{bc} = (\tilde{g}^{ab} + S^{ab})(\tilde{g}_{bc} + k_b k_c) = \delta^a{}_c \,,$$

which reduces to

$$(\tilde{g}^{ab}k_b + S^{ab}k_b)k_c + S^{ab}\tilde{g}_{bc} = 0\ . \tag{2.1}$$

The contraction of this equation with $\tilde{k}^c = \tilde{g}^{cd}k_d$, and the assumption that k_c is a null vector field of $\tilde{V}_4$ gives

$$S^{ab}\tilde{g}_{bc}\tilde{g}^{cd}k_d = S^{ab}k_b = 0\ .$$

From (2.1) we see that

$$S^{ab} = -\tilde{k}^a\tilde{k}^b$$

and further

$$k^a = g^{ab}k_b = \tilde{g}^{ab}k_b + S^{ab}k_b = \tilde{k}^a\ .$$

The contravariant form of (1.1) is therefore

$$g^{ab} = \tilde{g}^{ab} - \tilde{k}^a\tilde{k}^b = \tilde{g}^{ab} - k^ak^b\ , \tag{2.2}$$

and k_a is also a null vector field of V_4.

The metric relations (1.1) and (2.2) will imply a correspondence between quantities in V_4 and $\tilde{V}_4$. In particular, there will be relations between their respective affine connexions and curvature tensors.

We will denote by ";" covariant differentiation with respect to the connexion Γ^a_{bc} of V_4, and by "||" covariant differentiation with respect to $\tilde{\Gamma}^a_{bc}$ the connexion of $\tilde{V}_4$. A simple calculation gives

$$\Gamma^a_{bc} - \tilde{\Gamma}^a_{bc} = \tilde{g}^{ad}(k_bk_{[d||c]} + k_ck_{[d||b]}) + k^a(k_{(b||c)} + k_{(b}q_{c)}) \tag{2.3}$$

where $k_{(a||b)}$ and $k_{[a||b]}$ denote the symmetric and antisymmetric parts of $k_{a||b}$, and $q_c = k_{c||d}k^d$. We note also that

$$\Gamma^a_{bc}k^b = \tilde{\Gamma}^a_{bc}k^b + \tilde{g}^{ad}k_{(c}q_{d)} \tag{2.4}$$

$$\Gamma^a_{bc}k_a = \tilde{\Gamma}^a_{bc}k_a - k_{(b}q_{c)} , \tag{2.5}$$

$$\Gamma^a_{bc}k^b k^c = \tilde{\Gamma}^a_{bc}k^b k^c , \tag{2.6}$$

and

$$\Gamma^a_{ba} = \tilde{\Gamma}^a_{ba} . \tag{2.7}$$

It follows therefore that

$$q_c = k_{c||d}k^d = k_{c;d}k^d ; \quad \tilde{q}^c = q^c . \tag{2.8}$$

From equations (2.4)-(2.7) we deduce the following theorem.

Theorem 2.1. The expansion [2] of the null vector field k_a is invariant under the transformation $\tilde{V}_4$ V_4 given by (1.1). Further if k_a is a geodesic vector field of $\tilde{V}_4$, then it is also geodesic in V_4, and the shear and rotation [2] of k_a are also invariant.

Proof.

We have the relation

$$k_{a||b} = k_{a;b} + (\Gamma^d_{ab}k_d - \tilde{\Gamma}^d_{ab}k_d) ,$$

which with (2.5) implies

$$k_{a||b} = k_{a;b} - k_{(a}q_{b)} \, , \tag{2.9}$$

and

$$k^{a||b} = k^{a;b} - k^{[a}q^{b]} \, . \tag{2.10}$$

The expansion $\tilde{\theta}$ of k_a in V_4 is defined by $\tilde{\theta} = \frac{1}{2} k^{a}{}_{||a}$.
From (2.7) or (2.9) we have

$$\frac{1}{2} k^{a}{}_{||a} = \frac{1}{2} k^{a}{}_{;a} = \theta \, ,$$

and the first part of the theorem follows.

For the shear $\tilde{\sigma}$ and the rotation $\tilde{\omega}$ of k_a in $\tilde{V}_4$ we have

$$2\tilde{\omega}^2 = k_{[a||b]}k^{a||b}, \text{ and } 2\tilde{\sigma}^2 = k_{(a||b)}k^{a||b} - 2\tilde{\theta}^2 \, .$$

Now from (2.10)

$$k_{(a||b)}k^{a||b} = (k_{(a;b)} - k_{(a}q_{b)})(k^{a;b} - k^{[a}q^{b]})$$

$$= k_{(a;b)}k^{a;b} - k_{(a;b)}k^{a}q^{b} = k_{(a;b)}k^{a;b} - \frac{1}{2} q^{b}q_{b} \, .$$

We have already that $\tilde{\theta} = \theta$ and therefore

$$2\tilde{\sigma}^2 = k_{(a;b)}k^{a;b} - 2\theta^2 - \frac{1}{2} q^{b}q_{b} = 2\sigma^2 - \frac{1}{2} q^{b}q_{b} \tag{2.11}$$

Similarly from (2.9) we have for the rotations

$$2\tilde{\omega}^2 = 2\omega^2 - \frac{1}{2} q^{b}q_{b} \, . \tag{2.12}$$

The condition for k_a to define a geodesic congruence in $\tilde{V}_4$ is $q_a = \lambda k_a$; and from (2.8) this is invariant under $\tilde{V}_4$ V_4. The second part of the theorem now follows from (2.11) and (2.12).

3. The Generalisation of the Kerr-Schild Result. An elementary calculation gives the result

$$R_{ab}k^a k^b = \tilde{R}_{ab}k^a k^b + q^b q_b \tag{3.1}$$

from which follows

Theorem 3.1.

If $\tilde{V}_4$ and V_4 are vacuum space-times then k_a necessarily defines a null congruence of geodesics in both $\tilde{V}_4$ and V_4.

Proof.

From (3.1) $q^b q_b = 0$, and also from the definition of q_b, $k^b q_b = 0$. Since space-time is of signature $+2$, no two real null vectors can be orthogonal unless one is a nultiple of the other. Therefore we must have

$$q_a = k_{a||b}k^b = k_{a;b}k^b = \lambda k_a \, ,$$

and the congruences defined by k_a are necessarily geodesic.

Theorem 3.2.

If $\tilde{V}_4$ and V_4 are vacuum space-times and $\tilde{V}_4$ is algebraically special in the Pirani-Petrov classification with k_a as a double root

<u>of its Debever equation, then V_4 is necessarily algebraically special with k_a as a double root of its Debever equation.</u>

<u>Proof</u>.

The Goldberg-Sachs theorem [3] implies that k_a is a geodesic and shear-free congruence of $\tilde{V}_4$. The conditions of the theorem ensure that both these properties hold also for the congruence defined in V_4 by k_a. Hence by the Goldberg-Sachs theorem V_4 is algebraically special with k_a as a double root of its Debever equation.

The conditions on $\tilde{V}_4$ can be considerably weakened and yet preserve the result for the special Einstein space V_4. In fact if it is assumed that k_a defines a geodesic and shear-free congruence in $\tilde{V}_4$, ($\tilde{V}_4$ is not necessarily algebraically special) then it follows that it also defines a geodesic and shear-free congruence in V_4, and hence from the Goldberg-Sachs theorem V_4 is algebraically special.

REFERENCES

[1] Report to the International Meeting on General Relativity, Florence, Italy, September 1964.

[2] J. Ehlers & W. Kundt. Chapter 2 of GRAVITATION ed. L. Witten, John Wiley & Sons, Inc. 1962.

[3] W. Kundt & A. H. Thompson, Comptes Rendus <u>254</u>, p.4257-4259. 1962.