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# ON THE INTERMEDIATE ORBITS OF THE EARTH'S ARTIFICIAL SATELLITES

by

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# ON THE INTERMEDIATE ORBITS OF THE EARTH'S ARTIFICIAL SATELLITES*

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by E. P. Aksenov

## SUMMARY

This paper provides new forms for the solution of the problem of construction of intermediate orbits, which may be more practical for the calculation of satellite's coordinates and more useful during the indispensable expansion in series of the perturbation function.

* * *

## INTRODUCTION.-

At construction of the analytical theory of AES motion it is interesting to take for the intermediate orbit instead of the Keplerian ellipse a more complex curve, that would include the most substantial perturbations conditioned by the distinction between the Earth's gravitational field and the central. Such an orbit can be constructed on the basis of the generalized theory of two fixed centers, inasmuch as the force function of that problem differs from the terrestrial gravitation potential only by the terms of second order of smallness relative to Earth's contraction and admits a strict integration of the differential equations of motion [1, 2].

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* O PROMEZHUTOCHNYKH ORBITAKH ISKUSSTVENNYKH SPUTNIKOV ZEMLI

Formulas, describing such an intermediate orbit, were obtained by us earlier [3]. They express the satellite's conical coordinates as explicit functions of a certain intermediate variable, which is, in its turn, linked with time. However, the method of calculation of satellite coordinates by such formulas is not unique, and possibly not the most convenient. That is why we have decided to provide in the present work other forms of solution of the problem considered. On the one hand, these forms of solution may in some cases result more practical for the calculation of satellite coordinates, and they will, on the other hand, be more useful at expansion of the perturbation function in series, which must necessarily be carried out when accounting the perturbations from other perturbing factors.

## 1. - STATEMENT OF THE PROBLEM.

Let us select a rectangular system of coordinates Oxyz with origin at mass center of the Earth in such a way that the axis Oz be directed at the celestial pole, and the axis xy coincide with the Earth's equatorial plane. Then the differential equation of satellite's intermediate motion will be written :

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z}, \tag{1}$$

where the force function U is given by the formula

$$U = \frac{fm}{2}\left\{\frac{1}{\sqrt{x^2+y^2+(z-ic)^2}} + \frac{1}{\sqrt{x^2+y^2+(z+ic)^2}}\right\}, \tag{2}$$

in which $f$ is the gravitational constant, $m$ is the Earth's mass, $i=\sqrt{-1}$, $c$ is a certain constant, numerically equal to $\sim 210$ km.

As already noted, the equations (1) with the force function (2) are rigorously integrated in quadratures. When integrating these equations, it is appropriate to pass to new variables $\xi, \eta, w$, linked with x, y, z by the correlations

$$\begin{aligned} x &= \sqrt{(\xi^2+c^2)(1-\eta^2)}\cos w, \\ y &= \sqrt{(\xi^2+c^2)(1-\eta^2)}\sin w, \\ z &= \xi\eta. \end{aligned} \tag{3}$$

Moreover, instead of the time t it is appropriate to introduce a new variable $\tau$ according to the equality

$$dt = (\xi^2 + c^2\eta^2)\,d\tau. \qquad (4)$$

Formulas, expressing the spheroidal coordinates $\xi, \eta, w$ through $\tau$, and the relationship between the variable $\tau$ and the time $t$ were obtained by us in [3]. In the same work were introduced the elements of the intermediate orbit. Before bringing forth all the formulas necessary here, let us briefly recall the qualitative pattern of satellite motion and the geometrical meaning of some of the orbit elements.

The motion of the satellite takes place in such a fashion that

$$\begin{gathered} a(1-e) \leqslant \xi \leqslant a(1+e), \\ -s \leqslant \eta \leqslant +s \leqslant 1, \end{gathered} \qquad (5)$$

where a, e, s are constants, determined by the initial conditions. The region, in which the satellite moves, constitutes a certain toroidal space, bounded by two ellipsoids of revolution with minor semi-axes $a(1-e)$ and $a(1+e)$ and a hyperboloid of revolution, whose equation is

$$\frac{x^2+y^2}{c^2(1-s^2)} - \frac{z^2}{c^2s^2} = 1.$$

The major semi-axes of the bounding ellipsoids are respectively equal to $\sqrt{a^2(1-e)^2+c^2}$ and $\sqrt{a^2(1+e)^2+c^2}$. The constants a, e, s, thus define fully the region of satellite motion, and therefore, it is practical to use them for orbit elements.

According to the work [3], the variables $\xi, \eta, w$ are expressed through $\tau$ in the following fashion:

$$\begin{gathered} \xi = \frac{\bar{p}(1+\varkappa\cos\psi)}{1+\bar{e}\cos\psi}, \\ \eta = s\cdot\sin\varphi, \\ w = \operatorname{arc\,tg}(\cos i \operatorname{tg}\varphi) + \bar{\Omega}, \\ \bar{\Omega} = c_0\varphi + \bar{c}_0\psi + c_2\sin 2\varphi + \bar{c}_1\sin\psi + \bar{c}_2\sin 2\psi + \\ + \bar{c}_3\sin 3\psi + \bar{c}_4\sin 4\psi + c_5, \end{gathered} \qquad (6)$$

where $\bar{p}, \varkappa, \bar{e}, c_0, \bar{c}_0, c_2, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ are constants, depending on the elements a, e, s, and $c_5$ is an arbitrary integration constant.

Let us introduce the parameter $\varepsilon$ according to formula

$$\varepsilon = \frac{c}{a(1-e^2)}.$$

Inasmuch as $a(1-e)$ is greater than the Earth's radius $\varepsilon < \frac{1}{30}$. That is why all the constants, entering into the formula (6), may be expanded in series by powers $\varepsilon$. Rejecting in the expansion the terms of the order $\varepsilon^6$, we find [3]

$$\bar{p} = a(1-e\bar{e}),$$

$$\bar{e} = e\{1+\varepsilon^2(1-e^2)(1-2s^2)+\varepsilon^4(1-e^2)[3-16s^2+14s^4-2e^2(1-s^2)^2]\},$$

$$\varkappa = \varepsilon^2 e\{(1-2s^2)+\varepsilon^2[(3-16s^2+14s^4)-e^2(1-2s^4)]\},$$

$$c_0 = -\frac{1}{2}\varepsilon^2\cos i\left\{1-\frac{\varepsilon^2}{8}[(30-35s^2)+e^2(2+3s^2)]\right\}(1-e^2),$$

$$\bar{c}_0 = -\frac{1}{2}\varepsilon^2\cos i\left\{(2+e^2)+\frac{\varepsilon^2}{8}[(24-56s^2)-e^2(4+64s^2)-e^4(2+3s^2)]\right\},$$

$$c_2 = \frac{3}{22}\varepsilon^4(1-e^2)^2 s^2\cos i,$$

$$\bar{c}_1 = -2\varepsilon^2 e\cos i\left\{1+\frac{\varepsilon^2}{8}[(4-28s^2)-e^2(6+7s^2)]\right\},$$

$$\bar{c}_2 = -\frac{\varepsilon^2}{4}e^2\cos i\left\{1-\frac{\varepsilon^2}{2}[11+e^2(1+s^2)]\right\},$$

$$\bar{c}_3 = \frac{\varepsilon^4}{4}e^3\cos i(2-s^2),$$

$$\bar{c}_4 = \frac{\varepsilon^4}{64}e^4\cos i(2+s^2),$$

where

$$i = \arcsin s.$$

As to the variables $\varphi$ and $\psi$, they are linked with the variable $\tau$ by the formulas

$$\varphi = am[\sigma_1(\tau+c_3), k_1],$$
$$\psi = am[\sigma_2(\tau+c_4), k_2], \qquad (7)$$

where $c_3$ and $c_4$ are still two more arbitrary integration constants and the modules $k_1$, $k_2$ and the constants $\sigma_1$ and $\sigma_2$ are determined from the equalities

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$$k_1^2 = \varepsilon^2(1-e^2)s^2\{1-4\varepsilon^2(1-s^2)\},$$
$$k_2^2 = \varepsilon^2 e^2 s^2 - \varepsilon^4 e^2(1-10s^2+11s^4+e^2s^4), \tag{8}$$

$$\sigma_1 = \sqrt{fma(1-e^2)}\Big\{1+\frac{\varepsilon^2}{2}(3+e^2)(1-s^2)-$$
$$-\frac{\varepsilon^4}{8}[(9+2s^2-11s^4)+e^2(6+28s^2-34s^4)+e^4(1+2s^4-3s^4)]\Big\},$$
$$\sigma_2 = \sqrt{fma(1-e^2)}\Big\{1-\frac{\varepsilon^2}{2}(3-4s^2-e^2)- \tag{9}$$
$$-\frac{\varepsilon^4}{8}[(9-72s^2+64s^4)+e^2(2-40s^2+48s^4)+e^4]\Big\}.$$

Therefore, the coordinates of the satellite are certain combinations of elliptical functions of $\underline{\tau}$ with small modules (of the order $^1/_{30}$).

Formulas (3) — (9) will fully resolve the stated problem provided we find a formula, by which $\underline{\tau}$ may be computed for any given moment of time $\underline{t}$. Indeed, if the elements of satellite's orbit are known, and if $\underline{\tau}$ is known for the given $\underline{t}$, $\xi$, $\eta$, $w$ can be found with the aid of tables for elliptical functions, and then, the rectangular coordinates of the satellite can be computed with the help of formula (3).

However, for this purpose it is useful to have other formulas. Here we shall take advantage of the circumstance, that for elliptical functions there exist trigonometric series, very rapidly converging for sufficiently small values of their modules.

## 2. - EXPANSIONS FOR $\varphi$ AND $\psi$.

For am ($\tilde{\lambda}$, k) we have the following well known expression

$$am(\tilde{\lambda}, k) = \frac{\pi\tilde{\lambda}}{2K} + \sum_{n=1}^{\infty}\frac{2}{n}\frac{q^n}{1+q^{2n}}\sin\frac{\pi n\tilde{\lambda}}{K}, \tag{10}$$

where

$$K = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\dots\right\}, \tag{11}$$

$$q = \frac{k^2}{16}\left\{1+\frac{k^2}{2}+\frac{21}{64}k^4+\dots\right\}. \tag{12}$$

Rejecting the terms with $k^6$, we shall find from formulas (10) — (12)

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$$am(\widetilde{\lambda}, k) = \bar{\lambda} + \frac{k^2}{8}\left(1 + \frac{k^2}{2}\right)\sin 2\bar{\lambda} + \frac{k^4}{256}\sin 4\bar{\lambda}, \qquad (13)$$

where

$$\bar{\lambda} = \left(1 - \frac{1}{4}k^2 - \frac{5}{64}k^4\right)\widetilde{\lambda}. \qquad (14)$$

Substituting in (13) and (14) $k_1$ and $k_2$ in place of k, and then replacing them by the expressions (8) and (9), we shall obtain

$$\varphi = u + \frac{\pi}{2} - \frac{1}{8}\varepsilon^2(1-e^2)s^2\left\{1 - \frac{\varepsilon^2}{2}(8 - 9s^2 + e^2s^2)\right\}\sin 2u, \qquad (15)$$

$$\psi = v + \frac{1}{8}\varepsilon^2e^2\left\{s^2 - \frac{\varepsilon^2}{2}(2 - 20s^2 + 22s^4 - 3e^2s^4)\right\}\sin 2v, \qquad (16)$$

where

$$u = n_1\tau + \left(n_1c_3 - \frac{\pi}{2}\right), \quad v = n_2(\tau + c_4), \qquad (17)$$

with, at the same time,

$$n_1 = \sqrt{fma(1-e^2)}\left\{1 + \frac{\varepsilon^2}{4}[(6 - 7s^2) + e^2(2 - s^2)] - \right.$$

$$\left. - \frac{\varepsilon^4}{8}\left[\left(9 - 3s^2 - \frac{43}{8}s^4\right) + e^2\left(6 + 34s^2 - \frac{165}{4}s^4\right) + e^4\left(1 + s^2 - \frac{11}{8}s^4\right)\right]\right\},$$

$$n_2 = \sqrt{fma(1-e^2)}\left\{1 - \frac{\varepsilon^2}{4}[(6 - 8s^2) - e^2(2 - s^2)] - \right.$$

$$\left. - \frac{\varepsilon^4}{8}\left[(9 - 72s^2 + 64s^4) - e^2(23s^2 - 30s^4) + e^4\left(1 + s^2 - \frac{11}{8}s^4\right)\right]\right\}.$$

Note, that in formulas (15) and (16) all terms with amplitudes of the order $10^{-8}$ and above were rejected.

## 3. - EXPANSION for $\frac{1}{\xi}$, $\eta$ and $\bar{\Omega}$

For Jacobi elliptical functions we have

$$\operatorname{sn}\widetilde{\lambda} = \frac{2\pi}{kK}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\sin(2n-1)\frac{\pi\widetilde{\lambda}}{2K}, \qquad (20)$$

$$\operatorname{cn}\widetilde{\lambda} = \frac{2\pi}{kK}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}\cos(3n-1)\frac{\pi\widetilde{\lambda}}{2K}, \qquad (21)$$

where K and q are determined by the equalities (11) and (12).

That is why, preserving only the terms with amplitudes, greater than $10^{-9}$, we shall find

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$$\sin\varphi=\left(1+\frac{k_1^2}{16}+\frac{7}{256}k_1^4\right)\cos u-\frac{k_1^2}{16}\left(1+\frac{k_1^2}{2}\right)\cos 3u+\frac{k_1^4}{256}\cos 5u, \quad (22)$$

$$\cos\psi=\left(1-\frac{k_2^2}{16}-\frac{9}{256}k_2^4\right)\cos v+\frac{k_2^2}{16}\left(1+\frac{k_2^2}{2}\right)\cos 3v+\frac{k_2^4}{256}\cos 5v, \quad (23)$$

where u and $v$ are given by formulas (17).

Subsequently, we have

$$\cos 2\psi=-\frac{k_2^2}{8}+\cos 2v+\frac{k_2^2}{8}\cos 4v+\ldots, \quad (24)$$

$$\cos 3\psi=\cos 3v+\ldots, \quad (25)$$

$$\sin 2\psi=\sin 2v+\frac{k_2^2}{8}\sin 4v+\ldots, \quad (26)$$

$$\sin 3\psi=\sin 3v+\ldots, \quad (27)$$

$$\sin 4\psi=\sin 4v+\ldots. \quad (28)$$

From the first formula (6) we find

$$\frac{p}{\xi}=A_0+A_1\cos\psi+A_2\cos 2\psi+A_3\cos 3\psi, \quad (29)$$

where

$$p=a(1-e^2),$$

$$A_0=1+\frac{1}{2}\varepsilon^2e^2(1-2s^2)+\frac{1}{2}\varepsilon^4e^2[(3-16s^2+14s^4)-e^2(1-2s^2)],$$

$$A_1=e\left\{1-\frac{1}{4}\varepsilon^4e^2(1-2s^2)^2\right\}, \quad (30)$$

$$A_2=-\frac{1}{2}\varepsilon^2e^2(1-2s^2)-\frac{1}{2}\varepsilon^4e^2[(3-16s^2+14s^4)-e^2(1-2s^4)],$$

$$A_3=\frac{1}{4}\varepsilon^4e^3(1-2s^2)^2.$$

We rejected in formula (29) all the terms with amplitudes of the order $\varepsilon^6$ and higher.

Substituting the equalities (23), (24) and (25) into (29), we shall finally find

$$\frac{p}{\xi}=a_0+a_1\cos v+a_2\cos 2v+a_3\cos 3v+a_4\cos 4v, \quad (31)$$

where

$$a_0=1+\frac{\varepsilon^2}{2}e^2(1-2s^2)+\frac{\varepsilon^4}{16}e^2[(24-128s^2+112s^4)-e^2(8+s^2-18s^4)],$$

$$a_1 = e\left\{1 - \frac{\varepsilon^2}{16}e^2s^2 - \frac{\varepsilon^4e^2}{256}[(48 - 96s^2 + 80s^4) - 7e^2s^4]\right\}, \tag{32}$$
$$a_2 = -\frac{1}{2}\varepsilon^2e^2(1 - 2s^2) - \frac{\varepsilon^4e^2}{2}[(3 - 16s^2 + 14s^4) - e^2(1 - 2s^4)],$$
$$a_3 = \frac{\varepsilon^2e^3s^2}{16} + \frac{\varepsilon^4e^3}{32}(6 - 12s^2 + 10s^4 - e^2s^4),$$
$$a^4 = -\frac{\varepsilon^4e^4s^2}{16}(1 - 2s^2).$$

Let us now substitute (22) into the second formula (6). We then shall obtain

$$\eta = b_1 \cos u + b_3 \cos 3u, \tag{33}$$

where

$$b_1 = s\left\{1 + \frac{\varepsilon^2}{16}(1 - e^2)s^2 - \frac{\varepsilon^4}{256}(1 - e^2)s^2[64(1 - s^2) - 7s^2(1 - e^2)]\right\},$$
$$b_3 = -s^3\left\{\frac{\varepsilon^2}{16}(1 - e^2) - \frac{\varepsilon^4}{32}(1 - e^2) - [8(1 - s^2) - s^2(1 - e^2)]. \right. \tag{34}$$

We may analogously obtain the following formula for $\bar{\Omega}$:

$$\bar{\Omega} = \Omega + d_1 \sin v + d_2 \sin 2v + d_3 \sin 3v + d_4 \sin 4v + \bar{d}_2 \sin 2u,$$
$$\Omega = n_3\tau + c_5, \tag{35}$$

where

$$d_1 = -2\varepsilon^2e\cos i\left\{1 + \frac{\varepsilon^2}{16}[(8 - 56s^2) - e^2(12 + 13s^2)]\right\},$$
$$d_2 = -\frac{\varepsilon^2e^2}{4}\cos i\left\{1 - \frac{\varepsilon^2}{4}[(22 - 2s^2) + e^2(2 + s^2)]\right\},$$
$$d_3 = \frac{\varepsilon^4e^3}{8}\cos i\,(4 - 3s^2), \tag{36}$$
$$d_4 = \frac{\varepsilon^4e^4}{64}\cos i\,(2 - s^2),$$
$$\bar{d}_2 = -\frac{\varepsilon^4}{32}(1 - e^2)^2s^2\cos i,$$
$$n_3 = -\frac{3}{2}\varepsilon^2\sqrt{fmp}\cos i\left\{1 - \frac{\varepsilon^2}{3}[(6 + s^2) - e^2(4 - 26s^2)]\right\}. \tag{37}$$

## 4. - EXPRESSIONS FOR RECTANGULAR COORDINATES

From the third formula (6) we find

$$\sqrt{1 - \eta^2}\cos w = \cos\varphi\cos\bar{\Omega} - \cos i\sin\varphi\sin\bar{\Omega},$$
$$\sqrt{1 - \eta^2}\sin w = \cos\varphi\sin\bar{\Omega} + \cos i\sin\varphi\cos\bar{\Omega}. \tag{38}$$

That is why we shall have for satellite's rectangular coordinates x, y, z the following formulas :

$$x=\sqrt{\xi^2+c^2}(\cos\varphi\cos\bar{\Omega}-\cos i\sin\varphi\sin\bar{\Omega}),$$
$$y=\sqrt{\xi^2+c^2}(\cos\varphi\sin\bar{\Omega}+\cos i\sin\varphi\cos\bar{\Omega}),\qquad(39)$$
$$z=\xi\cdot s\cdot\sin\varphi.$$

Formulas (39) may serve for the computation of satellite's rectangular coordinates instead of formulas (3). They may be found to be more practical, for example, in the case when s is near the unity, that is, in the case of nearly polar orbits, when the third formula (6) is of little convenience for the calculation of $w$.

## 5. - CORRELATION BETWEEN $v$ AND t

From the equalities (17) and (35) we have

$$u=(1+\nu)v+\omega_0,$$
$$\Omega=\mu v+\Omega_0,\qquad(40)$$

where $\omega_0$ and $\Omega_0$ are constants, linked with the constants $c_3$, $c_4$ and $c_5$ by the correlations

$$\omega_0=n_1c_3-\frac{\pi}{2}-c_4n_1,$$
$$\Omega_0=c_5-n_3c_4,$$

while the constants $\nu$ and $\mu$ are determined from the formulas

$$1+\nu=\frac{n_1}{n_2},\ \mu=\frac{n_3}{n_2}$$

or

$$\nu=\frac{e^2}{4}(12-15s^2)+\frac{e^4}{64}[288-1296s^2+1035s^4-e^2(144+288s^2-510s^4)].\qquad(41)$$

$$\mu=-\frac{3}{2}\cos i-\frac{3}{16}(6-17s^2-24e^2s^2)\cos i.\qquad(42)$$

Evidently, we may take for the independent variable $v$ instead of $\tau$. Let us establish the relationship between $v$ and $t$. Since

$$dt=(\xi^2+c^2\eta^2)\,d\tau,$$

we have, on the basis of (17)

$$\frac{dt}{dv}=\frac{\xi^2+c^2\eta^2}{n_2}.\qquad(43')$$

Let us express the right-hand part of this equality through $v$. We then shall have

$$\frac{\xi^2+c^2\eta^2}{n^2}=\frac{1}{\bar{n}}\left\{\frac{(1-\bar{e}^2)^{3/2}}{(1+\bar{e}\cos v)^2}-\lambda+2\varepsilon^2(1-\varepsilon^2)^{3/2}\lambda_2\cos 2u+\right.$$

$$\left.+\frac{\varepsilon^2 e(1-e^2)^{3/2}}{(1+e\cos v)^3}(\bar{\lambda}_0+\bar{\lambda}_1\cos v+\bar{\lambda}_2\cos^2 v)\right\}, \tag{43}$$

where

$$\bar{n}=\sqrt{\frac{fm}{a^3}}\left\{1-\frac{3}{2}\varepsilon^2(1-e^2)(1-s^2)+\right.$$

$$\left.+\frac{3}{8}\varepsilon^4(1-e^2)(1-s^2)[(1+11s^2)-e^2(1-ss^2)]\right\},$$

$$\lambda=-\frac{\varepsilon^4}{16}(1-e^2)^{3/2}(24-96s^2+75s^4), \tag{44}$$

$$\bar{\lambda}_0=\frac{e}{4}(8-13s^2),$$

$$\bar{\lambda}_1=\frac{1}{4}[(8-12s^2)+e^2(8-11s^2)],$$

$$\bar{\lambda}_2=\frac{e}{2}(4-5s^2),\ \lambda_2=\frac{s^2}{4}.$$

In formula (43) we rejected all the terms which provide at integration periodical terms with amplitudes of the order $\varepsilon^4$ and higher.

Integrating (43'), we find

$$\bar{n}(t-t_0)+\bar{M}_0=2\,\mathrm{arc\,tg}\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\,\mathrm{tg}\,\frac{v}{2}-\bar{e}\sqrt{1-\bar{e}^2}\,\frac{\sin v}{1+\bar{e}\cos v}-\lambda v+$$

$$+\varepsilon^2(1-e^2)^{3/2}\lambda_2\sin 2u+\frac{\varepsilon^2 e(1-e^2)^{3/2}}{(1+e\cos v)^2}(\beta_0+\beta_1\cos v)\sin v, \tag{45}$$

where

$$\beta_0=2-3s^2,\ \beta_1=\frac{e}{4}(8-11s^2), \tag{46}$$

$\bar{M}_0$ being acceptable for the sixth arbitrary constant in place of $c_4$.

Applying the same approach for the solution of the equation (45) as was done in the work [3], we may express $v$ as a function of t. We shall forego here the details of operations, and we shall bring forth all formulas for the calculation of $v$ as a function of t.

$$v=\theta-\varepsilon^2\lambda_2(1+e\cos\theta)^2\sin 2(\theta+\omega)-\varepsilon^2 e(\beta_0+\beta_1\cos\theta)\sin\theta, \tag{47}$$

where

$$\omega=\nu\theta+\omega_0, \tag{48}$$

and $\theta$ is determined from the following equations:

$$\operatorname{tg}\frac{\theta}{2}=\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\operatorname{tg}\frac{E}{2}, \tag{49}$$

$$E-\bar{e}\sin E=M, \tag{50}$$

$$M=n(t-t_0)+M_0, \tag{51}$$

$$n=\frac{\bar{n}}{1-\lambda},\quad M_0=\frac{\bar{M}_0}{1-\lambda}. \tag{52}$$

Therefore, the relationship between $\theta$ and $t$ does not practically differ from the correlation between the true anomaly and the time in the Keplerian motion. Knowing $\theta$, we may easily find $v$ by formula (47).

Note that in deriving the formula (47) we rejected all the periodical terms proportional to $\varepsilon^4$, and all the terms of higher order. Although for most of practical problems the precision obtained here is quite sufficient, we shall point out, however, that if a higher precision is required, we may use the formulas (133) of [3], where the relationship between $\psi$ and $t$ is well established (in [3] the denotation $v$ was used instead of $\psi$). And $v$ can always be computed by the given $\psi$ using the formula

$$v=\psi+h_2\sin 2\psi+h_4\sin 4\psi, \tag{53}$$

where

$$h_2=-\frac{\varepsilon^2}{16}e^2s^2+\frac{\varepsilon^4e^2}{32}(2-20s^2+22s^4+e^2s^4),$$

$$h_4=\frac{3}{256}\varepsilon^4e^4s^4.$$

The last term in (53) was nearly always rejected, for it has the order $10^{-8}$.

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## CONCLUSIONS

In the present work we found all the formulas describing the intermediate motion of the satellite. These formulas depend on six arbitrary constants $a$, $e$, $s$ (or $i$), $\omega_0$, $\Omega_0$, $M_0$. When deriving all the formulas we conducted the expansions only by powers of small magnitude $\varepsilon^2$, but nowhere did we expand by powers $e$ or $s$, so that the obtained formulas are valid for any eccentricities and orbit inclinations. Wherever we conducted these expansions, we utilized series converging absolutely for any moments of time.

Note that numerous formulas were derived with a great reserve of precision, inasmuch as all periodical terms, proportional to $\varepsilon^4$ may, as a rule, by quietly dropped.

Assume now that the numerical values of elements $a$, $e$, $s$, $\omega_0$, $\Omega_0$, & $M_0$ are known to us (they may be found from observations or computed according to the initial data) and that we are required to compute the coordinates of the satellite for any moment of time. We shall, first of all, compute all the constants $\bar{n}$, $\bar{e}$, $\nu$, $\mu$, etc... Then we find M by the given t, using the equation (51). The solution of the equation (50) will give us E, after which utilizing formula (49) we shall find $\theta$. Having found the latter from formula (47), we shall compute $v$. Then, from correlations (40) we we shall determine u and $\Omega$.

The subsequent computations may be conducted according to various schemes. Let us indicate some of them:

1.- Using the formulas (15), (31) and (35) we shall find $\varphi$, $\psi$, $\bar{\Omega}$ and then, utilizing the equalities (38), we will be in a position to compute the rectangular coordinates of the satellite.

2.- We shall find $\varphi$, $\psi$, $\bar{\Omega}$, by the formulas (15), (16) and (35). After that the equalities (6) will give us $\xi$, $\eta$, $w$, and the formulas (3) — the rectangular coordinates.

3.- Knowing $v$, we may find $\tau$ from the second equality (17), and then, making use of the tables for elliptical functions, we will be able to find either $\varphi$, $\xi$, $\bar{\Omega}$ or $\xi$, $\eta$, $w$, and then x, y, z also.

**** THE END ****

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* GAISH stands for Gosudarstvennyy Astronomicheskiy Institut imeni Shternberga (State Astronom.Institute in the name of Sternberg)