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CALCULATION OF SMALL ROCKET TRAJECTORIES USING ONLY ROCKET-TO-LAUNCHER RADIAL VELOCITY DATA

by

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A very accurate method for obtaining rocket trajectories is the radio Doppler tracking system, known as DOVAP (Ref. 1 and 2). This system provides radial velocities (and by integration radial distances) between the rocket and the receiving site. In the DOVAP system, which was developed by the Ballistic Research Laboratory, a minimum of three well-separated ground stations are required. Three sets of radial distances are obtained and the rocket position is calculated by a triangulation technique. A simplified version of this system has been developed by J. C. Seddon (Ref. 3) in which only one station is required and in which a two-axis interferometer is used to supplement the radial velocity data. Since the Doppler measurement gives the radial distance, and the interferometer gives the direction cosines, their joint use provides the rocket position vector. It has been found, however, that a satisfactory interferometer requires a fairly elaborate instrumentation, and that its calibration is a very difficult procedure. Hence extremely skilled personnel are needed for its operation.

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In the present paper a method is developed for calculating rocket trajectories using only the Doppler data at one receiving site. With this method it is possible to calculate small rocket trajectories above the drag region ($Z>70$ km) with an accuracy of about 1 km. In practice only a few simple operations on a desk calculator are required in making use of this method. To justify the method, a detailed and lengthy discussion of its underlying theory is required. However the simplicity of the method is readily seen from the summary which is given after the theory. The calculations are based upon a measurement at the launching site of the rocket radial velocity $\dot{R}$ versus time using Doppler data at 73.6 Mc. If desired a counter can be used to determine $R = \int_0^t \dot{R}\, dt$ and this information can also be recorded during the rocket flight.

The method should be applicable to any small rocket trajectory with a peak altitude less than 200 kilometers; however since it has been checked mainly with typical Nike-Cajun data, the present report is based upon Nike-Cajun trajectories. Examination of Nike-Cajun performance data (Ref. 4) shows that, at the time of the second stage burnout, the horizontal range is of the order of 4 km. Assuming a maximum azimuth dispersion of the order of ± 30 degree, will place the "plane" of the trajectory at most 2 km away from the launch site. Since peak altitudes are typically in excess of 100 km, the radial distance R to the launch site is given with sufficient accuracy by:

$$R^2 = X^2 + Z^2 \tag{1}$$

where:

X = horizontal displacement

Z = altitude

Even if the Doppler site is as much as 4 km away from the plane of the trajectory, the approximation made in formula (1) introduces an error of less than 0.1% in the value of R at a radial distance of 100 km. For consistency with the system of coordinates introduced later in Section III, the Z and X directions are defined as the vertical direction and the direction of the velocity vector, respectively, at the peak of the trajectory. The origin of the coordinate system is the projection of the Doppler receiving site upon the XZ plane. In the simplified analysis (Sections I and II) the earth curvature and the Coriolis effects are neglected.

I. Calculation of Peak Time

Differentiating equation (1) with respect to time gives:

$$R\dot{R} = X\dot{X} + Z\dot{Z} \tag{2}$$

For the peak time (t_o), $\dot{Z} = 0$ and,

$$(R\dot{R})_{t_o} = (X\dot{X})_{t_o} \tag{3}$$

For the accuracy desired, the Coriolis acceleration can be neglected and we can assume that $\dot{X}$ is a constant. The error introduced by this approximation in the final trajectory determination is shown in section III to be less than 0.5 km for a typical Nike-Cajun flight. Applying equation (2) for two times $(t_o - t)$ and $(t_o + t)$ symmetrical with respect to the peak time t_o gives:

$$(R\dot{R})_{t_o-t} = (Z\dot{Z})_{t_o-t} + (X_{t_o} - \dot{X}t)\dot{X}$$

and,

$$(R\dot{R})_{t_o+t} = (Z\dot{Z})_{t_o+t} + (X_{t_o} + \dot{X}t)\dot{X}$$

and since $$(Z\dot{Z})_{t_o+t} = - (Z\dot{Z})_{t_o-t}$$

$$(R\dot{R})_{t_o-t} + (R\dot{R})_{t_o+t} = 2\, X_{t_o}\, \dot{X}$$

Making use of (3), this last equation can be written:

$$(R\dot{R})_{t_o-t} + (R\dot{R})_{t_o+t} = 2(R\dot{R})_{t_o}$$

or

$$(R\dot{R})_{-t} + (R\dot{R})_{+t} = 2(R\dot{R})_o \tag{4}$$

where the following simplifications in notation have been made in the subscripts: (t_o) changed to (o), $(t_o - t)$ changed to $(-t)$ and $(t_o + t)$ changed to $(+t)$. In this simplified notation X_o, Z_o, and R_o will represent values of X, Z and R corresponding to the peak time.

The calculation of peak time which is based upon formula (4) will be explained with the aid of Figure 1. This figure shows a curve of $R\dot{R}$ as a function of time for the free fall portion of a typical Nike-Cajun trajectory. The $R\dot{R}$ values were obtained from equation (2), using theoretical values of $Z\dot{Z}$ and $X\dot{X}$ for the trajectory defined by the parameters shown in Figure 1 and assuming no Coriolis effects. The ordinates A, B, and C correspond respectively to $(R\dot{R})_o$, $(R\dot{R})_{-t}$ and $(R\dot{R})_{+t}$. Thus:

$$B + C = 2A$$

It should be noted that in the illustration C is a negative quantity, as will always be the case for times $(t_o + t)$ remote from peak. Let the ordinates A', B' and C' correspond to times $(t_o + \Delta't)$, $(t_o - t + \Delta't)$ and $(t_o + t + \Delta't)$. This is equivalent to over-estimating the peak time. It is readily seen that:

$$B' + C' \approx B + C$$

but that

$$A' < A$$

Hence:

$$B' + C' > 2A', \text{ if peak time estimate is too large.} \qquad (5)$$

Conversely for ordinates A", B" and C", symmetrical with respect to $(t_o - \Delta''t)$:

$$B'' + C'' < 2A'', \text{ if peak time estimate is too small.} \qquad (5)'$$

The ordinates A", B" and C" are not shown on figure 1. A convenient form of equation (4) is as follows:

$$(R\dot{R})_{max} + (R\dot{R})_{min} = 2A \qquad (4)'$$

$$= 2(R\dot{R})_o$$

The determination of the peak time t_o can therefore be made in the fol'owing manner. First, the function $R\dot{R}$ is calculated and plotted. The values of $(R\dot{R})_{max}$ and $(R\dot{R})_{min}$ are read from the graph and added. This gives a very good estimate of the quantity 2A and consequently of the time t_o, which is the abscissa corresponding to the ordinate A. One could also obtain this estimate directly from interpolation between tabulated values of $R\dot{R}$ as a function of time. The graph is therefore mostly an aid in visualizing the operation and in smoothing the experimental data points. The value of t_o' obtained by this method is usually within a second of the correct value t_o.

The next step could be to select symmetrical times $(t_o' - t)$ and $(t_o' + t)$, such that corresponding ordinates B and C will be close to $(R\dot{R})_{max}$ and $(R\dot{R})_{min}$. Then one compares the value of (B + C) with the value of 2A, where A is now the ordinate corresponding to t_o', and use the criterion developed above (Formulas (5) and (5)') to determine whether the first estimate of t_o should be increased or decreased. By observing the manner in which the difference (B + C) - 2A is altered in successive trials, one can determine very rapidly (by extrapolation or interpolation) the value of t_o with an accuracy of ± 0.25 second. It should be noted that this is a rapidly convergent process, and consequently that the initial estimate of t_o need not be particularly accurate. Thus it is possible to begin with the theoretical maximum possible value of t_o, namely the value t_m at which R is maximum ($R\dot{R} = 0$). In practice this estimate would be too great by 5 to 30 seconds. In general the method based upon $(R\dot{R})_{max}$ and $(R\dot{R})_{min}$ yields the answer with sufficient accuracy.

It is also of interest to note that $R\dot{R}_{max}$ and $R\dot{R}_{min}$ occur at an altitude of no less than $\frac{2}{3}(Z_o)$, that is, above the drag region for a normal Nike-Cajun trajectory. The proof of this statement is as follows. The maximum and minimum values of $R\dot{R}$ occur when

$$\frac{d}{dt}(R\dot{R}) = \dot{Z}^2 + Z\ddot{Z} + \dot{X}^2 = 0$$

or $$Z = \frac{\dot{Z}^2}{-\ddot{Z}} + \frac{\dot{X}^2}{-\ddot{Z}} = \frac{\dot{Z}^2}{-\ddot{Z}}(1 + \alpha) \text{ where } \alpha = \frac{\dot{X}^2}{\dot{Z}^2}$$

Using the approximation $\ddot{Z} = g$, $\dot{Z} = gt$ and $h = \frac{1}{2}gt^2$ gives

$$Z_o - h = 2h(1 + \alpha)$$

or $$h = \frac{Z_o}{3 + 2\alpha}$$

showing that the distance dropped, h, is less than $\frac{Z_o}{3}$.

II. <u>Calculation of the Peak Altitude</u>

This calculation is based upon formulas (2) applied at a time $t_o + t_1$:

$$(R\dot{R})_1 = (X\dot{X})_1 + (Z\dot{Z})_1 \tag{6}$$

The left term of equation (6) is an experimentally determined quantity.

From equation (3): $$\dot{X} = \frac{(R\dot{R})_o}{X_o} \tag{7}$$

and from (6):

$$(R\dot{R})_1 = \dot{X}(X_o + \dot{X}t_1) + (Z_o - h_1)\dot{Z}_1 \tag{8}$$

Where h_1 is the distance fallen from peak during the time interval t_1. A method for calculating $\dot{Z}_1$ and h_1 is given in the appendix. From (7) and (8):

$$(R\dot{R})_1 - (R\dot{R})_o - \frac{(R\dot{R})_o^2}{X_o^2} t_1 - (Z_o - h_1)\dot{Z}_1 = 0$$

or:

$$(R\dot{R})_1 - (R\dot{R})_o - \frac{(R\dot{R})_o^2}{R_o^2 - Z_o^2} t_1 - (Z_o - h_1)\dot{Z} = 0 \qquad (9)$$

Equation (9) can be re-written as a third degree equation:

$$A_1 Z_o^3 + B_1 Z_o^2 + C_1 Z_o + D = 0 \qquad (10)$$

Where:

$$A_1 = \dot{Z}_1$$

$$B_1 = (R\dot{R})_o - (R\dot{R})_1 - h_1 \dot{Z}_1$$

$$C_1 = - R_o^2 A_1$$

$$D_1 = \left[(R\dot{R})_1 - (R\dot{R})_o\right] R_o^2 - t (R\dot{R})_o^2 + R_o^2 h_1 \dot{Z}_1$$

Then, Z_o is the solution of the equation:

$$f(z) = A_1 z^3 + B_1 z^2 + C_1 z + D = 0 \qquad (11)$$

The function f(z), for the same trajectory as was used in deriving figure 1, and for t = - 105 seconds is shown in figure 2, with coefficients based upon the correct peak time ($\Delta t = 0$) and also based upon $\pm$ 1 second errors in peak time ($\Delta t = +1$, and $\Delta t = -1$). It was found necessary to use four significant figures on $\dot{R}$ and five significant figures on R (Calculated points shown as circles) to define the f(z) function. This accuracy on R represents the very best that can be expected from a Doppler measurement, however, in practice $\dot{R}$ can at best be determined to three significant figures. The f(z) function was therefore re-calculated for the $\Delta t = 0$ curve using the best accuracy which can be achieved experimentally and the calculated values were shown as crosses. It is seen that in practice it should be quite difficult to define the f(z) function, using experimental data. It is also seen that for $\Delta t = -1$ sec there are two solutions ($Z_o = 168.4$ km and $Z_o = 173.9$ km), and that for $\Delta t = +1$ sec there are no solutions. Even for $\Delta t = 0$ the cubic (11) has two positive solutions very close to each other, one of which is the correct answer. A criterion for selecting the correct solution, can be derived from the sign of the tangent to the curve representing equation (11). It can be shown that the sign of the slope is, at the true solution, the same as the sign of the time t_1 or, in other words, it is negative for ascent values and positive for descent values. The above discussion suggests the undesirability of using equation (11) directly for obtaining Z_o unless extremely accurate values of $R\dot{R}$ are available.

It is of interest to note that, when the correct peak time is used, the minimum of the function f(z) occurs for a value of z close to Z_o, and that this minimum can be determined more accurately (by solving $f'(z) = 0$) than the f(z) function. This observation suggests that a fairly good value of Z_o can be obtained by solving $f'(z) = 0$. A study of the derivative formula showed that it could be obtained directly by proceeding as shown in the following analysis.

A very accurate second degree equation (whose solution requires less precise experimental values) can be obtained by assuming that

$$X_o = \dot{X}T \tag{12}$$

where T is the free fall time corresponding to the peak altitude Z_o, i.e. T is defined by the following equation:

$$Z_o = \frac{1}{2} g_o (1 + \alpha T^2)T^2 \tag{13}$$

in which the term αT^2 is a correction for the variation of g with altitude. The deviation of equation (13) is given in the appendix. The quantity $\dot{X}^2t$ which appeared in equation (8) can then be evaluated as follows:

From (12)

$$\dot{X}^2 = \frac{X_o^2}{T^2} = \frac{X_o^2\, g_o(1 + \alpha T^2)}{2Z_o}$$

$$\dot{X}^2 = \frac{X_o^2}{2Z_o} \quad \frac{-\dot{Z}_1}{g_o(1 + 2 \alpha t_1^2)t} \quad g_o(1 + \alpha T^2)$$

and $$\dot{X}^2t = - \frac{\dot{Z}_1 X_o^2}{2Z_o} \frac{1 + \alpha T^2}{1 + 2 \alpha t^2}$$

$$\dot{X}^2 t = - \frac{\dot{Z}_1 (\lambda X_o^2)}{2Z_o} \tag{14}$$

where $\lambda = \frac{1 + \alpha T^2}{1 + 2 \alpha t^2} \cong 1 + \alpha (T^2 - 2 t^2).$

For the example used in the text and for t =-105 seconds,

$\lambda = 1.002$

Equation (8) then becomes:

$$(R\dot{R})_1 = \dot{X}X_o + \dot{X}^2 t_1 + (Z_o - h_1)\dot{Z}_1$$

$$= (R\dot{R})_o - \frac{\dot{Z}_1 \lambda X_o^2}{2Z_o} + (Z_o - h_1)\dot{Z}_1 \tag{15}$$

Changing (15) to a quadratic in Z_o gives:

$$2 Z_o^2 \dot{Z}_1 - 2 \left[(R\dot{R})_1 - (R\dot{R})_o + h \dot{Z}_1\right] Z_o - \dot{Z}_1 \lambda X_o^2 = 0$$

$$2 Z_o^2 + \lambda Z_o^2 - 2 \left[\frac{(R\dot{R})_1 - (R\dot{R})_o}{\dot{Z}_1} + h\right] Z_o - \lambda Z_o^2 - \lambda X_o^2 = 0$$

$$Z_o^2 - \frac{2}{2+\lambda} \left[\frac{(R\dot{R})_1 - (R\dot{R})_o}{\dot{Z}_1} + h\right] Z_o - \frac{\lambda}{2+\lambda} R_o^2 = 0 \tag{16}$$

The solution of equation (16) is:

$$Z_o = P_1' + \sqrt{\frac{\lambda R_o^2}{2+\lambda} + P_1'^2} \tag{17}$$

where

$$p_1' = \frac{(R\dot{R})_1 - (R\dot{R})_o}{(2+\lambda)\dot{Z}_1} + \frac{h_1}{2+\lambda}$$

Since λ is extremely close to unity equation (17) can be written:

$$Z_o = p_1 + \sqrt{\frac{R_o^2}{3} + p_1^2} \tag{18}$$

where

$$p_1 = \frac{(R\dot{R})_1 - (R\dot{R})_o}{3\dot{Z}_1} + \frac{h_1}{3}$$

If λ is unity, equation (16) can be written

$$3\, Z_o^2\, \dot{Z}_1 - 2\left[(R\dot{R})_1 - (R\dot{R})_o + h_1\, \dot{Z}_1\right] - R_o^2\, \dot{Z}_1 = 0$$

and using the notation of equation (10)

$$3\, A_1\, Z_o^2 + 2\, B_1\, Z_o + C_1 = 0$$

This is the derivative of equation (10).

Thus the cubic of equation (10) has a minimum occurring for a value of Z_o very close to the correct answer. In other words the cubic has two roots very close in value, as seen in Fig. 2. The approximation made in equation (12) introduces a small error in Equation (15). Actually $X_o = \dot{X}T + \xi$ which is equivalent to changing X_o^2 in equation (14) to $(X_o - \zeta)^2$, or $(X_o^2 - 2\zeta X_o + \zeta^2)$. Thus equation (14) becomes:

$$\dot{X}^2 t = \frac{-\dot{Z}_1 \lambda X_o^2}{2Z_o} \left[1 - \frac{2\zeta}{X_o} + \frac{\zeta^2}{X_o^2}\right]$$

$$\cong \frac{-\dot{Z}_1 X_o^2}{2Z_o} \lambda \left[1 - \frac{2\zeta}{X_o}\right]$$

This introduces a small correction $(\cong \frac{2\zeta}{X_o})$ on the value of λ in equation (17). It can be shown that the value of Z_o obtained by using $\lambda = 1$ (instead of $\lambda = 1 - \frac{2\zeta}{X_o}$) is too large by an amount ϵ given by:

$$\epsilon \cong \left(\frac{2\zeta}{X_o}\right) \left(\frac{X_o^2}{4Z_o}\right)$$

For a Nike-Cajun trajectory, $\frac{2\zeta}{X_o}$, is always less than 1 per cent.

For the trajectory used to derive Figure 1, a one per cent change in λ introduced a change in Z_o of 0.1 km. Thus formula (18) is a very good approximation.

In formula (18) R_o^2, $(R\dot{R})_1$, and $(R\dot{R})_o$ are experimentally determined quantities; h_1 and $\dot{Z}_1$ are calculated using the A(t) and B(t) functions given in the appendix for the time t_1 corresponding to $(R\dot{R})_1$ and for the value of g corresponding to Z_o. Since Z_o is unknown, it is necessary to calculate (18) using first an estimated value of g which will give an estimate of Z_o. This estimate of Z_o can be used to re-calculate g and a better determination of Z_o is then possible. In practice only two attempts are sufficient since a good first estimate of g is given by:

$$g(R_o) < g(Z_o) < g\left(\sqrt{R_o^2 - (R\dot{R})_o t_p}\right)$$

in which t_p is the peak time. This criterion for estimating Z_o is based upon the fact that Z_o is less than R_o but greater than $\sqrt{R_o^2 - (R\dot{R})_o t_p}$. The second part of the preceding statement follows from the following relations:

$$Z_o^2 = R_o^2 - X_o^2; \; X_o \dot{X}_o = (R\dot{R})_o \text{ and } \dot{X}_o > \frac{X_o}{t_p} .$$

III. Errors Due to Approximations Made in the Theory

Discussion of errors

For more rigorous treatment considering the effects of earth curvature and rotation, it is convenient to use coordinate axis X', Y', and Z' defined as follows. The Z' axis is the vertical direction at peak and the X' axis is in the direction of the horizontal velocity V_H at peak. The Y' axis is chosen such that X', Y', Z' form a right-handed system of coordinates. The time t is measured from peak and it is positive for increasing X'. From this definition, the rocket coordinates at peak satisfy the following relations:

$$
\left| \begin{array}{l} X' = 0 \\ \\ Y' = 0 \\ \\ Z'_o = Z_o \end{array} \right. \qquad \left| \begin{array}{l} \dot{X}'_o = V_H \\ \\ \dot{Y}'_o = 0 \\ \\ \dot{Z}'_o = 0 \end{array} \right.
$$

Due to the earth's curvature, the Doppler station will be at a distance $\frac{+X_o^2}{2R_e}$ below the X'Y' plane, where X_o is defined as previously and R_e is the radius of the earth. The relationship between this system of coordinates and the one used previously is seen in Fig. (3) to be as follows:

$$X = X_o + X'$$

$$Y = d + Y'$$

$$Z = Z' + \frac{X_o^2}{2R_e}$$

Thus the main effect of the earth's curvature is to introduce the term $\frac{X_o^2}{2R_e}$ in the expression for Z. If X_o is less than 50 km, the term $\frac{X_o^2}{2R_e}$ is less than 0.25 km and usually negligible; however if X_o is 100 km, the term $\frac{X_o^2}{2R_e}$ is about 0.8 km and it may be desirable to take it into consideration.

The effect of the earth's rotation is to introduce a Coriolis acceleration which is defined in terms of the earth's rotation vector Ω. This vector has a magnitude of 0.7272×10^{-4} radians/sec and it is directed towards the

North Star. At a given location it has a component $\omega_Z = \omega \sin \theta$ in the vertical direction and $\omega_N = \omega \cos \theta$ in the North direction, where θ is the latitude (θ is positive in the Northerm hemisphere and negative in the Southern hemisphere). The Coriolis acceleration $\bar{a}$ is:

$$a = -2\, \bar{\omega} \times \dot{\bar{R}} = -2 \begin{vmatrix} i & \omega_X & \dot{X} \\ j & \omega_Y & \dot{Y} \\ k & \omega_Z & \dot{Z} \end{vmatrix}$$

and its components are:

$$a_X = 2\,(\omega_Z \dot{Y} - \omega_Y \dot{Z})$$

$$a_Y = 2\,(\omega_X \dot{Z} - \omega_Z \dot{X})$$

$$a_Z = 2\,(\omega_Y \dot{X} - \omega_X \dot{Y})$$

The effect of the Coriolis acceleration can be illustrated by using an eastward trajectory, in which case $\omega_Y = \omega_N$ and $\omega_X = 0$. Furthermore by assuming that the trajectory is at middle latitude in the Northern hemisphere, ω_Z and ω_N are positive quantities, having comparable magnitudes. This case will also represent a typical trajectory at Wallops Island. The Coriolis velocities are then:

$$V_X = +\, 2\int_0^t (\omega_Z \dot{Y} - \omega_N \dot{Z})dt$$

$$V_Y = -2\int_0^t \omega_Z \dot{X}\, dt$$

$$V_Z = 2\int_0^t \omega_N \dot{X}\, dt$$

These expressions can be further simplified considering that $\dot{Z} >> \dot{Y}$ and $\dot{X} \approx$ constant giving:

$$V_X = -2\,\omega_N (Z - Z_o) = +2\,\omega_N h$$

$$V_Y = -2\,\omega_Z \dot{X} t = -2\,\omega_Z X'$$

$$V_Z = +2\,\omega_N \dot{X} t = 2\,\omega_N X'$$

The corrections on X, Y and Z due to the Coriolis effect are

$$\Delta X \cong \omega_N \int_0^t g t^2 dt$$

$$\Delta X \cong \frac{2}{3}\,\omega_N h t$$

$$\Delta Y \cong -2\omega_Z \int_0^t \dot{X} t\, dt$$

$$\Delta Y \cong -\omega_Z \dot{X} t^2$$

$$\Delta Y \cong -\omega_Z X' t$$

$$\Delta Z \cong 2\omega_N \int_0^t \dot{X} t\, dt$$

$$\Delta Z \cong \omega_N X' t$$

For a typical small rocket trajectory the region of interest corresponds to values of t between -150 seconds and +150 seconds, representing the altitudes between the peak and 100 km below the peak. Taking the maximum values of $t = \pm 150$ seconds, $h = 100$ km, $X' = 50$ km, and $\omega_N = \omega_Z = \frac{\omega}{\sqrt{2}} \cong 0.5 \times 10^{-4}$ gives:

$\Delta X = +0.5$ km for +150 seconds and -0.5 km for -150 seconds.

$\Delta Y = -0.375$ km for $t = \pm 150$ and $\Delta Z = 0.375$ km for $t = \pm 150$.

It is seen that even the maximum values of $\Delta X, \Delta Y,$ and ΔZ are quite small. However, if desired, a correction can be made for the Coriolis effect, if the approximate direction of the plane of the trajectory is known. An accuracy of ± 10 degrees for this information is adequate to estimate the X', Y', Z' components of ω and hence calculate the Coriolis terms. From the above discussion it is seen that ΔX, ΔY, ΔZ, V_x, V_y, V_z are all expressed in terms of time and flight parameters derived from the approximate analysis for a plane non-rotating earth. The approximate analysis could therefore be refined by letting:

$$X = X_o + \dot{X}'_o t + \Delta X \qquad \dot{X} = \dot{X}_o + V_x$$

$$Z = Z' + \frac{X_o^2}{2R_e} + \Delta Z \qquad \dot{Z} = |g_o| B(t) + V_Z$$

and introducing corresponding correction terms in Equations 4 and 18. Thus Equation (4) would become:

$$(R\dot{R})_{+t} + (R\dot{R})_{-t} - 4X_o\omega_N h = 2(R\dot{R})_o \tag{19}$$

For the trajectory used in Fig. 1, the term $4X_o\omega_N h$ is equal to 0.8 km^2/sec which increased the peak time value by 0.28 second. Similarly p_1 in Equation (18) should be changed to p_1', where:

$$p_1' = p_1 - \frac{\Delta Z}{3} - \left[\frac{(R\dot{R})_1 - (R\dot{R})_o}{3\dot{Z}}\right] \frac{(V_Z)}{\dot{Z}} \tag{20}$$

where ΔZ and V_Z are the Coriolis terms. To evaluate the correction introduced by changing p_1 to p_1', the data in Figure 1 were re-calculated for an East trajectory and with the corresponding Coriolis terms. Starting with these more exact values of $R\dot{R}$ and using equation (18), it was found that the calculated value of Z_o was too low by 0.4 km. However using the value of p_1' given in formula (20) the error in Z_o was less than 0.1 km. In making this calculation the exact peak time was used, to insure that the results would show only the effect of the Coriolis acceleration.

In summary, the Coriolis terms introduce for the example given a correction of 0.28 second in peak time and 0.4 km in peak altitude. Thus a slight improvement in accuracy can be achieved if desired. However even with this correction it is believed that there will be an uncertainty of about 0.5 km in the final trajectory determination in view of the approximations made in formula (1), (18), and in the A(t) and B(t) functions.

IV Summary of Method

1. From the Doppler data calculate and plot a graph of $R\dot{R}$ similar to that shown in Figure 1.

2. The peak time t_p is obtained by calculating

$$(R\dot{R})_o = \tfrac{1}{2}\left[(R\dot{R})_{max} + (R\dot{R})_{min}\right] \qquad (4)'$$

and reading the time at which $R\dot{R} = (R\dot{R})_o$. One should attempt to read this time to within an accuracy of 0.25 second.

3. The peak altitude is obtained from:

$$Z_o = p_1 + \sqrt{\frac{R_o^2}{3} + p_1^2} \qquad (18)$$

where:

$$p_1 = \frac{h_1}{3} + \frac{(R\dot{R})_1 - (R\dot{R})_o}{3\dot{Z}_1}$$

In Equation (18) the quantity $(R\dot{R})_1$ corresponds to a convenient time t_1 of the order of -100 seconds, giving values of $R\dot{R}$ close to the maximum of the $R\dot{R}$ function, and h_1 and $\dot{Z}_1$ are given by:

$$h_1 = g_{Z_o} A(t_1)$$

$$\dot{Z}_1 = g_{Z_o} B(t_1),$$

where the A(t) and B(t) functions are tabulated in the appendix. A preliminary calculation of (18) will be required using an estimated value of g_{Z_o} based upon:

$$g(R_o) < g(Z_o) < g\left(\sqrt{R_o^2 - (R\dot{R})_o t_p}\right)$$

4. The horizontal distance at peak and the horizontal velocity are then determined respectively from

$$X_o = \sqrt{R_o^2 - Z_o^2}$$

and

$$\dot{X}_o = \frac{(R\dot{R})_o}{X_o}$$

5. The complete trajectory can then be calculated versus time from:

$$Z = Z_o - g(Z_o)\ A(t)$$

and

$$X = X_o + \dot{X}_o t$$

6. For a slightly better accuracy a correction for Coriolis effects can be made using the method given in Section III. However this correction is not required if an accuracy of the order of 1 km is satisfactory for the values of X and Z versus time.

Appendix I

If Coriolis effects are negligible, free fall calculations for small rocket trajectories can be performed with a very good accuracy by means of the formulas:

$$h = g_o \frac{t^2}{2} \left[1 + \alpha\, t^2\right] = g_o A(t)$$

$$\dot{h} = g_o t \left[1 + 2\alpha\, t^2\right] = g_o B(t)$$

where: h = distance fallen from peak

$\dot{h}$ = vertical velocity

g_o = value of g at the peak altitude Z_o.

t = time from peak

α = $2.42 \times 10^{-7}/\text{sec}^2$ for a typical small rocket trajectory.

If desired the value of h and $\dot{h}(=\dot{z})$ can subsequently be corrected for Coriolis effect as indicated in section III of the report. For convenience the A(t) and B(t) functions have been tabulated for a representative value of α equal to $2.42 \times 10^{-7}/\text{sec}^2$. The derivation of the formula for the A(t) and B(t) functions is as follows:

$$\text{Let } g_{(Z_o)} = g_e \frac{R_e^2}{(R_e + Z_o)^2}$$

where g_e = g at surface of the earth

and R_e = earth's radius.

Then

$$g(Z_o - h) = g_e \frac{R_e^2}{(R_e + Z_o - h)^2}$$

$$\cong g_e \frac{R_e^2}{(R_e + Z_o)^2} \left[1 + \frac{h}{R_e + Z_o}\right]^2$$

$$\cong g(Z_o) \left[1 + \frac{2h}{R_e + Z_o}\right]$$

$$\cong g(Z_o) + \frac{2g(Z_o)\ h}{R_e + Z_o}$$

Thus

$$g(Z_o - h) = g(Z_o) + kh$$

The value of k is relatively constant for values of Z_o ranging from 100 to 200 km as can be seen from the following table:

Z_o	100 km	150 km	200 km
k	2.94×10^{-6}	2.87×10^{-6}	2.81×10^{-6}

Thus the calculation of h and $\dot{h}$ consists in evaluating the integrals

$$\dot{h} = \int_0^t \left[g(Z_o) + kh\right] dt$$

and

$$h = \int_0^t \dot{h}\, dt$$

Since the contribution of the kh term is small, an estimated value of h given by $h = \frac{1}{2} g_{(Z_o)} t^2$ is adequate for calculating the correction term kh. Thus g can be written:

$$g = g(Z_o) \left[1 + \tfrac{1}{2} \kappa t^2 \right]$$

The integration yields readily:

$$\dot{h} = g(Z_o) t \left[1 + \frac{k}{6} t^2 \right]$$

and

$$h = \tfrac{1}{2} g(Z_o) t^2 \left[1 + \frac{k}{12} t^2 \right]$$

Thus the constant α in the expression for A(t) and B(t) is given by:

$$\alpha = \frac{k}{12}$$

and the values of α for the 100 to 200 km altitude range vary between 2.45×10^{-7} and 2.34×10^{-7}. A value of α of 2.42×10^{-7} which corresponds to an altitude of 125 km is considered representative of the free-fall portion of most Nike-Cajun trajectories and it was used in calculating the A(t) and B(t) functions.

It should be noted that if T is the total falling time from peak (assuming no drag), equation (21) gives:

$Z_o = \frac{1}{2} g_o (1 + a T^2)T^2$, which is the Equation (13) used in the calculation of Z_o. A plot of Equation (13) is given in Figure 4.

Values of $g(Z_o)$ for Wallops Island are given on the following table:

h (km)	g(h) km/sec^2
0	.0097986
60	.0096173
70	.0095874
80	.0095577
90	.0095281
100	.0094986
110	.0094693
120	.0094401
130	.0094110
140	.0093821
150	.0093533
160	.0093246
170	.0092961
180	.0092677
190	.0092394
200	.0092113

Table of A(t) and B(t) Functions

t	A(t)	h	B(t)	$\dot{Z}$
0	0		0	
5	12.50		5.00	
10	50.00		10.00	
15	112.51		15.00	
20	200.02		20.00	
25	312.55		25.01	
30	450.10		30.01	
35	612.68		35.02	
40	800.31		40.03	
45	1013.00		45.04	
50	1250.76		50.06	
55	1513.60		55.08	
60	1801.57		60.10	
65	2114.65		65.13	
70	2452.92		70.17	
75	2816.33		75.20	
80	3204.96		80.25	
85	3618.82		85.30	
90	4057.94		90.35	
95	4522.34		95.42	
100	5012.10		100.48	
105	5527.22		105.56	
110	6067.73		110.64	

t	A(t)	h	B(t)	$\dot{Z}$
115	6633.66		115.74	
120	7225.13		120.84	
125	7842.03		125.95	
130	8484.56		131.06	
135	9152.69		136.19	
140	9846.45		141.33	
145	10566.01		146.48	
150	11311.31		151.63	
155	12082.29		156.80	
160	12879.36		161.98	
165	13702.21		167.17	
170	14551.01		172.38	
175	15425.97		177.59	
180	16327.02		182.82	
185	17254.19		188.06	
190	18207.76		193.32	
195	19187.42		198.59	
200	20193.60		203.87	
205	21226.20		209.17	
210	22285.27		214.48	
215	23371.13		219.81	
220	24483.38		225.15	

REFERENCES

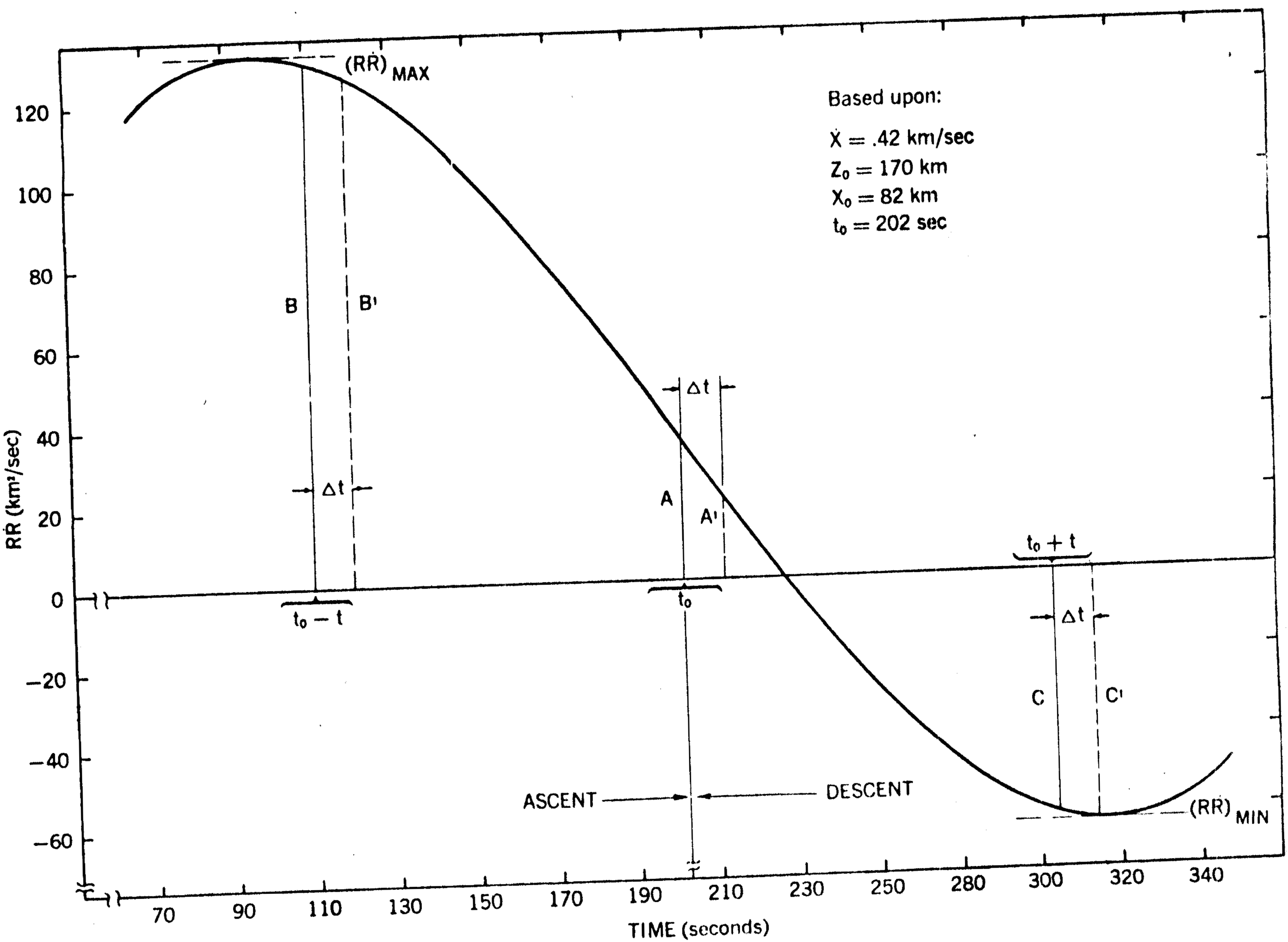
1. Zancanata, H. W. "Ballistic Instrumentation for the IGY Rocket Project at Fort Churchill," Aberdeen Proving Ground: BRL MR-1094, September 1957.

2. Zancanata, H. W. "Ballistic Instrumentation and Summary of Instrumentation Results for the IGY Rocket Project at Fort Churchill," Aberdeen Proving Ground: BRL R-1091, January 1960.

3. Seddon, J. C. "Preliminary Report on the Single Station Doppler Rocket Tracking Technique," NASA Tech. Note D-1344, 1962.

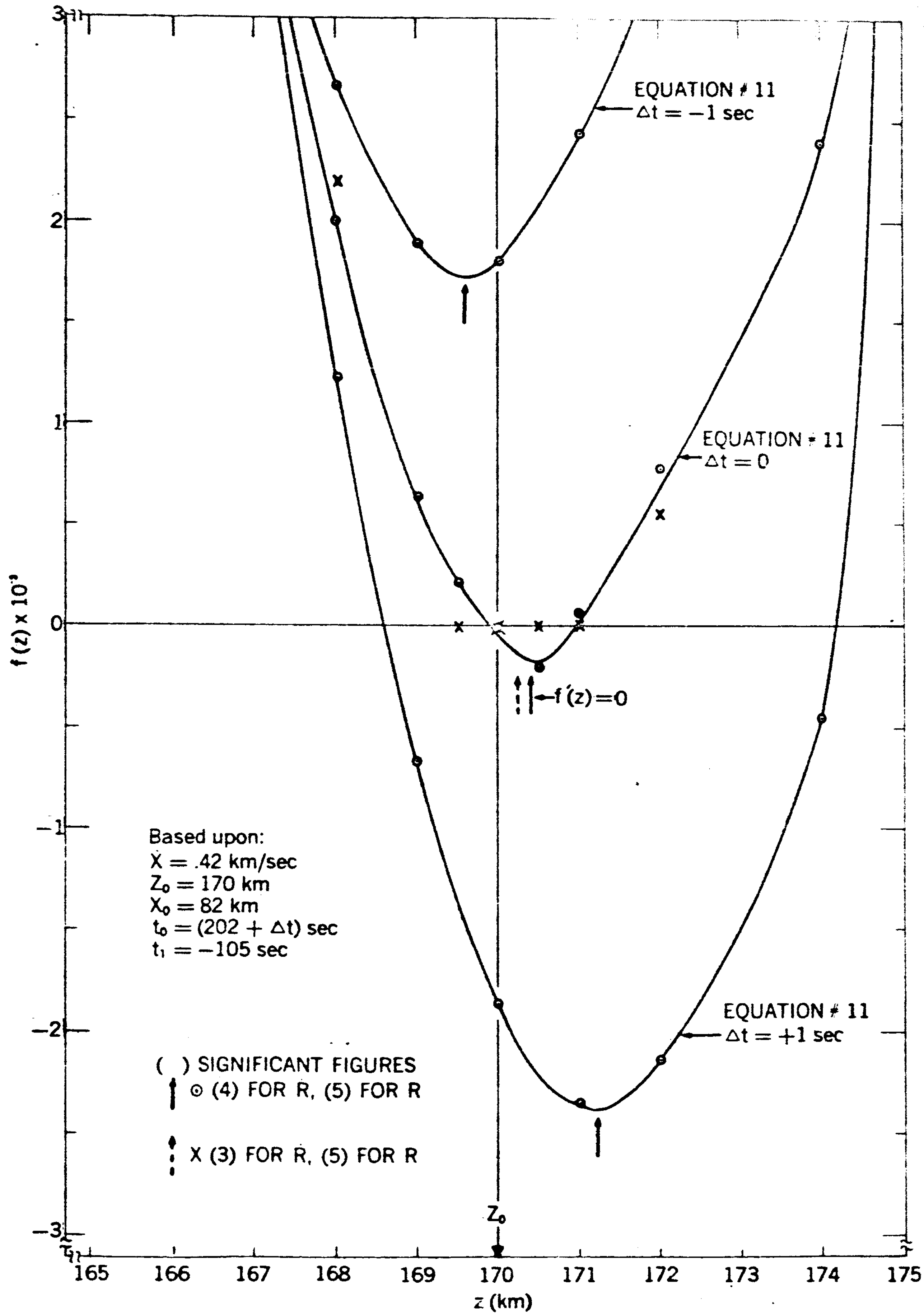
4. Lane, J. H. and Kemmerer, N. L. "Nike Cajun Sounding Rocket Performance," NASA Report X-616-62-66, April 1962.

LIST OF FIGURES



(RṘ) MAX
Based upon:
Ẋ = .42 km/sec
Z₀ = 170 km
X₀ = 82 km
t₀ = 202 sec
B
B'
Δt
A
A'
t₀ + t
t₀ − t
t₀
C
C'
ASCENT
DESCENT
(RṘ) MIN
RṘ (km²/sec)
120
100
80
60
40
20
0
−20
−40
−60
70
90
110
130
150
170
190
210
230
250
280
300
320
340
TIME (seconds)

FIGURE 1



EQUATION # 11
Δt = −1 sec
EQUATION # 11
Δt = 0
f′(z)=0
EQUATION # 11
Δt = +1 sec
Based upon:
Ẋ = .42 km/sec
Z₀ = 170 km
X₀ = 82 km
t₀ = (202 + Δt) sec
t₁ = −105 sec
( ) SIGNIFICANT FIGURES
⊙ (4) FOR R, (5) FOR R
X (3) FOR R, (5) FOR R
Z₀
f (z) x 10⁻³
z (km)

FIGURE 2

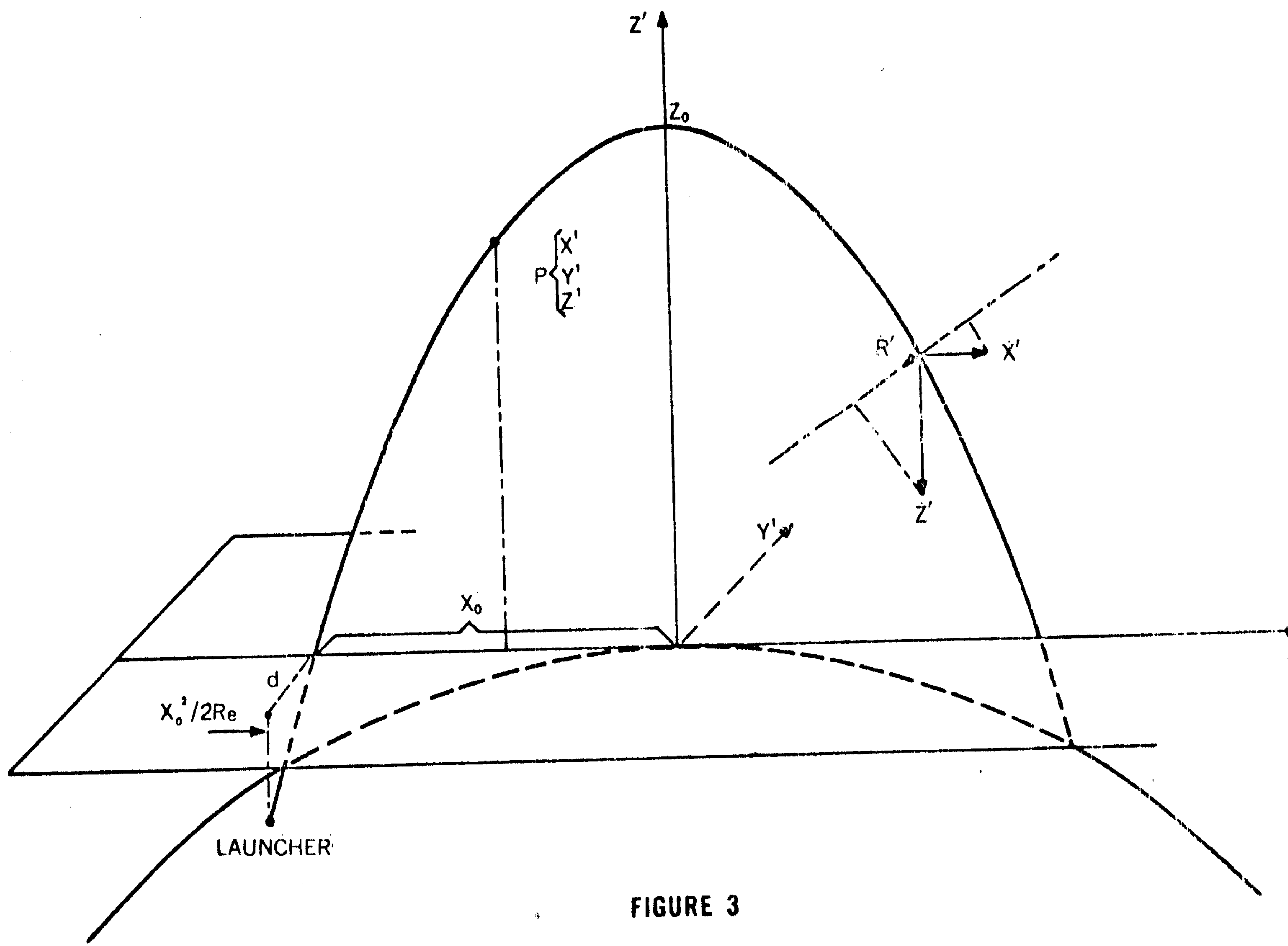


FIGURE 3

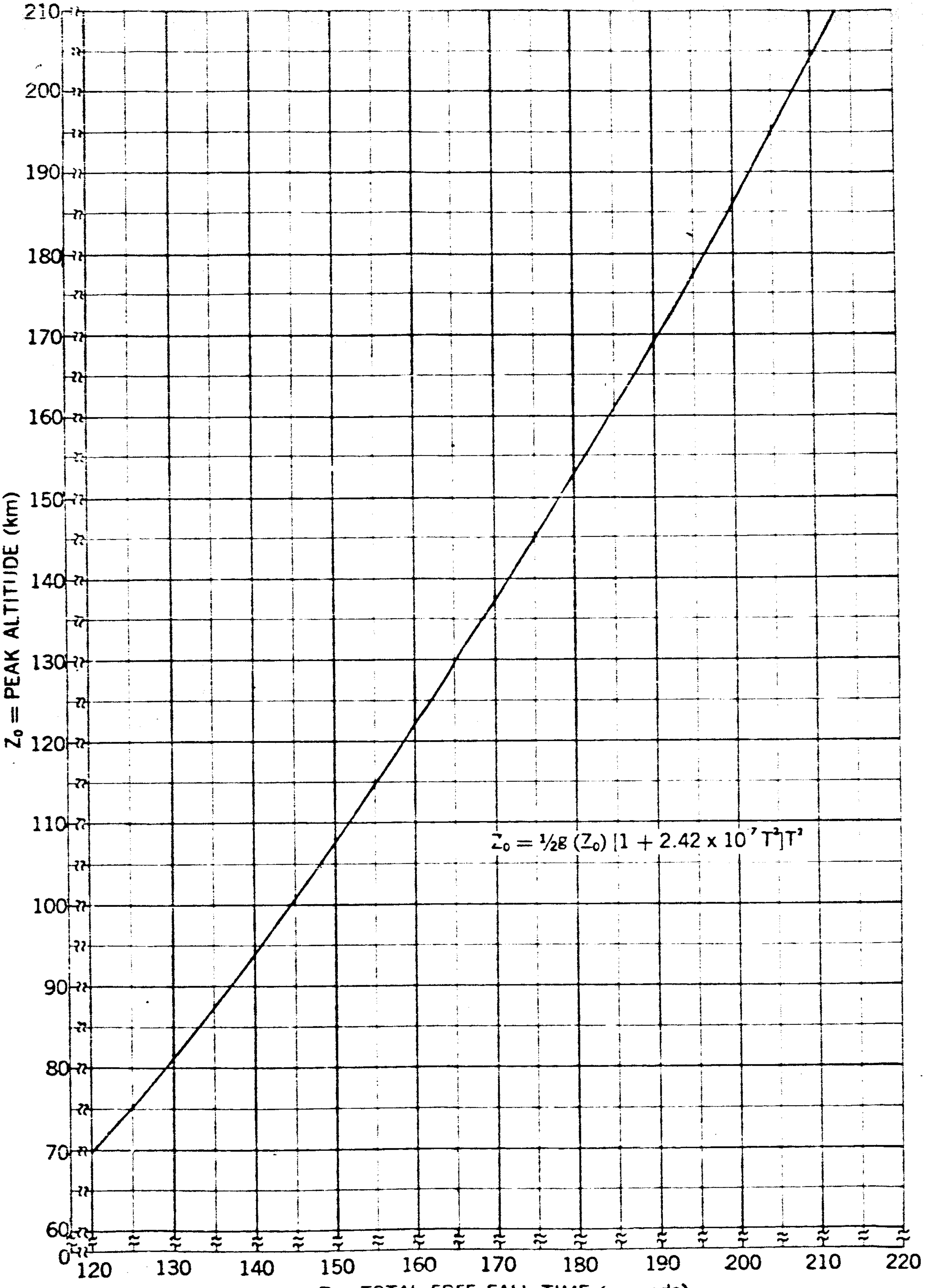

Z_0 = PEAK ALTITUDE (km)
T = TOTAL FREE FALL TIME (seconds)
$Z_0 = \frac{1}{2} g (Z_0) [1 + 2.42 \times 10^{-7} T^2] T^2$
210
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
0
120
130
140
150
160
170
180
190
200
210
220

FIGURE 4

where

$$p_1' = \frac{(R\dot{R})_1 - (R\dot{R})_o}{(2+\lambda)\dot{Z}_1} + \frac{h_1}{2+\lambda}$$

Since λ is extremely close to unity equation (17) can be written:

$$Z_o = p_1 + \sqrt{\frac{R_o^2}{3} + p_1^2} \qquad (18)$$

where

$$p_1 = \frac{(R\dot{R})_1 - (R\dot{R})_o}{3\dot{Z}_1} + \frac{h_1}{3}$$

If λ is unity, equation (16) can be written

$$3\, Z_o^2\, \dot{Z}_1 - 2\left[(R\dot{R})_1 - (R\dot{R})_o + h_1\, \dot{Z}_1\right] - R_o^2\, \dot{Z}_1 = 0$$

and using the notation of equation (10)

$$3\, A_1\, Z_o^2 + 2\, B_1\, Z_o + C_1 = 0$$

This is the derivative of equation (10).

Thus the cubic of equation (10) has a minimum occurring for a value of Z_o very close to the correct answer. In other words the cubic has two roots very close in value, as seen in Fig. 2. The approximation made in equation (12) introduces a small error in Equation (15). Actually $X_o = \dot{X}T + \zeta$ which is equivalent to changing X_o^2 in equation (14)

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