Stability of Helmholtz Flow in an Unstable Atmosphere*

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Report No. 66-011 August, 1966

*This work was supported by the Space Nuclear Propulsion Office of the National Aeronautics and Space Administration and the Atomic Energy Commission under Grant No. NsG 586.

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Abstract

The stability of the Helmholtz profile in an atmosphere with an unstable density gradient has been studied. It is found that the convective and shear instabilities are essentially separate. It is noted that the shear gives rise to an amplified mode at right angles to the direction of the mean flow.

1. Introduction

A previous study[1] of the stability of a system in which an unstable velocity profile is combined with unstable stratification concluded that, roughly speaking, the instability of the system could be considered to be the sum of the separate instabilities. The problem, however, led to a very complicated eigenvalue equation, which was solved numerically. These stability properties are brought into sharper focus in this paper by means of a simpler example, which can easily be understood analytically. For this purpose a velocity profile of the Helmholtz type is used in an atmosphere which is characterized by a constant (negative) Richardson number.

The velocity profile by itself would admit of a single unstable mode. It will be seen that this mode is somewhat modified by the convective instability. Since it is unbounded, the unstable atmosphere by itself would be associated with a continuum of unstable modes. The effect of this continuum is not altered by the presence of the shear flow. Thus, the total instability is simply the sum of the convective instability and the modified shear mode. It will be seen that for a perturbation of a given wavenumber k making an angle θ with the direction of the mean flow, the convective or shear instability dominates depending on whether $\cos^2\theta \lessgtr \frac{1}{2} J/k^2$ where $J > 0$ is the negative of the Richardson number.

A somewhat surprising result is that in the limit $\theta \to 90^\circ$, i.e., when the perturbation is at right angles to the shear, the discrete eigenvalue associated with the shear remains as an amplified mode. Physically it would be natural to expect that perturbations at right angles to the mean flow would be unaffected by it, in which case there would be no normal modes. However, in this limit the discrete mode is not the dominant instability.

2. The Perturbation Equations

We assume an inviscid, incompressible fluid in a constant gravitational field. The basic equations are the momentum equation, the incompressibility condition, and the continuity equation:

$$\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\nabla p + \vec{g} = 0$$

$$\frac{D\rho}{Dt} = 0 \qquad \nabla\cdot\vec{v} = 0$$

We assume that the equations have been written in dimensionless form by introducing suitable reference quantities for the length, velocity, and density. The quantities $\vec{v}$, p, and ρ are the dimensionless velocity, pressure, and density. The dimensionless gravitational constant is g and $\vec{g}$ is (0, 0, g). The basic equations are linearized about a mean velocity $(U(z), 0, 0)$, a mean pressure $\bar{p}(z)$, and a mean density $\bar{\rho}(z)$ by the relations

$$\vec{v} = (U + u', v', w')$$

$$p = \bar{p} + p' \qquad \rho = \bar{\rho} + \rho'$$

where the primed quantities are assumed small. If we assume that the primed quantites are of the form

$$q' = q(z,t)\ \exp\ i\,(\alpha x + \beta y)$$

and solve the linearized equations for w with the Boussinesq approximation of neglecting the variation of density except where it is multiplied by the gravitational constant we obtain

$$L^2 (w'' - k^2 w) - L(i \alpha U'' w) + k^2 J w = 0 \qquad (2.1)$$

where $$L = \frac{\partial}{\partial t} + i \alpha U$$

Here primes denote derivatives with respect to z, k is the total wavenumber of the perturbation ($k^2 = \alpha^2 + \beta^2$), and J is the negative of the local Richardson number

$$J = g \bar{\rho}' / \bar{\rho}$$

Thus, J is defined so that positive J implies convective instability.

Assuming the normal mode type of solution, we would set $w = \hat{w}(z) \exp - i \alpha ct$ in eq. (2.1) and obtain

$$\hat{w}'' - \left\{ k^2 + \frac{U''}{U-c} + \frac{k^2}{\alpha^2} \quad \frac{J}{(U-c)^2} \right\} \hat{w} = 0 \qquad (2.2)$$

Regarding the complex wave velocity c as the eigenvalue to be determined by imposing the appropriate boundary conditions, we have instability if $c_i > 0$.

3. The Static Stability

Let us first investigate the stability of the atmoshere without shear. We assume that J is positive and constant. Setting $U = 0$ in eq. (2.2) we see that a continuous eigenvalue spectrum $(\alpha c)^2 \leq -J$ is obtained. To determine the effect of this spectrum we solve the initial value problem using a technique developed by Case[2]. Taking the Laplace transform of eq. (2.1) we obtain

$$w_s'' - \gamma^2 w_s = \frac{F_1}{s} + \frac{F_2}{s^2} \tag{3.1}$$

where w_s is the transform of w

$$w_s = \int_0^{\infty} w(z,t)\, e^{-st}\, dt$$

and
$$F_1(z) = \left[\frac{\partial^2 w}{\partial z^2} - k^2 w \right]_{t=0}$$

$$F_2(z) = \left[\frac{\partial}{\partial t} \left(\frac{\partial^2 w}{\partial z^2} - k^2 w \right) \right]_{t=0}$$

$$\gamma^2 = k^2 (1 - J/s^2)$$

The branch cut for γ is taken along the real axis of the s - plane between $s = \pm \sqrt{J}$. The solution of eq. (3.1) is

$$w_s = \int_{-\infty}^{\infty} dz_o \quad G(z,z_o) \left\{ \frac{F_1(z_o)}{s} + \frac{F_2(z_o)}{s^2} \right\} \tag{3.2}$$

where $G(z, z_o)$ is the Green's function

$$G(z,z_o) = \frac{-1}{2\gamma} \exp - \gamma \left| z - z_o \right|$$

The asymptotic form of w for large values of t may be obtained from the inversion integral for w_s by making use of the appropriate Tauberian theorem[3] and expanding

w_s in powers of $(s - s_o)^{\frac{1}{2}}$ where s_o is the singularity furthest to the right in the s-plane. Here s_o is the branch point $s_o = \sqrt{J}$ and to first order we obtain

$$w \sim t^{-1/2} \exp \sqrt{J} \; t$$

4. Stability of the Helmholtz Profile in an Unstable Atmosphere

For the mean flow we now choose the dimensionless Helmholtz profile $U(z) = \text{sgn } z$ and we continue to assume that J is positive and constant. Let us first use the normal modes approach. Solutions of eq. (2.2) are easily written as exponentials. At the interface $z = 0$ we impose the continuity of $\hat{w}/(U - c)$ and $(U - c) \; \hat{w}' - U' \; \hat{w}$. These conditions are equivalent in the linearized theory to requiring that the interface be a streamline and requiring that the pressure be continuous across the interface[4]. After imposing the boundary conditions we obtain a pair of eigenvalues

$$(\alpha c_i)^2 = \alpha^2 + \frac{1}{2} J \qquad\qquad c_r = 0$$

which are complex conjugates, one being an amplified mode and the other a damped mode. If $J = 0$ these are exactly what we would have for the Helmholtz flow alone. Thus, the shear modes are modified by the stratification. It is interesting that when $\alpha = 0$ (perturbations at right angles to the mean flow) we still have a pair of normal modes which arise from the shear. By solving the initial value problem we shall see that these are true normal modes and not simply a part of the continuum of section 3.

We may solve the initial value problem as we did in the previous section. Eq. (2.1) becomes

$$\left(\frac{\partial}{\partial t} \pm i\alpha\right)^2 \left(\frac{\partial^2}{\partial z^2} - k^2\right) w + k^2 J\, w = 0 \qquad z \gtrless 0$$

Taking the Laplace transform, we obtain

$$w_s'' - \gamma_{\pm}^2 w_s = \frac{F_{\pm}}{(s \pm i\alpha)^2} \tag{4.1}$$

where $F_{\pm} = \left[\left(\frac{\partial^2}{\partial z^2} - k^2\right) \left\{ (s \pm 2i\alpha) w + \frac{\partial w}{\partial t} \right\} \right]_{t=0}$

$$\gamma_{\pm}^2 = k^2 \left\{ 1 - \frac{J}{(s \pm i\alpha)^2} \right\}$$

and the branch cuts for $\gamma_{\pm}$ are along

$$(\mathcal{R}e\, s)^2 \leq J \qquad \text{Im}\, s = \pm i\alpha$$

Solutions of eq. (4.1) are

$$W_{s\pm} = A_{\pm} e^{\mp \gamma_{\pm} z} \mp \frac{1}{2\gamma_{\pm}} \int_0^{\pm\infty} e^{-\gamma_{\pm} |z - z_0|} \frac{F_{\pm}}{(s \pm i\alpha)^2} \, dz_0 \tag{4.2}$$

The constants $A_{\pm}$ are determined by imposing boundary conditions at the interface. As before, we require the interface to be a streamline. Let

$$\gamma = h(x, y, t) = \eta(t) \exp i(\alpha x + \beta y)$$

be the equation of the interface. Then the condition is

$$\frac{D}{Dt}\left\{ z - h \right\} = 0 \qquad \text{or} \qquad \frac{Dh}{Dt} = \left[w \right]_{z = h}$$

The condition may be linearized by expanding w in a Taylor series about $z = 0$ to give

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) h = \left[w \right]_{z = 0}$$

For the present problem this becomes

$$\left(\frac{\partial}{\partial t} \pm i\alpha\right)\eta = \left[w \right]_{z = 0\pm} \qquad z \gtrless 0$$

Taking the Laplace transform,

$$(s \pm i\alpha)\,\eta_s = w_s(0\pm) + \eta(0)$$

from which we obtain

$$\frac{s + i\alpha}{s - i\alpha} = \frac{w_s(0+) + \eta(0)}{w_s(0-) + \eta(0)} \tag{4.3}$$

We also impose the continuity of the pressure at the interface, which in terms of w gives

$$(s + i\alpha)\, w_s'(0+) - w'(0+,0) = (s - i\alpha)\, w_s'(0-) - w'(0-,0) \tag{4.4}$$

Using eq. (4.2) and solving eq. (4.3) and eq. (4.4) for $A\pm$,

$$\delta A_{\pm} = \pm (s^2 + \alpha^2) \int_0^{\mp\infty} d\zeta_0 \, e^{\pm \gamma_{\mp} \zeta_0} \frac{F_{\mp}(\zeta_0)}{(s \pm i\alpha)^2}$$

$$+ \frac{1}{2\gamma_{\pm}} \left\{ \gamma_{-} (s - i\alpha)^2 - \gamma_{+} (s + i\alpha)^2 \right\} \int_0^{\pm\infty} d\zeta_0 \, e^{\mp \gamma_{\pm} \zeta_0} \frac{F_{\pm}(\zeta_0)}{(s \pm i\alpha)^2}$$

$$\pm 2 i\alpha (s \mp i\alpha) \eta(0) \gamma_{\mp} \quad - (s \pm i\alpha) \Delta$$

where $\qquad \delta = \gamma_{+} (s + i\alpha)^2 + \gamma_{-} (s - i\alpha)^2$

$$\Delta = w'(0+, 0) - w'(0-, 0)$$

The transform w_s, which is given by eq. (4.2), can now be inverted.

$$w = \frac{1}{2\pi i} \int_C ds \, w_s \, e^{st}$$

The contour C is parallel to the imaginary axis of the s - plane and to the right of all singularities of w_s. Among these are the branch points $s = \pm \sqrt{J} \pm i\alpha$ and the poles of $A_{\pm}$ (zeros of δ) which occur at $s^2 = \alpha^2 + \frac{1}{2} J$ and correspond to normal modes. There are also poles at $s = \pm i\alpha$ which correspond to neutral modes. (The latter would be obtained from an initial value treatment of the Helmholtz flow with no stratification.)

We now move the contour C to the left until the first singularity is encountered. This will be a pole if $\alpha^2 > J/2$ giving a term proportional to $\exp(\alpha^2 + J/2)^{\frac{1}{2}} t$. The next singularities are the branch points at $s_{\pm} = \sqrt{J} \pm i\alpha$. The contribution of the integral near these points can be estimated asymptotically as in section 3. We expand w_s in powers of $(s - s_+)^{\frac{1}{2}}$ and invert term by term. The largest contribution is from $(s - s_+)^{-\frac{1}{2}}$ giving terms proportional to $t^{-1/2} \exp(\sqrt{J} \pm i\alpha)\, t$. These terms are dominant if $\alpha^2 < J/2$, for then the branch points are the singularities furthest to the right in the s - plane. (It seems unlikely that one could construct an example in which the shear mode is dominant in the limit $\alpha \to 0$ because of the inequality for normal modes[1]:

$|\alpha c|^2 \leq \alpha^2 U_o + J$ where U_o is the half width of the velocity profile)

We have, then, contributions to the inversion integral from the poles, which correspond to the shear modes (modified by the presence of the J/2 term), and contributions from the branch cuts, which correspond to the continuum of eigenvalues associated with the stratification. The two effects are essentially separate; the only interaction is represented by the J/2 term in the shear mode. If we introduce the angle θ which the perturbation makes with the direction of the mean flow ($\alpha = k \cos\theta$) then convection or shear dominates according to

$$\cos^2\theta \lessgtr \frac{1}{2} J/k^2 .$$

5. Acknowledgment

This work was supported by the Space Nuclear Propulsion Office of the National Aeronautics and Space Administration and the Atomic Energy Commission under Grant No. NsG 586.

Footnotes

1. T. J. Eisler, Phys. Fluids 8, 1635 (1965).

2. K. M. Case, Phys. Fluids 3, 366 (1960).

3. B. van der Pol and H. Bremmer, Operational Calculus Based on the Two-Sided Laplace Integral (Cambridge University Press, London, 1950), Chap. 7.

4. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability (Oxford University Press, London, 1961), Chap. 11 .