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TECHNICAL REPORT

STUDY OF RANDOM PROCESS THEORY

By: Charles J. Greaves

CAL No. XM-1970-B-2

Prepared for:
National Aeronautics and Space Administration
Marshall Space Flight Center
Huntsville, Alabama

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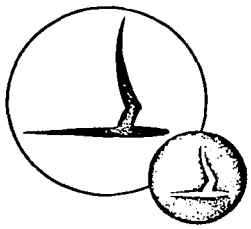

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BUFFALO, NEW YORK 14221

STUDY OF RANDOM PROCESS THEORY

BY:
CHARLES J. GREAVES

CONTRACT NO. NAS 8-11346

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PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA

APPROVED BY: George Bordner
G.W. Bordner, Head
Analysis and Simulation Section

APPROVED BY: W C Schultz
W.C. Schultz, Assistant Head
Avionics Department

ABSTRACT

This report is concerned with random process theory as applied to discrete-data processing. Both stationary and nonstationary data processing techniques are studied.

Methods for reducing the computational times, associated with the digital generation of correlation functions for stationary processes, are described. A theory for developing optimal spectral smoothing weights is discussed, and the experimental results obtained from the application of this theory are included.

The remainder of the report deals with the development of a practical discrete-data nonstationary correlation function detection theory. The experimental results achieved with the application of this correlation detector to Saturn launch vehicle vibration data are also presented.

PREFACE

The research described herein was supported in its entirety by the National Aeronautics and Space Administration, Marshall Space Flight Center, Huntsville, Alabama. This report which may be considered a Phase II report under Contract NAS 8-11346 covers the period from 1 July 1965 through 1 April 1966. The contract technical monitor was Mr. Jack Jones of the Computation Laboratory.

This research represents a continued effort to provide the Computation Laboratory with new or improved applications of statistics to random data processing. The investigations are directed towards the processing of vibration, acoustic, and meteorological type random phenomena. The major accomplishments of the research performed under Phase I were in the areas of stationary correlation function computation, digital filtering, optimal spectral smoothing, deterministic data processing, nonstationary spectrum analysis, and nonstationary correlation theory. A Final Summary Report which describes these accomplishments in detail was submitted to Marshall Space Flight Center on 24 June 1965.

The work, reported here, was carried out by the Cornell Aeronautical Laboratory, Inc. (CAL), primarily in the Avionics Department. The Project Engineer and Senior Investigator was Mr. Charles J. Greaves. Investigators were Dr. Thomas R. Benedict who assisted in the studies of the Half-Polarity Correlator with Added Noise, and Messrs. William D. Fryer and George W. Bordner who provided a significant portion of the material on the N-level correlators. The writer would also like to acknowledge the work of Dr. Walter W. Wierwille, who contributed to Phase I of this project.

The author wishes to express his thanks to Mr. Jones and his colleagues for their helpful participation in this project, and also to acknowledge the technical and administrative guidance provided by Mr. G. W. Bordner and Dr. W. C. Schultz of the Avionics Department at CAL.

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INTRODUCTION

The first two sections of this report are concerned with stationary data processing techniques. In the first section, emphasis has been placed on the development of highly efficient methods of digitally computing correlation functions.

The very significant time savings which can be achieved when a conventional correlation scheme is replaced by a half-polarity correlator is discussed in Reference 1. Also shown in Reference 1 is the fact that the half-polarity correlator is a biased estimator of the true underlying correlation function when the process under investigation has a zero mean bivariate Gaussian distribution. Two classes of non-Gaussian processes are developed, and the half-polarity correlator is shown to also be a biased estimator for any particular member of these classes.

To further expand the applicability of the half-polarity correlator, investigations into the effects of adding uncorrelated noise to the signal prior to processing were performed. These studies have yielded the best means for adding the noise, and also bounds on the strength of the noise to be added to the signal prior to processing by the half-polarity correlator.

Also discussed in Reference 1 is a method for computing the half-polarity correlator with the use of the summation by parts formula. This technique, commonly referred to as Stieltjes correlation, had yielded an additional savings of computer time. Thus the two-level Stieltjes correlator represents an extremely efficient method for computing the half-polarity correlation. With the objective of retaining a high degree of precision, while at the same time keeping computational time at a minimum, an N-level Stieltjes correlator is developed.

In the second section of this report, the theory for optimal spectral smoothing as presented in Reference 1 is briefly reviewed. A particular method of applying this theory, which was directed at obtaining greater accuracy in the spectral estimates, is developed. The experimental results,

which were obtained from analyzing a segment of NASA's vibration data with this new technique, are presented.

A greater percentage of the random process data being analyzed today is of a nonstationary nature. The application of stationary data processing techniques to nonstationary data will often lead to a misinterpretation of the process being analyzed. With this in mind, the last section of this report presents a theory for the detection of nonstationary discrete-data correlation functions. The detection is accomplished in an optimal manner from a single record pair. This correlator is an extension to the continuous data correlator as presented in Reference 1. The usefulness of this approach is demonstrated by applying this theory to several segments of NASA's Saturn launch vehicle vibration data. Methods by which the associated time varying power spectra may be generated are also described.

AUTOCORRELATION FUNCTIONS OF STATIONARY PROCESSES

The generation of power spectral density functions with the use of a digital computer is usually performed by first computing the correlation function. For any given (digital) processor designed to compute spectral densities, the computation of the correlation function generally consumes the major portion of the total time required to produce the power spectra. Thus, the economies provided by the application of rapid (digital) correlation techniques can result in a sizable overall savings in regards to the spectral analysis of random data.

The half-polarity correlator, which is described in Reference 1, when prcperly applied is shown to effect a 20:1 computer time advantage over the more standard correlation computation techniques. It is the purpose of this section to further extend the usefulness of the half-polarity correlator. This is accomplished, in the paragraphs below, by demonstrating the applicability of the half-polarity processor to other than Gaussian bivariate processes. It is also shown that, with the addition of a specified quantity of uncorrelated noise, the half-polarity correlator can be made to be an unbiased estimator of the particular function sought.

The final portion of this section is concerned with insuring the attainment of a high degree of precision in the spectral estimates derived from a particular correlation function, while still keeping the correlation computation time at a minimum. The two level correlator (half-polarity correlator) is extended to N-levels, and a technique for digitally computing the N-level correlator by the Stieltjes correlation method is presented.

Half-Polarity Correlation For Non-Gaussian Processes

Theoretical Background. - The half polarity correlator $R_3(m)$ yields an estimate of the true correlation by the use of one of the following forms:

$$R_3(m) = \frac{1}{N-u} \sum_{k=1}^{N-u} x_k \, \text{sign}\, x_{k+u} \quad , \quad u = |m| \tag{1}$$

or

$$R_3(m) = \frac{1}{N-u} \sum_{k=1}^{N-u} x_{k+u} \, \text{sign}\, x_k \quad , \quad u = |m| \tag{2}$$

Although the expressions (1) and (2) given for the half-polarity correlator $R_3(m)$ are not in general equal, they do possess similar statistical characteristics. It has been shown (References 1, 2) that when the sequence x_k represents N uniformly spaced samples from a zero mean Gaussian process, either form given for $R_3(m)$ is a biased estimator of the true underlying correlation function $\phi(m)$. Thus, the expected value of the half-polarity correlator $R_3(m)$ is proportional to the true correlation $\phi(m)$, that is

$$E\left[R_3(m)\right] = K\phi(m) \tag{3}$$

In the spectral analysis of random phenomena, interest is generally centered on obtaining a knowledge of the relative amplitudes of the spectral estimates. This being the case, a biased estimator provides sufficient information. For those situations where actual rather than relative values

of the spectrum are desired, the proportionality constant K can be estimated as $\hat{K}$. Then the statistic $R_3(m)/\hat{K}$ is to an approximation an unbiased estimator of the true correlation function $\phi(m)$.

The statistical properties which a random process must possess in order that the half-polarity correlator $R_3(m)$, as defined by either (1) or (2), be a biased estimator in the sense of (3), will now be considered. This desired property is illustrated in Figure 1.

Let the joint density function $f(x, y, m)$ of the process x be expanded in terms of a set of orthonormal polynomials θ_j , where the subscript indicates the degree of the polynomial.

$$f(x,y,m) = f(x) f(y) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}(m)\, \theta_i(x)\, \theta_j(y) \tag{4}$$

where $f(x)$ represents the first order density function. The orthonormality condition is given by

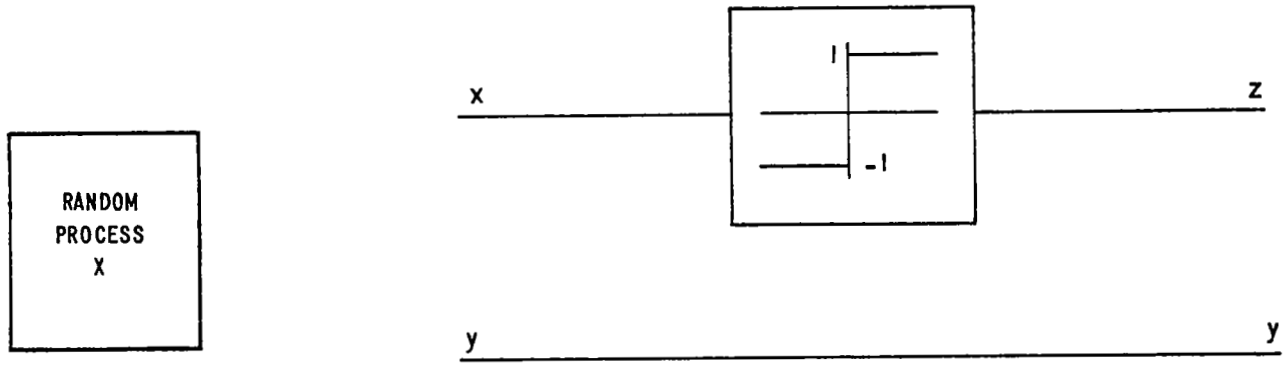
$$\int f(x)\, \theta_i(x)\, \theta_j(x)\, dx = \delta_{ij} \tag{5}$$

The integration in (5) is carried out over the range of the variable x , and δ_{ij} is the Kronecker delta. For the autocorrelation case $x = y$ and from (4) it can be shown that $a_{ij} = a_{ji}$. Using the definition (4) and the orthonormality condition of (5), the following expressions are readily obtained.

$$a_{ij}(m) = \int \int f(x,y,m)\, \theta_i(x)\, \theta_j(y)\, dx\, dy \tag{6}$$

$$\theta_0(x) = 1 \qquad a_{i0} = \delta_{i0}$$

$$\theta_1(x) = \frac{x - M}{\sigma} \qquad a_{11} = \frac{\phi_{xy} - M^2}{\sigma^2}$$



$x = y$

$$\Phi_{zy}(m) = K\Phi_{xy}(m)$$

Figure 1 A DESIRED AUTOCORRELATION PROPERTY

where M = mean value of process

σ = standard deviation of process

To find what restrictions must hold on the quantities a_{7j}, the correlation $\phi_{7y}(m)$ of Figure 1 is computed.

$$\phi_{7y}(m) = \int\int y(\text{sign } x)\, f(x, y, m)\, dx\, dy \tag{7}$$

The sign x is now expanded as follows

$$\text{sign } x = \sum_{j=0}^{\infty} c_j\, \theta_j(x) \tag{8}$$

where

$$c_j = \int f(x)(\text{sign } x)\, \theta_j(x)\, dx \tag{9}$$

When (8) is placed into (7), the following is obtained with the use of (4), (5) and (6)

$$\phi_{7y}(m) = \sigma \sum_{j=1}^{\infty} c_j\, a_{1j}(m) \tag{10}$$

If the half-polarity correlator is to be a biased estimator, (10) can be written as

$$\sigma \sum_{j=1}^{\infty} c_j\, a_{1j}(m) = K\, \phi_{xy}(m) \tag{11}$$

Equation (11) is as far as the analysis can be carried out without specifying either the marginal or joint densities or specific properties of these density functions.

The existence of many random processes, with various joint and marginal densities, for which the half-polarity correlator is a biased estimator is indicated by (11), although in somewhat abstract form. If the random process under consideration has zero mean and the property

$$a_{1j} = d_j\, a_{11} \tag{12}$$

then from the definition for a_{11} in (6), (11) is seen to hold for some K . This represents one possible solution.

It has been demonstrated (Reference 3) that a zero mean Gaussian density function can be expanded in the form given by (4) in terms of orthonormal Hermite polynomials. For this expansion $a_{ij} = 0$ for $i \neq j$. When this condition is placed into (11) it constitutes an additional proof that the half-polarity correlator is a biased estimator in the sense of (3) for zero mean Gaussian processes, though not as concise as the proof presented in References 2 or 1.

A Performance Measure. - Consider now the use of (11) as a performance measure in the sense of being an upper bound indicator of the applicability of the half-polarity correlator to a given process. Before this can be demonstrated, a general property which the a_{ij} must possess is needed. The following inequality is satisfied by all $a_{ij}(m)$ (Reference 4).

$$|a_{ij}(m)| \leq 1 \text{ for all } i\, j \tag{13}$$

As an example of the use of (11), suppose that for a given sequence x_n a histogram of the amplitude indicates that the marginal density is approximately uniform between $-a$ and a . This information along with (5) specifies the polynomials θ_j as the Legendre polynomials. From (9) and the symmetry properties of the Legendre polynomials, $C_j = 0$ for j even. An expression for C_j when j is odd is shown in Appendix A to be given by

$$C_{2N+1} = (-1)^N \frac{\sqrt{4N+3}\ (2N-1)!!}{2^{N+1}(N+1)!}, \quad N = 0, 1, 2, \cdots \tag{14}$$

where

$$K!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot K$$

Equation (11) may now be expressed approximately as

$$\sigma C_1 \frac{\phi_{xy}(m)}{\sigma^2} - \sigma C_3\, a_{13}(m) + \sigma C_5\, a_{15}(m) - \sigma C_7\, a_{17}(m) + \cdots \doteq K \phi_{xy}(m) \tag{15}$$

Using the values of C_j given by (14), the above equation when normalized with respect to σC_1 becomes

$$\rho_{xy}(m) - .381\, a_{13}(m) + .239\, a_{15}(m) - .175\, a_{17}(m) + \cdots \doteq K' \rho_{xy}(m) \qquad (16)$$

where $\rho_{xy}(m)$ is the normalized correlation. The greater the rate of convergence of the C_j for a given process, the greater the reliability in the spectral estimates obtained from the half-polarity correlation. Equation (16) is an upper bound indicator, since all of the a_{ij} may be zero or proportional to ρ_{xy} in which case (16) is exact regardless of the values of the C_j. In addition to these two possibilities, other combinations of the a_{ij} may exist which make (16) an exact expression.

Class I. - The first class $f_I(x, y, m)$ of non-Gaussian processes for which the half-polarity correlator is a biased estimator in the sense of (3) is defined as follows

$$f_I(x, y, m) = \sum_{j=1}^{\infty} C_j f_j(x, y, m)$$

where

$$f_j(x, y, m) = \frac{1}{2\pi \sigma_j^2 R_j} e^{-\frac{1}{2} \frac{x^2 - 2\rho_j xy + y^2}{\sigma_j^2 R_j^2}} \qquad (17)$$

$$R_j = \sqrt{1 - \rho_j^2}$$

$$\sigma_i = \sigma_j = \sigma \quad \text{for all } i, j$$

$$\rho_i(m) \neq \rho_j(m) \quad \text{for } i \neq j$$

$$\sum_{j=1}^{\infty} C_j = 1$$

$$f_I(x, y, m) \geq 0 \quad \text{for all } x, y, m$$

$$C_i C_j \neq 0 \quad \text{for some } i \neq j$$

Although every member in the class $f_I(x, y, m)$ is comprised of a weighted sum of Gaussian bivariate densities, no particular member represents a bivariate Gaussian process. The last condition of (17) insures that each member must be comprised of at least two terms, the correlation ρ_j being different, eliminates the possibility of any member representing a Gaussian bivariate process.

It is now demonstrated that the half-polarity correlator is a biased estimator in the sense of (3) for any particular process in $f_I(x, y, m)$. For the half-polarity correlator to be a biased estimator, the following property must hold

$$E[x\,y] = K E[x \operatorname{sign} y] \tag{18}$$

Taking any particular member of $f_I(x, y, m)$ and utilizing the half-polarity correlator property (Eq. 51) demonstrated in Reference 1, Equation (18) becomes

$$\sigma^2 \sum_{j=1}^{\infty} C_j \rho_j(m) = K\sqrt{\frac{2}{\pi}}\,\sigma \sum_{j=1}^{\infty} C_j \rho_j(m) \tag{19}$$

Equation (19) completes the proof, since it illustrates the expectations of (18) are related by a multiplicative constant.

Class II. - The second class $f_{II}(x, y, m)$ of non-Gaussian processes is defined in a similar manner as the first class, with the following changes

$$\begin{aligned} &\rho_i = \rho_j = \rho && \text{for all } i, j \\ &\sigma_i \neq \sigma_j && \text{for } i \neq j \end{aligned} \tag{20}$$

When (18) is applied to any member of this second class of non-Gaussian bivariate processes the following is obtained

$$\rho(m) \sum_{j=1}^{\infty} C_j \sigma_j^2 = K\sqrt{\frac{2}{\pi}}\,\rho(m) \sum_{j=1}^{\infty} C_j \sigma_j \tag{21}$$

Thus, for this second class $f_{II}(x, y, m)$ of densities, it is seen that for any member (18) is satisfied.

Although the definitions of Class I and Class II appear similar, there are significant differences between the members of these two classes. The joint densities in Class I may exhibit the characteristic of being composed of both continuous and discrete (along $x = y$) densities for a given m. On the other hand, a member from Class II may consist of either a continuous or a discrete density for a given m, but not both. The correlation for any member of Class II is proportional to the correlation of the individual component densities which make up that member. For members of Class I, however, the correlation is proportional to a weighted sum of the individual component correlations. As a final comparison between the members of these two classes, consider their marginal densities. It is interesting to note that though the joint densities in Class I are non-Gaussian, the associated marginal distributions are Gaussian. The associated marginal distributions for Class II, however, are non-Gaussian as is illustrated in the following example.

Example. - Consider the member of Class II which is given by

$$f_{II}(x, y, m) = 1.1 f_1(x, y, m) - .1 f_2(x, y, m) \tag{22}$$

where

$$\sigma_1 = 10 \sigma_2$$

To obtain the marginal density, integration of (22) with respect to either x or y is performed. This results in

$$f_{II\,M}(x) = \frac{1.1}{\sqrt{2\pi}\, 10\sigma_2} e^{-\frac{1}{2}\left(\frac{x}{10\sigma_2}\right)^2} - \frac{1}{\sqrt{2\pi}\, 10\sigma_2} e^{-\frac{1}{2}\left(\frac{x}{\sigma_2}\right)^2} \tag{23}$$

Figure 2 illustrates the non-Gaussian character of this density function.

Conclusions. - The properties which a random process must possess in order that the half-polarity correlator be a biased estimator have been derived and are expressed by Equation (11). The main objective of this

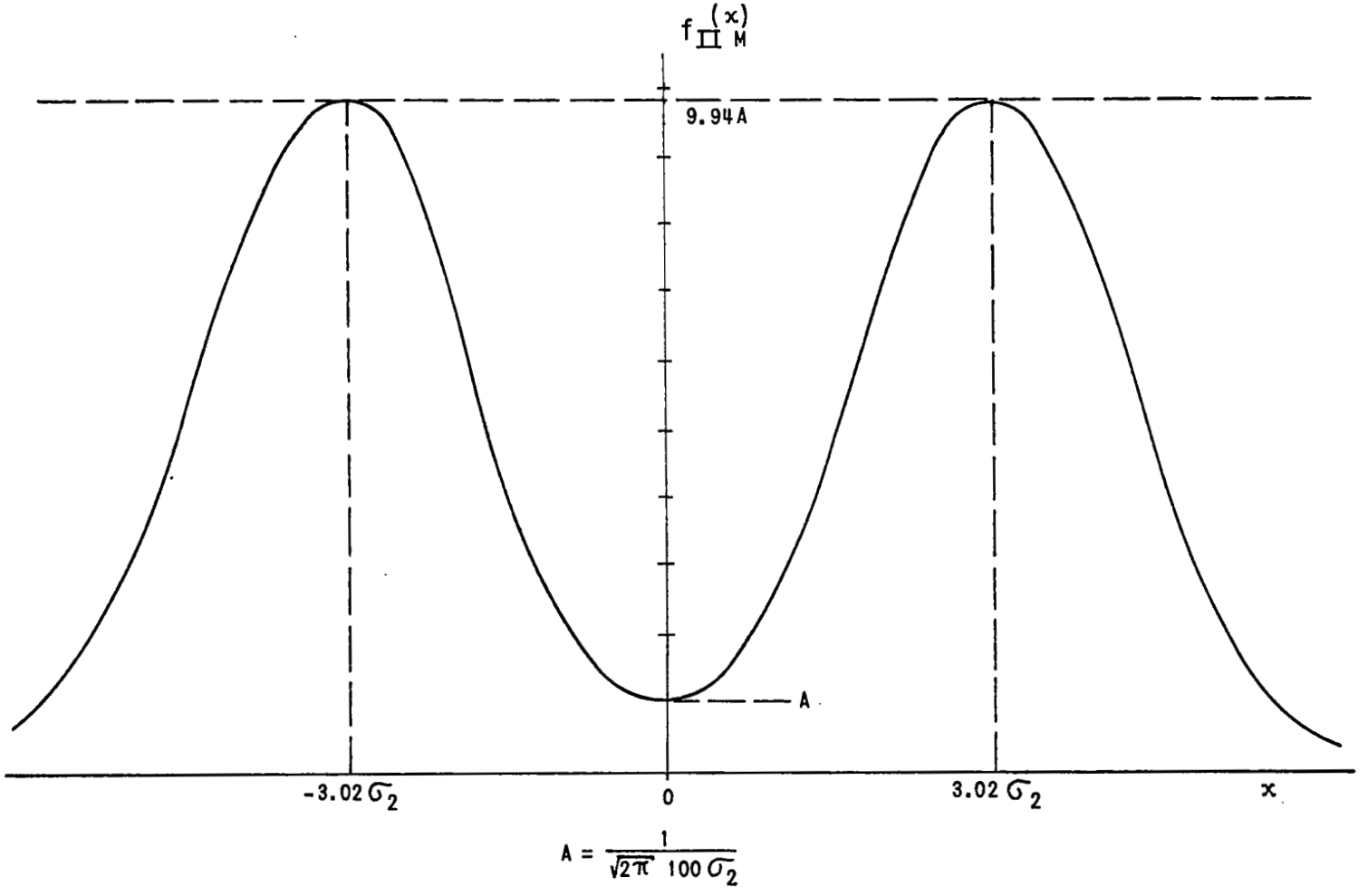


Figure 2 MARGINAL DENSITY FOR A MEMBER OF CLASS II

section, however, was the determination of processes other than joint Gaussian for which the half-polarity correlator would be an appropriate processor. This goal has been accomplished by the development of two such classes of random processes. Much of the physical phenomena encountered in practice is often approximated by a marginal density associated with the second class discussed.

It is believed that, in addition to the two classes developed here, there exist many other processes which possess the desired property. A more descriptive definition, other than (11), for these other processes has not yet been determined.

The problem of developing tests to determine if the half-polarity correlator is appropriate for a given process is a difficult one. Demonstrating the expanded usefulness of the half-polarity correlator has not simplified this problem area. The example given above demonstrates that a test for the applicability of the half-polarity correlator, to a given process, which is based upon a Gaussian criteria, is very restrictive. This is clearly the case, since every member of Class II has a non-Gaussian marginal distribution.

The possibility of using the expansion coefficients of sign χ as an upper bound measure of performance was discussed. The potential usefulness of this error measure will depend upon the existence of marginal densities which provide for rapid convergence of the expansion coefficients.

Half-Polarity Correlator with Added Noise

Theoretical Background. - The half-polarity correlator has been demonstrated to be applicable over a wider class of random processes. There are still, however, many processes for which this correlator, in its present form, will not yield sufficient accuracy. It is to this remaining class of random processes that the following paragraphs are devoted.

Methods for modifying the half-polarity correlator while still retaining the essential features which allow for rapid digital computation are described. The improvement that can be achieved by adding uncorrelated Gaussian noise to the signal prior to processing by the half-polarity correlator has been

experimentally demonstrated in Reference 1. It is the specific aim of this section to obtain analytical expressions which will provide information on the following:

1. Where should the noise be added ?
2. What type of noise should be added ?
3. How much noise should be added ?

In the analysis presented here, no assumptions are made regarding the underlying statistics of the signals being processed. It is also understood that all expectations imply averaging over the probability space of the added noise. The full implications of these statements are made clear in the paragraphs below.

A Performance Index. - Three possible arrangements for the addition of noise into the half-polarity correlator will be considered. Figure 3 illustrates the correlator modifications. For the first case to be considered $u = n$ that is, the same noise is added to each leg of the correlator. This particular correlator is the same as that discussed in Reference 1, and for the purpose of the present discussion it will be designated as $R_1(m)$. By definition

$$R_1(m) = \frac{1}{N-m} \sum_{k=1}^{N-m} (x_k + n_k)\, sign(y_{k+m} + n_{k+m}) \tag{24}$$

In the second correlator to be considered, the noise signals u and n are independent, and the correlator is defined as

$$R_2(m) = \frac{1}{N-m} \sum_{k=1}^{N-m} (x_k + u_k)\, sign(y_{k+m} + n_{k+m}) \tag{25}$$

The last case to be considered is obtained by setting the noise signal u equal to zero. An expression for this correlation is then given by

$$R_3(m) = \frac{1}{N-m} \sum_{k-1}^{N-m} x_k\, sign(y_{k+m} + n_{k+m}) \tag{26}$$

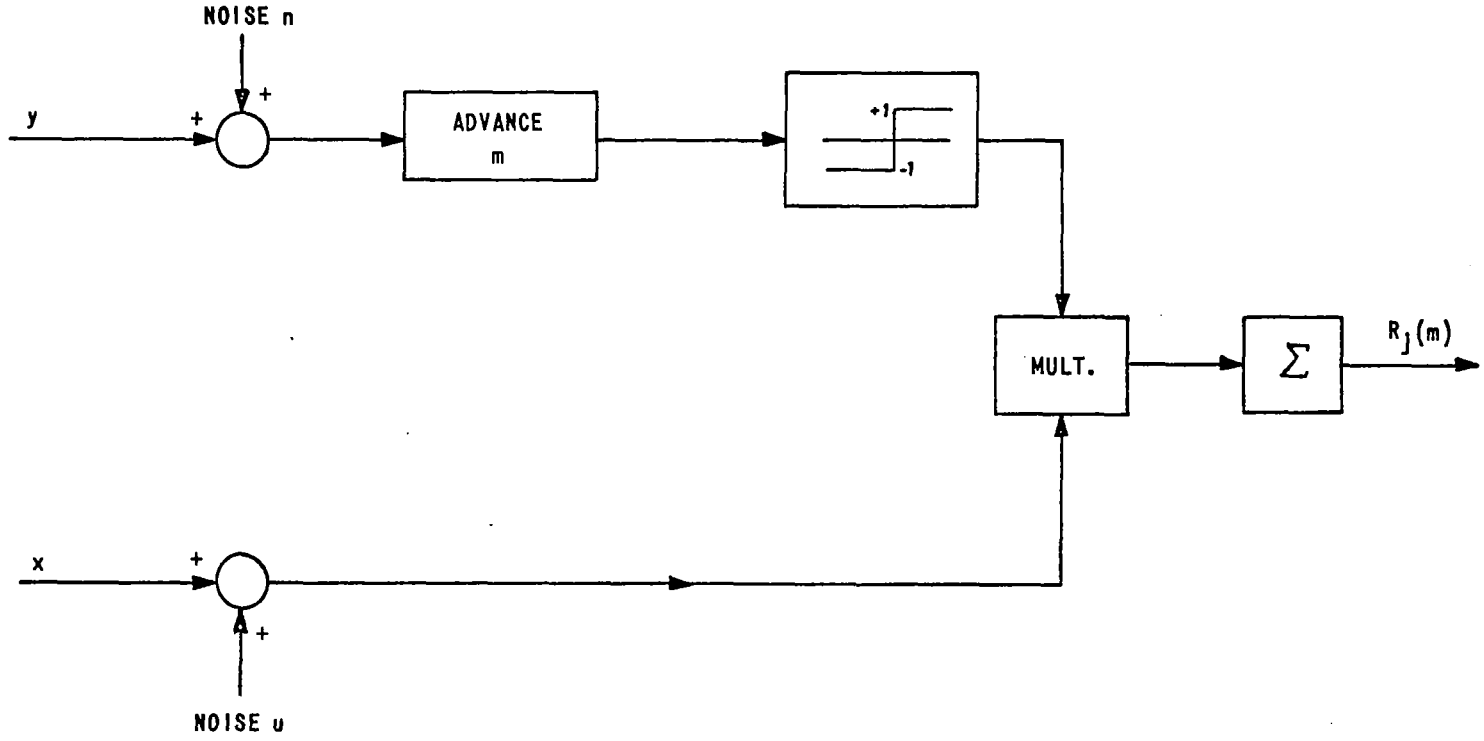


Figure 3 HALF-POLARITY CORRELATOR WITH ADDED NOISE

As a means of comparing the performance of the three correlators $R_1(m)$, $R_2(m)$, and $R_3(m)$, a performance index is developed. Let the error be defined as the difference between a constant times one of the noise added correlators and the actual estimate sought. That is

$$e_j(m) = K_j R_j(m) - R(m) \tag{27}$$

where

$$j = 1, 2, 3$$

$$R(m) = \frac{1}{N-m} \sum_{k=1}^{N-m} x_k y_{k+m}$$

The constant K_j is selected such that for some value of the strength of noise added, the expected value of the random variable $e_j(m)$ is equal to zero. The performance index chosen will be the mean squared value of error, and is given by

$$J_j = E\left[e_j^2(m)\right] \tag{28}$$

With the use of (27) the above equation may be expanded in a more convenient form as follows:

$$J_j = E\left[\left(K_j R_j(m) - R(m)\right)^2\right]$$

$$= E\left[K_j^2 R_j^2(m) - 2K_j R_j(m) R(m) + R^2(m)\right]$$

$$= K_j^2 E\left[R_j^2(m)\right] - 2K_j E\left[R_j(m)\right] R(m) + R^2(m)$$

$$= K_j^2 E\left[R_j^2(m)\right] - K_j^2 E^2\left[R_j(m)\right]$$
$$+ K_j^2 E^2\left[R_j(m)\right] - 2K_j E\left[R_j(m)\right] R(m) + R^2(m)$$

$$J_j = K_j^2 \, VAR\left[R_j(m)\right] + \left[K_j E\left[R_j(m)\right] - R(m)\right]^2 \tag{29}$$

It should be noted that both terms in the expression for the performance index (29) are positive quantities.

The idealized objective is to minimize J_j with respect to the types of correlators, types of noises and amount of noise strength added, the solution of which must not be dependent upon the detailed nature of the sequence being processed. Though the goal is a worthy one, it is somewhat overambitious. In the first place, the stated objective is somewhat contradictory in the sense that an exact mathematical minimum must, in general, depend upon the specific nature of the sequence being processed.

The analysis to follow will consider the three types of correlators discussed above, and two types of uncorrelated noise. General guides as to the amount of noise which should be added will be obtained rather than exact values.

Expected Value and Variance of Correlators. - Expressions for the expected value and variances of the three correlators as derived in Appendix B are given by

$$E\left[R_1(m)\right] = \frac{1}{N-m} \sum_{k=1}^{N-m} x_k f_{k+m} + \frac{\sigma_{om}}{N} \sum_{k=1}^{N} g_k \qquad (30)$$

where

$$f_k = E\left[\operatorname{sign}(y_k + n_k)\right]$$

$$g_k = E\left[n_k \operatorname{sign}(y_k + n_k)\right]$$

$$\begin{aligned} VAR\left[R_1(m)\right] = {} & \frac{\sigma^2}{N-m} + \frac{1}{(N-m)^2} \sum_{k=1}^{N-m} x_k^2 \left[1 - f_{k+m}^2\right] \\ & + \left[\frac{2}{(N-m)^2} \sum_{k=1}^{N-2m} x_k g_{k+m} f_{k+2m}\right](1-\delta_{om}) \\ & - \left[\frac{2}{N^2} \sum_{k=1}^{N} x_k f_k g_k + \frac{1}{N^2} \sum_{k=1}^{N} g_k^2\right] \delta_{om} \end{aligned} \qquad (31)$$

where σ^2 is the variance of the noise n

$$E\left[R_2(m)\right] = \frac{1}{N-m} \sum_{k-1}^{N-m} x_k f_{k+m} \qquad (32)$$

$$VAR\left[R_2(m)\right] = \frac{\sigma_u^2}{N-m} + \frac{1}{(N-m)^2}\sum_{k=1}^{N-m} x_k^2\left[1-f_{k+m}^2\right] \tag{33}$$

where σ_u^2 is the variance of the noise u

$$E\left[R_3(m)\right] = \frac{1}{N-m}\sum_{k=1}^{N-m} x_k f_{k+m} \tag{34}$$

$$VAR\left[R_3(m)\right] = \frac{1}{(N-m)^2}\sum_{k=1}^{N-m} x_k^2\left[1-f_{k+m}^2\right] \tag{35}$$

When (33) is compared with (35), it is seen that the variance of the $R_3(m)$ correlator is always less than that of the $R_2(m)$ correlator. Since the value for the constant K_j will only depend upon the expected value of the correlator, (29) indicates that the following relation will always hold

$$J_3 < J_2 \tag{36}$$

That is, the $R_3(m)$ correlator will provide a better estimate than the $R_2(m)$ correlator. In the following paragraphs no further consideration will be given to the $R_2(m)$ correlator.

Gaussian Noise. - When zero mean uncorrelated Gaussian noise is added to the $R_1(m)$ and $R_3(m)$ correlators, the functions f_k and g_k as defined above are found in Appendix C to be given by

$$f_k = \frac{2}{\sqrt{\pi}}\int_0^{\frac{y_k}{\sqrt{2}\sigma}} e^{-\beta^2}\,d\beta$$

$$= erf\left(\frac{y_k}{\sqrt{2}\sigma}\right)$$

$$g_k = \sigma\sqrt{\frac{2}{\pi}}\; e^{-\frac{1}{2}\left(\frac{y_k}{\sigma}\right)^2} \tag{37}$$

Using the above relations, the expected value of the $R_3(m)$ (or $R_1(m)$ for m not equal to zero) correlator becomes

$$E\left[R_3(m)\right] = \frac{1}{N-m} \sum_{k=1}^{N-m} x_k \frac{2}{\sqrt{\pi}} \int_0^{\frac{y_{k+m}}{\sqrt{2}\,\sigma}} e^{-\beta^2} d\beta \tag{38}$$

With a change of integration variable, this may also be expressed as

$$E\left[R_3(m)\right] = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \frac{1}{N-m} \sum_{k=1}^{N-m} x_k \int_0^{y_{k+m}} e^{-\frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2} d\alpha \tag{39}$$

If the constants K_j are selected to be

$$K_1 = K_3 = \sigma\sqrt{\frac{\pi}{2}} \tag{40}$$

Then from (39) the following limiting values are obtained

$$\begin{aligned} &\lim_{\sigma \to \infty} \quad K_1 E\left[R_1(m)\right] = R(m), \quad m \neq 0 \\ &\lim_{\sigma \to \infty} \quad K_3 E\left[R_3(m)\right] = R(m) \end{aligned} \tag{41}$$

With the use of the following property of the error function

$$\lim_{\alpha \to \infty} \quad erf(\alpha) = 1 \tag{42}$$

the variances of the $R_1(m)$ and $R_3(m)$ correlators are seen to go to zero as the strength of the noise σ is decreased.

The determination of an optimum or near-optimum choice of the noise strength σ in the sense of minimizing either of the indices J_1 or J_3 for all sequences x_k is a difficult, if not impossible, task. A useful lower bound, however, can be obtained for the noise strength σ. A value of σ greater than this lower bound will insure the better performance of the correlator. This lower bound will now be determined. If the $R_3(m)$ correlator is to provide greater accuracy, the following relationship between the performance indices must be satisfied

$$J_1 - J_3 > 0 \tag{43}$$

From the definition of the performance index given by (29) and the expression for the variances, (43) reduces for m not equal to zero to the following

$$K_1 \left[\frac{\sigma^2}{N-m} + \frac{2}{(N-m)^2} \sum_{k=1}^{N-2m} x_k \, \sigma \sqrt{\frac{2}{\pi}} \, e^{-\frac{1}{2}\left(\frac{y_{k+m}}{\sigma}\right)^2} \, erf\left(\frac{y_{k+2m}}{\sqrt{2}\,\sigma}\right) \right] > 0 \quad (44)$$

or

$$1 + \frac{4/\sqrt{2\pi}}{(N-m)} \sum_{k=1}^{N-2m} x_k \, \frac{e^{-\frac{1}{2}\left(\frac{y_{k+m}}{\sigma}\right)^2}}{\sigma} \, erf\left(\frac{y_{k+2m}}{\sqrt{2}\,\sigma}\right) > 0 \quad (45)$$

Assuming the worst possible conditions for (45), a situation not very likely to occur in practice, (45) reduces to

$$1 - \frac{1}{\sigma} \, \frac{4}{\sqrt{2\pi}\,(N-m)} \sum_{k-1}^{N-2m} |x_k| > 0 \quad (46)$$

or

$$\sigma > \frac{4}{\sqrt{2\pi}\,(N-M)} \sum_{k=1}^{N} |x_k|$$

where M is the largest value of lag to be computed. If the sum of the magnitudes is a quantity not normally available, Schwarz's inequality can be employed to obtain a lower bound which is in general, larger, and given by

$$\sigma \geq \frac{4}{2\pi} \sqrt{\frac{N}{N-M} \, \frac{\sum_{k=1}^{N} x_k^2}{N}} \quad (47)$$

When the following range of σ is used in (45)

$$\sigma > \frac{C}{.95} \quad (48)$$

where C is given by

$$C = \max_k \left\{ |x_k| \right\} \quad (49)$$

it can be demonstrated that the inequality is also satisfied. Thus, the expression given by (48) represents a different lower bound on the strength of the noise to be added. For many sequences, and especially when M is a large percentage of N, this second bound may prove to be lower than that of (46).

For the zero lag case when $m = 0$, the difference of the performance indices is no longer given by (45). This difference must be reformulated from the basic relations given by (29), (30), (31), (34), (35), for $m = 0$. With this reformulation, a value of $\sigma > C/.837$ is found to be a lower bound on the noise strength.

Returning to Equation (45), it will now be demonstrated that the $R_3(m)$ correlator is also better than the $R_1(m)$ correlator for small values of σ. Each term in the sum in the second member of the left hand side of (45) must contain a factor of the form $\frac{e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2}}{\sigma}$. This particular function of σ approaches zero as σ approaches zero, thus the inequality of (45) can always be satisfied by reducing the noise strength.

Uniform Noise. - With the addition of uncorrelated uniform noise, the functions f_k and g_k as shown in Appendix D are given by

$$f_k = \delta_1(y_k) + \frac{1}{\sqrt{3}\,\sigma}\, y_k\, \delta_2(y_k) - \delta_3(y_k) \tag{50}$$

$$g_k = \frac{3\sigma^2 - y_k^2}{2\sqrt{3}\,\sigma}\, \delta_2(y_k) \tag{51}$$

The $\delta(y)$ functions separate the range of y into three nonoverlapping regions and have a value (unity) only in their respective regions. A more complete description of these functions is given by Equation (D-1) of Appendix D.

If the maximum absolute value of the sequence y_k is again designated as C, then from (34) and (50) the expected value of the $R_3(m)$ correlator is seen to be

$$E\left[R_3(m)\right] = \frac{1}{\sqrt{3}\,\sigma}\, R(m), \quad \sigma \geq \frac{C}{\sqrt{3}} \tag{52}$$

thus, the constants K_1 and K_3 are evaluated as

$$K_1 = K_3 = \sqrt{3}\,\sigma \tag{53}$$

The shape of the performance index J_3 as a function of σ will now be obtained. Let the first term of the expression for J_3 (Equation (29)) be designated as $V(\sigma)$, and the second term as $B(\sigma)$.

Figure 4 illustrates the general shape of these functions for any sequence x_k. The functions $V(\sigma)$ and $B(\sigma)$ are decreasing and increasing functions of σ respectively, though not necessarily smoothly as shown in Figure 4. The optimum choice for the noise strength σ to be added to this correlator is seen to be bounded by the interval $(0, c/\sqrt{3})$. It must be emphasized that this result is not dependent upon a detailed knowledge of the sequence x_k being processed. This is a desirable characteristic.

If a lower bound is sought, as was done for the Gaussian noise case, to insure the superiority of the $R_3(m)$ correlator the following result would be obtained

$$\sigma > c \tag{54}$$

This lower bound on the noise strength is not very useful, since the optimum choice of σ for the $R_3(m)$ correlator has been demonstrated to lie in the interval $(0, c/\sqrt{3})$. The relative magnitudes of J_1 and J_3 are difficult to determine for general sequences when (54) is not satisfied. However, it can again be demonstrated that the index J_3 is less than J_1 for sufficiently small σ.

Conclusions. - From the analysis of three modified (noise added) half-polarity correlators, it can be concluded that the $R_3(m)$ correlator as defined by (26) performs better than the other two correlators considered. Lower bounds on the noise strength, to be added in order to insure the superiority of the $R_3(m)$ correlator, have been obtained. The nature by which these lower bounds were obtained would lead the author to conclude that for some interval below the stated bounds, the condition $J_3 < J_1$ is still valid for any sequence x_k whatever. Indeed, the $R_3(m)$ correlator may well be superior for any value of noise strength.

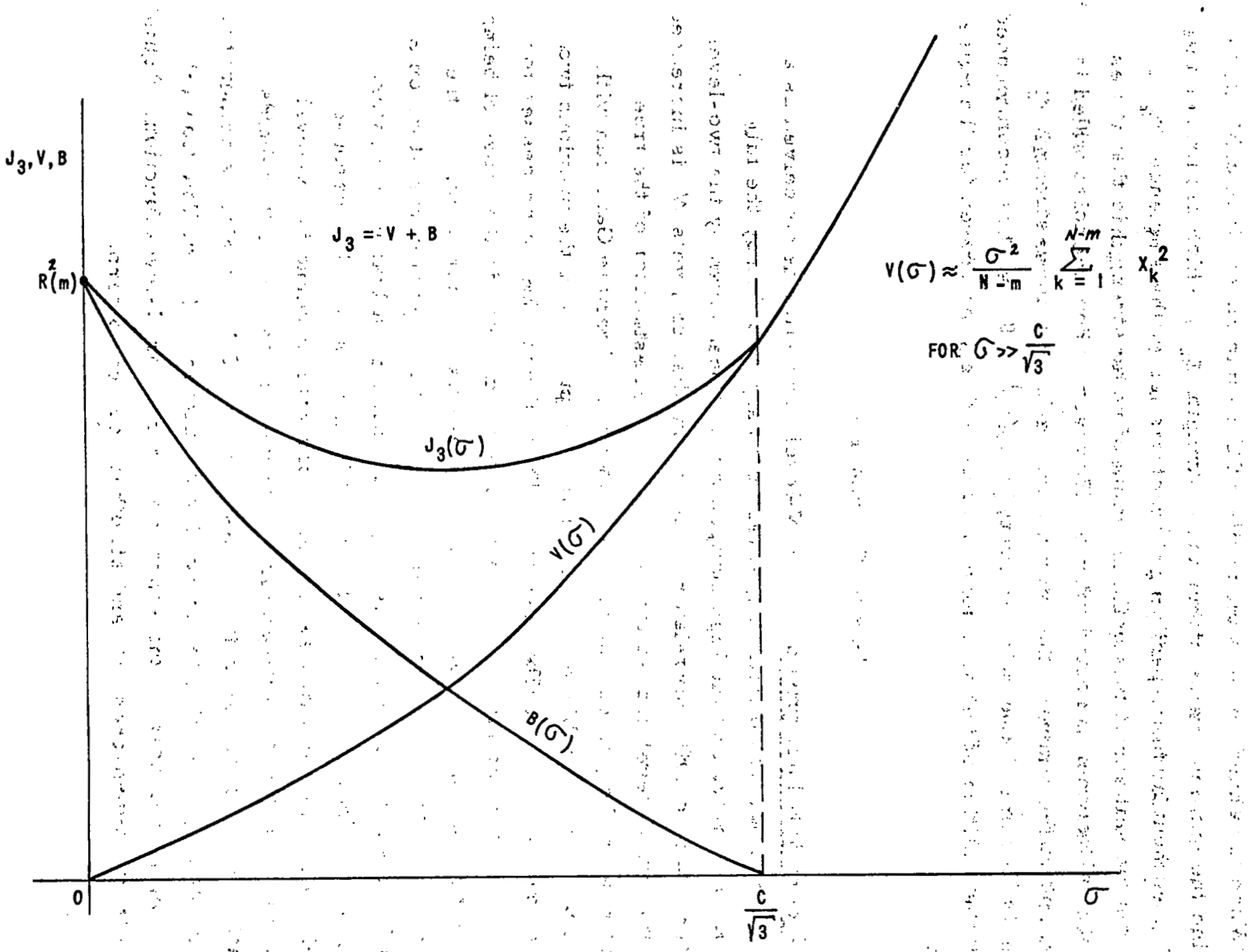


Figure 4 INDEX FOR THE R_3 (m) CORRELATOR

When the added noise is uncorrelated Gaussian noise, it is difficult to determine the optimum value of the noise strength σ_o , which minimizes the index J , without further specifying characteristics of the sequence x_k . All that can be said about this optimum value of noise strength is that it lies in the interval between zero and infinity. However, when the noise added is uncorrelated uniform noise, the optimum choice for the noise strength σ_o is known to lie somewhere in the interval $(0, c/\sqrt{3})$. Thus, it is recommended that with the addition of uniform noise a value of $c/2\sqrt{3}$ be chosen for the noise strength.

N-Level Correlators

Theoretical Background. - The N-level correlator is conceived as a means of combining the merits of greater accuracy afforded by the full precision correlator, and the high computational speed given by the two-level correlator (half-polarity correlator). As the number of levels N is increased beyond two, the N-level correlator is still a biased estimator of the true correlation when the process under investigation is bivariate Gaussian with zero mean. It is also a biased estimator for any process in the previous two non-Gaussian classes described earlier. Thus, the number of processes for which the N-level correlator is an appropriate processor, in the sense of being a biased estimator, apparently does not decrease as N increases. On the other hand, the accuracy obtained with the use of an N-level correlator, on a process for which the correlator is only approximately a biased estimator, increases as N increases. That this must be the case can be reasoned intuitively, since in the limit, as N increases without bound the N-level correlator approaches the full-precision correlator (assuming thresholds and relative levels decrease with increasing N). This can also be demonstrated mathematically by observing the expansion coefficients C_j of Equation (9) with the sign x replaced by the appropriate nonlinear N-level function. Again in the limit C_1 approaches σ and all other C_j go to zero.

Coding Scheme. - An efficient means for coding the N-level correlator on a general purpose digital computer will now be presented. The N-level correlator is given by

$$R_m = \frac{1}{N-m} \sum_{n=1}^{N-m} x_n \; y_{n+m} \;, \quad n = 0, 1, 2 \cdots \tag{55}$$

where in (55) y represents the quantized version of the x data. Roman numerals heading subsequent paragraphs refer to the corresponding blocks in Figure 5.

I. Preparation of x data

The mean value associated with meteorological, acoustic, or vibrational random phenomena is often zero. It is recommended that any bias in the data be removed, whether the bias is the result of instrumentation error or a true characteristic of the process. If the data z_i are integers, set $x_i = z_i - \left[\bar{z}\right]$ where $\left[\bar{z}\right]$ is the integerized average of the z values. However, in most cases the z_i are floating point numbers. For this case, the z_i should be scaled, then set $x_i = \left[z_i - \bar{z}_i\right]$, where $\left[\quad\right]$ denotes the integerization process. Integerization is performed to take advantage of the faster speed that certain fixed-point arithmetic instructions possess over their floating point equivalents.

II. Preparation of y data

For the present autocovariance problem x and y data are logically equivalent; however, a numerical difference is forced into the y data by quantizing rather coarsely compared with the x data. Thus

$$y_i = quant\,(x_i)$$

where quant ( ) implies quantization of the x data into, say, N-levels.

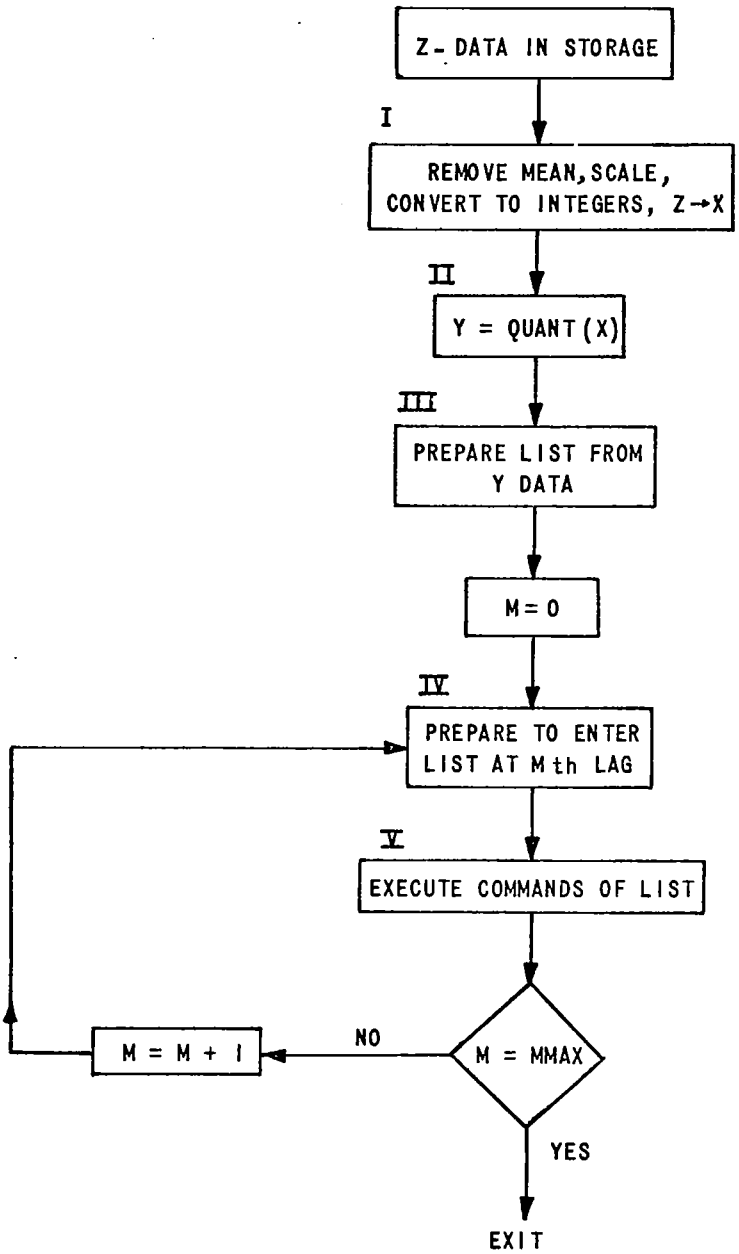


Figure 5 N - LEVEL CORRELATOR

III. Prepare list from y data

With a counter keeping track of n a LIST array is formed. IBM 7044 machine language is used here to represent the actual operation:

(a) If $y_n = 0$, increase n by 1 and go to the next y_n value

(b) If $y_n = +1$, add to the LIST array the command:
ADD $X+n$, 1

(c) If $y_n = -1$, add to the LIST array the command:
SUB $X+n$, 1

(d) If y_n has a positive value $k > 1$, add to the LIST array k commands:
ADD $X+n$, 1
ADD $X+n$, 1

ADD $X+n$, 1

(e) If y_n has a negative value greater than 1 in magnitude, follow step (d) except use SUB $X+n$, 1 commands.

What to do for the y_n values when $|y_n| > 1$, is a matter of choice. The simplest option, as shown above, is to insert the proper number of ADD or SUB commands directly into the list. If $|y_n| > 1$ occurs frequently, the length of the list may increase beyond desirable bounds; in this case, a transfer command could be inserted into the operable array which would give an exit to short subroutines, one for positive y_n , one for negative y_n , to save storage at a slight cost in terms of machine time.

IV. Prepare to enter list at proper point for lag M

In direct calculation of correlation sums, exactly one term is dropped as the lag number is increased by one. This one-to-one relation does not apply to the packed list, of course. One "direct" term may correspond to none, one, or more executable instructions, and hence, the point of entry may remain fixed, move ahead by one, or move ahead several cells depending upon the original data.

Logic to determine the proper entry point rapidly (with minimal auxiliary storage) was coded as follows:

(a) tentatively select entry point same as for previous lag number

(b) check effective address (of the ADD or SUB command in this cell) against the beginning of the data array:

(i) effective address = [true address] - [contents of index register]
= [address of some X(I)] - [lag number]

(ii) is effective address $\geq$ address of X(1) ?

(c) if answer to (ii) is no, the next cell is selected and step (b) test is applied. This process is repeated until the proper entry point is located.

Since the previous entry point equals, or is very close to, the correct new entry point, this search procedure requires little time and requires no extensive table of entry point values or other auxiliary storage.

V. Execute commands of list

With the correct entry point determined, the list commands are executed. The resultant sum in the accumulator is then proportional to the correlation at lag M .

The following table shows the x , y and List arrays that would be generated by the given original data Z array. Threshold levels are at ±5, ±15, ±25, ±35.

The N-Level Stieltjes Correlator. - The so-called Stieltjes correlator is an application of the summation-by-parts formula (analogous to integration-by-parts) to the calculation of correlation sums. The N-level correlator described above, and the Stieltjes correlator described here will both provide a very large speed advantage over the usual direct summation. Comparison between the N-level and Stieltjes N-level correlator will be made in later paragraphs. To be considered now are the modifications required in the above coding scheme in order to convert the program into a Stieltjes N-level correlator.

Table I EXAMPLE OF N-LEVEL CORRELATOR

n	z	x	y	LIST
1	29.1	29	3	ADD X+1,1
2	20.4	20	2	ADD X+1,1
3	4.1	4	0	ADD X+1,1
4	-3.9	-4	0	ADD X+2,1
5	-14.2	-14	-1	ADD X+2,1
6	-19.8	-20	-2	SUB X+5,1
7	-34.0	-34	-3	SUB X+6,1
.	.	.	.	SUB X+6,1
.	.	.	.	SUB X+7,1
.	.	.	.	SUB X+7,1
.	.	.	.	SUB X+7,1

By applying the summation-by-parts formula, (55) may be written as

$$R_m = \frac{1}{N-m} \left\{ C_{N-m}\ y_N - \sum_{n=1}^{N-m-1} C_n\ D_{n+m} \right\} \qquad (56)$$

where

$$C_n = \sum_{j=1}^{n} x_j$$

$$D_n = y_{n+1} - y_n$$

When the definition for C_n and D_n are placed into R_m above and expanded, it is readily demonstrated that Equations (55) and (56) are equivalent regardless of the form of the x , y data. The second term within the brackets of (56) is commonly referred to as a Stieltjes sum (Reference 6).

To modify the N-level correlator as shown in Figure 5, the following insertions are made. The calculations of C_n and D_n as given by (56) would be performed after Block II. The D_n array would replace the y_n array which is no longer needed (except for y_N), and the C_n array would replace the x_n array. In Block III the list array is now formed by using the D_n . The only other modification needed to convert the N-level correlator to an N-level Stieltjes correlator is the addition (with appropriate sign) of the

term $C_{N-m} y_N$. This addition can be performed after the exit when the scaling by $1/(N-m)$ is accomplished.

Quantizer Thresholds. - The actual coding of the quantizer is a minor problem. Several good fast methods can be developed by an average programmer, even if the threshold values are not uniformly spaced. And in any case, the required machine time will be small compared with that used in calculating correlation sums and spectra.

The purpose of this section is to document some principles for determining good threshold values -- quantitatively, for Gaussian data, and approximately for nearly Gaussian data.

Before this is accomplished, however, some of the theoretical aspects of threshold selection will be discussed. With the processing of random data, amplitude histograms are often computed, from which information on the marginal density can be obtained. In an earlier portion of this report, it was demonstrated that the marginal density specified the orthonormal expansion polynomials θ_j . If the particular N-level device selected is designated by $N(x)$ then the expansion coefficients C_j may be expressed as

$$C_j = \int N(x)\, f(x)\, \theta_j(x)\, dx \tag{57}$$

where $f(x)$ represents the marginal density for the process. A potentially useful technique for the selection of the thresholds $\{T\}$ associated with $N(x)$ would then be given by

$$\min_{\{T\}} \left\{ \frac{C_j}{C_1} \; ; \quad j = 2, 3, 4 \cdots \right\} \tag{58}$$

That is, the thresholds $\{T\}$ are selected so as to reduce the values of C_j/C_1 in Equation (11). The complexities which may be involved in obtaining solutions of (58) for the thresholds $\{T\}$ have not been fully explored. The convergence properties of the normalized coefficients $\frac{C_j}{C_1}$ must also be examined.

Return now to the problem of specifying the thresholds under the assumption that the data is Gaussian or nearly Gaussian.

Consider a correlation sum, the exact value of which is $R = \sum x_i\, y_i$, and which is to be approximated with

$$\hat{R} = \sum x_i\, \hat{y}_i$$

where $\hat{y}$ is a quantized value of the exact y_i value. Consider a particular quantum region, say

If $A \leq y \leq B$, assign the value v to y.

That is, a single value v is to act as a representative for all values of y between A and B. If the distribution of the y data is known, then it appears that a potential multiplicative bias could be removed, by choosing for the number v the average value of the class, i.e.,

$$\text{let } v = \frac{\int_A^B y\, p(y)\, dy}{\int_A^B p(y)\, dy} \quad , \quad \text{conditioned average for interval A, B} \qquad (59)$$

Note that this criterion would work for the half-polarity correlator, where the use of $V_+ = +1$ for positive y and $V_- = -1$ for negative y gives the correct expected value of correlation, multiplied by the constant $\sqrt{\frac{\pi}{2}}$ (Gaussian data, $\sigma = 1$). By the average-value criterion, one should use for positive y the value

$$v_+ = \frac{\int_0^\infty \frac{1}{\sqrt{2\pi}}\, y\, e^{-\frac{y^2}{2}}\, dy}{\int_0^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy} = \sqrt{\frac{2}{\pi}}$$

and by symmetry $-\sqrt{2}/\sqrt{\pi}$ for negative y , a choice which does indeed remove the multiplicative bias. (Agreed, one would do the actual coding with +1 and -1 for convenience, and put the multiplier in just once at the end.)

The following scheme resolves the multiplicative bias problem, while preserving integer values for the quantized y data, when the number of quantum levels is an odd number (3, 5, ...). Specifically, the v values are set to fixed integer values, and the set of equations (59) is used to solve for the set of threshold values.

To illustrate, assume a three-level quantizer, operating on Gaussian data with zero mean and unit variance with the following rule for quantizing:

If $-\infty \quad y \quad -A$ let $y = -1$

$-A \quad y \quad +A$ let $y = 0$

$A \quad y \quad \infty$ let $y = +1$

The middle requirement is automatically correct (the average of any symmetrical region about 0 is zero), and the first will, by symmetry, be satisfied if the third one is. Thus, the only unknown is A, which must, by (59) satisfy

$$\frac{\int_A^\infty \frac{1}{\sqrt{2\pi}}\, y\, e^{-\frac{y^2}{2}}\, dy}{\int_A^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy} = 1 \qquad (60)$$

Denoting P $(\cdot)$ as the cumulative distribution function for the standardized Gaussian distribution, then (60) may be written

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{A^2}{2}} = \left[1 - P(A)\right] \qquad (61)$$

This transcendental equation is easily solved by hand, since both the left hand side and the P function are tabulated functions; the numerical solution is

$$A = 0.3026$$

Thus, for a three-level quantizer, the thresholds should be ± 0.3026 (more generally $\pm 0.3026\,\sigma$), for Gaussian data.

If the number N of quantum levels is odd, say $2K+1$, there are K positive threshold values to be determined (negative ones follow by symmetry). The system of equations obtained by applying (59) comprises exactly K equations. Furthermore, these equations are easily solved, one variable at a time, when one begins with the most positive threshold. Threshold values for 3, 5, and 7 level correlators are tabulated below.

Table 2
THRESHOLD VALUES FOR N-LEVEL CORRELATORS

LEVELS	THRESHOLDS
3	$\pm 0.3026\sigma$
5	$\pm 0.5928\sigma$, $\pm 1.572\sigma$
7	$\pm 0.555\sigma$, $\pm 1.654\sigma$, $\pm 2.071\sigma$

If the number N of quantum levels is even, say 2K, there are K-1 positive threshold values to be determined, but there are K equations implied by the application of (59). Thus the unknowns are over-specified, and there is in general no solution. (One may observe that when N is odd, the middle interval (centered on the origin) satisfies its zero average requirement "for free", and the excess equation is avoided.)

Experimental Results. - The accuracy obtained with the N-level correlator, or Stieltjes N-level correlator, is the same since their mathematical forms are equivalent. The computational times, however, are not in general the same, but are dependent upon the nature of the data being processed. The computational time for a given set of data is dependent primarily upon the length of the list array.

An attempt will now be made to determine which of the two correlators would be most appropriate in a given situation. Assume that the frequency spectra of a quantized signal is reasonably similar to that of the unquantized signal. In many practical situations this can be shown to be the case. The list array commands for the Stieltjes correlator are determined by the forward first difference of the y data. The frequency characteristics of this data may be expressed as

$$D(z) = (z-1)\ Y(z) \tag{62}$$

Thus, the Stieltjes correlator modifies the frequency characteristics of the y data (which is used to generate the list in the N level correlator) by the factor (Z-1). The frequency characteristics of the magnitude of this factor is shown in Figure 6.

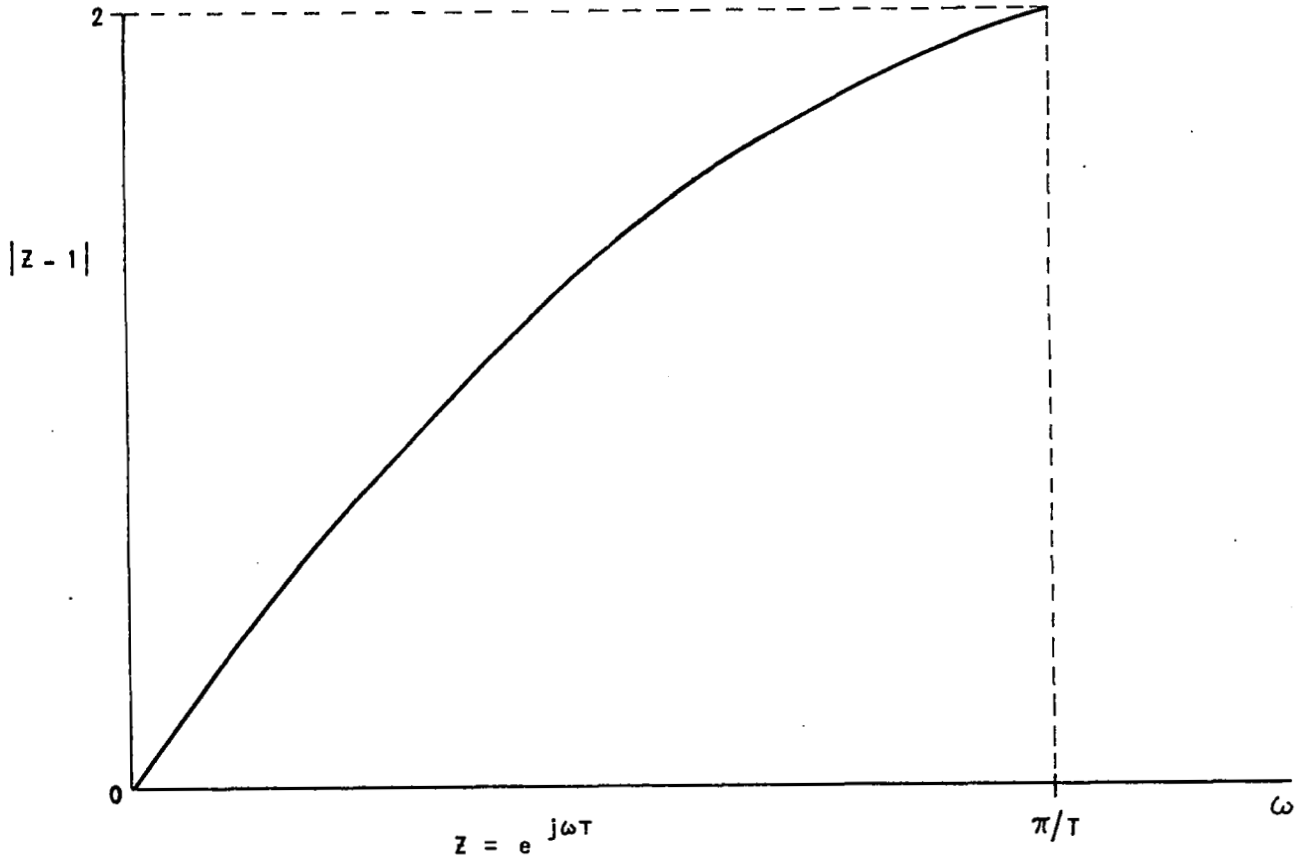


Figure 6 FREQUENCY CHARACTERISTICS OF |z-1|

From the sketch of Figure 6 it would be reasonable to assume that the Stieltjes correlator would not provide a time advantage over the N-level correlator when the energy of the original data being analyzed is about equally dispersed over the frequency interval $0, \pi/\tau$. The same would be true if a greater portion of the energy were in the higher frequency range of this interval. The high frequency content in the quantized data would tend to reduce the number of zeros of D_n , the first difference of the y data, thus creating a longer list array. On the other hand, if the significant portion of the energy lies in a lower frequency region, then the Stieltjes correlator would maintain the computational time advantage.

A segment of NASA's vibration data was used to compare the computational times of these two correlators. Three-level correlators were programmed. The full precision (Hanned) spectrum of the data used is that shown in Figure 10. The results of these experiments, along with the computational times of the full precision and Stieltjes two-level correlators for comparison, are tabulated below.

Table 3 COMPARISON OF COMPUTER TIME

CORRELATOR	COMPUTATIONAL TIME SEC.	LENGTH OF LIST ARRAY
STIELTJES 2 - LEVEL	12.4	4003
3 - LEVEL	16.5	5283
STIELTJES 3 - LEVEL	22.1	7230
FULL PRECISION	244.3	NA

TESTS CONDUCTED ON IBM 7044 MACHINE
CORRELATION FOR 700 LAGS AND 7000 DATA POINTS

Conclusions. - Two types of N-level correlators, both of which possess a computational time advantage over the full precision correlator, have been developed. A practical method for selecting the correlator threshold levels is also presented. A technique for determining when one of the two N-level correlators described will possess a computational time advantage over the other is discussed. Since the technique was based upon

assumptions and approximations, an experimental test was performed. The results of this test were in agreement with the expected results.

When the number of levels selected is about 21 or higher, the computational time advantage over the full precision correlator may not be significant. This estimate is based upon a multiply time of 17 cycles and an add or subtract time of 2 cycles, and also allows somewhat for the fact that only a portion of the data will be in the largest threshold region. The size of the list array produced from a 21-level correlator would, in general, be prohibitive without the use of transfer commands in the list array. Judging from the results of Reference 1, (where a two-level correlator was tested), however, sufficient accuracy should be obtained in general from a correlator with a low number of levels, say 3, 5, or 7.

A THEORY OF OPTIMUM SPECTRAL SMOOTHING FOR STATIONARY PROCESSES

Theoretical Background. - In Reference 1 a theory for optimal spectral smoothing was developed. The objective was the development of smoothing weights which are optimal with respect to the correlation function to which they will be applied. This approach is different from the usual selection of a standard smoothing weight, such as Hanning or Hamming, regardless of the nature of the data being analyzed.

The smoothing weights, Dj, were specified by minimizing the expected value of the following index with respect to the weights.

$$J = \int_{-\omega_c}^{\omega_c} \left| \Phi_a(\omega) - \Phi_i(\omega) \right|^2 d\omega \tag{63}$$

where in (63) the spectrum $\Phi_a(\omega)$ represents the actual spectrum achieved, and is of course directly dependent upon the weights Dj selected. The spectrum $\Phi_i(\omega)$ represents the true underlying spectra of the process being analyzed. The solution for the weights, as given by Equation (69) of Reference 1, are

$$D_j = \phi_j^2 \left[\phi_j^2 + \frac{1}{(N-j)^2} \sum_{k=-(N-j-1)}^{N-j-1} (N-j-|k|)(\phi_k^2 + \phi_{k+j}\,\phi_{k-j}) \right]^{-1}, \quad j<N \tag{64}$$

The terms ϕ_j in the above equation represent the value of the underlying correlation at lag j . Before the weights can be evaluated for a given process, a reasonable choice for the function ϕ_j must be made.

Optimum Weights. - The method for generating the weights for any particular process will now be considered in detail. Since the true underlying spectra $\Phi_i(\omega)$ is unknown, an estimate of this function, say $\hat{\Phi}_i(\omega)$, must be used. This estimate should contain the general characteristics of $\Phi_i(\omega)$. One particular characteristic of special significance is

$$\hat{\Phi}_i(\omega) \geq 0 \quad \text{for all } \omega \tag{65}$$

The importance of this particular characteristic stems from the nature of the specific index (63), upon which optimality is based. A mean square type of performance index will not insure the positiveness of the resultant spectrum $\hat{\Phi}_a(\omega)$. Thus, in order to keep the negative portion of $\hat{\Phi}_a(\omega)$ to a minimum, the condition given by (65) should be met.

Let the unsmoothed empirical correlation for a given process be designated as R_m. If the maximum lag M which is computed is less than the number of data points N, then the empirical spectra associated with this (unweighted) correlation will, in general, possess negative power. To provide a spectrum $\hat{\Phi}_i(\omega)$ which will have the same general characteristics as the underlying spectra of the process, while at the same time insuring that this spectra will have only positive power, the following weighting is used:

$$\hat{\phi}_m = \omega_m R_m \tag{66}$$

where

$$\omega_m = 1 - \frac{|m|}{M}$$

It will now be demonstrated that the spectra $\hat{\Phi}_i(\omega)$ associated with $\hat{\phi}_m$ must possess the property specified by (65).

Define the operation $*^+$ by

$$x *^+ x = \sum_{k=-\infty}^{\infty} x_k \, x_{m+k} \tag{67}$$

Then if x represents the data sequence being processed

$$R_m = \frac{1}{N} \, x *^+ x \tag{68}$$

also

$$\omega_m = \frac{1}{M} \, s *^+ s \tag{69}$$

where

$$s_k = \begin{cases} 1 & 1 \le k \le M \\ 0 & otherwise \end{cases}$$

The correlation in (66) may now be expressed as

$$\hat{\phi}_m = \frac{1}{MN} \, (s *^+ s) \cdot (x *^+ x) \tag{70}$$

Taking the $\mathcal{Z}$ transform of (70) yields

$$\hat{\Phi}_i(\omega) = \frac{1}{MN} \left| S(\omega) \right|^2 * \left| X(\omega) \right|^2 \qquad (71)$$

In (71) $S(\omega)$ and $X(\omega)$ are the $\mathcal{Z}$ transforms of s_k and x_k with Z replaced by $e^{i\omega T}$. The operator $*$ indicates convolution. Since the convolution of two positive functions must be positive (71) demonstrates that $\hat{\Phi}_i(\omega)$ is always greater than, or equal to, zero.

Experimental Results. - A segment of NASA's vibration data was used to test this theory. Correlation values up to 700 were computed from a data sequence of 7000 points. The folding frequency was 3500 cps. Results obtained for the correlation function and power spectrum, when (66) is placed into (64) and these optimum weights used instead of standard weights, are shown in Figures 8 and 11. Also shown for comparison are the unweighted correlation, the unsmoothed power spectrum and the Hanned power spectrum in Figures 7, 9, and 10, respectively.

The significant difference between the Hanned spectrum and the optimally smoothed spectrum is most dramatically demonstrated when their graphs are superimposed in front of a lighted background. The Hanning weights, while producing a very smooth spectrum, have also smoothed over spectra in the areas of 875 cps, 1400 cps, and 2100 cps which are more clearly defined in the optimally smoothed spectrum.

The computational time to compute the correlation (unweighted) for this data was 244.3 seconds. Since the nature of the optimum weights are more complex than the .25, .5, .25, Hanning weighting of the spectrum, additional computer time is required. This additional computer time for the particular data analyzed amounted to 51.6 seconds.

Conclusions. - The usefulness of an optimally derived set of smoothing weights has been demonstrated. If additional smoothing is desired, for any particular process, this can be achieved in an optimal manner by modifying the weighting function ω_m. Additional smoothing would be afforded by selecting a function ω_m which decreased more rapidly than the one selected. Care must be exercised, however, to insure that property (65) is maintained.

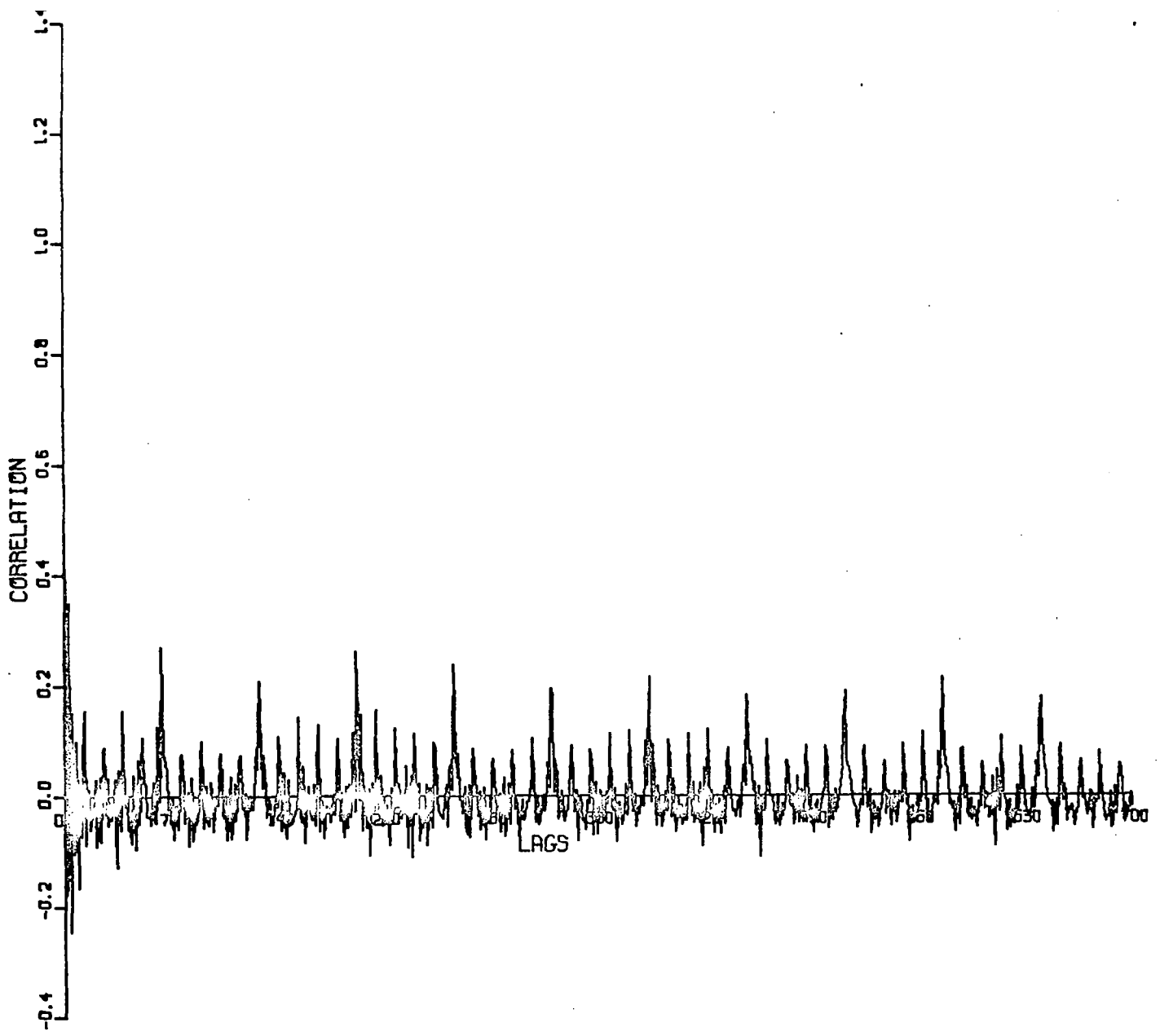


Figure 7 CORRELATION FUNCTION - UNWEIGHTED

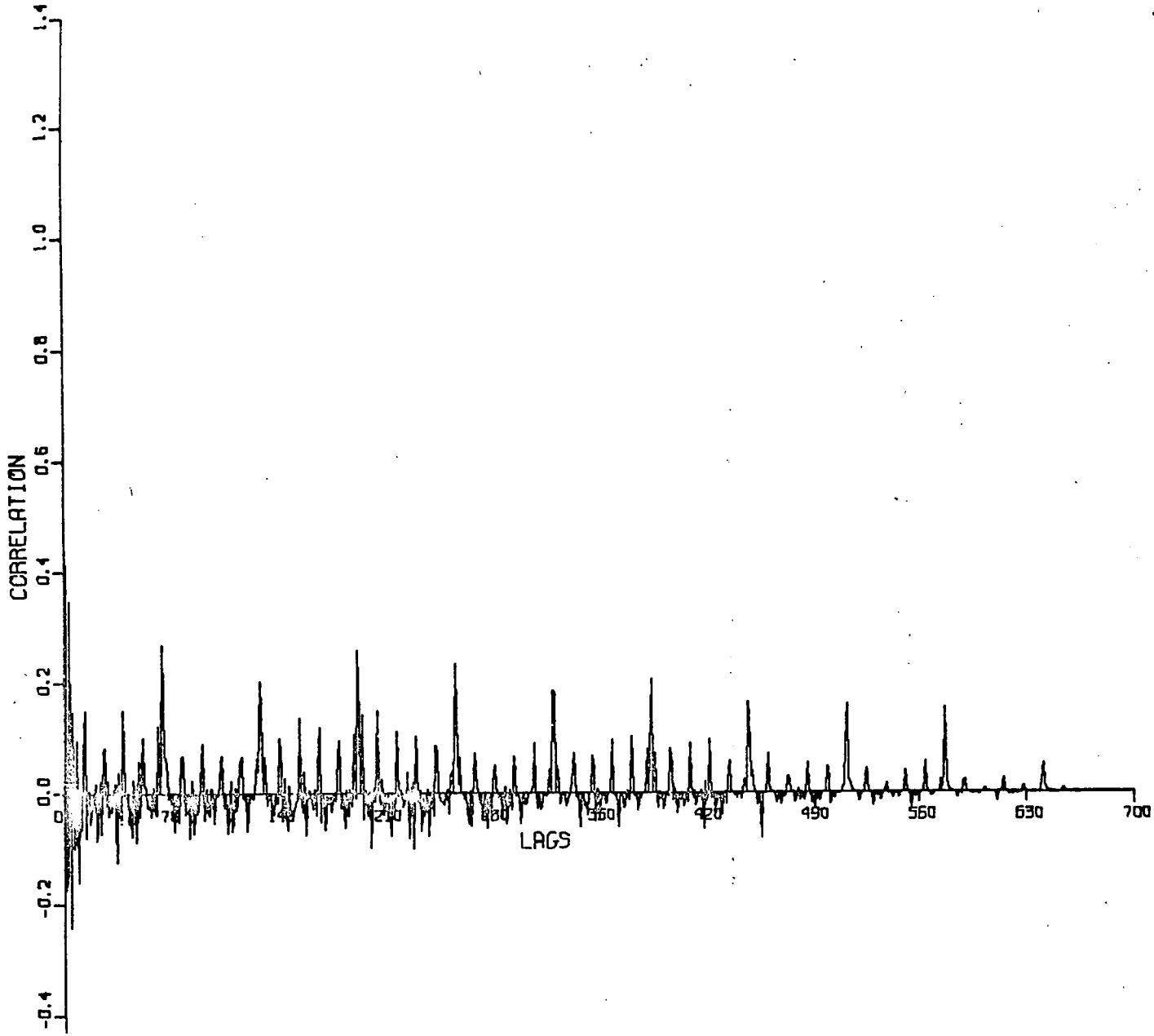


Figure 8 CORRELATION FUNCTION - WEIGHTED (Optimum Weights)

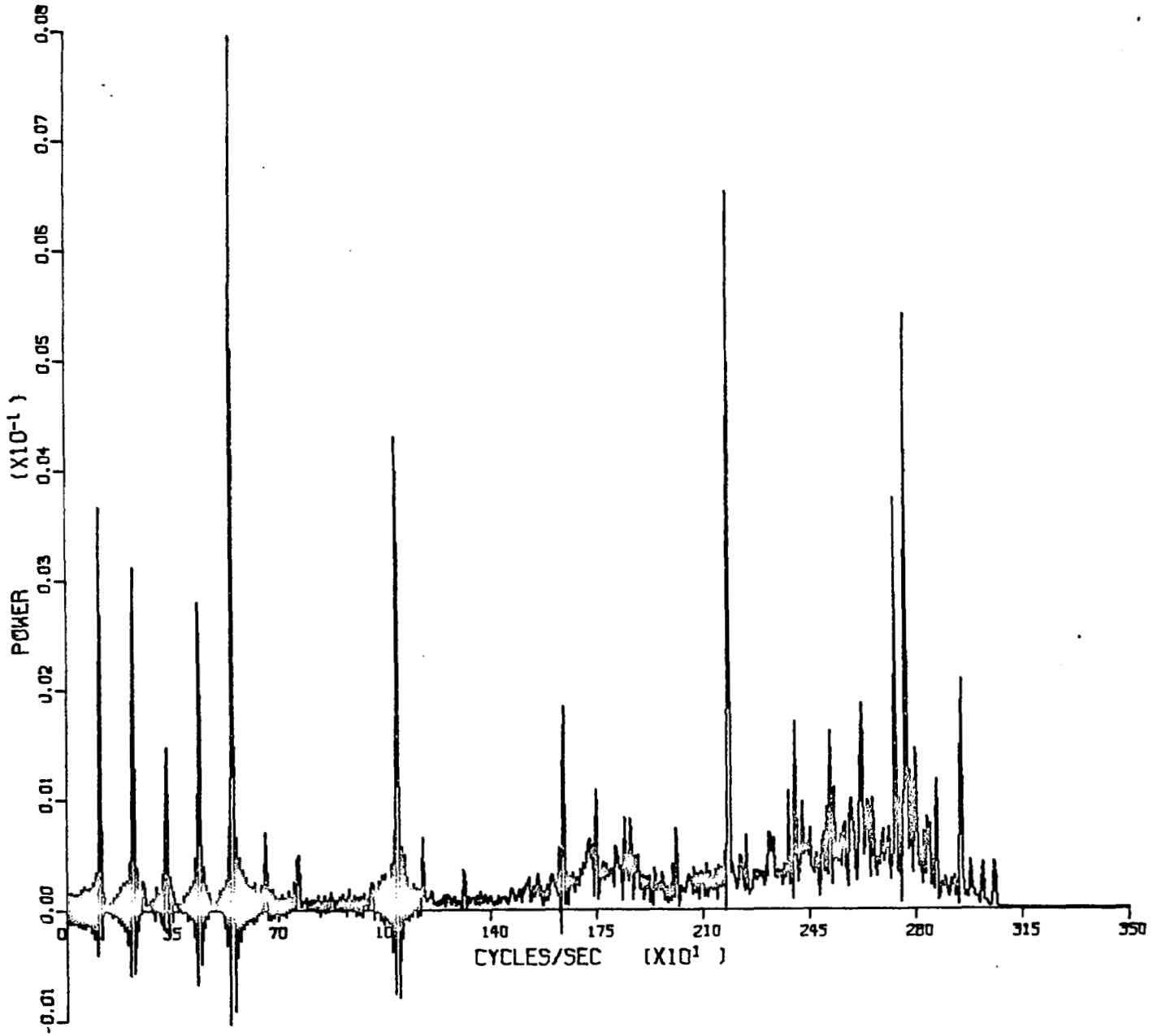


Figure 9 POWER SPECTRUM - UNSMOOTHED

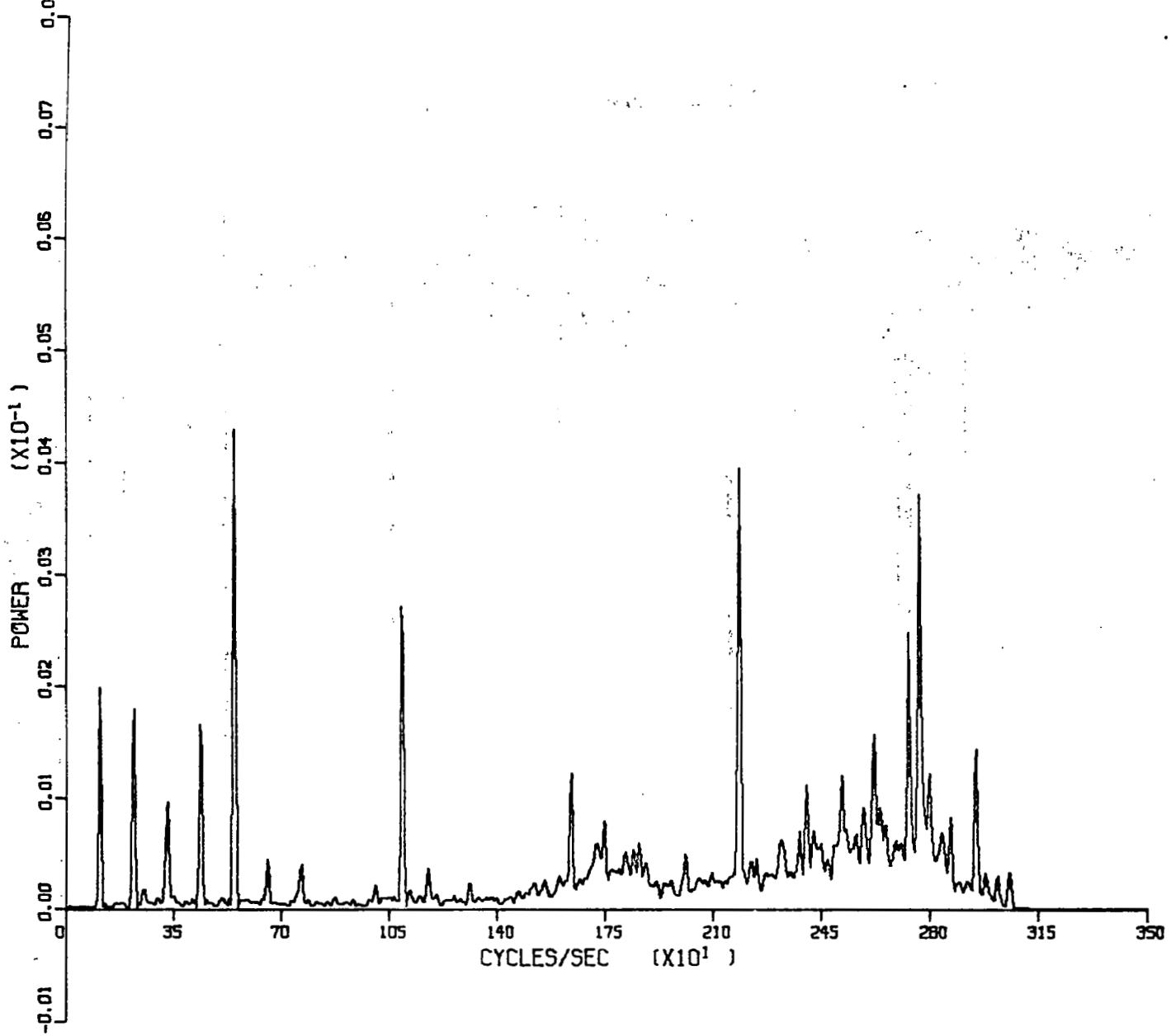


Figure 10 POWER SPECTRUM - SMOOTHED (Hanning Weights)

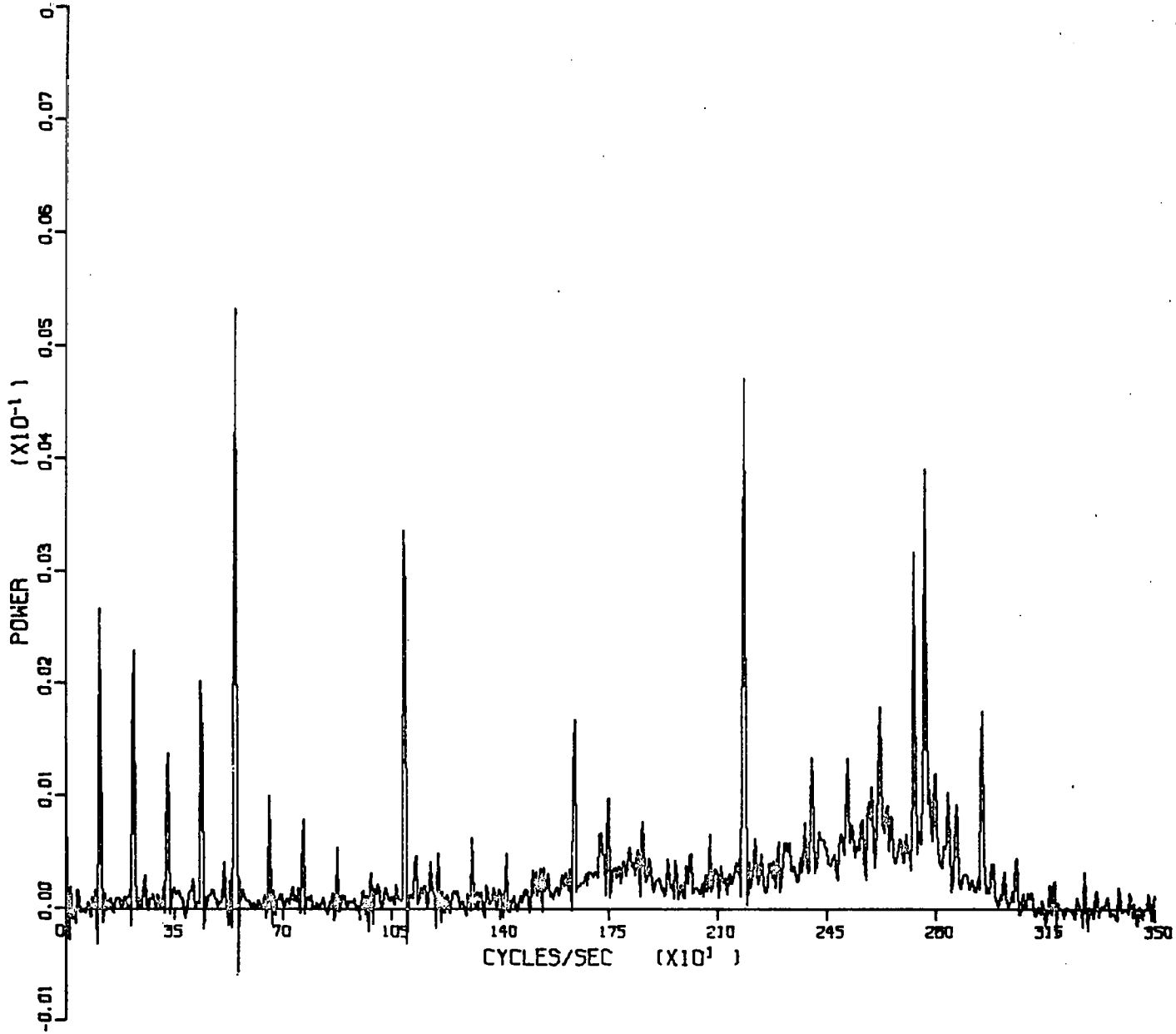


Figure 11 POWER SPECTRUM - SMOOTHED (Optimum Weights)

A THEORY OF NONSTATIONARY CORRELATION DETECTION

The discrete data nonstationary correlation function detection theory presented here has as its primary objectives the following:

(a) To increase the usefulness of the continuous data nonstationary correlation theory described in Reference 1, by developing an analogous discrete form of this theory.

(b) To improve this nonstationary correlation theory by developing performance measures not previously considered, and which will allow for fewer approximations in the development of the theory.

(c) To experimentally demonstrate the usefulness of the discrete nonstationary correlation detection theory which is developed.

As an alternative to developing the discrete data theory, the optimum filters for the continuous case could be transformed into discrete form. The discrete data filters produced in this manner, however, will in general no longer be optimum with respect to the analogous discrete performance index. The fact that there is no one universal technique for transforming an analog filter into discrete form is a verification of the above statement. One may argue that the filters generated in this fashion must minimize some discrete performance index. Though this may indeed be true, the fact that the form of the index and the relationships between index weights and filter parameter values remains unknown, would dictate the rejection of this approach. In addition, this approach would not, in general, give insight into the means for accomplishing the second objective listed above.

Figure 12 is a block diagram of a discrete time correlator which possesses the same topology as the continuous correlation function detector reported in Reference 1. This correlator configuration exhibits a separability property along the n and m axes that is analogous to the separability of the continuous correlator along the t and τ axes. As a matter of convenience, the n and m axes will still be referred to as the t and τ axes, respectively.

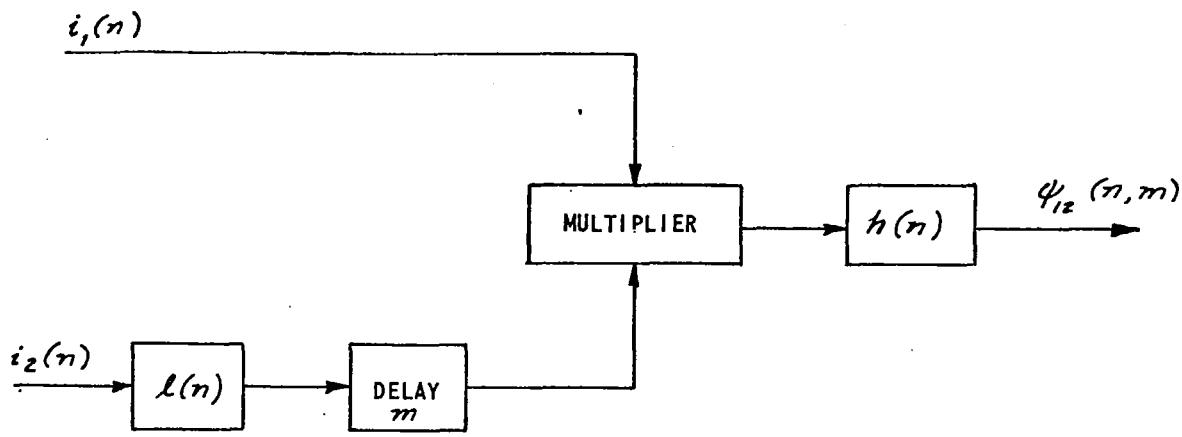


Figure 12 DISCRETE TIME CORRELATOR

The output of this correlator, when expressed in terms of the input signals, is given by

$$\psi_{12}(n,m) = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h(\beta)\ell(\alpha) i_1(n-\beta)\, i_2(n-\beta-\alpha-m) \tag{72}$$

where, in Equation (72), α and β are dummy integer variables. If it is assumed that the product of the input signal pair is composed of the true ensemble correlation function and added noise as shown in Equation (73),

$$i_1(n)\, i_2(n-m) = \phi_{12}(n,m) + \eta(n,m) \tag{73}$$

then the output of the correlator becomes

$$\psi_{12}(n,m) = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h(\beta)\ell(\alpha)\phi_{12}(n-\beta, \alpha+m) + \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h(\beta)\ell(\alpha)\eta(n-\beta, \alpha+m) \tag{74}$$

The objective is to force, by means of judicious selection of the filtering operations $\ell(n)$ and $h(n)$, the first term of the output of the correlator in expression (74) to follow as closely as possible the true correlation function, $\phi_{12}(n, m)$. In addition it is desired to reduce the second term in Equation (74) which represents the effect of the noise that is unrelated to the true ensemble correlation.

Performance Measure. - As an aid to selecting the filtering operations in an optimum manner, a performance index is developed which assesses the various errors involved in detecting the underlying ensemble correlation function. Minimization of this index will then yield the specification of the optimum filtering operations $\ell(n)$ and $h(n)$.

Consider a representative correlation test function to be given by

$$\phi_{12}(n, m) = \begin{cases} nT ; & n \geq 0 \\ 0 ; & n < 0 \end{cases} \quad ; \quad m \geq 0$$

An error measure of the distortion along the t axis may then be expressed as

$$e_t^2(n) = \left[\phi_{12}(n, m) - \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h(\beta)\ell(\alpha)\phi_{12}(n-\beta, \alpha+m)\right]^2$$

$$= \left[nT - \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h(\beta)\ell(\alpha)(n-\beta)T \right]^2 \qquad (75)$$

$$= \left[nT - A \sum_{\beta=0}^{n} h(\beta)(n-\beta)T \right]^2$$

where

$$A = \sum_{\alpha=0}^{\infty} \ell(\alpha)$$

Consider now the distortion along τ axis and select a representative correlation test function to be given by

$$\phi_{12}(n, m) = \begin{cases} MT - mT\,; & 0 \le m \le M \\ 0 & \text{otherwise} \end{cases}$$

then an error measure of the distortion along the τ axis may be expressed as

$$e_\tau^2(M-m) = \left[MT - mT - \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{M-m} h(\beta)\, l(\alpha)(MT - (\alpha+m)T) \right]^2$$

In order to simplify the above expression, the following variable substitution is made

$$M - m = n$$

$$e_\tau^2(n) = \left[nT - B \sum_{\alpha=0}^{n} l(\alpha)(n-\alpha)T \right]^2, \qquad (76)$$

$$0 \le n \le M$$

where

$$B = \sum_{\beta=0}^{\infty} h(\beta)$$

To obtain a measure of the noise contribution in the output of the correlator, select the test input signal pair to be members of uncorrelated stationary Gaussian processes with zero means. The following measure will be considered as representative of the noise distortion with the above signal pair applied.

$$E\left[(\psi_{12} - \phi_{12})^2 \right]$$

As shown in Appendix E, the above expression reduces to the following

$$\sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h^2(\beta)\, l^2(\alpha)$$

The complete performance index may now be formulated by combining the error measures developed above which are representative of the t axis distortion, τ axis distortion, and noise distortion,

$$\theta = \lambda_t \sum_{n=0}^{\infty} e_t^2(n) + \lambda_\tau \sum_{n=0}^{M} e_\tau^2(n) + \sum_{n=0}^{\infty} h^2(n) \sum_{m=0}^{\infty} \ell^2(m)$$

The λ coefficients in the above expression are positive weights which dictate the relative importance of the individual error indices. In order to make the solution of the problem manageable, the upper limit in the second term is allowed to be infinite. With this change the performance index becomes

$$\hat{\theta} = \lambda_t \sum_{n=0}^{\infty} e_t^2(n) + \lambda_\tau \sum_{n=0}^{\infty} e_\tau^2(n) + \sum_{n=0}^{\infty} h^2(n) \sum_{m=0}^{\infty} \ell^2(m) \quad (77)$$

From the form of θ and $\hat{\theta}$, it is apparent that $\theta \leq \hat{\theta}$ for all $\ell(n)$ and $h(n)$. This fact by itself does not allow the change of limits; however, it does add to the reasonableness of such an approach.

The minimization of $\hat{\theta}$ in Equation (77) will specify the optimum filters $\ell(n)$ and $h(n)$ to be used in the correlation function detector of Figure 12. Before extremizing this performance index, however, it will be helpful to examine some of its properties. When Equations (75), (76), and (77) are combined it becomes clear that the extremum of (77) is independent of the final values of the step response of the $\ell(n)$ and $h(n)$ filters providing that the product of these final values is unity. For simplicity and a desire to distribute the gains equally, these gains are selected to be unity. This selection dictates the following conditions:

$$A = B = 1$$

Since the specification of the optimum filters $\ell(n)$ and $h(n)$ is based upon the minimization of $\hat{\theta}$ rather than θ, a technique for improving these filters has been considered. If the form of the filters which are specified by the minimization of $\hat{\theta}$ are placed into θ, and this index minimized

with respect to the parameter values, a lower value of the index θ could be obtained. If this approach is taken, however, the filter parameters will be a function of M, as one would intuitively feel they should be.

Optimum Detector Filters. - Consider now the problem of minimizing the index presented by (77). Due to the symmetry of this index with respect to $\ell(n)$ and $h(n)$ it can be concluded that the form of the filters will be identical. If it is assumed that $h(n) = h_o(n)$ the optimal filter, then variations are to be considered only over the $\ell(n)$ filtering operation. The index, for the purpose of extremizing with respect to variations over $\ell(n)$ may then be considered to take on the following form:

$$\hat{\theta}_\ell = \sum_{n=0}^{\infty} \left\{ \lambda_\tau c_\tau^2(n) + \left(\sum_{n=0}^{\infty} h_o^2(n) \right) \ell^2(n) \right\} \tag{78}$$

Minimization of the index given in (78) as shown in Appendix F yields the following form for the optimum filters

$$L_o(z) = \frac{A_{\ell 2} z^2 + A_{\ell 1} z}{z^2 + b_{\ell 1} z + b_{\ell o}} \tag{79}$$

where

$$\begin{aligned} A_{\ell 1} &= b_{\ell 1} + 2 b_o \\ A_{\ell 2} &= 1 - b_{\ell o} \\ b_{\ell o} &= \frac{-b_{\ell 1}}{4 + b_{\ell 1}} \\ b_{\ell 1} &= \sqrt{2\sqrt{c_\ell^4 + 4c_\ell^2} - 2c_\ell^2} - 2 \\ c_\ell^2 &= \frac{T^2}{4} \frac{\lambda_\tau}{H} \\ H &= \sum_{n=0}^{\infty} h_o^2(n) \end{aligned} \tag{80}$$

From the conditions of (80) it is seen that the solution is considerably more complex than its continuous analog. Furthermore, the filter parameters cannot be decoupled analytically as they have been in the continuous case.

From the conditions given in (80) the following functional relationships may be established:

$$b_{\ell 1} = f\left(\frac{\lambda_\tau}{H}\right), \quad b_{h1} = f\left(\frac{\lambda_t}{L}\right) \tag{81}$$

An expression for H and L in terms of its filter parameters can be found in Reference 10. When the conditions in (80) are applied to this expression, the resultant general form can be achieved

$$G = \frac{b+2}{4}\left[\frac{3\sqrt{2}-4}{b+2(1+\sqrt{2})} - \frac{4+3\sqrt{2}}{b+2(1-\sqrt{2})}\right] \tag{82}$$

where in (82) the G becomes L or H depending upon whether $b_{\ell 1}$ or b_{h1} is used in its right hand member. Thus, the problem of completely specifying the filters $\ell(n)$ and $h(n)$ in terms of the weights λ_t and λ_τ selected reduces to one of decoupling Equations (81) and (82) by computer techniques. Although this can readily be achieved the filter specifications will be made in accordance with their frequency characteristics as will be described later.

Other Performance Measures. - In addition to the discrete forms of the indices presented in Appendix I of Reference 1, two other promising performance indices are considered. The first of these is formulated by considering the test function along the τ axis to be given by an exponential such as

$$\phi(n, m) = e^{-amT} \qquad \begin{matrix} a \geq 0 \\ m \geq 0 \end{matrix} \tag{83}$$

With this choice of test function, no difficulty would occur with the upper limit since all sums would have an infinite upper limit. Another approach may be the formulation of an index which incorporates a test function of the following form

$$\phi(n, m) = \phi_t(nT)\, \phi_\tau(mT), \quad m \geq 0 \tag{84}$$

With the use of (84), both the t and τ axes distortions are considered simultaneously. In general, the selection of the form of the index and the weights within the index should be predicated upon the best knowledge of the correlation function which is to be detected.

The form of the optimum filters, when an exponential rather than a triangular test function is used in the tau axis error measure, is now considered. One of the primary advantages of this new approach is that no approximation is required for the explicit solution of the filtering operations.

Using an exponential test function, the τ axis error measure is given by

$$e_\tau^2(m) = \left[\phi_{12}(n, m) - \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} h(\beta)\, \ell(\alpha)\, \phi_{12}(n-\beta, m+\alpha)\right]^2 \tag{85}$$

where

$$\phi_{12}(n, m) = \begin{cases} e^{-amT}, & m \geq 0 \\ 0, & m < 0 \end{cases} \tag{86}$$

Placing (86) into (85) the error measure reduces to

$$e_\tau^2(m) = e^{-2amT} \left[1 - B \sum_{\alpha=0}^{\infty} \ell(\alpha)\, e^{-a\alpha T}\right]^2 \tag{87}$$

Selecting the same t axis and noise error measures as described earlier, the total performance index is expressed as

$$\theta = \sum_{n=0}^{\infty} \left\{\lambda_t \left[nT - A \sum_{\beta=0}^{n} h(\beta)(n-\beta)T\right]^2 + \lambda_\tau e^{-2amT} \left[1 - B \sum_{\alpha=0}^{\infty} \ell(\alpha)\, e^{-a\alpha T}\right]^2 + \left(\sum_{m=0}^{\infty} h^2(m)\right) \ell^2(n)\right\} \tag{88}$$

where

$$A = \sum_{n=0}^{\infty} \ell(n)$$
$$B = \sum_{n=0}^{\infty} h(n) \tag{89}$$

The conditions on A and B are similar to those previously described; hence, the following selection is made

$$A = B = 1$$

Considering variations from the optimal filtering operations independently the performance index to be minimized may be written as

$$\theta_\ell = \sum_{n=0}^{\infty} \left\{ \lambda_\tau c_\tau^2(n) + H \ell^2(n) \right\} \tag{90}$$

Performing the indicated summation over the first term of (90) the index can be reduced to

$$\theta_\ell = C \left(1 - \sum_{\alpha=0}^{\infty} \ell(\alpha) e^{-a\alpha T} \right)^2 + H \sum_{n=0}^{\infty} \ell^2(n) \tag{91}$$

where

$$C = \frac{\lambda_\tau}{1 - e^{-2aT}} \tag{92}$$

Since it is also required that the step response be unity, this additional constraint must be included into the performance index. This condition may be expressed as

$$\sum_{n=0}^{\infty} \ell(n) - 1 = 0 \tag{93}$$

To consider variations in $\ell(n)$, but at the same time maintain the relationship given by (93), the following two-parameter family is introduced

$$\ell(n) = \ell_0(n) + \epsilon_1 \eta_1(n) + \epsilon_2 \eta_2(n) \tag{94}$$

and the boundary conditions are given by

$$\begin{aligned} \ell_o(0) &\qquad \text{Arbitrary} \\ \ell_o(\infty) &= \quad 0 \end{aligned} \tag{95}$$

After substituting (94) into (91) and adding the constraint (93), the index is minimized. As is shown in Appendix G, the minimization specifies the optimum filter to be of the form

$$L_o(z) = \left(1 - e^{-aT}\right) \frac{z}{z - e^{-aT}} \tag{96}$$

Since the t axis error measure was left unchanged, the form of the $h(n)$ filtering operation is unchanged and is given by (79), (80).

Equations (79) and (80) along with (96) completely specify the correlation detector. The selection of the constants a and b_1 can be made directly or indirectly through the specification of λ_t , λ_τ since a one-to-one mapping exists between these two pairs of constants. As an aid in the selection of a and b_1 , the filter frequency characteristics as a function of these parameters are investigated.

Filter Frequency Characteristics. - Consider first the τ axis filter which is given by

$$L(z) = \frac{a_1 z}{z + b_o} \tag{97}$$

where

$$\begin{aligned} a_1 &= 1 - e^{-aT} \\ b_o &= -e^{-aT} \end{aligned} \tag{98}$$

To obtain the frequency characteristics of this discrete filter, the following substitution is made

$$z = e^{j\omega T} \tag{99}$$

When (99) is placed into (97), the following expression for the frequency characteristics is obtained

$$L(\omega) = \frac{1 - e^{-aT}}{\sqrt{(1-e^{-aT})^2 + 2e^{-aT}(1-\cos \omega T)}} \Big/ -\tan^{-1} \frac{e^{-aT} \sin \omega T}{1 - e^{-aT} \cos \omega T} \qquad (100)$$

Recalling that the domain of interest is $\omega \in (0, \pi/T)$, the magnitude of the above frequency characteristics is seen to be monotonically decreasing in this domain. Consider the bandwidth to extend from zero to the point at which the magnitude of the gain is down by $\frac{1}{\sqrt{2}}$. An expression for e^{-aT} as a function of the bandwidth ω_0 is now obtained.

At the half-power point, the square of the magnitude of the gain is given by

$$\frac{(1-e^{-aT})^2}{(1-e^{-aT})^2 + 2e^{-aT}(1-\cos \omega_0 T)} = \frac{1}{2} \qquad (101)$$

Upon solving (101) explicitly for e^{-aT} , the final desired expression is achieved.

$$e^{-aT} = 2 - \cos \omega_0 T - \sqrt{(2 - \cos \omega_0 T)^2 - 1} \qquad (102)$$

To obtain an analytical expression for the t axis filter characteristics $H(\omega)$ in terms of b_1 , a simplifying bilinear transformation is applied to the filter $H(z)$. This approximation yields an accurate description of the frequency characteristics in the lower frequency portion of the domain of interest ($0, \pi/T$). In general this will provide a sufficient description for the t axis filter $h(n)$. The bilinear transformation to be applied is given by

$$z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}} \quad , \quad s = j\omega \qquad (103)$$

When (103) is placed into the expression for $H(z)$ which is given by (79) and the relationships between the coefficients in $H(z)$ are utilized, the following

expressions are achieved.

$$H(s) = \frac{\left(\frac{s}{s_1} + 1\right)\left(\frac{s}{s_2} + 1\right)}{\left(\frac{s}{\omega_D}\right)^2 + \frac{2\zeta_D}{\omega_D} s + 1} \tag{104}$$

where

$$s_1 = \frac{2}{T} \frac{2 + b_1}{2 - b_1}, \quad s_2 = \frac{2}{T}$$

$$\zeta_D = \frac{\sqrt{2}}{2} \frac{1}{\sqrt{1 + \left(\frac{T\omega_D}{2}\right)^2}} \doteq \frac{\sqrt{2}}{2} \tag{105}$$

and

$$b_1 = 2\left(-1 + \frac{\sqrt{2}\,\omega_D}{\sqrt{\frac{4}{T^2} + \omega_D}}\right) \doteq -2 + \sqrt{2}\,T\omega_D$$

With the use of standard Bode charts, the bandwidth of (104) is determined to be $2\omega_D$. Expression (102) along with the last equation of (105) characterize the detector filtering operations.

Correlation Detector Program. - The primary objective, for the development of a digital nonstationary correlation detector program, was to test the theory by obtaining experimental results. Secondary goals were to ascertain the nature of engineering problems which may arise in the application of the theoretical studies. With these objectives in mind the discrete data program was written in FORTRAN IV language. It is estimated that the processor computation time can be further reduced, with the use of machine language programming, by a factor of 2/3 to 1/2.

Table 4 summarizes the program computation times. Figures 13, 14, and 15 are provided to give an overall perspective of the nature of the program. The storage requirements dictated the need for two programs which have been designated the detector and plotter and are illustrated in Figures 13 and 14, respectively.

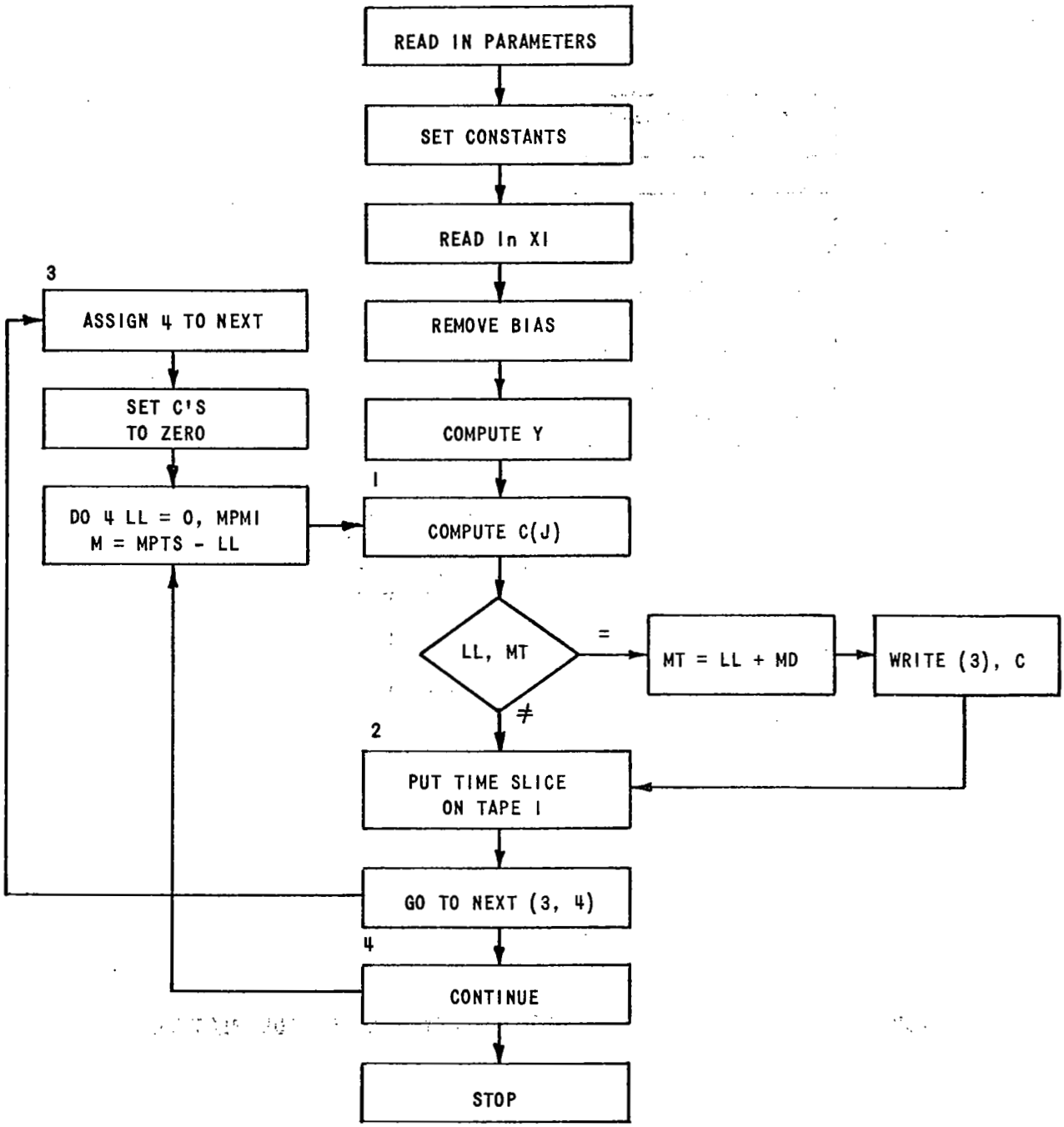


Figure 13 FLOW DIAGRAM - NONSTATIONARY CORRELATION DETECTOR

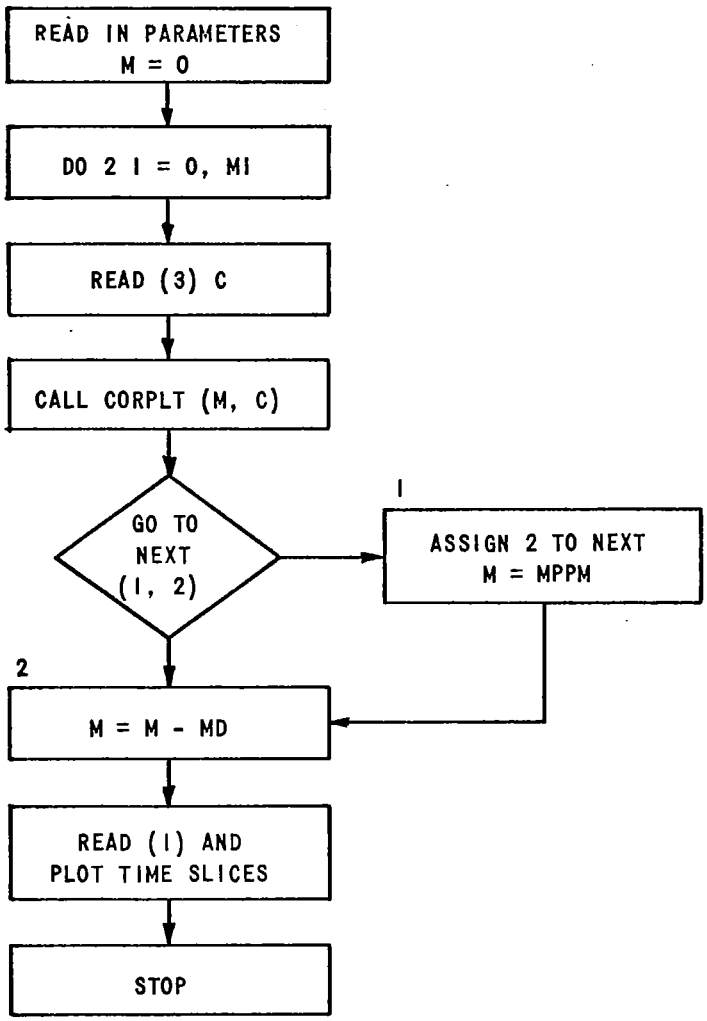


Figure 14 FLOW DIAGRAM - NONSTATIONARY CORRELATION PLOTTER

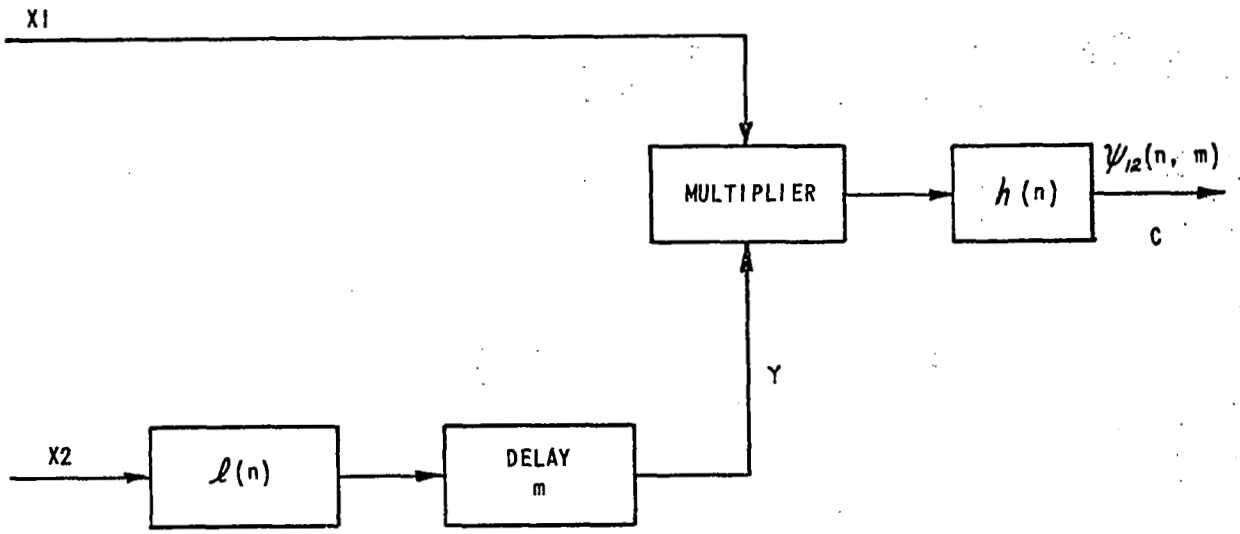


Figure 15 DISCRETE CORRELATION DETECTOR

Table 4

NONSTATIONARY CORRELATOR TIMING

FUNCTION	TIME-MINUTES	DATA POINTS	LAGS
DETECTOR	26	8000	800
PLOTTER	4	8000	800

Experimental Results. - The nonstationary autocorrelation functions which are presented here were obtained by processing two records of NASA's data from test SA-10-3-11. Each record of one second duration contained 8000 data points. As indicated by the flow charts of the previous section, the bias was removed from each record. This was done since it is unlikely that the bias is a true characteristic of the process under investigation, and was most probably produced by the particular instrumentation used.

The first of these records, which started at minus 1.81 seconds, was selected during the build up of the envelope of the random vibration signal. The second record, starting at 1.00 second, represents a portion of the signal where the envelope has come to a steady state condition.

The form of the filters used in the test are those given by (79) and (96) for the t and τ axis filtering operations, respectively.

The filter specifications must be chosen in relation to the noise power, since in essence, λ_t , λ_τ and 1 are the original weights in the performance index. Since the frequency content of the signal under investigation was known to extend to approximately 3000 cps, specification of the τ axis filtering operation $\ell(n)$ was based upon this value. Specification of $h(n)$ should be based upon the best knowledge of the nonstationarity. Two approaches are presented which provide guidelines for the selection of the bandwidth of $h(n)$. The first considers the approximate rise time of the envelope. With this approach a bandwidth of 6.7 rad/sec is obtained. The second approach uses a visual estimate of the predominant modulating frequencies and yields a bandwidth of 62.8 rad/sec. The bandwidth for $h(n)$ should be selected to be within this range and a choice of 12 rad/sec was made.

In general, greater knowledge concerning the particular process under investigation would allow a better choice to be made in the specification of the bandwidths. It should be recalled, however, that the detector filters are optimal, and any reasonable choice of the filter bandwidths should produce useful results.

Figures 16 through 34 illustrate the detected nonstationary autocorrelation functions for both records. The large relative value of correlation for zero lag portrays the large noise content which has been a characteristic of much of NASA's vibration data.

Nonstationary Power Spectrum. - In the analysis of stationary data the correlation function is generally developed as an aid in obtaining the power spectrum. The same condition can be stated regarding nonstationary data processing; however, the resulting nonstationary power spectrum must be properly defined. It has been shown (Reference 11) that definitions based upon intuitive insight may lead to a power spectrum with appreciable negative energy. In addition to non-negativity of the power estimates, it would be desirable to develop a time-varying spectrum which has useful physical significance.

Development of a nonstationary spectrum which is based upon the above nonstationary correlation theory will now be considered. In the derivation for the nonstationary correlation function $\psi(n,m)$ it is recalled that $\psi(n,m)$ was undefined for $m < 0$. Before an attempt can be made at obtaining the related power spectral density function the specification of $\psi(n,m)$ for $m < 0$ must be made. For the case of autocorrelation the following relationship holds $\phi(n,m) = \phi(n,-m)$, where $\phi(n,m)$ is the "true" underlying nonstationary correlation function for the random process. Thus for autocorrelation analysis a logical selection is given by

$$\psi(n,m) = \psi(n,-m) \tag{106}$$

As an alternative to the double frequency variable approach as presented in Reference 9, a technique for developing a time-varying spectrum is presented.

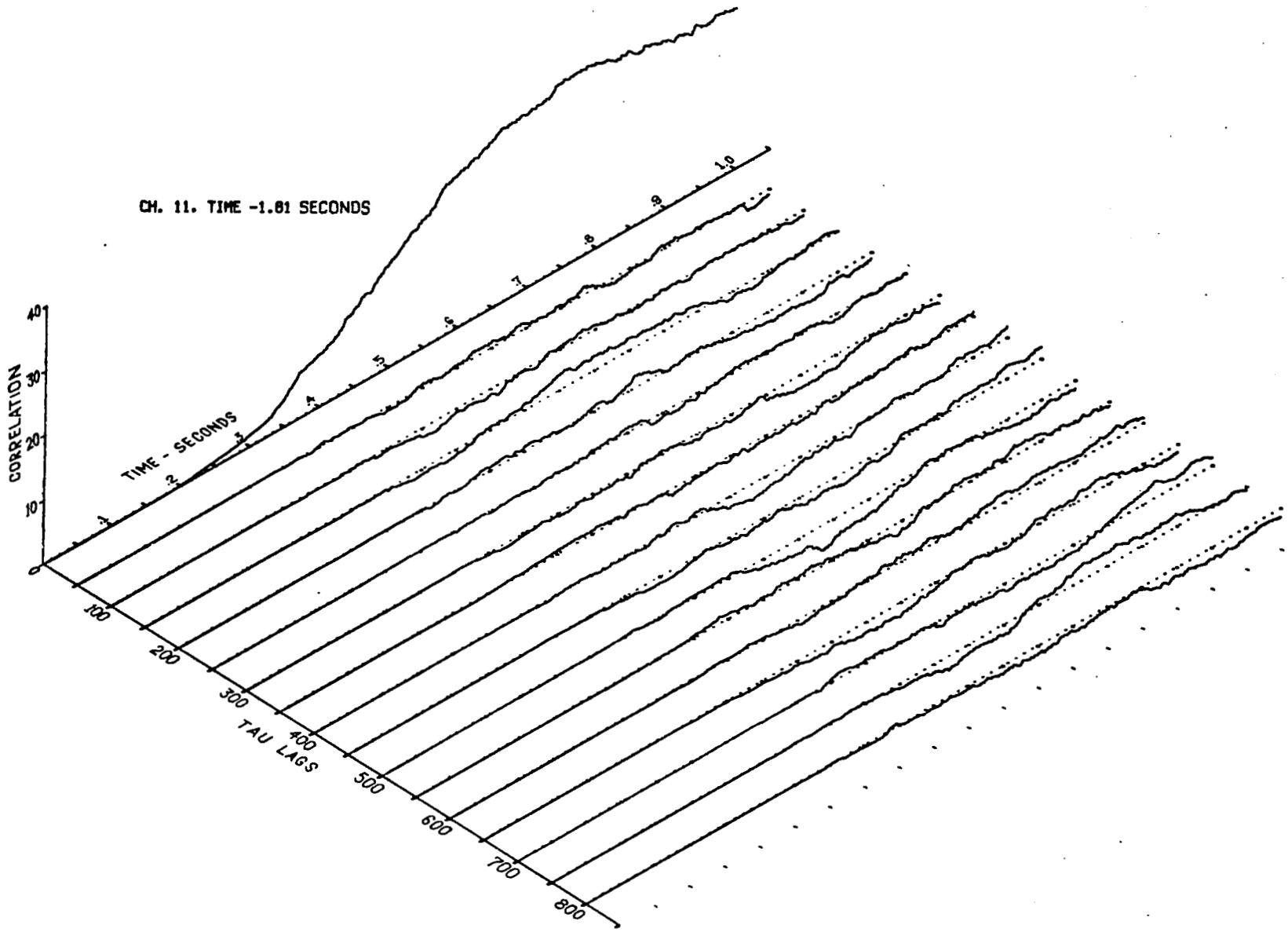


Figure 16 NONSTATIONARY CORRELATION DETECTOR OUTPUT RECORD SAIO-3-11 #1

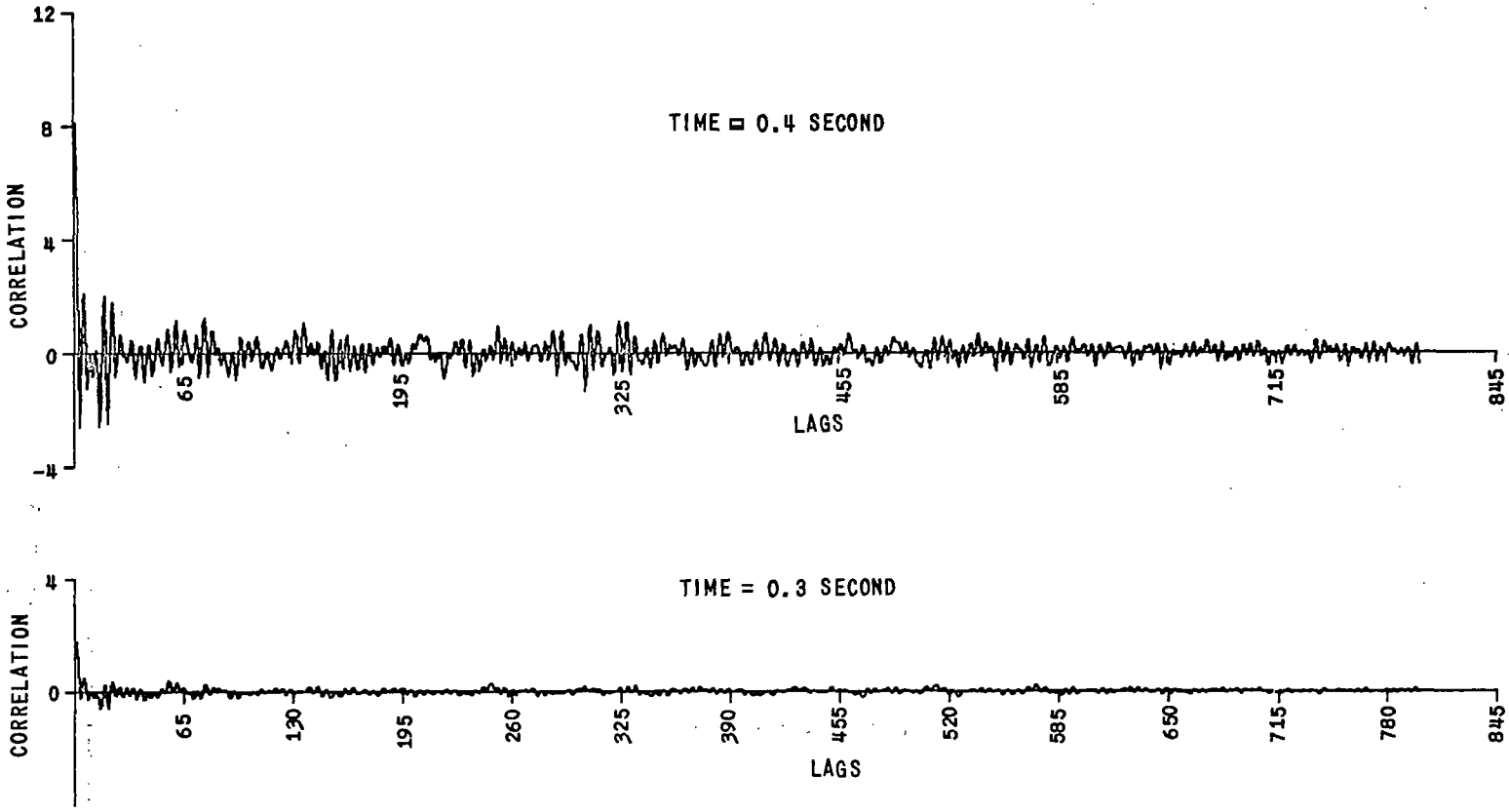


Figure 17 CORRELATION TIME SLICES FOR RECORD SA10-3-11 #1

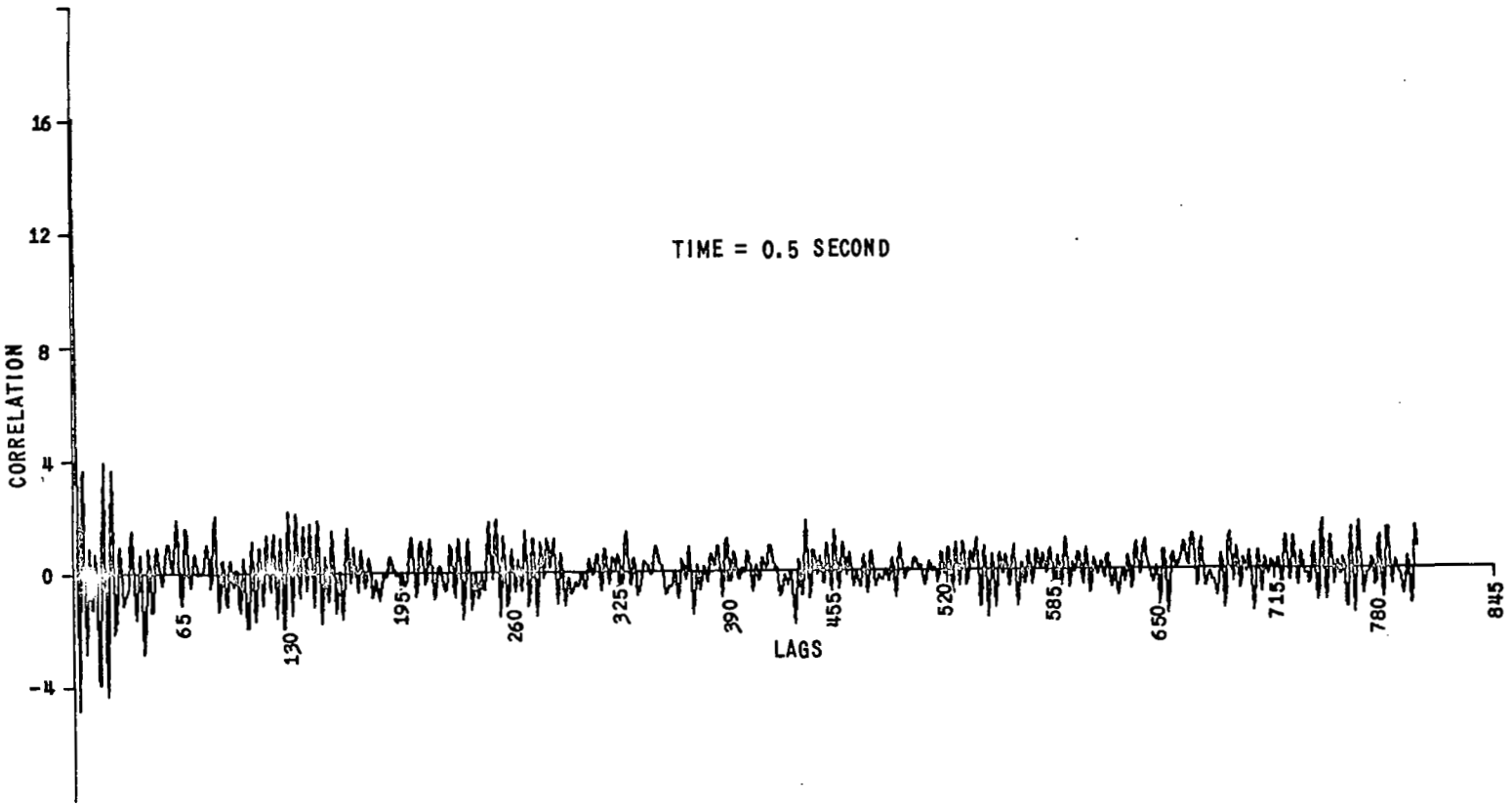


Figure 18 CORRELATION TIME SLICES FOR RECORD SA10-3-11 #1

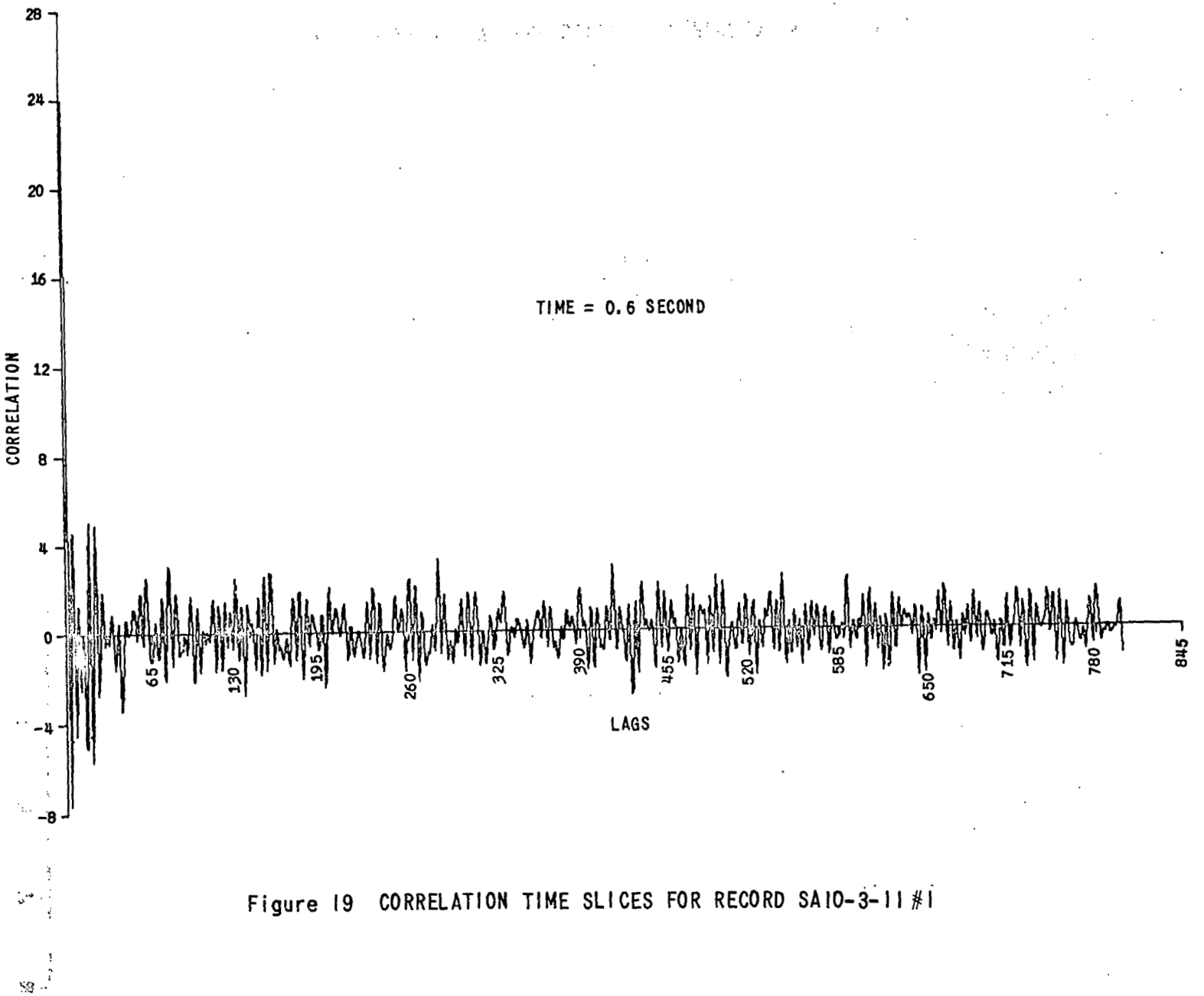


Figure 19 CORRELATION TIME SLICES FOR RECORD SA10-3-11 #1

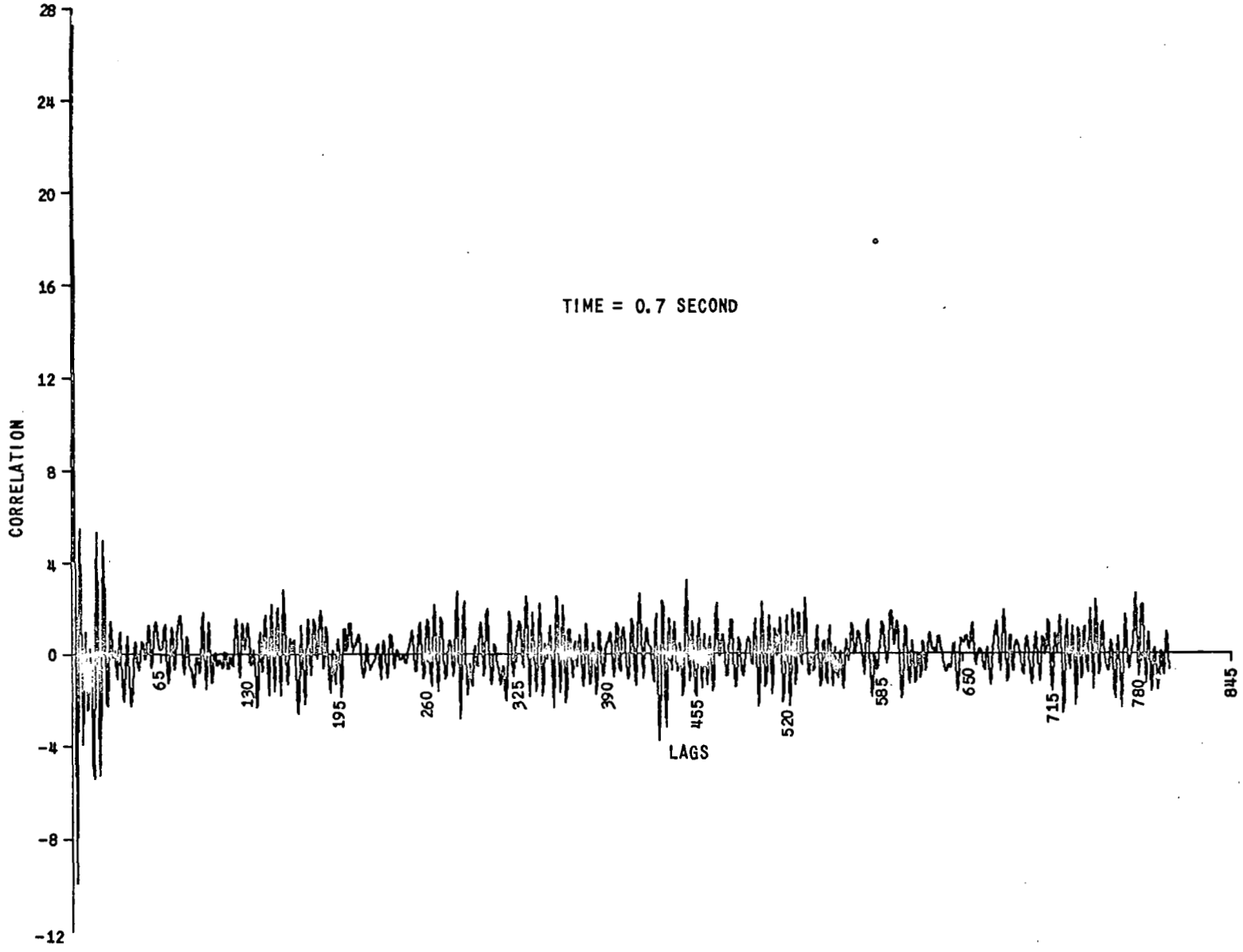


Figure 20 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #1

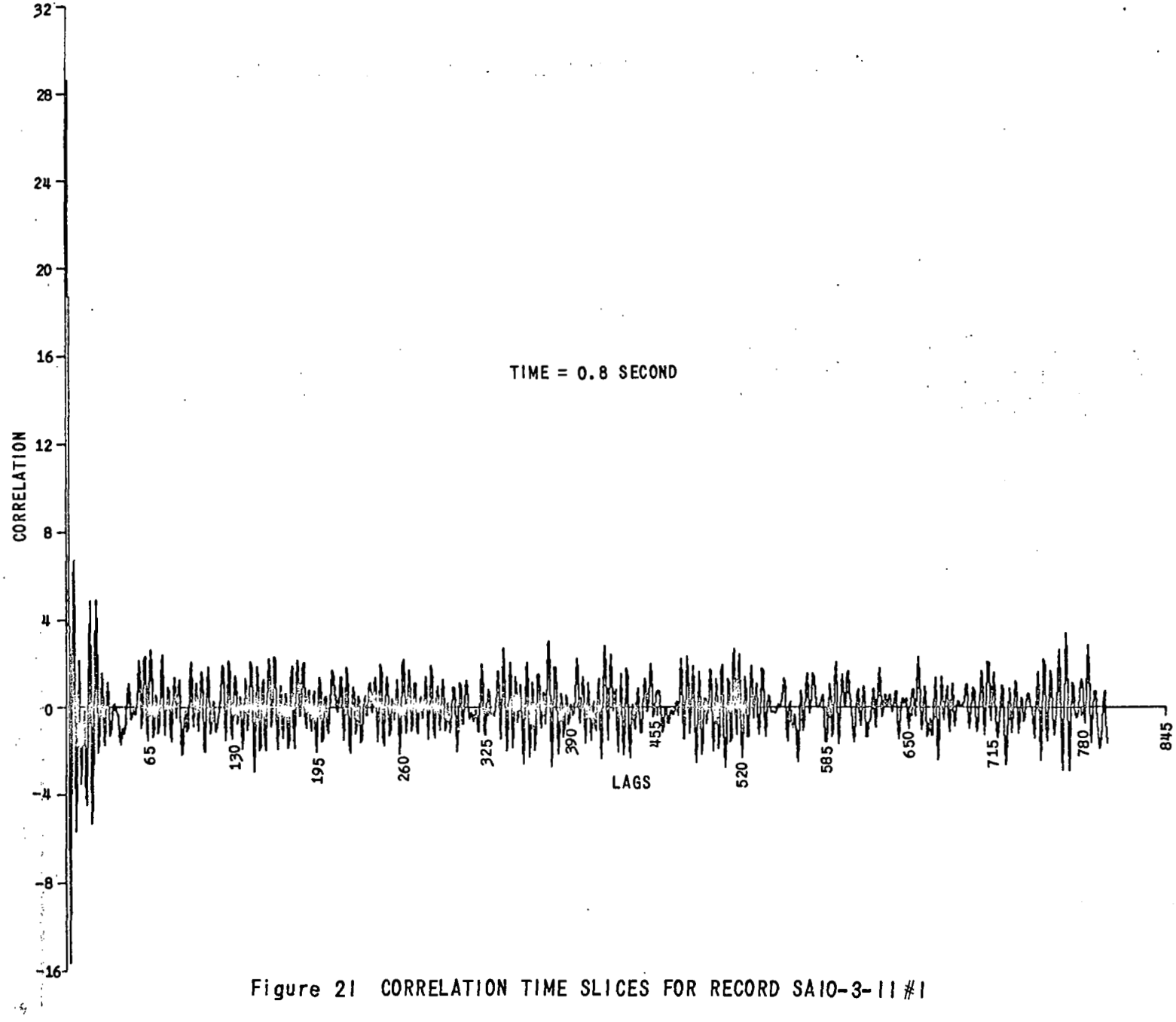


Figure 21 CORRELATION TIME SLICES FOR RECORD SA10-3-11#1

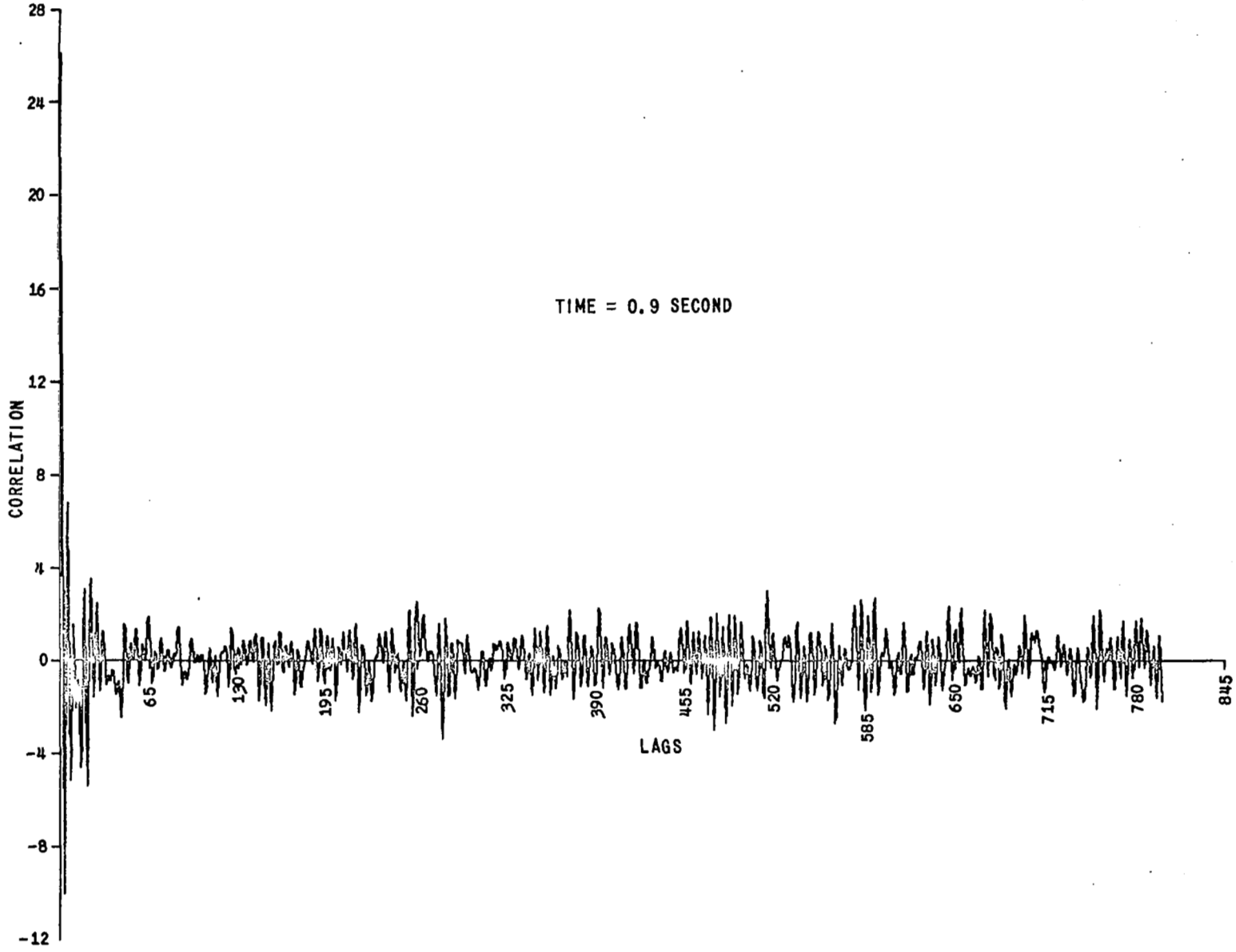


Figure 22 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #1

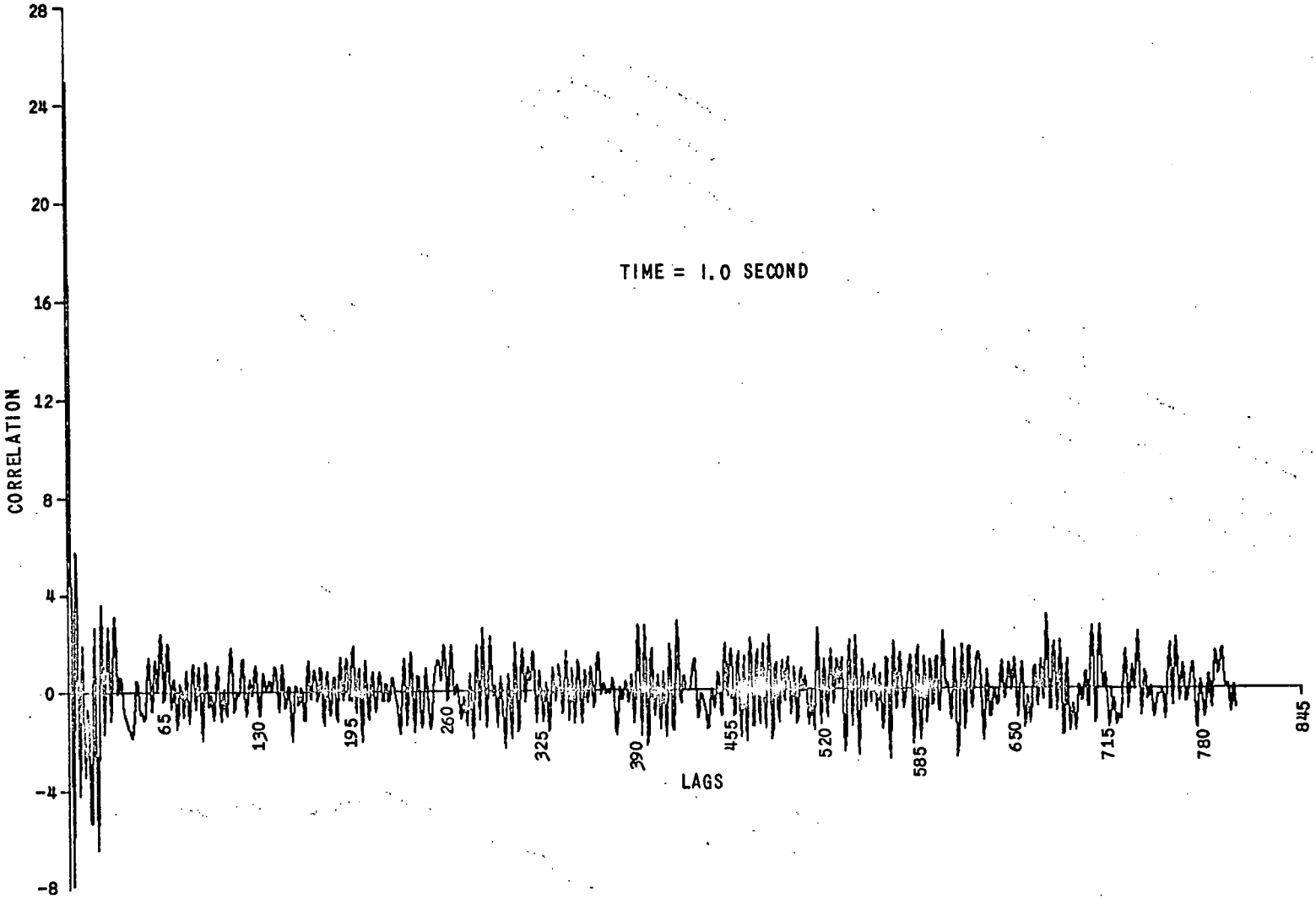


Figure 23 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #I

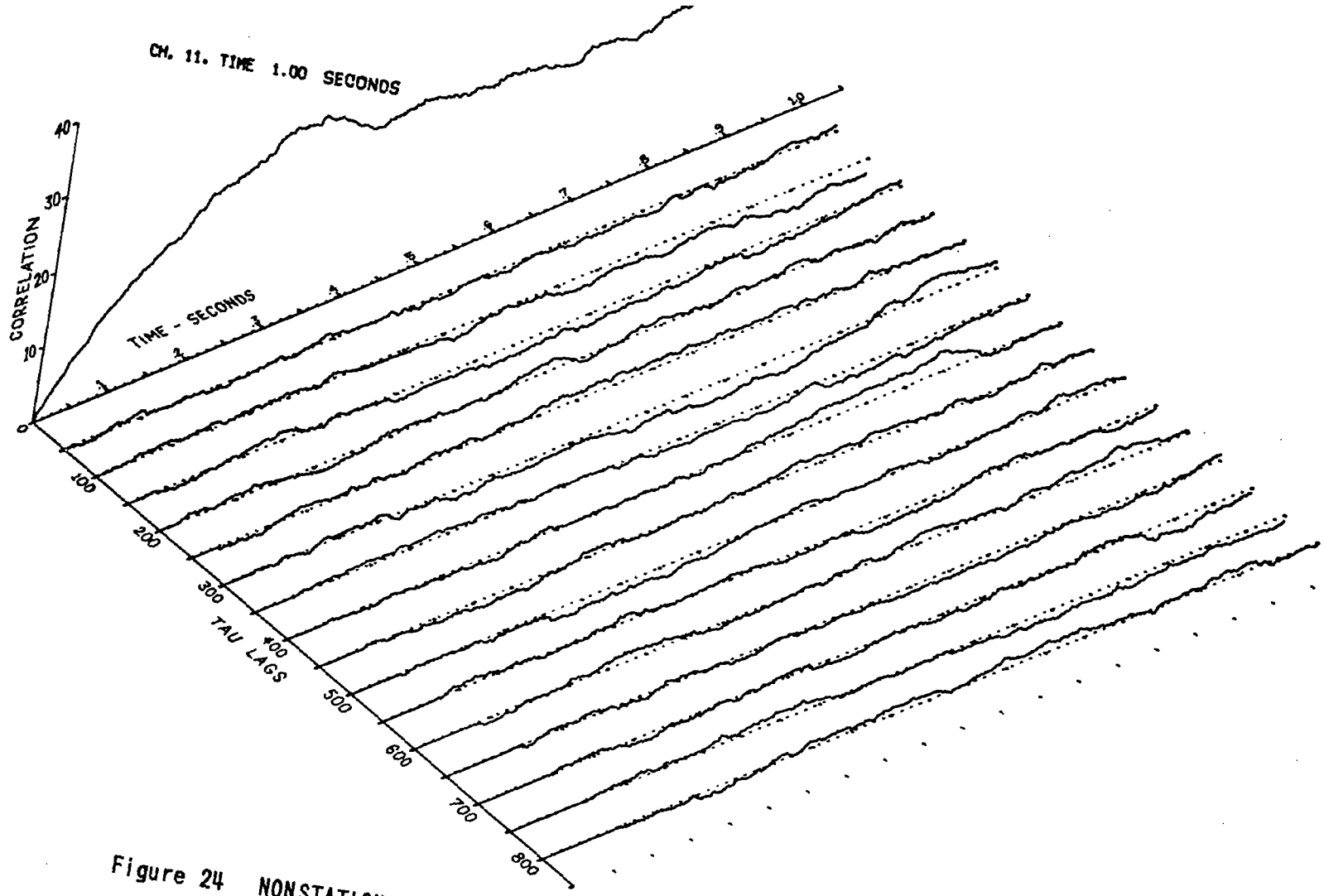


Figure 24 NONSTATIONARY CORRELATION DETECTOR OUTPUT RECORD SA10-3-11 #2

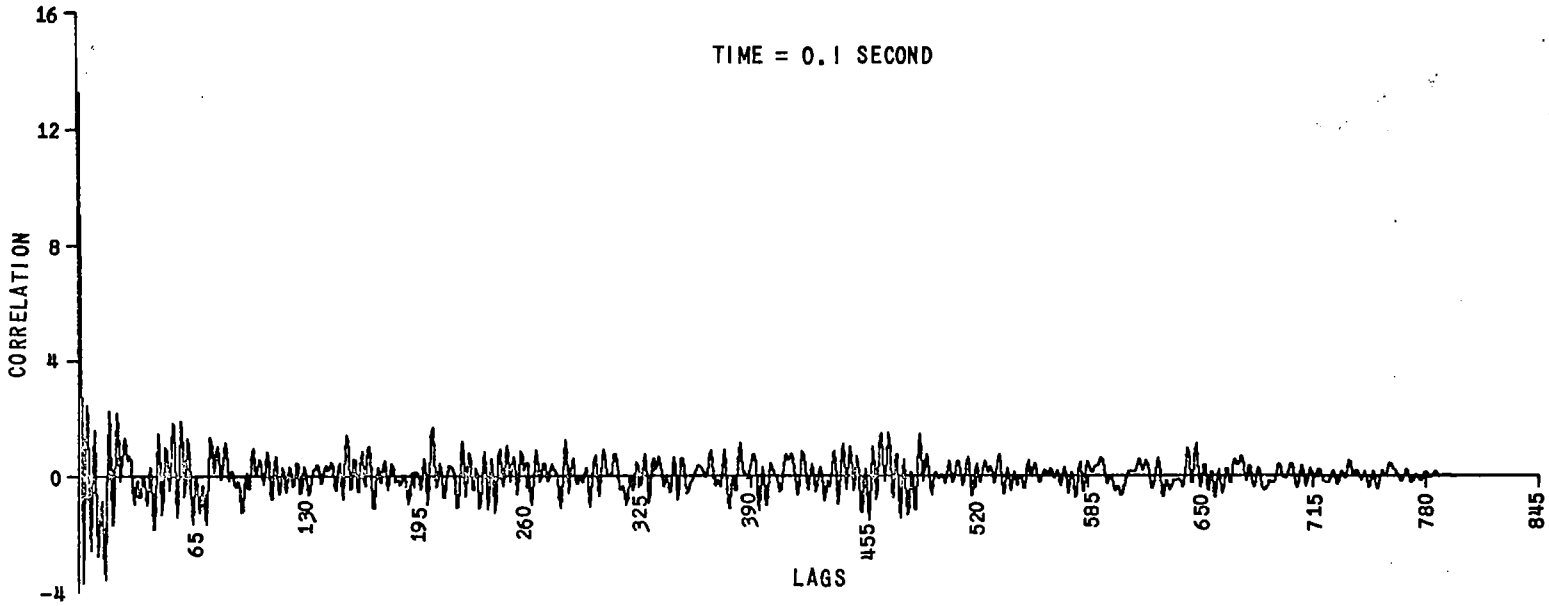


Figure 25 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

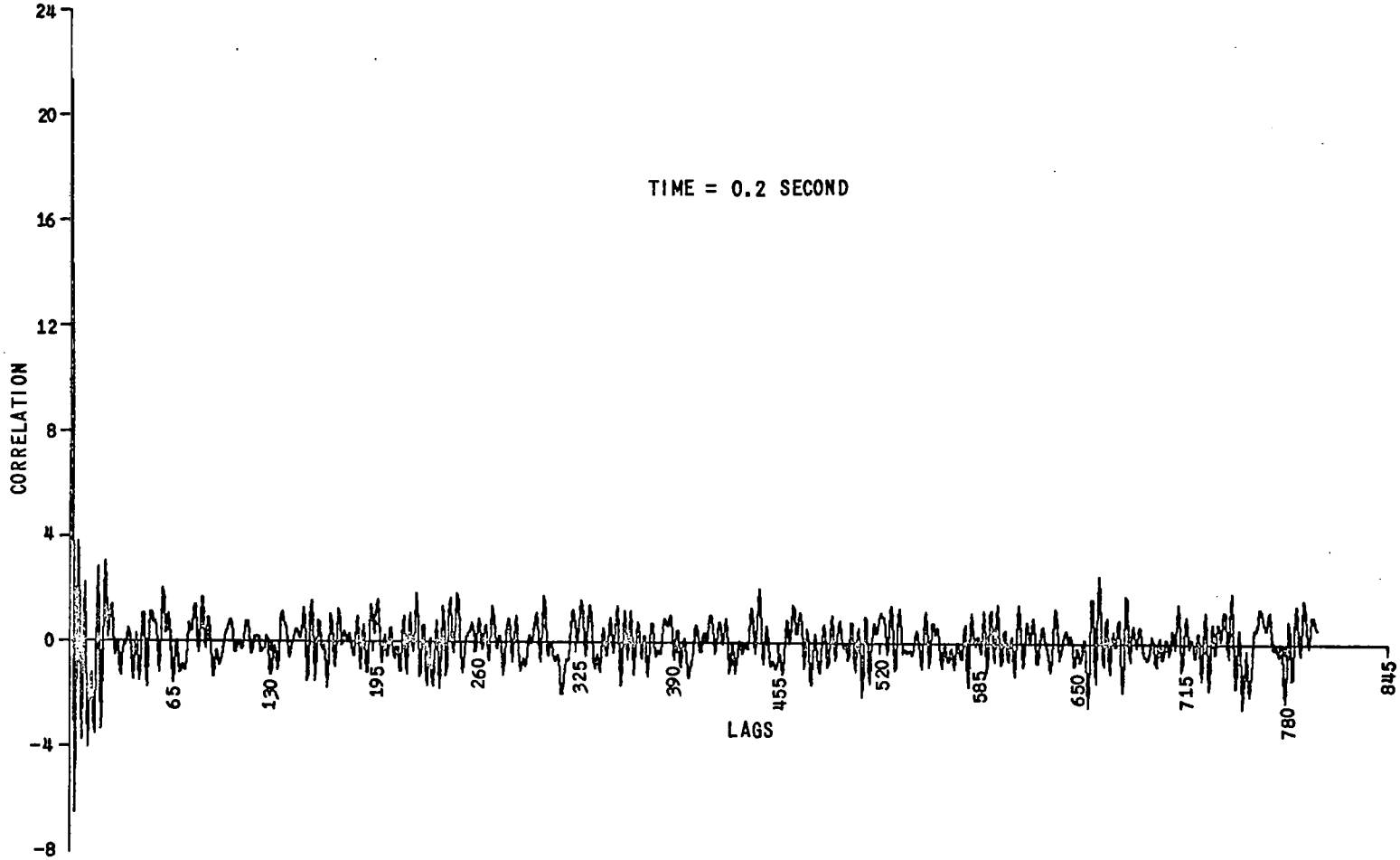


Figure 26 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

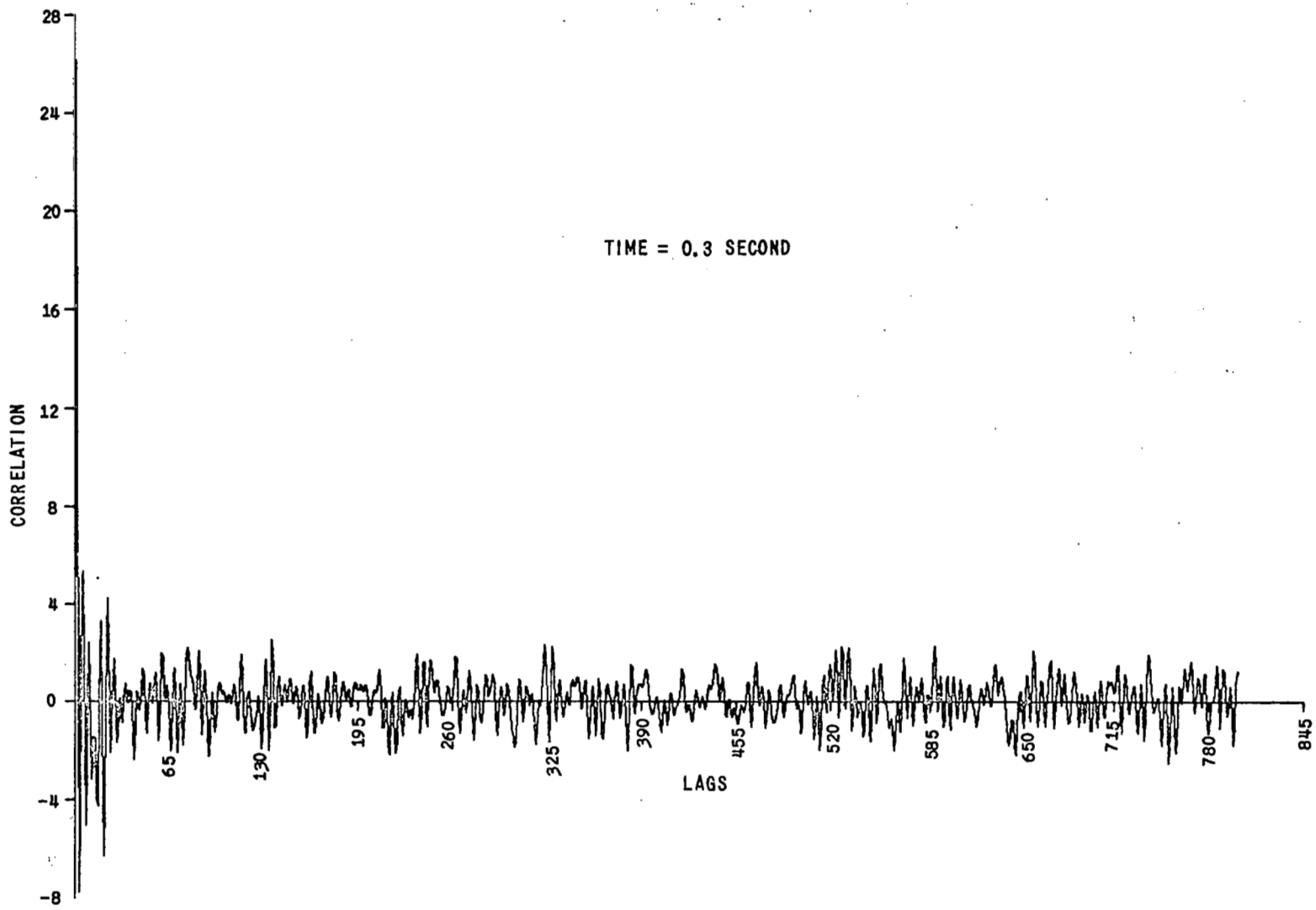


Figure 27 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

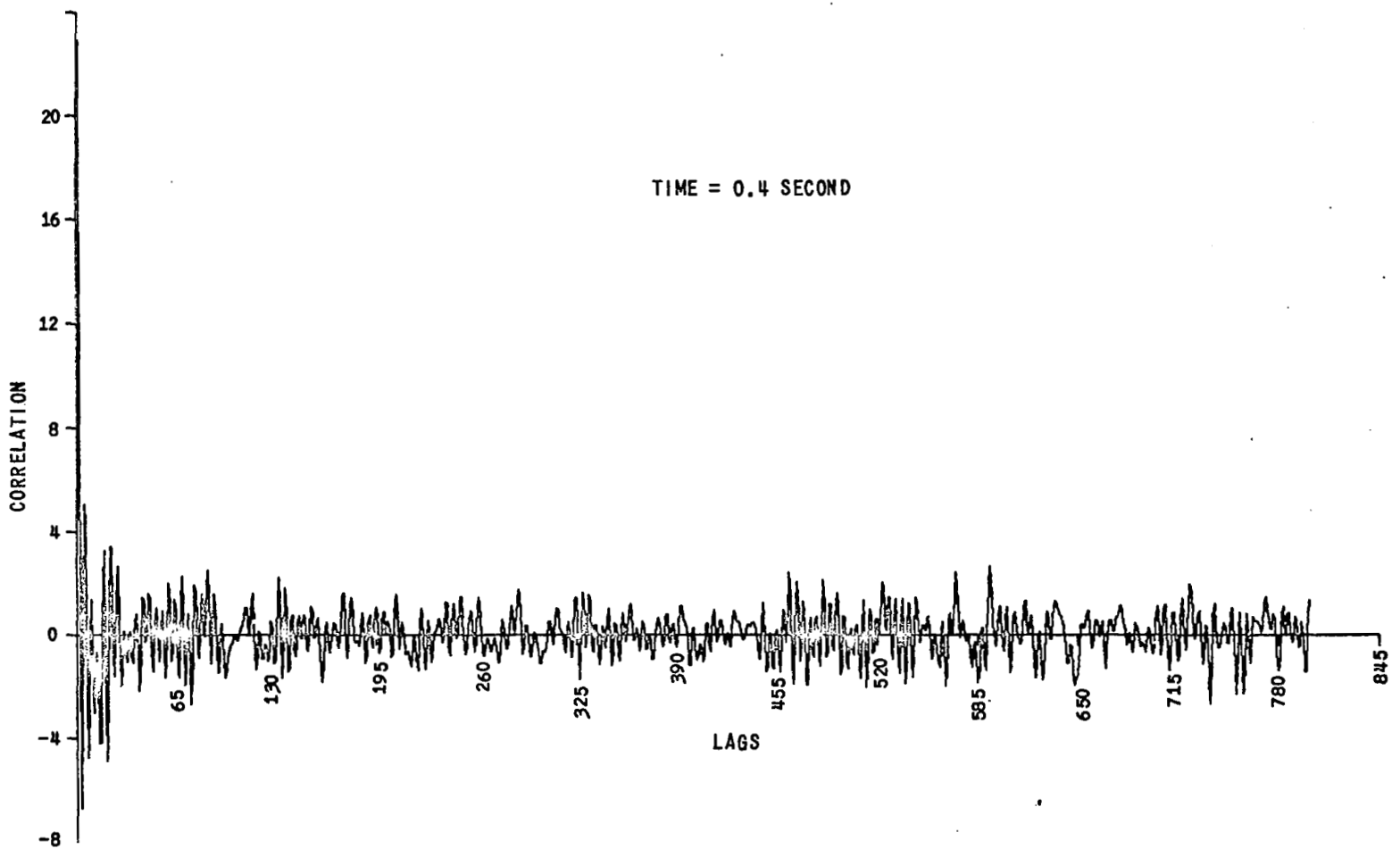


Figure 28 CORRELATION TIME SLICES FOR RECORD SAIO-3-11 #2

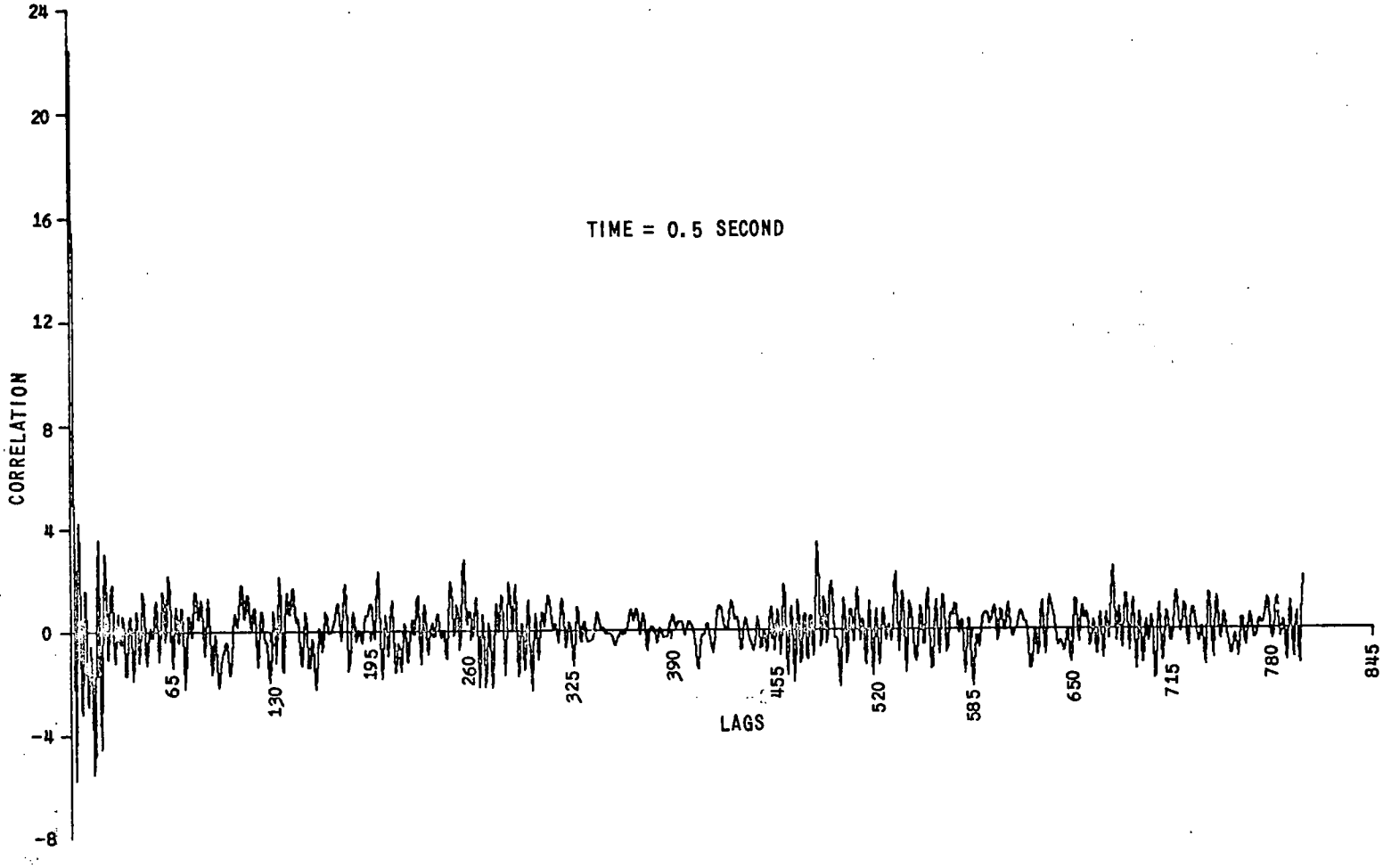


Figure 29 CORRELATION TIME SLICES FOR RECORD SA10-3-11 #2

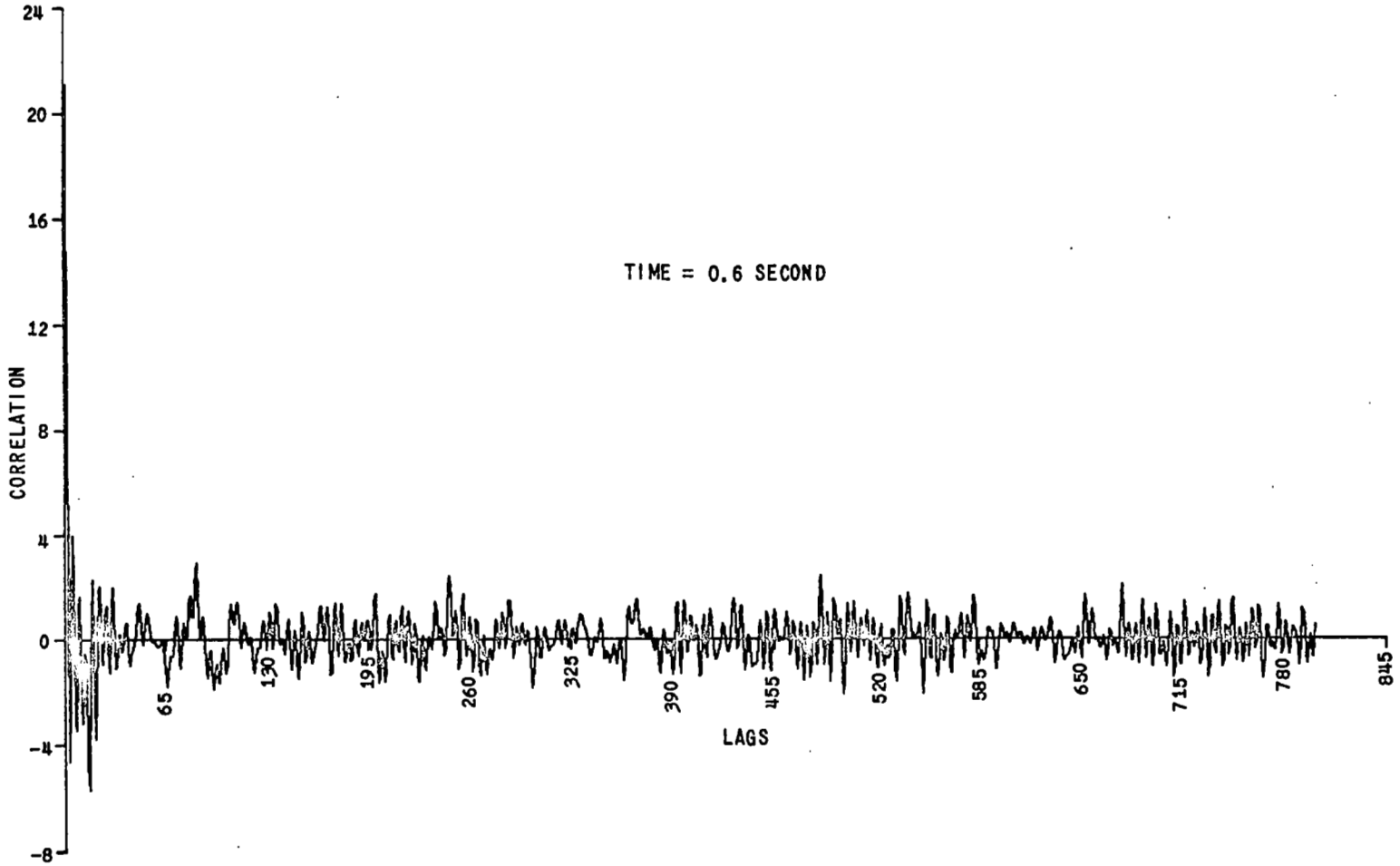


Figure 30 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

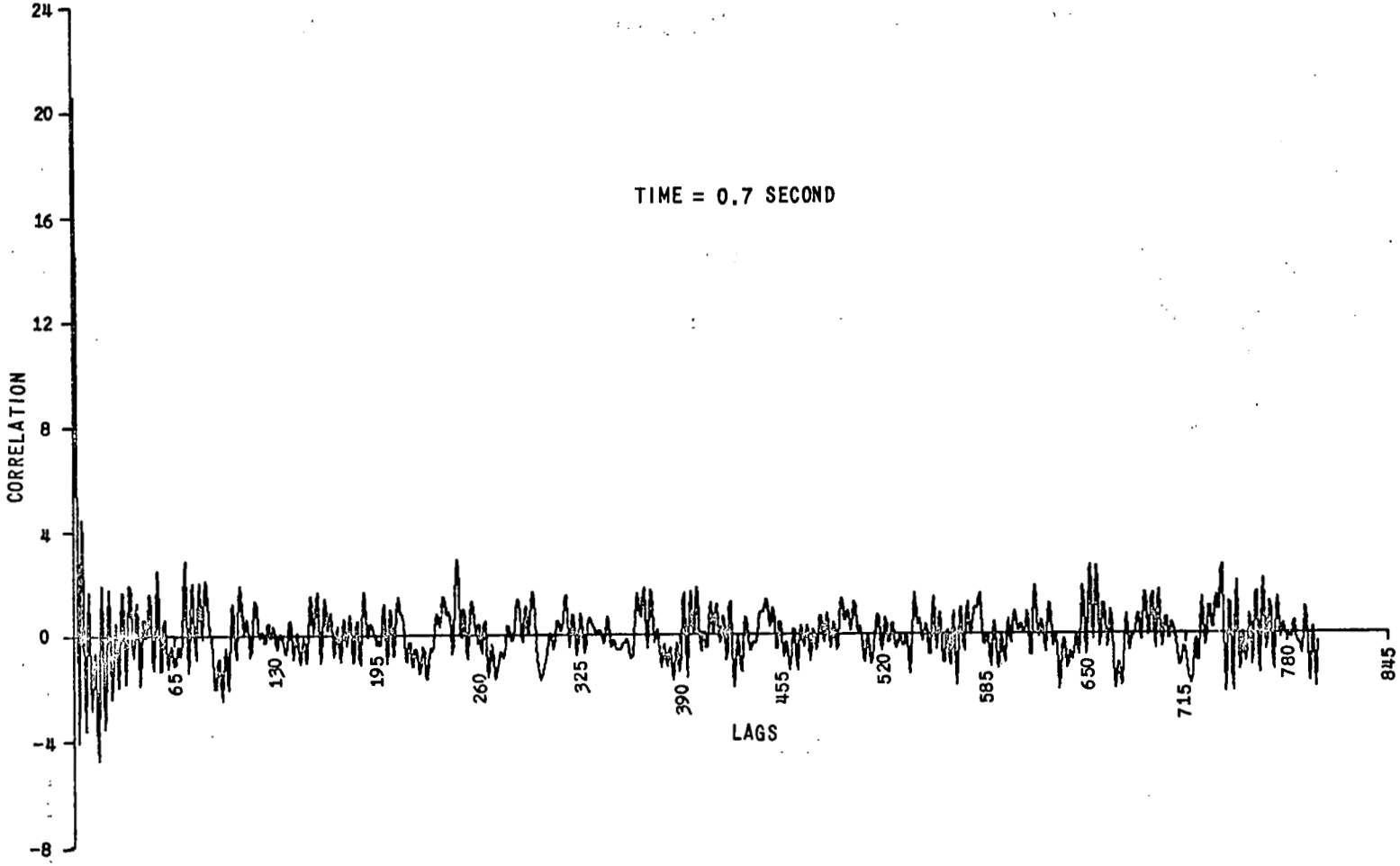


Figure 31 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

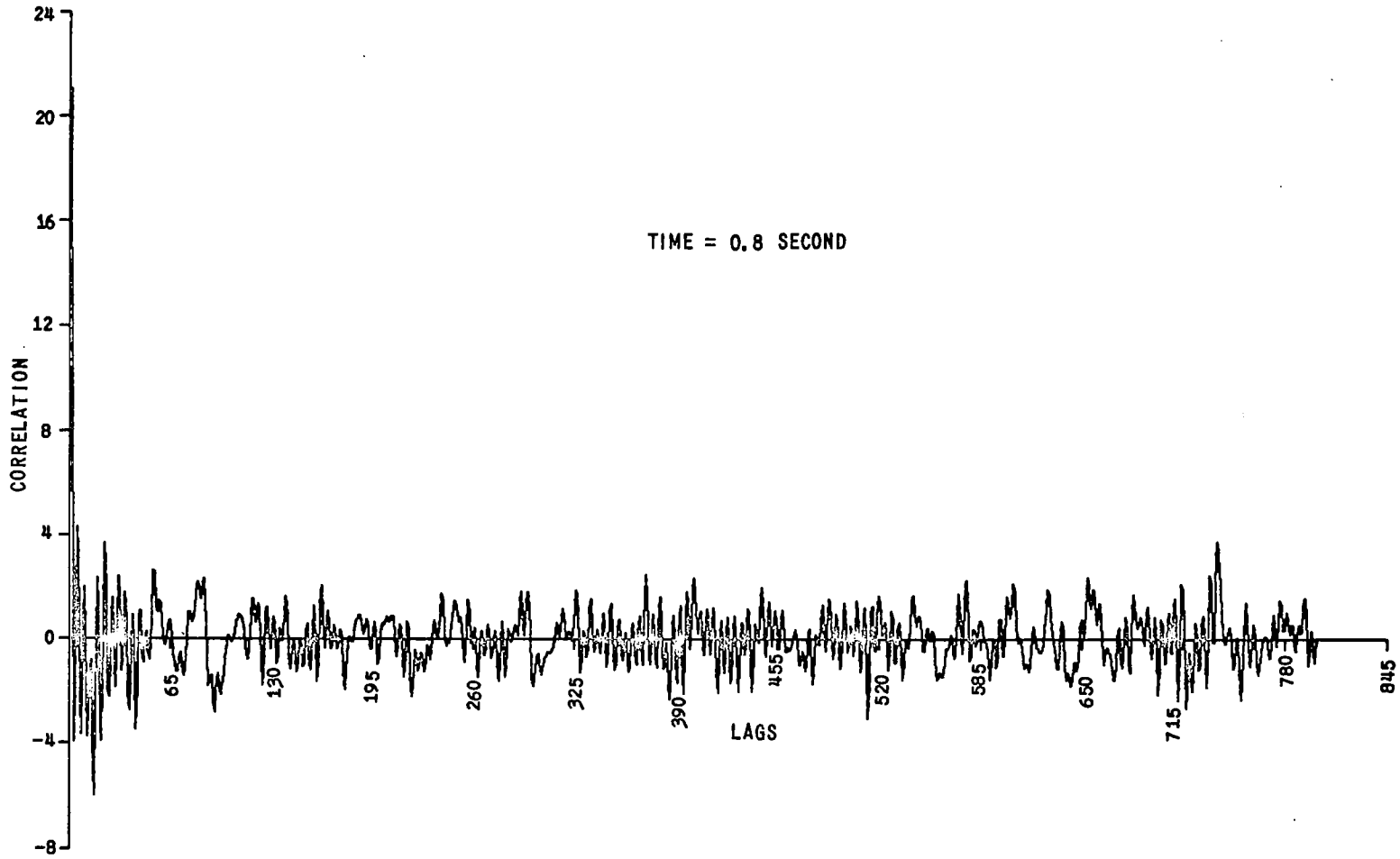


Figure 32 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

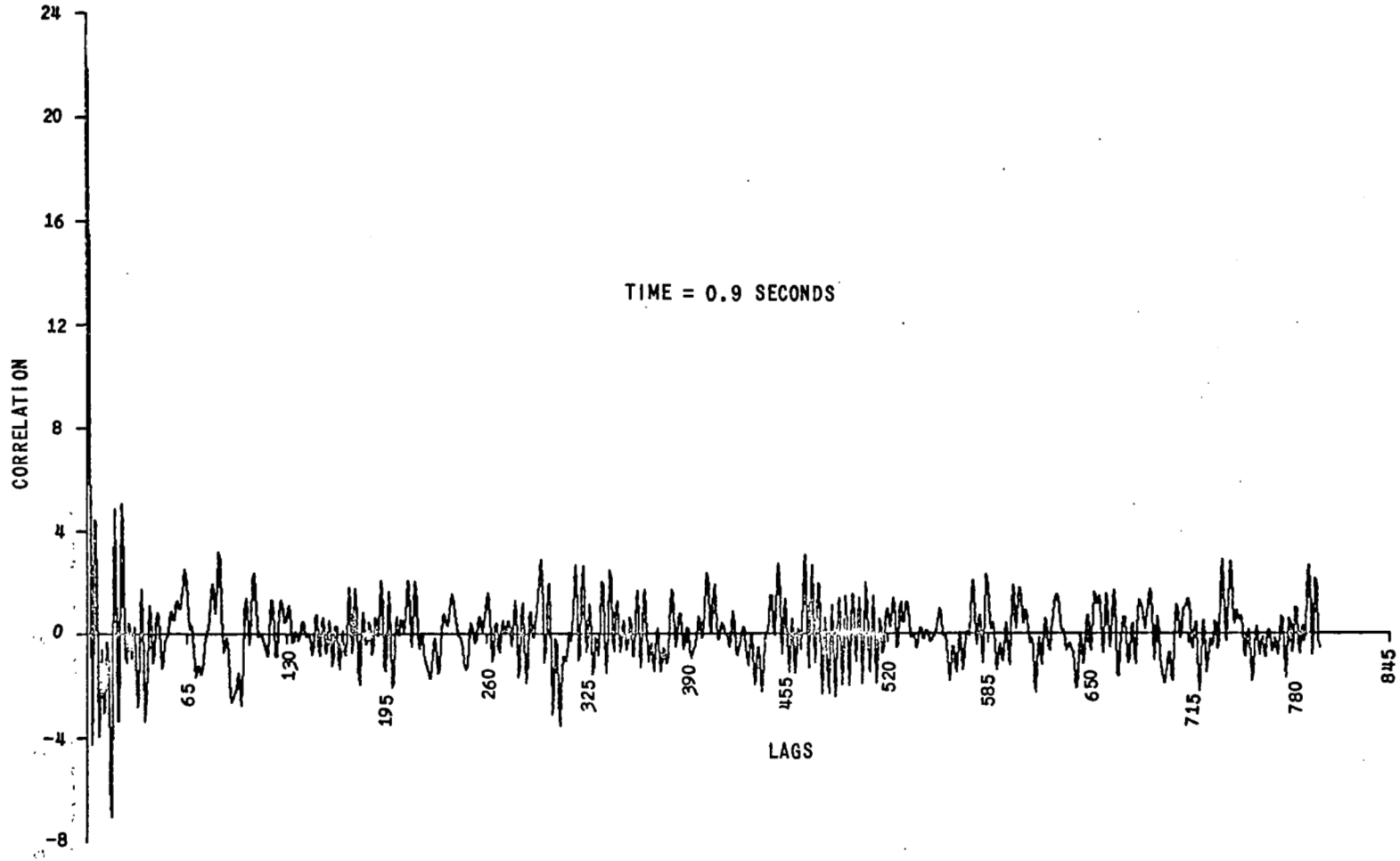


Figure 33 CORRELATION TIME SLICES FOR RECORD SAIO-3-II #2

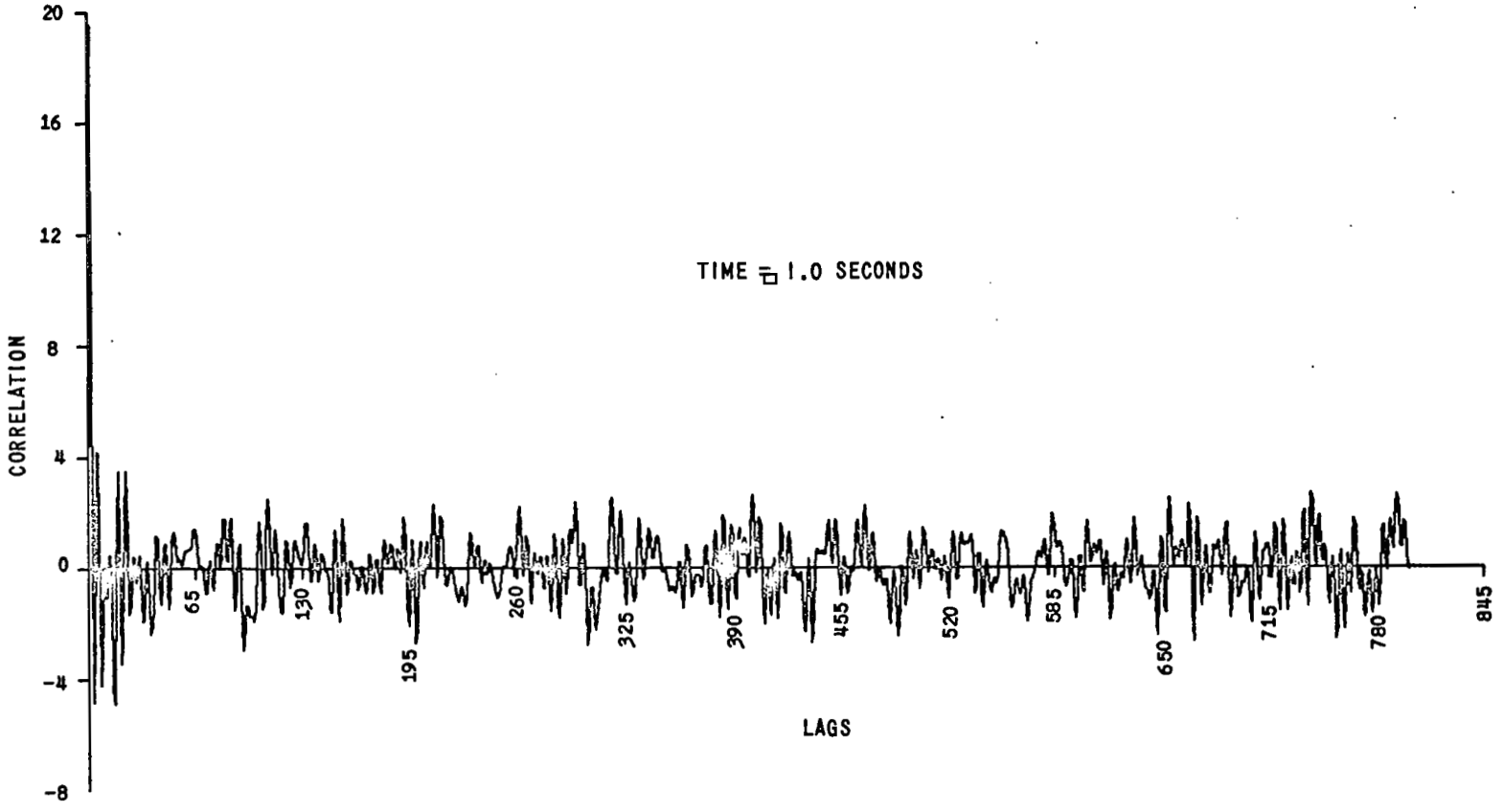


Figure 34 CORRELATION TIME SLICES FOR RECORD SA10-3-11#2

If the $\mathcal{Z}$ transform, with respect to m, of the detected correlation function $\psi(n,m)$ is obtained, then a meaningful spectrum from a physical viewpoint will be produced. There are strong indications that with the appropriate selection of the filter parameters a and b, a nonstationary spectrum which does not exhibit negative power will be generated. This condition should be experimentally verified. The variability of the spectrum with time thus obtained will yield insights into the vehicles response to excitation at different times. As in the case of stationary data processing additional smoothing of the spectral estimates may be required. This additional smoothing may possibly be incorporated in the tau axis filtering operation.

Conclusions. - By considering the individual sources of error, distortion along the t axis, distortion along the τ axis, and noise distortion, a performance index has been developed which accurately assesses the errors involved in discrete-data nonstationary correlation detection. The detector filters were then specified by minimizing this performance index. The minimization was accomplished with the use of $\mathcal{Z}$ transform theory and the theory of calculus of variations. The filters obtained in this manner are thus considered optimum.

It was demonstrated that a one-to-one correspondence existed between the performance index weights and the optimal filter bandwidths. As an aid to the selection of appropriate weights λ_t, λ_τ, for the performance index, the frequency characteristics of the optimal filters were determined.

A digital program was developed which was used to analyze portions of NASA's nonstationary vibration data. The experimental results have indicated the practicality of the discrete-data nonstationary correlation theory developed. It has also provided vital information on the computational times associated with the generation of the nonstationary correlation function. A means for obtaining the associated time varying spectra was also presented.

RECOMMENDATIONS

During the course of performing the research described herein, many new and unexplored areas, where further research would be beneficial, have become apparent. Summarized below are the particular areas where additional investigations would appear to be most fruitful for the Computation Laboratory.

(a) Develop a means for the evaluation of the expansion coefficients C_j by using marginal densities obtained from actual test data. This will provide one type of indicator as to how good the half-polarity (or N-level) correlator is for a given process.

(b) Verify the theories presented on the half-polarity correlation with added noise. That is, conduct actual tests to see if the type of noise and the amount of noise to be added are the same as the theory indicates.

(c) Study the possibility of developing optimal thresholds for an N-level correlator with the use of actual measured marginal densities. Compare the accuracy obtained with the use of optimum thresholds.

(d) Since a visual examination of the spectra produced by the N-level and full precision correlators (Reference 1) was remarkably similar, a mathematical performance index should be developed to ascertain exactly how much additional accuracy is being obtained from an N-level correlator as N increases.

(e) The combination of the material reported in Reference 1 and this present document has provided several sophisticated means, for achieving spectral estimates, that possess a substantial computational time advantage over more conventional techniques. Although the correlators discussed have wide application, they should be utilized with the particular

type of random data for which they are known to yield good results. Thus, investigations into the possibility of developing rapid tests to determine what processor should be utilized for a given data sequence, should be undertaken.

(f) Test of the optimum smoothing weights with various degrees of smoothing should be conducted by means of varying the function ω_m .

(g) Development of a nonreal-time theory for correlation analysis of nonstationary signals.

(h) Development of a nonstationary correlation theory utilizing ensemble averages.

(i) Development of methods for obtaining nonstationary spectra by transforming nonstationary correlation functions.

(j) Development of methods for improving the efficiency of nonstationary data processing.

(k) Development of a theory for detecting nonstationary power spectra, in an optimal manner, without directly computing the nonstationary correlation function.

APPENDIX A

EXPANSION COEFFICIENTS OF SIGN X

When a uniform marginal density is placed into the expression (Equation 9) for the expansion coefficients C_j the result becomes

$$C_j = \int_{-a}^{a} \frac{1}{2a} \; (sign\, x)\, \theta_j(x)\, dx \tag{A-1}$$

The Legendre polynomials $\theta_j(x)$ are even functions of χ when j is even, and they are odd functions of χ when j is odd. When this information is placed in (A-1) the following is obtained

$$\begin{aligned} C_j &= \frac{1}{a} \int_{o}^{a} \theta_j(x)\, dx \qquad && j \text{ odd} \\ C_j &= 0 && j \text{ even} \end{aligned} \tag{A-2}$$

let $\frac{x}{a} = \beta$ (A-2) now becomes

$$C_j = \int_{o}^{1} \theta_j(a\beta)\, d\beta \qquad j \text{ odd} \tag{A-3}$$

The orthonormality condition which the Legendre polynomials must satisfy is found from (5) to be given by

$$\int_{-a}^{a} \frac{1}{2a} \; \theta_i(x)\, \theta_j(x)\, dx = \delta_{ij} \tag{A-4}$$

Using the same change of variable as above (A-4) may be written as

$$\int_{-1}^{1} \frac{1}{2} \; \theta_i(a\beta)\, \theta_j(a\beta)\, d\beta = \delta_{ij} \tag{A-5}$$

From Equation 7.4112, 1 of Reference 5 the following relation is obtained

$$\int_{-1}^{1} \frac{1}{2} \sqrt{2i+1}\, P_i(x) \sqrt{2j+1}\, P_j(x)\, dx = \delta_{ij} \qquad \text{(A-6)}$$

where the $P_i(x)$ are also Legendre polynomials, but with a different standardization than the set $\theta_i(x)$.

From (A-5) and (A-6) it is seen that

$$\theta_j(a\beta) = \sqrt{2j+1}\; P_j(\beta) \qquad \text{(A-7)}$$

The expression for the coefficients given by (A-3) may now be expressed as

$$C_\sigma = \sqrt{2\sigma+1} \int_0^1 P_\sigma(x)\, dx \;,\; \sigma \text{ odd} \qquad \text{(A-8)}$$

With the expression for C_σ in its present form and the relation $P_0(x)=1$, Equation 7.113, 1 of Reference 5 can be used to evaluate the coefficients C_σ as

$$C_\sigma = \frac{\sqrt{2\sigma+1}\; \frac{\Gamma(\frac{1}{2})\,\Gamma(1+\frac{\sigma}{2})}{\Gamma(\frac{1}{2}+\frac{\sigma}{2})\,\Gamma(1)}\,(-1)\,j^{\sigma+1}}{\frac{1}{2}\,\pi\,\sigma(\sigma+1)} \;,\; \sigma \text{ odd} \qquad \text{(A-9)}$$

where in (A-9) $j = \sqrt{-1}$

To simplify the above expression, let $\sigma = 2N+1$, (A-9) then becomes

$$C_{2N+1} = (-1)^{2(N+1)} \frac{\sqrt{4N+3}\;\sqrt{\pi}\;\Gamma(1+N+\frac{1}{2})}{2^{-1}\pi\,(2N+1)(2N+2)\,\Gamma(N+1)} \qquad \text{(A-10)}$$

Finally, with the use of Equation 8.339,2 of Reference 5 (A-10) reduces to

$$C_{2N+1} = (-1)^N \frac{\sqrt{4N+3}\ (2N-1)!!}{2^{N+1}\,(N+1)!} \qquad \text{(A-11)}$$

where

$$N = 0, 1, 2, \cdots$$

$$(2N-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2N-1)$$

APPENDIX B

CORRELATOR STATISTICS

Expected Value of R_1 (m). - n_k are samples from uncorrelated zero mean noise

$$E\left[R_1(m)\right] = \frac{1}{N-m} \; E\left[\sum_{k=1}^{N-m} (x_k + n_k)\, sign(y_{k+m} + n_{k+m})\right]$$

for

$$m \neq 0$$

$$E\left[R_1(m)\right] = \frac{1}{N-m} \sum_{k=1}^{N-m} x_k \, f_{k+m}$$

where

$$f_{k+m} = E\left[sign(y_{k+m} + n_{k+m})\right]$$

for

$$m = 0$$

$$E\left[R_1(m)\right] = \frac{1}{N} \sum_{k=1}^{N} x_k \, f_k + \frac{1}{N} \sum_{k=1}^{N} g_k$$

where

$$g_k = E\left[n_k \; sign(y_k + n_k)\right]$$

The expected value for all m is then given by

$$E\left[R_1(m)\right] = \frac{1}{N-m} \sum_{k=1}^{N-m} x_k \, f_{k+m} + \frac{\sigma_{om}}{N} \sum_{k=1}^{N} g_k \qquad \text{(B-1)}$$

Variance of $R_1(m)$.

$$VAR\left[R_1(m)\right] = E\left[R_1^2(m)\right] - E^2\left[R_1(m)\right]$$

$$E\left[R_1^2(m)\right] = \frac{1}{(N-m)^2} E\left[\left(\sum_{k=1}^{N-m}(x_k+\eta_k)\, sign\left(y_{k+m}+\eta_{k+m}\right)\right)^2\right]$$

$$= \frac{1}{(N-m)^2} E\left[A^2 + 2AB + B^2\right]$$

where

$$A = \sum_{k=1}^{N-m} x_k\, sign\left(y_{k+m}+\eta_{k+m}\right)$$

$$B = \sum_{k=1}^{N-m} \eta_k\, sign\left(y_{k+m}+\eta_{k+m}\right)$$

$$E\left[R_1^2(m)\right] = \frac{1}{(N-m)^2}\left[E\left[A^2\right] + 2E\left[AB\right] + E\left[B^2\right]\right]$$

$$E\left[A^2\right] = E\left[\sum_{k=1}^{N-m} x_k^2 + \sum_{\substack{j,k \\ j\neq k}}^{N-m} x_j x_k\, sign\left(y_{k+m}+\eta_{k+m}\right) \times sign\left(y_{j+m}+\eta_{j+m}\right)\right]$$

$$E\left[A^2\right] = \sum_{k=1}^{N-m} x_k^2 + \sum_{\substack{j,k \\ j\neq k}}^{N-m} x_j x_k f_{j+m} f_{k+m}$$

$$E\left[AB\right] = E\left[\left(\sum_{j=1}^{N-m} x_j\, sign\left(y_{j+m}+\eta_{j+m}\right)\right)\left(\sum_{k=1}^{N-m}\eta_k\, sign\left(y_{k+m}+\eta_{k+m}\right)\right)\right]$$

Consider first the case for m = 0.

The main diagonal terms in the above product are zero, since n has zero mean. The off diagonal terms are $f_i\ g_k$. For the case $m \neq 0$ the main diagonal terms are still zero. All off diagonal terms are also zero except along the diagonal where $j + m = k$, since it is along this diagonal that there is correlation between the products. The individual terms are given by

$$E\left[x_j\, sign\left(y_{j+m}+\eta_{j+m}\right)\eta_k\, sign\left(y_{k+m}+\eta_{k+m}\right)\right]$$

where

$$j + m = k$$

From the above relation these terms can be expressed as $x_j \; f_{j+2m} \; g_{j+m}$
Combining the expressions for either case of m yields

$$E\left[AB\right] = \delta_{om} \sum_{\substack{j,k \\ j \neq k}}^{N-m} x_j f_j g_k + (1-\delta_{om}) \sum_{j=1}^{N-2m} x_j \, g_{j+m} \, f_{j+2m}$$

$$E\left[B^2\right] = E\left[\left(\sum_{k=1}^{N-m} n_k \; sign\left(y_{k+m} + n_{k+m}\right)\right)^2\right]$$

For m = 0, the main diagonal terms are σ^2 and the off diagonal terms are $g_j \, g_k$. When $m \neq 0$ the main diagonal terms are σ^2 and the off diagonal terms are zero. Combining these results, the above expression reduces to

$$E\left[B^2\right] = (N-m)\sigma^2 + \delta_{om} \sum_{\substack{j,k \\ j \neq k}}^{N-m} g_j \, g_k$$

The mean square value of the $R_1(m)$ correlator may now be expressed as

$$E = \left[R_1^2(m)\right] = \frac{1}{(N-m)^2}\left[\sum_{k=1}^{N-m} x_k^2 + \sum_{\substack{j,k \\ j \neq k}}^{N-m} x_j \; x_k \; f_{j+m} \; f_{k+m} + 2\,\delta_{om} \sum_{\substack{j,k \\ j \neq k}}^{N-m} x_j f_j g_k + 2(1-\delta_{om}) \sum_{j=1}^{N-2m} x_j \, g_{j+m} \, f_{j+2m} + (N-m)\sigma^2 + \delta_{om} \sum_{\substack{j,k \\ j \neq k}}^{N-m} g_j \, g_k\right]$$

From (B-1) the square of the mean may be expressed as

$$E^2\left[R_1(m)\right] = \frac{1}{(N-m)^2}\left[\sum_{k=1}^{N-m} x_k^2 \, f_{k+m}^2 + \sum_{\substack{j,k \\ j \neq k}}^{N-m} x_j \; x_k \; f_{j+m} \; f_{k+m} + 2\,\delta_{om}\left(\sum_{k=1}^{N} g_k\right)\left(\sum_{k=1}^{N} x_k \, f_k\right) + \delta_{om}\left(\sum_{k=1}^{N} g_k\right)^2\right]$$

Upon combining the mean square value, and the square of the mean, the variance becomes

$$VAR\left[R_1(m)\right] = \frac{\sigma^2}{(N-m)} + \frac{1}{(N-m)^2}\sum_{k=1}^{N-m} x_k^2\left[1-f_{k+m}^2\right]$$

$$+ \frac{(1-\delta_{om})}{(N-m)^2}\sum_{k=1}^{n-2m} x_k\, g_{k+m} f_{k+2m} \qquad \text{(B-2)}$$

$$- \frac{2\delta_{om}}{(N-m)^2}\sum_{k=1}^{N-m} x_k f_k g_k - \frac{\delta_{om}}{(N-m)^2}\sum_{k=1}^{N-m} g_k^2$$

Expected Value of $R_2(m)$.

$$E\left[R_2(m)\right] = \frac{1}{(N-m)}\, E\left[\sum_{k=1}^{N-m}(x_k + u_k)\, sign\,(y_{k+m} + \eta_{k+m})\right]$$

where u and n represent uncorrelated independent zero mean noise. The derivation for the expected value follows identically as that for the $R_1(m)$ correlator, with u_k replacing the appropriate η_k. The expected value is given by

$$E\left[R_2(m)\right] = \frac{1}{N-m}\sum_{k=1}^{N-m} x_k\, f_{k+m} \qquad \text{(B-3)}$$

Variance of $R_2(m)$. - The derivation of the variance of the $R_2(m)$ correlator also follows exactly that for the $R_1(m)$ correlator. When the appropriate η_k is replaced by u_k the following result is obtained

$$VAR\left[R_2(m)\right] = \frac{\sigma_u^2}{(N-m)} + \frac{1}{(N-m)^2}\sum_{k=1}^{N-m} x_k^2\left[1-f_{k+m}^2\right] \qquad \text{(B-4)}$$

Expected Value and Variance of the $R_3(m)$ Correlator. - These statistics are obtained directly from the $R_2(m)$ correlator statistics by setting $u = 0$. This results in

$$E\left[R_3(m)\right] = \frac{1}{N-m}\sum_{k=1}^{N-m} x_k\, f_{k+m} \qquad \text{(B-5)}$$

$$VAR\left[R_3(m)\right] = \frac{1}{(N-m)^2}\sum_{k=1}^{N-m} x_k\left[1-f_{k+m}^2\right] \qquad \text{(B-6)}$$

APPENDIX C

EVALUATION OF f_k AND g_k FOR GAUSSIAN NOISE

$$
\begin{aligned}
f_k &= E\left[\operatorname{sign}(y_k + n_k)\right] \\
&= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \operatorname{sign}(y_k + n)\, e^{-\frac{1}{2}\frac{n^2}{\sigma^2}}\, dn \\
&= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_{0}^{y_k} e^{-\frac{1}{2}\frac{n^2}{\sigma^2}}\, dn \\
&= \frac{2}{\pi} \int^{\frac{y_k}{\sqrt{2}\sigma}} e^{-\beta^2}\, d\beta = \operatorname{erf}\left(\frac{y_k}{\sqrt{2}\,\sigma}\right)
\end{aligned}
\qquad (C\text{-}1)
$$

$$
\begin{aligned}
g_k &= E\left[n_k \operatorname{sign}(y_k + n_k)\right] \\
&= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} n \operatorname{sign}(y_k + n)\, e^{-\frac{1}{2}\frac{n^2}{\sigma^2}}\, dn \\
&= \sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma} \int_{y_k}^{\infty} n\, e^{-\frac{1}{2}\frac{n^2}{\sigma^2}}\, dn \\
&= \sigma\sqrt{\frac{2}{\pi}}\ e^{-\frac{1}{2}\left(\frac{y_k}{\sigma}\right)^2}
\end{aligned}
\qquad (C\text{-}2)
$$

APPENDIX D

EVALUATION OF f_k AND g_k FOR UNIFORM NOISE

The noise density for n is uniformly distributed between $(-\sqrt{3}\sigma, \sqrt{3}\sigma)$. Before the calculation of f_k and g_k is made it will be convenient to define the following auxiliary functions.

$$\delta_1(y) = \begin{cases} 1, & -y < -\sqrt{3}\sigma \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_2(y) = \begin{cases} 1, & |y| \le \sqrt{3}\sigma \\ 0, & \text{otherwise} \end{cases} \qquad \text{(D-1)}$$

$$\delta_3(y) = \begin{cases} 1, & -y > \sqrt{3}\sigma \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_k &= E\left[n \operatorname{sign}(y_k + n_k)\right] \\ &= \frac{1}{2\sqrt{3}\sigma} \int_{-\sqrt{3}\sigma}^{\sqrt{3}\sigma} \operatorname{sign}(y_k + n)\, dn \\ &= \delta_1(y_k) + \frac{1}{\sqrt{3}\sigma}\, y_k\, \delta_2(y_k) - \delta_3(y_k) \end{aligned} \qquad \text{(D-2)}$$

$$\begin{aligned} g_k &= E\left[n_k \operatorname{sign}(y_k + n_k)\right] \\ &= \frac{1}{2\sqrt{3}\sigma} \int_{-\sqrt{3}\sigma}^{\sqrt{3}\sigma} n \operatorname{sign}(y_k + n)\, dn \\ &= \frac{\delta_2(y_k)}{\sqrt{3}\sigma} \int_{y_k}^{\sqrt{3}\sigma} n\, dn \\ &= \frac{3\sigma^2 - y_k^2}{2\sqrt{3}\sigma}\, \delta_2(y_k) \end{aligned} \qquad \text{(D-3)}$$

APPENDIX E

DETECTOR VARIANCE

It is desired to obtain an expression for the variance of the detector output in terms of the filtering operations $\ell(n)$ and $h(n)$ when the input signal pair are members of uncorrelated stationary Gaussian processes with zero means.

$$E\left[(\psi-\phi)^2\right] = E\left[\psi^2\right] - \phi^2$$
$$= E\left[\psi^2\right] \qquad \text{(E-1)}$$

The true correlation ϕ is zero since the processes are uncorrelated. The squared output of the correlation detector expressed as a function of the noise test signals N_1 and N_2 is given by

$$\psi^2(n,m) = \sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{b=0}^{\infty}\sum_{a=0}^{\infty} h(\beta)\,h(b)\,\ell(a)\,\ell(a) \times \qquad \text{(E-2)}$$
$$N_1(n-\beta)\,N_1(n-b)\,N_2(n-\beta-\alpha-m)\,N_2(n-b-a-m)$$

Taking the expected value of the above yields

$$E\left[\psi^2\right] = \sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{b=0}^{\infty}\sum_{a=0}^{\infty} h(\beta)\,h(b)\,\ell(\alpha)\,\ell(a) \times \qquad \text{(E-3)}$$
$$E\left[N_1(n-\beta)\,N_1(n-b)\,N_2(n-\beta-\alpha-m)\,N_2(n-b-a-m)\right]$$

Using a property of jointly distributed random variables, (see References 7, 8) the expectation in the above equation may be replaced by

$$E\left[N_1(n-\beta)\,N_1(n-b)\right]\;E\left[N_2(n-\beta-\alpha-m)\,N_2(n-b-a-m)\right] \qquad \text{(E-4)}$$
$$+\,E\left[N_1(n-\beta)\,N_2(n-\beta-\alpha-m)\right]\;E\left[N_1(n-b)\,N_2(n-b-a-m)\right]$$
$$+\,E\left[N_1(n-\beta)\,N_2(n-b-a-m)\right]\;E\left[N_1(n-b)\,N_2(n-\beta-\alpha-m)\right]$$

Since the signals N_1 and N_2 are uncorrelated, only the first term in the above expression contributes to the expression for the expected value of the squared output which is now given as

$$E\left[\psi^2\right] = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \sum_{b=0}^{\infty} \sum_{a=0}^{\infty} h(\beta)\, h(b)\, \ell(\alpha)\, \ell(a)\, \delta(\beta-b)\, \delta(\beta-b+\alpha-a) \qquad \text{(E-5)}$$

The Kronecker delta δ takes on its usual definition and is injected into the above expression for $E\left[\psi^2\right]$ when one applies the stationarity property introduced earlier. Summing over b yields

$$E\left[\psi^2\right] = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \sum_{a=0}^{\infty} h^2(\beta)\, \ell(\alpha)\, \ell(a)\, \delta(\alpha-a) \qquad \text{(E-6)}$$

Summing over a now yields the final expression sought.

$$E\left[(\psi-\phi)^2\right] = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} h^2(\beta)\, \ell^2(\alpha) \qquad \text{(E-7)}$$

APPENDIX F

DERIVATION OF THE OPTIMUM FILTERS

The optimum filters are specified by minimizing the form given by (78) which is repeated here as

$$\hat{\theta}_{\ell} = \sum_{n=0}^{\infty} \left\{ \lambda_{\tau} \; e_{\tau}^{2}(n) + H \, \ell^{2}(n) \right\} \tag{F-1}$$

where

$$H = \sum_{n=0}^{\infty} h_{o}^{2}(n)$$

The optimum weighting sequence $\ell_{o}(n)$ is that sequence which renders (F-1) a minimum. From (76), $e_{\tau}(n)$ may be expressed as

$$e_{\tau}(n) = nT - \sum_{\alpha=0}^{n} \ell(\alpha)(n-\alpha)T \tag{F-2}$$

where the D.C. gains of ℓ and h , A, B have been set to unity. With the following designations

$$\begin{aligned} k^{2} &= \frac{H}{\lambda_{\tau}} \\ \gamma(n) &= nT \end{aligned} \tag{F-3}$$

the index of (F-1) may be expressed as

$$\hat{\theta}_{\ell} = \lambda_{\tau} \sum_{n=0}^{\infty} \left\{ \left(\gamma(n) - \ell(n) * \gamma(n) \right)^{2} + k^{2} \, \ell^{2}(n) \right\} \tag{F-4}$$

Applying variational techniques $\ell(n)$ is expressed as

$$\ell(n) = \ell_{o}(n) + \epsilon \, \eta(n) \tag{F-5}$$

where $\ell_{o}(n)$ is an extremizing sequence with respect to $\hat{\theta}_{\ell}$ and ϵ a constant. The arbitrary variation $\eta(n)$ must possess the following boundary conditions

$$\eta(0) \quad \text{arbitrary}$$
$$\eta(\infty) = 0 \tag{F-6}$$

The first condition of (F-6) is a departure from the continuous theory. In the continuous case, the initial value of the step response of $\ell_o(t)$ had to be zero in order to provide a finite performance measure. For the discrete case, this restriction can be relaxed. The second boundary condition of (F-6) follows since $\ell_o(\infty)$ is specified at this point $\ell_o(\infty)$ must be zero for convergence of $\hat{\theta}_\ell$. When (F-5) is placed into (F-4) the following is obtained

$$\hat{\theta}_\ell(\epsilon) = \lambda_r \sum_{n=0}^{\infty} \left\{ \left(r(n) - \ell_o(n) * r(n) - \epsilon\, \eta(n) * r(n) \right)^2 + k^2 \left(\ell_o(n) + \epsilon\, \eta(n) \right)^2 \right\} \tag{F-7}$$

With the use of $\mathcal{Z}$ transform theory, the right hand member of the above equation may be expressed as

$$\hat{\theta}_\ell(\epsilon) = \frac{\lambda_r}{j2\pi} \oint \left\{ \left[R(z) - L_o(z) R(z) - \epsilon\, H(z) R(z) \right] \times \left[R(z^{-1}) - L_o(z^{-1}) R(z^{-1}) - \epsilon\, H(z^{-1}) R(z^{-1}) \right] + k^2 \left[L_o(z) + \epsilon\, H(z) \right] \left[L_o(z^{-1}) + \epsilon H(z^{-1}) \right] \right\} \frac{dz}{z} \tag{F-8}$$

For the condition of an extremum to exist, the following must be satisfied

$$\left. \frac{\partial \hat{\theta}_\ell(\epsilon)}{\partial \epsilon} \right|_{\epsilon = 0} = 0 \tag{F-9}$$

When the partial derivative of $\hat{\theta}_\ell(\epsilon)$ is taken and evaluated at $\epsilon = 0$ the following results are obtained

$$\frac{\lambda\tau}{j2\pi}\oint\Big\{\Big[R(z)-L_o(z)R(z)\Big]\Big(-H(z^{-1})R(z^{-1})\Big)$$
$$+\Big[R(z^{-1})-L_o(z^{-1})R(z^{-1})\Big]\Big(-H(z)R(z)\Big)$$
$$+\kappa^2\Big[L_o(z)H(z^{-1})+L_o(z^{-1})H(z)\Big]\Big\}\frac{dz}{z} \qquad \text{(F-10)}$$
$$=0$$

With the following change of integration variable $z^{-1}=P$ in the terms of (F-10) which contain $H(z)$ as a factor, the above reduces to

$$\frac{\lambda\tau}{j\pi}\oint\Big\{\Big[L_o(z)-1\Big]R(z)R(z^{-1})+\kappa^2 L_o(z)\Big\}H(z^{-1})\frac{dz}{z}=0 \qquad \text{(F-11)}$$

Since $H(z)$ has poles inside the unit circle (only stable variations are considered) $H(z^{-1})$ must have poles outside the unit circle. Due to the arbitrary nature of $H(z)$ it must be concluded that

$$\Big\{\Big[L_o(z)-1\Big]\;R(z)R(z^{-1})+\kappa^2 L_o(z)\Big\}\frac{1}{z}=\chi(z) \qquad \text{(F-12)}$$

where $\chi(z)$ is a function having poles only outside the unit circle. Equation (F-12) can be rearranged into the following form

$$L_o(z)\Big[\kappa^2+R(z)\;R(z^{-1})\Big]-R(z)\;R(z^{-1})=z\,\chi(z) \qquad \text{(F-13)}$$

Consider now the limit as z approaches zero, (F-13) reduces to

$$L_o(0)=0 \qquad \text{(F-14)}$$

since $\kappa\neq 0$ and $\chi(z)$ can have no poles within the unit circle. Let

$$\Big[\kappa^2+R(z)\;R(z^{-1})\Big]=Y(z)\,Y(z^{-1}) \qquad \text{(F-15)}$$

where $Y(z)$ has poles and zeros only inside the unit circle. Placing (F-15) into (F-13) and applying spectral factorization, the following expression for $L_o(z)$ is obtained

$$Y(z)\, L_0(z) - \left[\frac{R(z)\, R(z^{-1})}{Y(z^{-1})} \right]_+ = K \qquad \text{(F-16)}$$

where in (F-16) the symbol $[\quad]_+$ designates the expansion of the quantity inside the bracket into partial fractions and removing all the terms which have poles outside the unit circle. The constant K is yet to be determined.

An explicit expression for the z transform of the optimum impulse response becomes

$$L_0(z) = \frac{K}{Y(z)} + \frac{1}{Y(z)} \left[\frac{R(z)\, R(z^{-1})}{Y(z^{-1})} \right]_+ \qquad \text{(F-17)}$$

An expression will now be obtained for $Y(z)$.

$$Y(z) = \left\{ k^2 + \frac{T^2}{(z-1)^2 (z^{-1}-1)^2} \right\} \qquad \text{(F-18)}$$

Since there are poles on the unit circle, ϵ an infinitesimal positive constant is introduced to evaluate (F-18) as follows:

$$Y(z) = \left\{ \frac{k^2 (z-1)^2 (z^{-1}-1)^2 + T^2}{(z-1+\epsilon)^2 (z^{-1}-1+\epsilon)^2} \right\}^+$$

$$\text{let} \qquad \frac{T^2}{k^2} = (2C)^2 \qquad \text{(F-19)}$$

$$Y(z) = \left\{ k^2 \frac{(z-1)^2 (z^{-1}-1)^2 + (2C)^2}{(z-1+\epsilon)^2 (z^{-1}-1+\epsilon)^2} \right\}^+$$

From an inspection of the term within the braces on the right hand member of (F-19) $Y(z)$ may be expressed as

$$Y(z) = \left\{ k^2 \frac{(z^2 - 2R_e r_1 z + |r_1|^2)(z^{-2} - 2R_e r_2 z + |r_2|^2}{(z-1+\epsilon)^2 (z^{-1}-1+\epsilon)^2} \right\}$$

$$= \left\{ k \frac{|r_2|}{(z-1+\epsilon)^2} \left(z^2 - 2R_e r_1 z + |r_1|^2 k \frac{|r_2|}{(z^{-1}-1+\epsilon)^2} \left(z^{-2} - \frac{2R_e r_2}{|r_2|^2} z^{-1} + \frac{1}{|r_2|^2} \right) \right) \right\}^+ \qquad \text{(F-20)}$$

If the validity of the following two relationships are shown, then the evaluation of $Y(z)$ will be complete.

(a) $$|r_1|^2 = \frac{1}{|r_2|^2}$$

(b) $$R_e\, r_1 = \frac{R_e\, r_2}{|r_2|^2}$$

Condition (a) can be shown to be true by equating the constant terms in (F-19) and (F-20). Condition (b), which implies that the angles to the roots (upper or lower H.P.) are identical, can be shown to be valid by inserting the roots into (F-19) and using the relation given by (a) with the knowledge that the roots consist of two pairs of complex conjugate zeros. Thus $Y(z)$ becomes

$$Y(z) = k \frac{|r_2|}{(z-1)^2} (z^2 + b_1 z + b_0) \qquad \text{(F-21)}$$

where

$$b_1 = -2R_e\, r_1$$

$$b_0 = |r_1|^2$$

and the optimum filter becomes

$$L_0(z) = \frac{1}{k|r_2|(z^2+b_1 z+b_0)\cdot\frac{1}{(z-1)^2}} \left[\frac{\frac{T^2}{(z-1+\epsilon)^2}}{k|r_2|(z^{-2}+b_1 z^{-1}+b_0)}\right]_+ + \frac{K(z-1)^2}{k|r_2|(z^2+b_1 z+b_0)} \qquad \text{(F-22)}$$

At this point the form of the optimum filter can be seen to be

$$L_0(z) = \frac{a_2 z^2 + a_1 z}{z^2 + b_1 z + b_0} \qquad \text{(F-23)}$$

Rather than solving for a_2 and a_1 by expanding the term in brackets in (F-22) a more convenient technique presents itself. By recognizing from the form of the performance index that the ramp response of (F-23) must go to a ramp, the following relationships can be obtained:

$$a_2 = 1 - b_0$$

$$a_1 = b_1 + 2b_0$$

By equating the coefficient of Z in expressions (F-19) and (F-20), another useful relationship may be obtained which is given by

$$b_0 = \frac{-b_1}{4 + b_1}$$

Finally, the specification is completed by obtaining a relationship between b_1 and c this can be achieved by factoring the quartic in (F-19) and selecting the roots which lie within the unit circle to be r_1 and r_1^*. When this is done, the following relationship is achieved:

$$b_1 = \sqrt{2\sqrt{c^4 + 4c^2} - 2c^2} - 2$$

APPENDIX G

DERIVATION OF THE TAU AXIS FILTER

Combining Equations (91) and (94) and adding the constraint of (93), the performance index takes on the following form

$$\theta_\ell(\epsilon_1,\epsilon_2) = C\left(1-\sum_{\alpha=0}^{\infty}\left(\ell_0(\alpha)+\epsilon_1\eta_1(\alpha)+\epsilon_2\eta_2(\alpha)\right)e^{-a\alpha T}\right)^2 + H\sum_{\alpha=0}^{\infty}\left(\ell_0(\alpha)+\epsilon_1\eta_1(\alpha)+\epsilon_2\eta_2(\alpha)\right)^2 + \lambda\left(\sum_{\alpha=0}^{\infty}\left(\ell_0(\alpha)+\epsilon_1\eta_1(\alpha)+\epsilon_2\eta_2(\alpha)\right)-1\right) \quad \text{(G-1)}$$

Where in (G-1) λ is an undetermined Lagrangian multiplier. For the condition of an extremum, the following relation must be satisfied

$$\left.\frac{\partial\theta_\ell}{\partial\epsilon_j}\right|_{\epsilon_1=\epsilon_2=0} = 0 \quad \text{(G-2)}$$

Performing the derivative indicated in (G-2) the following equations are obtained:

$$\sum_{\alpha=0}^{\infty}\left\{2C\left(L_0(e^{aT})-1\right)e^{-a\alpha T} + 2H\ell_0(\alpha)+\lambda\right\}\eta_j(\alpha)=0 \quad \text{(G-3)}$$

where $L_0(e^{aT})$ is the $\mathcal{Z}$ transform of $\ell_0(n)$ evaluated at e^{aT}. Since both η_1 and η_2 are arbitrary, (G-3) reduces to the single equation

$$\ell_0(\alpha) = \frac{C}{H}\left(1-L_0(e^{aT})\right)e^{-a\alpha T} - \frac{\lambda}{2H} \quad \text{(G-4)}$$

Applying the final value condition on the step response of $\ell_0(n)$ to (G-4) the Lagrangian multiplier λ is found to be zero. The following explicit solution for $L_0(e^{aT})$ is obtained by taking the $\mathcal{Z}$ transform of (G-4)

$$L_0(e^{aT}) = \frac{\lambda_T}{H\left(1-e^{-2aT}\right)^2+\lambda_T} \quad \text{(G-5)}$$

Substituting (G-5) into (G-4) and taking the $\mathcal{Z}$ transform yields

$$L_0(z) = \frac{\lambda_T\left(1-e^{-2aT}\right)}{H\left(1-e^{-2aT}\right)^2+\lambda_T}\;\frac{z}{z-e^{-aT}} \quad \text{(G-6)}$$

Since the final value of the step response of (G-6) must be unity, the following functional relationship between a and λ_τ must be satisfied:

$$\frac{\lambda_\tau (1-e^{-2aT})}{H(1-e^{-2aT})^2 + \lambda_\tau} = 1 - e^{-aT} \qquad \text{(G-7)}$$

The condition of (G-7) is somewhat unexpected, since it states that both a and λ_τ cannot be selected independently. It can be shown, however, by a comparison of $\theta_\ell(\ell_o)$ and $\theta_\ell(\ell_o + \epsilon\eta)$ that a minimum in the calculus of variations sense does exist when condition (G-7) is satisfied. The optimum m axis filter is thus given by

$$L_o(z) = \frac{(1-e^{-aT})z}{z-e^{-aT}} \qquad \text{(G-8)}$$

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