NASA CONTRACTOR REPORT

NASA CR-567

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Prepared by
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Mass.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1966

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1. Introduction

The formulation of the spherical shell problem in terms of stress functions is applied to the case of a special non-axisymmetric boundary. The geometry of the boundary curves and the coordinates used in the solution of the problem are in relation to properties of the stereographic projection of a sphere. The boundary is described by four circular arcs whose stereographic projection, henceforth called projection, from pole P on the equatorial plane is a rectangle centered at the center of the sphere, Fig. 1. Only the interior state of stress is considered through use of the membrane and inextensional solutions.

The notation for spherical coordinates, stress resultants, stress couples, displacements, and rotations is indicated in Fig. 2.

2. Loading and Boundary Conditions

The shell is of constant thickness h and is subjected to a constant vertical load of intensity γh per unit area of the middle surface. γ may be considered as the specific weight of the material of the shell.

The shell is bounded by 4 edge beams along the boundary described earlier and is supported at 4 corners. It will be assumed that the edge beams are infinitely stiff in their own planes and infinitely flexible out of their planes. The boundary conditions for the membrane and inextensional solutions are

a) zero in plane stress resultant normal to the edge

b) zero component of displacement tangent to the edge.

Condition a) allows determining the stress resultants through the membrane solution independently of the displacements. The membrane solution is obtained as the sum of a particular solution and of the general solution of the homogeneous equations. The latter is completely separable from the inextensional bending (1).

3. Homogeneous Membrane Solution

If ψ is the general solution of the differential equation

$$(\Delta + 2)\psi = \psi'' + \cot\xi \; \psi' + \frac{\psi^{\cdot\cdot}}{\sin^2\xi} + 2\psi = 0 \qquad 1$$

where Δ is the Laplace operator on the sphere and

$$(\;)' = \frac{\partial(\;)}{\partial\xi} \qquad 2\text{-}1$$

$$(\;)^{\cdot} = \frac{\partial(\;)}{\partial\theta} \qquad 2\text{-}2$$

the membrane solution for zero surface load may be written in the form

$$N_\xi = -N_\theta = -a\gamma h(\psi + \psi'')$$

$$= a\gamma h\left(\frac{\psi^{\cdot\cdot}}{\sin^2\xi} + \cot\xi \; \psi' + \psi\right) \qquad 3\text{-}1$$

$$N_{\xi\theta} = N_{\theta\xi} = -a\gamma h\left(\frac{\psi}{\sin}\right)'^{\cdot} \qquad 3\text{-}2$$

$$u_\xi = -(1+\nu)\frac{a^2\gamma}{E}\psi' \qquad 3\text{-}3$$

$$u_\theta = -(1+\nu)\frac{a^2\gamma}{E}\frac{\psi^{\cdot}}{\sin\xi} \qquad 3\text{-}4$$

$$w = -(1+\nu)\frac{a^2\gamma}{E}\psi \qquad 3\text{-}5$$

where γ is an arbitrary constant chosen as the specific weight of the shell in order to non-dimensionalize ψ. The stress couples, transverse shear and rotations are zero. Eq. 1 may be solved in terms of a general harmonic function G by letting

$$\psi = (G\sin\xi)' = G'\sin\xi + G\cos\xi \qquad 4$$

where G satisfies

$$\Delta G = 0 \qquad 5$$

The problem may be treated in plane coordinates by making a change of independent variables from ξ and θ to the plane coordinates of the stereographic projection on the equatorial plane. Non-dimensional polar coordinates in that plane are, Fig. 3, θ and

$$\rho = \tan \frac{\xi}{2} \qquad 6\text{-}1$$

Non dimensional cartesian coordinates are, Fig. 3,

$$x = \rho \cos \theta \qquad 6\text{-}2$$

$$y = \rho \sin \theta \qquad 6\text{-}3$$

The Laplace operator on the surface of the sphere takes the form

$$\Delta = \frac{(1+\rho^2)^2}{4} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} \right) = \frac{(1+\rho^2)^2}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \qquad 7$$

Thus the function G harmonic on the sphere is also harmonic in the cartesian coordinates x and y, i. e.

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \qquad 8$$

The boundary of the shell in the plane (x, y) is the rectangle of sides $2b_1$ and $2b_2$ shown in Fig. 1. Some useful relations are

$$\sin \xi = \frac{2\rho}{1+\rho^2} \qquad 9\text{-}1$$

$$\cos \xi = \frac{1-\rho^2}{1+\rho^2} \qquad 9\text{-}2$$

$$\frac{\partial}{\partial \xi} = \frac{1+\rho^2}{2} \frac{\partial}{\partial \rho} \qquad 9\text{-}3$$

Eq. 4 takes the form

$$\psi = \rho \frac{\partial G}{\partial \rho} + \frac{1-\rho^2}{1+\rho^2} G = x \frac{\partial G}{\partial x} + y \frac{\partial G}{\partial y} + \frac{1-\rho^2}{1+\rho^2} G \qquad 10$$

G will be sought as a linear combination of harmonic polynomials $P_n(x, y)$ and $Q_n(x, y)$ defined by the relation

$$P_n + iQ_n = (x + iy)^n = \rho^n(\cos n\theta + i \sin n\theta) \qquad 11$$

where i is the imaginary number $\sqrt{-1}$.

In order to express eqs. 3 in terms of x and y, transformation formulas may be used. It is, however, simpler to take account of the following properties:

a) Eqs. 3 are valid at a point in any system of geographical coordinates;

b) The stereographic projection conserves the angles;

c) The ratio between an infinitesimal line element ds on the sphere and its stereographic projection $d\sigma$ is

$$\frac{d\sigma}{ds} = \frac{1 + \rho^2}{2} \qquad 12$$

Let N_x, N_y, N_{xy}, $= N_{yx}$ be the stress resultants defined with regard to the directions (x) and (y) on the sphere. These are orthogonal by property b) above.

In order to obtain N_x at a point Q we consider the line, parallel to the y axis and passing through the projection Q' of the point, as the projection of a parallel circle of a system of geographical coordinates (ξ', θ'), Fig. 4. Along this line, ξ' is constant and is related to x through the relation

$$\cot \xi' = - x$$

Using properties a, b, c above, eqs. 3-1, 2 are transformed into

$$N_x = - N_y = a\gamma h \left[\frac{1 + \rho^2}{2} \frac{\partial}{\partial y} \left(\frac{1 + \rho^2}{2} \frac{\partial \psi}{\partial y} \right) - x \frac{1 + \rho^2}{2} \frac{\partial \psi}{\partial x} + \psi \right]$$

$$N_{xy} = N_{yx} = - a\gamma h \frac{1 + \rho^2}{2} \frac{\partial}{\partial y} \left[\frac{1 + \rho^2}{2} \frac{\partial \psi}{\partial x} + x\psi \right]$$

or

$$N_x = - N_y = a\gamma h \frac{1+\rho^2}{2}\left[\frac{1+\rho^2}{2} \frac{\partial^2\psi}{\partial y^2} + y\frac{\partial\psi}{\partial y} - x\frac{\partial\psi}{\partial x}\right] + a\gamma h\ \psi$$

$$N_{xy} = N_{yx} = - a\gamma h \frac{1+\rho^2}{2}\left[\frac{1+\rho^2}{2} \frac{\partial^2\psi}{\partial x\partial y} + y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y}\right]$$

In terms of G and using the notation

$$\frac{\partial(\ )}{\partial x} = (\ ),_x \qquad 13\text{-}1$$

$$\frac{\partial(\ )}{\partial y} = (\ ),_y \qquad 13\text{-}2$$

obtain

$$N_x = - N_y = a\gamma h \frac{(1+\rho^2)^2}{4} (xG,_x + yG,_y + G),_{yy} \qquad 14\text{-}1$$

$$= - a\gamma h \frac{(1+\rho^2)^2}{4} (xG,_x + yG,_y + G),_{xx}$$

$$N_{xy} = N_{yx} = - a\gamma h \frac{(1+\rho^2)^2}{4} (xG,_x + yG,_y + G),_{xy} \qquad 14\text{-}2$$

Eqs. 3-3, 4, 5 are transformed into

$$u_x = - (1+\nu)\frac{a^2\gamma}{E} \frac{1+\rho^2}{2} \left(xG,_x + yG,_y + \frac{1-\rho^2}{1+\rho^2} G\right),_x \qquad 14\text{-}3$$

$$u_y = - (1+\nu)\frac{a^2\gamma}{E} \frac{1+\rho^2}{2} \left(xG,_x + yG,_y + \frac{1-\rho^2}{1+\rho^2} G\right),_y \qquad 14\text{-}4$$

$$w = - (1+\nu)\frac{a^2\gamma}{E} \left(xG,_x + yG,_y + \frac{1-\rho^2}{1+\rho^2} G\right) \qquad 14\text{-}5$$

The harmonic function G is sought in the form

$$G = \sum_{n=0}^{\infty} (g_n P_n + h_n Q_n) \qquad 15$$

where P_n and Q_n are the harmonic polynomials defined in eq. 11 and g_n and h_n are undetermined constants. The negative values of n yield harmonic functions with a singularity at the origin and need not be considered. It is easily shown using the defining equation of the harmonic polynomials that these satisfy the relations

$$P_n(-x, y) = (-1)^n P_n(x, y) \qquad 16\text{-}1 \qquad Q_n(-x, y) = -(-1)^n Q_n(x, y) \qquad 16\text{-}3$$

$$Q_n(x, -y) = P_n(x, y) \qquad 16\text{-}2 \qquad Q_n(x, -y) = -Q_n(x, y) \qquad 16\text{-}4$$

$$P_{n,x} = Q_{n,y} = nP_{n-1} \qquad 17\text{-}1$$

$$P_{n,y} = -Q_{n,x} = -nQ_{n-1} \qquad 17\text{-}2$$

The symmetry of the shell and of the loading with regard to the vertical planes passing through the x and y axes and the form of eqs. 14 require G to be an even function of x and y. From eqs. 16 there follows

$h_n = 0$ for all n's

$g_n = 0$ for $n = 1, 3, 5, \ldots$

and G takes the form

$$G = \sum_{n=0}^{\infty} g_n P_n \qquad n = 0, 2, 4, \ldots \qquad 18$$

For $n = 0$, $P_o = 1$. The corresponding term contributes zero stress resultants and a vertical rigid body displacement. Noting that P_n is a homogeneous function of x and y of degree n we can write Euler's relation

$$xP_{n,x} + yP_{n,y} = nP_n \qquad 19$$

Eqs. 14 take then the form

$$N_x = -N_y = -a\gamma h \frac{(1+\rho^2)^2}{4} \sum_{n=2,4}^{\infty} g_n (n+1)n(n-1)P_{n-2} \qquad 20\text{-}1$$

$$N_{xy} = N_{yx} = a\gamma h \frac{(1+\rho^2)^2}{4} \sum_{n=2,4}^{\infty} g_n (n+1)n(n-1)Q_{n-2} \qquad 20\text{-}2$$

$$u_x = -(1+\nu)\frac{a^2\gamma}{E}\frac{1+\rho^2}{2} \sum_{n=2,4}^{\infty} g_n \left[n(n + \frac{1-\rho^2}{1+\rho^2})P_{n-1} - \frac{4x\,P_n}{(1+\rho^2)^2} \right] \qquad 20\text{-}3$$

$$+ (1+\nu)\frac{a^2\gamma}{E}\frac{2x\,g_o}{1+\rho^2}$$

$$u_y = (1+\nu)\frac{a^2\gamma}{E}\frac{1+\rho^2}{2} \sum_{n=2,4}^{\infty} g_n \left[n(n + \frac{1-\rho^2}{1+\rho^2})\,Q_{n-1} + \frac{4yP_n}{(1+\rho^2)^2} \right] \qquad 20\text{-}4$$

$$+ (1+\nu)\frac{a^2\gamma}{E}\frac{2y\,g_o}{1+\rho^2}$$

$$w = -(1+\nu)\frac{a^2\gamma}{E} \sum_{n=2,4}^{\infty} g_n (n + \frac{1-\rho^2}{1+\rho^2})P_n - (1+\nu)\frac{a^2\gamma}{E}\frac{1-\rho^2}{1+\rho^2}\,g_o \qquad 20\text{-}5$$

It is convenient to let

$$C_{n-2} = g_n (n+1)n(n-1) \qquad 21$$

and to non-dimensionalize the stress resultants in the form

$$f_x = -f_y = \frac{4N_x}{a\gamma h(1+\rho^2)^2} \qquad 22\text{-}1$$

$$f_{xy} = \frac{4N_{xy}}{a\gamma h(1+\rho^2)^2} \qquad 22\text{-}2$$

then

$$f_x = -f_y = \sum_{n=0,2}^{\infty} -C_n P_n \qquad 23\text{-}1$$

$$f_{xy} = \sum_{n=0,2}^{\infty} C_n Q_n \qquad 23\text{-}2$$

4. <u>The Particular Solution</u>

A simple particular solution is that of a shell with a circular boundary, Fig. 5, for which the stress resultants and displacements may be written in the form

$$N_\xi = -\frac{a\gamma h}{1+\cos\xi} \qquad 24\text{-}1$$

$$N_\theta = a\gamma h\left(\frac{1}{1+\cos\xi} - \cos\xi\right) \qquad 24\text{-}2$$

$$N_{\xi\theta} = 0 \qquad 24\text{-}3$$

$$u_\xi = -\frac{(1+\nu)a^2\gamma}{E}\sin\xi\left(\frac{1}{2}\tan^2\frac{\xi}{2} - \log 2\cos^2\frac{\xi}{2}\right) + U_o\sin\xi \qquad 24\text{-}4$$

$$u_\theta = 0 \qquad 24\text{-}5$$

$$w = -\frac{(1+\nu)a^2\gamma}{E}\left(\frac{3}{2}\cos\xi - 1 + \cos\xi\log 2\cos^2\frac{\xi}{2}\right) + \frac{\nu a^2\gamma}{E}\cos\xi - \qquad 24\text{-}6$$

$$U_o\cos\xi$$

where U_o is an arbitrary vertical rigid body displacement.

N_x, N_y, N_{xy}, u_x and u_y may be obtained from eqs. 24 through the transformation formulas.

$$N_x = N_\xi \cos^2\theta + N_\theta \sin^2\theta$$

$$N_y = N_\xi \sin^2\theta + N_\theta \cos^2\theta$$

$$N_{xy} = (N_\xi - N_\theta) \sin\theta \cos\theta$$

$$u_x = u_\xi \cos\theta$$

$$u_y = u_\xi \sin\theta$$

Performing the above transformation and using eqs. 6 and 9 to express the result in terms of x and y obtain

$$N_x = -\frac{a\gamma h}{2}\left(x^2 - y^2 - 3 + 4\,\frac{1 + x^2}{1 + x^2 + y^2}\right) \qquad 25\text{-}1$$

$$N_y = -\frac{a\gamma h}{2}\left(y^2 - x^2 - 3 + 4\,\frac{1 + y^2}{1 + x^2 + y^2}\right) \qquad 25\text{-}2$$

$$N_{xy} = -a\gamma h\,xy\left(1 + \frac{2}{1 + x^2 + y^2}\right) \qquad 25\text{-}3$$

$$u_x = -(1 + \nu)\frac{a^2\gamma}{E}\,\frac{x}{1 + \rho^2}\left(\rho^2 + \frac{1}{2}\log\frac{1 + \rho^2}{2}\right) + \frac{2x}{1 + \rho^2}\,U_o \qquad 26\text{-}1$$

$$u_y = -(1 + \nu)\frac{a^2\gamma}{E}\,\frac{y}{1 + \rho^2}\left(\rho^2 + \frac{1}{2}\log\frac{1 + \rho^2}{2}\right) + \frac{2y}{1 + \rho^2}\,U_o \qquad 26\text{-}2$$

$$w = -(1 + \nu)\frac{a^2\gamma}{E}\left[\frac{1 - 5\rho^2}{2(1 + \rho^2)} - \frac{1 - \rho^2}{1 + \rho^2}\log\frac{1 + \rho^2}{2}\right] + \frac{1 - \rho^2}{1 + \rho^2}\left(\frac{\nu a^2\gamma}{E} - U_o\right)$$

26-3

The non-dimensional stress resultants f_x, f_y, and f_{xy} are defined as for the homogeneous solution through the relations

$$f_x = \frac{4N_x}{a\gamma h(1+\rho^2)^2} = 2\frac{y^2+3-x^2}{(1+x^2+y^2)^2} - 8\frac{1+x^2}{(1+x^2+y^2)^3} \tag{27-1}$$

$$f_y = \frac{4N_y}{a\gamma h(1+\rho^2)^2} = 2\frac{x^2+3-y^2}{(1+x^2+y^2)^2} - 8\frac{1+y^2}{(1+x^2+y^2)^3} \tag{27-2}$$

$$f_{xy} = \frac{4N_{xy}}{a\gamma h(1+\rho^2)^2} = -\frac{4xy}{(1+x^2+y^2)^2}\left(1+\frac{2}{1+x^2+y^2}\right) \tag{27-3}$$

5. Stress Boundary Condition

The normal stress resultant at the boundary obtained by superimposing the homogeneous and the particular membrane solutions must be zero. Due to symmetry it will be sufficient to satisfy the boundary condition along part ABC of the boundary, Fig. 1. Using eqs. 23-1 and 27-1,2 the boundary condition takes the form

$$\sum_{n=0,2}^{\infty} C_n P_n = \begin{cases} 2\dfrac{y^2+3-b_1^2}{(1+b_1^2+y^2)^2} - 8\dfrac{1+b_1^2}{(1+b_1^2+y^2)^3} & 0 \le y \le b_2 \\[2ex] -2\dfrac{x^2+3-b_2^2}{(1+b_2^2+x^2)^2} + 8\dfrac{1+b_2^2}{(1+b_2^2+x^2)^3} & 0 \le x \le b_1 \end{cases} \tag{28}$$

The constants C_n may be computed as Fourier coefficients of the expansion in terms of P_n of the right hand side of eq. 28 considered as a function of position along the boundary. Denoting this function by f(s) we obtain for the determination of the constants C_n an infinite system of linear algebraic equations of the form

$$\sum_{n=0,2}^{\infty} C_n \int_A^C P_n P_k \, ds = \int_A^C f P_k \, ds \qquad k = 0, 2, \ldots \tag{29}$$

where the integrals are taken along the boundary between points A and C. Let

$$R_{kn} = \int_A^C P_k P_n \, ds \tag{30}$$

$$S_k = \int_A^C f \, P_k \, ds \tag{31}$$

Eqs. 29 may then be written in the form

$$\sum_{n=0,2}^{\infty} R_{kn} C_n = S_k \qquad k = 0, 2, \ldots \tag{32}$$

The coefficient matrix R_{kn} is symmetric as may be seen through its definition, eq. 30. Questions pertaining to the existence of a solution of an infinite system of equations such as 32 are discussed in ref. 2. For the present problem it is deemed sufficient to solve the first N equations in the first N unknowns and to examine critically the results in the light of their physical meaning. An improvement on the method of solution described here may be made by taking account separately of the discontinuity at the corner B of the right hand side of eq. 28.

6. Computation

a) Coefficient matrix

From eq. 30 R_{kn} may be computed according to

$$R_{kn} = I_{kn} + J_{kn} \tag{33}$$

where

$$I_{kn} = \int_0^{b_1} P_k(x, b_2) \, P_n(x, b_2) \, dx \tag{34-1}$$

$$J_{kn} = \int_0^{b_2} P_k\,(b_1, y)\; P_n\,(b_1, y)\; dy \qquad 34\text{-}2$$

Recurrence formulas for the computation of I_{kn} and J_{kn} are established through integration by parts and use of eqs. 17

$$J_{kn} = \int_0^{b_1} P_k P_n\, dx = \frac{P_k P_{n+1}}{n+1}\Bigg|_0^{b_1} - \frac{k}{n+1}\int_0^{b_1} P_{k-1}\, P_{n+1}\, dx$$

$$= \frac{P_k P_{n+1}}{n+1}\Bigg|_0^{b_1} - \frac{k}{(n+1)(n+2)}\, P_{k-1} P_{n+2}\Bigg|_0^{b_1} + \frac{k(k-1)}{(n+1)(n+2)}\int_0^{b_1} P_{k-2} P_{n+2}\, dx$$

letting

$$\pi_{kj} = P_i(b_1,\ b_2) P_j(b_1,\ b_2) \qquad 35\text{-}1$$

I_{kn} obeys the recurrence relation

$$I_{kn} = \frac{\pi_{k,\ n+1}}{n+1} - \frac{k}{(n+1)(n+2)}\ \pi_{k-1,\ n+2} + \frac{k(k-1)}{(n+1)(n+2)}\ I_{k-2,\, n+2} \qquad 36\text{-}1$$

$(k, n = 2, 4, \ldots)$

A relation for J_{kn} is established similarly

$$J_{kn} = \frac{\Phi_{k,\ n+1}}{n+1} - \frac{k}{(n+1)(n+2)}\ \Phi_{n+2,\ k-1} + \frac{k(k-1)}{(n+1)(n+2)}\ J_{k-2,\ n+2} \qquad 36\text{-}2$$

$(k, n = 2, 4, \ldots)$

where

$$\Phi_{ij} = P_i(b_1,\ b_2)\ Q_j(b_1,\ b_2) \qquad 37$$

Thus R_{kn} satisfies the recurrence relation

$$R_{kn} = \frac{\Pi_{k,\ n+1} + \Phi_{k,\ n+1}}{n+1} - \frac{k(\Pi_{k-1,\ n+2} + \Phi_{n+2,\ k-1})}{(n+1)(n+2)} + \frac{k(k-1)R_{k-2,\ n+2}}{(n+1)(n+2)} \qquad 38$$

The computation of the harmonic polynomials is made through their expressions in polar coordinates, eq. 11. Then

$$\Pi_{ij} = \rho_o^{i+j} \cos i\,\theta_o \cos j\,\theta_o \qquad 39\text{-}1$$

$$\Phi_{ij} = \rho_o^{i+j} \cos i\,\theta_o \sin j\,\theta_o \qquad 39\text{-}2$$

where

ρ_o, θ_o are the polar coordinates of the corner B, Fig. 3. For $k = 0$, $P_k = 1$. Eq. 30 yields

$$R_{on} = \frac{P_{n+1}(b_1,\ b_2) + Q_{n+1}\ (b_1,\ b_2)}{n+1} = \rho_o^{n+1}\ \frac{\cos(n+1)\,\theta_o + \sin(n+1)\,\theta_o}{n+1} \qquad 40$$

The first row of the R_{kn} matrix i.e. R_{on} may be computed through formula 40. By using the recurrence relation 38 on the remaining elements may be computed according to the scheme shown in Fig. 6.

A check on the round off error propagation in the use of the recurrence relation is afforded by continuing the computation beyond the main diagonal and examining the symmetry of the matrix obtained. If an R_{kn} matrix of N x N elements is desired the method of computing R_{kn} described above requires a number of elements in the first row equal to 2N-1. This may be avoided by computing directly N-1 elements in the last column of the matrix and using them together with the N elements of the first row to start the recursive computation. A formula for the direct computation of I_{kn} may be obtained by using successively the result of the first integration by parts that lead to eq. 36-1. The result is

$$I_{kn} = \sum_{i=1}^{k+1} (-1)^{i+1} p_{ikn} \Gamma_{k-i+1,\ n+i} \qquad 41$$

where

$$p_{ikn} = \frac{k!\,n!}{(k-i+1)!\,(n+i)!} \qquad 42$$

Similarly J_{kn} is obtained in the form

$$J_{kn} = \sum_{i=1,3}^{k+1} p_{ikn} \Phi_{k-i+1,\ n+i} - \sum_{i=2,4}^{k} p_{ikn} \Phi_{n+i,\ k-i+1} \qquad 43$$

b) Constant vector

The computation of

$$S_k = \int_A^C fP_k \, ds$$

where f is the right hand side of eq. 28, by exact methods, is theoretically possible but becomes extremely tedious as k increases. The method used below consists in replacing f by series developments in y and x along the boundary sides AB and AC, respectively. These series are of the form

$$f = -\sum_{n=0,2}^{\infty} A_n y^n \qquad x = b_1 \quad , \quad 0 \leq y \leq b_2 \qquad 44\text{-}1$$

$$f = \sum_{n=0,2}^{\infty} B_n x^n \qquad y = b_2 \quad , \quad 0 \leq x \leq b_1 \qquad 44\text{-}2$$

The series converge in their respective domains if

$$\frac{y^2}{1 + b_1^2} < 1 \qquad \text{and} \qquad \frac{x^2}{1 + b_2^2} < 1 \qquad 45$$

i.e., choosing $b_2 > b_1$, if

$$\frac{b_2^2}{1 + b_1^2} < 1$$

The above inequality is certainly satisfied if the corners of the shell are above the equator since in that case b_1, $b_2 < 1$. S_k is computed through the relations

$$S_k = \sum_{n=0,2}^{\infty} (B_n K_{kn} - A_n L_{kn}) \qquad 46$$

where

$$K_{kn} = \int_0^{b_1} P_k(x, b_2)\, x^n\, dx \qquad 47\text{-}1$$

$$L_{kn} = \int_0^{b_2} P_k(b_1, y)\, y^n\, dy \qquad 47\text{-}2$$

A_n and B_n are obtained in the form

$$A_n = (-1)^{\frac{n}{2}}\, 2\, \frac{(1 + b_1^2)(n+1) + n(\frac{n}{2} + 1)}{(1 + b_1^2)^{\frac{n}{2}+2}} \qquad 48\text{-}1$$

$$B_n = (-1)^{\frac{n}{2}}\, 2\, \frac{(1+b_2^2)(n+1) + n(\frac{n}{2} + 1)}{(1 + b_2^2)^{\frac{n}{2}+2}} \qquad 48\text{-}2$$

Recurrence relations for the computation of K_{kn} and L_{kn} may be obtained by integration by parts. They have the same coefficients as eqs. 36.

$$K_{kn} = \frac{\Gamma_{k,\ n+1}}{n+1} - \frac{k}{(n+1)(n+2)}\,\Gamma_{k-1,\ n+2} + \frac{k(k-1)}{(n+1)(n+2)}\,K_{k-2,\ n+2} \qquad 49\text{-}1$$

$$(k, n = 2, 4, \ldots)$$

$$L_{kn} = \frac{\Delta_{k,\ n+1}}{n+1} + \frac{k}{(n+1)(n+2)}\,\Delta'_{k-1,\ n+2} - \frac{k(k-1)}{(n+1)(n+2)}\,L_{k-2,\ n+2} \qquad 49\text{-}2$$

$$(k,\ n = 2, 4, \ldots)$$

where

$$\Gamma_{kn} = P_k(b_1,\ b_2)b_1^n \qquad 50\text{-}1$$

$$\Delta_{kn} = P_k(b_1,\ b_2)b_2^n \qquad 50\text{-}2$$

$$\Delta'_{kn} = Q_k(b_1,\ b_2)b_2^n \qquad 50\text{-}3$$

For $k = 0$ obtain

$$K_{on} = \frac{b_1^{n+1}}{n+1} \qquad 51\text{-}1$$

$$L_{on} = \frac{b_2^{n+1}}{n+1} \qquad 51\text{-}2$$

Formulas for the direct computation of K_{kn} and L_{kn} may be established similarly to eqs. 41 and 43.

$$K_{kn} = \sum_{i=1}^{k+1} (-1)^{i+1}\ p_{ikn}\,\Gamma_{k-i+1,\ n+i}$$

$$L_{kn} = \sum_{i=1,3}^{k+1} (-1)^{\frac{i-1}{2}}\ p_{ikn}\,\Delta_{k-i+1,\ n+i} + \sum_{i=2,4}^{k} (-1)^{\frac{i-2}{2}}\ p_{ikn}\,\Delta'_{k-i+1,\ n+i} \qquad 52\text{-}2$$

7. Determination of the Displacements, Inextensional Bending.

The inextensional solution of a spherical shell may be expressed in terms of a stress function ψ through the relations (1).

$$M_{\xi} = - M_{\theta} = - \frac{\gamma h^3}{24} (\psi + \psi'') \qquad 53\text{-}1$$

$$M_{\xi\theta} = M_{\theta\xi} = - \frac{\gamma h^3}{24} \left(\frac{\psi}{\sin \xi}\right)' \qquad 53\text{-}2$$

where ψ satisfies the same differential equation as the function ψ of the membrane solution. The displacements and rotations are expressed in terms of the real and imaginary parts F_r and F_i of an analytic function of $\rho e^{i\theta}$, $F(\rho e^{i\theta})$, related to ψ through the relation

$$\psi = (F_r \sin \xi)' \qquad 54$$

Comparing eq. 54 to eq. 4 it is seen that F_r plays the same role as the harmonic function G in the membrane solution. The displacements and rotations are expressed in the form (1)

$$u_{\xi} = - (1 + \nu) \frac{a^2\gamma}{2E} \sin\xi \, F_r \qquad 55\text{-}1$$

$$u_{\theta} = - (1 + \nu) \frac{a^2\gamma}{2E} \sin \xi \, F_i \qquad 55\text{-}2$$

$$w = (1 + \nu) \frac{a^2\gamma}{2E} \psi = (1 + \nu) \frac{a^2\gamma}{2E} (\sin\xi \, F_r)' \qquad 55\text{-}3$$

$$\beta_{\xi} = \frac{1}{a} (u_{\xi} - w') \qquad 55\text{-}4$$

$$\beta_{\theta} = \frac{1}{a} \left(u_{\theta} - \frac{w}{\sin \xi}\right) \qquad 55\text{-}5$$

In the systems of coordinates x and y eq. 54 and eqs. 53 take the form

$$\Psi = xF_{r,x} + yF_{r,y} + \frac{1-\rho^2}{1+\rho^2} F_r \qquad 56$$

$$M_x = -M_y = \frac{\gamma h^3}{96} (1 + \rho^2)^2 \left[xF_{r,x} + yF_{r,y} + F_r \right]_{,yy}$$

$$= -\frac{\gamma h^3}{96} (1 + \rho^2)^2 \left[xF_{r,x} \, X \, yF_{r,y} + F_r \right]_{,xx} \qquad 57\text{-}1$$

$$M_{xy} = M_{yx} = -\frac{\gamma h^3}{96} (1 + \rho^2)^2 \left[xF_{r,x} + F_{r,y} + F_r \right]_{,xy} \qquad 57\text{-}2$$

The displacements and rotation components in the directions x and y are obtained in the form

$$u_x = -(1 + \nu) \frac{\gamma a^2}{E} \frac{1}{1+\rho^2} (xF_r - yF_i) \qquad 58\text{-}1$$

$$u_y = -(1 + \nu) \frac{\gamma a^2}{E} \frac{1}{1+\rho^2} (yF_r + xF_i) \qquad 58\text{-}2$$

$$w = (1 + \nu) \frac{\gamma a^2}{2E} (xF_{r,x} + yF_{r,y} + \frac{1-\rho^2}{1+\rho^2} F_r) \qquad 58\text{-}3$$

$$\beta_x = \frac{1}{a} (u_x - \frac{1+\rho^2}{2} w,_x) \qquad 58\text{-}4$$

$$\beta_y = \frac{1}{a} (u_y - \frac{1+\rho^2}{2} w,_y) \qquad 58\text{-}5$$

The analytic function $F = F_r + iF_i$ is sought in the form

$$F = \sum_{n=0}^{\infty} f_n (x + iy)^n = \sum_{n=0}^{\infty} f_n (P_n + iQ_n) \qquad 59$$

where P_n and Q_n are the harmonic polynomials defined earlier and the f_n's are arbitrary complex constants. Considerations of symmetry require F_r to be an even function in x and y and F_i to be an odd function in x and y. This is satisfied if the constants f_n are real and if n is restricted to even values. F_r and F_i take then the form

$$F_r = \sum_{n=0,2}^{\infty} f_n P_n \qquad 60\text{-}1$$

$$F_i = \sum_{n=0,2}^{\infty} f_n Q_n \qquad 60\text{-}2$$

Taking account of the relation $(x + iy)(P_n + iQ_n) = P_{n+1} + iQ_{n+1}$, eqs. 57 and 58-1, 2, 3 take the form

$$M_x = - M_y = - \frac{\gamma h^3}{96} (1 + \rho^2)^2 \sum_{n=2,4}^{\infty} f_n (n+1)n(n-1) P_{n-2} \qquad 61\text{-}1$$

$$M_{xy} = M_{yx} = \frac{\gamma h^3}{96} (1 + \rho^2)^2 \sum_{n=2,4}^{\infty} f_n (n+1)n(n-1) Q_{n-2} \qquad 61\text{-}2$$

$$u_x = - (1 + \nu) \frac{\gamma a^2}{E} \frac{1}{1 + \rho^2} \sum_{n=2,4}^{\infty} f_n P_{n+1} - (1 + \nu) \frac{\gamma a^2}{E} \frac{x f_o}{1+\rho^2} \qquad 61\text{-}3$$

$$u_y = - (1 + \nu) \frac{\gamma a^2}{E} \frac{1}{1 + \rho^2} \sum_{n=2,4}^{\infty} f_n Q_{n+1} - (1 + \nu) \frac{\gamma a^2}{E} \frac{y f_o}{1 + \rho^2} \qquad 61\text{-}4$$

$$w = (1 + \nu) \frac{\gamma a^2}{2E} \sum_{n=2,4}^{\infty} f_n \left(n + \frac{1 - \rho^2}{1 + \rho^2}\right) P_n + (1 + \nu) \frac{\gamma a^2}{2E} \frac{1 - \rho^2}{1 + \rho^2} f_o \qquad 61\text{-}5$$

The constant f_o contributes a vertical rigid body translation.

The constants f_n are determined by satisfying the condition that the component of displacement tangent to the boundary and obtained through superposition of the membrane and inextensional solution is zero.

8. Results

Numerical results for the 4 cases described below are plotted in Figs. 7, 8, 9, 10. Two of the shells have rectangular stereographic projections of same proportions but different sizes. The other two have square projections of different sizes. The meridional angles at points A, B, and C, Fig. 3, are denoted by ξ_A, ξ_B and ξ_C, respectively. They are chosen such that 2 of the shells are relatively deep and the other 2 are relatively shallow.

Case 1, Fig. 7 — Deep shell with rectangular projection

$b_1 = 0.50$, $b_2 = 0.75$

$\xi_A = 53.1°$, $\xi_B = 84.1°$, $\xi_C = 73.7°$

Case 2, Fig. 8 — Shallow shell with rectangular projection

$b_1 = 0.10$, $b_2 = 0.15$

$\xi_A = 11.4°$, $\xi_B = 20.4°$, $\xi_C = 17.1°$

Case 3, Fig. 9 — Deep shell with square projection

$b_1 = b_2 = 0.4082$

$\xi_A = \xi_C = 44.4°$, $\xi_B = 60°$

Case 4, Fig. 10 — Shallow shell with square projection

$b_1 = b_2 = 0.1895$

$\xi_A = \xi_C = 21.5°$, $\xi_B = 30°$

The in-plane stress resultants N_x, N_y and N_{xy} are plotted as functions of x and y along lines y = constant and x = constant, respectively. For the two shells with a square projection only N_x and N_{xy} are plotted.

The number of simultaneous equations used in the four cases is N = 20. The number of terms A_i and B_i used in the computation of the right hand side of each equation is M = 30. Fewer terms are actually needed because with M = 30, 16 significant digits were obtained.

A check on the global equilibrium of the shell and of rectangular cut-outs symmetrical with regard to the x and y axes was performed. Due to symmetry only vertical equilibrium was checked by comparing the weight of the shell or of the cut out with the vertical resultant of the boundary stress resultants. The results are shown in the following table where the numbers are coefficients of $\gamma a^2 h$.

		Weight of Shell	Vertical Reaction
Case 1	Entire shell	3.940	3.952
	Cut at $x = \pm 0.30$, $y = \pm 0.45$	1.812	1.813
	Cut at $x = \pm 0.25$, $y = \pm 0.225$	0.837	0.837
Case 2	Entire shell	0.235	0.242
	Cut at $x = \pm 0.08$, $y = \pm 0.105$	0.133	0.133
	Cut at $x = \pm 0.05$, $y = \pm 0.09$	0.0715	0.0715
Case 3	Entire shell	2.185	2.211
	Cut at x, $y = \pm 0.245$	0.889	0.890
	Cut at $x = \pm 0.204$, $y = \pm 0.326$	0.972	0.972
Case 4	Entire shell	0.548	0.562
	Cut at $x = \pm 0.152$, $y = \pm 0.057$	0.136	0.136
	Cut at $x = \pm 0.114$, $y = \pm 0.095$	0.170	0.170

In the computation of the elements of the coefficient matrix R and of the constant vector S, eq. 32, special attention was given to the propagation of round off errors caused by the repeated use of recurrence relations 38, 49-1, and 49-2. In the upper half of the R matrix the coefficients of the recurrence relation 38 are of order of magnitude unity and round-off errors are negligible. In the lower half, which is symmetrical of the upper half, the coefficients of the recurrence relation increase significantly beyond unity and cause "instability" of the computation. While this is of no consequence for the computation of the symmetrical R matrix, the same behavior of the recurrence relations in the computation of the S vector through the K and L matrices, eq. 46, will make meaningless the numerical value of S_k for k large enough. The same behavior takes place if the direct relations 52 are used. In that case a small sum is computed by combining numbers of large magnitude. For the case of 20 equations round off errors are negligible with regard to the truncation error. If, however, 40 equations were to be used a different procedure for the computation of the S vector would be necessary. Numerical integration with a proper estimate of the truncation error should be a practical and accurate method. The number of simultaneous equations used is equal to the number of polynomials P_n used to represent the boundary function f(s) of the boundary value problem, eq. 28. This provides a criterion for a rational choice of the number of equations. It also makes apparent the fact that the discontinuity of f(s) at the corner B is a cause of slow convergence. This singularity is expected however to be of a weak nature and could be analytically isolated from the boundary value problem. Finally, numerical results for the displacements were not available at the time of writing. The inextensional bending is expected to be negligible. It should be of interest however to study its behavior and to compare the interior solution obtained in this report with an interior solution taking account more realistically of the stiffness of the edge beams.

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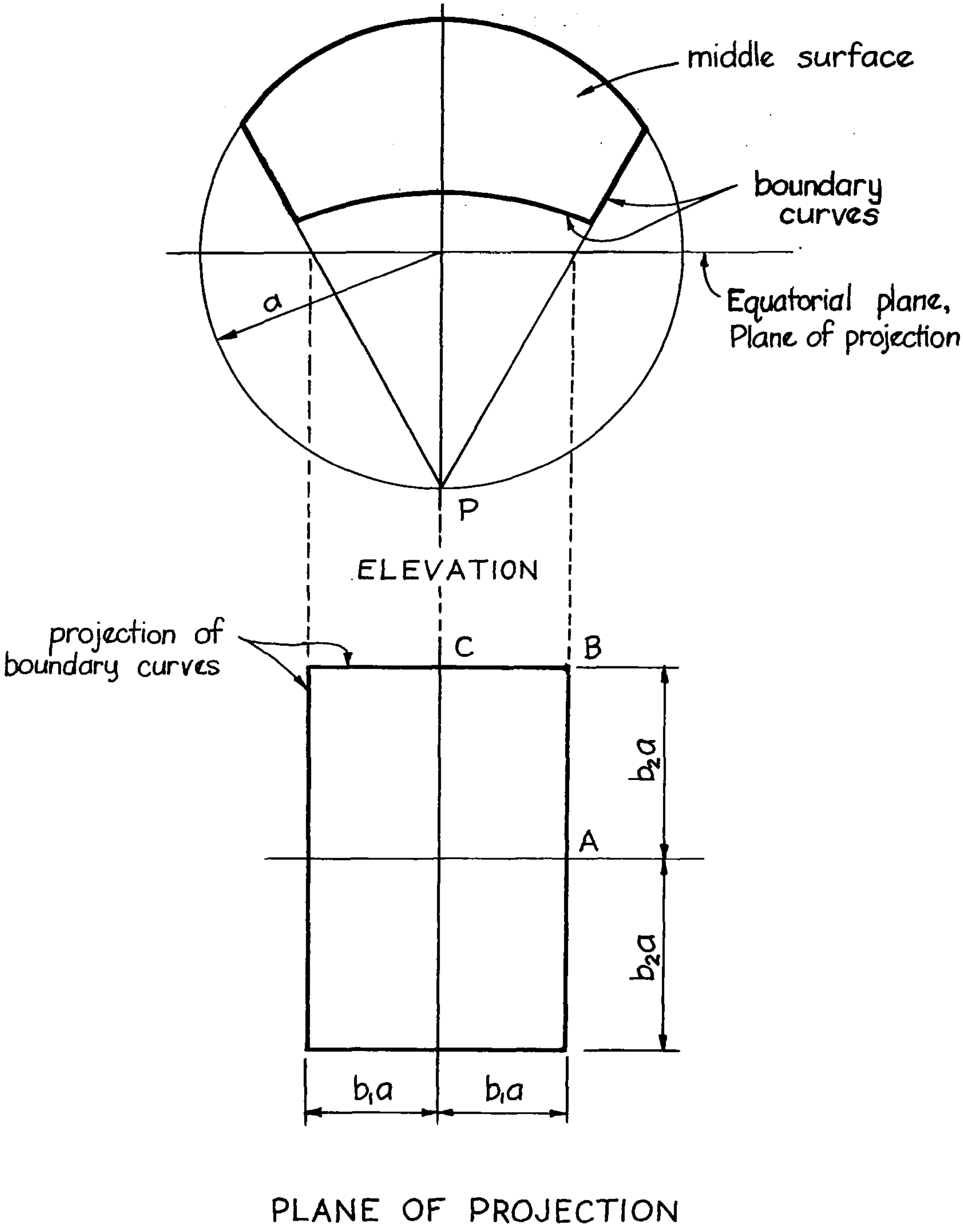
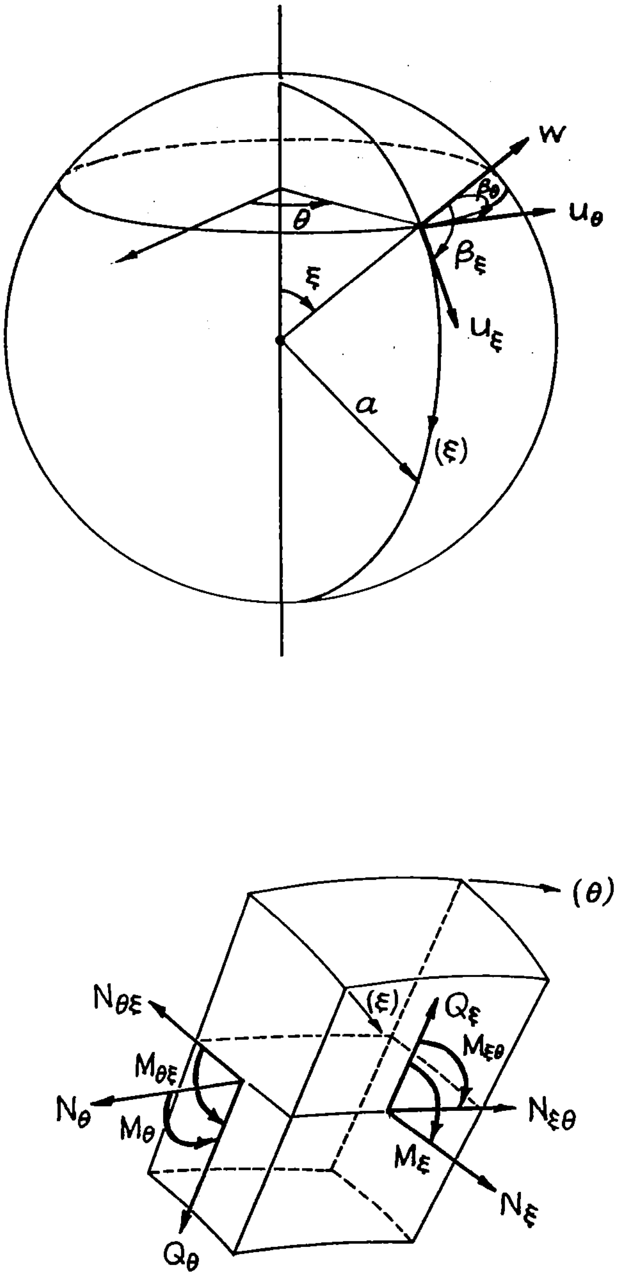
2. Kantorovitch and Krylov, "Approximate Methods of Higher Analysis", P. Noordhoff, 1958.



middle surface
boundary curves
a
Equatorial plane, Plane of projection
P
ELEVATION
projection of boundary curves
C
B
A
b_2a
b_2a
b_1a
b_1a
PLANE OF PROJECTION

FIG. 1


W
β_θ
u_θ
θ
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ξ
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a
(ξ)
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$M_{\theta\xi}$
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M_θ
Q_θ
(ξ)
Q_ξ
$M_{\xi\theta}$
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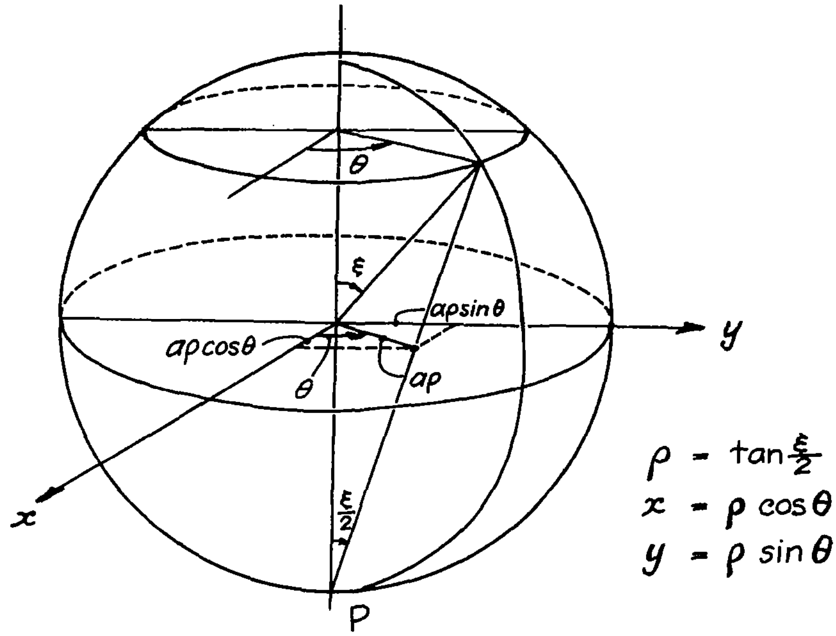
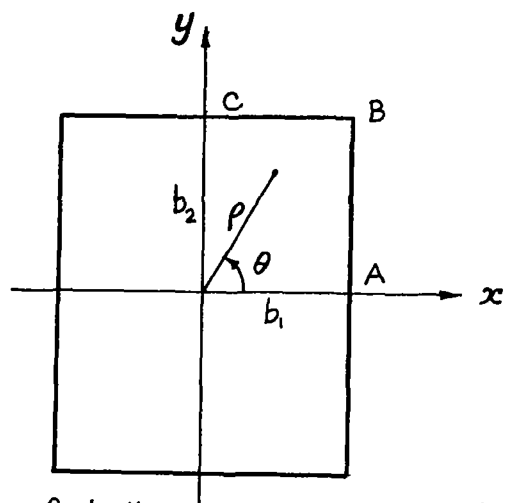
M_ξ
N_ξ

FIG. 2





Boundary of shell in x, y coordinates

FIG. 3

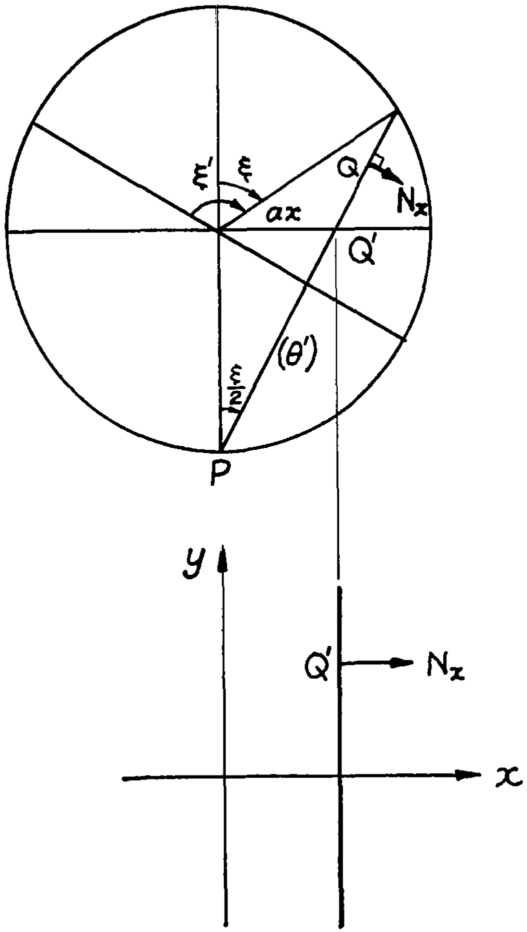


FIG. 4

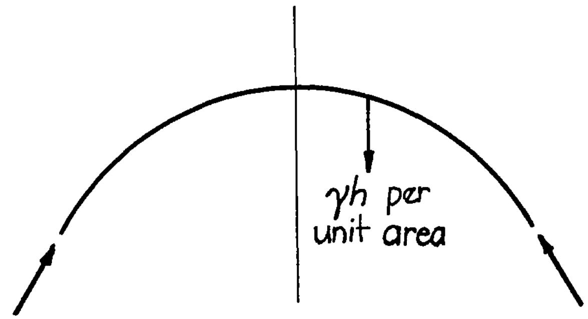


FIG. 5

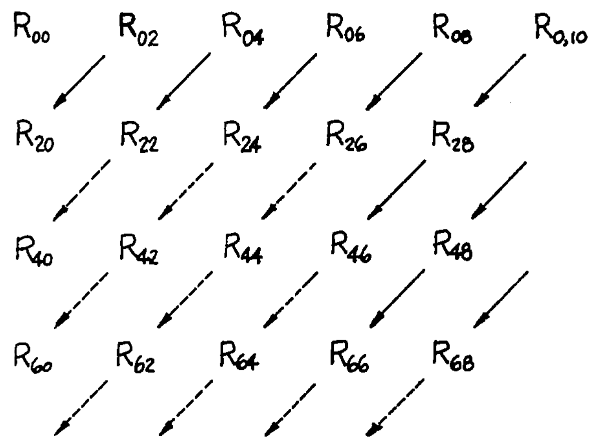


FIG. 6

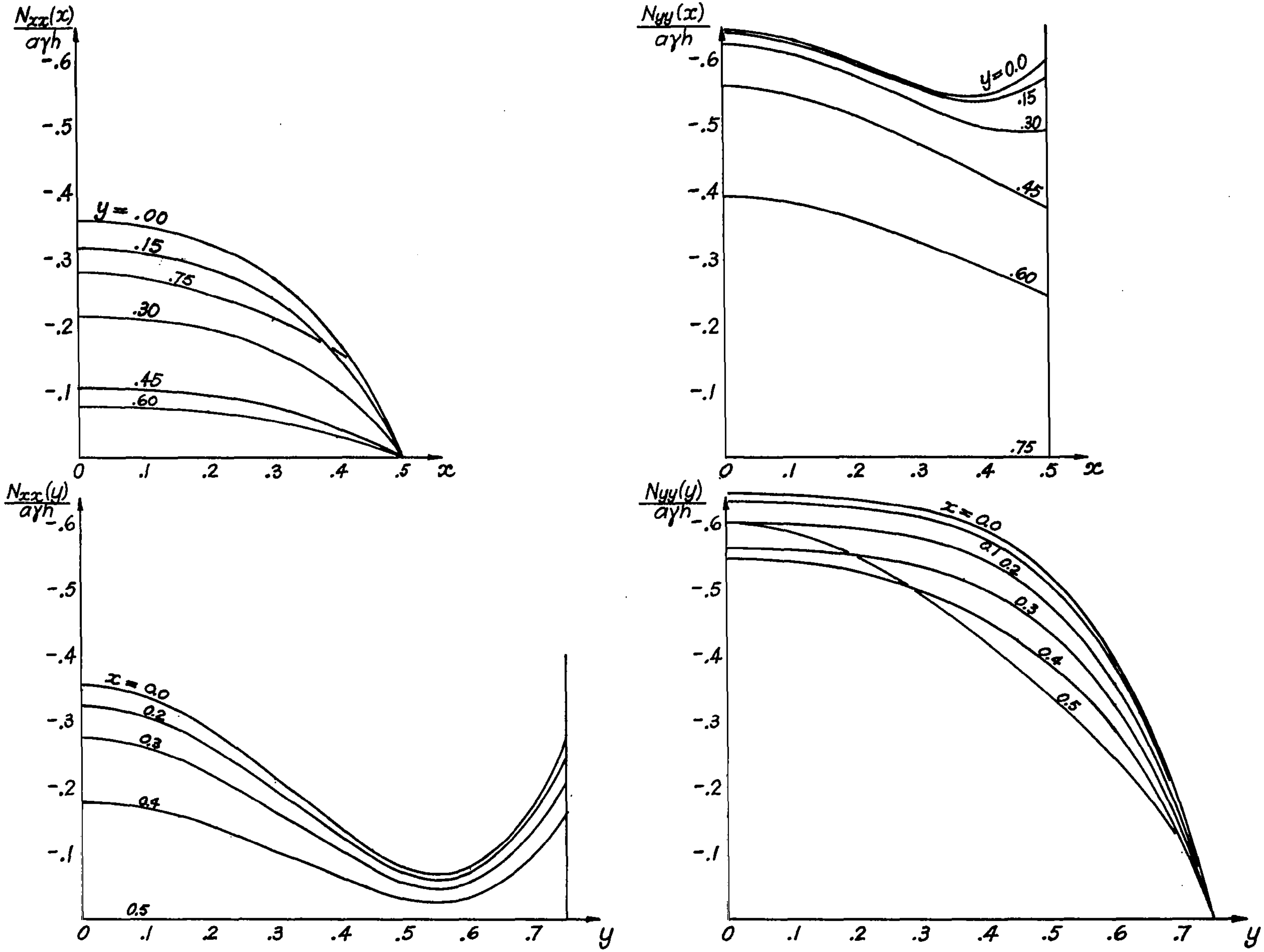


FIG. 7. Shell with $b_1 = 0.50$, $b_2 = 0.75$, $\xi_A = 53.1°$, $\xi_B = 84.1°$, $\xi_C = 73.7°$

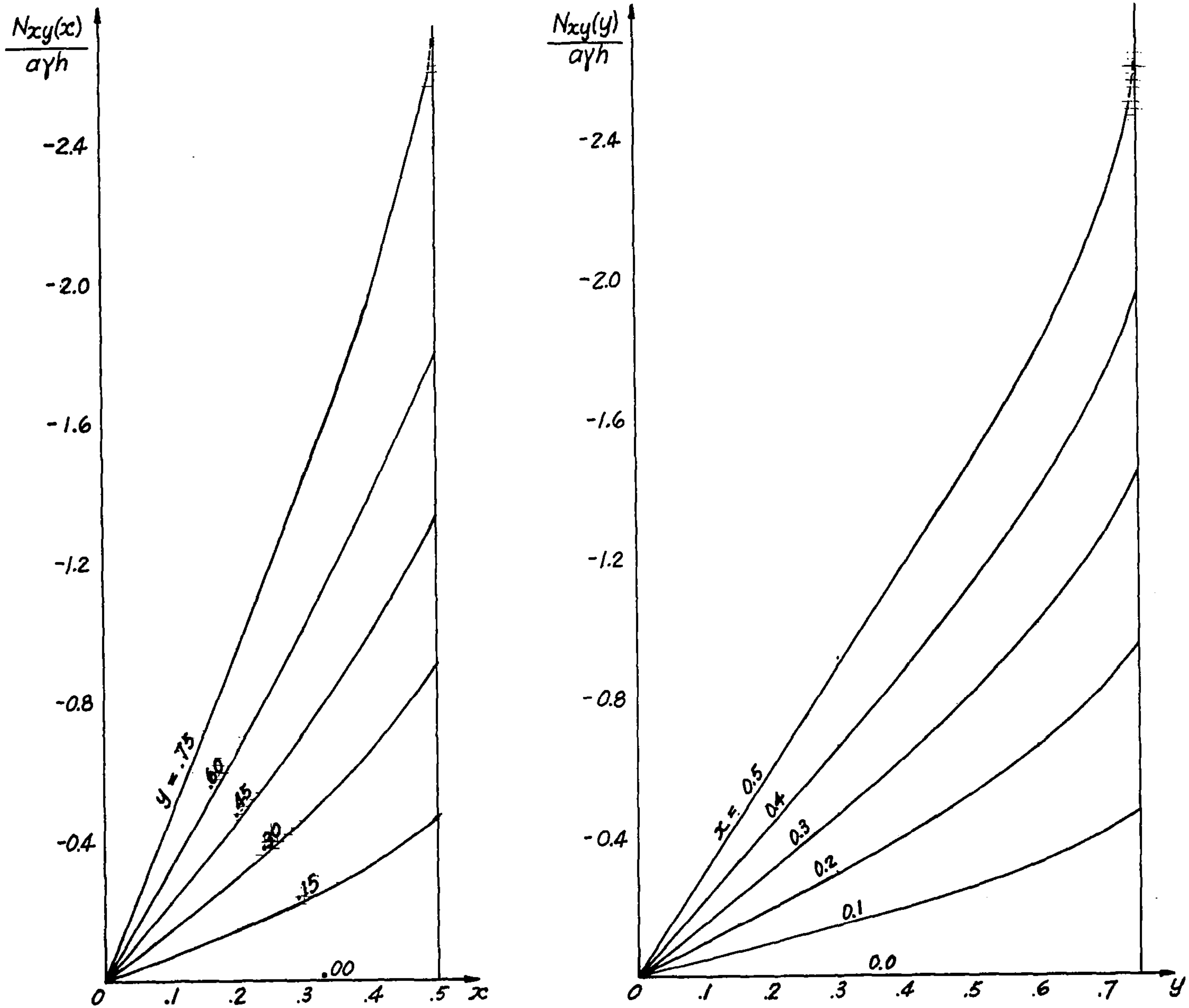


FIG. 7. Continued.

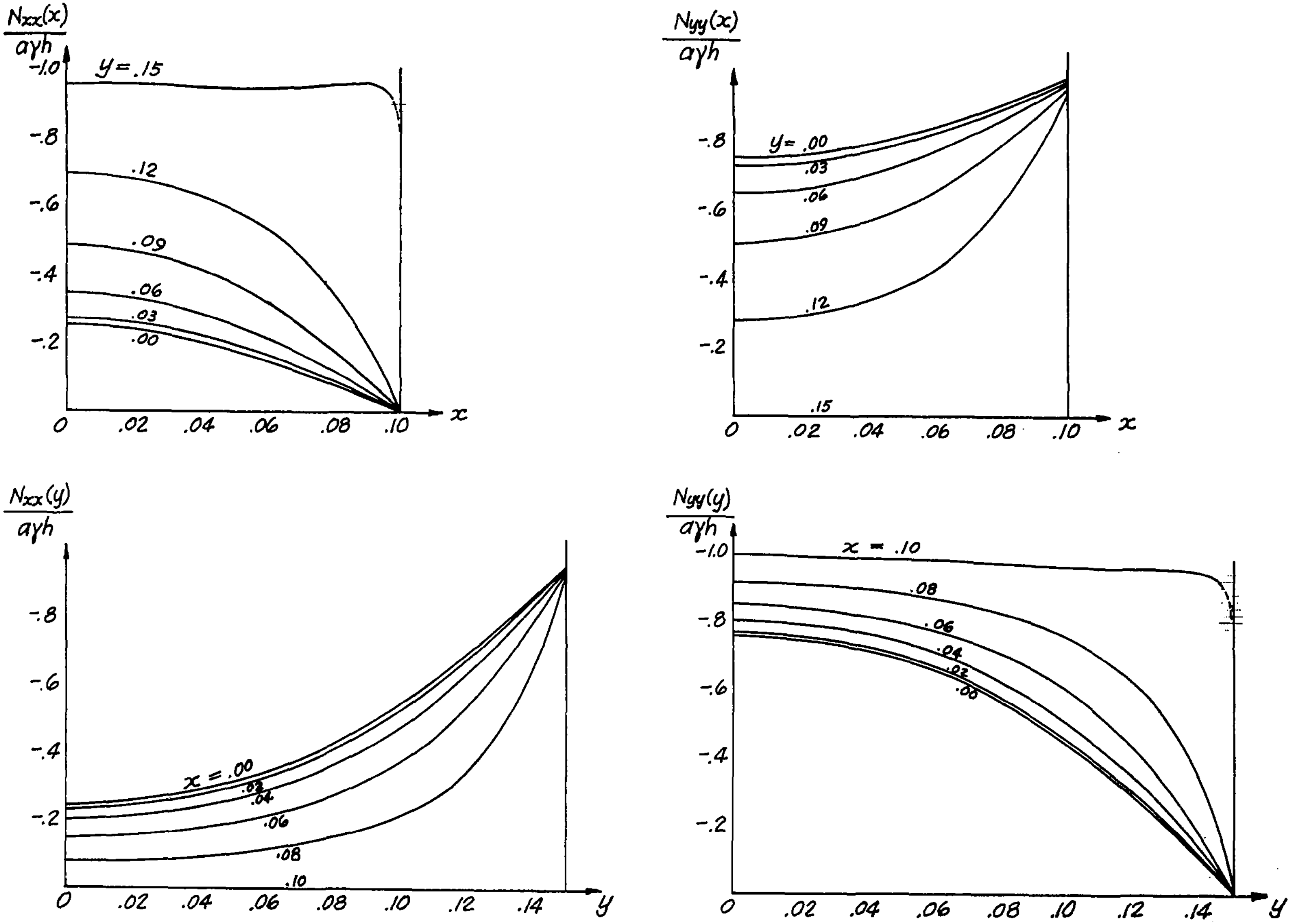


FIG. 8. Shell with $b_1 = 0.10$, $b_2 = 0.15$, $\xi_A = 11.4^\circ$, $\xi_B = 20.4^\circ$, $\xi_C = 17.1^\circ$

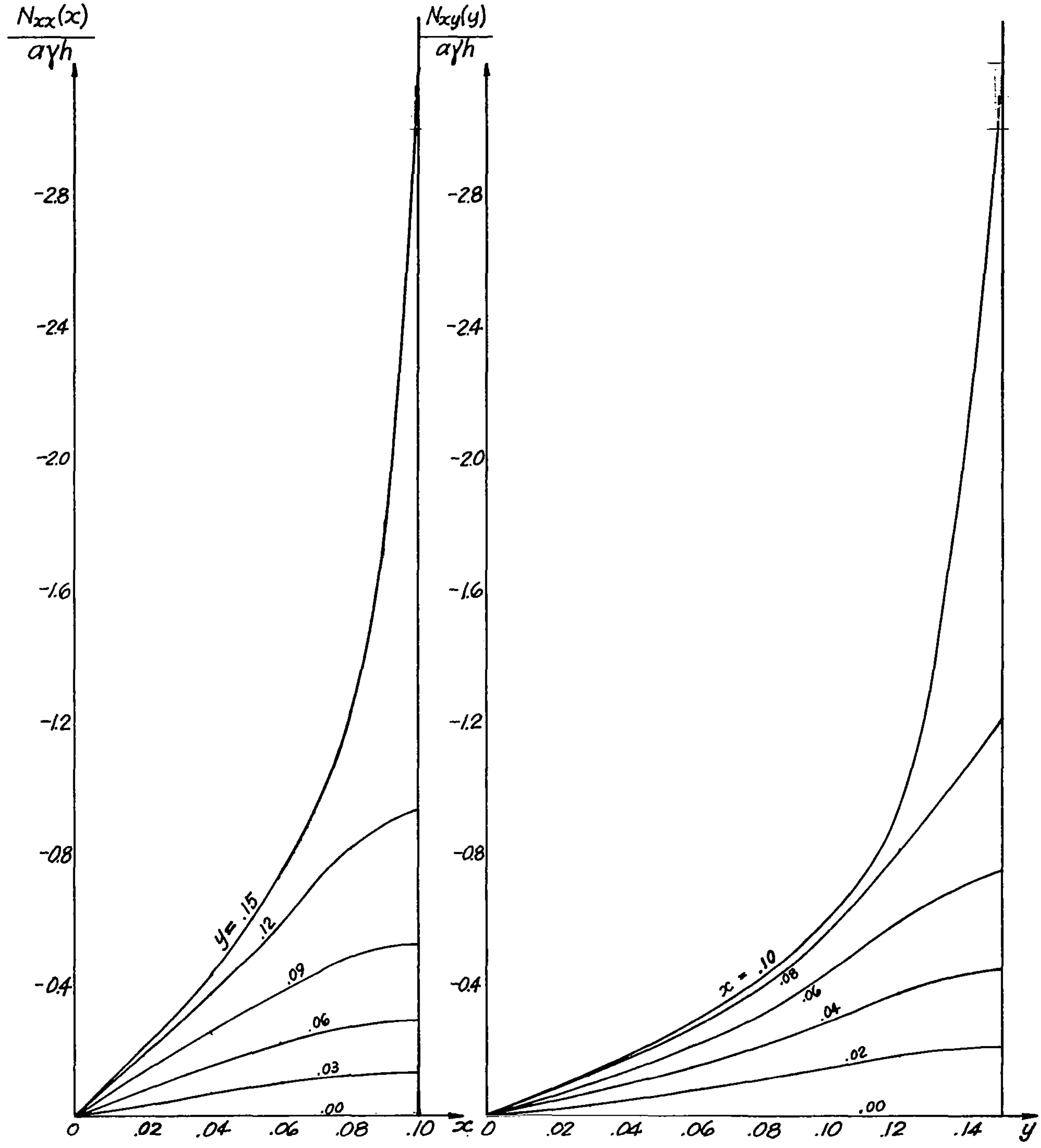


FIG. 8. Continued.

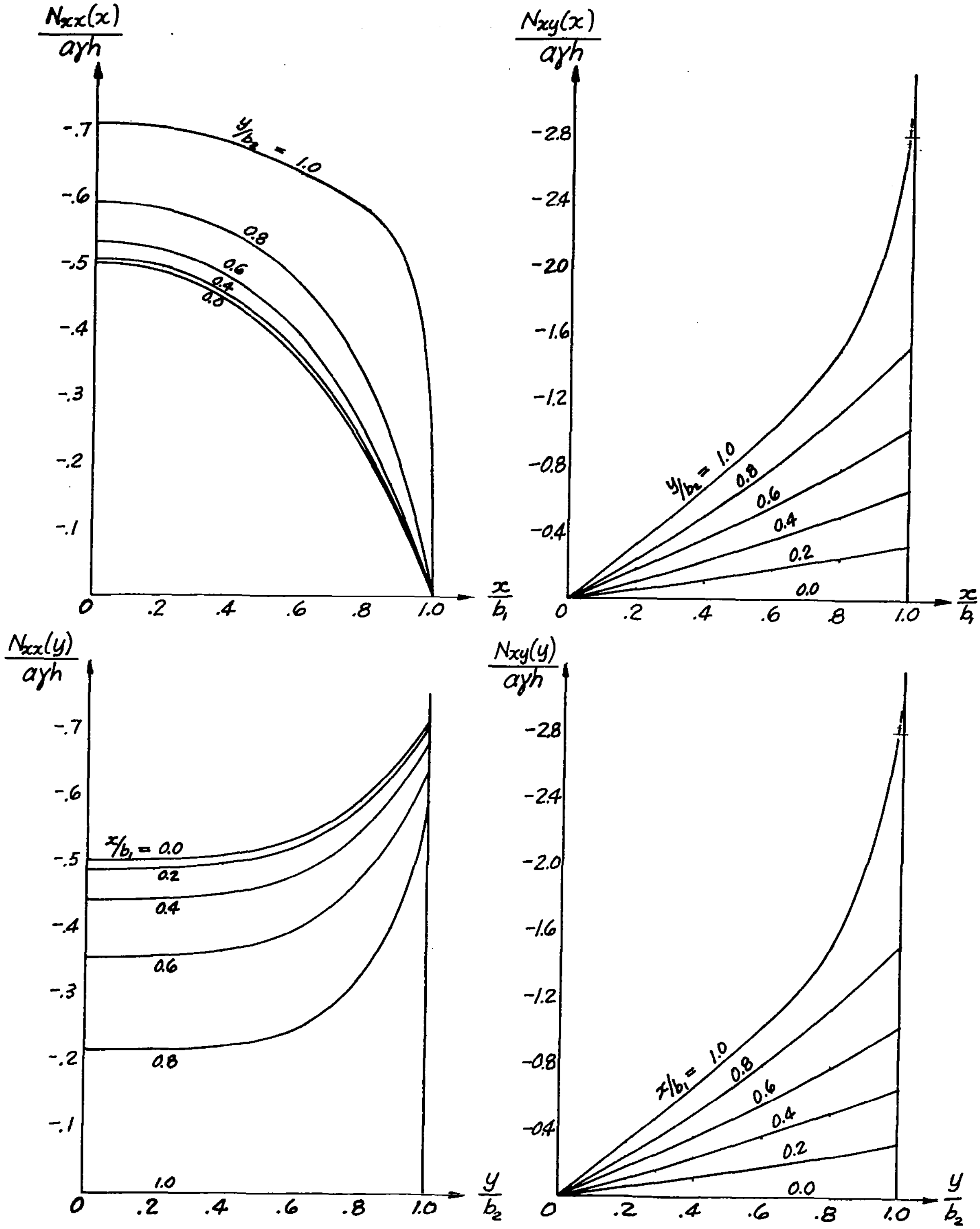


FIG. 9. Shell with $b_1 = b_2 = 0.4082$, $\xi_A = \xi_C = 44.4°$, $\xi_B = 60.0°$

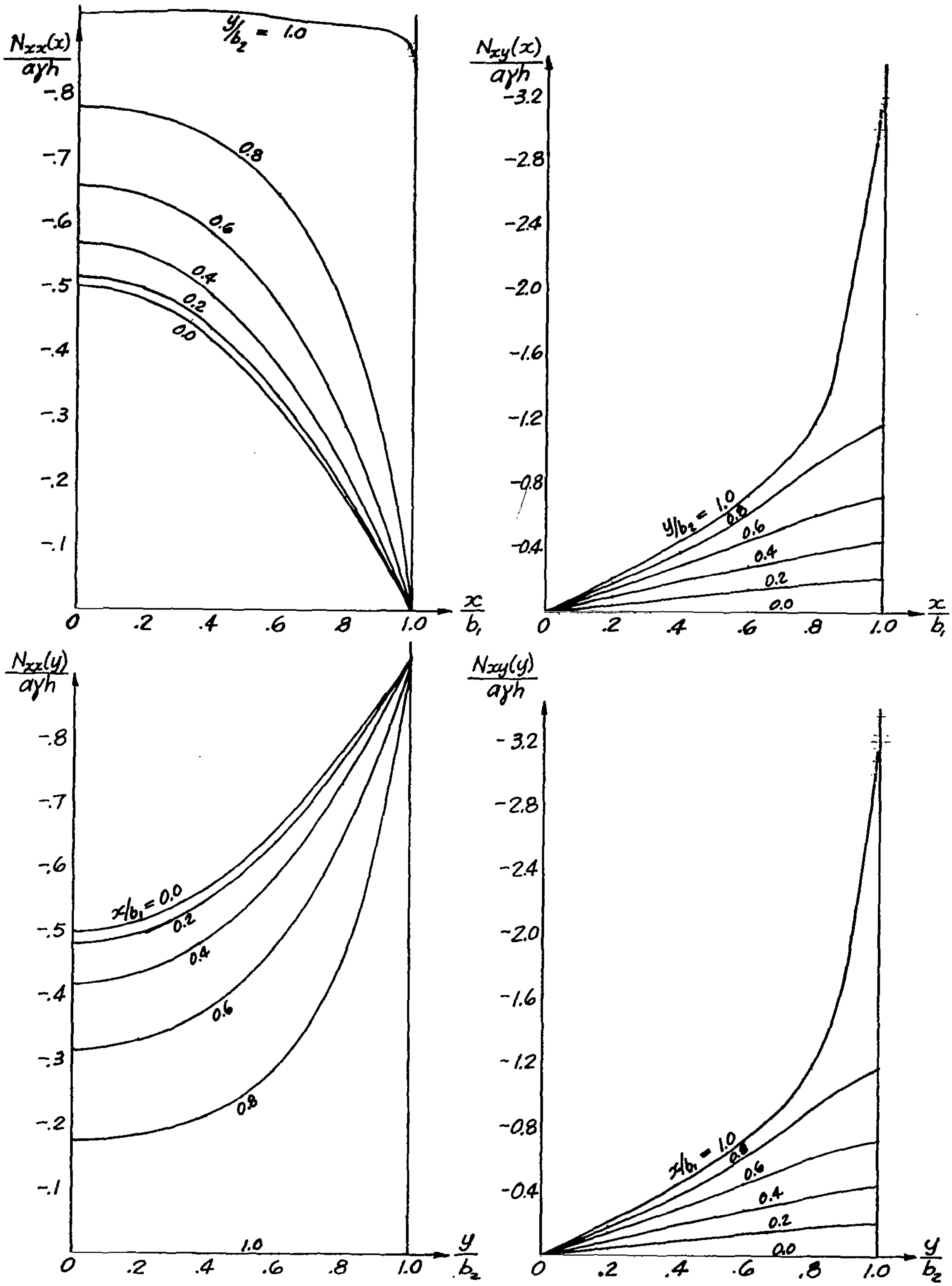


FIG. 10. Shell with $b_1 = b_2 = 0.1895$, $\xi_A = \xi_C = 21.5°$, $\xi_B = 30°$