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by James P. Avery

Prepared by
ARIZONA STATE UNIVERSITY
Tempe, Ariz.
for

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INTRODUCTION:

This investigation, supported by NASA Research Grant Number NGR-03-001-013, has been concerned with the development of a Green's function approach to the vibration problem of thin spherical shell segments. The class of problems considered has been limited by the following assumptions.

1. The vibration is undamped.
2. The vibration is of small amplitude and hence, considered linear.
3. The shell is thin, permitting use of a simplified elastic law for the flexure of thin shells, in which the thickness-to-radius ratio is ignored when compared with unity.

It should be noted, however, that the approach, while subject to the above restrictions, is adaptable to vibration analysis of spherical segments with arbitrary boundary conditions prescribed along the segment edge. The edge, in turn, may be defined by an arbitrary closed contour in a spherical surface. Thus a generality is achieved in both the shape and boundary conditions of segment.

GREEN'S FUNCTION FORMULATION:

The vibration problem to be considered is first replaced by an equivalent static problem. A static load proportional to displacement is substituted for the inertial loading of the vibrating shell. Additionally, the artifice of an elastic foundation is introduced such that the foundation reaction is proportional to displacement but in opposite sense. Thus, if the applied load is proportional to displacement, then also the net load on the shell (applied load together with foundation reaction) adheres to this proportionality. This manner of reacting applied loads permits a simplifying symmetry for the required Green's functions and yet does not disturb the proportionality of load to displacement which is necessary to simulate the vibration problem.

The relationship between the posed vibration problem and the equivalent static problem may be expressed symbolically:

(a) For the vibration problem

$$\vec{q} = \mu \, \omega^2 \, \vec{u}$$

where,

$\vec{q}$ = inertial shell force per unit surface area

μ = shell mass per unit surface area

ω = natural angular frequency

$\vec{u}$ = displacement vector of the middle surface

(b) For the shell on the elastic foundation

$$\vec{q} = \frac{1}{\lambda} \vec{u} - k\vec{u} = \left(\frac{1}{\lambda} - k\right) \vec{u}$$

where

$\vec{q}$ = net load on shell (per unit surface area)

λ = proportionality factor between applied load and displacement

k = foundation modulus

Consequently, if $(\frac{1}{\lambda} - k)$ is made equal to $\mu\,\omega^2$, the static problem is equivalent to the vibration problem.

The equivalent static problem is next formulated in terms of fundamental influence functions (Green's function). In brief, if the displacement and stress fields are known for a unit load (and unit couple) applied to a point on the complete sphere (on an elastic foundation), then through superposition the required relationships may be written satisfying boundary conditions (specified along a given contour) as well as the condition that the applied load be proportional to displacement.

For condensed notation, let:

A(m,n)	be the displacement vector at point <u>m</u> (on the sphere) due to a base load vector applied at point <u>n</u> on the sphere.
B(s,n)	be the boundary condition "residual" vector* (four dimensional) at point <u>s</u> on the contour, C, due to a unit load vector at point <u>n</u>.
C(s,t)	be the boundary condition residual vector at point <u>s</u> on the contour C due to corrective load system base vector** (four dimensional) at point <u>t</u> on the contour, C.
D(m,t)	be the displacement vector at point <u>m</u> on the sphere due to the corrective load base vector at point <u>t</u> on the contour, C.
$\vec{q}\,(n)$	be the applied load (intensity) vector at point n.
$\vec{u}\,(m)$	be the displacement vector at point m.
$\vec{L}\,(t)$	be the contour corrective load system at point <u>t</u> on the contour, C.

Then employing superposition, the specified boundary conditions along C are satisfied if the boundary condition "residuals" vanish; that is,

$$\int_S\!\!\int B(s,n)\,\vec{q}(n)\,d\sigma + \int_C C(s,t)\,\vec{L}(t)\,ds = 0 \qquad (1)$$

---

* The boundary condition residuals refer to deviations from four specified boundary conditions along the contour, C.

** The corrective load vector includes four components of a line load system (three force components and one force-couple) applied to the sphere along the contour, C.

The applied load-displacement relationship is also obtained from superposition,

$$\iint_S A(m,n)\ \vec{q}(n)\ d\sigma + \int_C D(m,t)\ \vec{L}(t)\ ds = u(m) \qquad (2)$$

The function $\vec{L}(t)$ may be eliminated between equations (1) and (2) (by means of inverse operators), and the resulting equation could be symbolized by:

$$\iint_S G(m,n)\ \vec{q}(n)\ d\sigma = \vec{u}(m) \qquad (3)$$

Then for $\vec{q}(n) = \frac{1}{\lambda}\ \vec{u}(n)$, equation (3) becomes

$$\iint_S G(m,n)\ \vec{u}(n)\ d\sigma = \lambda\ \vec{u}(m) \qquad (4)$$

which defines the eigenvalue problem for the general static equivalent problem, and hence, for the original vibration problem.

A more detailed discussion of the Green's function formulation appears in Appendix I, in which a free edge boundary condition is used for illustrative purposes and formulations for other boundary conditions are indicated.

FUNDAMENTAL PROBLEMS

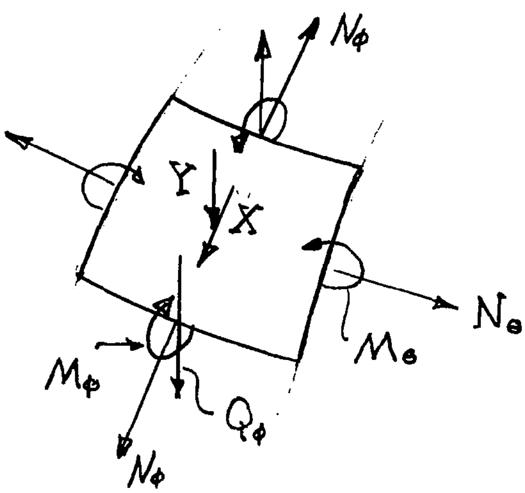
The required Green's functions, as defined in the foregoing discussion consist of displacements and stress resultants resulting from a unit force (or force couple) applied to the complete spherical shell on an elastic foundation. We now argue that only two fundamental problems require solution to provide the necessary Green's functions.

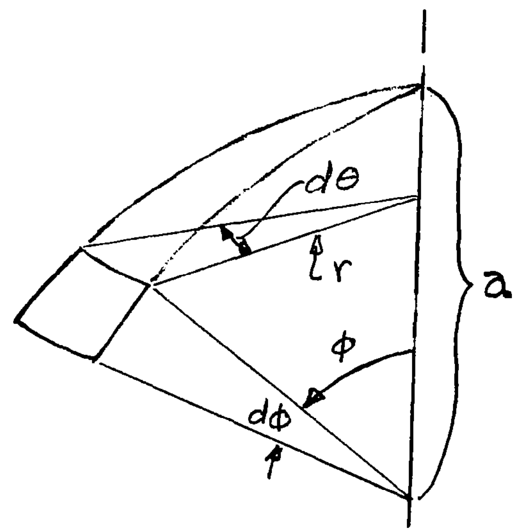
The two fundamental problems are defined by the application to the spherical shell of: (1) a unit normal load and (2) a unit tangential load. In each problem both displacements and stress resultants are sought. To obtain the Green's functions in which the stimulus is a force, such a force may be resolved into components, normal and tangential to the sphere, permitting direct use of fundamental problems (1) and (2). For those Green's functions in which the stimulus is a unit couple applied to the sphere, Betti's reciprocal theorem may be employed in conjunction with problems (1) and (2). The rotation of a tangent to the midsurface of the shell due to a unit load may be equated to a displacement due to a unit force couple.

We now consider separately the two fundamental problems.

FUNDAMENTAL PROBLEM I

The first fundamental problem is that associated with a spherical shell on an elastic foundation subjected to an applied unit normal load. Selecting the polar axis such that the unit load acts at the pole $(\phi = 0)$, the problem possesses polar symmetry. Employing the notation and sign conventions used by Timoshenko in "Theory of Plates and Shells", the stress resultants for the polar symmetric problem are as they appear in the sketch below:





Equilibrium of the element leads to:

$$(N_\phi r)^\circ - N_\theta\, a \cos\phi - (Q_\phi r_0) = -r\, aY$$
$$(N_\phi r) + N_\theta\, a \sin\phi + (Q_\phi r_0) = -r\, aZ \qquad (5\text{ a-c})$$
$$Q_\phi r\, a = (M_\phi r)^\circ - M_\theta a \cos\phi$$

where $(\ )^\circ$ denotes $\frac{d(\ )}{d\phi}$.

For the symmetric problem, stress-strain and strain-displacement relationships may be combined to obtain the elastic law for the thin spherical shell:

$$N_\phi = \frac{K}{a}\left[(\overset{\circ}{v} - w) + \nu(v \cot \phi - w)\right]$$

$$N_\theta = \frac{K}{a}\left[(v \cot \phi - w) + \nu(\overset{\circ}{v} - w)\right] \qquad (6\text{ a-d})$$

$$M_\phi = -\frac{D}{a}\left[\overset{\circ}{X} + \nu X \cot \phi\right]$$

$$M_\theta = -\frac{D}{a}\left[X \cot \phi + \nu X\right]$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$, $K = \dfrac{Eh}{(1-\nu^2)}$, $X = \left(v + \overset{\circ}{w}\right)$ ν = Poisson's ratio and w, v are displacement components.

The elastic foundation reaction loads are:

$$Y = -kv \quad , \qquad Z = -kw \qquad (\underline{k} \text{ is the foundation modulus.}) \qquad (7)$$

Some manipulation is required to obtain the governing differential equation for Problem I. We start by solving equations (6 a) and (6 b) for the quantities, $(\overset{\circ}{v} - w)$ and $(v \cot \phi - w)$:

$$(\overset{\circ}{v} - w) = \frac{a}{K(1-\nu^2)} \left[N_\phi - \upsilon N_\theta\right] \qquad (8\text{ a})$$

$$(v \cot \phi - w) = \frac{a}{K(1-\upsilon^2)} \left[N_\theta - \upsilon N_\phi\right] \qquad (8\text{ b})$$

Differentiating the latter equation, we obtain:

$$\overset{\circ}{v} \cot \phi - v \frac{1}{\sin^2 \phi} - \overset{\circ}{w} = \frac{a}{K(1-\upsilon^2)} \left[\overset{\circ}{N_\theta} - \upsilon \overset{\circ}{N_\phi}\right] \qquad (8\text{ c})$$

Eliminating $\overset{\circ}{v}$ and w from the three equations (8 a-c) yields:

$$(v + \overset{\circ}{w}) = \frac{a}{K(1-\nu^2)} \left[\nu \overset{\circ}{N_\phi} - \overset{\circ}{N_\theta} + (1 + \nu)(N_\phi - N_\theta) \cot \phi\right]$$

or

$$K(1-\nu^2)X = \left[\nu \overset{\circ}{N_\phi} - \overset{\circ}{N_\theta} + (1 + \nu)(N_\phi - N_\theta) \cot \phi\right] \tag{9}$$

Next we consider the equilibrium of the spherical shell that lies above a latitudinal plane, ϕ equals a constant. (We consider the pole, $\phi = 0$, to be at the top of the sphere). The forces acting upon this free body include the edge forces (Q_ϕ , N_ϕ), the foundation reaction forces over the surface, and the applied unit load at the pole. The resulting equilibrium equation when solved for N_ϕ yields:

$$N_\phi = - Q_\phi \cot \phi - \frac{1}{2\pi \, a \sin^2 \phi} + \frac{ka}{\sin^2 \phi} \int_0^\phi \left[v \sin \phi + w \cos \phi\right] \sin \phi \, d\phi \tag{10 a}$$

Upon substituting this equation into (5 b) where $Z = - kw$ we obtain:

$$N_\theta = - \overset{\circ}{Q_\phi} + \frac{1}{2\pi \, a \sin^2 \phi} - \frac{ka}{\sin^2 \phi} \int_0^\phi \left[v \sin \phi + w \cos \phi\right] \sin \phi \, d\phi + kaw \tag{10 b}$$

When equations (10 a, b) are substituted into equation (9) the latter reduces to:

$$K (1-\nu^2) X = L (Q_\phi) + \nu Q_\phi + (1+\nu) kav - kaw^\circ \tag{11}$$

where the operator L is defined by:

$$L (\ ) = (\ )^{\circ\circ} + (\ )^{\circ} \cot \phi - (\ ) \cot^2 \phi$$

We now assume that the displacement v is small in comparison with $\overset{\circ}{w}$ since the rotation of shell is a primary consequence of bending in the vicinity of the load while the displacement v is a secondary effect. Then a function $\bar{k}$ defined as,

$$\bar{k} = k \left[1 - \frac{(2+\nu)v}{v+\overset{\circ}{w}}\right]$$

is essentially equal to k and may be treated as a constant over small ranges in ϕ.

The displacement terms on the right side of equation (11) are then:

$$(1+\nu)\, kav - ka\overset{\circ}{w} = ka\left[(2+\nu)kv - (v+\overset{\circ}{w})\right]$$

$$= -\bar{k}a^2 X$$

Hence equation (11) becomes:

$$\left[K\,(1-\nu^2) + \bar{k}a^2\right] X = L\,(Q_\phi) + \nu Q_\phi \tag{12 a}$$

Now substituting the elastic law equations (6 c, d) into the third equilibrium equation (5 c) we obtain an expression for Q_ϕ in terms of X:

$$-\frac{a^2}{D}\, Q_\phi = L\,(X) - \nu X \tag{12 b}$$

Finally, X may be eliminated between (12 a) and (12 b) to obtain the governing fourth order differential equation:

$$LL\,(Q_\phi) + 4\,\varkappa^4\, Q_\phi = 0 \tag{13}$$

where:

$$4\varkappa^4 = \frac{(1-\nu^2) + \beta - \nu^2\alpha}{\alpha}$$

$$\alpha = \frac{D}{Ka^2} = \frac{h^2}{12a^2}$$

$$\beta = \frac{\bar{k}a^2}{K}$$

Equation (13) may be factored to yield two second order differential equations whose solutions satisfy (13)

$$L(Q_\phi) \pm 2\,i\,\varkappa^2\, Q_\phi = 0 \tag{14}$$

A substitution for the dependent variable,

$$Q_\phi = \frac{Z}{\sqrt{\sin \phi}} ,$$

reduces equations (14) to a simpler form:

$$Z^{\cdot\cdot} + \left[\frac{2-3 \cot^2\phi}{4} \pm 2i\varkappa^2\right] Z = 0 \tag{15}$$

Considering a boundary to be defined at $\phi = \phi_o$, a new independent variable ψ may be introduced such that:

$$\psi = \phi - \phi_o$$

Expanding $\cot^2\phi$ in a Talyor series about ϕ_o we obtain,

$$\cot^2\phi = \cot^2\phi_o - 2 \frac{\cot \phi_o}{\sin^2\phi_o} \psi + \ldots$$

Retaining the first two terms* this may be substituted into equation (15) to give:

$$Z^{oo} = (A_o + A_1 \psi) Z \tag{16}$$

where,

$$A_o = \frac{1}{4} (3 \cot^2\phi_o - 2 \mp 2 i\varkappa^2)$$

$$A_1 = - \frac{3}{2} \frac{\cot \phi_o}{\sin^2\phi_o}$$

We now attempt a solution of the form

$$Z = \exp \left[a, \psi + a_2\psi^2 + \ldots\right]$$

which when substituted into (16) leads to:

$$(a_1 + 2 a_2\psi + \ldots)^2 + (2a_2 + 6a_3\psi + \ldots) = A_o + A_1\psi$$

* The justification for retention of only two terms is that the solution of the equation is to be applied for only small values of ψ.

Upon equating coefficients of corresponding powers in ψ we obtain:

$$a_1^2 + 2a_2 = A_o \tag{17}$$

$$4a_1 \, a_2 + 6a_3 = A_1$$

Consistant with the truncated Talyor series for $\cot^2 \phi$, we retain only two terms in the series for log Z. Then the complex numbers a_1 and a_2 may be obtained through the solution of equations (17). We now let

$$(a_1 + a_2 \, \psi) \, \psi = (\pm \xi \pm \eta \, i) \, \psi.$$

If only those solutions with negative real exponents are retained, the required decay behavior of functions of ϕ is assured. The solution for Q_ϕ is then expressible as:

$$Q_\phi = \frac{e^{-\xi \psi}}{\sqrt{\sin \phi}} \left[C_1 \cos \eta \, \psi + C_2 \sin \eta \, \psi \right] \tag{18}$$

which solution is valid for small ψ.

Employing the differential equations (14) in Q_ϕ, in conjunction with equation (12a) (that expresses X in terms of Q_ϕ) we obtain:

$$X = \frac{a^2 \, e^{-\xi \cdot \psi}}{D(4 \, \gamma \, e^4 + \nu^2)\sqrt{\sin \phi}} \left[\left(\nu \, C_1 - 2 \, \gamma \, e^2 \, C_2\right) \cos \eta\psi + \left(\nu C_2 + 2 \, \gamma \, e^2 C_1\right) \sin \eta \, \psi \right] \tag{19}$$

Expressions (6a-d), (8a,b), and (10a,b) together with the solutions (18) and (19) provide the basis for a quasi-numeric integration of the governing differential equations for the First Fundamental Problem. Functions of ψ assume the general form:

$$f = \frac{C}{\sqrt{\sin \phi}} \; e^{-\xi \psi} \left[\left(f_1 + f_2 X\right) \cos \, \eta\psi + \left(f_3 + f_4 \psi\right) \sin \, \eta\psi \right]$$

in which it should be noted ξ and η are linear functions of ψ: that is,

$\xi = \xi_1 + \xi_2 \psi$ and $\eta = \eta_1 + \eta_2 \psi$

Then the derivative of f becomes:

$$g = f^o = \frac{c}{\sqrt{\sin \phi}} \quad e^{-\xi\psi} \left[(g_1 + g_2 \psi) \cos \eta\psi + (g_3 + g_4 \psi) \sin \eta\psi \right] \tag{20}$$

where

$$g_1 = -\left(\xi_1 + \frac{1}{2} \cot \phi_o\right) f_1 + f_2 + \eta_1 f_3$$

$$g_2 = -\left(\xi_2 - \frac{1}{2 \sin^2 \phi_o}\right) f_1 - \left(\xi_1 + \frac{1}{2} \cot \phi_o\right) f_2 + \eta_2 f_3 + \eta_1 f_4$$

and with corresponding expressions for g_3 and g_4.

To achieve a step-wise integration, we assume at the outset an appropriate start in the form of a solution near the apex of the sphere. (This is subsequently discussed). The operations to be performed at each stage follow:

1. Starting at $\phi = \phi_o$, with all functions known for ϕ less than or equal to ϕ_o, the constants C_1 and C_2 are evaluated from expressions (18) and (19). Also, the value of $\mathcal{K}$ and the constants ξ_1, ξ_2, η_1, η_2 are evaluated (employing equation 17).

2. The variable ψ is considered to range from $\underline{0}$ to an appropriate small value $\underline{\Delta_\phi}$. At the end of this range; that is, for $\psi = \Delta_\phi$, the functions Q_ϕ and X are evaluated, again from (18) and (19).

3. Displacement and stress resultant functions are evaluated at the beginning of each step (and at selected intermediate points for larger steps).

4. The starting value ϕ_o is next incremented by $\Delta\phi$.

Operations (1) through (4) are repeated successively until functions have decayed sufficiently. It may be verified that in matching Q_ϕ and X at the boundaries of each step in ϕ, all functions of ϕ will likewise match.

Operation (3) above deserves further discussion. The sub-operations to obtain displacements and stress resultants at each stage are outlined below:

(a) Employing relations (10a, b), the membrane forces N_ϕ and

N_θ may be obtained. However, initially the definite integral, containing $\underline{v}$ and $\underline{w}$ must be approximated in range from ϕ_o to $\phi_o + \Delta\phi$, by means of extrapolated values of $\underline{v}$ and $\underline{w}$. (The definite integral is evaluated most conveniently by numerical quadratures.) The error introduced by approximating $\underline{v}$ and $\underline{w}$ in range of the new increment of ϕ is small as this increment, $\Delta\phi$, is small.

(b) Eliminating $\underline{w}$ between (6a) and (6b), we have the expression for v,

$$\left(\frac{v}{\sin\phi}\right)^o = \frac{a}{K(1-\nu)} \frac{[N_\phi - N_\theta]}{\sin\phi}$$

which may be integrated numerically to obtain $\underline{v}$.

(c) Since X is now known, and also

$$X = \frac{v + w^o}{a}$$

$\underline{w}$ may be found by numerical integration.

(d) Then with improved values for both v and w steps (a), (b) and (c) may be repeated.

(e) The bending moments are obtainable directly from moment-curvature relationships (6c, d).

In the above operations a sub-routine for differentiation based upon expression (20) would be used whenever differentiation is indicated.

Turning attention now to the solution near the pole ($\phi = 0$), the differential equations (14) may be simplified, with the introduction of only small error, by considering only the first term in a Laurent series for cot ϕ:

$$\cot\phi \doteq \frac{1}{\phi}$$

Then

$$LQ_\phi \pm 2i\,\varkappa^2 Q_\phi = 0$$

becomes:

$$Q_\phi^{oo} + \frac{1}{\phi} Q_\phi^{o} - \frac{1}{\phi^2} Q_\phi \pm 2i\varkappa^2 Q_\phi = 0 \qquad (21)$$

Letting $x = \varphi\sqrt{2}$ in equations (21) and combining appropriate conjugate complex solutions, we obtain (ref. 1)

$$Q_\phi = B_1 \ker{}'x + B_2 \operatorname{kei}'x \qquad \text{(Kelvin functions)}$$

The boundary conditions are then enforced as ϕ approaches the singular point $\phi = 0$, to obtain:

$$Q_\phi = \frac{\varkappa}{\sqrt{2}\,\pi\, a} \left[\operatorname{Ker}'x - \frac{\nu}{2\varkappa^2} \operatorname{kei}'x \right] \tag{22}$$

and

$$X = \frac{-a}{2\sqrt{2}\,\pi\, D\varkappa} \operatorname{kei}'x \tag{23}$$

Ignoring $\underline{v}$ near the pole and integrating equation (23) we find:

$$w = -\frac{a^2}{4\pi\, D\varkappa^2} \operatorname{kei} x \tag{24}$$

(If the radius-to-thickness ratio is allowed to become very large, this solution reduces to the deflection of a plate on an elastic foundation acted upon by unit concentrated normal load.)

Employing equations (22), (23), (24) in expressions (6a - d), (8a, b) and (10a, b), a starting solution is provided for the stepwise integration outlined above.

Fundamental Problem II

The development of the governing differential equations (in terms of displacements) for the unit tangential load, follows a pattern similar in some respects to that for Fundamental Problem I.

The equilibrium of the element leads to three equilibrium equations, expressed in terms of stress resultants. Substitution of the elastic law, yields three equations in the displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$. (ref. 1) These may be further simplified by recognizing that a solution of the form:

$$\begin{aligned} \bar{u} &= u\,(\phi)\,\sin\theta \\ \bar{v} &= v\,(\phi)\,\cos\theta \\ \bar{w} &= w\,(\phi)\,\cos\theta \end{aligned} \tag{25}$$

is consistent with the singuarity associated with the unit tangent load at $\phi = 0$, which load is contained in the meridianal plane, $\theta = 0$.

Upon introducing expressions (25) into the first two of the equilibrium equations (the latitudinal and meridianal force summations) we obtain:

$$(1 + \alpha)\left[\frac{1-\nu}{2}(u^{\circ\circ}\sin\phi + u^{\circ}\cos\phi) - u\left\{\frac{1}{\sin^2\phi} - \frac{1-\nu}{2}(1 - \cot^2\phi)\sin\phi\right\}\right.$$

$$\left. - \frac{1+\nu}{2}\,v^{\circ} - \frac{3-\nu}{2}\,v\cot\phi + (1+\nu)\,w\right] \tag{26 a}$$

$$- \alpha\left[w^{\circ\circ} + w^{\circ}\cot\phi + w\left(2 - \frac{1}{\sin^2\phi}\right)\right] = \beta\,u\sin\phi$$

$$(1 + \alpha)\left[\frac{1+\nu}{2}\, u^{\circ} - \frac{3-\nu}{2}\, u \cot\phi + v^{\circ\circ}\sin\phi + v^{\circ}\cos\phi \right. \tag{26 b}$$

$$- \frac{\nu}{\sin\phi}\left(\cos^2\phi + \alpha\sin^2\phi + \frac{1-\nu}{2}\right) - \left.(1+\nu)\, w^{\circ}\sin\phi\right]$$

$$+ \alpha\left[w^{\circ\circ\circ}\sin\phi + w^{\circ\circ}\cos\phi + w^{\circ}\left(2 - \frac{3}{\sin^2\phi}\right)\sin\phi + 2w\,\frac{\cos\phi}{\sin^2\phi}\right] =$$

$$\beta\, v \sin\phi$$

If equation (26a) is differentiated with respect to ϕ, multiplied by $\sin\phi$ and then added to (26b), we succeed in eliminating <u>w</u> and obtain upon reduction:

$$\tau^{\circ\circ} - \tau^{\circ}\cot\phi - \rho\tau = 0 \tag{27}$$

where:

$$\tau = (u \sin\phi)^{\circ} + v$$

$$\rho = \frac{2\beta}{(1-\nu)(1+\alpha)} - 2$$

Equation (27) can be further reduced by a substitution for the independent variable. If we let

$$s = 1 - \cos\phi \qquad (\ )' = \frac{d(\ )}{ds}$$

then equation (27) becomes,

$$(2s - s^2)\,\tau'' - \rho\tau = 0 \tag{28}$$

Before attending to the solution of this equation in τ, we proceed to eliminate v for equation (26a) by introducing τ. Then we obtain:

$$H(u\sin\phi) - (1+\nu)(u\sin\phi) - \frac{\beta}{1+\alpha}(u\sin\phi) = \frac{1+\nu}{2}\left[\tau^\circ - \tau\cot\phi\right] + 2\tau\cot\phi + \frac{\alpha}{1+\alpha}H(w) - (1+\nu)w \tag{29}$$

where:

$$H(\ ) = (\ )^{\circ\circ} + (\ )^\circ\cot\phi + \left(2 - \frac{1}{\sin^2\phi}\right)(\ )$$

This may be simplified if the new independent variable s is introduced along with the dependent variables:

$$y = u\sin^2\phi\ , \qquad x = w\sin\phi$$

Then equation (29) becomes:

$$(2s - s^2)y'' + \left(1 - \nu - \frac{\beta}{1+\alpha}\right)y = \frac{1+\nu}{2}(2s + s^2)\,\tau' + \frac{3-\nu}{2}(1-s)\tau + \frac{\alpha}{1+\alpha}\left[(2s - s^2)\,x'' - 2\right] - (1+\nu)\,x \tag{30}$$

Since from the Betti reciprocal theorem w in Fundamental problem II is equal to v in Fundamental Problem I, we may consider w (and hence x) known as we approach a solution to equations (28) and (30). The series solution of the differential equation (28), in τ, (since the roots of the indicial equation are: $r = 0, 1$) may take the form:

$$\tau = \sum_{n=0}^{\infty}(a_n + \bar{a}_n \log s)\, s^n$$

Substitution of this solution into equation (28) and equating coefficients of like powers of $\underline{s}$ (separately for the log and non-log terms) leads to two recurrence relations:

$$J_1 \bar{a}_n = J_2 \bar{a}_{n-1}$$

$$J_1 a_n = J_2 a_{n-1} + J_3 \bar{a}_n + J_4 \bar{a}_{n-1} \qquad (31\ a,\ b)$$

Where

$$J_1 = 2n\ (n-1)$$

$$J_2 = (n-1)(n-2) + \rho$$

$$J_3 = -2(2n-1)$$

$$J_4 = 2n-3$$

A similar form of solution for $\underline{y}$ in equation (30) leads to additional recurrence formula. If we use

$$y = \sum (b_n + \bar{b}_n \text{ Log } s)\ s^n$$

$$x = \sum d_n\ s^n$$

Then:

$$J_1 \bar{b}_n = J_5 \bar{b}_{n-1} + J_6 \bar{a}_{n-1} + J_7 \bar{a}_{n-2}$$

$$J_1 b_n = J_5 b_{n-1} + J_3 \bar{b}_n + J_4 \bar{b}_{n-1} \qquad (32\ a,\ b)$$

$$+ J_6 a_{n-1} + J_7 a_{n-2} + J_8 \bar{a}_{n-1} + J_9 \bar{a}_{n-2}$$

$$+ J_{10} d_n + J_{11} d_{n-1}$$

where:

$$J_5 = (n-1)(n-2) - (1-\nu- \frac{\beta}{1+\alpha})$$

$$J_6 = (1 + \nu)(n-1) + \frac{3 - \nu}{2}$$

$$J_7 = -(n-2)\,\frac{1 + \nu}{2} - \frac{3 - \nu}{2}$$

$$J_8 = 1 + \nu$$

$$J_9 = -\,\frac{1 + \nu}{2}$$

$$J_{10} = \frac{\alpha}{1 + \alpha}\quad 2n\,(n-1)$$

$$J_{11} = \frac{\alpha}{1 + \alpha}\left[(n-1)(n-2) - 2\right] - (1+\nu)$$

From a consideration of the first few orders of the recurrence relations (31 a, b) and (32 a, b), it may be verified that:

$\bar{a}_o$ and $\bar{b}_o$ are zero

The constants $\bar{a}_1$, a_1 , $\bar{b}_1$, b_1 are arbitrary. All other coefficients are expressible in terms of these four arbitrary constants.

To evalute these constants by means of boundary conditions it is first necessary to express $\underline{v}$ in a comparable series form:

$$v = \sum{}' \left(c_n + \bar{c}_n \operatorname{Log} s\right) s^n$$

Then from the definition of τ,

$$v = \tau - (u^\circ \sin \phi)$$

and hence upon substitution of series solutions.

$$2\bar{c}_n = \bar{c}_{n-1} + 2\bar{a}_n - \bar{a}_{n-1} - (2n + 1)\,\bar{b}_{n+1} + (n-1)\bar{b}_n$$

$$2c_n = c_{n-1} + 2a_n - a_{n-1} - (2n + 1)\,b_{n+1} + (n - 1)\,b_n - 2\bar{b}_{n+1} + \bar{b}_n \qquad (33\ a,\ b)$$

From which the coefficients $\bar{c}_n$ and c_n are expressible in terms of the arbitrary constants $\bar{a}_1$, a_1, $\bar{b}_1$, and b_1.

To complete the solution to Fundamental Problem II four boundary conditions are required. Care must be used in considering boundary conditions near the singularity, $\phi = 0$, as the simplified definition of strain (implicit in the elastic law of shell) loses physical significance. To define extensional strain as the change in length divided by the original length produces anomalies when strains become very large. For this reason, boundary conditions will be enforced near the pole, but not at the pole. Considering the equilibrium of a free body consisting of a disc of radius $\underline{\epsilon}$ containing the pole, we obtain two equations: Summing forces in tangential direction of $\theta = 0$ leads to:

$$1 = a \int_0^{2\pi} \left[N_\phi \, \theta \sin \theta - N_\theta \cos \theta \right] \sin \epsilon \, d\theta \tag{34a}$$

Moment equilibrium about the axis, $\theta = \frac{\pi}{2}$ yields:

$$0 = \int_0^{2\pi} \Big[(Q_\phi \, a \sin \epsilon - M_\phi + N_\phi \, a \sin^2 \epsilon) \cos \theta + M_\phi \, \theta \sin \theta \Big] d\theta \tag{34b}$$

Employing the elastic law and the lower order terms of the series solutions for u and v (w may be ignored in region near the pole as may be verified from the solution near the apex of Problem I), we find from boundary conditions (34 a, b):

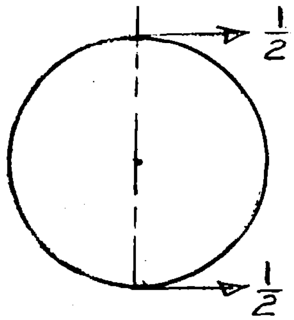
$$\frac{1}{\pi K} = -\left(\frac{1 + \nu}{4} \right) \frac{b_o}{s} + \bar{b}_1 - \left[\frac{3 - \nu}{4} \bar{b}_2 + \frac{5 + \nu}{2} \bar{c}_1 \right] s \log s \tag{35 a}$$

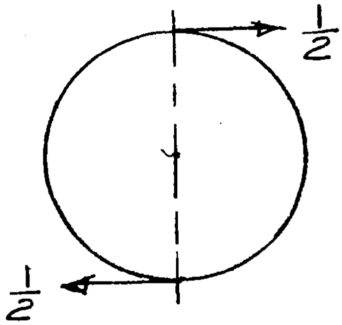
and:

$$0 = \frac{2 b_o}{s} + b_1 + 2c_o - 2\bar{b}_1 \tag{35 b}$$

where $s = 1 - \cos \epsilon$. For numerical computation $\underline{\epsilon}$ is taken as suitably small, but not so small that strains become excessive.

Two additional boundary conditions are required. For these we resolve the original Fundamental Problem II into symmetric and anti-symmetric component problems illustrated below:





At the equator, corresponding to $s = 1$ (or $\phi = 90°$) we have the respective boundary conditions:

For the symmetric problem:

$$\left.\begin{aligned} \tau &= 0 \\ N_\phi &= 0 \end{aligned}\right\} \quad \text{at } s = 1 \tag{36}$$

For the anti-symmetric problem:

$$\left.\begin{aligned} y &= 0 \\ N_\phi &= 0 \end{aligned}\right\} \quad \text{at } s = 1 \tag{37}$$

Associated with each set of equatorial boundary conditions (36) and (37), are the boundary conditions near the apex, (35 a, b) (however, with concentrated load equal to 1/2). Consequently each component problem may now be solved independently as four suitable boundary conditions are specified (for each) leading to two sets of simultaneous equations in the four arbitrary constants. (It may be verified by the ratio test that the series are convergent at $s = 1$).

Finally, the desired solution to the Fundamental Problem II is to be found in the superposition of the two component solutions.

MECHANICS OF SOLUTION

The Green's function approach to the vibration analysis in a specific problem entails a number of computational operations. In the discussion of these operations it is convenient to consider separately : (1) the mechanics of solution of the integral equations in which the Green's functions are assumed known and (2) the computational steps in obtaining the required Green's functions.

Solution of Integral Equations

The governing integral equations (1) and (2) are more explicitly presented in Appendix I as equations (1a) and (2a). If these equations, are replaced by finite difference approximations to the equations, the integral operators become rectangular matrices and the functions u_α, q_α, and L_i become column matrices (or "vectors").

Let the surface S (enclosed by the contour C) be subdivided into **N** elements, the n^{th} element denoted by $\Delta\sigma_n$.

Also let the contour C be subdivided into **M** segments, with Δs_m denoting the m^{th} segment.

The integrals of equation (1a) may then be approximated by mechanical quadratures: for example,

$$\iint_S B_{i\beta}\, q_\beta\, d\sigma \doteq \sum_{n=1}^{N} B_{i\beta}\,(m,\, n)\; q_\beta(n)\quad \Delta\sigma_n$$

where $B_{i\beta}$ (m, n) is $B_{i\beta}$ evaluated at the central points of the m^{th} segment on contour C and the n^{th} surface element.

$q_\beta(n)$ is the central value of q_β for surface element $\Delta\sigma_n$.

Next let the symbol [B] denote a 4M by 3N rectangular matrix whose element in row [4(m-1) + i] and column [3(n-1) + β] is $B_{i\beta}(m,\, n)\, \Delta\sigma_n$. Also let $\vec{q}$ be a column matrix whose element in row [3(n-1) + β] is $q_\beta(n)$. Then:

$$\sum_{n=1}^{N} B_{i\beta}\ (m,n)\ q(n)\ \Delta\sigma_n \ = \ [B]\ \vec{q}$$

We define other rectangular and column matrices correspondingly to represent the other integral operators and functions appearing in equations (1a) and (2a):

Matrix	Element	Row	Column
[C]	$C_{ij}(m,\ p)\Delta s_p$	$4(m-1)\ +\ i$	$4(p-1)\ +\ j$
[A]	$A_{\alpha\beta}(r,\ n)\Delta\sigma_n$	$3(4-1)\ +\ \alpha$	$3(n-1)\ +\ \beta$
[D]	$D_{\alpha j}(4,\ p)\Delta s_p$	$3(r-1)\ +\ \alpha$	$4(p-1)\ +\ j$
$\vec{L}$	$L_j(p)$	$4(p-1)\ +\ j$	
$\vec{u}$	$u_\alpha(r)$	$3(r-1)\ +\ \alpha$	

The mechanical quadrature approximation to equations (1a) and (2a) is then:

$$[B]\ \vec{q} \quad + \quad [C]\ \vec{L} \ = \ 0 \qquad (1b)$$

$$[A]\ \vec{q} \quad + \quad [D]\ \vec{L} \ = \ u \qquad (2b)$$

If $[C^{-1}]$ denotes the inverse of matrix [C], then from equation (1b):

$$\vec{L} \ = \ -[C^{-1}]\ [B]\ \vec{q} \qquad (3b)$$

This may be substituted into equation (2b) to obtain:

$$[A]\vec{q} - [D][C^{-1}]\ [B]\ \vec{q} = \vec{u}$$

or

$$\left\{[A]\ -[D]\ [C^{-1}]\ [B]\right\} \vec{q} = \vec{u} \tag{4b}$$

Finally, if $\vec{q} = \frac{1}{\lambda}\vec{u}$, we may write:

$$[G]\ \vec{u} = \lambda\vec{u} \tag{5b}$$

where

$$[G] = \left\{[A]\ -[D]\ [C^{-1}]\ [B]\right\}$$

The eigenvalue problem symbolized by equation (5b) may be solved conveniently by the Vianello-Stadola iterative method. Upon obtaining each eigenvalue and mode shape, the matrix [G] is then purified of that mode shape characteristic such that higher order modes and eigenvalues will emerge.

Development of the Green's Functions:

To facilitate the development of specific Green's function, two spherical polar coordinate systems are introduced, one fixed in space to serve as a reference, the other oriented with respect to two selected points on the sphere. The fixed coordinates, ϕ and θ serve to locate sample points for numerical integration and are used in defining the contour C enclosing the spherical segment under consideration. The "relative" coordinate system, as the second system may be termed, has variables denoted by $\hat{\phi}$ and $\hat{\theta}$. The pole, $\hat{\phi} = 0$ is located at a "stimulus" point defined as a sample point at which a unit force or unit couple is considered to be applied. Also the meridianal plane, $\hat{\theta} = 0$, is oriented to contain a "response" point, that is, a sample point at which a displacement or stress resultant is sought.

The Green's functions (or for computational purposes, the elements of the influence matrices) are responses at one point on the sphere due to unit stimuli at a second point, with the location of points and surface vector (or tensor) components expressed with respect to the fixed reterence frame. However, the solutions to the Fundamental Problems I and II, (which provide the basis for the Green's functions) are expressed in terms of the relative coordinate system. Then vector (or tensor) transformations of components are required at the stimulus point and again at the response point. (The details of these transformations together with the required spherical surface geometry are contained in Appendix II).

As a demonstration example, consider the development of the Green's functions for a shell segment with fixed (built-in) supports. The residuals in this case would be the displacements w, v, u, and the rotation $\frac{dw}{dn}$ (respectively R_1, R_2, R_3, R_4,).

The required displacement functions from the solutions of the Fundamental Problems I and II are denoted by the following:

For Fundamental Problem I, let

$$w_1 = w\,(\hat{\phi}) \qquad v_1 = v\,(\hat{\phi})$$
$$w_1^{\circ} = w^{\circ}\,(\hat{\phi}) \qquad v_1^{\circ} = v^{\circ}(\hat{\phi})$$
$$w_1^{\circ\circ} = w^{\circ\circ}\,(\hat{\phi})$$

For Fundamental Problem II, let

$$v_2 = v(\hat{\phi}, \hat{\theta}=0)$$

$$u_2 = u(\hat{\phi}, \hat{\theta}=\frac{\pi}{2})$$

Introducing the notation:

γ = the angle between the tangent to the meridian circle at the stimulus point toward the pole and the tangent to a great circle at the stimulus point toward the response point.

δ = similar to γ, with the roles of stimulus point and response point interchanged.

We obtain from spherical geometry (Appendix II)(and from Betti's reciprocal theorem) the following expressions for the Green's functions: $A_{\alpha\beta}$, $B_{i\beta}$, C_{ij}, and $D_{\alpha j}$ (defined earlier)* (Functions of $\bar{\phi}$, $\bar{\theta}$, for stimulus point and ϕ, θ, for response point):

$$A_{11} = w_1$$
$$A_{12} = -v_1 \cos\gamma$$
$$A_{13} = -v_1 \sin\gamma$$
$$A_{21} = -v_1 \cos\delta$$
$$A_{22} = -v_2 \cos\gamma \cos\delta - u_2 \sin\gamma \sin\delta$$
$$A_{23} = -v_2 \sin\gamma \cos\delta + u_2 \cos\gamma \sin\delta$$
$$A_{31} = v_1 \sin\delta$$
$$A_{32} = v_2 \cos\gamma \sin\delta - u_2 \sin\gamma \cos\delta$$
$$A_{33} = v_2 \sin\gamma \sin\delta + u_2 \cos\gamma \cos\delta$$

*(For sign conventions see Appendix II)

$B_{\alpha\beta} = A_{\alpha\beta}$ for $\alpha = 1, 2, 3$, and $\beta = 1, 2, 3$ also,

$B_{41} = -w_1^\circ \cos\delta$

$B_{42} = v_1^\circ \cos\gamma\cos\delta - v_1 \sin\gamma\sin\delta\csc\hat{\phi}$

$B_{43} = v_1^\circ \sin\gamma\sin\delta + v_1 \cos\gamma\sin\delta\csc\hat{\phi}$

$C_{i\beta} = B_{i\beta}$ $\quad i = 1, 2, 3, 4$, and $\beta = 1, 2, 3$

also,

$C_{14} = -w_1^\circ \cos\gamma$

$C_{24} = v_1^\circ \cos\gamma\cos\delta - v_1 \sin\gamma\sin\delta\csc\hat{\phi}$

$C_{34} = -v_1^\circ \cos\gamma\sin\delta - v_1 \sin\gamma\cos\delta\csc\hat{\phi}$

$D_{\alpha j} = B_{j\alpha}$ from the reciprocal theorem.

The quantities $\hat{\phi}$, γ, δ, are functions of $\bar{\phi}$, $\bar{\theta}$, ϕ, θ, and are computed from relationships developed in Appendix II.

When the response and stimulus points are coincident the above expressions do not apply but rather singular values of displacements or residuals are used, which values are found directly from the solutions of the Fundamental Problems or from an appropriate limiting process, with statically equivalent distributed loads.

The mechanics in obtaining the Green's functions for other boundary conditions would follow a similar pattern.

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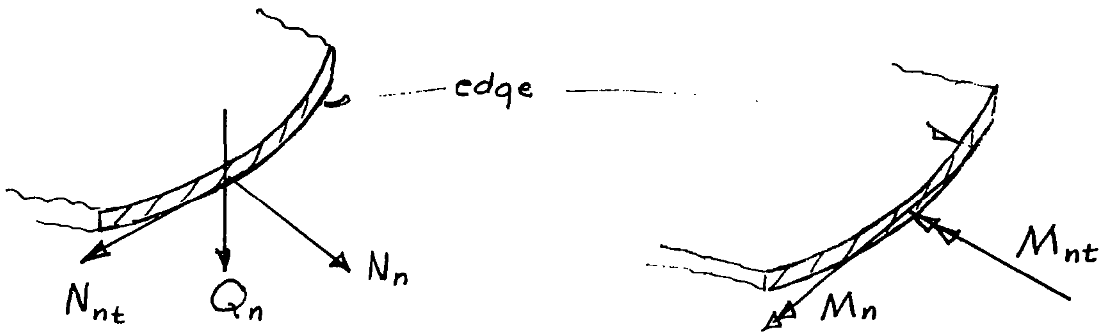
APPENDIX I FORMULATION FOR GREEN'S FUNCTION SOLUTION

The natural frequencies and mode shapes are sought for a segment of a spherical shell subject to specified boundary conditions. As indicated in the report text, a substitute static equivalent problem is instead considered, in which a complete spherical shell on an elastic foundation is loaded over that portion of the spherical surface enclosed by a contour C (corresponding to the edge of the segment) and loaded by line loads and moments, along the contour C. The conditions to be met in order to make the two problems equivalent are (1) the static distributed surface force must be proportional to displacement (2) the desired boundary conditions must be enforced along the contour C.

To simplify the present discussion, we consider the case of a free boundary (along contour C); that is, the stress resultants (per unit length) along the edge must be zero. According to Kirchhoff's formulation, V_n, N_n, S_{nt}, and M_n must each vanish at the boundary, where referring to the sketches,



Forces/unit length   Force couples/unit length

N_n & N_{nt} are membrane stress resultants

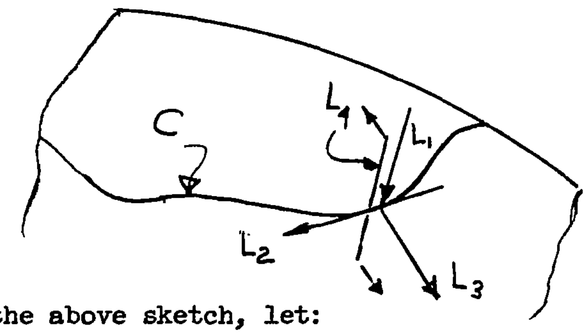
M_{nt} is the twisting moment

$V_n = (Q_n + \frac{\partial M_{nt}}{\partial s})$, a static equivalent normal edge reaction.

$S_{nt} = (N_{nt} + \frac{M_{nt}}{a})$, a static equivalent to tangential edge reaction.

If we denote the four quantities V_n, N_n, S_{nt}, M_n by R_1, R_2, R_3, R_4 and refer to them as boundary condition "residuals", we note the boundary conditions are met if, $R_i = 0$ along the contour C.

Next consider possible "line loads" applied to the complete sphere along the contour, which loads are to assist in meeting the boundary conditions.



Referring to the above sketch, let:

L_1, L_2, L_3 be rectangular components of force (per unit length) which are respectively normal to the surface, tangential to the curve C, and in the third orthogonal direction.

L_4 be a force couple (per unit length) with axis tangential to the curve C.

If we introduce a reference polar axis for the complete sphere, surface points may be located by the polar coördinates ϕ and θ, the latitudinal and meridinal angles, respectively, (ϕ measured from the pole).

Let two three-dimensional vectors q_α, u_α (Greek indices for three dimensions) be defined such that:

q_1, q_2, q_3 are components of the applied surface force (per unit surface area) in normal direction, tangential to the meridian circle, and tangential to the latitude circle, respectively.

u_1, u_2, u_3 are the displacement components of the shell middle surface (in the normal, and two tangential directions, respectively).

Next we define the Green's functions:

$A_{\alpha\beta}(\phi,\theta,\bar{\phi}\bar{\theta})$ as the displacement u_α (at point ϕ,θ) due to an applied force vector p_β (at point $\bar{\phi},\bar{\theta}$) whose components are each unity.

$B_{i\beta}(s,\bar{\phi},\bar{\theta})$ as the boundary condition residuals R_i (at point $\underline{s}$ on the contour C) due the same applied force vector p_β(at $\bar{\phi},\bar{\theta}$)

$C_{ij}(s,t)$ as the boundary condition residuals R_i (at point $\underline{s}$ on contour C) due to an applied line load system K_j (at point $\underline{t}$ on contour C) whose components are each unity.

$D_{\alpha j}(\phi,\theta,t)$ as the displacement u_α (at point ϕ,θ) due to the same line load system K_j (at point $\underline{t}$).

Then for the applied force, q_β acting over the surface element $\underline{d\sigma}$, we may express the resulting contribution to the boundary condition residuals:

$$dR_i = \sum_{\beta=1}^{3} q_\beta \, d\sigma \; B_{i\beta}$$

or employing the summation convention (for repeated indices):

$$dR_i(t) = q_\beta \, B_{i\beta} \, d\sigma \quad \text{(a function of } \underline{t}\text{)}.$$

Similarly, for an applied force system, L_i acting over the arc length ds (of the contour C), the contribution to residuals is:

$$dR_i = L_j \, C_{ij} \, ds$$

Thus through superposition, the requirement that residuals vanish along contour C is met if,

$$\iint_S B_{i\beta} \; (t,\bar{\phi},\bar{\theta}) \; q_\beta \; (\bar{\phi},\bar{\theta}) \; d\sigma + \int_C \; C_{ij} \; (t,s) \; L_j(s) \; ds = 0 \qquad \text{(1a)}$$

In a similar fashion, the contributions to displacement, from the applied surface load q_α (acting on $d\sigma$) and the line load L_i (acting over ds) are respectively:

$$du_\alpha = q_\beta A_{\alpha\beta} \, d\sigma \qquad ; \qquad du_\alpha = L_j \, D_{\alpha j} \, ds$$

and hence, the displacement at surface point (ϕ,θ) is:

$$\iint_S A_{\alpha\beta}(\phi,\theta,\bar{\phi},\bar{\theta}) \, q_\beta(\bar{\phi},\bar{\theta}) \, d\sigma + \int_C D_{\alpha j}(\phi,\theta,s) \, L_j(s) \, ds = u_\alpha(\phi,\theta) \qquad (2a)$$

The formulation of the problem is completed with the further requirement that

$$q_\alpha(\phi,\theta) = \frac{1}{\lambda} \, u_\alpha(\phi,\theta) \qquad (3b)$$

where λ is the eigenvalue of the problem.

Were the boundary conditions specified entirely in terms of displacements, as in a fixed edge support, then w, v, u and $\frac{dw}{dn}$ must vanish along the contour C. The formulation would remain the same except that the boundary condition residuals, R_i, would in this case be defined as

$$R_1 = w \, , \quad R_2 = v \quad , \quad R_3 = u \quad , \quad R_4 = \frac{dw}{dn}$$

where:

w, v, u are displacement components along the contour.

$\frac{dw}{dn}$ is the normal derivative of w along the contour.

In a similar fashion, other boundary conditions (for example: simply supported or elastically supported edges) may guide the specific nature of the formulation.

APPENDIX II SPHERICAL GEOMETRY

In the development of the Green's functions (discussed under Mechanics of Solution in the report text), use is made of two surface coordinate systems, one fixed in space, the other oriented with respect to a given stimulus and a given response point. To develope geometric relationships between these two coordinate systems and the transformation relationships for surface vectors (or tensors), a third coordinate system is introduced: a rectangular cartesian coordinate system with an origin at the center of the sphere, y-axis along the fixed polar axis and x-axis in the fixed meridianal plane, $\theta = 0$.

As the Fundamental Problems I and II, which form the basis for the Green's functions, are expressed in terms of the relative coordinate system and in particular the polar angle $\hat{\varphi}$ in that system, an expression is required for $\hat{\varphi}$.

Let point P be the stimulus point and have coordinates, $\bar{\varphi}$, $\bar{\theta}$ in the fixed system. Also let point Q be the response point having coordinates φ , θ in the fixed system. In the relative system P is at the pole and $\hat{\Phi}$ is the polar angle of Q. For convenience, we assume a unit sphere. Then the position vectors of P and Q in the r.c.c. system have components:

$$\vec{p} = \sin \bar{\Phi} \cos \bar{\theta}\, \vec{i} + \cos \bar{\Phi}\, \vec{j} + \sin \bar{\Phi} \sin \bar{\theta}\, \vec{k}$$
$$\vec{q} = \sin \Phi \cos \theta\, \vec{i} + \cos \Phi\, \vec{j} + \sin \Phi \sin \theta\, \vec{k} \qquad (1c)$$

respectively.

The angle between these vectors is also the polar angle $\hat{\varphi}$ in the relative system. Hence,

$$\cos \hat{\Phi} = \vec{p} \cdot \vec{q} , \quad \sin \hat{\Phi} = \sqrt{1 - \cos^2 \hat{\Phi}} \qquad (2)$$

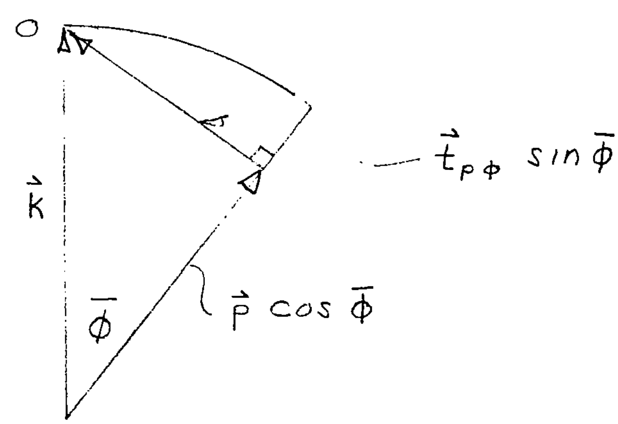
Using equations (1c) and (2c) in concert, then $\hat{\varphi}$ and its functions may be found.

At point P, stimuli may appear as either scalar or vector (two dimensional) quantities. The unit normal load is a scalar while the tangential force has two components, as does the unit moment about a tangential axis. The Green's function stimuli are in terms of the fixed coordinate system with positive tangential directions (1) toward the pole and (2) in the increasing θ direction. To transform these vector quantities from the fixed system to the relative system so that the Fundamental Problem solutions may be applied, the angle γ (between the tangent at P toward the fixed pole O and the tangent at P toward Q) is required.

A unit tangent vector at P towards O is given by:

$$\vec{t}_{p\varphi} = (\vec{k} - \cos\bar{\varphi}\,\vec{p})\csc\bar{\phi} \tag{3c}$$

as may be verified from vector addition and reference to the sketch below.



In a similar fashion the unit tangent vector at P toward Q is:

$$\vec{t}_{pq} = (\vec{q} - \text{cox}\,\hat{\phi}\,\vec{p})\csc\hat{\phi} \tag{4c}$$

The inner product of these two vectors then yields:

$$\cos\gamma = \vec{t}_{p\phi}\cdot\vec{t}_{pq} \quad\text{and}\quad \sin\gamma = \sqrt{1-\cos^2\gamma} \tag{5c}$$

Then also,

$$\cos\bar{\delta} = \vec{t}_{q\phi}\cdot\vec{t}_{qp} \quad\text{and}\quad \sin\delta = \sqrt{1-\cos^2\delta} \tag{6c}$$

where:

$$\vec{t}_{q\phi} = (\bar{k} - \cos\phi\,\vec{q})\csc\varphi$$
$$\vec{t}_{qp} = (\vec{p} - \cos\hat{\phi}\,\vec{p})\csc\hat{\phi}$$

In the relative coordinate system let the positive senses for vector components be (1) towards Q and (2) in the direction defined by the unit vector $\vec{p} \times \vec{t}_{pq}$. Then it may be verified that the transformation law is:

$$S_i' = a_{ij} S_j \qquad \text{(summation convention)} \tag{7c}$$

where

S_i' are stimuli components in the relative system

S_j are the stimuli components in the fixed system

$$a_{ij} \sim \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

At point P ; responses may be either scalar, vector, or second order tensor quantities (two dimensional). (An example of the latter is the moment tensor, M_{ij}.) In the relative coordinate system the positive senses for components is (1) towards P and (2) in the direction defined by the unit vector $\vec{q} \times \vec{t}_{qp}$.

Then the transformation laws become:

$$T_i = b_{ij} T_j' \tag{8c}$$

$$T_{ij} = b_{ij} b_{jm} T_{km}'$$

where

T_i, T_{ij} are first and second order response tensors in the fixed system

T_j', T_{km}' are the same tensors in the relative system

$$b_{ij} \sim \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix}$$

Employing transformation relations (7c) and (8c) the Green's functions may be expressed directly in terms of solutions of the Fundamental Problems I and II.