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VARIATIONAL PROBLEMS AND THEIR SOLUTION BY THE ADJOINT METHOD

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ABSTRACT

A variational problem typical of those encountered in flight mechanics is posed. The adjoint technique is then developed in a manner to indicate its application to general two-point boundary value problems. It is then specifically applied to the variational problem. Finally, the practicality of the adjoint method is illustrated by solving three typical problems of exo-atmospheric flight and one problem involving an ascent through the atmosphere to low earth orbit.

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SUMMARY

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I. INTRODUCTION

Many flight mechanical problems require that some "cost" criterion be extremized. A typical criterion is that the scientific payload of a rocket booster be maximized. Besides maximizing the payload, certain functional relationships among the "state" variables of the vehicle must invariably be satisfied at the end point. The calculus of variations, a branch of classical mathematics under development for over 250 years, forms the basis for handling optimization problems of this type. Unfortunately, this mathematical treatment leads to two-point boundary value problems which in themselves are important stumbling blocks to a straightforward application of the theory. The adjoint method has been developed as a tool to solve the two-point boundary value problem which in turn allows the ideas and results of the classical calculus of variations to be applied in a great variety of interesting and practical problems. The remainder of this text is devoted to formulating the variational problem, developing the adjoint method of solution to the split boundary value problem, and applying the resultant theory numerically to some typical trajectory problems. No claim for mathematical rigor is made. Necessary continuity requirements and the existence of derivatives of the required order are assumed. The treatments and developments of the adjoint method are for the most part formal in nature but their utility and worth have been borne out by the solution of a number of practical problems.

II. THE VARIATIONAL PROBLEM

Consider a set of ordinary first order differential equations:

$$\dot{x}_i = f_i(x_1, \ldots, x_m, u_1, \ldots, u_n) \quad i = 1, \ldots, m \tag{2.1}$$

$$= f_i(x,u)$$

where the x_i defining the "state" of the system are the dependent variables and the $u_1, \ldots, u_n$ are the control or forcing variables which are implicit functions of t, the independent variable, which here is taken as time. A solution of this system in an interval $t_o \leqq t \leqq t_f$ is given by m functions, $x_i(t)$, and n functions, $u_j(t)$, such that their substitution reduces equations (2.1) to identities. The system is said to have n degrees of freedom. The state values at t_o, together with t_o, are termed the initial boundary and the state values at t_f together with t_f are called the terminal boundary. The variational problem consists of extremizing a functional

$$J = \emptyset(x_1, \ldots, x_m)_{t_f} + \int_{t_o}^{t_f} f_o(x_1, \ldots, x_m, u_1, \ldots, u_n)\, dt \tag{2.2}$$

$$= \emptyset(x)_{t_f} + \int_{t_o}^{t_f} f_o(x,u)\, dt$$

subject to the differential constraints (2.1) and the terminal constraints

$$\psi_\ell\, (x_1, \ldots, x_m)_{t_f} - E_\ell = 0 \quad \ell = 1, \ldots, q \tag{2.3}$$
$$q \leqq m + 1$$

where the E_ℓ are given constants. Employing the method of Lagrange, an extremal requires that the first variation of the following functional vanish:

$$J = \emptyset(x_1, \ldots, x_m)_{t_f} + \int_{t_o}^{t_f} \Big[f_o(x_1, \ldots, x_m, u_1, \ldots, u_n)$$

$$+ q_i \Big(f_i(x, \ldots, x_m, u_1, \ldots, u_n) - \dot{x}_i \Big) \Big] dt + \nu_\ell \Big[\psi_\ell(x_1, \ldots, x_m)_{t_f} \Big] \tag{2.4}$$

where it is understood that a repeated subscript implies summation over its range, the $q_i(t)$ are Lagrange multiplier functions, and ν_ℓ are Lagrange multiplier constants.

The vanishing of δJ leads to the following necessary conditions:

$$\dot{q}_i = - \left.\frac{\partial H}{\partial x_i}\right)_{u,q} \qquad i = 1, \ldots, m \tag{2.5}$$

$$\left.\frac{\partial H}{\partial u_j}\right)_{x,q} = 0 \qquad j = 1, \ldots, n \tag{2.6}$$

where

$$H(x,u,q) = f_o(x,u) + q_i f_i(x,u) \tag{2.7}$$

is the Hamiltonian,

$$q_i(t_f) = \left.\frac{\partial \emptyset}{\partial x_i}\right)_{t_f} - \nu_\ell \left.\frac{\partial \psi_\ell}{\partial x_i}\right)_{t_f} \tag{2.8}$$

$$\Big[f_o(x,u) + q_i f_i(x,u) \Big]_{t_f} = \Big[\nu_\ell \frac{\partial \psi_\ell}{\partial t} - \frac{\partial \emptyset}{\partial t} \Big]_{t_f} . \tag{2.9}$$

$$i = 1, \ldots, m$$
$$\ell = 1, \ldots, q$$

Notice that H is a constant of the system since

$$\frac{dH}{dt} = \left.\frac{\partial H}{\partial x_i}\right)_{u,q} \dot{x}_i + \left.\frac{\partial H}{\partial q_i}\right)_{x,u} \dot{q}_i + \left.\frac{\partial H}{\partial u_j}\right)_{x,q} \dot{u}_j .$$

Substituting (2.1), (2.5), and (2.6) further yields

$$\frac{dH}{dt} = -\dot{q}_i\dot{x}_i + f_i\dot{q}_i + 0 = -\dot{q}_i\dot{x}_i + \dot{q}_i\dot{x}_i = 0.$$

The necessary conditions given have implicitly assumed that the initial state boundary is known. Equations (2.5) and (2.6) are the usual Euler-Lagrange equations. Equations (2.8) and (2.9) express the transversality conditions at the terminal boundary. A solution to the variational problem consists of finding functions x(t), u(t) and q(t) in the interval $t_o \leqq t \leqq t_f$ such that equations (2.1) and (2.5) are reduced to identities, equation (2.6) is satisfied and the boundary conditions, equations (2.3), (2.8) and (2.9) are satisfied. The u(t) are implicitly defined by (2.6) which means that solutions of the differential system (2.1) and (2.6) are determined by 2m + 2 boundary values of which m + 1 values are provided by the assumption that the initial state boundary is known and the remaining m + 1 values are obtained from the m + q + 1 terminal conditions, (2.3), (2.8), and (2.9) by eliminating $\nu_1, \ldots, \nu_q$.* Thus, an extremal requires the solution of a two-point boundary value problem. The nature of such a problem, as well as the fact that (2.1) and (2.5) are normally highly nonlinear, means that an analytic solution is rarely possible and a numerical solution must be sought. Numerical solutions require that one complete boundary be known. The classical technique of solution is to guess the unknown initial boundary values and to adjust them until the terminal boundary conditions are satisfied. This transforms the split boundary value problem into an initial value problem. The manner of solution described is called the indirect method since the control variables are obtained implicitly as functions of time and are not specified explicitly and then modified in a manner such that (2.3) is satisfied while simultaneously extremizing (2.2). Steepest descent techniques are of this latter type which are called direct methods.

*If (2.3) are m in number, then m of (2.8) and (2.9) are superfluous. If (2.3) are m + 1 in number, both (2.8) and (2.9) are unnecessary.

III. THE ADJOINT SOLUTION TO TWO-POINT BOUNDARY VALUE PROBLEMS

Consider a differential system of the following type:

$$\dot{z}_\alpha = F_\alpha(z_1, \ldots, z_{2m}) \quad \alpha = 1, \ldots, 2m \tag{3.1}$$

$$= F_\alpha(z).$$

The F_α are supposed to be functions of the differential dependent variables only and the comments concerning (2.1), except for the control variables, are applicable. Let

$$\delta z_\alpha(t) = z_\alpha(t) - \bar{z}_\alpha(t), \tag{3.2}$$

where $\bar{z}_\alpha(t)$ is a nominal solution of (3.1), $z_\alpha(t)$ is a varied solution and the concept of a variation symbolized by δ is discussed in Appendix I. The variations in z(t) obviously induce (to first order) a variation in the $F_\alpha(t)$ as follows:

$$\delta F_\alpha = F_\alpha(z + \delta z) - F_\alpha(z) = \frac{\partial F_\alpha}{\partial z_\beta}\,\delta z_\beta. \quad \beta = 1, \ldots, 2m \tag{3.3}$$

Combining (3.1) and (3.3) yields

$$\delta(\dot{z}_\alpha) = \frac{d}{dt}\,(\delta z_\alpha) = \delta F_\alpha = \frac{\partial F_\alpha}{\partial z_\beta}\,\delta z_\beta$$

or

$$\frac{d}{dt}\,(\delta z_\alpha) = \frac{\partial F_\alpha}{\partial z_\beta}\,\delta z_\beta \tag{3.4}$$

where the interchange of d/dt and δ are justified in Appendix I. The equations adjoint to (3.4) are defined as

$$\dot{p}_\alpha = - \frac{\partial F_\beta}{\partial z_\alpha} p_\beta, \quad \alpha, \beta = 1, \ldots, 2m \tag{3.5}$$

where the p_α are the adjoint variables. The usefulness of the adjoint variables (also called the influence functions) is indicated by the following:

$$\begin{aligned} \frac{d}{dt} (p_\alpha \, \delta z_\alpha) &= \dot{p}_\alpha \, \delta z_\alpha + p_\alpha \frac{d}{dt} (\delta z_\alpha) \\ &= - \frac{\partial F_\beta}{\partial z_\alpha} p_\beta \, \delta z_\alpha + p_\alpha \frac{\partial F_\alpha}{\partial z_\beta} \delta z_\beta \\ &= 0 \end{aligned} \tag{3.6}$$

where (3.4) has been substituted and the fact that α and β are dummy summation symbols has been used. Since $p_\alpha \, \delta z_\alpha$ is a constant of the system, obviously

$$\left. p_\alpha \, \delta z_\alpha \right)_{t_o} = \left. p_\alpha \, \delta z_\alpha \right)_{\bar{t}_f}, \tag{3.7}$$

where $\bar{t}_f$ is the nominal terminal time. Herein lies the usefulness of the adjoint variables; they relate variations at the nominal terminal boundary to variations at the initial boundary. Again let boundary conditions of the following form be imposed at the initial and terminal boundaries:

$$\omega^{(i)} (z)_{t_o} = 0 \qquad i = 1, \ldots, p \tag{3.8a}$$

$$\Omega^{(j)} (z)_{t_f} = 0. \qquad j = p + 1, \ldots, 2m - p + 1 \tag{3.8b}$$

Boundary values on the adjoint variables have not been imposed. A meaningful interpretation can be attached to the following choice for the j^{th} set of boundary values at the nominal terminal time:

$$p_\alpha^{(j)}\Big)_{\bar{t}_f} = \frac{\partial\Omega^{(j)}}{\partial z_\alpha}\Big)_{\bar{t}_f} . \qquad \begin{array}{l} j = p + 1, \ldots, 2m - p + 1 \\ \alpha = 1, \ldots, 2m \end{array} \tag{3.9a}$$

As shown in Appendix I, at the nominal terminal time,

$$dz_\alpha\Big)_{\bar{t}_f} = \Big(\delta z_\alpha + \dot{z}_\alpha \, dt\Big)_{\bar{t}_f} , \tag{3.10}$$

where dz_α is the total differential change in z_α at $\bar{t}_f$. Thus,

$$p_\alpha^{(j)} \, \delta z_\alpha\Big]_{\bar{t}_f} = \Big[p_\alpha^{(j)} \, (dz_\alpha - \dot{z}_\alpha \, dt)\Big]_{\bar{t}_f} = \Big[\frac{\partial\Omega^{(j)}}{\partial z_\alpha} \, (dz_\alpha - \dot{z}_\alpha \, dt)\Big]_{\bar{t}_f} . \tag{3.11}$$

Substituting the relations

$$\begin{aligned} d\Omega^{(j)} &= \frac{\partial\Omega^{(j)}}{\partial z_\alpha} \, dz_\alpha \\ \dot{\Omega}^{(j)} &= \frac{\partial\Omega^{(j)}}{\partial z_\alpha} \, \dot{z}_\alpha \end{aligned} \tag{3.12}$$

into (3.11) results in

$$p_\alpha^{(j)} \, \delta z_\alpha\Big]_{\bar{t}_f} = \Big[d\Omega^{(j)} - \dot{\Omega}^{(j)} \, dt\Big]_{\bar{t}_f} . \tag{3.13}$$

Substituting (3.13) into (3.7) yields

$$p_\alpha^{(j)} \; \delta z_\alpha \Big]_{t_o} = \Big[d\Omega^{(j)} - \dot{\Omega}^{(j)} \; dt \Big]_{\bar{t}_f}$$

or

$$p_\alpha^{(j)} \; \delta z_\alpha \Big]_{t_o} + \dot{\Omega}^{(j)} \; dt \Big]_{\bar{t}_f} = d\Omega^{(j)} \Big]_{\bar{t}_f} \; . \tag{3.14a}$$

p of the $\delta z)_{t_o}$ are eliminated in (3.14a) by using (3.8a). Assuming the $d\Omega^{(j)}_{\bar{t}_f}$ are specified, (3.14a) represents 2m - p + 1 equations for one unknown dt_f and 2m - p unknowns, $\delta z_i)_{t_o}$. Equations (3.14a) are generated by 2m - p + 1 integrations of the adjoint equations. This number may be reduced by one if the nominal terminal time is determined by one of the terminal boundary conditions being satisfied, say $\Omega^{(2m-p+1)}(z)_{\bar{t}_f} = 0$. Then,

$$d\Omega^{(2m-p+1)}(z)_{\bar{t}_f} = \frac{\partial \Omega^{(2m-p+1)}}{\partial z_\alpha} \; dz_\alpha \Big]_{\bar{t}_f} = 0. \qquad i = 1, \ldots, 2m$$

If the adjoint equations are integrated with boundary conditions,

$$p_\alpha^{(j)} \Big)_{\bar{t}_f} = \left[\frac{\partial \Omega^{(j)}}{\partial z_\alpha} - \frac{\dot{\Omega}^{(j)}}{\dot{\Omega}^{(2m-p+1)}} \left(\frac{\partial \Omega^{(2m-p+1)}}{\partial z_\alpha} \right) \right]_{\bar{t}_f} , \tag{3.9b}$$

a straightforward calculation shows that (3.14a) reduce to

$$p_\alpha^{(j)} \; \delta z_\alpha \Big]_{t_o} = d\Omega^{(j)} \Big]_{\bar{t}_f} . \qquad j = 1, \ldots, 2m - p \tag{3.14b}$$

Of course, these last operations are not possible if the designated terminal boundary condition cannot be satisfied from the initial boundary.

The implementation of the adjoint method is an iterative process. This means that each successive step is dependent on the previous step. The process is said to have converged when the differential equations and the terminal boundary conditions are reduced to identities by the solution functions. When t_o is fixed, a nominal solution $\bar{z}(t)$ is generated by assuming a nominal $\bar{t}_f$, guessing the missing initial boundary values, and integrating (3.1) forward in time until $t = \bar{t}_f$. Normally the

$$d\Omega^{(j)}\Big)_{\bar{t}_f} \neq 0,$$

and changes in $\bar{t}_f$ and the free initial boundary values are necessary. These changes are computed by using the initial conditions of (3.9a) to integrate the adjoint equations (3.5) backward in time until $t = t_o$. This yields the

$$p^{(j)}(t_o)$$

needed for (3.14a). Since (3.14a) is generally the result of a linear treatment of a nonlinear system, it is unreasonable to expect it to predict the whole correction necessary during the early steps of the convergence process. Consequently, only fractional portions of the terminal boundary condition violations are used in (3.14a) initially; i.e., the

$$d\Omega^{(j)}\Big)_{\bar{t}_f}$$

are replaced by

$$C_k d\Omega^{(j)}\Big)_{\bar{t}_f}, \qquad k = 1, \ldots, 2m - p + 1$$

where $0 < C_k \leqq 1$. As convergence is attained, the C_k are increased since the linear approximation is becoming better and better. Using these

$$C_k d\Omega^{(j)}\Big)_{\bar{t}_f},$$

(3.14a) is used to calculate the new initial boundary as

$$z_i\Big)_{t_o}^{new} = z_i\Big)_{t_o}^{old} + \delta z_i\Big)_{t_o}$$

and the new terminal time as

$$\bar{t}_f^{new} = \bar{t}_f^{old} + d\bar{t}_f.$$

The state equations are again integrated forward in time until $t = \bar{t}_f^{new}$ Now the

$$d\Omega^{(j)}\Big)_{\bar{t}_f^{new}}^{new}$$

will have either increased or decreased.

(1) If they have increased, the changes calculated are too large and the linearity assumptions have obviously been violated. This is corrected by halving the computed changes and reintegrating the state equations forward. This secondary iteration is done as many times as required to decrease the terminal boundary condition violations. When they have finally been reduced, the adjoint equations are again integrated backwards according to the previous step.

(2) If they have decreased, the adjoint equations are immediately integrated backwards and the previous step repeated.

This process continues until the terminal boundary conditions are satisfied to within some tolerance. The secondary iteration described above is essential since it can reduce the number of times the adjoint equations are integrated. This results in a large time saving since these equations are usually rather lengthy and involved.

IV. THE ADJOINT SOLUTION OF THE VARIATIONAL PROBLEM

Seeking to limit the scope of the discussion while at the same time illustrating the application of the adjoint technique to a problem of some interest, a particular type of variational problem is considered here: the time minimal problem. The functional to be minimized is, thus,

$$J = \int_{t_o}^{t_f} dt = t_f - t_o, \tag{4.1}$$

i.e., $f_o(x_1, \ldots, x_m, u_1, \ldots, u_n) = 1$. Further assumptions are that the initial boundary and the terminal boundary (except for t_f) are known. The problem, then, is to move from an initial known state to a terminal known state in the least time. In order that the scheme outlined in Section III be applicable to this variational problem, it is necessary that the dependence of the differential equations on the control variables be removed. This can be done in principle by using (2.6) where the relations are made explicit as

$$u_j = u_j(x,q). \qquad j = 1, \ldots, n \tag{4.2}$$

The differential equations of this variational problem become

$$\dot{x}_i = f_i\Big(x,\ u(x,q)\Big) \tag{4.3}$$

$$\dot{q}_i = g_i\Big(x,\ u(x,q),\ q\Big) = -\, q_k \frac{\partial f_k}{\partial x_i} \qquad \begin{matrix} k,i = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix} \qquad (4.4$$

$$q_i\left[\frac{\partial f_i}{\partial u_j}\right]_x = 0, \tag{4.5}$$

subject to the boundary conditions

$$\Big[\psi_\ell(x) - E_\ell\Big]_{t_f} = 0 \qquad \ell = 1, \ldots, q \tag{4.6}$$

$$\left[q_i + \nu_\ell \frac{\partial\psi_\ell}{\partial x_i}\right]_{t_f} = 0 \qquad (4.7)$$

$$\left[1 + q_i f_i\Big(x,\ u(x,q)\Big)\right]_{t_f} = 0. \qquad (4.8)$$

Equations (4.3) and (4.4) can be written in the form (3.1) by making the identifications

$$\begin{aligned} z_\alpha &\equiv x_i, \quad \dot{z}_\alpha \equiv f_i \qquad & \alpha &= 1, \ldots, m \\ z_\alpha &\equiv q_i, \quad \dot{z}_\alpha \equiv g_i. \qquad & \alpha &= m+1, \ldots, 2m \end{aligned} \qquad (4.9)$$

Then,

$$\dot{z}_\alpha = F_\alpha(x,q). \qquad \alpha = 1, \ldots, 2m.$$

Boundary conditions (4.6), (4.7), and (4.8) are m + 1 in number if the $\nu_1, \ldots, \nu_q$ are eliminated and correspond to (3.8) which are m + 1 in number since the initial state boundary has been assumed known. Consequently, the formulation of Section III is applicable. The adjoint equation (3.5) may be conveniently rewritten with indices varying from 1 to m as

$$\dot{p}_i^x = -\left.\frac{\partial f_k}{\partial x_i}\right)_q p_k^x - \left.\frac{\partial g_k}{\partial x_i}\right)_q p_k^q \qquad (4.10a)$$

$$\dot{p}_i^q = -\left.\frac{\partial f_k}{\partial q_i}\right)_x p_k^x - \left.\frac{\partial g_k}{\partial q_i}\right)_x p_k^q \qquad (4.10b)$$

where the superscripts have been used to indicate the correspondence between the subscripts of (3.5) and the identifications in (4.9).

To calculate the partial derivatives in (4.10), the following differentials of (4.2), (4.3) and (4.4) are required:

$$du_j = \left.\frac{\partial u_j}{\partial x_i}\right)_q dx_i + \left.\frac{\partial u_j}{\partial q_i}\right)_x dq_i \tag{4.11}$$

$$df_i = \left.\frac{\partial f_i}{\partial x_k}\right)_u dx_k + \left.\frac{\partial f_i}{\partial u_j}\right)_x du_j \qquad \begin{matrix} i,k = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix} \tag{4.12}$$

$$dg_i = \left.\frac{\partial g_i}{\partial x_k}\right)_{u,q} dx_k + \left.\frac{\partial g_i}{\partial u_j}\right)_{x,q} du_j + \left.\frac{\partial g_i}{\partial q_k}\right)_{x,u} dq_k. \tag{4.13}$$

Substituting (4.11) into (4.12) and (4.13) and rearranging yields

$$df_i = \left[\left.\frac{\partial f_i}{\partial x_k}\right)_u + \left.\frac{\partial f_i}{\partial u_j}\right)_x \left.\frac{\partial u_j}{\partial x_k}\right)_q\right] dx_k + \left[\left.\frac{\partial f_i}{\partial u_j}\right)_x \left.\frac{\partial u_j}{\partial q_k}\right)_x\right] dq_k \tag{4.14}$$

$$dg_i = \left[\left.\frac{\partial g_i}{\partial x_k}\right)_{u,q} + \left.\frac{\partial g_i}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial x_k}\right)_q\right] dx_k + \left[\left.\frac{\partial g_i}{\partial q_k}\right)_{x,u} + \left.\frac{\partial g_i}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial q_k}\right)_x\right] dq_k. \tag{4.15}$$

By holding x or q constant as required, (4.14) and (4.15) can be used to calculate the partial derivatives in (4.10). The results are

$$\dot{p}_i^x = -\left[\left.\frac{\partial f_k}{\partial x_i}\right)_u + \left.\frac{\partial f_k}{\partial u_j}\right)_x \left.\frac{\partial u_j}{\partial x_i}\right)_q\right] p_k^x - \left[\left.\frac{\partial g_k}{\partial x_i}\right)_{u,q} + \left.\frac{\partial g_k}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial x_i}\right)_q\right] p_k^q \tag{4.16}$$

$$\dot{p}_i^q = -\left[\left.\frac{\partial f_k}{\partial u_j}\right)_x \left.\frac{\partial u_j}{\partial q_i}\right)_x\right] p_k^x - \left[\left.\frac{\partial g_k}{\partial q_i}\right)_{x,u} + \left.\frac{\partial g_i}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial q_k}\right)_x\right] p_k^q. \tag{4.17}$$

The only unknowns in the last two equations are

$$\left.\frac{\partial u_j}{\partial x_i}\right)_q \quad \text{and} \quad \left.\frac{\partial u_j}{\partial q_i}\right)_x .$$

These partial derivatives may be calculated from (4.5) as follows:

$$h_r(x,u,q) = q_i \left.\frac{\partial f_i}{\partial u_r}\right)_x = 0$$

$$dh_r = \left.\frac{\partial h_r}{\partial x_k}\right)_{u,q} dx_k + \left.\frac{\partial h_r}{\partial u_j}\right)_{x,q} du_j + \left.\frac{\partial h_r}{\partial q_k}\right)_{x,u} dq_k = 0. \qquad (4.18)$$

Holding q constant in (4.18) yields

$$\left.\frac{\partial h_r}{\partial x_i}\right)_{u,q} + \left.\frac{\partial h_r}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial x_i}\right)_q = 0, \qquad \begin{array}{l} i,k = 1, \ldots, m \\ j,r = 1, \ldots, n \end{array} \qquad (4.19)$$

which represents n equations in the unknowns

$$\left.\frac{\partial u_j}{\partial x_i}\right)_q .$$

Consequently, if the array

$$\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q} -$$

where r is the row index and j is the column index - is nonsingular,

$$\left.\frac{\partial u_j}{\partial x_i}\right)_q = - \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial x_i}\right)_{u,q} \qquad (4.20)$$

Holding x constant in (4.18) yields

$$\left.\frac{\partial h_r}{\partial q_i}\right)_{x,u} + \left.\frac{\partial h_r}{\partial u_j}\right)_{x,q} \left.\frac{\partial u_j}{\partial q_i}\right)_x = 0$$

or

$$\left.\frac{\partial u_j}{\partial q_i}\right)_x = - \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial q_i}\right)_{x,u} . \tag{4.21}$$

With the aid of (4.20) and (4.21), the adjoint equations (4.16) and (4.17) become

$$\dot{p}_i^x = - \left[\left.\frac{\partial f_k}{\partial x_i}\right)_u - \left.\frac{\partial f_k}{\partial u_j}\right)_x \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial x_i}\right)_{u,q}\right] p_k^x$$

$$- \left[\left.\frac{\partial g_k}{\partial x_i}\right)_{u,q} - \left.\frac{\partial g_k}{\partial u_j}\right)_{x,q} \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial x_i}\right)_{u,q}\right] p_k^q \tag{4.22}$$

$$\dot{p}_i^q = - \left[- \left.\frac{\partial f_k}{\partial u_j}\right)_x \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial q_i}\right)_{x,u}\right] p_k^x$$

$$- \left[\left.\frac{\partial g_k}{\partial q_i}\right)_{x,u} - \left.\frac{\partial g_i}{\partial u_j}\right)_{x,q} \left[\left.\frac{\partial h_r}{\partial u_j}\right)_{x,q}\right]^{-1} \left.\frac{\partial h_r}{\partial q_k}\right)_{x,u}\right] p_k^q . \tag{4.23}$$

Equations (4.3) - (4.8), (4.22) and (4.23) can be expressed more succintly in terms of the Hamiltonian which, for this time minimal problem, is

$$H(x,u,q) = 1 + q_i f_i . \tag{4.24}$$

These equations can then be written

$$\dot{x}_i = \left.\frac{\partial H}{\partial q_i}\right)_{x,u} \tag{4.25}$$

$$\dot{q}_i = -\left.\frac{\partial H}{\partial x_i}\right)_{q,u} \tag{4.26}$$

$$\left.\frac{\partial H}{\partial u_j}\right)_{x,q} = 0 \tag{4.27}$$

$$\left.\psi_\ell(x) - E_\ell\right)_{t_f} = 0 \tag{4.28}$$

$$\left.q_i + \nu_\ell \frac{\partial \psi_\ell}{\partial x_i}\right)_{t_f} = 0 \tag{4.29}$$

$$H)_{t_f} = 0 \tag{4.30}$$

$$\begin{aligned}\dot{p}_i^x = &- \left[\left.\frac{\partial^2 H}{\partial x_i\, \partial q_k}\right)_{x,u,q} - \left.\frac{\partial^2 H}{\partial u_j\, \partial q_k}\right)_{u,x,q} \left[\frac{\partial^2 H}{\partial u_j\, \partial u_r}\right]^{-1}_{x,q,x} \left.\frac{\partial^2 H}{\partial x_i\, \partial u_r}\right)_{x,q,u}\right] p_k^x \\ &- \left[-\left.\frac{\partial^2 H}{\partial x_i\, \partial x_k}\right)_{u,q,u} + \left.\frac{\partial^2 H}{\partial u_j\, \partial x_k}\right)_{u,q,x} \left[\frac{\partial^2 H}{\partial u_j\, \partial u_r}\right]^{-1}_{x,q,x} \left.\frac{\partial^2 H}{\partial x_i\, \partial u_r}\right)_{x,q,u}\right] p_k^q\end{aligned} \tag{4.31}$$

$$\dot{p}_i^q = -\left[\left.\frac{\partial^2 H}{\partial u_j\, \partial q_k}\right)_{u,x,q} \left[\frac{\partial^2 H}{\partial u_j\, \partial u_r}\right]^{-1}_{x,q,x} \left.\frac{\partial^2 H}{\partial u_r\, \partial q_i}\right)_{u,q,x} \right] p_k^x$$

$$- \left[- \left.\frac{\partial^2 H}{\partial q_i \partial x_k}\right)_{q,u,x} + \left.\frac{\partial^2 H}{\partial u_j\, \partial x_i}\right)_{u,q,x} \left[\frac{\partial^2 H}{\partial u_j\, \partial u_r}\right]^{-1}_{x,q,x} \left.\frac{\partial^2 H}{\partial u_r\, \partial q_k}\right)_{u,q,x} \right] p_k^q, \tag{4.32}$$

where the nonstandard triple subscript notation indicates, for example, that

$$\left.\frac{\partial^2 H}{\partial x_i \partial q_k}\right)_{x,u,q}$$

is to be evaluated by finding the partial derivative of H with respect to q_k holding x and u constant and then finding the partial derivative of this result with respect to x_i by holding u and q constant. Equations (4.24) - (4.32) are convenient when an analytic derivative calculator such as IBM's FORMAC system is available, since all differential equations are essentially generated by the Hamiltonian.

A numerical solution of these equations proceeds as discussed in Section III and will not be repeated here. However, several modifications of the basic variational problem will be mentioned here. First, inequality constraints on the state and/or control variables can be introduced and handled by the adjoint technique. This will be the subject of a later report. Second, it can happen that some or all of the ν_ℓ's in (4.29) are all but impossible to eliminate analytically. If this happens they can be made part of the convergence process. Assuming all q of the ν_ℓ's are to be found, suppose q additional differential equations of the form $\dot{\nu}_1 = 0, \ldots, \dot{\nu}_q = 0$ are added to the basic set of state differential equations. Their solutions are, of course, constants whose correct values are those which reduce (4.29) to identities. It is easily verified that the Lagrange multipliers and adjoint variables introduced by these new equations are also constants. Consequently, numerical solutions of additional new differential equations are not required. However, formally, there are now a total of m + 2q + 1 terminal conditions of which q are of no interest; i.e., q of them are the final values of new Lagrange multipliers which do not enter into the problem solution at all. Therefore, there are m + q + 1 terminal boundary conditions to be satisfied. This requires q more integrations of the adjoint equations where boundary conditions are given by

$$p_{\alpha}^{(\ell)} = \frac{\partial \Omega^{(\ell)}}{\partial z_{\alpha}} \qquad \begin{aligned} &\ell = 2m - p + 1, \ldots, 2m - p + 1 + q \\ &\alpha = 1, \ldots, 2m + q \end{aligned}$$

and (3.14a) can again be used to calculate changes in the guessed ν_{ℓ}'s. Lastly, the adjoint technique can be used to calculate controls for neighboring extremals which can be useful in guidance analysis.

V. EXAMPLES

To reduce complexity, the differential equations of motion of the flight mechanical problems considered for these examples will be written in two dimensions. Further, the reference planet is considered spherical and thrust levels are constant. Under these conditions, the equations governing the motion of a rocket vehicle operating in a vacuum (i.e., exo-atmospheric) can be written

$$\dot{v} = f_1 = \frac{F}{m} \cos \alpha - \frac{GM}{r^2} \cos \vartheta$$

$$\dot{\vartheta} = f_2 = \frac{F}{mv} \sin \alpha + \left(\frac{GM}{r^2 v^2} - \frac{1}{r}\right) v \sin \vartheta \tag{5.1}$$

$$\dot{r} = f_3 = v \cos \vartheta$$

$$\dot{m} = f_4 = k$$

where

F is the rocket thrust,

GM is the product of the universal gravitational constant and the reference planet mass, v is the velocity,

m is the instantaneous rocket mass,

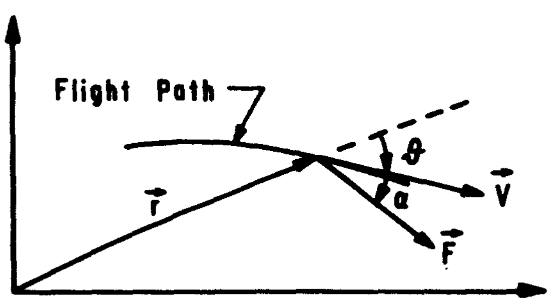
k is a given constant,

r is the distance of the rocket from the center of the reference planet,

ϑ is the angle between the radius vector $\vec{r}$ and the velocity vector $\vec{v}$,

α is the angle between the thrust vector $\vec{F}$ and the velocity vector $\vec{v}$.

Geometrical relations between these quantities are illustrated below.



There are a total of 8 variables, namely, r, v, ϑ, m, F, GM, k, and α of which 3 - F, GM, K - are known constants. Since v, ϑ, r, and m are determined from (5.1), there is one free variable α, the control variable, which can be varied to minimize the time of transfer between the given initial state and the given terminal state. In the development of the Euler-Lagrange equations and the adjoint equations, the following correspondences for the subscripts and variables are made:

1 => v	x_1 => v
2 => ϑ	x_2 => ϑ
3 => r	x_3 => r
4 => m	x_4 => m

which are read. Subscript 1 is replaced by v, subscript 2 is replaced by ϑ, etc. The Hamiltonian (4.24) becomes

$$H = 1 + q_v \left[\frac{F}{m}\cos\alpha - \frac{GM}{r^2}\cos\vartheta\right] + q_\vartheta \left[\frac{F}{mv}\sin\alpha + \left(\frac{GM}{r^2v^2} - \frac{1}{r}\right) v \sin\vartheta\right]$$

$$+ q_r(v\cos\vartheta) + q_m k. \tag{5.2}$$

The Euler-Lagrange equations (4.26) are

$$\begin{aligned}
\dot{q}_v &= \left(\frac{F}{mv^2}\sin\alpha + \frac{GM}{r^2v^2}\sin\vartheta + \frac{1}{r}\sin\vartheta\right) q_\vartheta - \cos\vartheta\, q_r \\
\dot{q}_\vartheta &= -\frac{GM}{r^2}\sin\vartheta\, q_v - \left(\frac{GM}{r^2v^2} - \frac{1}{r}\right) v\cos\vartheta\, q_\vartheta + v\sin\vartheta\, q_r \\
\dot{q}_r &= -\frac{2GM}{r^3}\cos\vartheta\, q_v + \left(\frac{2GM}{r^2v^2} - \frac{1}{r}\right)\frac{v\sin\vartheta}{r}\, q_\vartheta \\
\dot{q}_m &= \frac{F\cos\alpha}{m^2}\, q_v + \frac{F}{m^2v}\sin\alpha\, q_\vartheta.
\end{aligned} \tag{5.3}$$

The equation for the control variable (4.27) is

$$-\frac{F}{m}\sin\alpha\, q_v + \frac{F}{mv}\cos\alpha\, q_\vartheta = 0$$

or

$$\tan\alpha = \frac{q_\vartheta}{vq_v}. \tag{5.4}$$

The terminal boundary conditions (4.28) are

$$\Omega^{(1)} = v - V_f\Big)_{t_f} = 0$$

$$\Omega^{(2)} = \vartheta - \theta_f\Big)_{t_f} = 0 \tag{5.5}$$

$$\Omega^{(3)} = r - R_f\Big)_{t_f} = 0.$$

The terminal boundary conditions (4.29) and (4.30) are

$$q_v + \nu_v\Big)_{t_f} = 0$$

$$q_\vartheta + \nu_\vartheta\Big)_{t_f} = 0 \tag{5.6}$$

$$q_r + \nu_r\Big)_{t_f} = 0$$

$$q_m\Big)_{t_f} = 0 \tag{5.7}$$

$$H\Big)_{t_f} = 0. \tag{5.8}$$

Since (5.1) and (5.3) are a total of eight differential equations, ten boundary conditions are required. Five of these are given by the assumption that the initial boundary is known. Apparently, the remaining five come from (5.5) - (5.8). Equations (5.5) must be satisfied so that the remaining two conditions can be picked from (5.6) - (5.8). Actually, equations (5.6) yield no new information. Hence, the five additional boundary conditions occur on the terminal boundary and are given by (5.5), (5.7) and (5.8). A solution of the complete differential system proceeds by guessing $\bar{t}_f$ and the missing initial boundary values and varying them until the integration of (5.1) and (5.3), using (5.4) for the control, satisfies (5.5), (5.7) and (5.8) at some $\bar{t}_f = t_f$.

In the just outlined solution procedure, $\bar{t}_f$ and all four initial multiplier values were needed. Two artifices can be used to reduce these five initial guesses to three guesses, namely, $\bar{t}_f$ and two of the initial multiplier values. This can be done by noting that, first, the Euler-Lagrange equations are homogeneous in the multipliers and, second, that the first three of the Euler-Lagrange equations are independent of the fourth. Homogeneity means that for any solution set $\{(q_v(t), q_\vartheta(t), q_r(t), q_m(t)\}$, $\{kq_v(t), kq_\vartheta(t), kq_r(t), kq_m(t)\}$ is also a solution set where $k \neq 0$ is any real number. Consequently, for the multipliers solving the problem, there exists a k such that one of their initial values may be fixed at some number N. (N is often given the value 1.) For example, $\lambda(t_o) \neq 0$ of a solution set becomes 1 if $k = 1/\lambda(t_o)$. Requiring this condition at the outset reduces the initial guesses by 1. The second observation above, along with relaxing the requirement that (5.7) be satisfied, means that the initial value of $q_m(t_o)$ may be fixed at some convenient value, say 1. The motivation for doing so is that now there are three arbitrary parameters, $\bar{t}_f$, $q_v(t_o)$ and $q_\vartheta(t_o)$, say, to be picked (if a solution exists) such that the three boundary conditions (5.5) are satisfied. This handling of q_m introduces the simplification that now equations (3.14a) of the adjoint technique are three in number and easy to solve for the required initial corrections. Additionally, $q_m(t)$ can still be used in computing the integration constant H which is an aid in checking the accuracy of the integration algorithm. (Note that now $H \neq 0$.) The multiplier q_m could have been eliminated entirely from consideration, since by assumption m is a known function of time, but this would have introduced an explicit time dependency into the state differential equations which is not allowed in the formulation in Section II. (However, t can be introduced explicitly into (2.1) and (3.1) with no resultant difficulty in the analysis whatever.) It should be remembered that these contrivances for reducing the initial guesses from 5 to 3 are for convenience and expediency in solving the problem and that formally all five should be used to satisfy (5.5), (5.7) and (5.8). (It could be said that the three-guess solution finds incorrectly scaled multipliers while yielding the correct control program and the minimum transfer time.)

The adjoint equations (4.31) and (4.32) become, after some obvious cancellation and rearranging (p_m^x and p_m^q are ignored for the reasons given above):

$$\dot{p}_v^x = -\left[\left(\frac{1}{q_v\ v\ \cos\ \alpha + q_\vartheta\ \sin\ \alpha}\right)\left(\frac{F}{mv}\right) \sin\ \alpha\ \cos\ \alpha\right] p_v^x + \left[\frac{F}{mv^2}\ \sin\ \alpha\right.$$

$$\left. + \frac{GM}{r^2v^2}\ \sin\ \vartheta + \frac{\sin\ \vartheta}{r} + q_\vartheta \left(\frac{1}{q_v\ v\ \cos\ \alpha + q_\vartheta\ \sin\ \alpha}\right)\left(\frac{F}{mv^2}\right) \cos^2\alpha\right]\ p_\vartheta^x$$

$$- [\cos\ \vartheta]\ p_r^x + \left[\frac{2q_\vartheta}{v^3}\left(\frac{F}{m}\ \sin\ \alpha + \frac{GM}{r^2}\ \sin\ \vartheta\right)\right.$$

$$\left. + q_\vartheta^2 \left(\frac{1}{v\ q_v\ \cos\ \alpha + \sin\ \alpha\ q_\vartheta}\right)\left(\frac{F}{mv^3}\right) \cos^2\alpha\right] p_v^q - \left[q_\vartheta \left(\frac{GM}{r^2v^2} + \frac{1}{r}\right) \cos\ \vartheta\right.$$

$$\left. + q_r\ \sin\ \vartheta\right] p_\vartheta^q + \left[q_\vartheta \left(\frac{2GM}{r^2v^2} + \frac{1}{r}\right) \frac{\sin\ \vartheta}{r}\right] p_r^q. \tag{5.9a}$$

$$\dot{p}_\vartheta^x = -\left[\frac{GM}{r^2}\ \sin\ \vartheta\right] p_v^x - \left[\left(\frac{GM}{r^2v^2} - \frac{1}{r}\right) v\ \cos\ \vartheta\right] p_\vartheta^x + [v\ \sin\ \vartheta]\ p_r^x$$

$$- \left[q_\vartheta \left(\frac{GM}{r^2v^2} + \frac{1}{r}\right) \cos\ \vartheta + q_r\ \sin\ \vartheta\right] p_v^q + \left[q_v\ \frac{GM}{r^2}\ \cos\ \vartheta\right.$$

$$\left. - q_\vartheta \left(\frac{GM}{r^2v^2} - \frac{1}{r}\right) v\ \sin\ \vartheta - q_r\ v\ \cos\ \vartheta\right] p_\vartheta^q - \left[q_v\ \frac{2GM}{r^3}\ \sin\ \vartheta\right.$$

$$\left. + q_\vartheta \left(\frac{2GM}{r^2v^2} - \frac{1}{r}\right) \frac{v\ \cos\ \vartheta}{r}\right] p_r^q. \tag{5.9b}$$

$$\dot{p}_r^x = -\left[\frac{2GM}{r^3}\cos\vartheta\right]p_v^x + \left[\left(\frac{2GM}{r^2v^2}-\frac{1}{r}\right)\frac{v\sin\vartheta}{r}\right]p_\vartheta^x + \left[q_\vartheta\left(\frac{2GM}{r^2v^2}+\frac{1}{r}\right)\frac{\sin\vartheta}{r}\right]p_v^q$$

$$-\left[\frac{2q_\vartheta\,GM}{r^3}\sin\vartheta + q_\vartheta\left(\frac{2GM}{r^2v^2}-\frac{1}{r}\right)\frac{v\cos\vartheta}{r}\right]p_\vartheta^q$$

$$+\left[q_\vartheta\left(\frac{6GM}{r^4v^2}-\frac{2}{r^3}\right)v\sin\vartheta - \frac{6q_v\,GM}{r^4}\cos\vartheta\right]p_r^q\,. \tag{5.9c}$$

$$\dot{p}_v^q = -\left[\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)\left(\frac{Fv}{m}\right)\sin^2\alpha\right]p_v^x$$

$$+\left[\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)(F/m)\sin\alpha\cos\alpha\right]p_\vartheta^x + \left[\frac{GM}{r^2}\sin\vartheta\right]p_\vartheta^q$$

$$+\left[\frac{2GM}{r^3}\cos\vartheta\right]p_r^q + \left[q_\vartheta\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)\left(\frac{F}{mv}\right)\sin\alpha\cos\alpha\right]p_v^q. \tag{5.9d}$$

$$\dot{p}_\vartheta^q = \left[\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)(F/m)\sin\alpha\cos\alpha\right]p_v^x$$

$$-\left[\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)\left(\frac{F}{mv}\right)\cos^2\alpha\right]p_\vartheta^x - \left[\frac{F}{mv^2}\sin\alpha\right.$$

$$\left.+\left(\frac{GM}{r^2v^2}+\frac{1}{r}\right)\sin\vartheta + q_\vartheta\left(\frac{1}{q_v\,v\cos\alpha + q_\vartheta\sin\alpha}\right)\left(\frac{F}{mv^2}\right)\cos^2\alpha\right]p_v^q$$

$$+\left[\left(\frac{GM}{r^2v^2}-\frac{1}{r}\right)v\cos\vartheta\right]p_\vartheta^q + \left[\left(-\frac{2GM}{r^3v^2}+\frac{1}{r^2}\right)v\sin\vartheta\right]p_r^q. \tag{5.9e}$$

$$\dot{p}_r^q = [\cos \vartheta]\, p_v^q - [v \sin \vartheta]\, p_\vartheta^q . \tag{5.9f}$$

As mentioned earlier, even for the simple problem here, the adjoint equations are fairly lengthy. A solution can proceed by fixing $q_r(t_o)$, guessing values of $\bar{t}_f$, $q_v(t_o)$ and $q_\vartheta(t_o)$ and integrating (5.1) and (5.3) forward in time until $t = \bar{t}_f$. At this time, (5.5) will normally not be satisfied and (5.9) along with (5.1) are integrated backward in time until $t = t_o$. The starting values for the adjoint variables at $t = \bar{t}_f$ are calculated from (3.9a) to be

$$\begin{aligned}
&\overset{(1)}{p}{}_v^x = 1, \quad \overset{(1)}{p}{}_\vartheta^x = 0, \quad \overset{(1)}{p}{}_r^x = 0, \quad \overset{(1)}{p}{}_v^q = 0, \quad \overset{(1)}{p}{}_\vartheta^q = 0, \quad \overset{(1)}{p}{}_r^q = 0 \\
&\overset{(2)}{p}{}_v^x = 0, \quad \overset{(2)}{p}{}_\vartheta^x = 1, \quad \overset{(2)}{p}{}_r^x = 0, \quad \overset{(2)}{p}{}_v^q = 0, \quad \overset{(2)}{p}{}_\vartheta^q = 0, \quad \overset{(2)}{p}{}_r^q = 0 \\
&\overset{(3)}{p}{}_v^x = 0, \quad \overset{(3)}{p}{}_\vartheta^x = 0, \quad \overset{(3)}{p}{}_r^x = 1, \quad \overset{(3)}{p}{}_v^q = 0, \quad \overset{(3)}{p}{}_\vartheta^q = 0, \quad \overset{(3)}{p}{}_r^q = 0.
\end{aligned} \tag{5.10}$$

At time $t = t_o$, the integrated values of the adjoint variables corresponding to the initial values of (5.10) are substituted into equations (3.14a), which become

$$\begin{aligned}
&\left.\overset{(1)}{p}{}_v^q\, \delta q_v\right)_{t_o} + \left.\overset{(1)}{p}{}_\vartheta^q\, \delta q_\vartheta\right)_{t_o} + \left.\dot{v}\, dt_f\right)_{\bar{t}_f} = \left[\left. v - V_f\right)_{\bar{t}_f}\right] \cdot C_1 \\
&\left.\overset{(2)}{p}{}_v^q\, \delta q_v\right)_{t_o} + \left.\overset{(2)}{p}{}_\vartheta^q\, \delta q_\vartheta\right)_{t_o} + \left.\dot{\vartheta}\, dt_f\right)_{\bar{t}_f} = \left[\left. \vartheta - \theta_f\right)_{\bar{t}_f}\right] \cdot C_2 \\
&\left.\overset{(3)}{p}{}_v^q\, \delta q_v\right)_{t_o} + \left.\overset{(3)}{p}{}_\vartheta^q\, \delta q_\vartheta\right)_{t_o} + \left.\dot{r}\, dt_f\right)_{\bar{t}_f} = \left[\left. r - R_f\right)_{\bar{t}_f}\right] \cdot C_3
\end{aligned} \tag{5.11}$$

where $0 < C_1, C_2, C_3 \leqq 1$.

The right-hand sides of (5.11) represent a fractional portion of the difference between the desired terminal state values and the actual terminal state values for this particular integration. Equations (5.11) are solved for the increments $\delta q_V(t_o)$, $\delta q_\vartheta(t_o)$, $d\bar{t}_f$ and a forward integration is attempted with new initial values

$$\begin{aligned} \overset{\text{new}}{q_V(t_o)} &= \overset{\text{old}}{q_V(t_o)} + \delta q_V(t_o) \\ \overset{\text{new}}{q_\vartheta(t_o)} &= \overset{\text{old}}{q_\vartheta(t_o)} + \delta q_\vartheta(t_o) \end{aligned} \tag{5.12}$$

and a new cut-off time

$$\overset{\text{new}}{\bar{t}_f} = \overset{\text{old}}{\bar{t}_f} + d\bar{t}_f. \tag{5.13}$$

If the fractional terminal state violations have not been reduced by this forward integration, the increments just computed are reduced by some factor, say 1/2, (5.12) and (5.13) are recomputed and a new forward integration is made. This step is done as many times as required to achieve a reduction in the terminal state violations. (If the problem has a solution, there will be a reduction after finitely many steps.) If equations (5.5) are again not satisfied, a new cycle of backward and forward integration is initiated, the whole process being done as many times as necessary to achieve convergence. (C_1, C_2, and C_3 are increased as the linear predictions become increasingly more accurate.)

The first numerical example considered is that of transferring a rocket vehicle from a 100 nm (185.2 km) altitude circular orbit to a 300 nm (555.6 km) altitude circular orbit with the earth as the only gravitational body. The assumed vehicle and earth model parameters are given in Table 1.

TABLE 1

Vehicle
$F = 50$ lbf
Initial Weight = 500 lbf
Isp = 400 sec
Earth
Radius = 6370 km
$GM = 3.98059389 \times 10^5 \text{ km}^3/\text{s}^2$

Practically speaking, the engineering units as typified in Table 1 are not the best units in which to solve the problem. In order that the various elements of the terms in the adjoint equations be taken account of most accurately, it is advantageous to scale the units so that the problem variables are of the same order of magnitude. Ordinarily this is accomplished by writing the equations in an appropriate nondimensional form. Here, however, the nondimensionalized form is achieved by properly scaling the various input variables. In this way an engineering solution can also be obtained by unscaling these input variables. A convenient scaling is achieved by adopting the earth's radius as the unit of length and requiring GM = 1 in the scaled units. This results in 805.81475 sec as the unit of time. The vehicle's initial mass can be used as the unit of mass. With these scaling factors, the initial state is

$$r_o = \frac{6555.2}{6370} = 1.0290738, \quad v_o = \sqrt{1/r_o} = .98577260$$

$$\vartheta_o = 90°, \quad m_o = 1.$$

The terminal state is

$$r_f = \frac{6925.6}{6370} = 1.0872214, \quad v_f = \sqrt{1/r_f} = .95904947, \quad \vartheta_f = 90°,$$

m_f to be maximized; $q_r(t_o)$ and $q_m(t_o)$ were kept fixed at $q_r(t_o) = +1$ and $q_m(t_o) = +1$. The initial values of q_v and q_ϑ were guessed at $q_v(t_o) = +1$ and $q_\vartheta(t_o) = +1$. The cutoff time was initially guessed as .7 scaled units. The C's of equation (5.11) were initially set at .03 and were doubled or halved as the iteration proceeded according to whether it took more or less than three forward integrations per cycle to reduce the terminal boundary violations. The terminal boundary conditions were achieved to 8 decimal places within 10 trials. Using an integration time step corresponding to 20 sec, the Hamiltonian was constant to 5 decimal places using Runge-Kutta 4th order integration formulas. The final cutoff time was 1.3718816 scaled units indicating that the initial guess was wrong by a factor of .5. Figure 1 summarizes the control history as the trials progressed. Figure 2 summarizes the r - v history of the trials. The converged values of q_v and q_ϑ were $q_v(t_o) = .38828166$ and $q_\vartheta(t_o) = -.54702046$. Both figures indicate that the initial guesses were quite bad.

The second numerical example involves this same vehicle in an escape mission. The initial boundary is the same and the terminal state was chosen to be $r_f = 2.0175824$, $v_f = 1.6$, $\vartheta_f = 45°$, m_f maximized. The energy of this terminal state is typical of a low energy mission to Mars. The initial values of q_v and q_ϑ were guessed as +1 each, and the cutoff time was guessed as 2.23. The C's were initially set at .015 and subsequently halved or doubled as before. This time terminal boundary conditions were achieved to 8 decimal places in 18 trials. The Hamiltonian was constant to 5 decimal places using a time step equivalent to 40 seconds. The final cutoff time was 4.4778898, while the converged values of q_v and q_ϑ were $q_v(t_o) = +.23594021$ and $q_\vartheta(t_o) = -.49928473$. The control histories are summarized in Figure 3 and r - v plots are given in Figure 4.

A third numerical example, picked from the literature, is an Earth-Mars interplanetary transfer mission. Earth and Mars are assumed to be in coplanar, circular orbits about the sun. Adopting the astronomical unit as the unit of length, and requiring the heliocentric GM to equal 1 in the scaled system leads to a time unit of 58.13504 days. The vehicle parameters are shown in Table II. The initial state is $r_o = 1$, $v_o = 1$,

TABLE II

Vehicle
F = .127 lbf
Initial Weight = 1500 lbf
Isp = 5700 sec

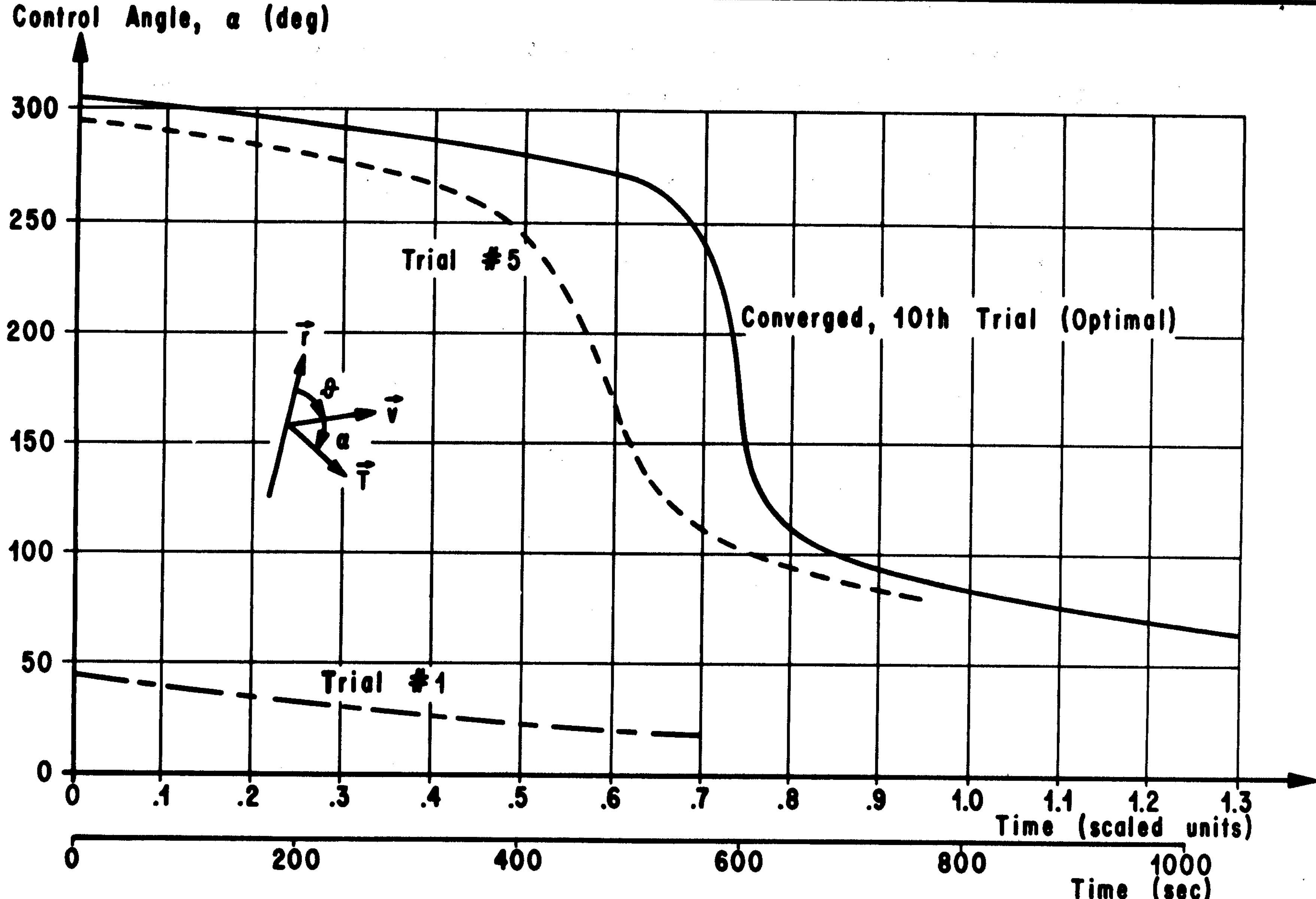


FIG. 1. CONTROL PROGRAMS FOR TRANSFER FROM 100 NAUTICAL MILES (185.2 KM) TO 300 NAUTICAL MILES (555.6 KM) CIRCULAR ORBITS

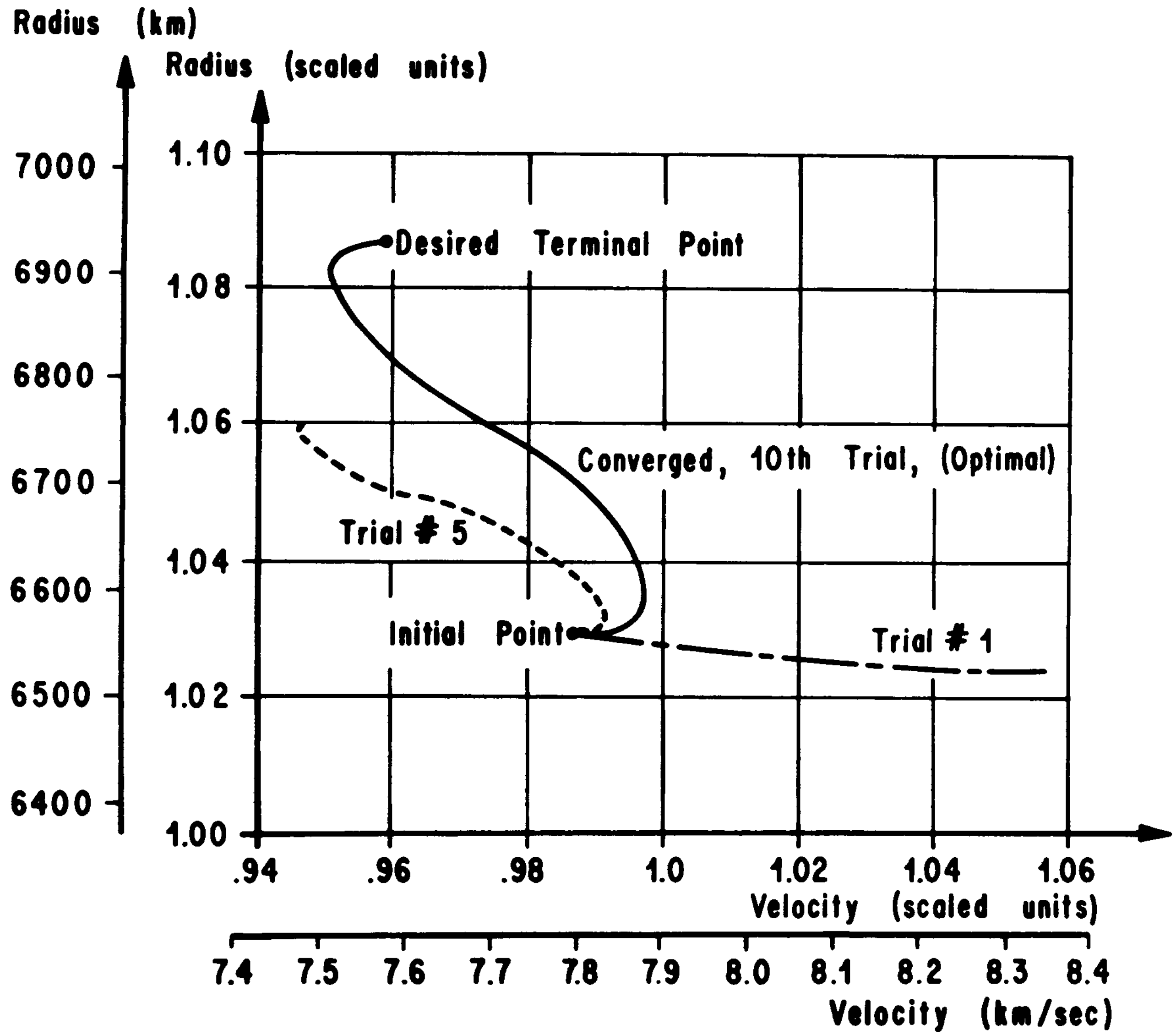


FIG. 2. RADIUS VERSUS VELOCITY PLOTS FOR TRANSFER FROM 100 NAUTICAL MILES (185.2 KM) TO 300 NAUTICAL MILES (555.6 KM) CIRCULAR ORBITS

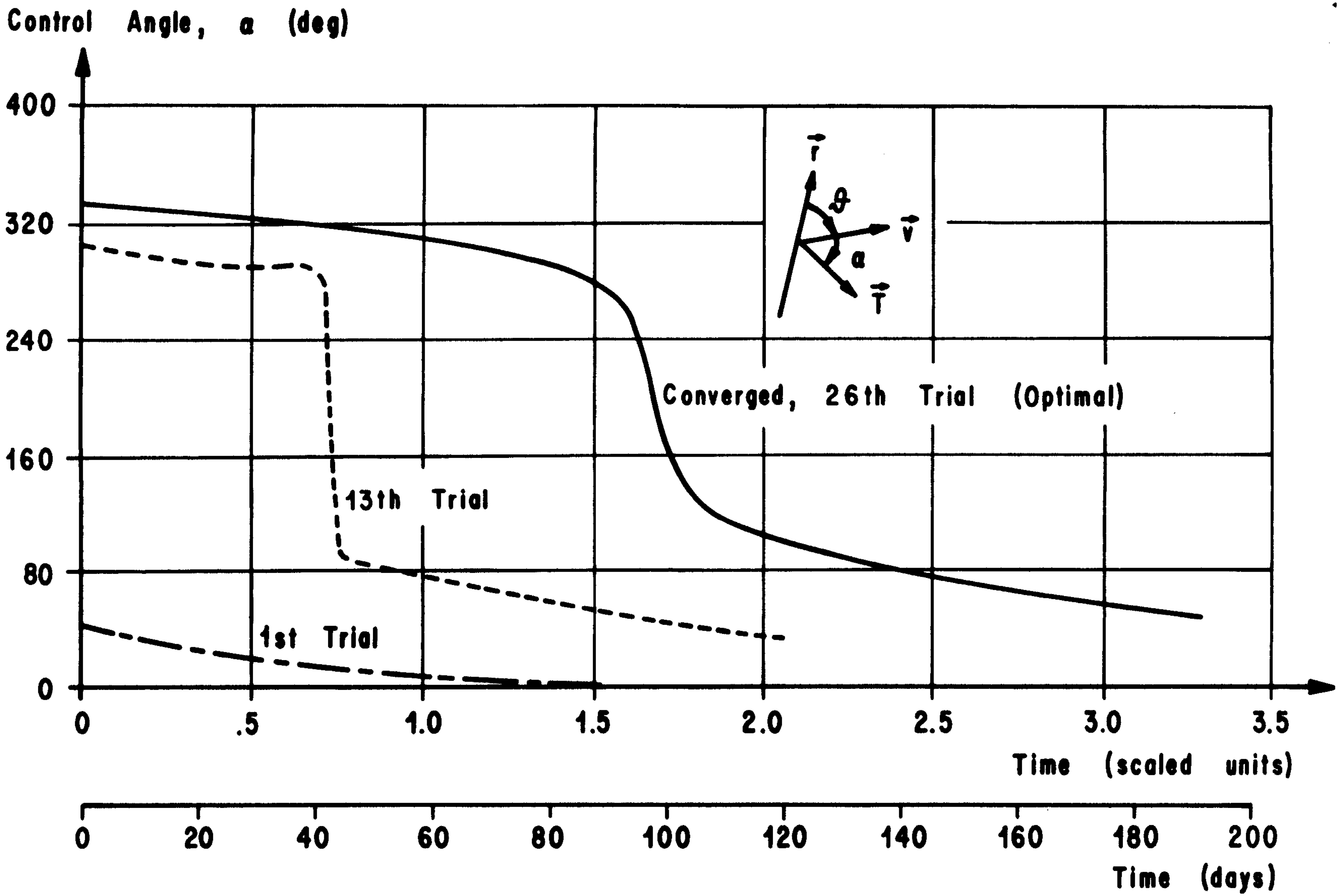


FIG. 3. CONTROL PROGRAMS FOR HELIOCENTRIC TRANSFER FROM EARTH'S ORBIT (ASSUMED CIRCULAR) TO MARS' ORBIT (ASSUMED CIRCULAR)

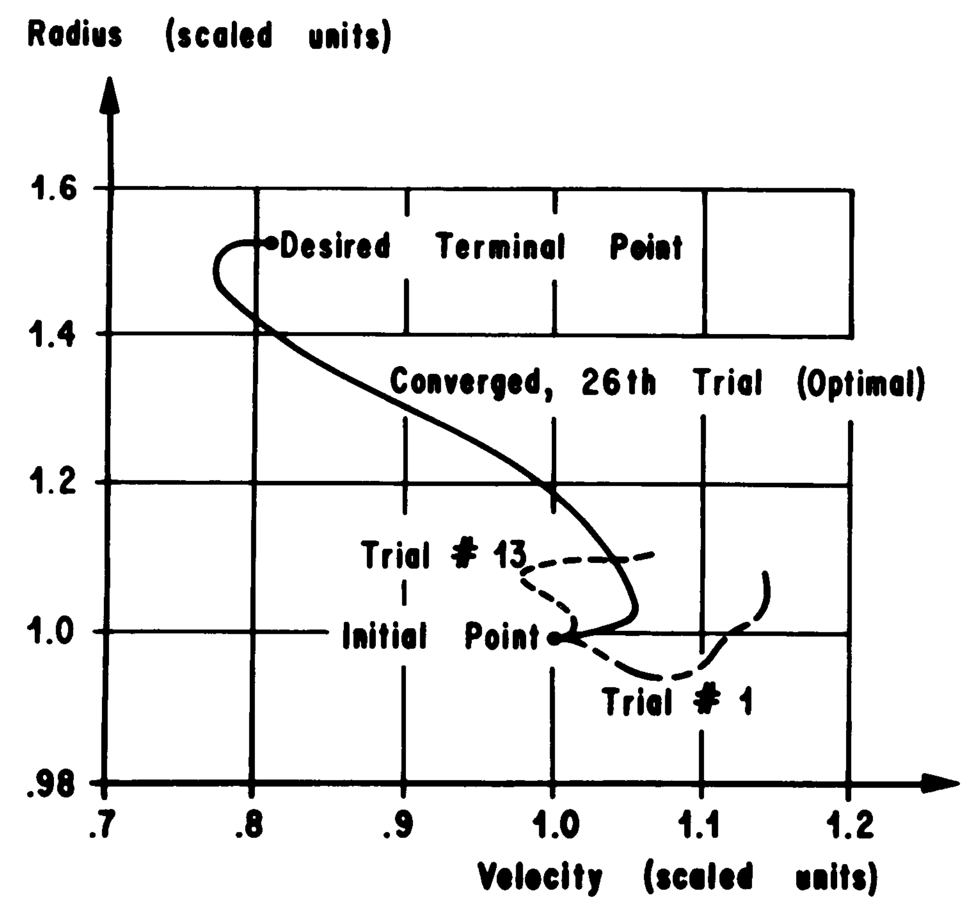


FIG. 4. RADIUS VERSUS VELOCITY PLOTS FOR HELIOCENTRIC TRANSFER FROM EARTH'S ORBIT (ASSUMED CIRCULAR) TO MARS' ORBIT (ASSUMED CIRCULAR)

$\vartheta_o = 90°$, $m_o = 1$. The terminal state is $r_f = 1.525$, $v_f = .8098$, $\vartheta_f = 90°$, and m_f maximized. Again, q_v and q_ϑ were initially set equal to +1 each and the cutoff time guessed as 1.66. The C's were initially set at .00015 and the integration time step was set equivalent to 5 days. Boundary conditions were achieved to 8 decimal places in 26 trials with the Hamiltonian being held constant to 4 decimal places. The final cutoff time was 3.3194012 with the converged values of q_v and q_ϑ being $q_v(t_o) = 1.0784030$ and $q_\vartheta(t_o) = -.49498598$. Typical control histories are shown in Figure 5, and r - v plots are shown in Figure 6.

The fourth and last numerical example involves launching a two-stage vehicle into a low circular earth orbit. The equations of motion are written in a two-dimensional earth-fixed coordinate system. The earth is assumed spherical and its atmosphere is modeled by an exponential function. The drag and lift coefficients of the vehicle are assumed to be constants. The differential equations are

$$\dot{v} = f_1 = \frac{F}{m} \cos \alpha - \frac{GM}{r^2} \cos \vartheta + \left[\omega'^2 r \cos \vartheta - \frac{D}{m} \cos \alpha - \frac{N}{m} \sin \alpha \right]$$

$$\dot{\vartheta} = f_2 = \frac{F}{mv} \sin \alpha + \left(\frac{GM}{r^2 v^2} - \frac{1}{r} \right) v \sin \vartheta + \left[\frac{N}{mv} \cos \alpha - \frac{D}{mv} \sin \alpha \right.$$

$$\left. - \frac{\omega'^2 r}{v} \sin \vartheta - 2\omega' \right] \tag{5.14}$$

$$\dot{r} = f_3 = v \cos \vartheta$$

$$\dot{m} = f_4 = k,$$

where

D = axial force = $1/2\ C_{d_o} \rho v^2 A_s$

N = normal force = $1/2\ C_{n_o} \rho v^2 A_s \alpha$

ρ = atmosphere density = $\rho_o e^{-Q(r-r_e)}$

$\omega' = \omega \cos \varphi \sin A_z$

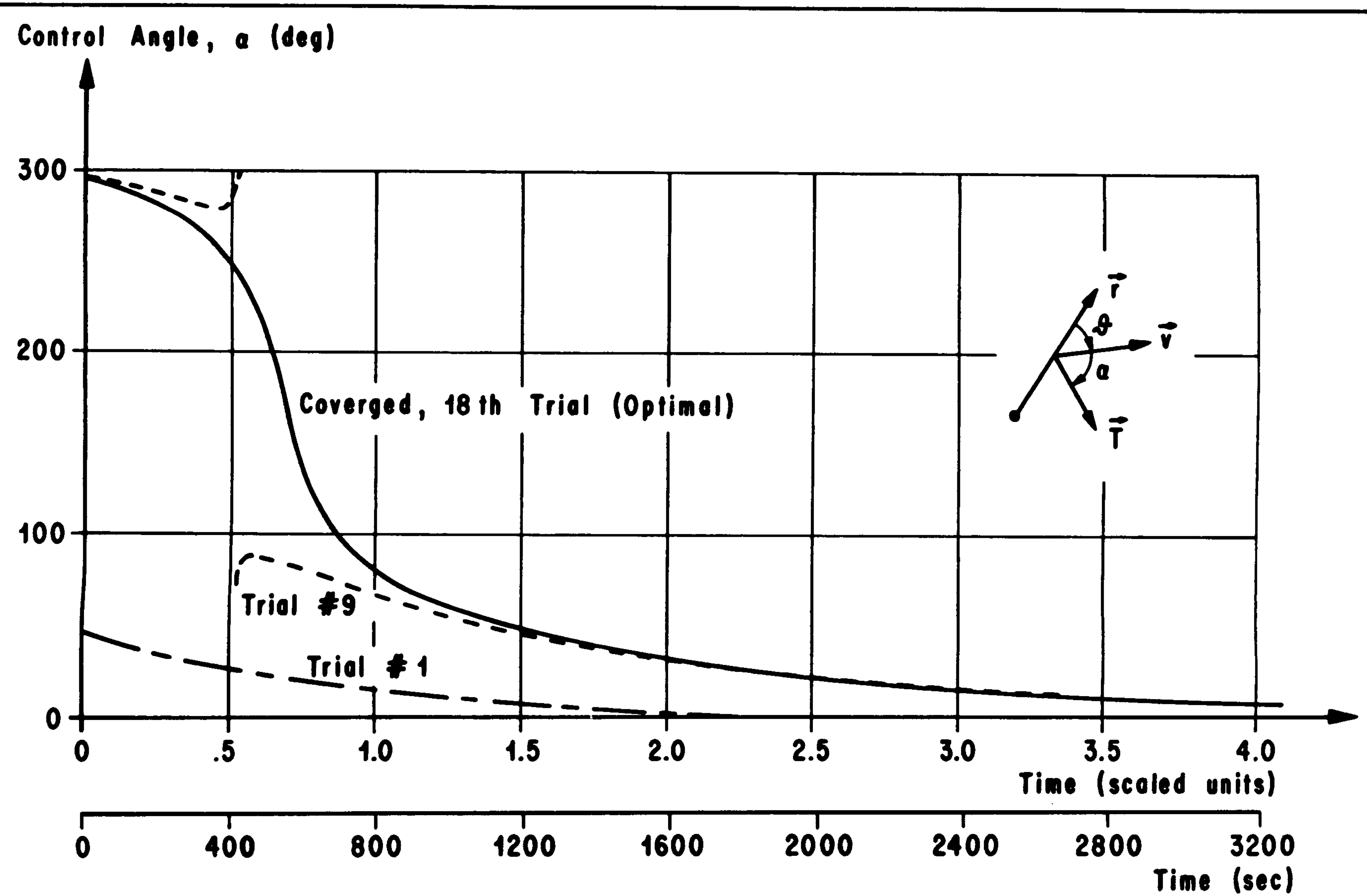


FIG. 5. CONTROL PROGRAMS FOR ESCAPE AT 3500 NAUTICAL MILES (6482 KM) STARTING FROM AN INITIAL CIRCULAR ORBIT OF 100 NAUTICAL MILES (185.2 KM)

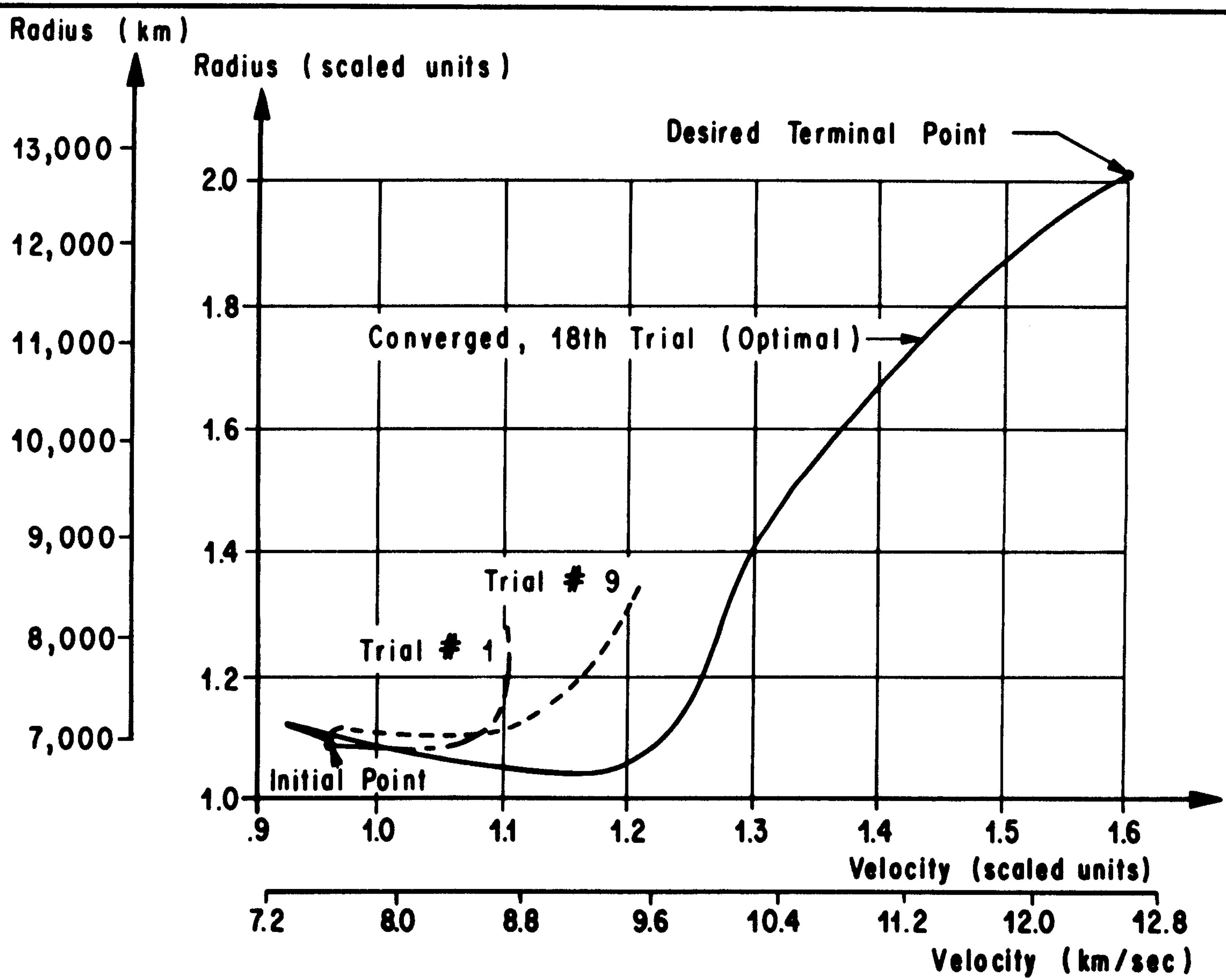


FIG. 6. RADIUS VERSUS VELOCITY PLOTS FOR ESCAPE AT 3500 NAUTICAL MILES (6482 KM) STARTING FROM AN INITIAL CIRCULAR ORBIT OF 100 NAUTICAL MILES (185.2 KM)

and

C_{d_o} is the axial force coefficient

C_{n_o} is the normal force coefficient

A_s is the reference frontal area of the vehicle

ρ_o is the atmospheric density at the earth's surface

Q is the inverse of the scale height

ω is the earth's rotational rate

r_e is the earth's radius

φ is the launch latitude

A_z is the launch azimuth

and the other terms are as defined in (5.1). There are a total of 17 variables - r, v, ϑ, m, F, GM, ω', φ, A_z, Cd_o, Cn_o, A_s, ρ_o, Q, r_e, k, α - of which 12 (F, GM, ω', φ, A_z, Cd_o, Cn_o, A_s, ρ_o, Q, r_e, k) are known constants. Equations (5.14) determine r, v, ϑ, m so that again one free variable α is available to minimize the transfer time from the given initial state to the given terminal state. Using the subscript notation defined previously, the Hamiltonian (4.24) becomes

$$H = 1 + q_v\left(\frac{F}{m}\cos\alpha - \frac{GM}{r^2}\cos\vartheta + \omega'^2 r\cos\vartheta - \frac{D}{m}\cos\alpha - \frac{N}{m}\sin\alpha\right)$$

$$+ q_\vartheta\left[\frac{F}{mv}\sin\alpha + \left(\frac{GM}{r^2v^2} - \frac{1}{r}\right)v\sin\vartheta + \frac{N}{mv}\cos\alpha - \frac{D}{mv}\sin\alpha\right.$$

$$\left. - \frac{\omega'^2 r}{v}\sin\vartheta - 2\omega'\right] + q_r(v\cos\vartheta) + q_m(k). \tag{5.15}$$

The Euler-Lagrange equations (4.26) can be written

$$\dot{q}_v = \left(\frac{F}{mv^2}\sin\alpha + \frac{GM}{r^2v^2}\sin\vartheta + \frac{1}{r}\sin\vartheta\right)q_\vartheta - \cos\vartheta\, q_r$$

$$+ \Bigg[(C_{d_o}\cos\alpha + C_{n_o}\alpha\sin\alpha)\left(\frac{vA_s\rho}{m}\right)q_v$$

$$+ \left\{(C_{d_o}\sin\alpha - C_{n_o}\alpha\cos\alpha)\ \frac{A_s\rho}{2m} - \frac{\omega'^2 r\sin\vartheta}{v^2}\right\}q_\vartheta\Bigg]$$

$$\dot{q}_\vartheta = -\frac{GM}{r^2}\sin\vartheta\, q_v - \left(\frac{GM}{r^2v^2} - \frac{1}{r}\right)v\cos\vartheta\, q_\vartheta + v\sin\vartheta\, q_r$$

$$+ \left[\omega'^2 r\sin\vartheta\, q_v + \frac{\omega'^2 r\cos\vartheta}{v}q_\vartheta\right]$$

$$(5.16)$$

$$\dot{q}_r = -\frac{2GM}{r^3}\cos\vartheta\, q_v + \left(\frac{2GM}{r^2v^2} - \frac{1}{r}\right)\frac{v\sin\vartheta}{r}q_\vartheta + \Bigg[\Bigg\{(-C_{d_o}\cos\alpha$$

$$- C_{n_o}\alpha\sin\alpha)\ \frac{Qv^2A_s\rho}{2m} - \omega'^2\cos\vartheta\Bigg\}q_v + \Bigg\{\frac{\omega'^2\sin\vartheta}{v} + (C_{n_o}\alpha\cos\alpha$$

$$- C_{d_o}\sin\alpha)\left(\frac{QvA_s\rho}{2m}\right)\Bigg\}q_\vartheta\Bigg]$$

$$\dot{q}_m = \frac{F\cos\alpha}{m^2}q_v + \frac{F}{m^2v}\sin\alpha\, q_\vartheta + \Bigg[-(C_{d_o}\cos\alpha + C_{n_o}\alpha\sin\alpha)\left(\frac{v^2A_s\rho}{2m^2}\right)q_v$$

$$+ (-C_{d_o}\sin\alpha + C_{n_o}\alpha\cos\alpha)\left(\frac{vA_s\rho}{2m^2}\right)q_\vartheta\Bigg].$$

The control, α, is obtained from (4.27) which becomes

$$- \frac{F}{m} \sin \alpha \, q_v + \frac{F}{mv} \cos \alpha \, q_\vartheta + \left[\left\{ (Cd_o - Cn_o) \sin \alpha - Cn_o \alpha \cos \alpha \right\} \left(\frac{v^2 A_s \rho}{2m} \right) q_v \right.$$

$$\left. + \left\{ (Cn_o - Cd_o) \cos \alpha - Cn_o \alpha \sin \alpha \right\} \left(\frac{v A_s \rho}{2m} \right) q_\vartheta \right] = 0. \qquad (5.17)$$

Notice that (5.14), (5.16) and (5.17) are identical with (5.1), (5.3) and (5.4) except for the bracketed terms which represent the effect of the earth's rotation, the assumed atmosphere and vehicle aerodynamics. The terminal boundary conditions are given by (5.5). The adjoint equations are much too lengthy to be listed here. However, the partials necessary for substitution into their general formulas, (4.10), are listed in Appendix II.

The vehicle, aerodynamic, and launch parameters are given in Table III. The formulation so far has not allowed for discontinuities in any of the variables, and thus is not directly applicable to a vehicle having stages with different characteristics. The vehicle described by Table III has discontinuities in thrust and mass at the fixed staging times which cause the differential equations involving these two variables to also be discontinuous at these times. However, these types of discontinuities introduce no difficulties since it has been demonstrated in the literature that the Lagrange multipliers are continuous across fixed staging times when these times are independent of the state. The Hamiltonian is discontinuous at these staging times, but the amount is unimportant and it still is a constant for each stage. The demonstration of these facts is straightforward and starts by defining functions dependent on the state variables and time which determine the staging times and the magnitude of the discontinuities at these staging times. These functions are adjoined to (2.4) with new Lagrange multipliers, and the integral appearing in (2.4) is divided into parts over which its arguments are continuous. The first variation of the new (2.4) obtained is set equal to zero, and an interpretation of the result yields the preceding statements for the special case of fixed staging times. The main result of this discussion is that the solution technique already discussed holds. Care simply must be taken at the staging points that the discontinuous differential equations are handled correctly.

TABLE III

Vehicle	
1st Stage	
Initial Weight	1,000,000 lbf
F	1,600,000 lbf
I_{sp}	300 sec
Staging Time	105 sec
2nd Stage (Coast)	
F	0 lbf
Staging Time	115 sec
Weight Drop	140,000 lbf
3rd Stage	
F	200,000 lbf
I_{sp}	420 sec
Aerodynamic	
A_s	25 m^2
ρ_o	.13133546 (kgm/m^3)
Q	.13623243 x 10^{-3} (1/km)
Cd_o	.5
Cn_o	5
Launch	
ω	.72921157 x 10^{-4} (rad/sec)
ϕ	28.28°
A_z	72°

The principal difficulty introduced by adding the atmosphere is the solution of (5.17) for the control. It has not been mentioned thus far, but the correct value of α computed from (4.27) is the value that maximizes the Hamiltonian. This statement can be proven by an application of the Pontryagin maximum principle or the Weierstrass E-function test. Since the solutions of (5.4) are periodic with period π, at most two solutions are possible in the interesting range $-\pi < \alpha \leqq \pi$. One of these maximizes the Hamiltonian, the other minimizes it. For the vacuum flights (5.4) does indeed maximize the Hamiltonian (as long as $q_r(t_o) > 0$). The situation for (5.17) is quite different. First, solutions are not periodic with period π (except in the limit for large α). However, there are multiple solutions in the range $-\pi < \alpha \leqq \pi$. The details of the solution of (5.17) will be discussed next.

Equation (5.17) is of the form

$$(A + B\alpha) \tan \alpha = C + D\alpha \tag{5.18}$$

where A, B, C, D are functions of v, m, F, C_{d_o}, C_{n_α}, A_s, q_v and q_ϑ. The left-hand side (L.H.S.) of (5.18) is the product of a linear function and a trigonometric function. The linear term is called the magnification factor since $(A + B\alpha)$ has the principal effect of increasing $\tan \alpha$ in absolute value. A graphical representation of the L.H.S. of (5.18) divides essentially into four different cases: $A \geqq 0$, $B \lesseqgtr 0$ and $A < 0$, $B \lesseqgtr 0$. An important subcase occurs when $A + B\alpha = 0$ and $|\alpha| = \pi/2$ simultaneously. A typical representation for $A > 0$ and $B \lesseqgtr 0$ is shown in Figures 7a and 7b. A solution of (5.18) is thus represented by the

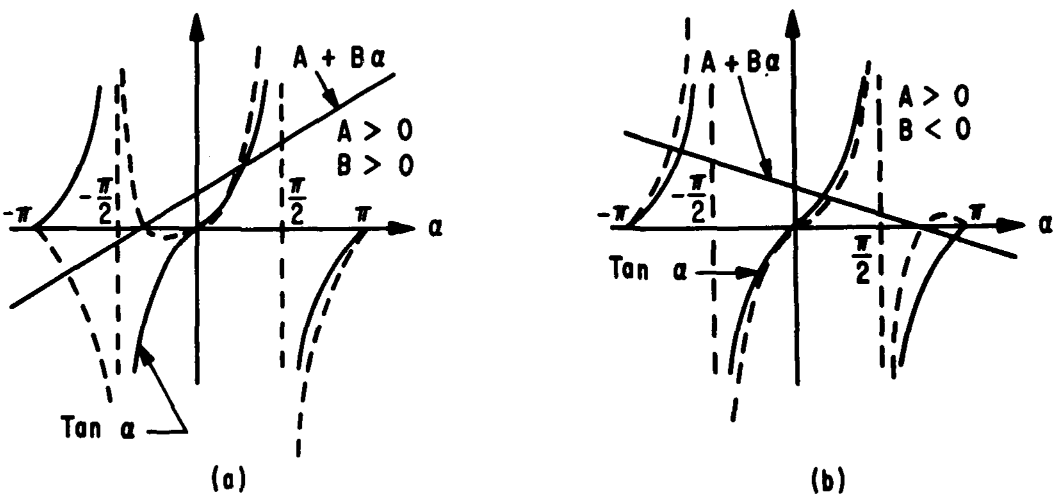


Figure 7

intersection of a straight line (the right-hand side (R.H.S.) of (5.18)) and the dotted lines (the L.H.S. of (5.18)) in Figure 7. Depending on the values assumed by C and D, there are at most four solutions. Further, there are at most two solutions per quadrant. Notice that always there are two solutions. The special case $A + B\alpha = 0$ and $|\alpha| = \pi/2$ requires special treatment. Taking $\alpha = -\pi/2$, then $A + B(-\pi/2) = 0$, implies $A = B(\pi/2)$. Therefore, $(A + B\alpha) \tan \alpha$ becomes

$$B\left(\frac{\pi}{2} + \alpha\right) \tan \alpha.$$

This is an indeterminant of the form $0 \cdot -\infty$ as

$$\alpha \to -\frac{\pi^+}{2}.$$

However, its limit does exist and is evaluated by L'Hospital's rule as follows:

$$\lim_{\alpha \to -\frac{\pi^+}{2}} B\left(\frac{\pi}{2} + \alpha\right) \tan \alpha = \lim_{\alpha \to -\frac{\pi^+}{2}} \frac{B\left(\frac{\pi}{2} + \alpha\right)}{\cot \alpha} = \lim_{\alpha \to -\frac{\pi^+}{2}} \frac{B}{-\frac{1}{\sin^2\alpha}}$$

$$= -B \lim_{\alpha \to -\frac{\pi^+}{2}} \sin^2\alpha = -B.$$

A similar evaluation for

$$\alpha \to -\frac{\pi^-}{2}$$

yields

$$\lim_{\alpha \to -\frac{\pi}{2}^-} B\left(\frac{\pi}{2} + \alpha\right) \tan \alpha = -B.$$

Thus,

$$B\left(\frac{\pi}{2} + \alpha\right) \tan \alpha$$

possesses a removable discontinuity at $\alpha = -\pi/2$ and

$$B\left(\frac{\pi}{2} + \alpha\right) \tan \alpha$$

is continuous everywhere in $[-\pi, 0]$. Thus, Figure 7a would become Figure 8. A similar result would hold for Figure 7b. The conclusions from Figure 7 still hold. A solution of (5.17) proceeds on the basis of

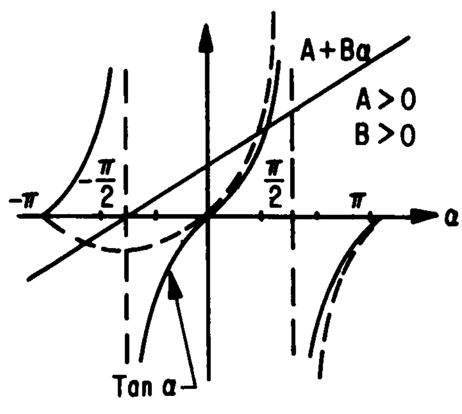


Figure 8

these conclusions. $\partial H(\alpha)/\partial\alpha$, i.e., (5.17), is evaluated at increments of $\pi/2$ in the range $[-\pi, \pi]$. If its sign changes at the end points of any of the subintervals, a root exists within this subinterval. If the sign of $\partial H/\partial\alpha$ does not change, the sign of $\partial^2 H(\alpha)/\partial\alpha^2$ is examined. If it changes, there are two roots in that subinterval. These two roots are placed in appropriate smaller subintervals by evaluating $\partial H/\partial\alpha$ within this subinterval. A modified Newton-Raphson procedure is then used to find roots within the intervals in which they have been isolated. The modification is as follows. The normal Newton-Raphson iteration for a root has the form

$$\alpha_{n+1} = \alpha_n - \frac{\frac{\partial H}{\partial \alpha_n}}{\frac{\partial^2 H}{\partial \alpha_n^2}} .$$

The modified form is

$$\alpha_{n+1} = \alpha_n - K_1 K_2 \frac{\frac{\partial H}{\partial \alpha_n}}{\frac{\partial^2 H}{\partial \alpha_n^2}} ,$$

where $K_1 = \pm 1$ and $0 < K_2 \leqq 1$. K_1 and K_2 can be given geometric interpretations, but it will suffice here to say that K_2 limits the sequence of α's to lie within the interval the root has been isolated within and K_1 causes the sign of the derivative to agree with the sign of the finite increment slope found in the process of isolating the root. In this manner, all the roots are found in the interval $-\pi < \alpha \leqq \pi$. Each is substituted into the Hamiltonian and the one maximizing H is picked as the control. A typical variation of $H(\alpha)$ is shown in Figure 9.. In use, the foregoing procedure is very rapid and the three roots indicated in Figure 9 would be found to eight significant figures in four or five iterations per root.

The mission for the two-stage vehicle described in Table III is the attainment of a circular orbit at 194.6 km altitude. Using the scaling given in the first numerical example, the initial state is $r_o = 1.0004223$, $v_o = .020240245$, $\vartheta_o = 6°$, and $w_o = 1$. The terminal state is $r_f = 1.0305494$, $v_f = .98506657$, $\vartheta_f = 90°$ and w_f maximized. The initial values of q_v and q_ϑ were guessed as +.1 and 0, respectively, and the cutoff time was guessed as .67. The C's were initially set at .0001. Terminal boundary conditions

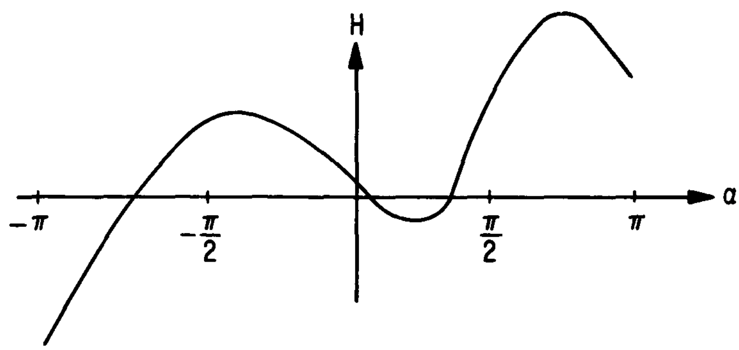


Figure 9

were achieved to 8 decimal places in 29 trials with the Hamiltonian being held constant to 5 decimal places. A time-step equivalent to 10 seconds was used. The final cutoff time was .7906737, and the converged values of q_v and q_ϑ were $q_v(t_o) = .074358687$ and $q_\vartheta(t_o) = .0032090726$. Control histories are given in Figure 10, and r - v plots are shown in Figure 11.

VI. CONCLUSIONS

The adjoint method has been shown to be a powerful tool in the solution of two-point boundary value problems. In particular, for variational problems treated via the Lagrange multiplier technique, solutions are freed quite drastically from dependence on the initial multiplier values. With respect to the numerical examples of the text, the number of trials necessary for convergence in the second and third examples could have been easily cut in half by guessing even remotely reasonable cutoff times. For the atmospheric example, the number of trials could have been reduced considerably by using a smaller time step in the integration algorithm since terminal boundary conditions had been satisfied to 2 decimal places after 15 trials. Generally, the convergence process proceeds more rapidly the smaller the integration time step. This is because the influence functions are obtained more accurately. The time step required for accurate forward integration is generally not the same as that for accurate backward integration. There is a trade-off between how accurately the influence functions need to be obtained to solve the problem and how many trials are required. Usually, the time-step for a backward integration needs to be smaller than for a forward integration for equivalent degrees

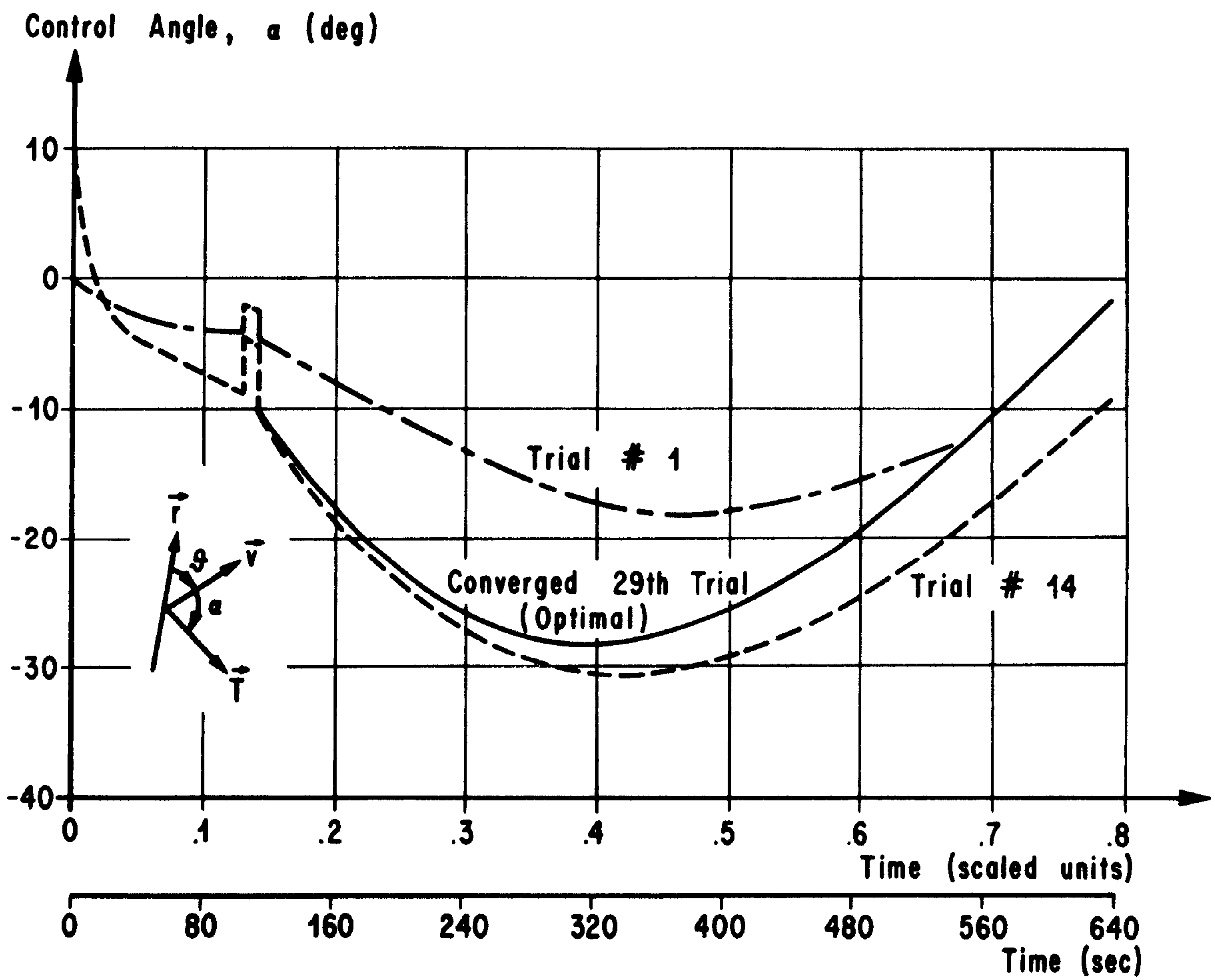


FIG. 10. CONTROL PROGRAMS FOR ASCENT THROUGH AN EXPONENTIAL ATMOSPHERE TO A 105 NAUTICAL MILE (194.6 KM) CIRCULAR ORBIT

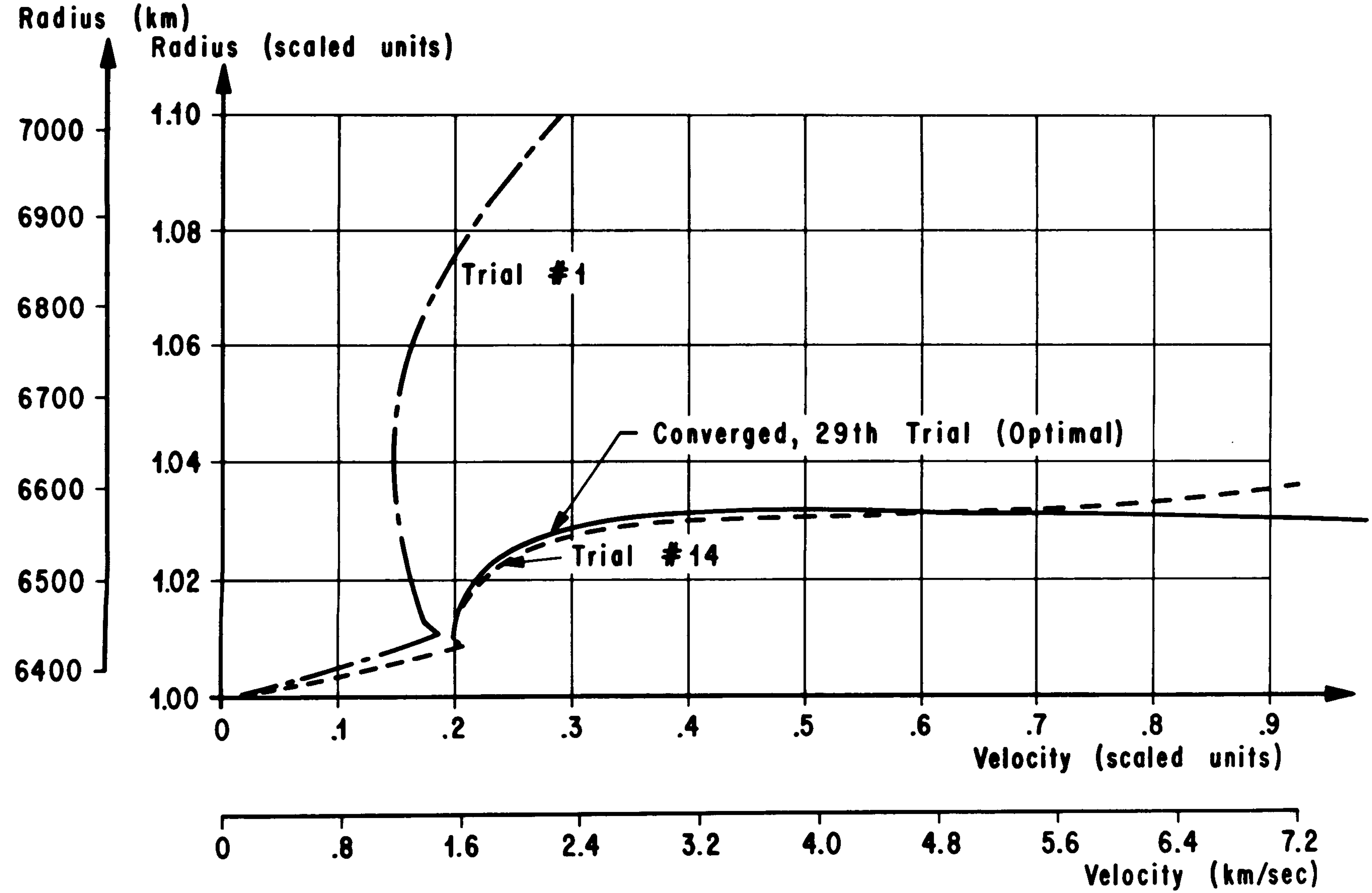


FIG. 11. RADIUS VERSUS VELOCITY PLOTS FOR ASCENT THROUGH AN EXPONENTIAL ATMOSPHERE TO A 105 NAUTICAL MILE (194.6 KM) CIRCULAR ORBIT

of accuracy. Two disadvantages of the adjoint technique are the slight programming problem caused by the backward integration and the time penalty incurred, because the state equations must be integrated backward as well as forward (or their forward values saved on tape and then interpolated during the backward integration). An almost one-third computational time saving can be achieved in launch and ascent problems by using velocity as the cut-off criteria. Velocity is an acceptable function for this purpose since it is usually a monotonically increasing function of time in this type of problem.

APPENDIX I

In the ordinary differential calculus, the notation dy denotes an infinitesimal, i.e., a change in the variable which can be made as small as desired. In the calculus of variations, δy denotes an infinitesimal change in the function y; i.e., δy is a variation of y(x). If the original function is denoted $\bar{y}(x)$ and the changed function denoted y(x), then $y(x) = \bar{y}(x) + \delta\bar{y}$. The change (variation) can be made more explicit by saying $y(x) = \bar{y}(x) + k\phi(x)$, where k is an arbitrary constant and $\phi(x)$ is independent of k and continuous over the same range that $\bar{y}(x)$ is continuous. For example, the straight line $\bar{y}(x) = x$, $0 \leqq x \leqq 1$, can be deformed into the parabola $y(x) = x^2$, $0 \leqq x \leqq 1$, by defining $\phi(x) = x^2 - x$. Then, $y(x) = x + k(x^2 - x)$. Thus, when $k = 0$ the original function is retained. When $k = 1$, the varied function is the parabola. This little example illustrates how a given function can be deformed into another given function. Ideas are fixed more firmly with the following definitions:

Def. 1 Let $\bar{y}(x)$ and y(x) be uniformly continuous functions in the interval $x_0 \leqq x \leqq x_1$. Then y(x) is said to lie in a <u>strong</u> neighborhood N_ϵ of $\bar{y}(x)$ if and only if for every $\epsilon > 0$, $|y(x) - \bar{y}(x)| \leqq \epsilon$.

Def. 2 Let $\bar{y}(x)$ and y(x) be differentiable, uniformly continuous functions in the interval $x_0 \leqq x \leqq x_1$. Then y(x) is said to lie in a <u>weak</u> neighborhood N_ϵ of $\bar{y}(x)$ if and only if for every $\epsilon > 0$, $|y(x) - \bar{y}(x)| \leqq \epsilon$ and $\left|\frac{d}{dx} y(x) - \frac{d}{dx} \bar{y}(x)\right| \leqq \epsilon$.

Strong variations are associated with Def. 1 and weak variations are associated with Def. 2. Futhermore, variations are (1) special if the independent variable is not varied and (2) general if the independent variable is varied. Almost exclusively special, weak variations (although very often this is not stated explicitly) are used in deriving the necessary conditions for extremals in the variational calculus.

Now consider a function of more than one argument, say $\zeta(x, \bar{y}_1, \ldots, \bar{y}_m, \bar{y}_1', \ldots, \bar{y}_m')$, where $\bar{y}_i' = \frac{d}{dx} y_i$. Consider the effect of replacing each of the arguments by its varied value; i.e., $\bar{y}_i$ is replaced by $\bar{y}_i + k\phi_i(x)$, $\bar{y}_i'$ is replaced by $\bar{y}_i' + k\phi_i'(x)$, etc. If $\zeta(x, \bar{y}_1, \ldots, \bar{y}_m, \bar{y}_1', \ldots, \bar{y}_m')$ has a series representation, then

$$\zeta(x, \bar{y}_i + k\phi_i, \bar{y}_i' + k\phi_i') = \zeta(x, \bar{y}_i, \bar{y}_i') + \delta\zeta + \frac{1}{2}\delta^2\zeta + \ldots$$

$$+ \frac{1}{n!}\delta^n\zeta + \ldots,$$

where $\delta\zeta$, $\delta^2\zeta$, ..., $\delta^n\zeta$ are the 1^{st}, 2^{nd}, and n^{th} variations of ζ and symbolically

$$\delta\zeta = \left(k\phi_j \frac{\partial}{\partial\bar{y}_j} + k\phi_j' \frac{\partial}{\partial\bar{y}_j'}\right)\zeta,$$

$$\delta^2\zeta = \left(k\phi_j \frac{\partial}{\partial\bar{y}_j} + k\phi_j' \frac{\partial}{\partial\bar{y}_j'}\right)^2\zeta,$$

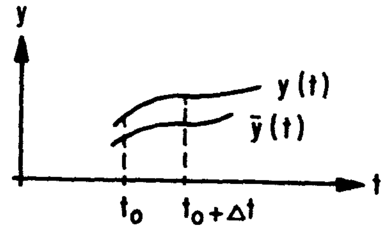
$$\delta^n\zeta = \left(k\phi_j \frac{\partial}{\partial\bar{y}_j} + k\phi_j' \frac{\partial}{\partial\bar{y}_j'}\right)^n\zeta.$$

Substituting successively $\bar{y}_1$, ..., $\bar{y}_m$, $\dot{\bar{y}}_1$, ..., $\dot{\bar{y}}_m$ for ζ yields

$$\begin{array}{llll}
\delta\bar{y}_1 = k\phi_1 & \delta^2\bar{y}_1 = 0 & \cdots & \delta^n\bar{y}_1 = 0 \\
\vdots & \vdots & & \vdots \\
\delta\bar{y}_m = k\phi_m & \delta^2\bar{y}_m = 0 & \cdots & \delta^n\bar{y}_m = 0 \\
\delta\dot{\bar{y}}_1 = k\dot{\phi}_1 & \delta^2\dot{\bar{y}}_1 = 0 & \cdots & \delta^n\dot{\bar{y}}_1 = 0 \\
\vdots & \vdots & & \vdots \\
\delta\dot{\bar{y}}_m = k\dot{\phi}_m & \delta^2\dot{\bar{y}}_m = 0 & \cdots & \delta^n\dot{\bar{y}}_m = 0
\end{array}$$

from which follows easily that $\delta\dot{\bar{y}}_j = k\dot{\phi}_j = \frac{d}{dt}(k\phi_j) = \frac{d}{dt}(\delta\bar{y}_j)$.

When the end point of a variational problem is not fixed but instead is allowed a general variation, the total change in the end point value is found as follows. Referring to the drawing, $y(t)$ is the varied curve and $\bar{y}(t)$ is the original curve; $\bar{y}(t)$ must be extended over the interval Δt. Then, since $y(t)$ and $\bar{y}(t)$ are assumed differentiable,



$$y(t_o + \Delta t) = y(t_o) + \dot{y}(t_o)\,\Delta t + \epsilon_1 \Delta t \qquad \text{(I-1)}$$

where $\lim\limits_{\Delta t \to 0} \epsilon_1 = 0.$

Since Def. 2 holds,

$$|\dot{y}(t_o) - \dot{\bar{y}}(t_o)| < \epsilon$$

or

$$\dot{y}(t_o) = \dot{\bar{y}}(t_o) + \epsilon_2 \qquad \text{(I-2)}$$

where $\lim \epsilon_2 = 0.$

Substituting (I-2) into (I-1) yields

$$y(t_o + \Delta t) = y(t_o) + \dot{\bar{y}}(t_o)\,\Delta t + (\epsilon_1 + \epsilon_2)\,\Delta t.$$

Adding and subtracting $\bar{y}(t_o)$ yields

$$y(t_o + \Delta t) = y(t_o) - \bar{y}(t_o) + \bar{y}(t_o) + \dot{\bar{y}}(t_o)\,\Delta t + (\epsilon_1 + \epsilon_2)\,\Delta t$$

or

$$y(t_o + \Delta t) - \bar{y}(t_o) = \delta y(t_o) + \dot{\bar{y}}(t_o)\,\Delta t + (\epsilon_1 + \epsilon_2)\,\Delta t, \qquad \text{(I-3)}$$

where $\delta y(t_o) = y(t_o) - \bar{y}(t_o)$ by definition. (I-3) shows that the principal part of the total change in the end point is

$$\Delta y = y(t_o + \Delta t) - \bar{y}(t_o) = \delta y(t_o) + \dot{\bar{y}}(t_o)\,\Delta t.$$

APPENDIX II

The differential equations for the influence functions are obtained by substituting the following partials into (4.22) and (4.23):

$$\frac{\partial \dot{v}}{\partial v} = (C_{d_o} \cos \alpha + C_{n_o}\alpha \sin \alpha) \left(\frac{vA_s \rho}{m}\right).$$

$$\frac{\partial \dot{v}}{\partial \vartheta} = \frac{GM}{r^2} \sin \vartheta - \omega'^2 r \sin \vartheta.$$

$$\frac{\partial \dot{v}}{\partial r} = \frac{2GM}{r^3} \cos \vartheta + (C_{d_o} \cos \alpha + C_{n_o}\alpha \sin \alpha) \left(\frac{Qv^2 A_s \rho}{2m}\right) + \omega'^2 \cos \vartheta.$$

$$\frac{\partial \dot{v}}{\partial m} = - \frac{F \cos \alpha}{m^2} + (C_{d_o} \cos \alpha + C_{n_o}\alpha \sin \alpha) \left(\frac{v^2 A_s \rho}{2m^2}\right).$$

$$\frac{\partial \dot{v}}{\partial \alpha} = - \frac{F}{m} \sin \alpha + \left((C_{d_o} - C_{n_o}) \sin \alpha - C_{n_o}\alpha \cos \alpha\right)\left(\frac{v^2 A_s \rho}{2m}\right).$$

$$\frac{\partial \dot{\vartheta}}{\partial v} = - \left(\frac{F}{mv^2} \sin \alpha + \frac{GM}{r^2 v^2} \sin \vartheta + \frac{1}{r} \sin \vartheta\right) - \left[(C_{d_o} \sin \alpha - C_{n_o}\alpha \cos \alpha) \left(\frac{A_s \rho}{2m}\right) - \frac{\omega'^2 r \sin \vartheta}{v^2}\right].$$

$$\frac{\partial \dot{\vartheta}}{\partial \vartheta} = \left(\frac{GM}{r^2 v^2} - \frac{1}{r}\right) (v \cos \vartheta) - \frac{\omega'^2 r \cos \vartheta}{v}.$$

$$\frac{\partial \dot{\vartheta}}{\partial r} = - \left(\frac{2GM}{r^2 v^2} - \frac{1}{r}\right)\left(\frac{v \sin \vartheta}{r}\right) - \left[\frac{\omega'^2 \sin \vartheta}{v} + (C_{n_o}\alpha \cos \alpha - C_{d_o} \sin \alpha) \left(\frac{QvA_s \rho}{2m}\right)\right].$$

$$\frac{\partial \dot{\vartheta}}{\partial m} = - \frac{F}{m^2 v} \sin \alpha - (-C_{d_o} \sin \alpha + C_{n_o} \alpha \cos \alpha) \left(\frac{vA_s \rho}{2m^2}\right).$$

$$\frac{\partial \dot{\vartheta}}{\partial \alpha} = \frac{F}{mv} \cos \alpha + \left((C_{n_o} - C_{d_o}) \cos \alpha - C_{n_o} \alpha \sin \alpha\right) \left(\frac{vA_s \rho}{2m}\right).$$

$$\frac{\partial \dot{r}}{\partial v} = \cos \vartheta$$

$$\frac{\partial \dot{r}}{\partial \vartheta} = - v \sin \vartheta$$

$$\frac{\partial \dot{r}}{\partial r} = 0$$

$$\frac{\partial \dot{r}}{\partial m} = 0$$

$$\frac{\partial \dot{r}}{\partial \alpha} = 0$$

$$\frac{\partial^2 H}{\partial \alpha^2} = - \frac{F}{m} \cos \alpha \, q_v - \frac{F}{mv} \sin \alpha \, q_\vartheta + \left[\left((C_{d_o} - 2C_{n_o}) \cos \alpha\right.\right.$$

$$\left.\left. + C_{n_o} \alpha \sin \alpha\right) \left(\frac{v^2 A_s \rho}{2m}\right)\right] q_v + \left[\left((C_{d_o} - 2C_{n_o}) \sin \alpha\right.\right.$$

$$\left.\left. - C_{n_o} \alpha \cos \alpha\right) \left(\frac{vA_s \rho}{2m}\right)\right] q_\vartheta .$$

$$\frac{\partial^2 H}{\partial\alpha\partial v} = -\frac{F}{mv^2}\cos\alpha\, q_\vartheta - \left[\left((C_{n_o} - C_{d_o})\sin\alpha + C_{n_o}\alpha\cos\alpha\right)\left(\frac{vA_s\rho}{m}\right)\right] q_v$$

$$+\left[\left((C_{n_o} - C_{d_o})\cos\alpha - C_{n_o}\alpha\sin\alpha\right)\left(\frac{A_s\rho}{2m}\right) + \frac{\omega'^2 r\sin\vartheta}{v^2}\right] q_\vartheta .$$

$$\frac{\partial^2 H}{\partial v^2} = \left(\frac{2F}{mv^3}\sin\alpha + \frac{2GM}{r^2v^3}\sin\vartheta\right) q_\vartheta - \left[(C_{d_o}\cos\alpha + C_{n_o}\alpha\sin\alpha)\left(\frac{A_s\rho}{m}\right)\right] q_v$$

$$-\frac{2\omega'^2 r\sin\vartheta}{v^3} q_\vartheta .$$

$$\frac{\partial^2 H}{\partial v\partial\vartheta} = -\left(\frac{GM}{r^2v^2} + \frac{1}{r}\right)\cos\vartheta\, q_\vartheta - \sin\vartheta\, q_r + \frac{\omega'^2 r\cos\vartheta}{v^2} q_\vartheta .$$

$$\frac{\partial^2 H}{\partial v\partial r} = \left(\frac{2GM}{r^2v^2} + \frac{1}{r}\right)\left(\frac{\sin\vartheta}{r}\right) q_\vartheta + \left[(C_{d_o}\cos\alpha + C_{n_o}\alpha\sin\alpha)\left(\frac{QvA_s\rho}{m}\right)\right] q_v$$

$$+\left[(C_{d_o}\sin\alpha - C_{n_o}\alpha\cos\alpha)\left(\frac{QA_s\rho}{2m}\right) + \frac{\omega'^2\sin\vartheta}{v^2}\right] q_\vartheta .$$

$$\frac{\partial^2 H}{\partial\alpha\partial\vartheta} = 0$$

$$\frac{\partial^2 H}{\partial\alpha\partial r} = \left[\left((C_{n_o} - C_{d_o})\sin\alpha + C_{n_o}\alpha\cos\alpha\right)\left(\frac{Qv^2A_s\rho}{2m}\right)\right] q_v$$

$$+\left[\left((C_{d_o} - C_{n_o})\cos\alpha + C_{n_o}\alpha\sin\alpha\right)\left(\frac{QvA_s\rho}{2m}\right)\right] q_\vartheta .$$

$$\frac{\partial^2 H}{\partial \vartheta^2} = \frac{GM}{r^2} \cos \vartheta \, q_v - \left(\frac{GM}{r^2 v^2} - \frac{1}{r}\right) v \sin \vartheta \, q_\vartheta - v \cos \vartheta \, q_r - \omega'^2 r \cos \vartheta \, q_v$$

$$+ \frac{\omega'^2 r \sin \vartheta}{v} q_\vartheta .$$

$$\frac{\partial^2 H}{\partial \vartheta \partial r} = - \frac{2GM}{r^3} \sin \vartheta \, q_v - \left(\frac{2GM}{r^2 v^2} - \frac{1}{r}\right) \frac{v \cos \vartheta}{r} q_\vartheta - \omega'^2 \sin \vartheta \, q_v - \frac{\omega'^2 \cos \vartheta}{v} q_\vartheta .$$

$$\frac{\partial^2 H}{\partial r^2} = - \frac{6GM}{r^4} \cos \vartheta \, q_v + \left(\frac{6GM}{r^4 v^2} - \frac{2}{r^3}\right) v \sin \vartheta \, q_\vartheta - \left[(C_{d_o} \cos \alpha \right.$$

$$\left. + C_{n_o} \alpha \sin \alpha) \left(\frac{Q^2 v^2 A_s \rho}{2m}\right)\right] q_v - \left[(C_{d_o} \sin \alpha - C_{n_o} \alpha \cos \alpha) \left(\frac{Q^2 v A_s \rho}{2m}\right)\right] q_\vartheta .$$

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VARIATIONAL PROBLEMS AND THEIR SOLUTION BY THE ADJOINT METHOD

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This document has also been reviewed and approved for technical accuracy.

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