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Volume I

CONTROL SYSTEM DESIGN BY STATE VARIABLE FEEDBACK TECHNIQUES

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by

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CHAPTER I

1.1 Introduction and Outline of Report

The impact of modern control theory has yet to make itself felt in a large percentage of industrial, military, and space applications. One reason for this is the fact that those in responsible positions had already graduated from the Universities before the state space methods were introduced. Since state space methods have been identified with the time domain and integral performance indices, these methods are not familiar to any except recent graduates.

This report outlines a method of linear system synthesis using state variable feedback that does not require a use of the time domain or of vector, matrix equations. The state variables are identified as the real, physical quantities that actually exist in the system, rather than as abstract mathematical inventions. The design criteria is the desired closed loop response, and only algebra is needed to complete the synthesis.

This report differs markedly from the previous reports that have been submitted on this contract. This report attempts to reduce the application of modern control theory to practice, while previous reports have been more theoretical in nature. This is in line with the policy agreed upon with the contracting agency, namely that some of our reports should be directed to the practicing engineer rather than the researcher.

The H equivalent method of linear system synthesis makes up the body of the report, and this is contained in Chapter 2. The further effort suggested by this research is briefly mentioned in Chapter 3.

Chapter 2 has a number of sections. After a brief introduction in Section 2-1, alternate means of representing state variable feedback systems are discussed in Section 2.2. The approach of this chapter is related to the familiar technique of Guillemin and Truxal in the following section. The H equivalent method is introduced in Section 2-4, and applied to the simplest case. Modifications that are possible when all the state variables are not available are discussed in Section 2.5, and the general case is treated in Section 2.6. Sections 2, 4, 5, and 6 include definite design procedures that may be applied in practice today. A number of less difinitive alternate approaches are considered in Section 2.7, and the Chapter is concluded in the final section.

The topics mentioned in the chapter on further study are those topics that have been suggested by this report. Other topics not related to this report are included in the other volumes.

CHAPTER II

A NEW METHOD OF LINEAR SYSTEM SYNTHESIS

2.1 Introduction

This Chapter outlines a new method of system synthesis for single input, single output, linear control systems. The method is based on the feedback of all of the state variables through constant gain elements. If all of the state variables are not available, these are effectively simulated by introducing dynamics in the feedback paths of those variables that are available. In the case when only the output variable is available, which is never, the method reduces to that of Guillemin and Truxal. The last section of this Chapter alludes to work that is now in progress to extend this approach to nonlinear systems and to multiple input, multiple output systems.

Although this work is based directly upon a joint application of the Second Method of Liapunov and the Maximum Principle of Pontryagin, the language of this Chapter is almost exclusively the Laplace transform rather than vector, matrix algebra. This is in line with the policy stated in Chapter I, namely that the report shall be written so as to convey the greatest amount of information to the greatest number of readers in the least painful form. In a few cases an equivalent matrix equation is given, but the material is completely consistent without these matrix equations. Because of this somewhat artificial restriction that the authors have placed upon themselves, the results presented here are lacking in proof. On the basis of a few examples, a general conclusion is stated, although a reference is always given. This approach may be somewhat contrary to the accepted scientific method, but experience has shown that it is pedagogically sound. The interested reader may find a more detailed and extensive treatment in the references.

2.2 System Representation

A conventional feedback control configuration is shown in Fig. 2.2-1. In this block diagram the transfer function of the plant being controlled is labled $G_p(s)$. Usually this plant is unalterable, and the dynamics of the overall closed loop system are controlled by the use of feedback, in conjunction with series and feedback compensation networks. The transfer function of the series compensation network is designated $G_c(s)$, and $H(s)$ is the transfer function of the feedback compensation elements. The Laplace transform of the input and output are $R(s)$ and $C(s)$ respectively. This notation is common in texts on control engineering.

In the pages that follow we continue to refer to the fixed plant as $G_p(s)$. The transfer function of the overall closed loop system, $C(s)/R(s)$, is simply referred to as the transfer function of the system. That portion of the system that is not included in the plant is designated as the controller. In Fig. 2.2-1, for instance, the controller would consist of $G_c(s)$, $H(s)$, and the summer immediately preceeding $G_c(s)$.

A conclusion of modern control theory is that in order to minimize an integral type performance index involving a quadratic function of the state variables and the control effort, it is necessary to feedback all of the state variables through constant gain elements. In this Chapter we are not interested in integral type performance indices. The criteria to which we shall design is a statement of the desired closed loop frequency response. However, we make use of the important conclusion of modern control theory, namely that all of the state variables should be fed back. If $C(s)/R(s)$ has an n th order characteristic equation, the system is describable in terms of n state variables.

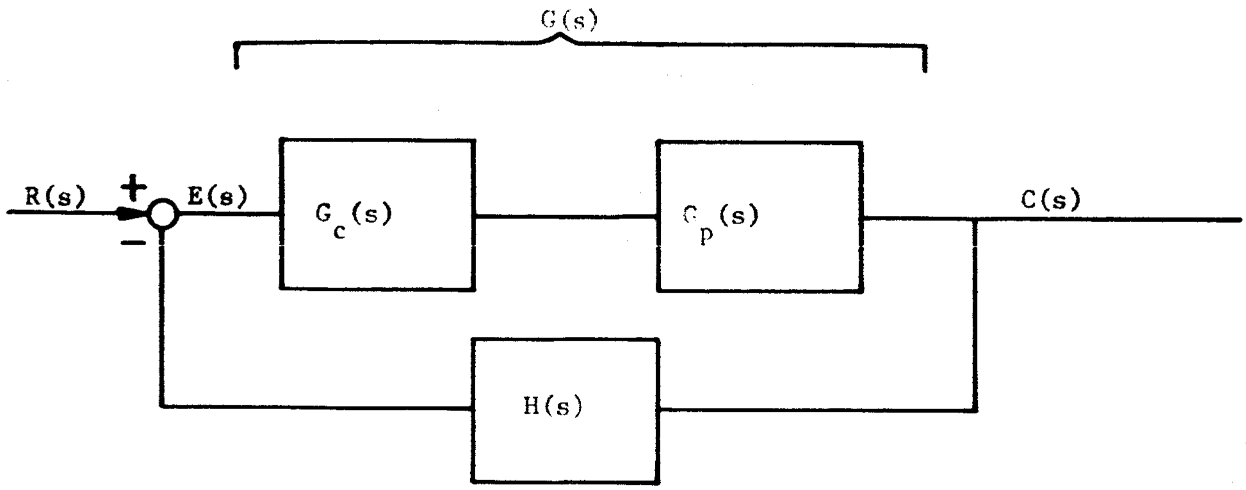

Fig. 2.2-1 A conventional feedback control configuration, where

$G_p(s)$ = Transfer function of the fixed plant.
$G_c(s)$ = Transfer function of series compensator.
$H(s)$ = Transfer function of feedback compensator.
$G(s)$ = Transfer function of the forward path,
$G(s) = G_c(s)G_p(s)$

These state variables are not unique, and any n variables which are capable of completely specifying the system behavior are satisfactory. Here we concentrate on one specific set of state variables, the real, physical, variables in which the plant and compensator differential equations are written. The point is that one need not make a big issue over state variables -- they are present in the describing differential equations.

Thus far we have accepted only one conclusion from modern control theory -- namely, "feedback all the state variables." In order to picture how this might be accomplished, the conventional control configuration of Fig. 2.2-1 is not adequate. The inadequacy of Fig. 2.2-1 stems from the "transfer function" approach to control system analysis and design that has become so popular in the past twenty years. The philosophy of the "transfer function" approach is this. The physical system to be controlled, the plant, is described by differential equations. These differential equations are transformed and the result described in block diagram form. Now the physical origins of the problem are no longer important. It does not matter if the physical system being controlled is a reactor or a positioning servomechanism. The problem has been reduced to a standard form, for which a variety of analysis and synthesis methods have been developed.

This general idea of making all problems look alike has met with remarkable success over the years. For example, a specialist in positioning systems can work on reactor control, once he has been given the governing equations. However, the strength of this approach is also its weakness. By making all systems look alike in their transformed form, the physical origins of the problem are no longer apparent. But if all of the state variables are to be fed back, then the variables whose identity has been obscured by the transfer function approach must be important. This is the basic difference between the modern design philosophy and the conventional design philosophy. The modern approach requires a maximum of detail in the system representation, so that all of the state variables may be recognized. Hopefully, these variables are also available for measurement and control.

Modern control theory, particularly the Maximum Principle, makes no provision for increasing the order of the plant to be controlled by the addition of series compensation. The design procedure advocated here does make use of series compensation, in addition to state variable feedback through constant gain elements. This provision is anticipated in Fig. 2.2-1 by labeling the forward transfer function, $G_c(s)G_p(s)$ as simply $G(s)$. To allow for series compensation in the final design technique, we shall discuss control of the modified plant, $G(s)$, rather than the plant itself. Of course, $G(s)$ is just the plant if no series compensation is needed. Fig.2.2-2 defines the notation that is used to represent the transfer function $G(s)$, as well as the designation of the system state variables, x_1 to x_n. Note that on the diagram the state variables are not designated as $x_i(s)$ or as $x_i(t)$. Strictly speaking, they should be designated as $x_i(s)$, since this is a block diagram. But these variables are real, physical system variables that exist in time, hence the unusual notation. Whether $x_i(s)$ or $x_i(t)$ is intended is always clear from the context.

Several other comments about Fig. 2.2-2 are in order. The state variables indicated are assumed to be actual physical variables, related to each other by definite gains and time constants that are inherent to the system being controlled. This is not always a realistic assumption, as with complex conjugate poles, for instance. However, this assumption will be maintained here in order to initiate the discussion.

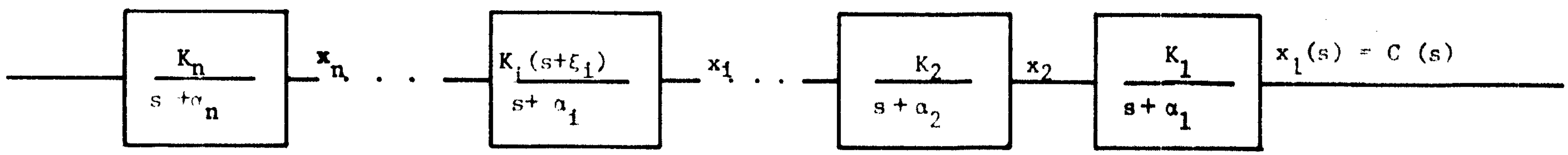

Fig. 2.2-2. Pictorial representation of G(s), indicating state variable locations, where G(s) is defined as

$$G(s) = G_c(s)G_p(s)$$

$$= \frac{K_1K_2 \cdots K_n (s+\xi_i)\cdots(s+\xi_n)}{s^n + a_n s^{n-1} + \cdots a_1}$$

$$= \frac{K_1}{(s+\alpha_1)} \frac{K_2 \cdots K_n (s+\xi_i)\cdots(s+\xi_n)}{(s+\alpha_2)\cdots(s+\alpha_n)}$$

The closed loop state variable feedback configuration associated with Figure 2.2-2 is indicated in Fig. 2.2-3. Here the k_i's are all assumed to be constant. The overall design problem that faces us is now clear in terms of Fig. 2.2-3. If $G_p(s)$ is of order p, how shall we pick the order of G(s) and how shall we evaluate all of the feedback coefficients, k_i, in order to realize a desired system behavior? This chapter is devoted to answering this question.

In terms of modern control theory, the system of Fig. 2.2-3 is overly restrictive, in the sense that the state variables are specifically indicated. A particular designer may wish to use the output and its n-1 derivatives as an alternate choice of state variables. This is, of course, a possibility, but not one that is considered here.

The system of Fig. 2.2-3 may appear overly restrictive in another way, since only one zero is indicated in the i th block. This is meant to imply that a zero may appear in any block, that is, i can be any number from 1 to m. In the general case there are m of these zeroes, and there is no restriction on which block any of the m zeroes may occupy.

The system configuration of Fig. 2.2-3 is not a convenient one with which to work because of the many inner loops. These inner loops may be eliminated by block diagram reduction. Two systematic methods of block diagram reduction immediately suggest themselves, and these are referred to as the $G_{eq}(s)$ reduction and the $H_{eq}(s)$ reduction, as indicated in Fig. 2.2-4. This is read as the G equivalent reduction and the H equivalent reduction. Both of these transfer functions are referred to as being "equivalent" because neither exists physically in the actual system. These block diagram reductions

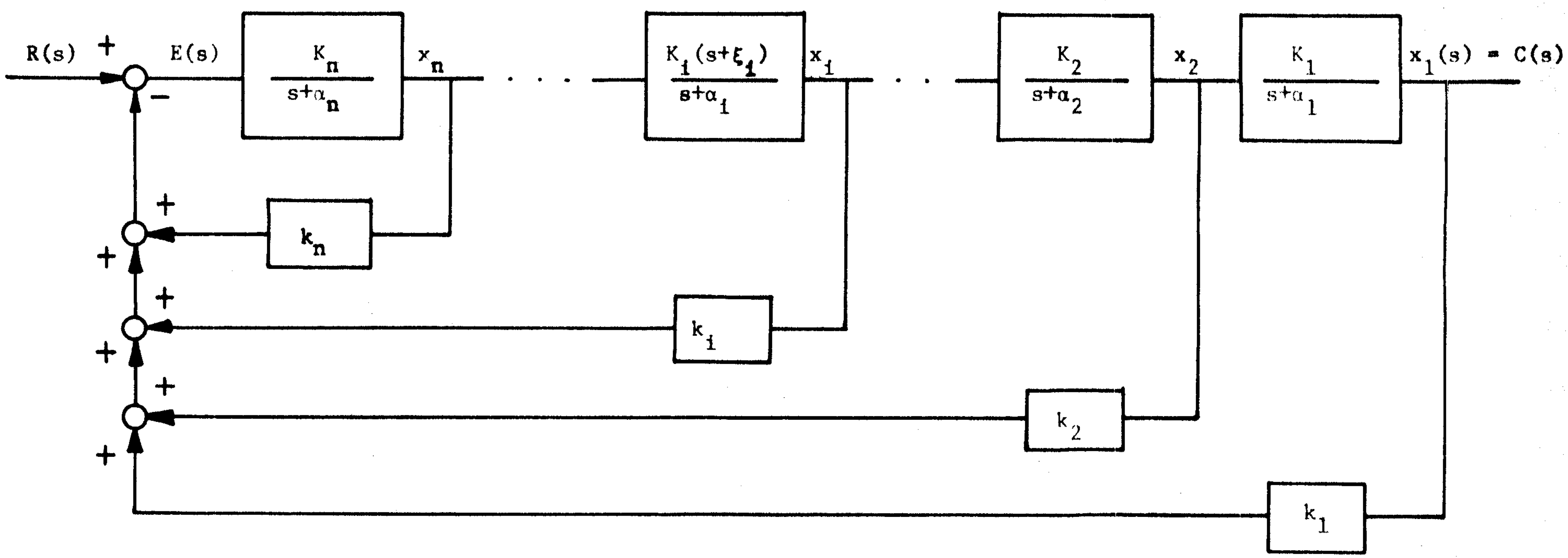

Fig. 2.2-3 A feedback control system with all state variables fed back. (Although only one zero is shown, in general m zeroes may be present, $m<n$.)

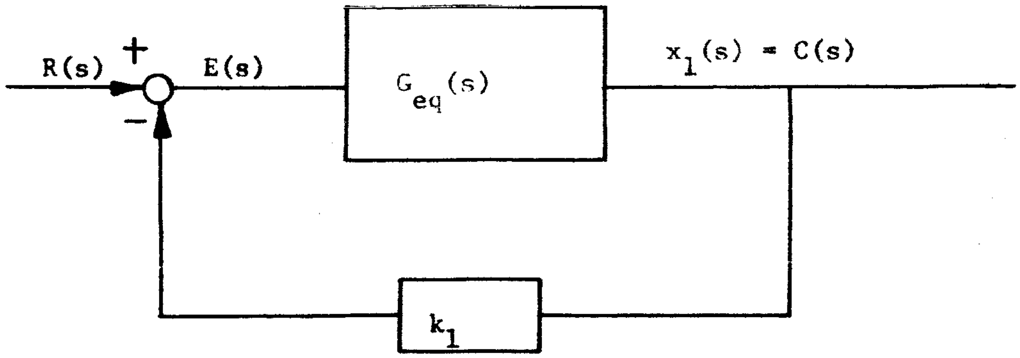

Fig. 2.2-4a The $G_{eq}(s)$ representation of Fig. 2.2-3.

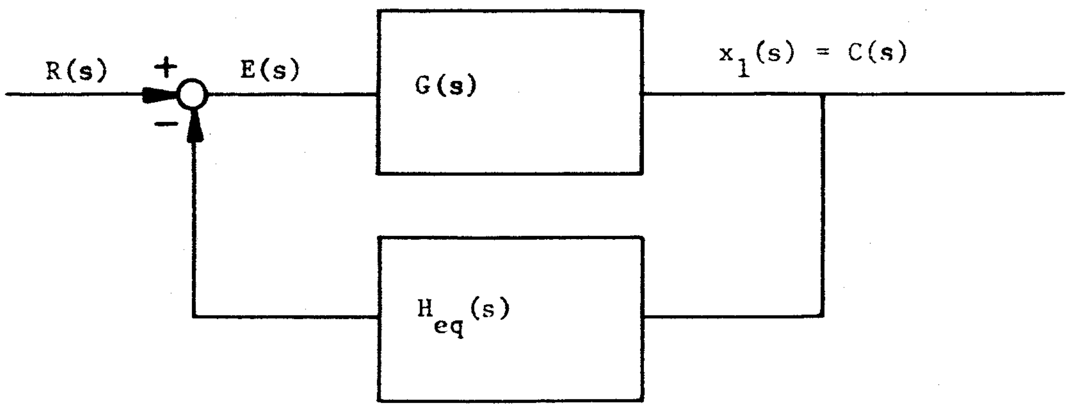

Fig. 2.2-4b. The $H_{eq}(s)$ representation of Fig. 2.2-3.

Fig. 2.2-4. Alternate simplified block diagram reductions of Fig. 2.2-3.

are just convenient ways for us to view the problem.

The $G_{eq}(s)$ reduction is accomplished by starting with the innermost loop, and combining the feedback coefficient k_n with the transfer function $K_n/s + \alpha_n$. Once this is accomplished, k_{n-1} forms the feedback coefficient for what is now the innermost loop. This procedure is repeated until no inner loops exist. The resulting zeroes of $G_{eq}(s)$ are identical to those of G(s), but the pole locations are different.

The $H_{eq}(s)$ reduction also begins with the feedback coefficient k_n. This feedback coefficient is moved to the right on the block diagram by multiplying it by the reciprocal of the transfer function $x_{n-1}(s)/x_n(s)$, and then combining the result with k_{n-1}. Each of the feedback coefficients is successively moved forward, until all are ultimately combined with k_1 to yield $H_{eq}(s)$. In the final block diagram, G(s) appears unaltered.

As an example of a block diagram reduction to both the $G_{eq}(s)$ form and the $H_{eq}(s)$ form, consider the position control system of Fig. 2.2-5, shown in block diagram form. In order to give a physical connotation to the state variables, gains, and time constants indicated on Fig. 2.2-5, one may assume that the physical system from which this block diagram was derived contained a field controlled DC motor as a power element. In this particular case the state variables and time constants may be easily identified on the block diagram of Fig. 2.2-5. If the output variable is actually θ, then x_1 is θ, x_2 is $d\theta/dt$, and x_3 is the motor field current. The gain $K_2 = 2$ is a physical property of the motor that relates the motor velocity to the motor field current, and the pole at $s = -3$ is associated with the mechanical time constants of the system. The gain $K_3 = 3$ is also an inherent motor characteristic relating field current to voltage input to the field. The pole at $s = -10$ is associated with the electrical time constant of the field

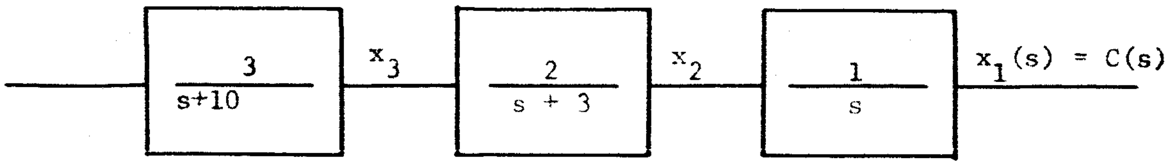

Fig. 2.2-5a The plant to be controlled.

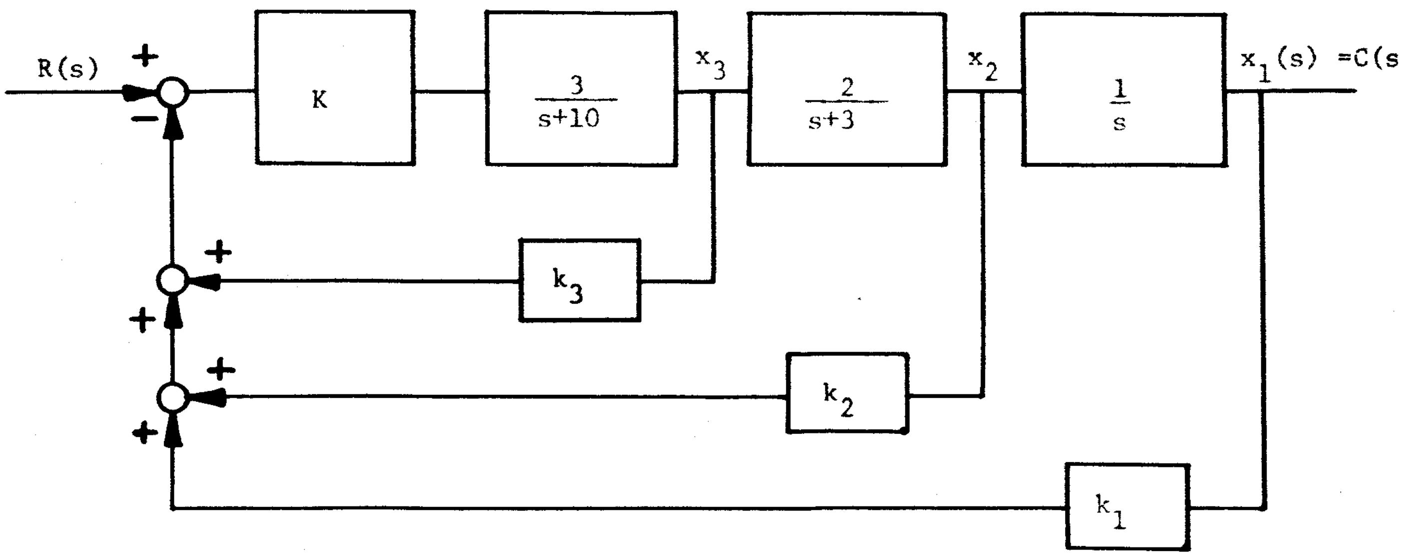

Fig. 2.2-5b. The system, including the plant and the controller.

Fig. 2.2-5. A simple positioning servomechanism using state variable feedback.

circuit. The gain K is an ideal gain constant associated with an electronic amplifier.

The procedure for reducing the third order system of Fig. 2.2-5 to the $G_{eq}(s)$ form is indicated in a step-by-step fashion in Fig. 2.2-6, where $G_{eq}(s)$ is found to be

$$G_{eq}(s) = \frac{6K}{s[s^2 + (13+3Kk_3)s + 30 + 9Kk_3 + 6Kk_2]} \qquad (2.2\text{-}1)$$

Note that the right hand pole of Fig. 2.2-5, that is, the pole at $s = 0$ remains unaltered in $G_{eq}(s)$, and the two remaining poles are specified in terms of a second order polynomial in s. If it is assumed for the time being that K is fixed, then in the second order polynomial there are still two free coefficients, k_2 and k_3. By specifying particular values of k_2 and k_3 the open loop poles of $G_{eq}(s)$ may be placed at any desired location on the s plane.

The closed loop transfer function $C(s)/R(s)$ is also indicated in Fig. 2.2-6b to be

$$C(s)/R(s) = \frac{6K}{s^3 + (13+3Kk_3)s^2 + (30+9Kk_3+6Kk_2)s + 6Kk_1} \qquad (2.2\text{-}2)$$

Here the denominator of $C(s)/R(s)$ is a cubic in s, now with 3 unspecified constants, k_1, k_2, and k_3. Thus the closed loop poles of the system may be located anywhere on the s plane by suitable choice of the three feedback coefficients. This is a significant feature of state variable feedback. The following general statement is true for the n th order case.

> In the n th order case, n of the system poles, the poles of $C(s)/R(s)$ may be located anywhere on the s plane by suitable choice of the n feedback coefficients, k_i.

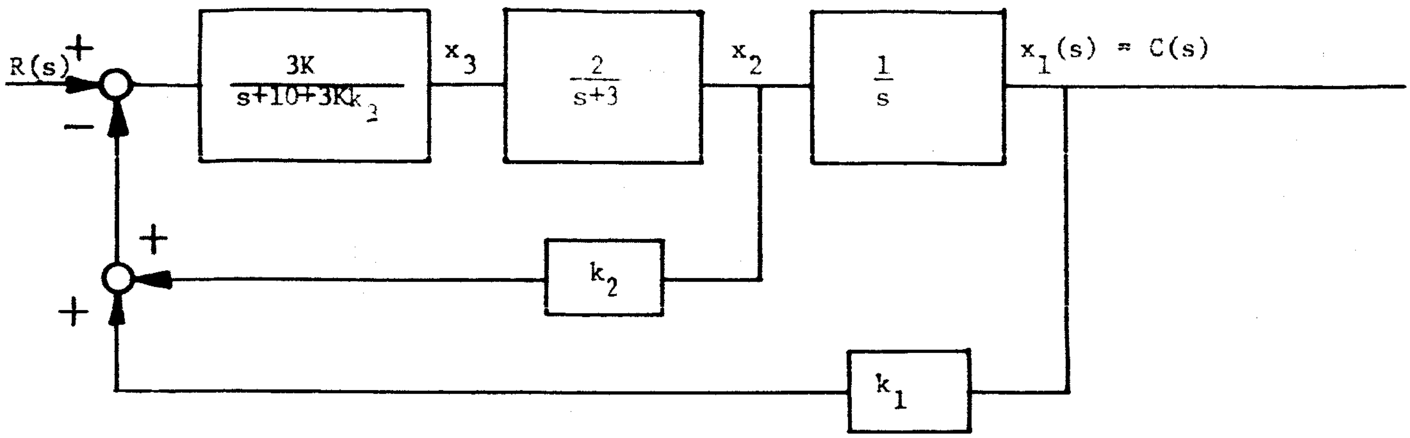

Fig. 2.2-6a. First step in the reduction of Fig. 2.2-5 to the $G_{eq}(s)$ form.

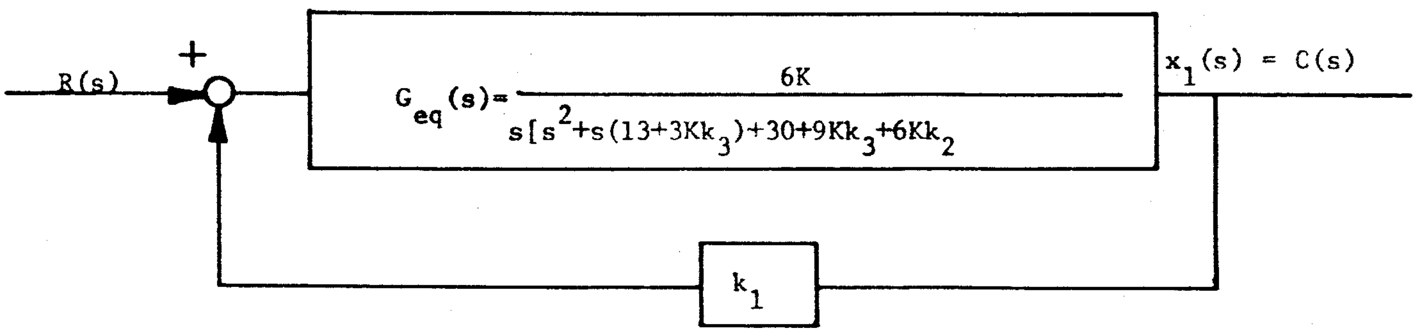

Fig. 2.2-6b. The $G_{eq}(s)$ form of Fig. 2.2-5.

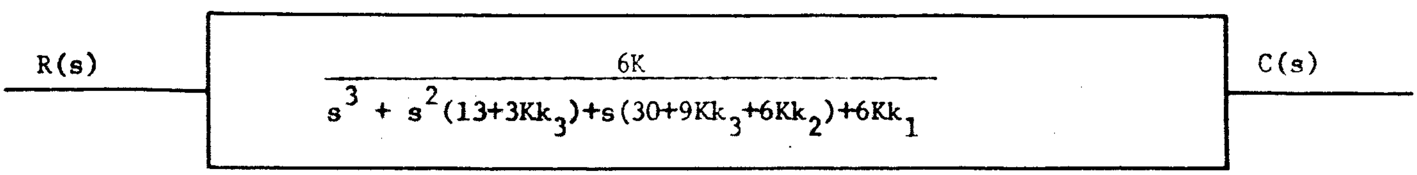

Fig. 2.2-6c. The closed loop transfer function for Fig. 2.2-5.

Fig. 2.2-6. Steps in the block diagram reduction of Fig. 2.2-5 to the $G_{eq}(s)$ form, and ultimately to the form of $C(s)/R(s)$.

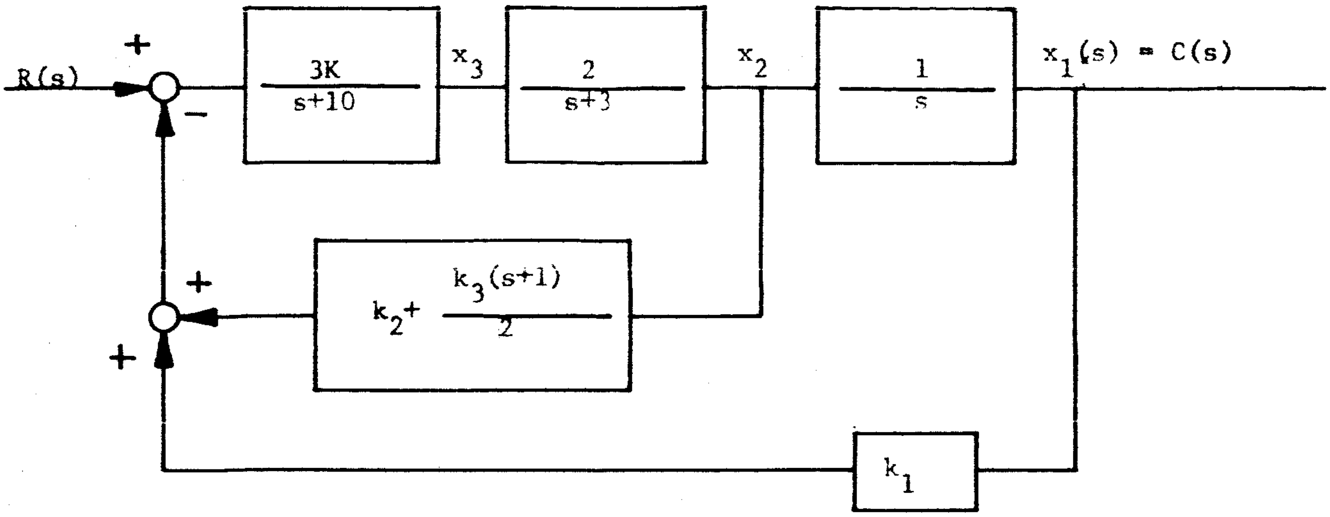

Fig. 2.2-7a. First step in the reduction of Fig. 2.2-5 to the $H_{eq}(s)$ form.

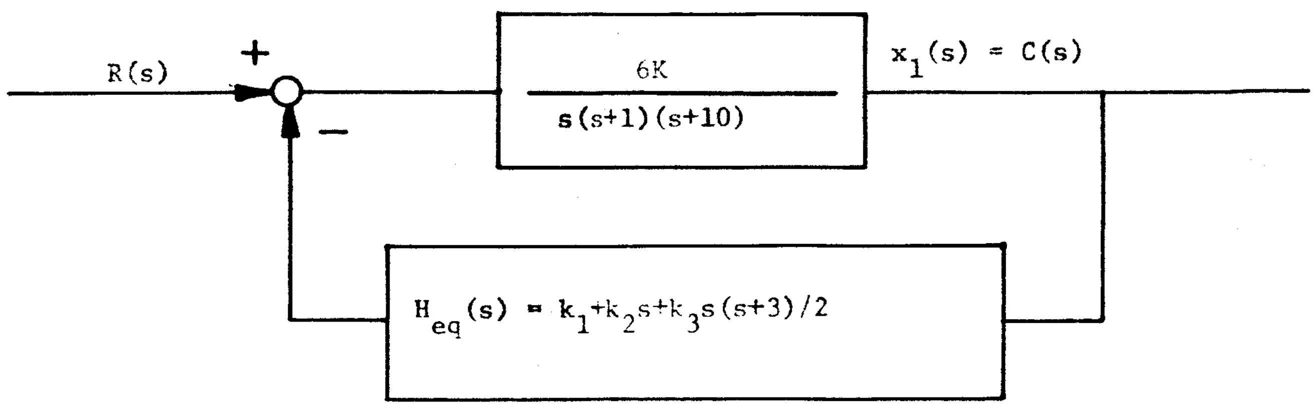

Fig. 2.2-7b. The $H_{eq}(s)$ form of Fig. 2.2-5.

Fig. 2.2-7. The reduction of Fig. 2.2-5 to the $H_{eq}(s)$ form.

This statement is proved by Brockett (196) and discussed at some length by Schultz and Melsa (1966). The importance of this statement is difficult to overemphasize.

In practice, k_1 is usually specified to insure zero steady state error. For example, in the positioning system of Fig. 2.2-5, or in any case with an integrator in $G_p(s)$, k_1 is made equal to 1. The ability to position all n poles is retained if the amplifier gain K is assumed to be adjustable. This is often true in practice.

On the basis of the discussion thus far, we already have at least a tentative design procedure. For a given G(s), pick a desired C(s)/R(s), with a compatible number of poles and zeroes. Equate C(s)/R(s) in terms of the k_i's and K to the desired C(s)/R(s), and force the coefficients of like powers of s to be equal. Often, however, G(s) is not given, only $G_p(s)$, with $G_c(s)$ unspecified. Root locus techniques prove to be helpful in choosing the form and tentative pole and zero locations that must be included in $G_c(s)$. Unfortunately, the $G_{eq}(s)$ formulation is not easily interpreted in terms of root locus methods, because the pole locations are a function of K. This is evident in the example being discussed here from Eq. 2.2-1. Hence we turn our attention to the $H_{eq}(s)$ formulation.

The equivalent feedback compensator, $H_{eq}(s)$, for the problem of Fig. 2.2-5 is given in Fig. 2.2-7 as

$$H_{eq}(s)= k_1+k_2s+k_3s(s+3)/2 = \frac{k_3s^2+s(2k_2+3k_3)+2k_1}{2} \qquad (2.2\text{-}3)$$

Here the numerator of $H_{eq}(s)$ is a second order polynomial in s and contains three feedback coefficients, k_1, k_2, and k_3, and $H_{eq}(s)$ is not a function of K. If k_1 is chosen to insure zero steady state error for step inputs, the two constants k_2 and k_3 are still available to insure that the roots of the second order polynomial may be any desired value.

Observe that in this example $H_{eq}(s)$ has two zeroes, and these are not accompanied by any poles. This is a highly desirable result in terms of stability, since the open loop transfer function $G(s)H_{eq}(s)$ now has only one more pole than zero. Clearly it would be impossible to build a device with a transfer function $H_{eq}(s)$, but recall that $H_{eq}(s)$ does not exist as such anywhere in the system. An effect equivalent to using a series compensator with the transfer function $H_{eq}(s)$ is accomplished by means of state variable feedback.

The following statements are true not only for the example under discussion, but for the general n th order case. (Schultz and Melsa, 1966.)

1. $H_{eq}(s)$ has $(m - 1)$ zeroes
2. If $G(s)$ has no zeroes, then $H_{eq}(s)$ has no poles.
3. If $G(s)$ has m zeroes, then
   a. if all of the zeroes of $G(s)$ are to the right of x_n on Fig. 2.2-3, $H_{eq}(s)$ also has m poles, and these poles coincide with the zeroes of $G(s)$. The resulting open loop transfer function, $G(s)H_{eq}(s)$, then has n poles and $(n - 1)$ zeroes.
   b. if one zero is to the left of x_n in Fig. 2.2-3, then $H_{eq}(s)$ has $(m - 1)$ poles, and these are coincident with the $(m - 1)$ zeroes of $G(s)$ that lie to the right of x_n on Fig. 2.2-3. The resulting $G(s)H_{eq}(s)$ has n poles and n zeroes.
4. $H_{eq}(s)$ is not a function of K.

The full significance of these statements is not apparent at this time, since we have made no reference to stability, relative stability, or sensitivity. The statements are simply a consequence of the $H_{eq}(s)$ representation of a control system using state variable feedback. The following sections of this chapter develop a synthesis procedure that depends on the truth of the above statements with respect to $H_{eq}(s)$.

2.3 The Guillemin-Truxal Method

The design procedure advocated in this chapter is based upon the specification of a desired system transfer function, $C(s)/R(s)$. Fortunately, this is not a new idea, as this is also the basis for the Guillemin-Truxal method of system synthesis. The motivation and advantages of such an approach are well covered in Chapter 5 of Truxal's book "Control System Synthesis," (Truxal, 1955). Here only a few salient points are covered in order to relate the Guillemin-Truxal method to that of state variable feedback.

The Guillemin-Truxal method is based upon the block diagram of Fig. 2.3-1. Here again $G_p(s)$ is the fixed plant, and $G_c(s)$ is to be determined, according to the following three step procedure.

1. A desired closed loop transfer function, $C(s)/R(s)$, is established from the system specifications. This is expressed as a ratio of polynomials, as

$$\frac{C(s)}{R(s)} = \frac{p(s)}{q(s)} \tag{2.3-1}$$

2. The required $G_c(s)$ is determined by solving the equation

$$\frac{C(s)}{R(s)} = \frac{p(s)}{q(s)} = \frac{G(s)}{1+G(s)} = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \tag{2.3-2}$$

for $G_c(s)$.

3. The required $G_c(s)$ is synthesized, usually through the use of passive elements, as resistors and condensers.

Let us comment briefly on each of these three steps, and illustrate the method by a simple example. The selection of a desired $C(s)/R(s)$ is equivalent to picking a specific time response for a given $r(t)$. Alternately, it is equivalent to picking the frequency response of the closed loop system.

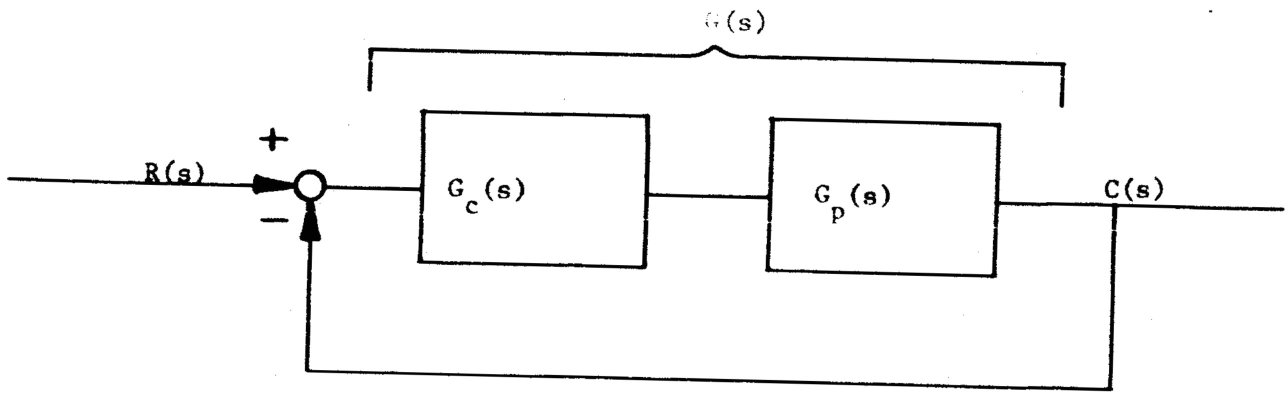

Fig. 2.3-1. The basic system configuration associated with the Guillemin-Truxal method.

Hence, it is not important whether system specifications are given in the frequency or in the time domain. Either domain is a suitable basis for deciding upon a desirable $C(s)/R(s)$.

Step two requires only algebra. Once the system transfer function is specified as a ratio of polynomials, $G(s)$ is uniquely defined. If the plant is given and unalterable, then $G_c(s)$ is determined from the equation

$$G(s) = G_c(s)G_p(s) \tag{2.3-3}$$

The synthesis techniques necessary to realize a given $G_c(s)$ in step three are more network problems than control system problems, and they are not considered here. The subject is also well covered in Truxal.

As an illustration of the Guillemin-Truxal method, consider once again the control of the plant given in Fig. 2.2-5a. Assume that the desired closed loop performance is given as

$$\frac{C(s)}{R(s)} = \frac{160}{[(s+2)^2+2^2](s+20)} = \frac{160}{s^3+24s^2+88s+160} \tag{2.3-4}$$

Equation 2.3-2 may be solved for $G(s)$, with the result that $G(s)$ is

$$G(s) = \frac{160}{s(s^2+24s+88)}$$

The resulting $G_c(s)$ from Eq. 2.3-3 is

$$G_c(s) = \frac{(s+3)(s+10)}{s^2+24s+88} = \frac{(s+3)(s+10)}{(s+4.6)(s+19.4)}$$

The final closed loop configuration is pictured in Fig. 2.3-2a. The outstanding feature of the final result is the cancellation of the poles of $G_p(s)$ by the zeroes of $G_c(s)$. In the ideal case this cancellation would be perfect, and the order of the closed loop system is the same as the order

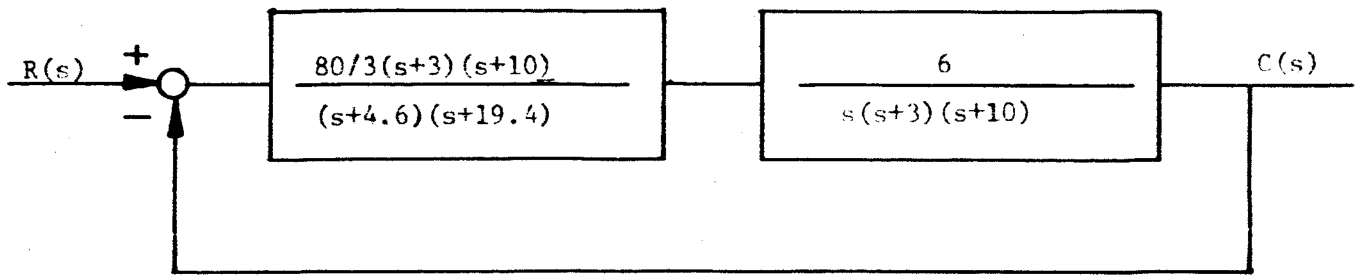

Fig. 2.3-2a. The Guillemin-Truxal realization of a specific C(s)/R(s).

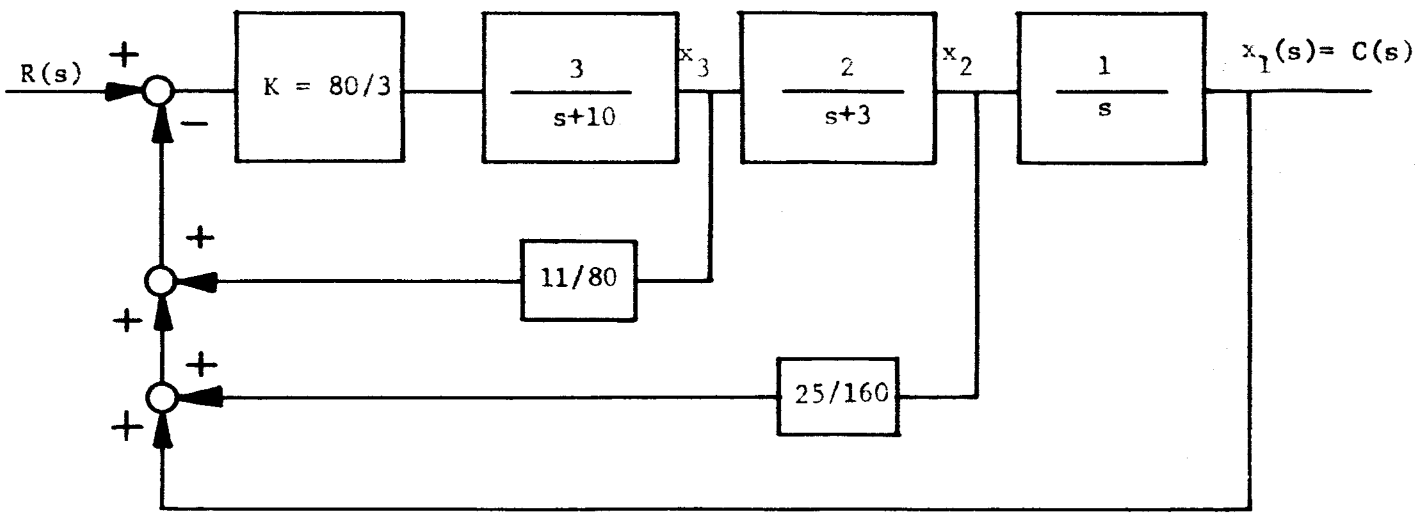

Fig. 2.3-2b. A state variable feedback realization that yields the same C(s)/R(s) as in Fig. 2.3-2a.

Fig. 2.3-2. Alternate methods of realizing $C(s)/R(s) = 160/s^3+24s^2+88s+160$, given the fixed plant of Fig. 2.2-5a.

of the plant being controlled. In this sense, the Guillemin-Truxal approach is much like that of optimal control theory, and different from those series compensation methods that do not rely on pole-zero cancellation. It will be shown later that in some cases it is highly desirable to increase the order of the system over that of the unalterable plant.

Fig. 2.3-2b is the realization of the same transfer function of Eq. 2.3-4 through the use of state variable feedback. The use of state variable feedback alters pole locations through the use of feedback, rather than cancelling them with zeroes. Because the mechanism by which new pole locations are realized is basically different, one might suspect that the state variable feedback design procedure would not suffer from the limitations that are imposed upon the Guillemin-Truxal method. This is indeed the case.

One obvious restriction of the Guillemin-Truxal method is a limitation to stable $G_p(s)$. If $G_p(s)$ has poles in the right half s plane, these cannot be cancelled with zeroes. Because state variable feedback simply alters pole locations rather than cancelling them, $G_p(s)$ is not restricted to being stable when state variable feedback is used.

The Guillemin-Truxal method requires that C(s)/R(s) be chosen not only to meet desired system specifications, but also to insure that all of the poles of the open loop transfer function G(s) lie on the negative real axis. The following is a quote from Truxal, page 297.

> The imposition of the additional constraint that all poles of the open loop transfer function lie on the negative real axis is not only necessary if the synthesis (of $G_c(s)$) is to be simple, but also frequently desirable to ensure that the transfer functions of the compensation networks be realizable by RC networks.

Thus it is seen that the difficulty in the Guillemin-Truxal method stems largely from the necessity to realize a rather complicated $G_c(s)$. This practical difficulty does not arise in the use of state variable feedback. In

Fig. 2.3-2b, for instance, no $G_c(s)$ at all was needed to realize the desired system response. Even in cases where series compensation needs to be added, the basic building block has the transfer function

$$\frac{K_i(\tau_i s+\gamma_i)}{(s+\beta_i)}$$

where τ_i is either 1 or 0. This is the familiar lead or lag circuit, but it is always used with feedback. As many of these basic elements as are needed may be used, although seldom are more than two required.

The basic theoretical limitation of the state variable feedback method is the same as that of the Guillemin-Truxal Method. The closed loop transfer function C(s)/R(s) cannot have a smaller pole-zero excess than that that originally existed in $G_p(s)$.

2.4 The Heq Method -- The Simplest Case

In a previous section two methods of system representation were outlined which might serve as the basis for a design procedure. The $G_{eq}(s)$ representation indicated how the use of state variable feedback might be considered as altering the pole locations of the open loop transfer function. The $H_{eq}(s)$ representation supplied an alternate interpretation of the effects of state variable feedback. It was shown that state variable feedback might be considered as introducing (n - 1) zeroes in the feedback path, while G(s) is left unaltered. Regardless of which method of system representation is used, the resulting C(s)/R(s) is the same, and the poles of this system transfer function may be located as desired.

In the remainder of this chapter we shall use the $H_{eq}(s)$ representation exclusively. The reason is not that it provides an easier means of system synthesis, but rather the $H_{eq}(s)$ formulation provides a convenient method of critically evaluating the results of state variable feedback. The $G_{eq}(s)$ formulation does not prove suitable for evaluation purposes on the root locus diagram simply because the pole locations of $G_{eq}(s)$ are functions of the amplifier gain, K. In the discussion of the $H_{eq}(s)$ representation, it was specifically noted that $H_{eq}(s)$ is not a function of K, and hence root locus methods might be effectively used. The preference for root locus methods is a personal preference of the authors.

The design procedures outlined in this section apply only to the simplest case when no series compensation need be added before the design procedure is started. In this case $G_c(s)$ is 1, and $G(s) = G_p(s)$. Such a situation arises when

1. $G_p(s)$ has the correct number of poles required in C(s)/R(s).
2. The zeroes of $G_p(s)$, if any, are the desired zeroes of C(s)/R(s).

It is possible to conjure up other situations in which no series compensation is needed, as when only some of the zeroes are in the desired locations, and C(s)/R(s) is to have less poles than exist originally in $G_p(s)$. Cases like this are considered as pathological cases. Only cases for which conditions 1 and 2 are satisfied are discussed here. The pathological cases present no theoretical problems, but they do complicate the presentation.

The H_{eq} method of system synthesis for the simplest case consists of the five following steps.

1. Assume all state variables are available, and represent the final closed loop system as in Fig. 2.2-3.
2. Choose the desired closed loop response, C(s)/R(s).
3. From the block diagram of 1, find C(s)/R(s) in terms of the k_i's, preferably by the use of the $H_{eq}(s)$ block diagram reduction.
4. Equate the answers from 2 and 3, and solve for the k_i's by equating coefficients of like powers of s.
5. If all of the state variables are not available, use the known values of the k_i's to determine suitable series or minor loop compensation.

In cases where the plant contains an integration, it is usually assumed at the outset that $k_1 = 1$, and the above procedure is carried out in terms of the k_i, $i = 2, 3, \ldots n$, and K. By setting $k_1 = 1$, this insures zero steady state position error for step inputs.

As an illustration of the above design procedure, consider the system of Fig. 2.2-5. This figure is already drawn in the form required by Step 1. In line with Step 2, let us assume that the desired closed loop response is given by

$$C(s)/R(s) = \frac{160}{[(s+2)^2 + 2^2]\ (s+20)}$$

$$= \frac{160}{s^3 + 24s^2 + 88s + 160} \qquad (2.4\text{-}1)$$

This is a closed loop response with zero steady state error for step inputs, as $C(0)/R(0) = 1$. The response is dominated by a set of complex conjugate poles at $s = -2 \pm j2$. H_{eq} (s) for this system may be written down by inspection as

$$H_{eq}(s) = h_3 s\ (s+3)/2 + k_2 s + 1$$

$$= \frac{k_3 s^2 + (3k_3+2k_2)\ s + 2}{2} \qquad (2.4\text{-}2)$$

Here k_1 has been assumed to be 1 to insure the zero steady state error. The transfer function in the forward loop of the $H_{eq}(s)$ representation is just $G(s)$, here equal to $G_p(s)$, since no series compensation is added in the simplest case. Thus $C(s)/R(s)$ may be written in terms of k_2, k_3, and K as

$$C(s)/R(s) = \frac{\dfrac{6K}{s\ (s+3)\ (s+10)}}{1 + \dfrac{6K}{s(s+3)\ (s+10)} \times \dfrac{k_3 s^2 + (3k_3+2k_2)\ s + 2}{2}}$$

$$= \frac{6K}{s^3 + (13+3Kk_3)\ s^2 + [30+3K\ (3k_3+2k_2)]\ s + 6K} \qquad (2.4\text{-}3)$$

This completes step 3.

Step 4 is accomplished by equating coefficients of like powers of s in the denominators of Equation 2.4-1 and 2.4-3. The following 3 algebraic equations result.

$$6K = 160$$

$$13 + 3Kk_3 = 24$$

$$30 + 3K\ (3k_3+2k_2) = 88$$

These algebraic equations are easily solved to give

$$K = 80/3$$
$$k_3 = 11/80 \qquad (2.4\text{-}4)$$
$$k_2 = 25/160$$

The final system is pictured back in Fig. 2.3-2b, where it is compared with the Guillemin Truxal realization of Fig. 2.3-2a. Before going to step 5, let us continue with the comparison by examining both systems on the root locus diagram. A systematic method of evaluation of the state variable feedback system involves the following steps.

a. Use the values of the k_i's to find $H_{eq}(s)$.

b. Determine the zeroes of $H_{eq}(s)$.

c. Locate the poles and zeroes of $G(s)\,H_{eq}(s)$ on the s plane. Also locate on the s plane the closed loop poles of $C(s)/R(s)$, as an aid in drawing the root locus.

d. Sketch the root locus to insure that the desired stability and sensitivity benefits that can be achieved through state variable feedback actually have been realized.

For the problem under discussion, $H_{eq}(s)$ is found by substituting the now known values of k_2 and k_3 into Eq. 2.4-2. The result is

$$H_{eq}(s) = \frac{11\ (s^2+5.285+14.5)}{160} = 11/160\ [(s+2.64+j2.75)(s+2.64-j2.75)]$$

Steps c and d are indicated on Fig. 2.4-1, with the closed loop poles enclosed in square boxes. The root locus for the corresponding Guillemin-Truxal realization is given in Fig. 2.4-2. It should be stressed that at the desired gain, the two systems are identical from an input-output point of view. If viewed from the point of view of varying gain, the state variable feedback

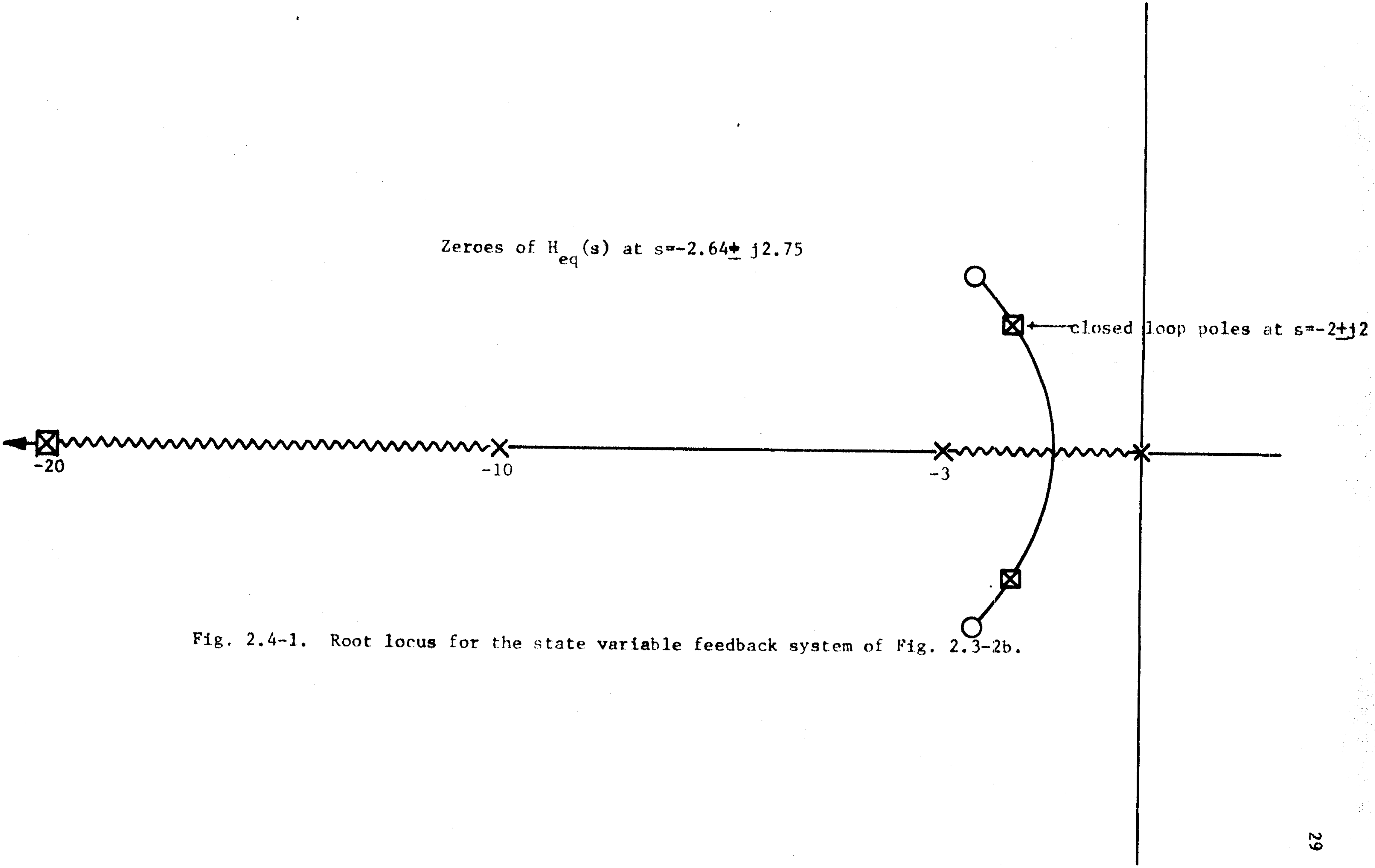

Fig. 2.4-1. Root locus for the state variable feedback system of Fig. 2.3-2b.

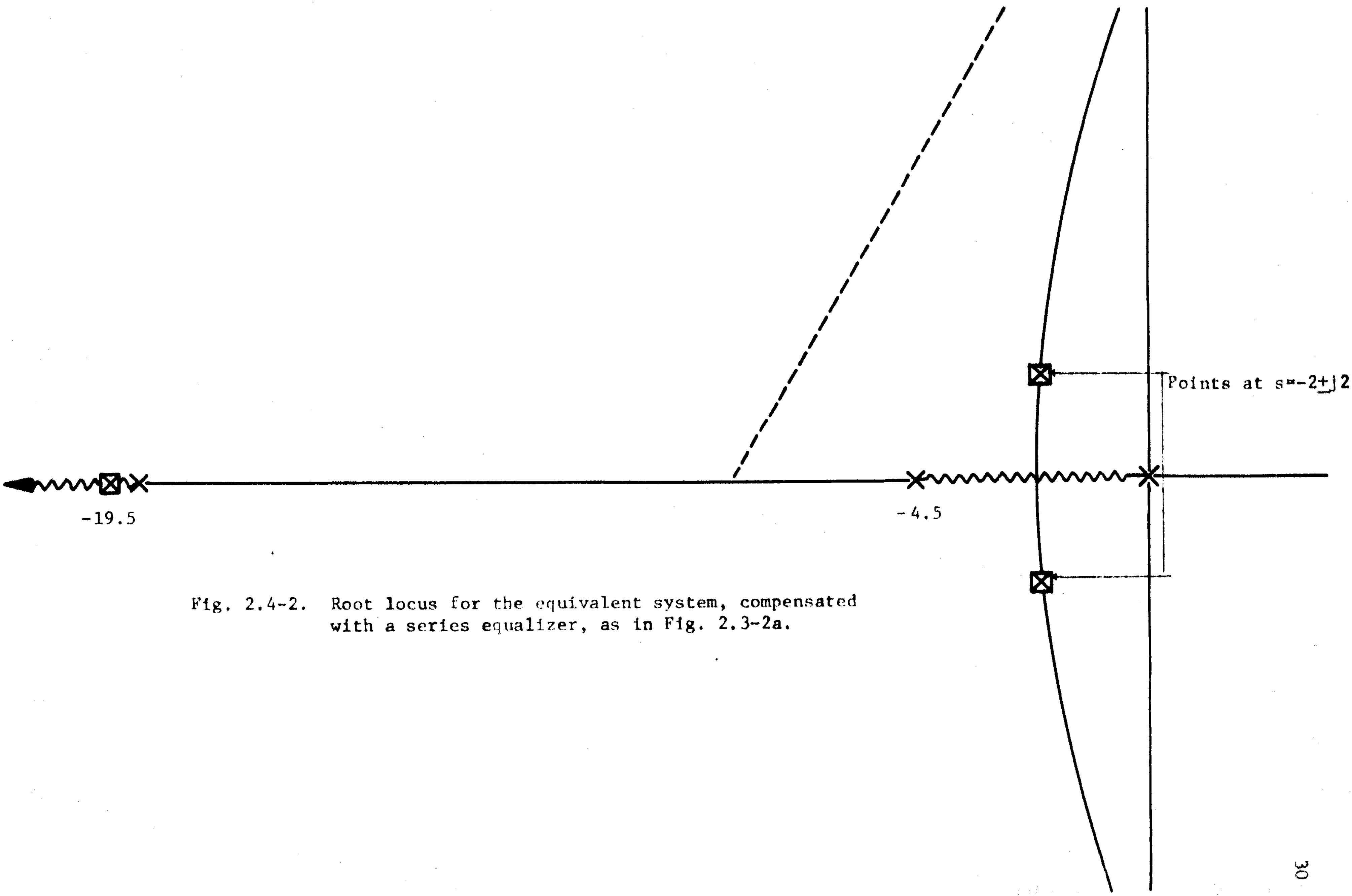

Fig. 2.4-2. Root locus for the equivalent system, compensated with a series equalizer, as in Fig. 2.3-2a.

system has obvious advantages. As the gain is varied from 160 to infinity, the dominant poles of the state variable feedback system change very little, and the damping ratio stays almost constant. The system is insensitive to gain changes as far as response or stability is concerned. Such is not the case for the comparison system using series compensation.

The method of system evaluation that was used above is based upon the use of $H_{eq}(s)$, and strictly speaking, this is not part of the design procedure. The design procedure was complete once the values of k_2, k_3, and K were specified in Eq. 2.4-4. However, the analysis of the final design is considered so important, that here the two are treated almost as one under the title of the $H_{eq}(s)$ method.

In order to demonstrate further the superiority of the state variable feedback method over the series compensation method, let us pursue the analysis of this example a bit further. While the root locus of Fig. 2.4-1 indicates the extreme insensitivity to gain changes that has been realized through state variable feedback, no indication is given as to the resulting insensitivity for changes in pole locations. Insensitivity to pole location is quite important, either because the initial pole location is not known exactly, or because the pole locations may actually change during the operating life of the system. This happens in aircraft control systems, for instance, where the damping due to the atmosphere changes radically as the plane flys from sea level to 30,000. The change is slow enough so that the system need not be considered as time varying, but the change in pole location can be critical.

Assume that in the example being considered that the design is complete, and that k_2, k_3, and K are as specified in Eq. 2.4-4. Assume further that the pole at $s = -3$ is allowed to take on different values. For sake of generality, this pole location may be designated as being at α, rather than at -3. This case is pictured in Fig. 2.4-3a. A variety of questions may be asked now, depending upon the amount of detail desired in the answer. One might ask is the system stable for all α, or given a fixed new pole position at $\alpha = \alpha_1$, is the system still relatively insensitive to gain changes, as previously, or if both α and K are allowed to vary, is stability still maintained. We will answer all of these questions for the system of Fig. 2.4-3a, not because these are questions that need to be answered in every problem, but because the answers to these questions demonstrate the desirable qualities that have been realized through the use of state variable feedback.

First consider the case where K is fixed and α is allowed to vary from 0 to infinity. The question of stability or location of the closed loop poles for any specific value of α may be answered by plotting a root locus diagram vs. α rather than K. This is a standard procedure, discussed by Kuo, (Kuo, 1962) for example, under the heading of "Root Locus Plots with Variable Other than Gain". If the characteristic equation

$$1 + G(s)\ Heq(s) = 0$$

is expressed with α written as a gain term, the result is that

$$\frac{\alpha(s)\ (s+21)}{s^3 + 21\ s^2 + 25s + 160} = -1 = 1\ \underline{/180^\circ} \qquad (2.4\text{-}5)$$

The poles of Eq. 2.4-5 are located at $s = -20.1$ and $s = -.45 \pm j2.6$. The root locus vs. α is plotted in Fig. 2.4-4, and, of course, for $\alpha = 3$, this locus goes through the desired pole locations of C(s)/R(s), as required in Eq. 2.4-1. The closed loop system is stable for all $0 \leq \alpha < \infty$.

As a further bit of analysis concerened with this same problem, assume that the pole at α is actually located at the origin, rather than at the

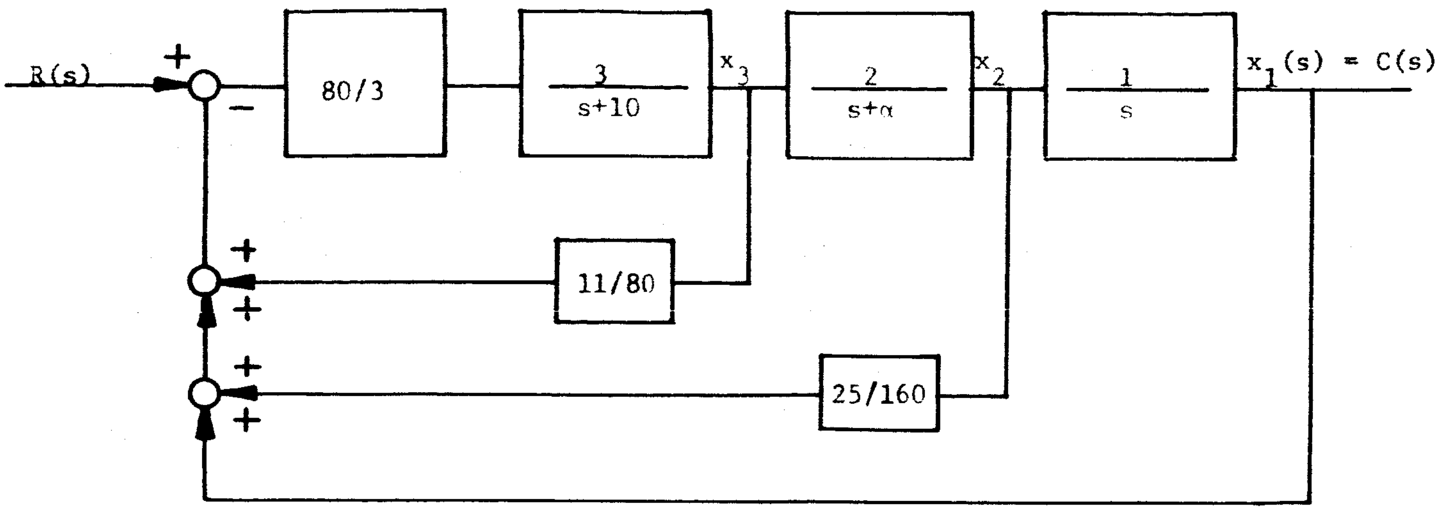

Fig. 2.4-3a. The example of **Fig.** 2.2-5 and Fig. 2.3-2b redrawn with the pole at s=-3 relocated at s=-α.

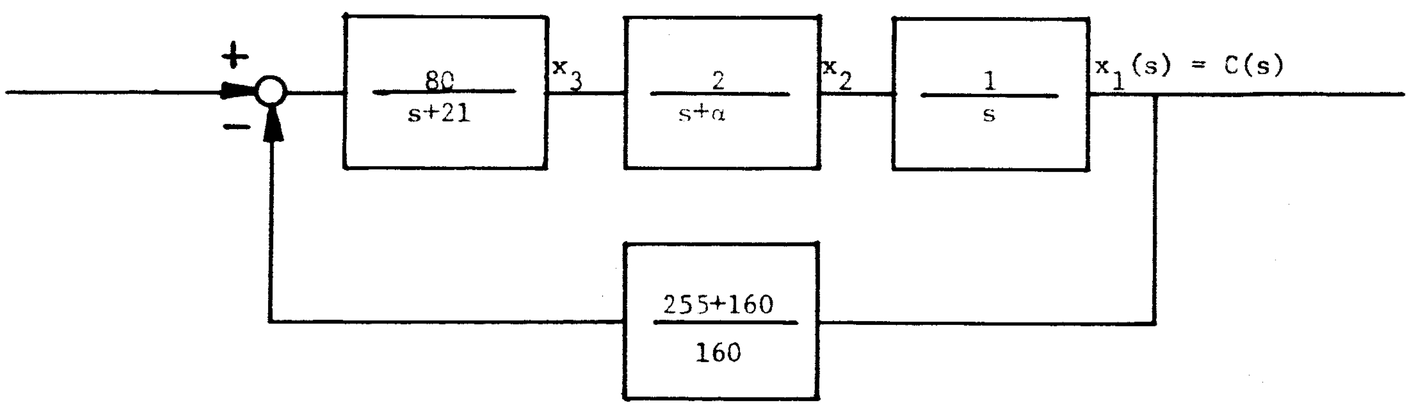

Fig. 2.4-3b. The system of Fig. 2.4-3a redrawn to leave the pole at α.

Fig. 2.4-3. Example system in which the pole at s=-3 is assumed to lie at an arbitrary point α.

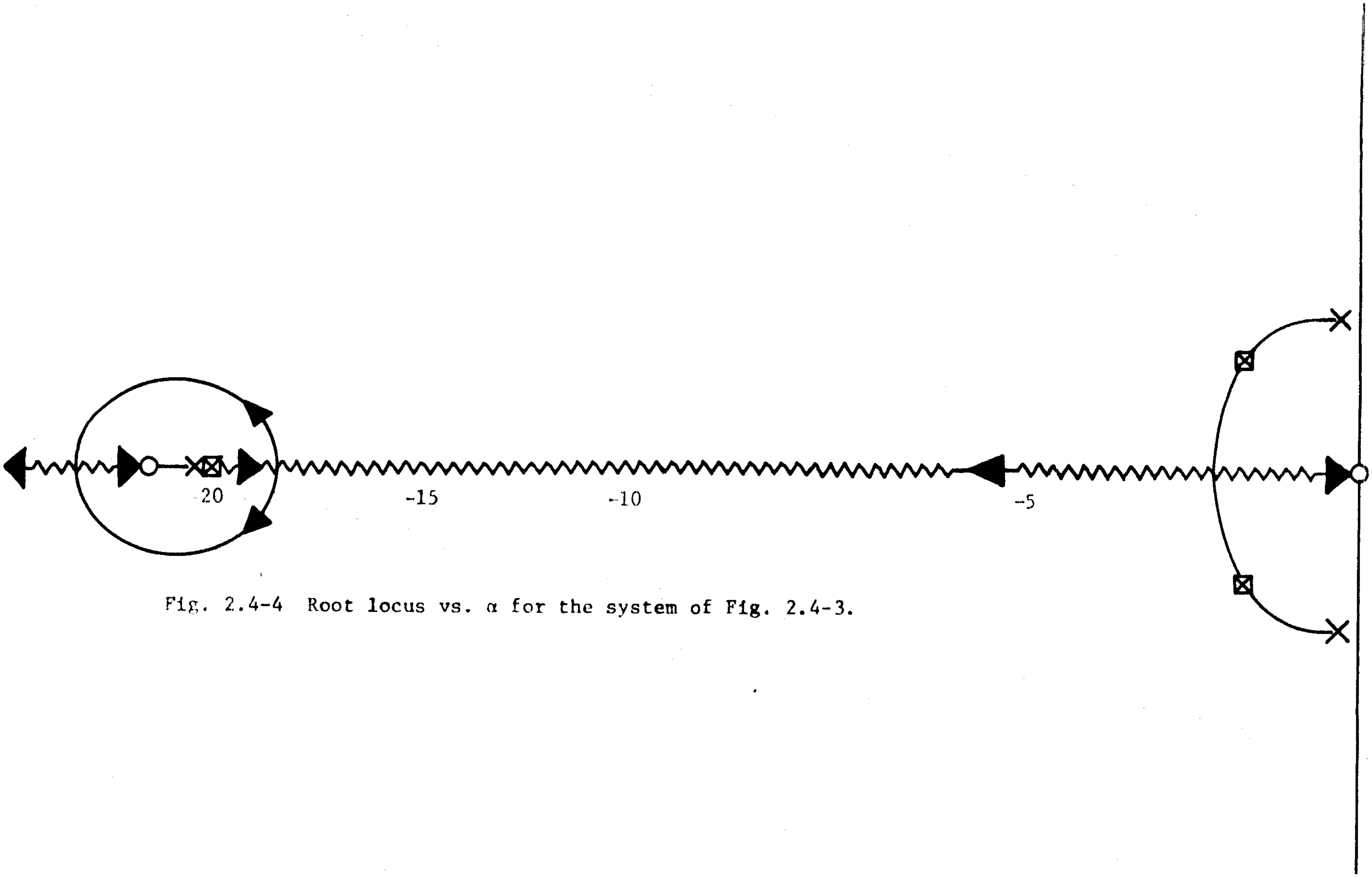

Fig. 2.4-4 Root locus vs. α for the system of Fig. 2.4-3.

design point of s = -3. This case is easy to consider, since we already know the closed loop pole locations when K is 80/3 from Fig. 2.4-4. Let us examine a normal root locus diagram and see once again how the damping ratio varies when K varies. When $\alpha = 0$, G(s) is

$$G(s) = \frac{160}{s^2\ (s+10)}$$

and Heq(s) is

$$Heq(s) = \frac{11s^2 + 255 + 160}{160}$$

$$= 11/160\ (s+1.13+j2.87)\ (s+1.13-j2.87) = 11/160\ [(s+1.13)^2+2.87^2]$$

The root locus diagram for this system is given in Fig. 2.4-5. Note that although the damping ratio is decreased by a significant amount, it remains relatively constant for a wide range of K. And this is despite the fact that k_2, k_3, and K were chosen for a pole location of s = -3 rather than zero.

Finally, it is possible to show that this system is stable for any α and any K. This is easily done by redrawing the final system so that the pole at α is not moved, and so that α does not appear any where else in the open loop transfer function. This may be done in terms of a pseudo $G_{eq}(s)$ and a pseudo $H_{eq}(s)$, as in Fig. 2.4-3b. The open loop transfer function is now

$$\frac{K\ (s+6.4)}{s\ (s+21)\ (s+\alpha)}$$

Here K is left arbitrary, since we wish to indicate stability for any K. If an asymptotic Bode diagram is now drawn for the amplitude of the open loop transfer function, it is noted that no value of α can be chosen such that the slope on the Bode diagram is never greater than -2, or 40 db. This is sufficient, though not necessary, to indicate that the phase shift never exceeds 180 degrees, and hence the system is stable for any gain and any α. The comparison system using the series equalizer is not stable for

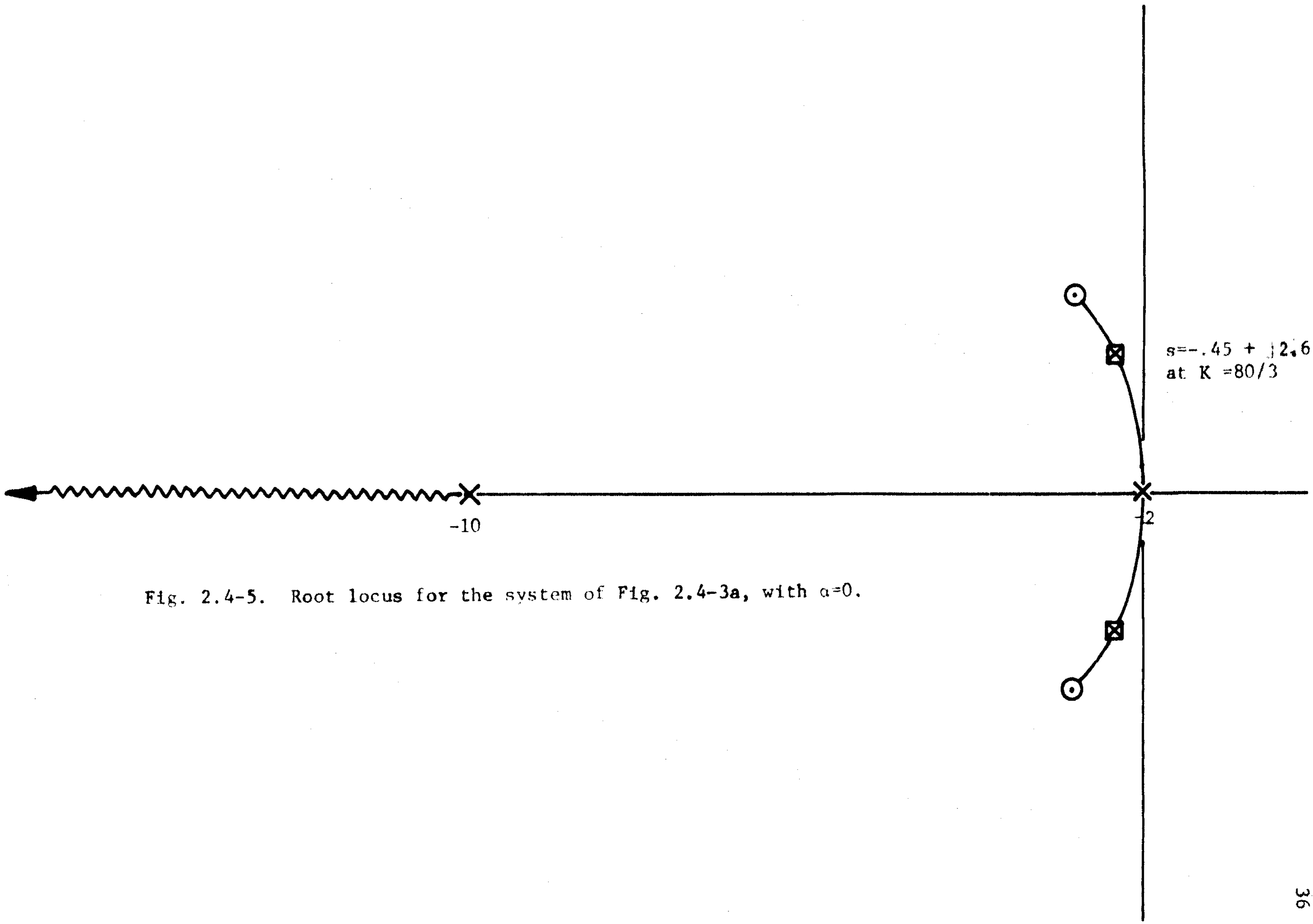

Fig. 2.4-5. Root locus for the system of Fig. 2.4-3a, with α=0.

any K even if the pole at α is located at the design value, s = -3.

So much for this example, which is meant to show that even in the simplest case surprising benefits may be realized by using state variable feedback. As yet we have still assumed that all of the state variables are available, so that we have not had to use step 5 of the design procedure. This is the subject of the next section, what to do when all of the state variables are not available. Before proceeding to that section however, it is important to indicate that the highly desirable stability and sensitivity properties exhibited by the previous example are not necessarily inherent in the state variable feedback approach. A simple example serves to illustrate the point. The plant is given in Fig. 2.4-6 as

$$G_p(s) = G(s) = \frac{1}{s(s+4)}$$

and the desired closed loop response is assumed to be

$$C(s)/R(s) = \frac{2}{(s+1)^2 + 1^2} = \frac{2}{s^2 + 2s + 2}$$

Only one feedback coefficient is present, and $H_{eq}(s)$ is

$$H_{eq}(s) = k_2 s + 1$$

C(s)/R(s) in terms of k_2 and K

$$C(s)/R(s) = \frac{K}{s^2 + (4+Kk_2) s + K}$$

and the resulting values of K and k_2 are

$$K = 2$$

$$k_2 = -1$$

Thus Heq(s) is

$$Heq(s) = -(s-1)$$

and G(s) Heq(s) is

$$G(s)\ Heq(s) = \frac{-2(s-1)}{s(s+4)} \qquad (2.4\text{-}6)$$

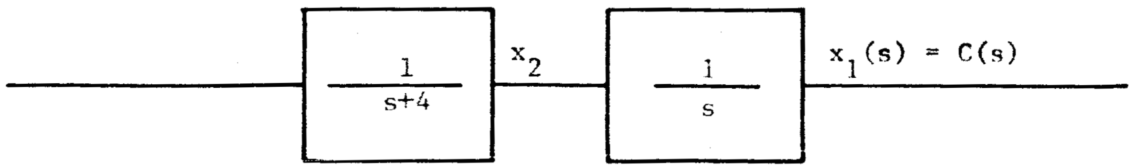

Fig. 2.4-6a. The given plant to be controlled.

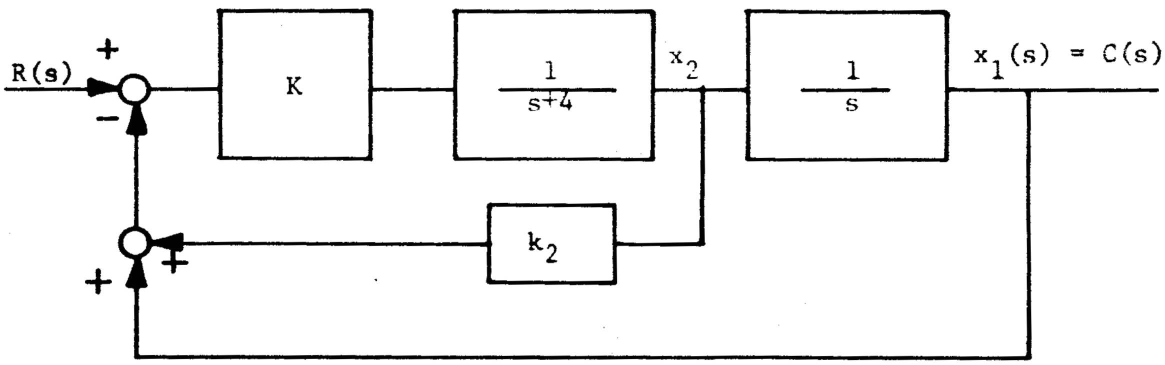

Fig. 2.4-6b. The state variable feedback configuration used to control the plant of Fig. 2.4-6a.

Fig. 2.4-6. A system in which stability is not realized for all gain.

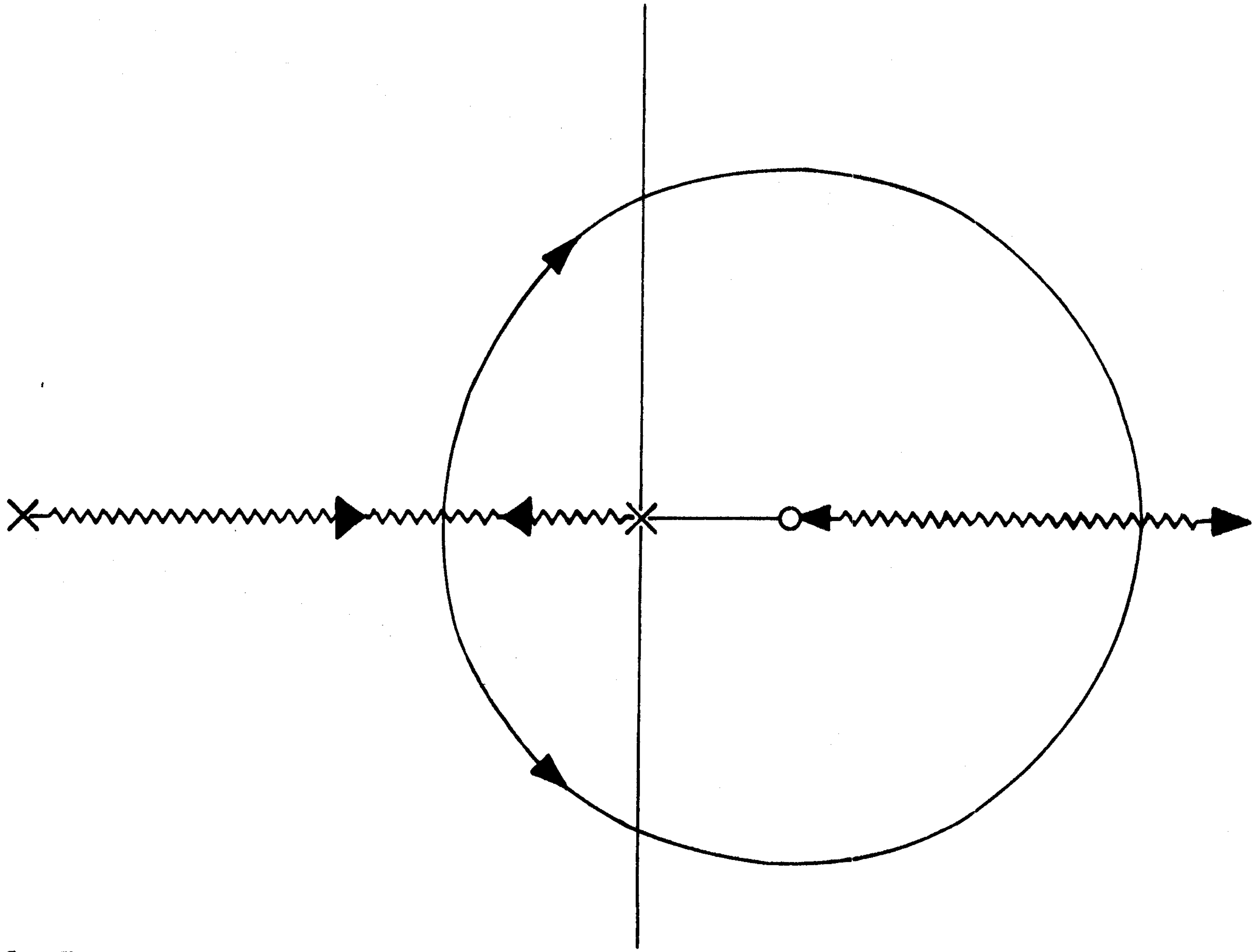

Fig. 2.4-7. The root locus for the closed loop system of Fig. 2.4-6b.

The root locus for Eq. 2.4-6 is plotted in Fig. 2.4-7. For high gain the system is unstable.

The reason for this undesirable result is that the system is being asked to respond with a time constant much longer than the inherent time constant of the system. We return to this example later after the general design procedure has been given and show there that it is not only possible to realize the desired closed loop poles at $s = -1 \pm j1$, and at the same time keep the system stable for all K, but the closed loop poles at $s = -1 \pm j1$ remain at exactly that point for all K.

2.5 All State Variables Not Available-The Simplest Case.

The previous section outlined a design procedure that was called the $H_{eq}(s)$ method. This procedure consisted of five steps, and applied only to the simplest case. The simplest case was defined as that restricted class of problems in which no series compensation need be added in order to realize the desired C(s)/R(s). In such instances the form of the unalterable plant, $C_p(s)$, is compatable with the form of the desired closed loop response.

Not only was the previous section limited to the simplest case, it was also assumed that all of the state variables were available for measurement and control. In this section we continue to treat the simplest case, but now assume that all of the state variables are not available. Two general methods are available to deal with this problem. The first method utilizes minor loop equalization, and the second uses series compensation. It may seem a contradiction to say that series compensation can be used in the simplest case, when we initially assumed that the simplest case required no series compensation. It is true that the simplest case needs no series compensation in order to make the given plant compatible with the required C(s)/R(s), and thus no series compensation need be added prior to the calculation of the k_i's. However, once the k_i's have been calculated under the assumption that all of the state variables are available, series compensation may need to be added if this assumption is to be violated.

The simplest method is the use of minor loop compensation. Instead of feeding back the state variables through constant elements, the k_i's, minor loop compensation makes use of dynamic elements in the feedback paths. Consider the case when the i th state variable is not available, as in Fig. 2.5-1a. The direct approach is to generate the unknown state variable from the previous one, as indicated in Fig. 2.5-1b. This generated state variable must still be fed back through the same constant element, k_i. But now there

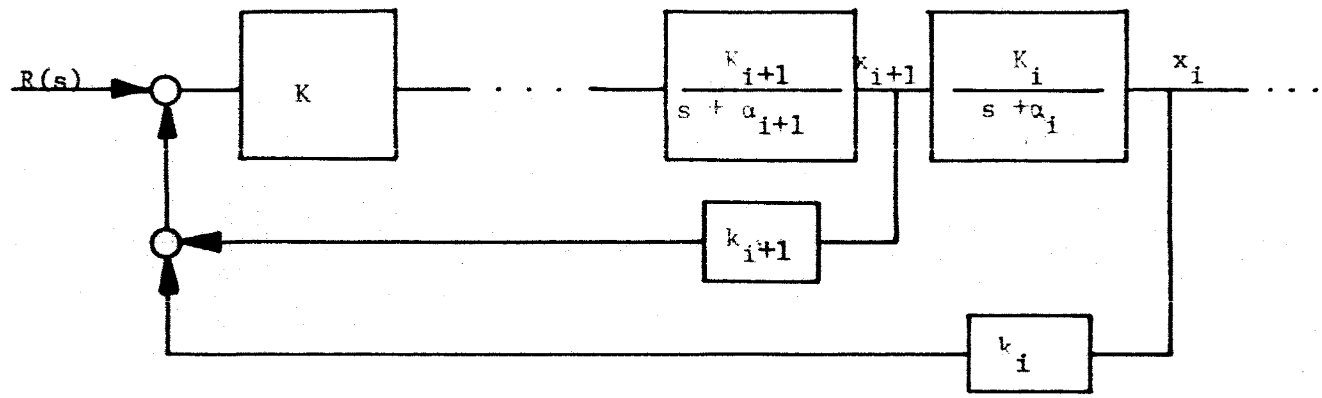

Fig. 2.5-1a. The general feed back structure associated with the state variables x_i and x_{i+1}.

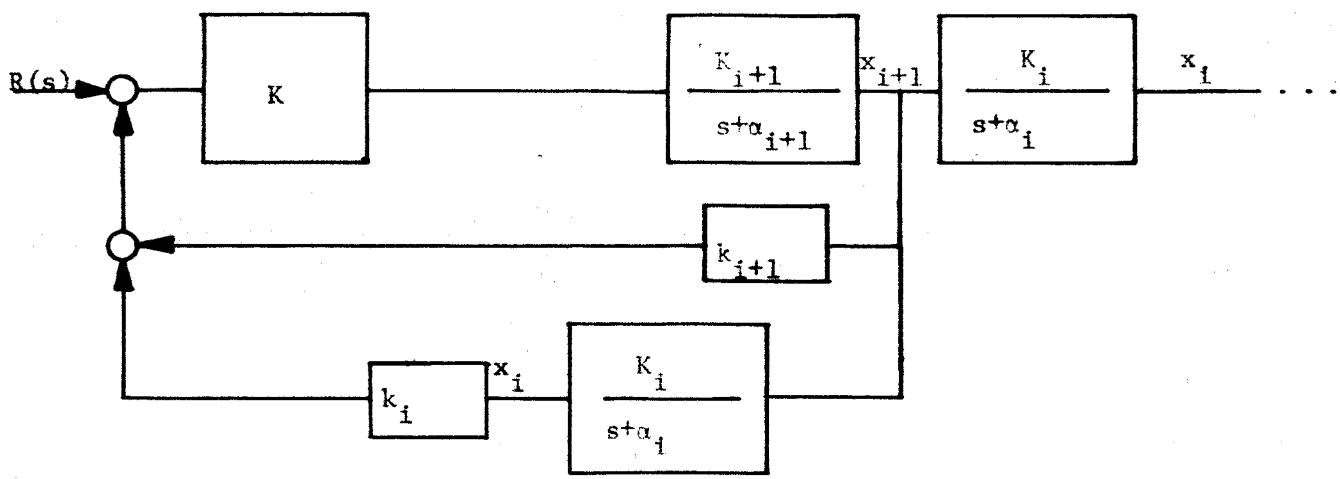

Fig. 2.5-1b. The generation of the state variable x_i.

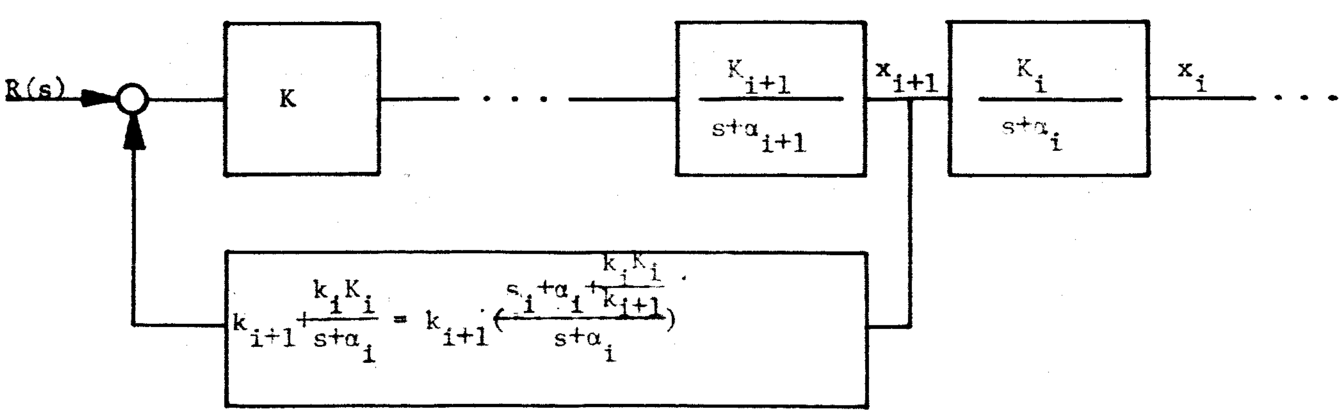

Fig. 2.5-1c. A block diagram equivalent to Fig. 2.5-1b.

Fig. 2.5. Procedure used to design the minor loop compensator when a state variable at x_i is not available.

are two feedback paths in parallel that start at the same state variable and end at the summer. These may be added, with the result as indicated in Fig. 2.5-1c. Note that the resulting compensation is just a simple pole, zero pair, as a typical lead or lag network.

The use of minor loop compensation is illustrated with the same example that has been used in the past. The plant to be controlled is pictured in Fig. 2.2-5a, and the desired closed loop transfer function is given in Eq. 2.2-4. The final design, assuming that all state variables are available is given in Fig. 2.2-2b. Step five of the design procedure states "If all of the state variables are not available, use the known values of the k_i's to determine suitable series or minor loop compensation." Here we are interested specifically in minor loop compensation, and the starting point of our design is Fig. 2.3-2b, which is repeated here for convenience as Fig. 2.5-2a.

Assume initially that the state variable x_2 is not available. Using the procedure illustrated in Fig. 2.5-1, Fig. 2.5-2a is redrawn to indicate the required minor loop compensator. This is almost a trivial step, and the desired closed loop poles are realized as before.

$H_{eq}(s)$ is still

$$H_{eq}(s) = \frac{11\ (s^2+5.285+14.5)}{160} \tag{2.5-1}$$

and the root locus of Fig. 2.4-7 still applies.

Once again let us emphasize that the results of <u>any</u> procedure that is used to realize C(s)/R(s) are identical, as long as all parameters are those assumed at the outset of the problem. Here if the pole at s = -3 is at this assumed location, then $H_{eq}(s)$ is as before when all state variables were available. If the pole at s = -3 is actually at some other point, then the state variable x_2 is only approximately generated, and the result is not the same.

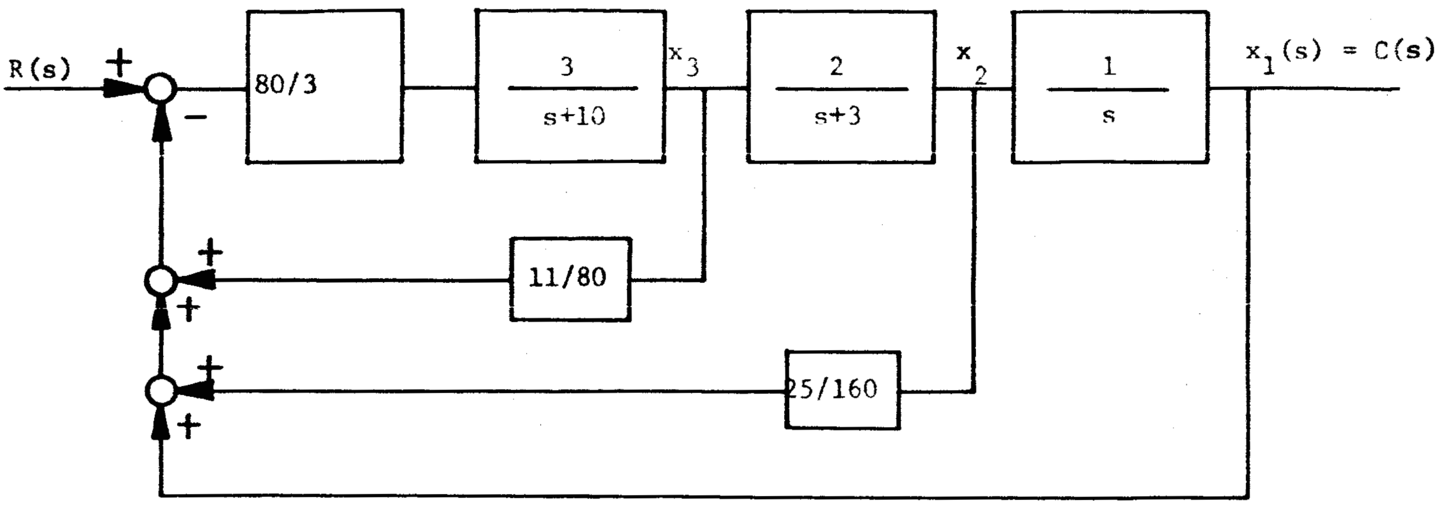

Fig. 2.5-2a. The final design, assuming all state variables available.

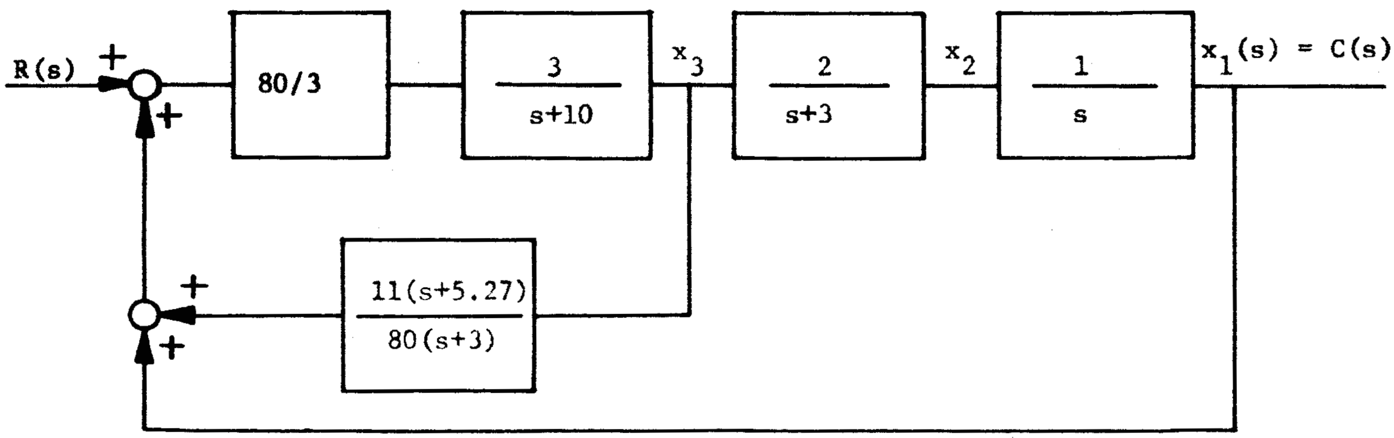

Fig. 2.5-2b. The final design, assuming x_2 is not available.

Fig. 2.5-2. An example illustrating a minor loop compensator when the state variable at x_2 is not available.

If the state variable x_3 is not available, it may be generated by feedback around the external gain element, K. It is for this reason that we have consistently separated the external gain from that which is inherent in the plant that is being controlled. The resulting minor loop compensated system is illustrated in Fig. 2.5-3a. Here the feedback element around K may be combined with K to determine an equivalent series equalizer. This is done in Fig. 2.5-3b. The $H_{eq}(s)$ associated with Fig. 2.5-3b has only one zero, while that associated with Fig. 2.5-3b has two zeroes, and is, in fact, given by Eq. 2.5-1. The author's preference is clearly the feedback case of Fig. 2.5-3a, and hopefully the reader concurrs. However, the series compensation of Fig. 2.5-3b does serve to introduce the idea of series compensation. It is possible to realize the series compensator of Fig. 2.5-3b as in Fig. 2.5-3c, where once again feedback is employed. Here, however, no dynamic elements are included in the feedback path.

Before considering the use of series compensation, let us conclude the discussion of minor loop compensation by considering the case when both x_2 and x_3 are not available. Again the procedure is very simple. The first step is accomplished in Fig. 2.5-2b, and all that remains to be done is to shift this compensator to the left by one block. This is done in Fig. 2.5-4. Once again $H_{eq}(s)$ is still specified by Eq. 2.5-1, and the root locus of Fig. 2.4-1 is still applicable. This time the compensator in the feedback path in Fig. 2.5-4 has a second order denominator, but this is not difficult to realize. Had the system been nth order, and had only the output variable been available, then the denominator of the feedback compensator transfer function would have been nth order. This might be quite difficult to realize, but no more so than the insertion of the series compensator that would be

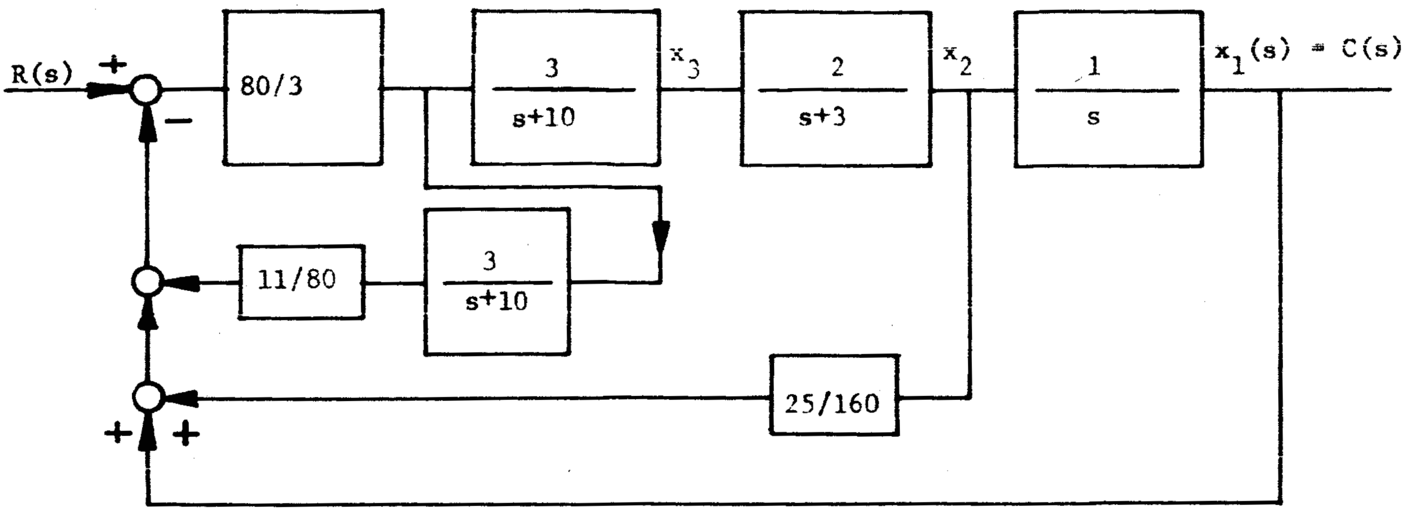

Fig. 2.5-3a. A minor loop design when x_3 is not available.

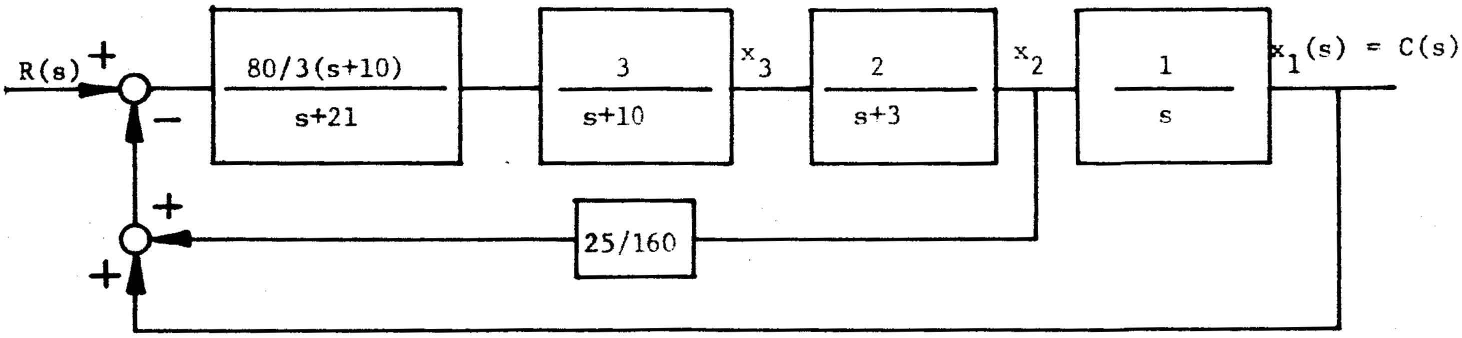

Fig. 2.5-3b. A series equalizer design, determined from Fig. 2.5-3a.

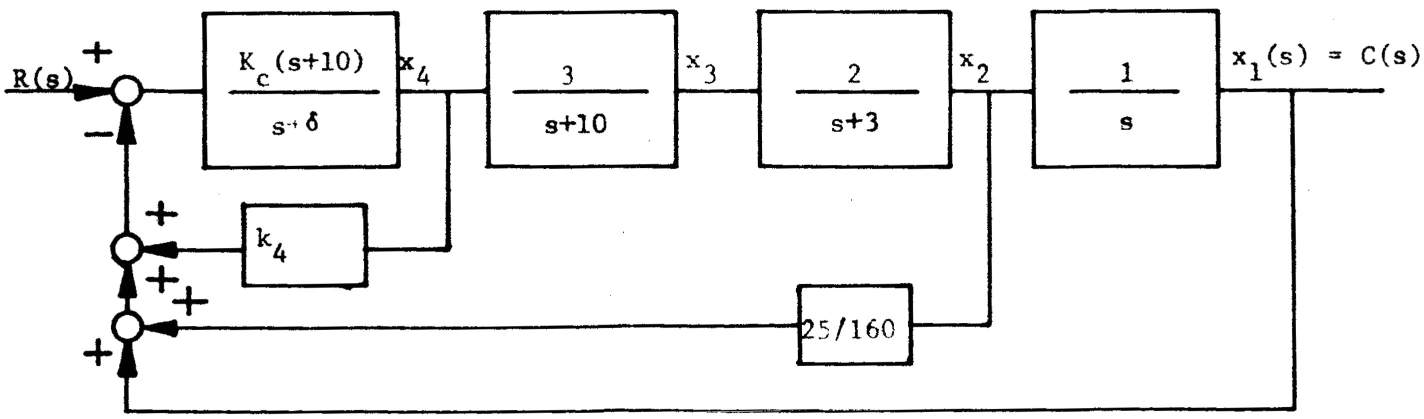

Fig. 2.5-3c. A method equivalent to a) and b) above, using an augmented state variable, x_4.

Fig. 2.5. Compensation methods when x_3 is not available.

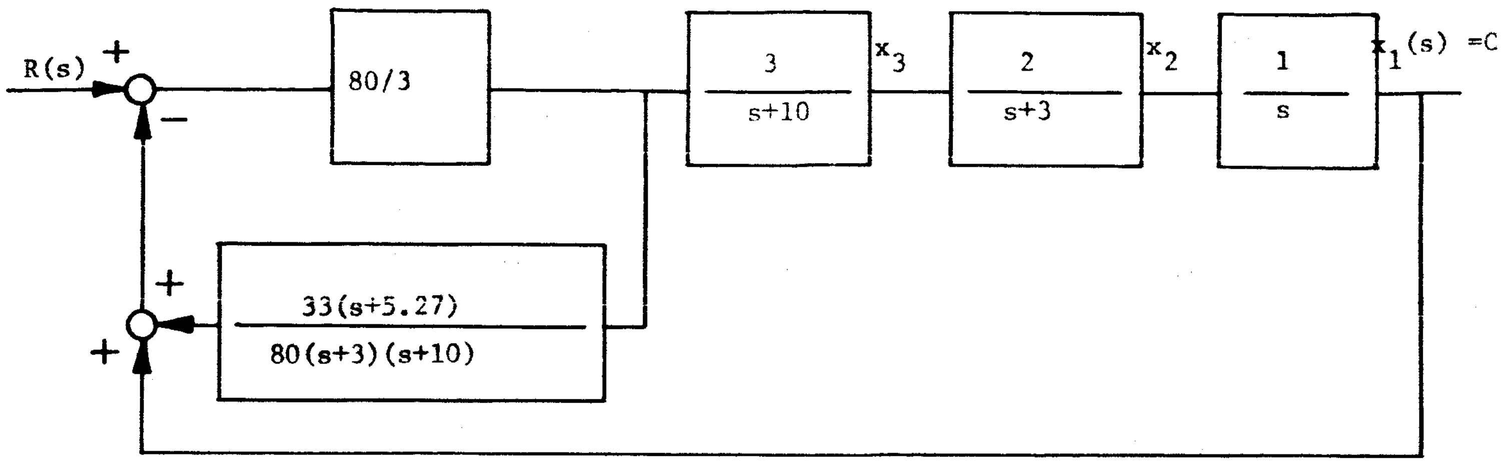

Fig. 2.5-4. The minor loop equalization when both x_2 and x_3 are not available.

required by the use of the Guillemin-Truxal technique.

This example points out an important feature of feedback compensation. Here feedback is around the gain element, and since $H_{eq}(s)$ is still specified by Eq. 2.5-1, and since the root locus of Fig. 2.4-1 still applies, it is clear that this system is still relatively insensitive to gain. Feedback always decreases the sensitivity of the output to variations in those elements that have feedback around them. This is the fundamental reason for the use of feedback.

This last example serves as another introduction to series compensation. If the feedback compensator is lumped in with the gain element, the Guillemin-Truxal realization of Fig. 2.3-2a results. The aim of the remainder of this section is to discuss means by which such series compensation can be realized by feeding back the state variables associated with the compensator itself. The methods discussed here will then serve as an introduction to the general case, which is treated in the next section.

A study of the methods of realizing a series compensation network by feeding back the variables associated with the compensator itself is complicated by two facts. Frrst the realization is not unique, and second, the equations involved are not linear algebraic equations. A systematic method of approach is necessary to avoid being bogged down in algebra. The three drawings of Fig. 2.5-3 serve to illustrate the problem. A series compensator equivalent to the feedback compensation system of Fig. 2.4-3a is illustrated in Fig. 2.5-3 . We wish to realize this transfer function by the scheme indicated in Fig. 2.5-3c. Why? Because if k_4 is anything other than zero, then a reduction of the system to the $H_{eq}(s)$ form results in an $H_{eq}(s)$ that has 3 zeroes. By a wise choice of k_4, it may be possible to insure that these zeroes remain in the left half s plane, so that the root locus of the resulting system never crosses into the right half plane for any value of gain.

At the same time that k_4 is being chosen to give the overall system desired properties, one must insure that the choice of k_4 does not place unreasonable demands on either K_c or δ, where these quantities are defined in Fig. 2.5-3c. Let us examine the requirements on K_c, δ, and k_4, as specified by the desired series compensator. The constants K_c, δ, and k_4 must be chosen to insure that

$$\frac{\dfrac{K_c\ (s+10)}{s+\delta}}{1+\dfrac{K_c\ k_4\ (s+10)}{s+\delta}} = \frac{80/3\ (s+10)}{(s+21)} \qquad (2.5\text{-}2)$$

The two are equal if the following are satisfied.

$$\frac{K_c}{1+K_c k_4} = 80/3 \qquad (2.5\text{-}3)$$

and

$$\frac{\delta + 10\ K_c\ k_4}{1+K_c\ k_4} = 21 \qquad (2.5\text{-}4)$$

Only two equations must be satisfied, and three constants are available. Thus one of these unspecified constants is arbitrary, and our first goal is to pick a resonable value for one constant. If Eq. 2.4-3 is solved for K_c, the result is

$$K_c = \frac{80/3}{1-\dfrac{80\ k_4}{3}}$$

This places an upper bound on k_4 of 3/80, if K_c is to be kept a positive number.

It is possible to place a second bound on k_4 by examining the expression for $H_{eq}(s)$. $H_{eq}(s)$ may be written in terms of k_4 directly from Fig. 2.4-3c as

$$H_{eq}(s) = k_4\ \frac{s(s+10)(s+3)}{6} + \frac{255}{160} + 1$$

$$= \frac{k_4\, s^3 + 13\, k_4\, s^2 + (30\, k_4 + .938)\, s + 6}{6}$$

To insure that the zeroes of $H_{eq}(s)$ remain in the left half plane, it is only necessary that the numerator of $H_{eq}(s)$ be a Hurwitz polynomial. Application of either the Routh or Hurwitz criteria indicates that the only limitation on k_4 is that

$$k_4 \geq 0$$

Thus k_4 is bounded on both sides, as

$$0 \leq k_4 < 3/80$$

Because of this range of possible values of k_4, an infinite number of solutions to this problem exist. Let us rather arbitrarily choose k_4 to be 2/80. Then the resulting K and s are

$$K_c = 80$$

$$s = 43$$

The final design is pictured in Fig. 2.5-5. For this system $H_{eq}(s)$ is

$$H_{eq}(s) = \frac{s^3 + 13s^2 + 67.5\, s + 240}{240}$$

$$= \frac{(s+8.4)\ [(s+2.3)^2 + (3.4)^2]}{240}$$

The root locus diagram corresponding to Fig. 2.5-5 is given in Fig. 2.5-6. This is the first time we have fed back a state variable other than those inherently associated with the system. The state variable x_4 is always available, since it is associated with the compensator. It is for this reason that the author maintains that the case never exists in which the Guillemin-Truxal series compensation should be used. In the series compensator there are always available state variables which can be fed back.

On the basis of this single example, let us postulate a design procedure to be used in the design of series compensation networks. Here it is assumed

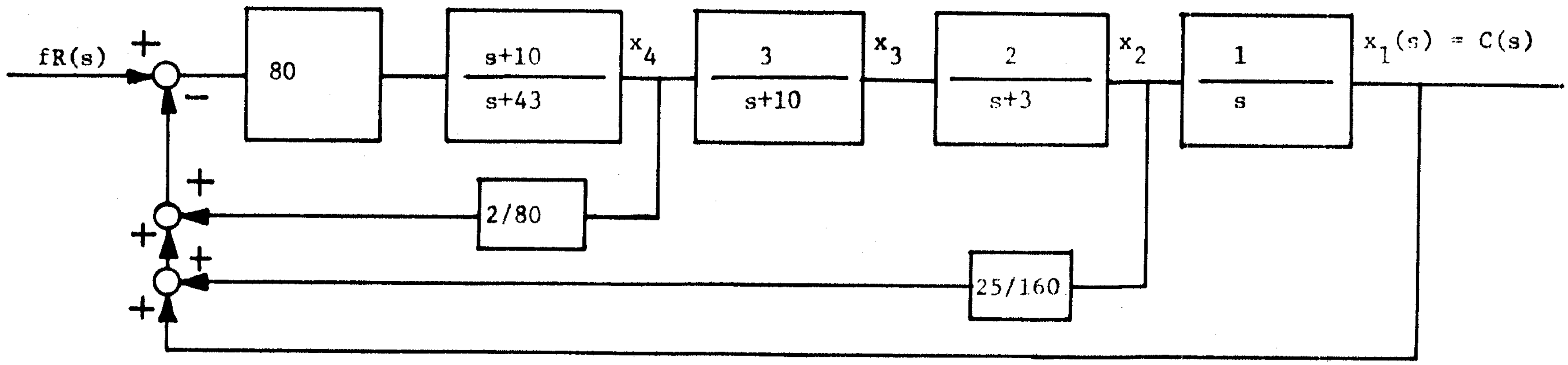

Fig. 2.5-5. A series compensation method used when x_3 is not available.

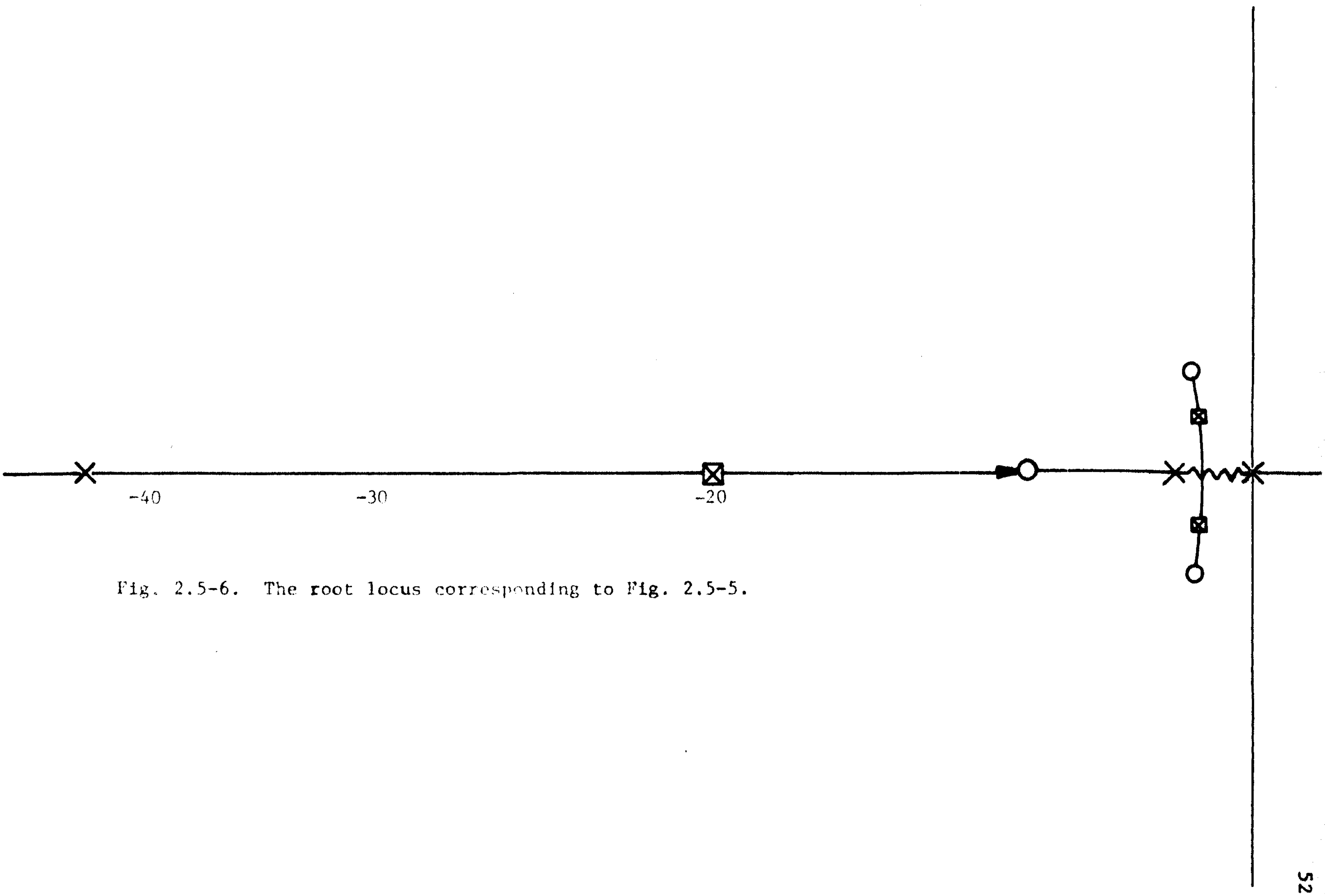

Fig. 2.5-6. The root locus corresponding to Fig. 2.5-5.

that one has already determined what the desired compensation network must be.

1. Establish a feedback configuration to realize the desired compensation network. This feedback configuration will contain an unknown gain, unknown pole locations, and unknown feedback coefficients.
2. Initially let all of the feedback coefficients be 0 except k_{n+1}.
3. Obtain limits on the k_{n+1} by examining the expression for gain, K_c, as a function of k_{n+1} and by examining requirements of $H_{eq}(s)$ to insure that all of the zeroes lie in the left half plane.
4. Choose k_{n+1}, which establishes K_c and the remaining unknown poles in the compensation network.
5. Repeat the procedure in order to determine k_{n+2}, and a new K_c.

In the example just solved, only the $k_{n+1} = k_4$ coefficient existed, and it was not necessary to repeat the procedure. If both x_2 and x_3 are not available, then it is necessary to repeat the procedure.

Let us now consider the case in which both x_2 and x_3 are not available. The required feedback compensation network is given in Fig. 2.5-5. If the inner loop is reduced, the result is the Guillemin-Truxal realization of Fig. 2.3-2a. Thus the series compensation network that is to be realized is taken from this latter figure to be

$$\frac{x_4(s)}{E(s)} = \frac{80/3\ (s+3)\ (s+10)}{s^2 + 245 + 88}$$

This is shown in Fig. 2.5-7a. Fig. 2.5-7b is the feedback configuration involving only the coefficient k_4 that might be used to realize the series compensator of Fig. 2.5-7a. This feedback configuration is shown in open loop form in Fig. 2.5-7c, and of course the transfer functions of Figs. 2.5-7a and 2.5-7b must be equal. These are equal if the following equations are

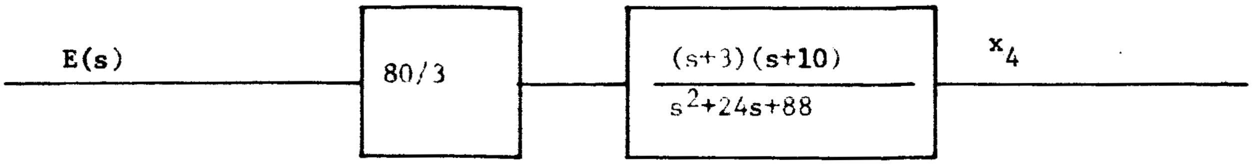

Fig. 2.5-7a. Series compensation network to be realized.

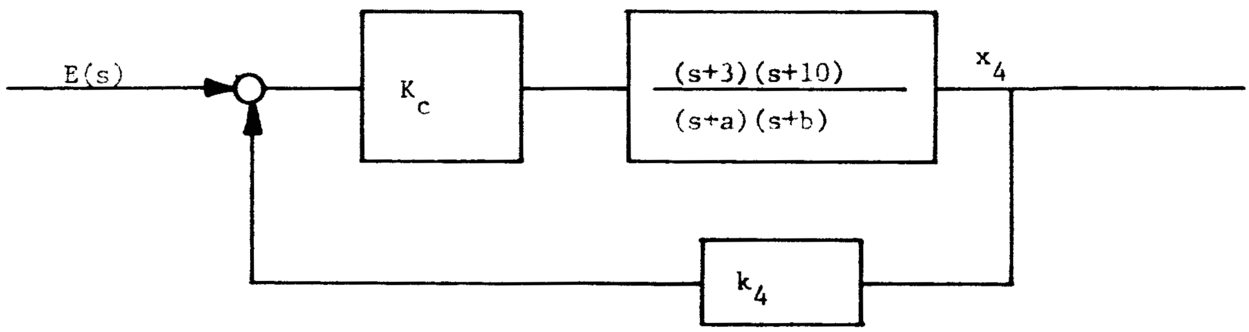

Fig. 2.5-7b. Step 1 of the series compensation design procedure.

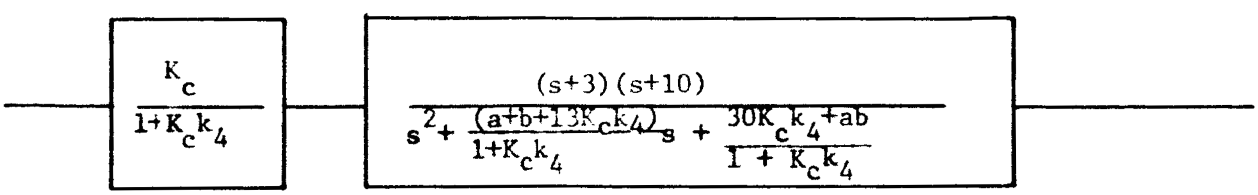

Fig. 2.5-7c. The open loop equivalent of Fig. 2.5-7b.

Fig. 2.5-7. Alternate representations of the required series compensation network.

satisfied.

$$\frac{K_c}{1 + K_c\ k_4} = 80/3$$

$$\frac{a + b + 13K_c\ k_4}{1 + K_c\ k_4} = 24 \qquad (2.5\text{-}5)$$

$$\frac{30\ K_c\ k_4 + ab}{1 + K_c\ k_4} = 88$$

Step 1 of the series compensation design procedure is satisfied by Fig. 2.5-7. Step two may be partially satisfied by solving the first equation above for the compensation gain, K_c, in terms of k_4. The result is

$$K_c = \frac{80/3}{1 - k_4\ 80/3}$$

and, as before, k_4 may not exceed 3/80 if the compensator gain is to be positive. The other bound on k_4 must be established from $H_{eq}(s)$. The overall system is pictured in Fig. 2.5-8, and for this system $H_{eq}(s)$ is

$$H_{eq}(s) = k_4\ \frac{(s+10)\ (s+3)\ (s)}{6} + 1$$

$$= \frac{k_4\ s^3 + 13\ k_4\ s^2 + 30\ k_4\ s + 6}{6}$$

Application of the Routh or Hurwitz criteria indicates the numerator of $H_{eq}(s)$ has zeroes in the left half plane for $k_4 > 1/65$. The bounds on k_4 have now been established as

$$1/65 < k_4 < 3/80$$

A k_4 in this range must be selected, so let us again pick $k_4 = 2/80$. For this value of k_4, $K_c = 80$, and even though Eqs. 2.5-5 are highly nonlinear, their solution for a and b is quite easy once k_4 and K_c are known. The pole locations at a and b turn out to be

$$a = 5 \text{ or } 41$$

$$b = 41 \text{ or } 5$$

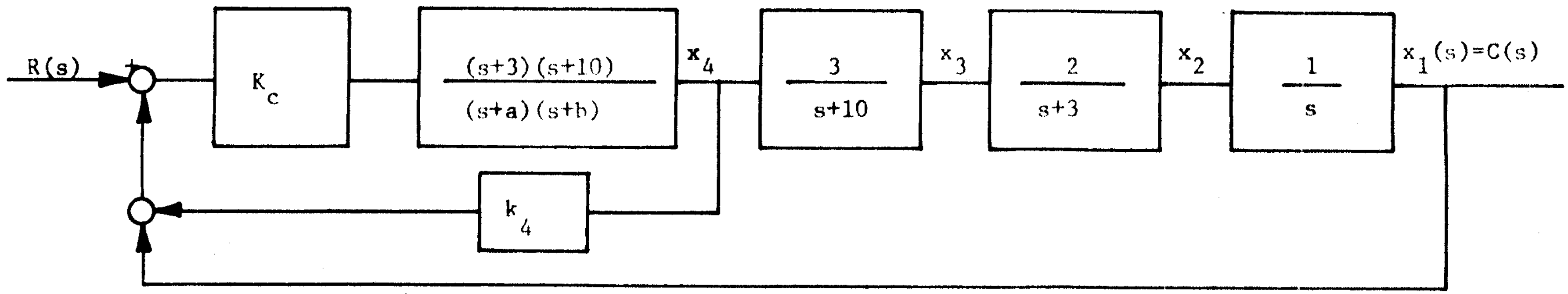

Fig. 2.5-8. The overall system, with k_4 as yet undefined.

Hence the compensator of Fig. 2.5-7b is completely specified. This compensator is shown in Fig. 2.5-9a.

The compensator of Fig. 2.5-9a has an internal state variable, since it is described by a second order polynomial in s, or equivalently, by a second order differential equation. All that remains to be done is to determine a suitable feedback coefficient associated with this state variable. That is, the series compensation procedure must be repeated again, in order to determine a new compensator gain and a feedback coefficient, k_5. The block diagram of Fig. 2.5-9b illustrates the final form of the feedback realization of the compensator network. Fig. 2.5-9c illustrates that portion of the Fig. 2.5-9c which is yet unspecified. From a comparison of Fig. 2.5-9a and 2.5-9b results the equality that is indicated in Fig. 2.5-9c. This equivalence is expressed in the equation

$$\frac{\dfrac{K_c^1}{1 + K_c^1 k_5}(s+10)}{s + \dfrac{\alpha + 10\, K_c^1 k_5}{1 + K_c^1 k_5}} = \frac{80\ (s+10)}{s + 5}$$

Here the notation K_c^1 is used to indicate the new gain that is needed. The above equation yields the following

$$\frac{K_c^1}{1 + K_c^1\, k_5} = 80 \tag{2.5-6}$$

$$\frac{\alpha + 10\, K_c^1\, k_5}{1 + K_c^1\, k_5} = 5 \tag{2.5-7}$$

Eq. 2.5-6 may be solved for K_c^1, as

$$K_c^1 = \frac{80}{1 - 80\ k_5} \tag{2.5-8}$$

To insure that K_c^1 is positive,

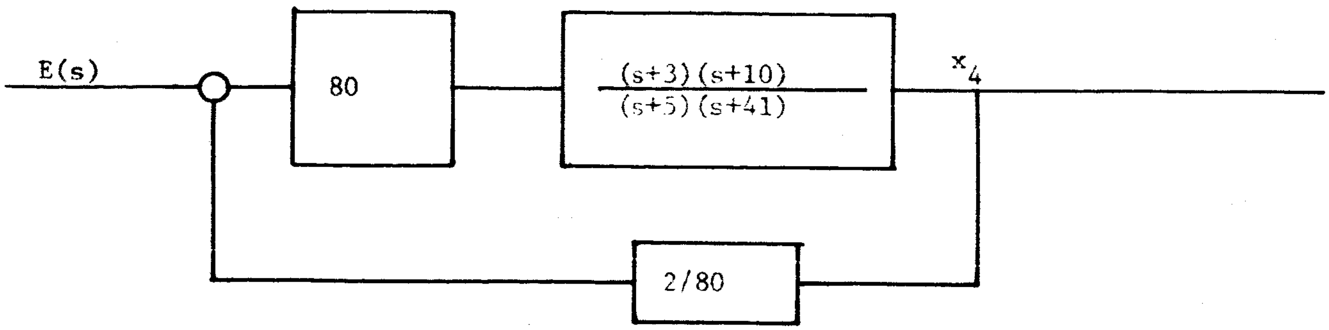

Fig. 2.5-9a. The first iteration on a feedback realization for the desired compensation network.

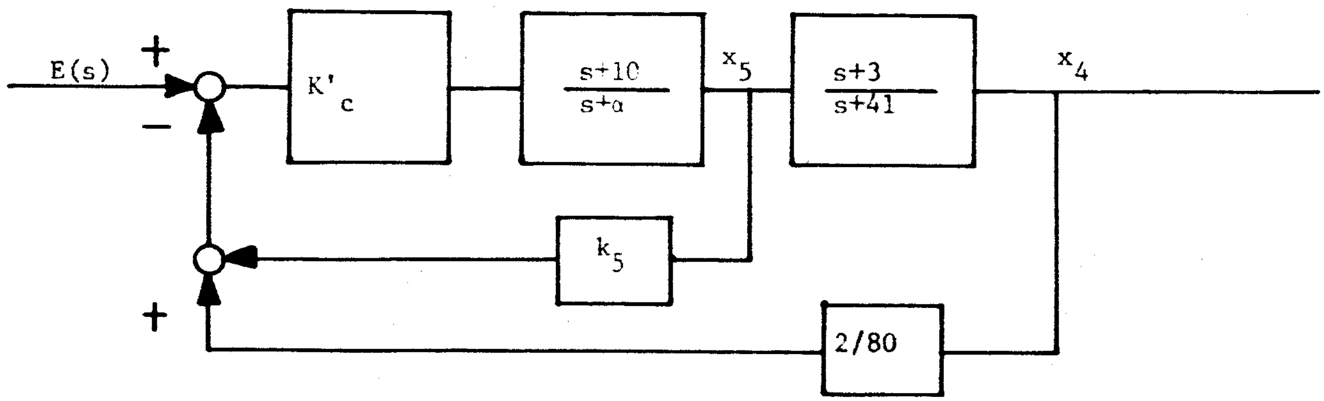

Fig. 2.5-9b. The final form of the feedback compensator.

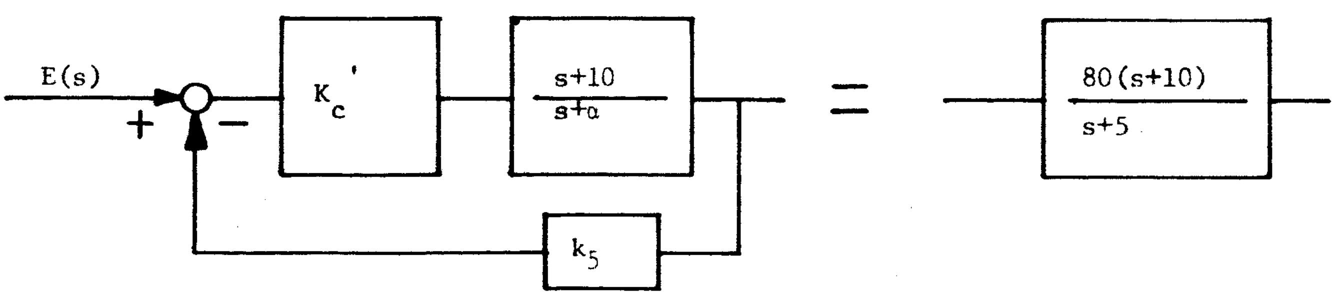

Fig. 2.5-9c. The problem yet to be solved so that the feedback compensator is completely specified.

Fig. 2.5-9. Feedback realization of the required series compensator.

$k_5 < 1/80$

The determination of the other bound on k_5 depends on $H_{eq}(s)$. The block diagram of Fig. 2.5-10 illustrates the overall system from which $H_{eq}(s)$ must be calculated. The resulting $H_{eq}(s)$ is quite complicated, but the zeroes of $H_{eq}(s)$ remain in the left half s plane even if k_5 goes negative. Let us take 0 as the lower bound, so that k_5 is bounded by

$0 \leq k_5 < 1/80$

As an arbitrary choice, let $k_5 = 1/160$. Then from Eq. 2.5-8, K_c' is 160, and from Eq. 2.5-7, α is found to be zero. The final system is that of Fig. 2.5-10, a$_1$ th $K_c^1 = 160$, $k_5 = 1/160$ and $\alpha = 0$. $H_{eq}(s)$ is

$$H_{eq} = \frac{s^3 + 20.6s^2 + 106\ s + 192}{192}$$

$$= \frac{(s+14)\ [(s+3.3)^2 + 1.7^2]}{192}$$

G(s) is now

$$G(s) = \frac{960}{s^2\ (s+41)}$$

and the root locus for G(s) $H_{eq}(s)$ is plotted in Fig. 2.5-11. The sensitivity and stability benefits realized when x_2 and x_3 were both present have been returned.

Two features of this example problem merit more discussion. In Fig. 2.5-9b the reader may have wondered what prompted the author to realize the transfer function 80 (s+10)/(s+5) with the feedback configuration involving k_5. Why not split up the gain, and enclose (s + 3)/(s+41) with k_5, or one of the two other possible pole zero combinations. In realizing lead lag compensators, that is two poles and two zeroes, the largest zero and the smallest pole are always put into the first block. This insures that $H_{eq}(s)$ will have zeroes in the left half s plane for the largest range of the last feedback coefficient.

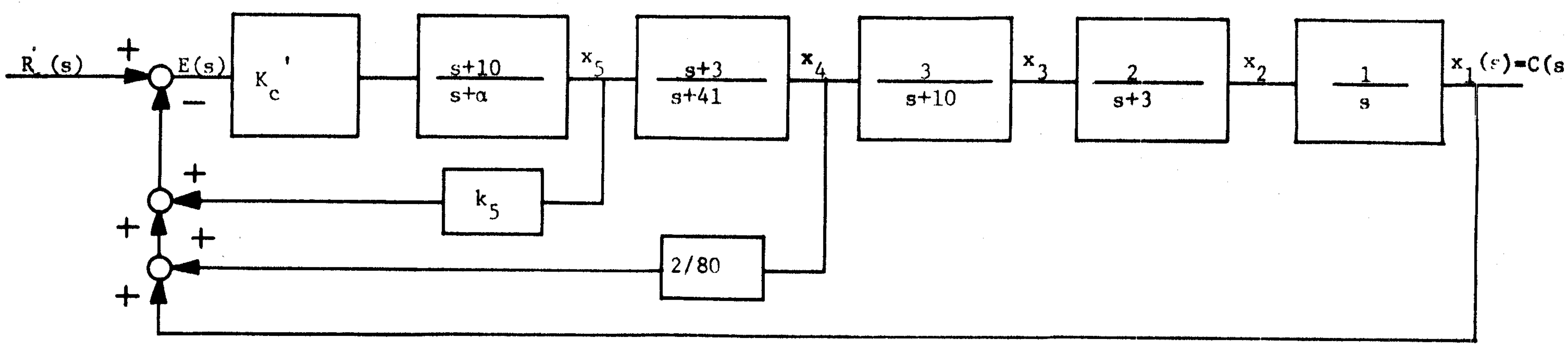

Fig. 2.5-10. The overall system configuration from which $H_{eq}(s)$ must be calculated. The final value of K_c^1 and k_5 are

$$K_c' = 160$$
$$k_5 = 1/160$$

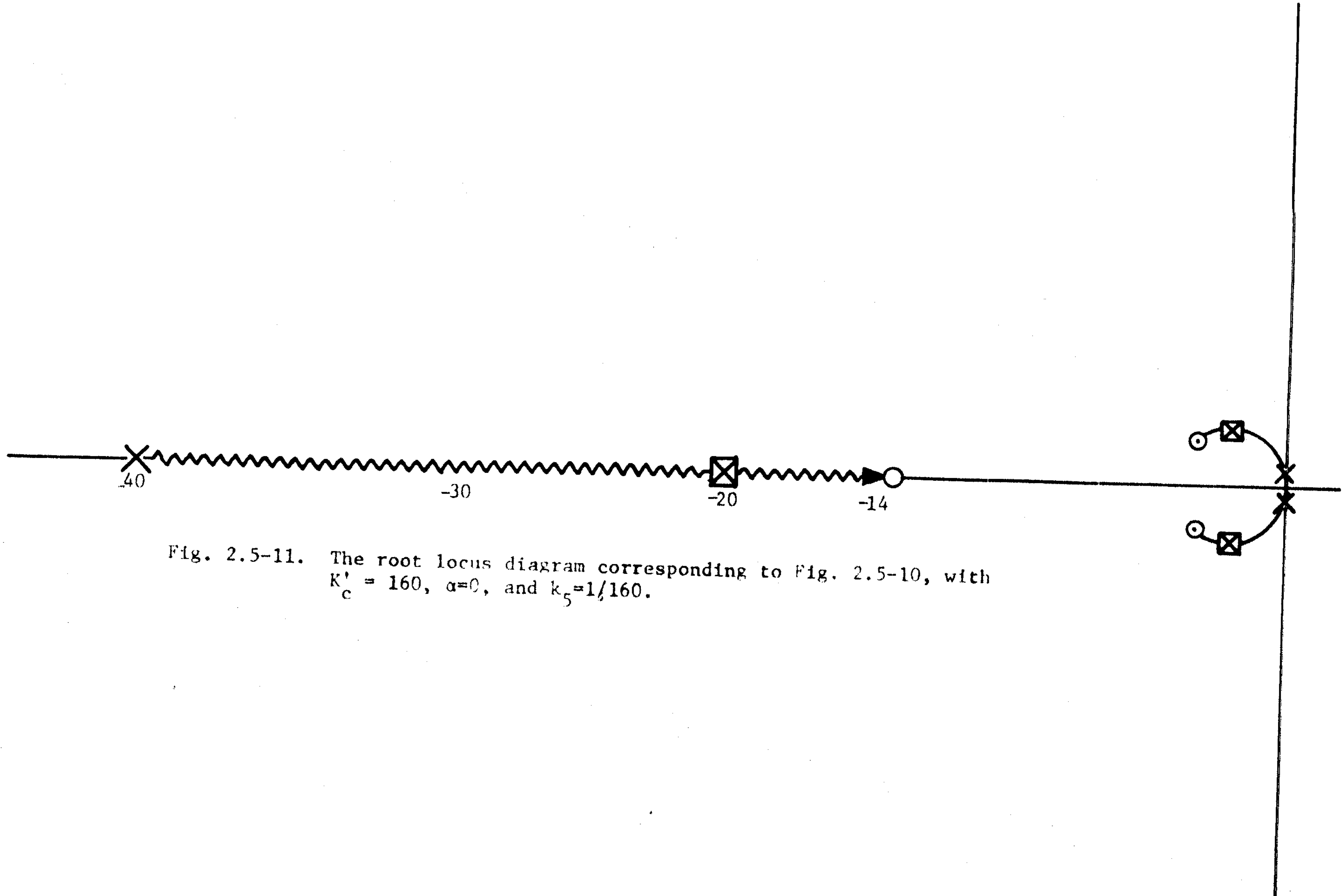

Fig. 2.5-11. The root locus diagram corresponding to Fig. 2.5-10, with $K_c' = 160$, $\alpha=0$, and $k_5=1/160$.

The second fact that needs discussion is the freedom of choice of k_4 and k_5 to a range of values. In the example the author picked k_4 and k_5 in a rather arbitrary manner, simply because no criteria have yet been established for their choice. It is felt that k_4 and k_5 should have been picked with a view toward a minimization of the effects of the most likely parameter variation. Work in this area is continuing.

This concludes the discussion of the simplest case. In the general case, when all of the state variables are not available, a combination of minor loop compensation and series compensation with feedback of the compensator state variables is used to realize the desired C(s)/R(s).

2.6 The General Case

The general case is concerned with systems in which the given $G_p(s)$ is not adequate to realize a desired $C(s)/R(s)$. The most common case in which this situation arises is the case in which zeroes are required in the closed loop transfer function. Usually $G_p(s)$ does not contain zeroes, or if they are present, they are not located in the desired place. Then series compensation must be added to realize the required zeroes. This increases the order of the open loop transfer function $G_c(s)G_p(s)$, since $G_c(s)$ is now no longer one. This situation is very similar to that encountered in the previous section when all of the state variables were not available, and series compensation was used with feedback to realize the desired closed loop response.

Because of the detail that was presented in the last two sections in the discussion of the special case, the discussion of the general case is made considerably easier. In the discussion that follows it is assumed that $G_p(s)$ has p poles and z zeroes, and that none of these zeroes are located in the positions required by the desired $C(s)/R(s)$. If unwanted zeroes are to be eliminated and not replaced by other zeroes, this may be accomplished in the following way: z of the poles of $C(s)/R(s)$ can be located at the zero positions, so that the resulting $C(s)/R(s)$ would have no zeroes and $(p - z)$ poles. If the desired denominator of $C(s)/R(s)$ is to be of higher order than $(p - z)$, then series compensation of the form $1/(s+\delta_1)$ must be added for each additional pole that is needed.

Often it is desired to add zeroes, either because of a desired frequency response requirement or because a high value of velocity error coefficient is required. In the presentation thus far it has always been assumed that k_1 was chosen to insure an infinite position error coefficient. In the examples, we have always considered systems with one integrator, so that the value of k_1 is always 1. An infinite velocity error coefficient may also be realized quite easily.. If the poles of the closed loop transfer function are designated by λ_i, and the zeroes by γ_i, Truxal (1955) has shown that an infinite velocity error coefficient is realized if

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots \frac{1}{\lambda_m} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots \frac{1}{\gamma_m} \tag{2.6-1}$$

Hence zeroes are often added in such a position so as to satisfy equation 2.6-1. Zeroes cannot be added without adding a pole at the same time. The basic compensation unit in the case that zeroes are to be added or are to replace unwanted zeroes is $(s+\gamma_i)/(s+\delta_i)$, where γ_i is known. This is just the usual lead or lag network, and is of the form considered in the previous section.

The previous paragraphs can be summarized quite briefly. Zeroes in $C(s)/R(s)$ can be eliminated by placing poles under them, and they can be added by using a series compensator of the form $(s+\gamma_i)/(s+\delta_i)$. The zero of this compensator appears as a zero in the closed loop, and thus its location is specified in advance. Additional poles are added by adding a compensator of the form $1/(s+\delta_i)$. The position of the n poles of $C(s)/R(s)$ may then be controlled by the adjustment of the k_i's, K, and the poles of the series compensators. The design procedure is thus the following.

1. Choose a desired $C(s)/R(s)$.
2. On the basis of the unalterable $G_p(s)$ and the required $C(s)/R(s)$, specify the form of the necessary $G_c(s)$. $G_c(s)$ need contain only compensation of the form $1/(s+\delta_i)$ or $(s+\gamma_i)/(s+\delta_i)$. Only the poles p of the compensation elements are unknown.
3. Determine the values of K, the k_i's, and the poles of the compensation networks necessary to realize the desired closed loop response. The k_i's determine a tentative $H_{eq}(s)$ with (p-1) zeroes, designated as $H^t_{eq}(s)$.

4. Synthesize the series compensation network by feeding back its own state variables, according to the procedure specified in the previous section. This specifies the desired $H_{eq}(s)$ with (n - 1) zeroes.
5. Account for any unavailable state variables with the use of minor loop compensation.

This general design procedure reduces to that of the special case in the situation when $G_p(s)$ and $C(s)/R(s)$ are compatible. In that instance, step two is automatically satisfied, and in step 3 the desired closed response is used to find the values of K and the k_i's, as there are no unknown values of the compensator poles, since there is no compensator. In case some of the state variables are unavailable, they may be realized either by the synthesis of a series network or by minor loop compensation.

The reader may now appreciate why so much time was spent in the previous section on the series and minor loop compensation methods. Both are used in the general case, in order to avoid the necessity of realizing an unwieldly series network. For example, if two zeroes need to be added and two state variables are unavailable, then a second order series compensator and a second order minor loop compensator would be used. The series compensator would be realized with feedback, however.

A more rigorous approach to this problem using a more modern matrix approach assures us that the design procedure outlined above always works. The proof amounts to showing that there are as many equations as there are unknowns, and hence a solution exists. But while the solution always exists, it may not always be a solution that appeals to the designer. That is, zeroes may appear in the right half plane, and the sign of $H_{eq}(s)$ may even be negative, so that it is necessary to draw a 0 degree root locus in order to investigate the results. These are handicaps that are rather easily overcome, but the ability to overcome these handicaps relies upon a good understanding of root locus techniques. In this section we apply the general design procedure as it is given, and accept the results as they are. After all, the results are indeed excellent, if one can choose any $C(s)/R(s)$. There is not even a problem of realizing zeroes in the right half plane, since they are only equivalent zeroes that do not exist in the physical system.

As a first example, consider the plant to be controlled as that given in Fig. 2.6-1. Here $G_p(s)$ is given as

$$G_p(s) = \frac{4(s+2)}{s(s+4)(s+8)}$$

Fig. 2.6-1 contains more information than simply the transfer function of Eq. 2.6-1, as the figure indicates the pole with which the zero is associated, assuming as before that we have used actual physical state variables. The object of this design is to simply remove the zero at $s = -2$, and at the same time insure that the closed loop response is specified by

$$C(s)/R(s) = \frac{450}{[(s+3)^2 + 3^2](s+25)} \tag{2.6-2}$$

In order that this might be done it is necessary to add a series compensation element of the form $1/(s+\delta_1)$, and actually design to a $C(s)/R(s)$ given as

$$C(s)/R(s) = \frac{450(s+2)}{(s+2)[(s+3)^2 + 3^2](s+25)}$$

$$= \frac{450(s+2)}{s^4 + 33s^3 + 230s^2 + 786s + 900} \tag{2.6-3}$$

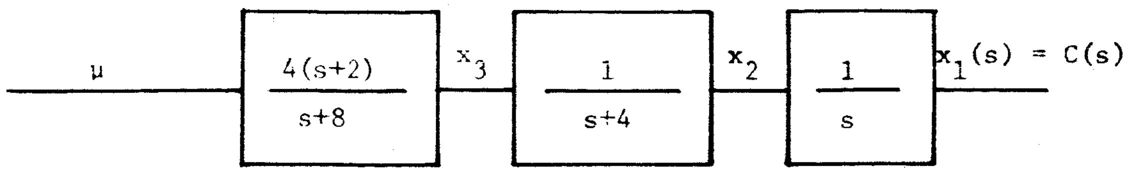

Fig. 2.6-1a. The given plant to be controlled, such that $C(s)/R(s) = \frac{450}{[(s+3)^2+3^2]\,[s+25]}$

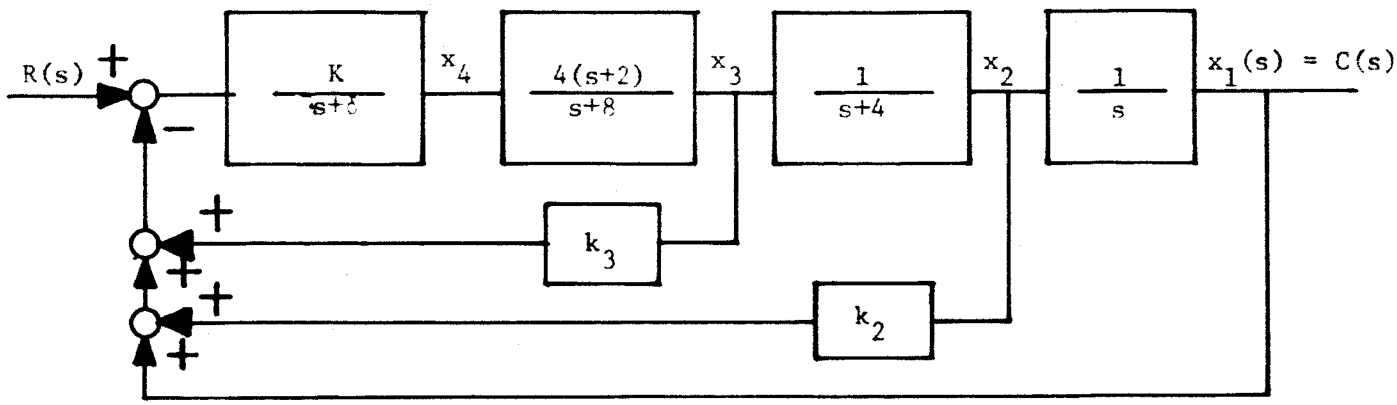

Fig. 2.6-1b. Means by which the plant of Fig. 2.6-1a may be controlled to realize the desired C(s)/R(s).

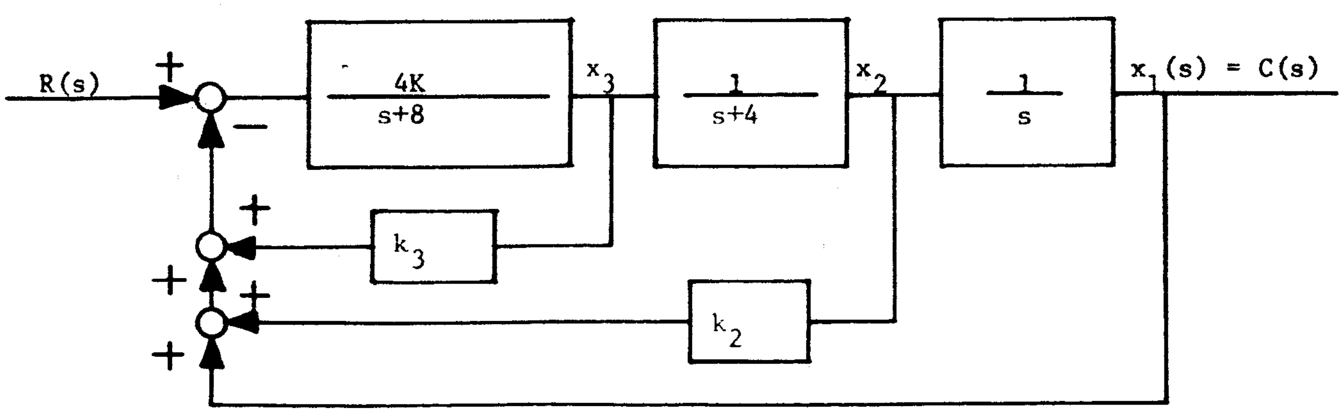

Fig. 2.6-1c. An obvious simplification of Fig. 2.6-1b, with δ=2.

Fig. 2.6-1. The first example of Section 2.6-1.

Figure 2.6-1b illustrates the configuration of the system with the series compensation element in place. It is apparent that this problem is somewhat artificial--a rather obvious choice of the series compensator would be one with a pole at $s = -2$, which would simply cancel the zero. The system would then be effectively that of Fig. 2.6-1c, which could then be treated as the special case. Let us proceed with this problem for purposes of illustration, and later we will examine the more difficult case when the zero at $s = -2$ is associated with the pole at $s = -4$.

Step 1 of the procedure has already been accomplished with the specification of $C(s)/R(s)$, and step two is also complete with the drawing of Fig. 2.6-1b. It is now necessary to solve for K, k_2, k_3, and δ.

For the system of Fig. 2.6-1b $H_{eq}(s)$ is

$$H_{eq}(s) = k_3 s^2 + (4k_3+k_2)\, s + 1$$

and $G(s)$ is

$$G(s) = \frac{4K(s+2)}{s(s+4)(s+8)(s+\delta)}$$

so that the resulting $C(s)/R(s)$ is

$$c(s)/R(s) = \frac{4K(s+2)}{s^4 + As^3 + Bs^2 + Cs + D}$$

where

$$\begin{aligned} A &= 12 + \delta + 450k_3 \\ B &= 32 + 12\delta + 4K(4k_3+k_2) + 900k_3 \\ C &= 32\delta + 4K + 8K(4k_3 + k_2) \qquad (2.6\text{-}4) \\ D &= 8K \end{aligned}$$

It is apparent from Eq. 2.6-3 that the following equalities must be satisfied

$$A = 33 \qquad C = 786$$

$$B = 230 \qquad D = 900$$

The simultaneous solution of Eqs. 2.6-4 results in the following values of the unknown system elements

$$K = 450/4$$

$$k_2 = 60/450$$

$$k_3 = 19/45$$

$$\delta = 2$$

Our initial suspicion has proved to be correct, and δ is 2. For these values of the parameters, the tentative $H^t_{eq}(s)$, $H_{eq}(s)$ is

$$H^t_{eq}(s) = 19/450\ [s^2 + 17.7s + 23.6]$$

$$= 19/45\ [(s+3.58)^2 + 3.26^2]$$

and the resulting root locus diagram is indicated in Fig. 2.6-2. This appears to be a completely satisfactory result. The zeroes of $H^t_{eq}(s)$ lie in the left half plane, and are in fact very close to the desired pole locations, indicating insensitivity to gain variations. The asymptote goes off at -180°, and there is no reason to proceed with step 4. If one or more of the state variables were unavailable, this shortcoming could be overcome with minor loop compensation, as in the preceeding section.

The above problem is quite trivial, but it does illustrate the method by which additional poles are added by the use of the simplest type series compensator, $1/(s+\delta)$. In this case the pole was added right under a zero, to cancel the zero. Because the zero appeared in the left hand block of the given plant, cancellation compensation resulted from the application of the general design procedure.

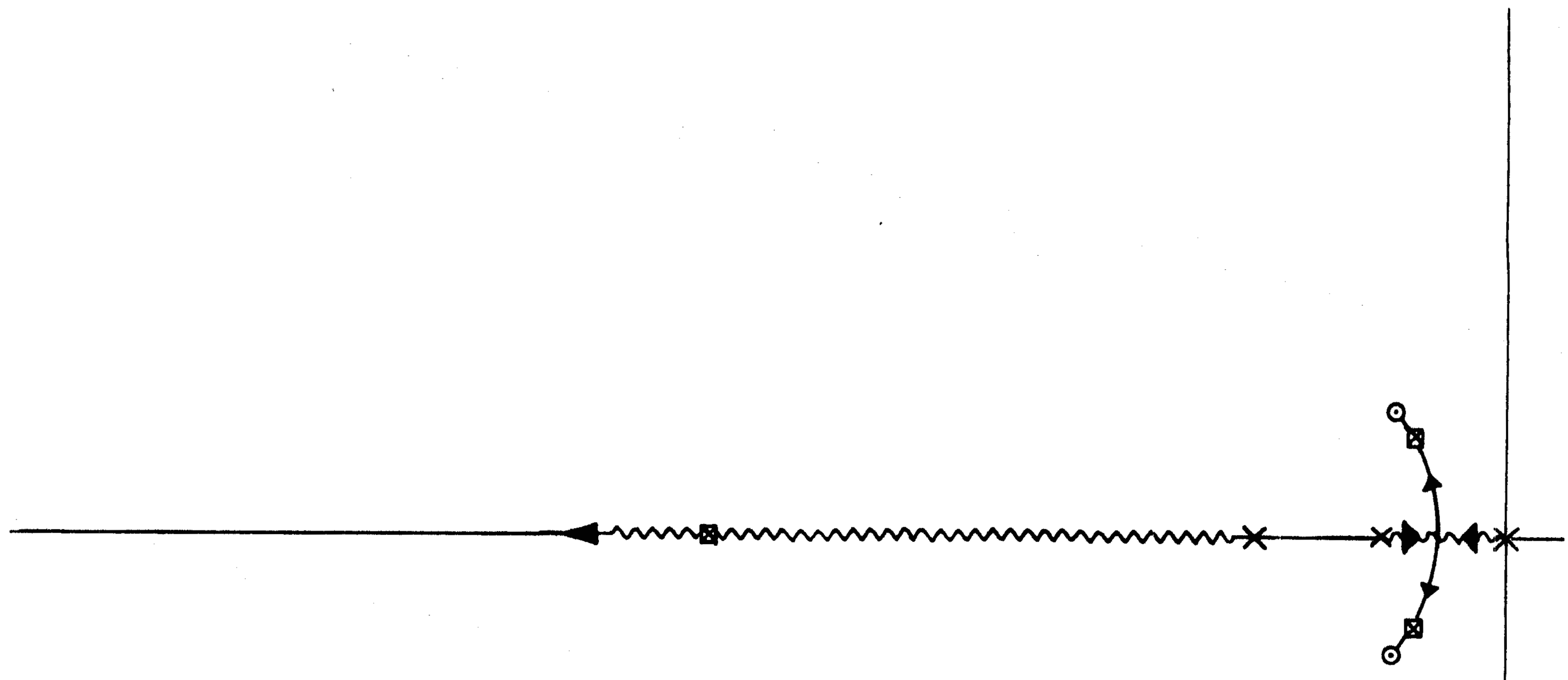

Fig. 2.6-2. The root locus corresponding to the system of Fig. 2.6-1b, with $4K=450$, $k_2=60/450$, $k_3=19/450$ and $\delta=2$.

The plant to be controlled in the second example is indicated in Fig. 2.6-3a. This example involves the substitution of a desired zero for an unwanted one. The desired zero is chosen so as to insure that the final closed loop system has an infinite velocity error coefficient. A second order example has been chosen here to simplify the algebra, as we eventually hope to add additional series compensation in this problem. The zero has not been placed in the left hand box to insure that cancellation compensation does not result. The desired C(s)/R(s) for this second example is

$$C(s)/R(s) = \frac{6\ (s+3)}{(s+3)^2 + 3^2} \tag{2.6-5}$$

Here the gain has been chosen so that $C(0)/R(0) = 1$, to insure zero position error for step inputs, and the zero position at $s = -3$ has been chosen to satisfy Eq. 2.6-1. This results in a zero steady state velocity error for ramp inputs. Since the given plant already had a zero which must be eliminated, the C(s)/R(s) that must actually be realized is

$$C(s)/R(s) = \frac{6\ (s+2)\ (s+3)}{(s+2)\ [(s+3)^2 + 3^2]}$$

$$= \frac{6\ (s+2)\ (s+3)}{s^3 + 8s^2 + 30s + 36}$$

From Fig. 2.6-3b, G(s), $H^t_{eq}(s)$, and C(s)/R(s) may all be written in terms of the unknowns, K_1 k_2, and δ. These expressions are

$$G(s) = \frac{2K\ (s+2)\ (s+3)}{s\ (s+\alpha)\ (s+4)}$$

$$H^t_{eq}(s) = \frac{s\ (k_2+1) + 2}{s + 2}$$

$$C(s)/R(s) = \frac{2K\ (s+2)\ (s+3)}{s^3 + As^2 + Bs + C}$$

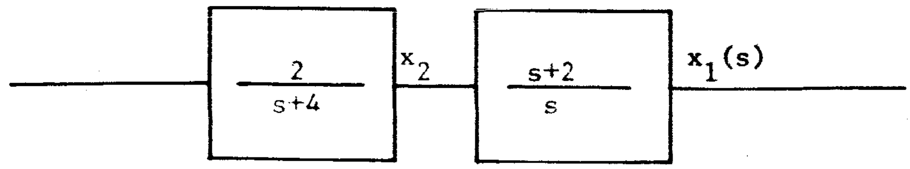

Fig. 2.6-3a. The given plant to be controlled such that $C(s)/R(s) = \frac{6(s+3)}{(s+3)^2+3^2}$

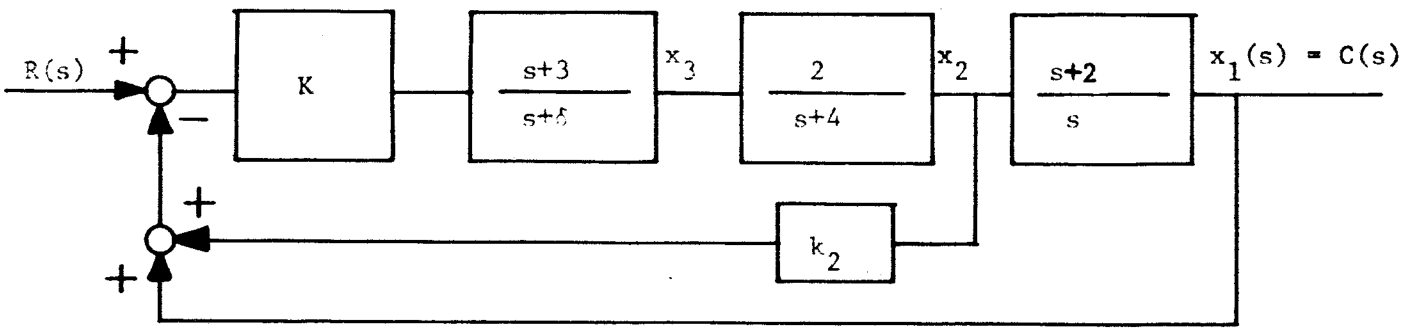

Fig. 2.6-3b. Means by which the plant of Fig. 2.6-3a may be controlled to realize the desired C(s)/R(s).

Fig. 2.6-3. The system for the second example.

where

$$A = \delta + 4 + 2K\ (k_2+1)$$

$$B = 4\delta + 4K + 6K\ (k_2 + 1)$$

$$C = 12\ K$$

By equating coefficients of equal powers of s in Eqs. 2.6-5 and 2.6-6, the following values result for the unknown system parameters.

$$K = 3$$

$$k_2 = -4/3$$

$$\delta = 6$$

and $H^t_{eq}(s)$ is

$$H^t_{eq}(s) = \frac{-1/3\ (s-6)}{s + 2}$$

Both G(s) and $H^t_{eq}(s)$ are known, and the root locus corresponding to $G(s)\ H^t_{eq}(s)$ may be drawn before steps 4 and 5 are complete. The root locus for this example is plotted in Fig. 2.6-4, and because of the negative sign in $H^t_{eq}(s)$, a zero degree locus must be drawn. Observe that in this figure $H^t_{eq}(s)$ has a zero in the right half plane, and two asymptotes go toward zeroes at infinity. The realization of the compensation network with feedback can be used to eliminate these difficulties, hence let us proceed to step 4. This is, in fact, a major reason for realizing the compensation network with feedback.

The final system configuration is given in Fig. 2.6-5. In order to make an intelligient choice of k_3, it is necessary to determine $H_{eq}(s)$, in order that one bound may be placed on k_3. By simple block diagram reduction of Fig. 2.6-5, $H_{eq}(s)$ is found to be

$$H_{eq}(s) = \frac{k_3\ s^2 + (4k_3-2/3)\ s + 4}{2\ (s+2)} \qquad (2.6\text{-}7)$$

Routh's criteria requires k_3 to be greater than 1/6.

$$G(s)H^{t}_{eq}(s) = \frac{2K\,(s+2)\,(s+3)}{s\,(s+4)\,(s+6)} \times \frac{-1/3\,(s-6)}{s+2}$$

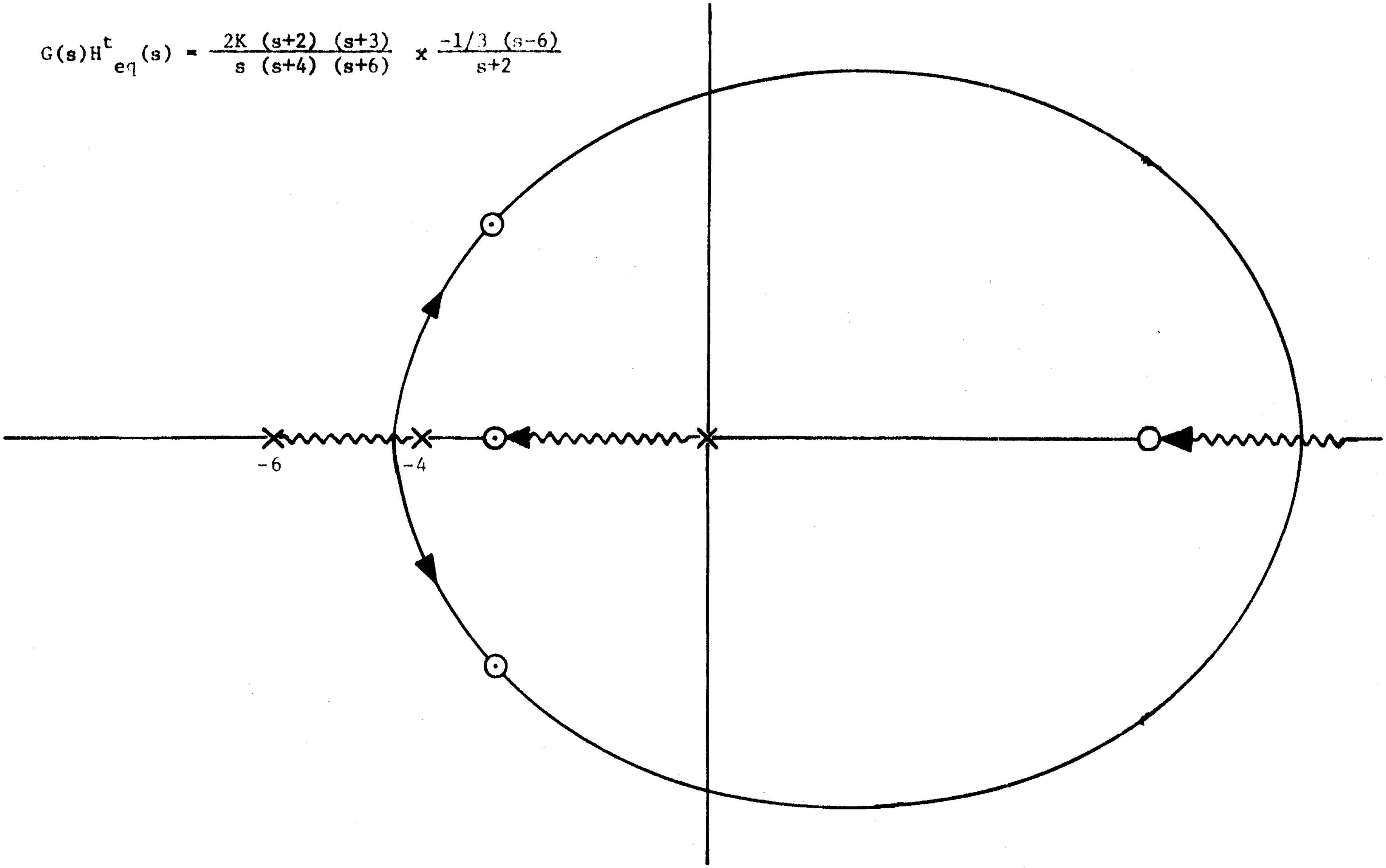

Fig. 2.6-4. The root locus of $G(s)H^{t}_{eq}(s)$ for the second example of the general case.

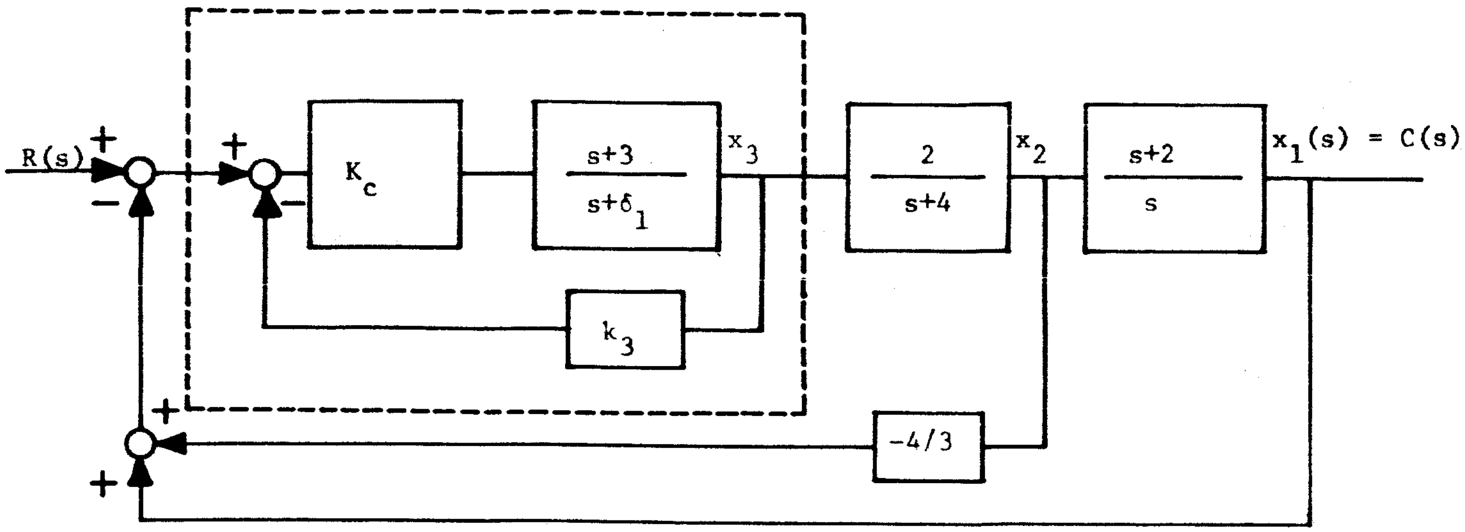

Fig. 2.6-5. The final system configuration, indicating that K_c, δ_1 and k_3 have yet to be specified. These are later found to be

$K_c = 12$

$\delta_1 = 15$

$k_3 = 1/4$

The dotted portion of Fig. 2.6-5 indicates the portion of the system that goes to realize the series compensation network, 3(s+3)/(s+6). The equation that must be satisfied here is

$$\frac{\dfrac{K_c(s+3)}{s+\delta_1}}{1+\dfrac{K_c\,k_3(s+3)}{s+\delta_1}} = \frac{3\,(s+3)}{s+6}$$

or

$$\frac{\dfrac{K_c}{1+K_c k_3}\,(s+3)}{s+\dfrac{\delta_1+3K_c\,k_3}{1+K_c\,k_3}} = \frac{3\,(s+3)}{s+6}$$

By equating gains it is seen that

$$\frac{K_c}{1+K_c\,k_3} = 3$$

or

$$K_c = \frac{3}{1-3k_3}$$

For K_c to be positive, k_3 must be less than 1/3. Thus k_3 is bounded by

$$1/6 < k_3 < 1/3$$

Let us assume a middle value of k_3 as equal to 1/4. Then $H_{eq}(s)$ from Eq. 2.6-7 is

$$H_{eq}(s) = \frac{s^2+4/3\,s+16}{8\,(s+2)} = \frac{(s+2/3)^2+(3.96)^2}{8\,(s+2)}$$

and K_c and δ_1 are easily found to be

$$K_c = 12$$

$$\delta_1 = 15$$

Once again, the final system is pictured in Fig. 2.6-5, where all system parameters are now known. $G(s)\,H_{eq}(s)$ is now known to be

$$G(s)H_{eq}(s) = \frac{24\ (s+2)(s+3)}{s\ (s+4)(s+15)} \times \frac{[(s+2/3)^2 + (3.96)^2]}{8\ (s+2)}$$

and the corresponding root locus is indicated in Fig. 2.6-6. The desired response has been realized, with infinite position and velocity error coefficients, and the resulting closed loop system is stable for all gain.

If it is assumed that the desired closed loop response is a stable response, then the general design procedure outlined in this section and illustrated by the last two examples always works. The truth of this statement is discussed under the heading of the "Design of High Gain Systems". What is important here is that the application of the general design procedure is tedious. A lot of algebra is involved, particularly in finding the bounds on the feedback coefficients that are associated with the compensator state variables. Then, in order to evaluate the answer, $H_{eq}(s)$ must be determined, and this involves the factoring of a polynomial of order (n-1). If the reader has any doubt about the amount of algebra involved, he is urged to work the first example of this section in the case when the zero is associated with the pole at s = -4, rather than in the given location.

A second drawback is that the designer has little idea of what effect the arbitrary choice of the compensator feedback coefficients actually has on the over all $H_{eq}(s)$. In the examples here, we assumed some middle value simply for lack of something better to do.

In defense of the general design procedure of this section, it should be pointed out that all of the problems that have to be solved involve linear algebraic equations, and these are easily programmed. Further more, the general design procedure always works. In the following section an alternate design procedure is outlined that may appeal to those with a classical control background.

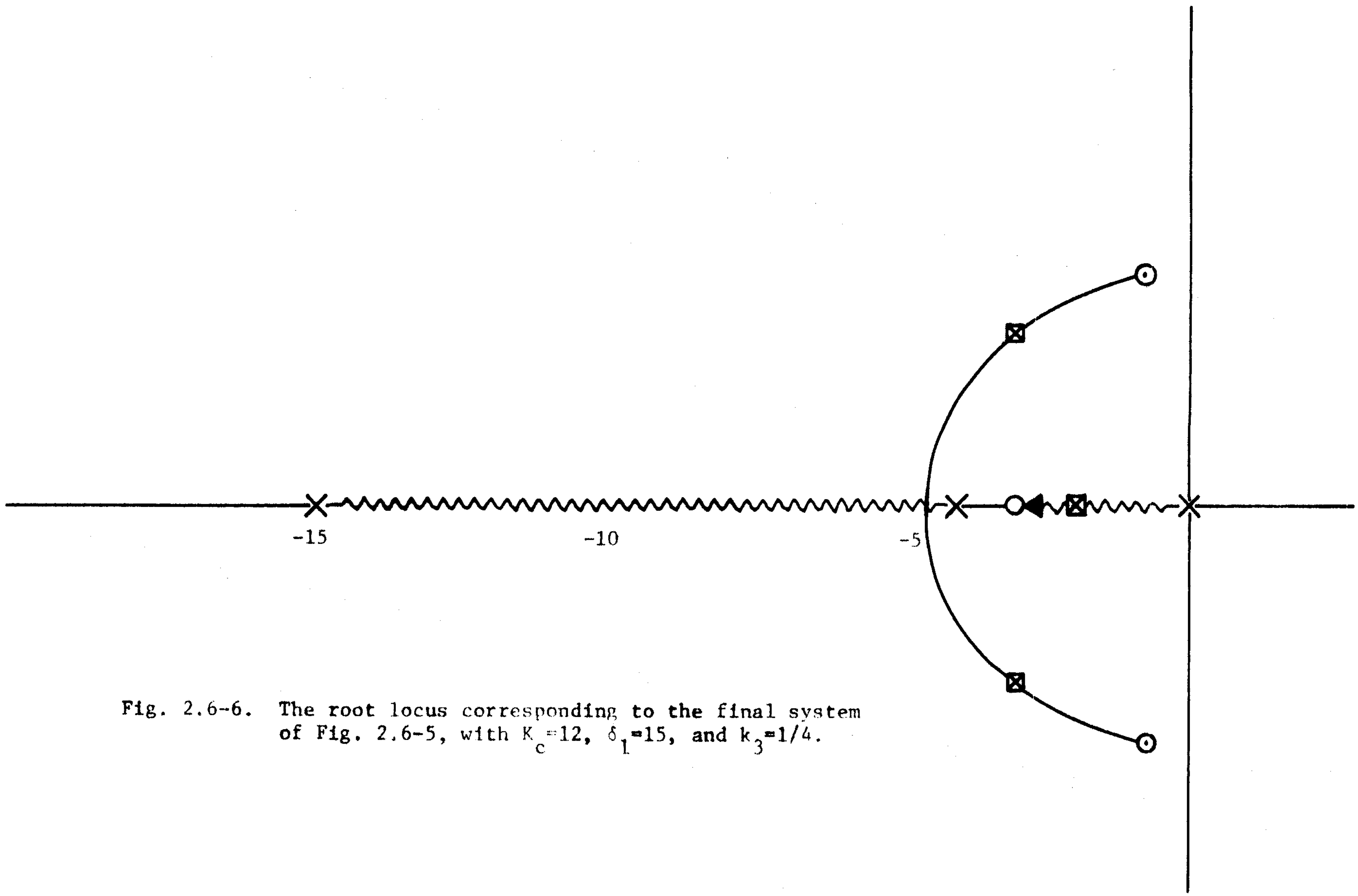

Fig. 2.6-6. **The root locus** corresponding **to the final** system **of Fig.** 2.6-5, with $K_c = 12$, $\delta_1 = 15$, **and** $k_3 = 1/4$.

2.7 Alternate Procedures

The previous section details a formal procedure for realizing exactly a desired $C(s)/R(s)$ in the general case in which the given plant is not compatible with the desired closed loop response. The compensation that was added was dictated by the differences between the pole and zero requirements of the desired response and of the given $G_p(s)$. Often this results in a zero term appearing in the left hand most block of the resulting system. The presence of this zero is unfortunate for two reasons. The zero is never cancelled by the poles of $H_{eq}(s)$, and hence this zero appears on the s plane plot of the poles and zeroes of $G(s)H_{eq}(s)$. Also the presence of this zero results in nonlinear algebraic equations that must be solved to evaluate the unknown system parameters. In terms of the state space representation, in physical variables it is no longer possible to describe the system as

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x}+\underset{\sim}{b}u$$

since a u dot term appears in the last equation. This would prevent a transformation of variables from any other system of variables in which the u dot term does not appear. Hence one of the basic motivations of this section is the elimination of the zero in the left hand block of $G(s)$.

A second motivation of this section is the desire to choose at the outset the poles of the compensation network. If these pole locations could be specified in advance, then the problem would be reduced to that of the simplest case. Here it is advocated that the pole positions be chosen on the basis of the knowledge of the desired closed loop response and on the techniques for drawing-the root locus diagram.

It would be wrong to imply that the methods outlined in this section actually constitute a design procedure. The contents of this section might be regarded more as tricks that have proved to be helpful, and that have yet to be incorporated into any formal design technique. In a sense, then, this section and the obvious inadequacies of this section begin to point up the further work that must be done to complete the $H_{eq}(s)$ method.

The approaches of this section depend to a large extent upon the designers knowledge of root locus methods. A few comments with regard to the root locus method are in order here. Normally $C(s)/R(s)$ has zeroes where $G(s)$ has zeroes and where $H(s)$ has poles. Our $H(s)$ is $H_{eq}(s)$, and its poles are all zeroes of $G(s)$, however $C(s)/R(s)$ does not have two sets of zeroes, one from $G(s)$ and one from the poles of $H_{eq}(s)$. The only zeroes of $C(s)/R(s)$ that appear are the zeroes of $G(s)$.

If a zero of $H_{eq}(s)$ is placed at the same location as a pole of $G(s)$, this does not mean that the pole and zero cancel. Rather, a branch of the root locus always lies between this pole and zero, so that the placing of a zero of $H_{eq}(s)$ on top of a pole of $G(s)$ guarantees a closed loop pole at that location. In the examples that follow, the zeroes of $H_{eq}(s)$ and the compensator poles are chosen so that they coincide, and hence specify one of the closed loop pole locations. At the same time, the factoring of $H_{eq}(s)$ is made simpler, since one of its zeroes is known. The idea is to shape the open loop transfer function before state variable compensation is used.

In the previous section we considered the removal of a zero in the plant of Fig. 2.6-1a. The application of the general procedure resulted in cancellation compensation, since the zero to be removed was located in the left hand most block of G (s). As a first example of this section, let us consider the removal of the zero when cancellation compensation is not quite so obvious. The open loop plant is described in Fig. 2.7-1a, where it is seen that $G_p(s)$ is

$$G_p(s) = \frac{4}{s+8} \frac{s+2}{s+4} \frac{1}{s}$$

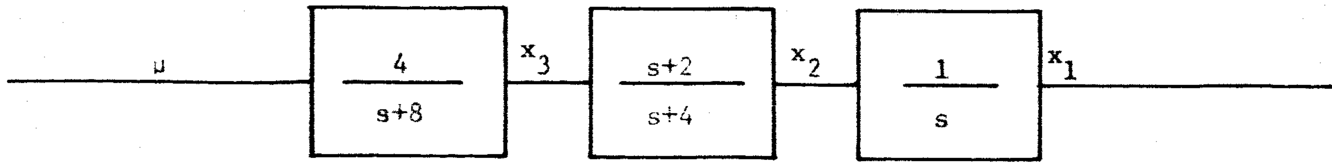

Fig. 2.7-1a. The given plant to be controlled such that Eq. 2.7-1 may be satisfied.

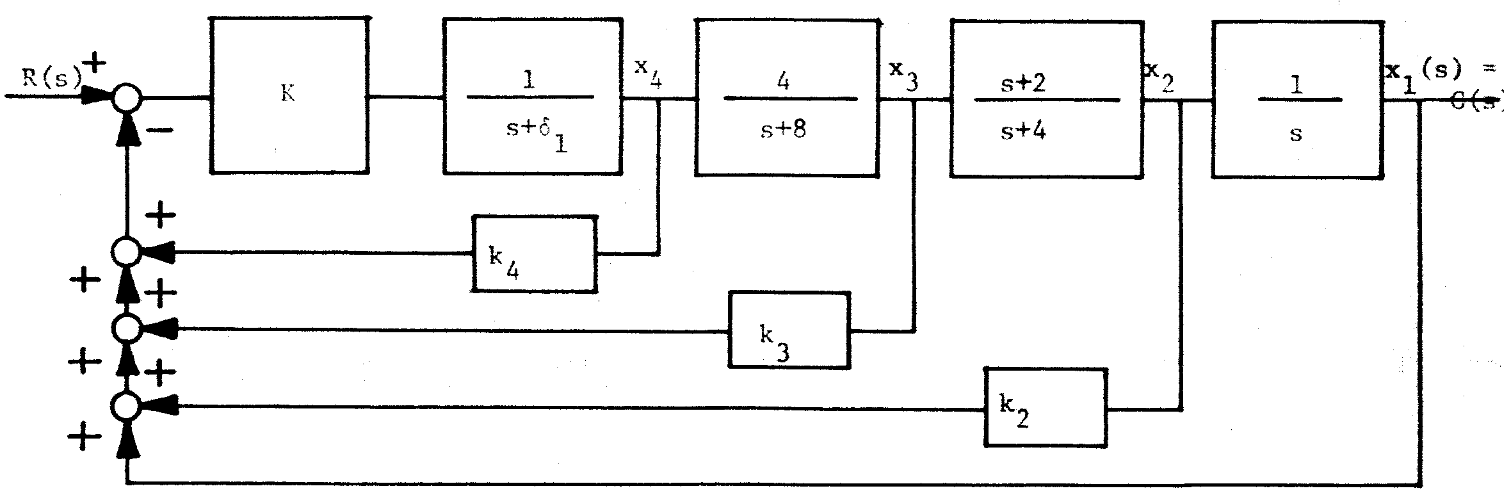

Fig. 2.7-1b. The means by which the plant of Fig. 2.7-1a may be controlled to realize the desired $C(s)/R(s)$.

The desired C(s)/R(s) is still

$$C(s)/R(s) = \frac{450}{[(s+3)^2 + 3^2](s+25)} \tag{2.7-1}$$

but in order to eliminate the zero at s = -2, the C(s)/R(s) that must be realized is again

$$C(s)/R(s) = \frac{450(s+2)}{(s+2)[(s+3)^2 + 3^2](s+25)}$$

$$= \frac{450(s+2)}{s^4 + 33s^3 + 230\ s^2 + 786s + 900} \tag{2.7-2}$$

The ultimate design configuration is shown in Fig. 2.7-1b. We desire to pick δ_1 at the outset, so that the simpler procedures associated with the simplest case can be used in the determination of all of the k_i's and K. Let us begin to plot the root locus of $G(s)H_{eq}(s)$. The open loop poles of $G_p(s)$ are known, and the desired closed loop poles are also known. The zeroes of $G_p(s)$ are always cancelled by the poles of $H_{eq}(s)$, and hence these zeroes do not appear on the s plane plot of $G(s)H_{eq}(s)$. The open loop poles of $G_p(s)$ and the closed loop poles of C(s)/R(s) are indicated on Fig. 2.7-2a. We have yet to choose δ_1. If δ_1 is chosen to be -2, then the only way in which a closed loop pole can also be located at -2 is to have a zero of $H_{eq}(s)$ at s = -2. This is indicated in Fig. 2.7-2b. It is immaterial that G(s) now has a pole and a zero both at s = -2, and this may represent a confusion factor, since $H_{eq}(s)$ will also have a pole and zero at s = -2. The primary point being made is that to insure a closed loop pole at s = -2, this may be done by putting a pole of the compensator at that same place. In order to satisfy the root locus requirements, a zero of $H_{eq}(s)$ must also lie in the same place.

In Fig. 2.7-1b, all of the poles of the open and closed loop transfer functions are indicated. As far as the drawing of the root locus is concerned,

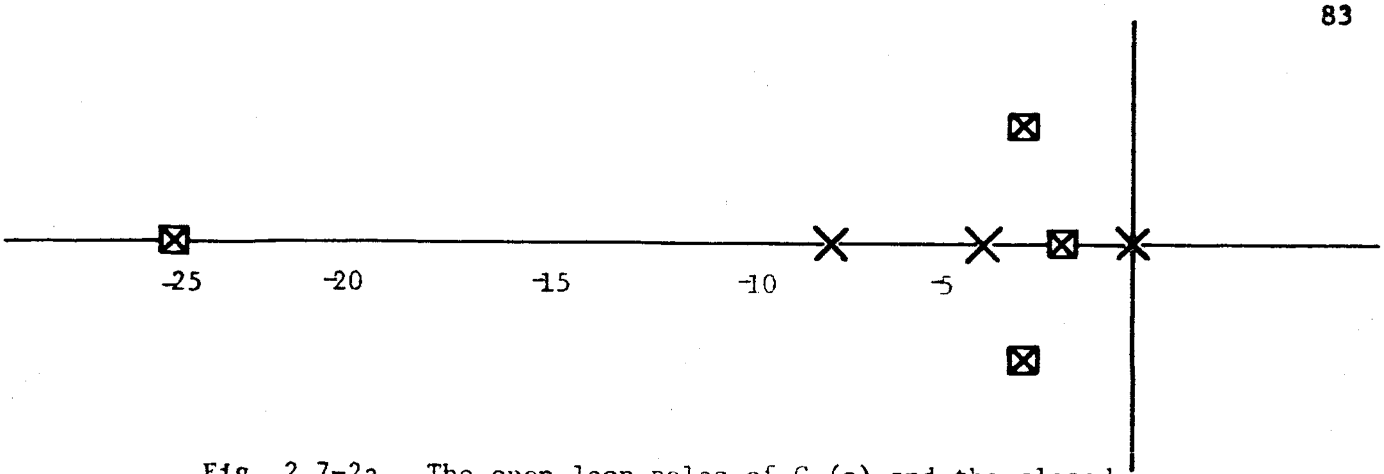

Fig. 2.7-2a. The open loop poles of $G_p(s)$ and the closed loop poles of $C(s)/R(s)$.

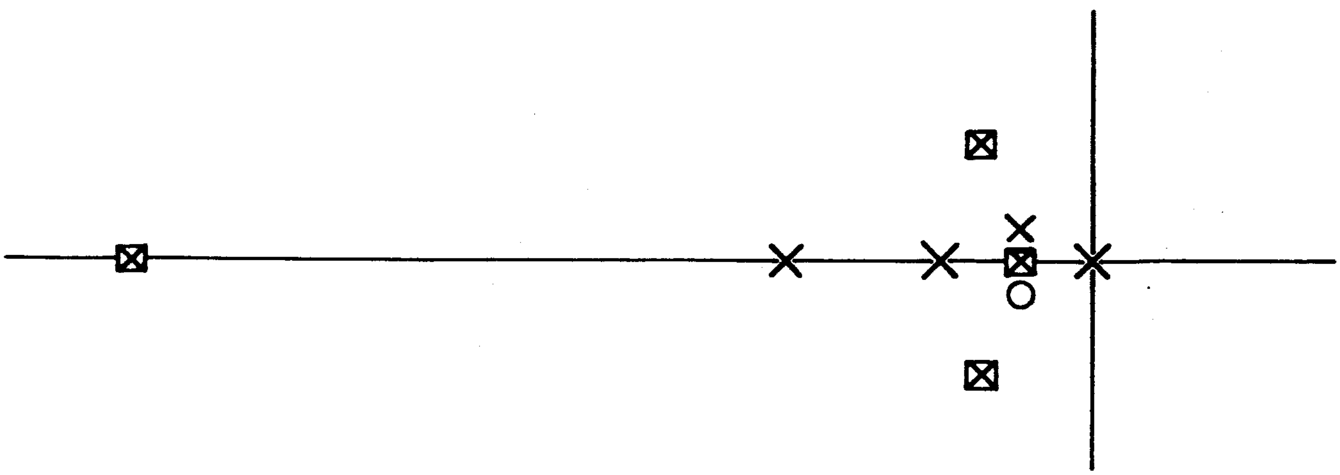

Fig. 2.7-2b. The addition of the pole δ_1 at $s = -2$, and the necessary zero of $H_{eq}(s)$.

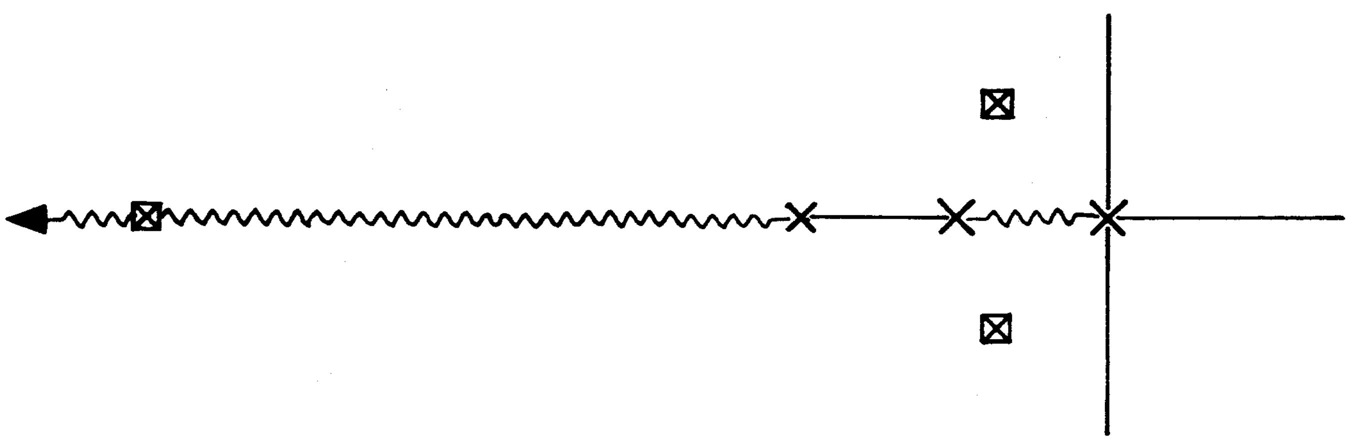

Fig. 2.7-2c. Known branches of the root locus. Branches must pass through the points $s=-3 \pm j3$.

the pole zero pair at $s = -2$ do cancel each other, although a closed loop pole does exist at that location. From a knowledge of the root locus method, the branches are indicated on Fig. 2.7-2c.

Thus far the design may be considered as tentative. We have chosen the pole of the compensator at $s = -2$, and we are simply looking to see if this is a logical choice. In order to complete the root locus of Fig. 2.7-2c, two more zeroes of $H_{eq}(s)$ must be added. In order to insure that the branches of the root locus pass through the points $s = -3 \pm j3$, it seems reasonable that a set of complex conjugate zeroes must exist somewhere in the vicinity of $s = -3 \pm j3$. The exact location is not known, and will not be until $H_{eq}(s)$ is determined. But it is concluded that the choice of the compensator pole at $s = -2$ was a reasonable one, and the system has been reduced to the simplest case.

Two methods of proceeding are now apparent. We may treat the problem as a problem in the simplest case, and thus realize the desired $C(s)/R(s)$ exactly. An alternate approximate approach would be to pick the remaining zeroes of $H_{eq}(s)$. Then $H_{eq}(s)$ would be specified completely, and K would be chosen for zero steady state error. The closed loop poles of $C(s)/R(s)$ would not be exactly as desired, depending on the guess of the remaining zeroes of $H_{eq}(s)$. In this case it should not be too difficult to guess the approximate location of the zeroes of $H_{eq}(s)$, particularly after a number of problems of this type have been worked. Both solutions are included below.

Let us treat the problem as one of the simplest case first. Then K and the k_i's must be found to realize $C(s)/R(s)$ exactly. For the configuration of Fig. 2.7-1b G(s) is

$$G(s) = \frac{4K}{s(s+4)(s+8)}$$

and $H_{eq}(s)$ is

$$H_{eq}(s) = \frac{1}{4(s+2)}\left[k_4 s^3 + (12k_4 + 4k_3 + 4k_2)\, s^2 + (32k_4 + 16k_3 + 8k_2 + 4)\, s + 8\right]$$

so that

$$C(s)/R(s) = \frac{4K(s+2)}{s^4 + As^3 + Bs^2 + Cs + D} \qquad (2.7\text{-}3)$$

where

$$A = 14 + Kk_4$$

$$B = 56 + K(12k_4 + 4k_3 + 4k_2)$$

$$C = 64 + K(32k_4 + 16k_3 + 8k_2 + 4)$$

$$D = 8K$$

By equating coefficients of like powers of s in Eqs. 2.7-2 and 2.7-3, the resulting values of K and the k_i's are

$$K = 450/4$$

$$k_2 = 60/450$$

$$k_3 = -114/450$$

$$k_4 = 76/450$$

For these values of the k_i's, $H_{eq}(s)$ is

$$H_{eq}(s) = \frac{76s^3 + 696s^2 + 288s + 3600}{4 \times 450(s+2)}$$

$$= \frac{19(s+2)[(s+3.58)^2 + (3.3)^2]}{450(s+2)}$$

The resulting root locus is shown in Fig. 2.7-3. The closed loop pole at s = -2 is not shown on this diagram since it does not appear in C(s)/R(s), as it is cancelled out. This is an important point.

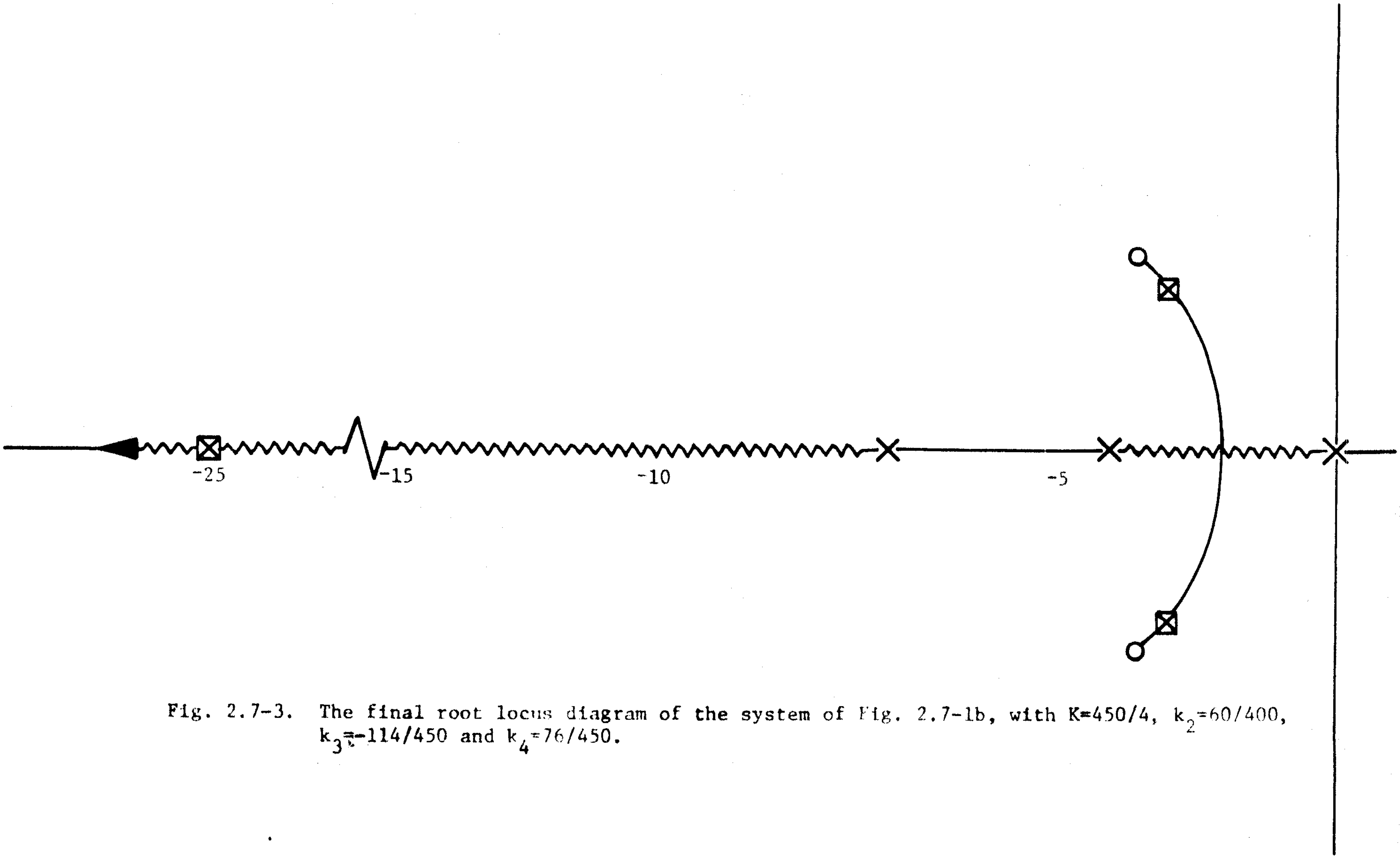

Fig. 2.7-3. The final root locus diagram of the system of Fig. 2.7-1b, with $K=450/4$, $k_2=60/400$, $k_3=-114/450$ and $k_4=76/450$.

Let us examine the magnitude of the error that would have occurred if the pole locations of $H_{eq}(s)$ had been assumed to be at $s = -4 \pm j4$, in addition to the pole at $s = -2$. This is now a root locus problem, and not much else. Since the pole and zero, and closed loop pole, at $s = -2$ do not affect the drawing of the root locus, these are not indicated on the s plane. The root locus is indicated in Fig. 2.7-4. The complex conjugate closed loop poles are chosen to be those near the desired closed loop poles. A number of choices are possible. On the figure we have chosen the complex conjugate poles to have the same real part as the desired closed loop poles, and the imaginary part is 3.4. This is higher than the desired value, but not much. If this deviation is too great, then one might choose a different set of zero locations for $H_{eq}(s)$. For the closed loop poles as indicated on Fig. 2.7-4, the gain is determined from the root locus diagram to be 11.3, and the remaining closed loop pole is at $s = -17.5$, determined graphically. These values seem significantly different from the desired ones, found exactly just above, to dictate a redesign. This may be done quite simply, and even without choosing new values for the zero locations of $C(s)/R(s)$. Just increase the gain until the gain is a desired value, or until the pole at $s = -25$ is realized. Note that increasing the gain here gives a higher damping ratio and a still faster response, so that the usual problems that are associated with increasing gain are not encountered in this problem.

This last problem brings out two important points. One need not use the $H_{eq}(s)$ method at all, and one need not design to a specified $C(s)/R(s)$. Any design procedure with which the designer is familiar may be used. In addition to the poles of the given plant, $(n - 1)$ zeroes are available from $H_{eq}(s)$. Any procedure whatever may be used to position these zeroes in an advantageous fashion. The preceeding solution made use of the root locus diagram, because

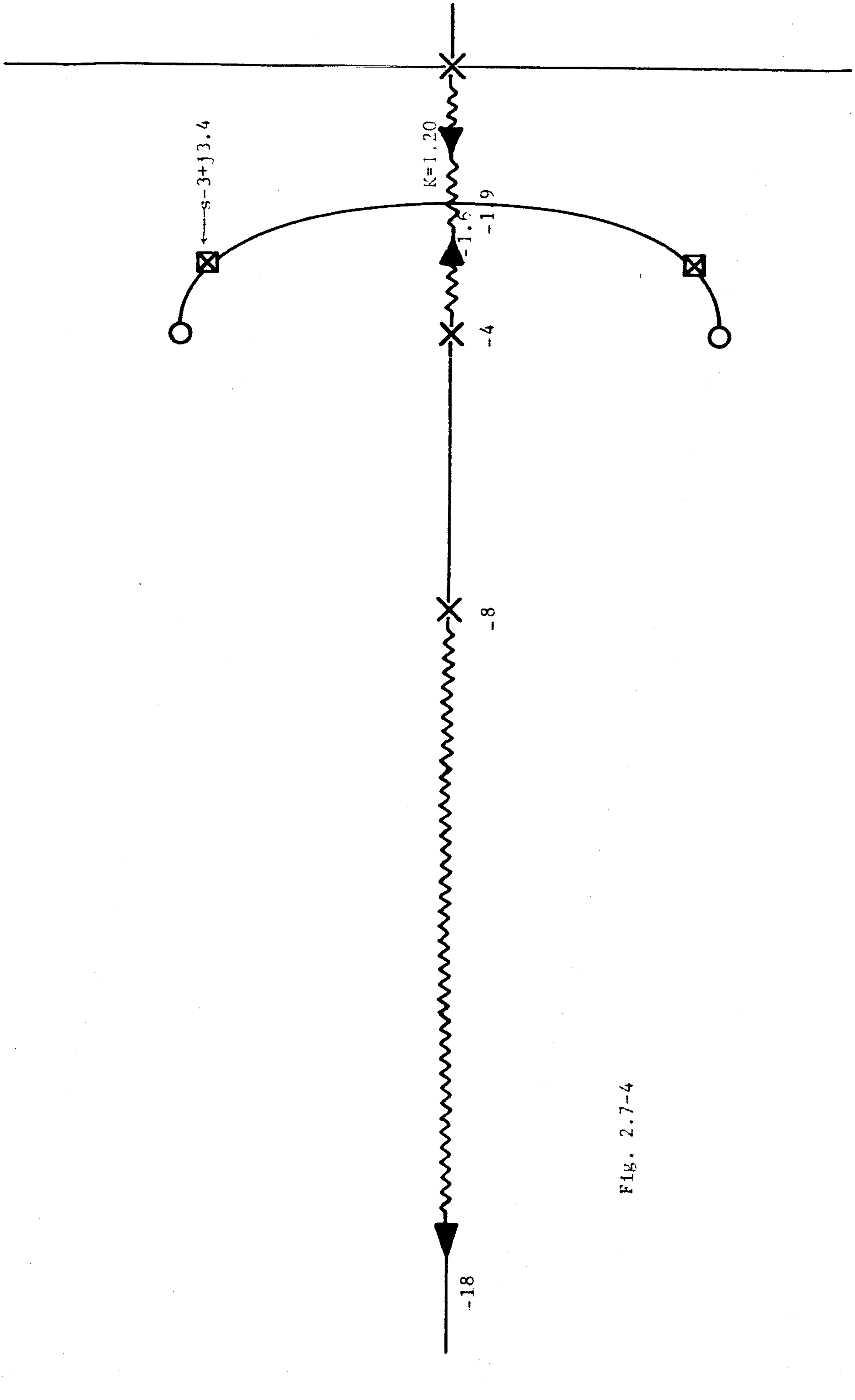

Fig. 2.7-4

we were still thinking in terms of closed loop poles. The design might have been completed on the Bode, Nyquist, or Nicholes chart as well, to any design criteria that is suitable. The presence of the (n - 1) zeroes makes the realization of the system specifications immeasurably easier than if an additional pole had to be added every time a zero was needed.

The second point is that the design started by initially altering the open loop transfer function of the system. Specifically, an open loop pole was located at exactly the spot where a closed loop pole was desired. In the limit, one might locate all of the open loop poles where the closed loop poles are desired, and then place the zeroes of $H_{eq}(s)$ at these same locations. Then the closed loop poles would remain at a fixed location, independent of gain. This is the procedure used in the next example.

The addition of series compensation, even poles without zeroes, has no destabilizing effect, as long as the compensator state variables are fed back. Because of this fact, any amount of series compensation may be added.

This example was considered earlier in Section 2.4 to show that stability for all gain is not necessarily an inherent characteristic of state variable feedback. The plant is given as

$$G_p(s) = \frac{1}{s(s+4)}$$

and is pictured again in Fig. 2.7-5a. The desired closed loop response is

$$C(s)/R(s) = \frac{2}{s^2 + 2s + 2} = \frac{2}{[(s+1)^2 + 1^2]} \qquad (2.7\text{-}4)$$

As shown in Section 2.4, this response can be realized with no series compensation by placing a zero of $H_{eq}(s)$ in the right half s plane. The object here is to introduce series compensation to modify the plant characteristics so that right half plane zeroes are not necessary. The open loop poles are to be located at the desired closed loop pole locations, so that the zeroes of

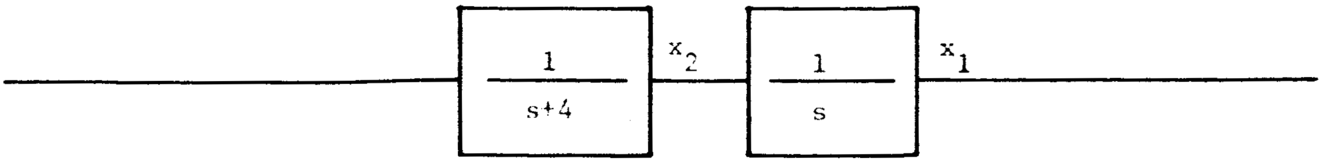

Fig. 2.7-5a. The plant to be controlled such that C(s)/R(s) is as given in Eq. 2.7-4

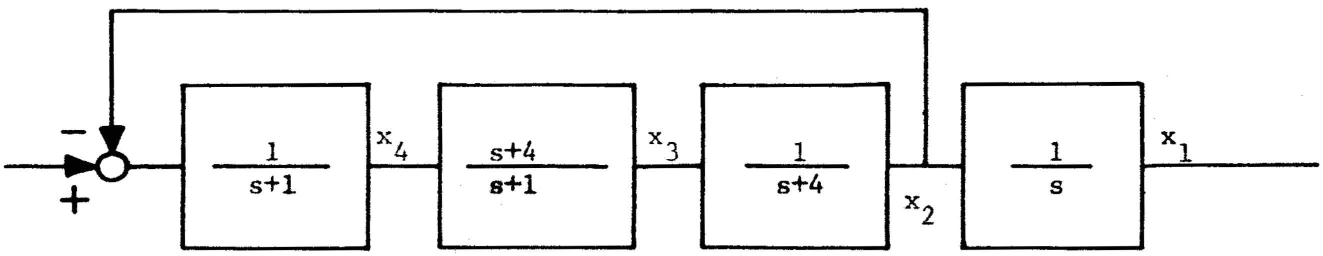

Fig. 2.7-5b. Modification of the open loop transfer function to realize the desired open loop transfer function.

Fig. 2.7-5. Modification of the plant open loop transfer function.

$H_{eq}(s)$ may be located at the same place.

In order to locate the open loop poles at $s = -1 \pm j1$, series compensation is introduced as in Fig. 2.7-5b. The state variable at x_3 is redundant, and the open loop transfer function from the input to x_2 is $1/(s+1)^2$. Feedback around this element results in the complex conjugate roots $s = -1 \pm j1$, as desired.

It is now necessary to use state variable feedback to realize the desired closed loop response. This is done in Fig. 2.7-6. For the system of Fig. 2.7-6, $H_{eq}(s)$ is

$$H_{eq}(s) = k_4 s(s+1) + k_2 s + 1$$

$$= s^2 + \frac{k_3 + k_2}{k_3} s + \frac{1}{k_3}$$

Since the zeroes of $H_{eq}(s)$ are to lie on the desired closed loop poles, $H_{eq}(s)$ must equal Eq. 2.7-4. This is easily accomplished if

$$k_2 = \frac{1}{2}$$

$$k_4 = \frac{1}{2}$$

The effective system, with all inner loops removed and $H_{eq}(s)$ in evidence is pictured in Fig. 2.7-7. The root locus for this system is shown in Fig. 2.7-8. The final closed loop poles are located at $s = -1 \pm j1$, and an additional pole exists along the negative real axis. The effect of this pole may be made negligible by using large gain K. The location of the complex conjugate poles is independent of K.

It is interesting to examine the Nyquist diagram for $G(s)H_{eq}(s)$. This is shown in Fig. 2.7-9. Notice that the return difference is always greater than 1.

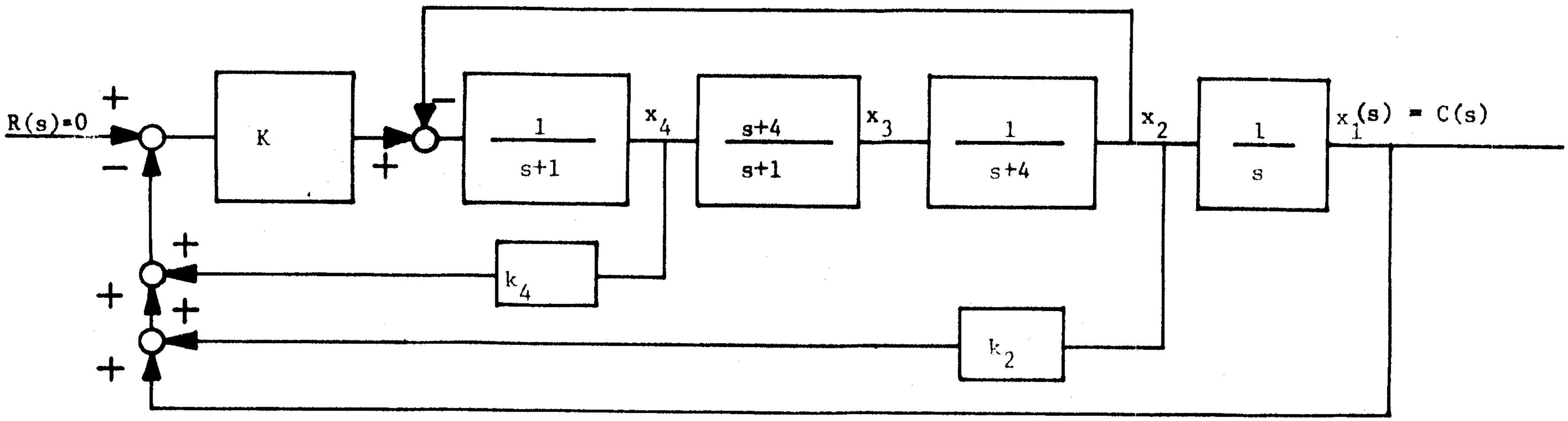

Fig. 2.7-6. The final system configuration for the controlled plant. The values of k_4 and k_2 each prove to be 1/2.

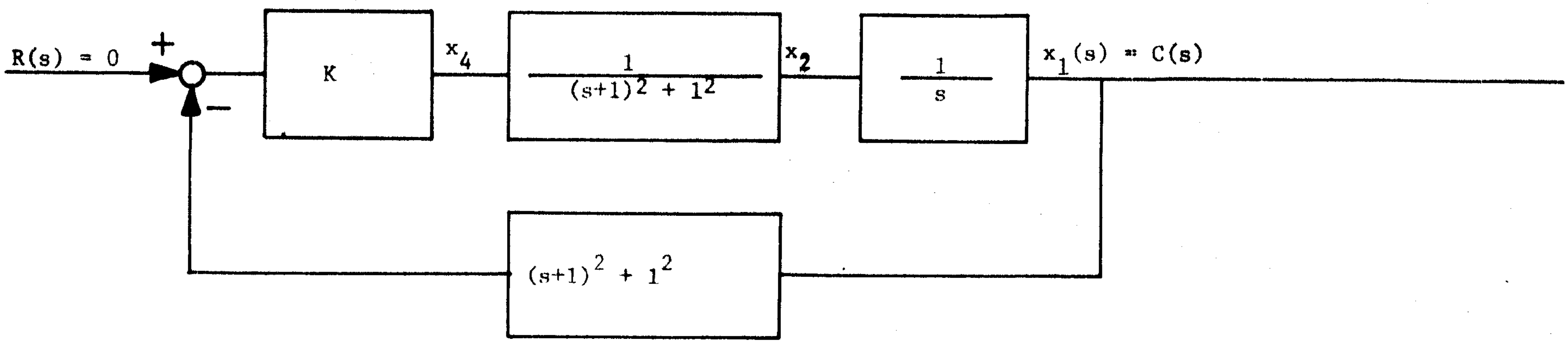

Fig. 2.7-7. The effective system of Fig. 2.6-6, where k_4 and k_2 have been set equal to 1/2.

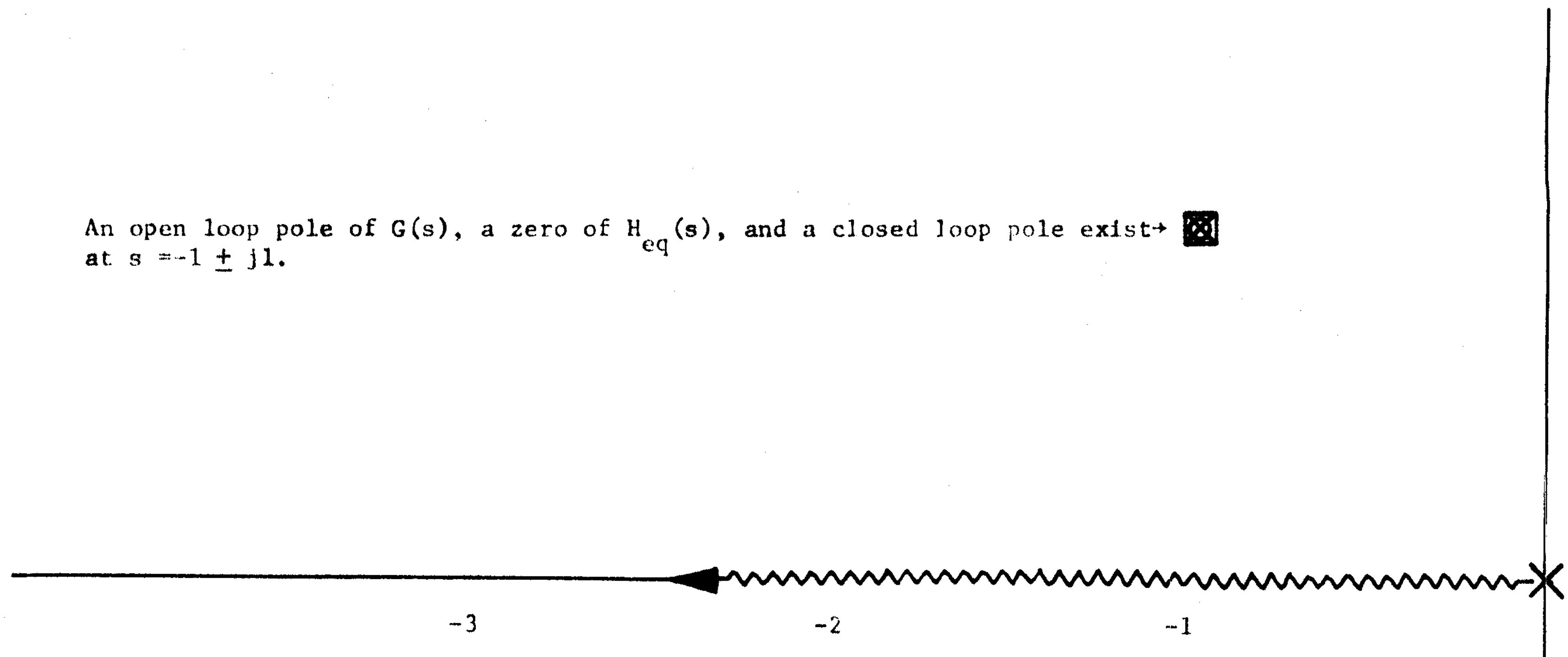

Fig. 2.7-8. The root locus corresponding to the system in Fig. 2.7-7. The closed loop pole on the negative real axis depends on K.

In view of the above Nyquist diagram, consider the case in which the gain K is not a linear gain, but is nonlinear. Then, in order to use the Popov stability criteria, it is necessary to draw a modified Nyquist diagram, where the real part of the function being plotted, $W(j\omega)$, is equal to the real part of $G(j\omega)H_{eq}(j\omega)$, and the imaginary part of $W(j\omega)$ is equal to $\omega G(j\omega)H_{eq}(j\omega)$. The shape of the resulting plot of $W(j\omega)$ is identical to that given in Fig. 2.7-9. In short, it is possible to draw a Popov line indicating global asymptotic stability for any single valued nonlinearity lying in the first and third quadrant, and whose value for zero input is zero. Thus, the state variable feedback used here has proved to be effective in realizing the desired response in the linear system, and it has also proved to be effective in stabilizing a class of nonlinear systems. This serves to introduce the subject of the next chapter, namely the outline of future work.

2.8 Conclusion

This chapter has outlined a design procedure called the H equivalent method. The H equivalent method utilizes state variable feedback, and is characterized by the following properties:

1. The design criteria is the desired closed loop response, $C(s)/R(s)$.
2. The desired closed loop response can be realized exactly.
3. The only mathematical tools necessary are the usual tools of the control engineer, the Laplace transform and the frequency domain.
4. The implementation of the design procedure requires that all of the variables be fed back through constant elements. Fortunately, these feedback coefficients are often less than one.
5. A method of minor loop equalization, in which dynamics are included in the feedback paths, results if one or more of the state variables are unavailable.

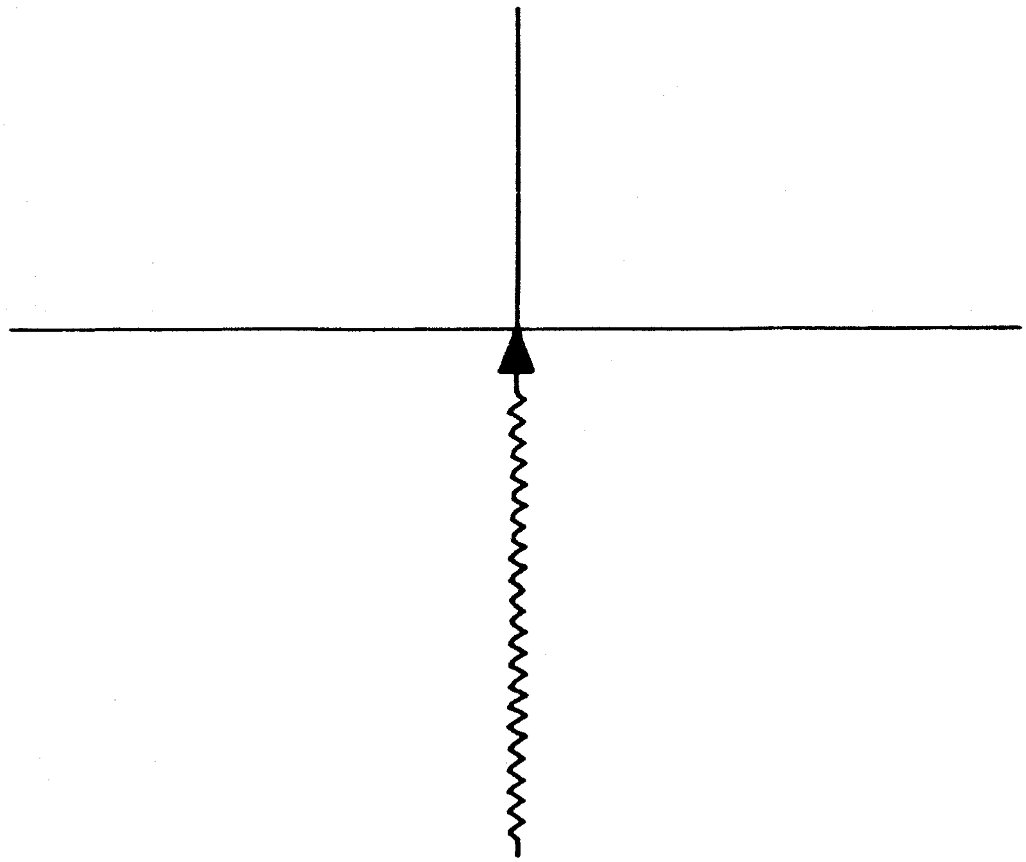

Fig. 2.7-9. The Nyquist diagram corresponding to $G(s)\ H_{eq}(s)$ of Fig. 2.7-7. A plot at the Popov function $W(jw)$, is identical in shape, that is, it lies along the negative imaginary axis.

6. If none of the state variables are available except the output, the method reduces to that of Guillemin and Truxal.

The H equivalent design procedure is significantly different from many conventional design procedures. Often conventional design procedures require manipulation of the open loop transfer function in order to force a desired closed loop response. This may be accomplished on the Nyquist, Bode, Nichols, or root locus diagram. Sometimes the criteria upon which the modifications of the open loop transfer functionare based are also "open loop" criteria, such as gain or phase margin, and the realization of a stable system is of prime importance. In contrast, the H equivalent method deals directly with the closed loop system specification, and the open loop transfer function is never altered simply to insure stability requirements. These are always satisfied, assuming, of course, that the desired closed loop system response is a stable response.

For these reasons and because the 6 properties listed above are highly desirable from either a practical or a theoretical point of view, it is the author's feeling that design procedures as outlined in this report, or closely related to the ideas presented here, will soon replace the conventional cut and try procedures of picking series equalization.

CHAPTER III

FUTURE STUDY

This report has outlined a method for the synthesis of single input, single output, linear control systems through the use of state variable feedback. Perhaps the most important contribution made here is utilization of the familiar frequency domain techniques to implement the command of modern control theory, that is to feed back all of the state variables. The outstanding feature of state variable feedback is the complete lack of concern for the stability of the resulting design. And not only is the resulting system stable, but any $C(s)/R(s)$ that does not require a smaller pole-zero excess than exists in the plant to be controlled may be realized. Since $(n - 1)$ zeroes are available in $H_{eq}(s)$, it is possible to always insure the return difference, $1 + G(s)H_{eq}(s)$, is always greater than 1. In terms of modern control theory, this is necessary if the resulting system is to be optimum for a quadratic type integral performance index.

The design procedure outlined in Chapter 2 is predicated upon the desire to realize a specified $C(s)/R(s)$, and at the same time insure stability for any gain, K, where K is the gain element located in the left hand most block of the open loop system. It is important to point out that the resulting system is not necessarily stable for changes in gain in other parts of the system. In a conventional control system with only output feedback, the location of gain within the loop is not important. Here, where many feedback paths are utilized, the location of gain is important, as clearly the gains of the various blocks influence the locations of the zeroes of $H_{eq}(s)$. Hence, it is now felt that an inordinate amount of attention was paid to the realization of stability for any K. More attention must be directed to design procedures that result in satisfactory response for variations in other of the system gains, as the K_i's, or the k_i's. Of course this is just a part of an overall sensitivity problem, where any of the system parameters may be assumed to vary. The general area of sensitivity of the system response to variations in plant parameters is thus an important area for future investigations, and, in fact, work in this area is already underway.

It was noted above that the concentration on the realization of a stable system for any K may have been unfortunate. In another sense, it was quite fortunate, as the realization of stability for any gain serves to introduce the use of state variable feedback techniques into the design of nonlinear systems. It was noted at the end of Chapter 2 that the presence of (n - 1) zeroes in $H_{eq}(s)$ made it quite easy to satisfy the Popov stability criteria. This suggests the design of intentionally nonlinear systems, where, for example, the input may be forced to saturate in order to avoid saturation phenomena of the power elements in the loop. If the input saturating element has a very high gain, this element then looks like a relay, with a switching hypersurface defined in n space by the feedback coefficients of all of the state variables. A PhD thesis concerned with the synthesis of nonlinear systems through the use of state variable feedback will be submitted as a part of the next report.

One further area of interest is evident from the problem statement given as the first sentence of this chapter. The ultimate concern is not for single input, single output systems, but for systems with multiple inputs and outputs. In the nuclear rocket problem, for instance, the material of this report may serve as a basis for the design of many of the controllers that are included within the main loop. However, the overall system being controlled has two inputs and two outputs. It is felt that work on the multiple input, output system should be guided at least in part by the results of the sensitivity investigations mentioned above.