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ESTIMATION IN THE NEGATIVE BINOMIAL DISTRIBUTION

By A. CLIFFORD COHEN, JR.
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

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ABSTRACT

Practical simplified procedures are developed in this paper for calculating estimates of parameters of the negative binomial distribution with probability function

$$f(x) = \frac{\Gamma(x + k)}{x! \; \Gamma(k)} p^k (1 - p)^x; \; x = 0, 1, 2\ldots$$

where $0 < p < 1$ and $k > 0$. Moment estimators, maximum likelihood estimators, and estimators based on moments and frequencies in selected classes are given both for the complete and for the truncated (with missing zero class) distribution. To facilitate calculation of the various estimators given, a table of the function $-p \ln p/(1 - p)$ with entries to six decimals at intervals of 0.001 for the argument p, is included. Illustrative examples are also included.

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By

A. Clifford Cohen, Jr.

TERRESTRIAL ENVIRONMENT GROUP
AEROSPACE ENVIRONMENT OFFICE
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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ESTIMATION IN THE NEGATIVE BINOMIAL DISTRIBUTION

SUMMARY

Practical simplified procedures are developed in this paper for calculating estimates of parameters of the negative binomial distribution with probability function

$$f(x) = \frac{\Gamma(x + k)}{x!\, \Gamma(k)} p^k (1 - p)^x \; ; \; x = 0, 1, 2...$$

where $0 < p < 1$ and $k > 0$. Moment estimators, maximum likelihood estimators, and estimators based on moments and frequencies in selected classes are given both for the complete and for the truncated (with missing zero class) distribution. To facilitate calculation of the various estimators given, a table of the function $-p \ln p/(1 - p)$ with entries to six decimals at intervals of 0.001 for the argument p, is included. Illustrative examples are also included.

I. INTRODUCTION

The negative binomial distribution is used extensively for the description of data that are too heterogeneous to be fitted by a Poisson distribution. Since much of the data collected in studies of atmospheric phenomena exhibit marked heterogeneity, this distribution is of particular interest to aerospace scientists. It has been considered by numerous investigators, among whom are Greenwood and Yule [6], Fisher [5], Haldane [7], Anscombe [1], and Bliss and Fisher [2]. Samples from this distribution when zero observations are missing have been studied by David and Johnson [4], Sampford [10], Rider [9], Hartley [8], and by Brass [3]. This paper is primarily concerned with estimation in the truncated negative binomial distribution from which the zero observations are missing. Consideration is also given to estimation in this distribution when there is no truncation. Tables of the function, $-p \ln p/(1 - p)$, which are useful in calculating estimates in both cases, are included.

II. THE PROBABILITY FUNCTION AND ITS MOMENTS

The probability function of the negative binomial distribution may be written as

$$f(x) = \frac{\Gamma(x + k)}{x!\ \Gamma(k)} p^k (1 - p)^x; \quad x = 0, 1, 2 \ldots, \tag{1}$$

so that $f(0) = p^k$. The form in which this function was considered by Fisher [5] follows from (1) when we make the transformation $q = (1 - p)/p$. A form considered by Anscombe [1] follows upon making the transformation $m = k(1 - p)/p = kq$. For the purposes of this paper, the form given in (1) is considered preferable.

When the zero observations are removed, the probability function for the resulting truncated distribution becomes

$$f_T(x) = \frac{\Gamma(k + x)}{x!\ \Gamma(k)} \frac{p^k (1 - p)^x}{(1 - p^k)}; \quad x = 1, 2, 3 \ldots . \tag{2}$$

The factorial moments of the truncated distribution are

$$\mu_{[j]} = \frac{\Gamma(k + j)}{\Gamma(k)(1 - p^k)} \left(\frac{1 - p}{p}\right)^j . \tag{3}$$

From (3) and from (2) it follows that

$$\mu_1' = \mu = \left(\frac{k}{1 - p^k}\right) \left(\frac{1 - p}{p}\right),$$

$$\mu_{[2]} = \frac{k(k + 1)}{1 - p^k} \left(\frac{1 - p}{p}\right)^2 = \mu(k + 1)\left(\frac{1 - p}{p}\right), \tag{4}$$

$$f_T(1) = P = \frac{kp^{k+1}}{1 - p^k} \left(\frac{1 - p}{p}\right) = \mu p^{k+1}.$$

III. ESTIMATION IN THE TRUNCATED DISTRIBUTION

Since estimating equations which result from equating the first two sample moments to corresponding distribution moments do not lead to explicit solutions, David and Johnson [4] considered explicit estimators based on the first three sample moments, but found that they were quite inefficient. Sampford [10] subsequently developed a reasonably rapid iterative technique for solving the two-moment estimating equations, but ultimately concluded that the values thereby obtained could often serve only as first approximations for use in an iterative solution to the maximum likelihood estimating equations. Later Brass [3] derived explicit estimators based on the first two moments and the density of ones, which turned out to be reasonably efficient for most combinations of distribution parameters. The Brass estimators follow when the last equation of (4) is solved for p^k to obtain

$$p^k = P/\mu p, \tag{5}$$

and this value is substituted in turn into the first and the second equation of (4). On equating the sample mean and variance to the distribution mean and variance and the relative frequency of ones in the sample to $f_T(1) = P$, the Brass estimators become

$$p^* = \frac{\bar{x}}{s^2}\left(1 - \frac{n_1}{n}\right) \text{ and } k^* = \frac{p^*\bar{x} - n_1/n}{1 - p^*}, \tag{6}$$

where

$$\bar{x} = \sum_{x=1}^{R} x n_x / n,$$

$$(7)$$

$$s^2 = \sum_{x=1}^{R} (x - \bar{x})^2 n_x / (n - 1),$$

in which n_x is the number of sample observations for which the random variable $X = x$, n is the total number of sample observations and R is the largest sample observation.

Alternate estimators based on the first two moments and the density of ones follow when we take logarithms of the third equation of (4), solve for $(k + 1)$ to obtain

$$(k + 1) = \frac{\ln(P/\mu)}{\ln p}, \tag{8}$$

and subsequently substitute this value into the second equation of (4). When $\bar{x}$ and s^2 are equated to μ and μ_2, respectively, and n_1/n is equated to P, the resulting equations become

$$k^{**} = \frac{p^{**}}{1 - p^{**}} \left[\frac{s^2 + \bar{x}^2 - \bar{x}}{\bar{x}}\right] - 1. \tag{9}$$

Although the estimator given above for p is not in explicit form, linear inverse interpolation in the accompanying table of the function $-p \ln p/(1 - p)$ quickly yields the required estimate to as many as four decimal places.

The estimators given in (9) and the Brass estimators given in (6) utilize information provided by sample values $\bar{x}$, s^2 and n_1/n, but the precise manner in which this information is employed differs in the two cases. Actually, with only two parameters to be estimated, sufficient information is provided by $\bar{x}$ and n_1/n. As we demonstrate below, it is unnecessary to use the sample variance s^2.

When the expression for p^k given in equation (5) is substituted into the first equation of (4), we simplify to obtain

$$k = \frac{\mu p - P}{1 - p},$$

$$k + 1 = \frac{(\mu - 1)p + (1 - P)}{1 - p}. \tag{10}$$

On taking logarithms of the third equation of (4) and solving for $(k + 1)$, we have

$$(k + 1) = \frac{\ln(P/\mu)}{\ln p}. \tag{11}$$

On setting the right side of (4) equal to the right side of the second equation of (10) and letting $\mu = \bar{x}$ and $P = n_1/n$, we obtain the following estimating equation in p alone.

$$G(p) = \left[\bar{x} - 1 + \left(\frac{n - n_1}{np}\right)\right]\left[\frac{-p \ln p}{1 - p}\right] = \ln\left(\frac{n\bar{x}}{n_1}\right). \tag{12}$$

With the aid of the accompanying table of $-p \ln p/(1 - p)$, it is relatively simple to solve this equation for the required estimate p^{***} using linear interpolation as indicated below. We need only find two consecutive tabled values of p such that $\ln(n\bar{x}/n_1)$ is in the interval (G_i, G_{i+1}).

p	G(p)
p_i	$G(p_i)$
p^{***}	$\ln(n\bar{x}/n_1)$
p_{i+1}	$G(p_{i+1})$

With p^{***} thus determined, we employ the first equation of (10), with $\mu = \bar{x}$ and $P = n_1/n$ to calculate

$$k^{***} = \frac{p^{***}\bar{x} - n_1/n}{1 - p^{***}}. \qquad (13)$$

The form of k^{***} is the same as the Brass estimator given in (6), the only difference being that here p^{***} is a root of estimating equation (12), whereas the Brass estimator, p^*, is given by the first equation of (6). To distinguish between the different estimators considered here, a single asterisk denotes the Brass estimators, double asterisks denote the alternate estimators of equation (9) and triple asterisks denote estimators based on the sample mean and the sample proportion of ones.

Of the three estimators considered, those given by Brass enjoy the advantage of being easily calculated. The sampling properties of the estimators, based on the sample mean and the observed proportion of ones, require further investigation, but they are relatively easy to calculate with the aid of the accompanying table of $-p \ln p/(1 - p)$, and they might be expected to be asymptotically more efficient and perhaps less affected by bias than the other two estimators. Certainly, any of these estimators would be satisfactory as first approximations in an iterative solution of the maximum likelihood estimating equations.

Maximum Likelihood Estimation

The likelihood function for a random sample of size n from the truncated distribution is

$$L = \left[\frac{p^k}{1 - p^k}\right]^n \prod_{i=1}^{n} \frac{\Gamma(x_i + k)}{x_i!\,\Gamma(k)} (1 - p)^{x_i}. \qquad (14)$$

On taking logarithms of (14), differentiating with respect to p and k in turn, and equating to zero, we obtain the estimating equations

$$\frac{\partial \ln L}{\partial p} = \frac{nk}{p} + \frac{nkp^{k-1}}{1 - p^k} - \frac{n\bar{x}}{1 - p} = 0,$$

$$\frac{\partial \ln L}{\partial k} = n \ln p + \frac{np^k \ln p}{1 - p^k} + \sum_{x=1}^{R} n_x \sum_{j=1}^{x} (k + j - 1)^{-1} = 0. \tag{15}$$

Following Haldane [7] and Sampford [10], these equations may be more conveniently rewritten as

$$\frac{k(1 - p)}{p(1 - p^k)} = \bar{x},$$

$$\frac{-p \ln p}{1 - p} = \frac{k}{n\bar{x}} \sum_{x=1}^{R} (k + x - 1)^{-1} \sum_{i=x}^{R} n_i. \tag{16}$$

It is of interest that the first equation of (16) equates the distribution mean given by the first equation of (4) to the sample mean, $\bar{x}$. The usual maximum-likelihood iterative procedures can be employed to arrive at solutions, but by taking advantage of the table of $-p \ln p/(1 - p)$ and following a procedure of Sampford [10], the computational labor involved can be greatly reduced. In some instances, the computational labor can be still further reduced by modifying Sampford's procedure. Begin with an initial approximation $k_{(1)}$, which might be obtained using any of the estimators previously discussed. Evaluate the right side of the second equation of (16) and interpolate in the table of $-p \ln p/(1 - p)$ to obtain a first approximation $p_{(1)}$. Rewrite the first equation of (16) as

$$H(k,\ p) = \frac{k(1 - p)}{p(1 - p^k)} - \bar{x} = 0. \tag{17}$$

The problem of solution is now reduced to that of finding two values $k_{(i)}$ and $k_{(i+1)}$ in a sufficiently narrow interval with H_i and H_{i+1} of opposite signs. Once such values have been found, the required estimates follow by linear interpolation as indicated below.

k	p	H
k_i	p_i	H_i
$\hat{k}$	$\hat{p}$	0
k_{i+1}	p_{i+1}	H_{i+1}

The symbol (^) serves to designate estimators obtained by the principle of maximum likelihood.

IV. ESTIMATION IN THE COMPLETE DISTRIBUTION

Although numerous estimators for parameters of the negative binomial distribution have been proposed, we shall examine here only estimators based on (1) the first two moments, (2) the first moment and the proportion of zero readings, (3) the first moment and the proportion of ones, and (4) the method of maximum likelihood.

In the parametric form considered by Anscombe [1], which follows from equation (1) on setting $m = k(1 - p)/p$, the mean and second central moment of the complete negative binomial distribution are, respectively,

$$\mu_1' = m, \qquad \mu_2 = m(1 + \frac{m}{k}). \tag{18}$$

Moment Estimators.

The usual moment estimators obtained by equating sample moments to distribution moments then follow as

$$m^* = \bar{x}, \qquad k^* = \frac{\bar{x}^2}{s^2 - \bar{x}}. \tag{19}$$

It has been pointed out by Fisher [2] that $\bar{x}$ is a fully efficient estimator of m, but that the efficiency of k^* is somewhat low for some combinations of m and k. In general the efficiency of k^* is high for small values of the mean and large values of k. More precise statements concerning the efficiency of k^* are given on page 185 of the 1941 paper by Fisher [5], and on page 371 of Anscombe's paper [1].

Estimators Based on Mean and Proportion of Zeros.

Anscombe [1] found the efficiency of estimators based on the mean and the proportion of zeros to be reasonably high for appropriate combinations of m and k. The higher efficiencies occur with the smaller values of m and the smaller values of k. The estimating equation for k in this case is

$$(1 + \frac{\bar{x}}{k^{**}})^{-k^{**}} = \frac{n_o}{n}, \tag{20}$$

and the estimator for m is the sample mean as in the usual moment estimators, previously discussed. Bliss [2] writes this equation in the form

$$k^{**}\ln(1 + \bar{x}/k^{**}) = \ln(n/n_o), \tag{21}$$

and suggests solving for k^{**} by a trial-and-error procedure.

If we adhere to the parametric form of the negative binomial probability function given here in (1), the estimating equations in the case under consideration assume a form which permits a rapid and simple solution by linear inverse interpolation in the table of $-p \ln p/(1-p)$. With the sample mean and the sample proportion of zeros equated to corresponding distribution values, the estimating equations, when equation (1) is the density, are

$$\begin{aligned} n_o/n &= p^k, \\ \bar{x} &= k(1 - p)/p. \end{aligned} \tag{22}$$

On taking logarithms of the first equation of (12) and solving for k, we have

$$k = \frac{\ln(n_o/n)}{\ln p}. \tag{23}$$

On solving the second equation of (22) for k, we have

$$k = p\bar{x}/(1 - p). \tag{24}$$

Equate the right side of (23) to the right side of (24) and simplify to obtain

$$\frac{-p \ln p}{1 - p} = \frac{\ln(n/n_o)}{\bar{x}}. \tag{25}$$

In this form, it is a simple matter to evaluate the right side of (25), and read the estimate p^{**} from the accompanying table of $-p \ln p/(1 - p)$. With p^{**} thus determined, the corresponding estimate of k follows from (23) as

$$k^{**} = \frac{p^{**}\,\bar{x}}{1 - p^{**}}, \tag{26}$$

with considerable saving in labor over the computational procedures otherwise necessary.

<u>Estimators Based on Mean and Proportion of Ones</u>.

Estimators based on the mean and the proportion of ones seem likely to be preferred over estimators based on the mean and the proportion of zeros, when $n_1 > n_0$. The properties of these estimators are being investigated further, but on the basis of preliminary studies, their inclusion here seems warranted. In this case the estimating equations are

$$\begin{aligned} n_1/n &= kp^k(1 - p), \\ \bar{x} &= k(1 - p)/p. \end{aligned} \tag{27}$$

Divide the first of these equations by the second, and we have

$$n_1/n\bar{x} = p^{k+1}. \tag{28}$$

Take logarithms of (28) and solve for k = 1 to obtain

$$k + 1 = \frac{\ln(n_1/n\bar{x})}{\ln p}. \tag{29}$$

Solve the second equation of (27) for $(k + 1)$, and we find

$$k + 1 = \frac{p(\bar{x} - 1)}{1 - p} + \frac{1}{1 - p}. \tag{30}$$

On equating the right side of (30) to the right side of (29) and simplifying, we have

$$\frac{-p \ln p}{1 - p} \left(\bar{x} + \frac{1 - p}{p}\right) = \ln(n\bar{x}/n_1). \tag{31}$$

This equation is only slightly more difficult to solve than (25). With the aid of the table of $-p \ln p/(1 - p)$, it is quite easy to find consecutive values of p such that p^{***} is in the interval (p_i, p_{i+1}). Once p^{***} has been determined, k^{***} follows from (30) as

$$k^{***} = \frac{p^{***} \bar{x}}{1 - p^{***}}. \tag{32}$$

Note that k^{***} differs from k^{**} given in (26) only in the substitution of p^{***} for p^{**}.

Maximum Likelihood Estimators.

Although maximum likelihood estimation in the complete negative binomial distribution has been quite fully discussed by Fisher [5], Anscombe [1], Bliss [2] and others, applicable estimators are included here as a matter of convenience. With the estimating equations obtained by equating to zero, the partial derivatives of the logarithm of the likelihood function with respect to p and k are

$$\frac{\partial \ln L}{\partial p} = \frac{nk}{p} - \frac{n\bar{x}}{1 - p} = 0,$$

$$\frac{\partial \ln L}{\partial k} = n \ln p + \sum_{x=1}^{R} n_x \sum_{j=1}^{x} (k + j - 1)^{-1} = 0. \tag{33}$$

These equations reduce to

$$\frac{k(1 - p)}{p} = m = \bar{x},$$

$$\frac{-p \ln p}{1 - p} = \frac{k}{n\bar{x}} \sum_{x=1}^{R} (k + x - 1)^{-1} \sum_{i=x}^{R} n_i. \tag{34}$$

Equations (34) can be solved by standard iterative procedures, and here again the accompanying table of $-p \ln p/(1 - p)$ is useful. As was indicated in the truncated case, we might begin with a first approximation $k_{(1)}$ and employ the second equation and the table of $-p \ln p/(1 - p)$ to obtain a first approximation $p_{(1)}$. Write the first equation of (34) as

$$Q(k,p) = \frac{k(1 - p)}{p} - \bar{x} = 0, \tag{35}$$

and our problem is reduced to finding approximations $k_{(i)}$ and $k_{(i+1)}$ such that $Q_{(i)}$ and $Q_{(i+1)}$ are of opposite signs. Final estimates are then obtained by linear interpolation as indicated provided the interval between $k_{(i)}$ and $k_{(i+1)}$ is sufficiently narrow.

k	p	$Q(k,p) = \frac{k(1-p)}{p} - \bar{x}$
$k_{(i)}$	$p_{(i)}$	$Q_{(i)}$
$\hat{k}$	$\hat{p}$	0
$k_{(i+1)}$	$p_{(i+1)}$	$Q_{(i+1)}$

Any of the methods previously described will serve to provide a satisfactory first approximation $k_{(1)}$.

V. ILLUSTRATIVE EXAMPLES

Complete Negative Binomial Distribution. To illustrate the calculation of estimates in the complete negative binomial distribution, we consider a sample reported by P. Garman on the counts of red mites on apple leaves, which was previously examined by Bliss [2]. These data are given below.

No. mites per leaf	x	0	1	2	3	4	5	6	7	8+
No. leaves observed	n_x	70	38	17	10	9	3	2	1	0

For this sample, $n = 150$, $n_o = 70$, $n_1 = 38$, $\sum_{x=0}^{7} x\, n_x = 172$,

$\sum_{x=0}^{7} x^2 n_x = 536$, $\bar{x} = 1.14667$, and $s^2 = 2.27365$.

<u>Estimates based on the first two moments</u> follow from equations (19) as

$$m^* = 1.14667,$$

$$k^* = \frac{(1.14667)^2}{2.27365 - 1.14667} = 1.16670.$$

<u>Estimates based on the first moment and the proportion of zeros</u> follow equation (25), which for the data given here becomes

$$\frac{-p \ln p}{1 - p} = \frac{\ln(150/70)}{1.14667} = 0.6646\ 5509.$$

Interpolating from the accompanying table, we have

$$p^{**} = 0.46391,$$

and from equation (26)

$$k^{**} = \frac{.46391}{.53609}(1.14667) = 0.9923,$$

which agrees with the value more laboriously computed by Bliss without benefit of the table employed here.

Estimates based on the mean and the proportion of ones follow on solving equation (31), which for our sample becomes

$$J(p) = \frac{-p \ln p}{1 - p}\left[1.14667 + \frac{1 - p}{p}\right] = \ln\left(\frac{172}{38}\right) = 1.50990832.$$

With the aid of the accompanying table, it is quickly established that $.480 < p^{***} < .481$. The final estimate is determined by linear interpolation as shown below.

p	J(p)
.4810	1.50967381
.4808	1.50990832
.4800	1.51084732

Accordingly, $p^{***} = 0.4808$, and from equation (32)

$$k^{***} = \frac{0.4808}{0.5192}(1.14667) = 1.0619.$$

Maximum likelihood estimates can be calculated from equations (34) with the aid of the table of $-p \ln p/(1 - p)$ or by following the technique of Bliss and Fisher [2]. To four decimal places, both procedures lead to the estimate

$$\hat{k} = 1.0246.$$

Calculations based on equations (34) and the table of $-p \ln p/(1 - p)$ are sketched below. As a first approximation to k, we select $k^{**} = 0.9923$ (based on the mean and the proportion of zeros) which we round off for ease of calculation to $k_{(1)} = 1.0000$. Accumulating on a desk calculator, with k = 1, we obtain

$$\sum_{x=1}^{7} (x)^{-1} \sum_{i=x}^{7} n_i = 114.9261904.$$

On substituting this value into the second equation of (34) we calculate

$$\frac{-p \ln p}{1 - p} = \frac{1}{172} (114.9261904) = 0.66817552.$$

Interpolating in the table of $-p \ln p/(1 - p)$, we have as a first approximation to the required estimate of p,

$$p_{(1)} = 0.46829.$$

When these values for $k_{(1)}$ and $p_{(1)}$ are substituted into the first equation of (34), we have

$$\frac{1(1 - 0.46829)}{0.46829} = 1.13543 < \bar{x} = 1.14667.$$

Since both the moment estimate and the estimate based on the mean and the proportion of ones exceed our first approximation, it seems appropriate that our second approximation should be greater than the first. Accordingly, we select $k_{(2)} = 1.03$. Again accumulating on a desk calculator, but this time with $k = 1.03$, we calculate

$$\sum_{x=1}^{7} (x + 0.03)^{-1} \sum_{i=x}^{7} n_i = 112.1650693.$$

As with the previous approximation, this value is substituted into the right side of the second equation of (34) and on interpolating in the table of $-p \ln p/(1 - p)$, we find as the second approximation

$$p_{(2)} = 0.47267.$$

Our final estimate, $\hat{k} = 1.0246$, is arrived at from the first equation of (34) by interpolation as shown below.

k	$k(1 - p)/p$
1.0300	1.14911
$\hat{k} =$ 1.0246	1.14667 $= \bar{x}$
1.0000	1.13543

To the number of decimals given, the value obtained here for $\hat{k}$ is in agreement with that calculated (perhaps more laboriously) by Bliss.

Auxiliary tabulations involved in the above calculations are included in the following table.

x	n_x	$\sum_{i=x}^{7} n_i$	$(k + x - 1)^{-1}$	
			$k = 1$	$k = 1.03$
0	70	150		
1	38	80	1.00000000	0.97087378
2	17	42	.50000000	.49261083
3	10	25	.33333333	.33003300
4	9	15	.25000000	.24813895
5	3	6	.20000000	.19880715
6	2	3	.16666667	.16583747
7	1	1	.14285714	.14224751
$\sum_{x=1}^{7} (x + k - 1)^{-1} \sum_{i=x}^{7} n_i$			114.9261904	112.1650693

A summary of the various estimates for the particular sample under consideration is contained in the following table.

Parameters	Estimates			
	M.L.	Moments	Mean and Freq. of Zeros	Mean and Freq. of Ones
k	1.02459	1.16670	0.9923	1.0619
m	1.14667	1.14667	1.14667	1.14667
p	0.47189	0.50433	0.46391	0.4808
q	1.11915	0.98283	1.15557	1.07983

In this example, estimates based on the mean and the frequency of zeros are in closest agreement with the maximum likelihood estimates while the moment estimates differ by the greatest amount.

Truncated Negative Binomial Distribution. To illustrate estimation in the truncated negative binomial distribution, we consider a sample of chromosome breakage which was originally presented by Sampford [10]. Data for this sample follows.

No. Breaks	x	1	2	3	4	5	6	7	8	9	10	11	12	13
No. Observations	n_x	11	6	4	5	0	1	0	2	1	0	1	0	1

For these data, $n = 32$, $n_1 = 11$, $\sum_{x=1}^{13} x\, n_x = 110$, $\sum_{x=1}^{13} x^2 n_x = 686$,

$\bar{x} = 3.4375$, and $s^2 = 9.9315$.

Estimates based on the first two moments as computed by Sampford are

$$\bar{k} = 0.633 \text{ and } \bar{p} = 0.2346.$$

Brass estimates based on the first two moments and the proportion of ones follow from equations (6) as

$$p^* = \frac{3.4375}{9.9315}\left(1 - \frac{11}{32}\right) = 0.2345,$$

$$k^* = \frac{0.2345\ (3.4375) - (11/32)}{0.7655} = 0.6040.$$

Alternate estimators based on the first two moments and the proportion of ones follow from equations (9). For these data, the first equation of (9) becomes

$$\frac{-p \ln p}{1 - p} = \left[\frac{3.4375}{9.9315 + 3.4375^2 - 3.4375}\right] \ln\left(\frac{110}{11}\right) = 0.439814.$$

Inverse linear interpolation in our table yields the required estimate

$$p^{**} = 0.2307.$$

From the second equation of (9), we have

$$k^* = \frac{0.2307}{0.7693} [5.2363636] - 1 = 0.570.$$

Estimates based only on the mean and the proportion of ones follow from equations (12) and (13). For those data, equation (12) becomes

$$G(p) = \left[2.4375 + \frac{21}{32p}\right] \left[\frac{-p \ln p}{1 - p}\right] = \ln \left(\frac{110}{11}\right) = 2.30258509.$$

With the aid of the table of $-p \ln p/(1 - p)$, we quickly determine that the required estimate p^{***} is in the interval (0.202 to 0.203), and we interpolate for the final estimate as indicated.

p	G(p)
0.2030	2.30291889
0.2025	2.30258509
0.2020	2.30226898

With $p^{***} = 0.2025$, we substitute in equation (13) and compute

$$k^{***} = \frac{0.2025\ (3.4375)\ -\ 11/32}{.7975} = 0.4418.$$

Maximum likelihood estimates can be computed from equations (16) with the aid of the table of $-p \ln p/(1 - p)$ as described by Sampford [10] in much the same manner as maximum likelihood estimates were calculated in this paper for the complete negative binomial distribution. Alternately, the technique described by Hartley [8] might be used. In either case the final estimates for the sample under consideration are $\hat{p} = 0.2113$ and $\hat{k} = 0.493$.

A summary of the various estimates for the sample discussed is presented in the following table.

Parameters	Estimates				
	M.L.	Moments	Brass	Alternate	Mean and Prop. Ones
k	0.493	0.633	0.6040	0.570	0.4418
m	1.8402	2.0652	1.9717	1.9007	1.7399
p	0.2113	0.2346	0.2345	0.2307	0.2025
q	3.7326	3.2626	3.2644	3.3346	3.9383

Attention is invited to the close agreement exhibited here between estimates based on the mean and the proportion of ones with the maximum likelihood estimates in contrast with the rather wide discrepancies between the moment estimates and the maximum likelihood estimates.

VI. SOME REMARKS ON RELIABILITY OF ESTIMATES

Asymptotic variances of estimates in the complete negative binomial distribution have been given by Anscombe [1], and by Bliss and Fisher [2]. Similar results in the truncated case were given by Sampford [10] and by Hartley [8]. In the interest of completeness, these results are presented here without proof.

In the complete negative binomial distribution,

$$V(\bar{x}) \doteq \left(m + \frac{m^2}{k}\right)/n,$$

$$V(k^*) \doteq \frac{2k(k+1)}{n(1-p)^2}. \tag{36}$$

$$V(k^{**}) \doteq \frac{p^{-k} - 1 - k(1-p)}{n[-\ln p - (1-p)]^2}. \tag{37}$$

$$V(\hat{k}) \doteq \frac{2k(k+1)}{n(1-p)^2} \bigg/ \left\{1 + \frac{4(1-p)}{3(k+2)} + \frac{3(1-p)^2}{(k+2)(k+3)} + \ldots\right\} \tag{38}$$

The variances of k*** and p*** in the complete negative binomial distribution remain to be determined. In the preceding variances, a single asterisk (*) denotes a moment estimate, double asterisks (**) denote an estimate based on the mean and the proportion of ones, the circumflex (^) denotes maximum likelihood estimates, while triple asterisks (***) denote estimates based on the mean and the proportion of ones.

In the truncated negative binomial distribution, variances and covariances of the maximum likelihood estimates are obtained in the usual manner by inverting the information matrix in which the components are expected values of the quantities

$$\frac{-\partial^2 \ln L}{\partial p^2} = \frac{nk[1 - (k + 1)\, p^k]}{p^2(1 - p^k)^2} + \frac{n\bar{x}}{(1 - p)^2},$$

$$\frac{-\partial^2 \ln L}{\partial p\, \partial k} = -\frac{n[1 - (1 - k \ln p)\, p^k]}{p(1 - p^k)^2}, \tag{39}$$

$$\frac{-\partial^2 \ln L}{\partial k^2} = \sum_{x=1}^{R} (k + x - 1)^{-2} \sum_{i=x}^{R} n_i - \frac{n(\ln p)^2\, p^k}{(1 - p^k)^2}.$$

It is usually satisfactory to use these quantities themselves rather than their expected values.

Variances of the ordinary moment estimators (based on the first two moments) are given by Sampford [10], while Brass [3] gave variances of his estimators. In both of these cases the expressions obtained are rather complicated and are not repeated here.

For the distribution of red mites on apple leaves, considered in Section V, it follows from (36) that

$$V(\bar{x}) \doteq \frac{1.14667 + 1.28330}{150} = 0.01620,$$

$$V(k*) \doteq \frac{2(1.1667)\ (1.1667 + 1)}{150(0.49567)^2} = 0.1372.$$

Accordingly, $s_{\bar{x}} \doteq 0.1273$ and $s_{k*} \doteq 0.370$.

From equation (38) it follows that

$$V(\hat{k}) \doteq 0.07614, \text{ and } s_{\hat{k}} \doteq 0.276.$$

In the example of chromosome breakage employed to illustrate estimation in the truncated negative binomial distribution, Sampford [10] calculated moment estimate variances and covariances to be

$$V(p^*) \doteq 0.015091,\ Cov(p^*,\ k^*) \doteq 0.08125, \text{ and } V(k^*) \doteq 0.4983.$$

Corresponding values for maximum likelihood estimates were found to be

$$V(\hat{p}) \doteq 0.009863,\ Cov(\hat{p},\ \hat{k}) \doteq 0.04719, \text{ and } V(\hat{k}) \doteq 0.2763.$$

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THE FUNCTION $\phi(p) = -p \ln p/(1-p)$

p	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009	p
0.00	0.0000 0000	0.0069 1467	0.0124 5412	0.0174 7987	0.0221 7454	0.0266 2471	0.0308 8126	0.0349 7776	0.0389 3801	0.0427 7979	0.00
.01	.0465 1687	.0501 6022	.0537 1881	.0572 0008	.0606 1032	.0639 5490	.0672 3848	.0704 6512	.0736 3839	.0767 6148	.01
.02	.0798 3720	.0828 6812	.0858 5653	.0888 0451	.0917 1397	.0945 8665	.0974 2416	.1002 2795	.1029 9940	.1057 3978	.02
.03	.1084 5024	.1111 3190	.1137 8576	.1164 1280	.1190 1389	.1215 8990	.1241 4161	.1266 6976	.1291 7508	.1316 5822	.03
.04	.1341 1983	.1365 6049	.1389 8079	.1413 8127	.1437 6244	.1461 2479	.1484 6880	.1507 9492	.1531 0358	.1553 9518	.04
0.05	0.1576 7012	0.1599 2878	0.1621 7152	0.1643 9869	0.1666 1062	0.1688 0764	0.1709 9004	0.1731 5814	0.1753 1222	0.1774 5255	0.05
.06	.1795 7941	.1816 9304	.1837 9371	.1858 8164	.1879 5707	.1900 2024	.1920 7134	.1941 1061	.1961 3824	.1981 5442	.06
.07	.2001 5936	.2021 5323	.2041 3623	.2061 0851	.2080 7027	.2100 2166	.2119 6284	.2138 9398	.2158 1521	.2177 2670	.07
.08	.2196 2858	.2215 2100	.2234 0409	.2252 7799	.2271 4283	.2289 9873	.2308 4583	.2326 8423	.2345 1406	.2363 3544	.08
.09	.2331 4847	.2399 5326	.2417 4993	.2435 3858	.2453 1930	.2470 9221	.2488 5739	.2506 1495	.2523 6497	.2541 0756	.09
0.10	0.2558 4279	0.2575 7076	0.2592 9154	0.2610 0525	0.2627 1194	0.2644 1170	0.2661 0460	0.2677 9074	0.2694 7018	0.2711 4299	0.10
.11	.2728 0926	.2744 6905	.2761 2243	.2777 6947	.2794 1025	.2810 4482	.2826 7326	.2842 9560	.2859 1195	.2875 2234	.11
.12	.2891 2685	.2907 2552	.2923 1842	.2939 0561	.2954 8715	.2970 6308	.2986 3346	.3001 9835	.3017 5780	.3033 1186	.12
.13	.3048 6058	.3064 0402	.3079 4222	.3094 7522	.3110 0309	.3125 2586	.3140 4358	.3155 5630	.3170 6406	.3185 6691	.13
.14	.3200 6488	.3215 5803	.3230 4640	.3245 3001	.3260 0893	.3274 8318	.3289 5281	.3304 1786	.3318 7836	.3333 3436	.14
0.15	0.3347 8588	0.3362 3297	0.3376 7566	0.3391 1400	0.3405 4801	0.3419 7772	0.3434 0318	0.3448 2442	0.3462 4147	0.3476 5437	0.15
.16	.3490 6314	.3504 6783	.3518 6844	.3532 6503	.3546 5762	.3560 4625	.3574 3094	.3588 1172	.3601 8863	.3615 6168	.16
.17	.3629 3092	.3642 9636	.3656 5804	.3670 1598	.3683 7021	.3697 2076	.3710 6765	.3724 1091	.3737 5057	.3750 8665	.17
.18	.3764 1917	.3777 4816	.3790 7365	.3803 9565	.3817 1421	.3830 2932	.3843 4103	.3856 4936	.3869 5431	.3882 5593	.18
.19	.3895 5423	.3908 4924	.3921 4097	.3934 2945	.3947 1469	.3959 9673	.3972 7557	.3985 5125	.3998 2378	.4010 9318	.19
0.20	0.4023 5948	0.4036 2268	0.4048 8282	0.4061 3991	0.4073 9397	0.4086 4501	0.4098 9307	0.4111 3815	0.4123 8027	0.4136 1946	0.20
.21	.4148 5573	.4160 8910	.4173 1958	.4185 4720	.4197 7196	.4209 9390	.4222 1302	.4234 2934	.4246 4287	.4258 5365	.21
.22	.4270 6167	.4282 6696	.4294 6953	.4306 6940	.4318 6658	.4330 6109	.4342 5295	.4354 4217	.4366 2876	.4378 1274	.22
.23	.4389 9412	.4401 7292	.4413 4916	.4425 2284	.4436 9399	.4448 6261	.4460 2872	.4471 9233	.4483 5350	.4495 1212	.23
.24	.4506 6983	.4518 2208	.4529 7341	.4541 2233	.4552 6884	.4564 1296	.4575 5470	.4586 9407	.4598 3109	.4609 6577	.24
0.25	0.4620 9812	0.4632 2815	0.4643 5588	0.4654 8132	0.4666 0447	0.4677 2536	0.4688 4399	0.4699 6037	0.4710 7451	0.4721 8644	0.25
.26	.4732 9615	.4744 0366	.4755 0897	.4766 1211	.4777 1308	.4788 1290	.4799 0856	.4810 0309	.4820 9550	.4831 8578	.26
.27	.4842 7397	.4853 6006	.4864 4406	.4875 2599	.4886 0585	.4896 8365	.4907 5942	.4918 3314	.4929 0484	.4939 7453	.27
.28	.4950 4221	.4961 0789	.4971 7158	.4982 3329	.4992 9304	.5003 5082	.5014 0666	.5024 6055	.5035 1250	.5045 6254	.28
.29	.5056 1065	.5066 5686	.5077 0117	.5087 4360	.5097 8414	.5108 2280	.5118 5961	.5128 9456	.5139 2766	.5149 5892	.29
0.30	0.5159 8834	0.5170 1595	0.5180 4174	0.5190 6572	0.5200 8791	0.5211 0830	0.5221 2691	0.5231 4374	0.5241 5880	0.5251 7211	0.30
.31	.5261 8366	.5271 9346	.5282 0153	.5292 0787	.5302 1248	.5312 1537	.5322 1656	.5332 1605	.5342 1384	.5352 0994	.31
.32	.5362 0437	.5371 9712	.5381 8820	.5391 7763	.5401 6540	.5411 5153	.5421 3602	.5431 1887	.5441 0010	.5450 7971	.32
.33	.5460 5771	.5470 3410	.5480 0890	.5489 8210	.5499 5371	.5509 2374	.5518 9220	.5528 5910	.5538 2443	.5547 8820	.33
.34	.5557 5043	.5567 1112	.5576 7026	.5586 2788	.5595 8397	.5605 3855	.5614 9161	.5624 4316	.5633 9321	.5643 4177	.34
0.35	0.5652 8884	0.5662 3442	0.5671 7853	0.5681 2116	0.5690 6232	0.5700 0203	0.5709 4028	0.5718 7708	0.5728 1243	0.5737 4635	0.35
.36	.5746 7883	.5756 0988	.5765 3951	.5774 6772	.5783 9452	.5793 1991	.5802 4390	.5811 6649	.5820 8769	.5830 0751	.36
.37	.5839 2594	.5848 4299	.5857 5868	.5866 7300	.5875 8595	.5884 9755	.5894 0780	.5903 1670	.5912 2426	.5921 3048	.37
.38	.5930 3537	.5939 3893	.5948 4117	.5957 4209	.5966 4170	.5975 4000	.5984 3699	.5993 3269	.6002 2709	.6011 2019	.38
.39	.6020 1202	.6029 0256	.6037 9182	.6046 7981	.6055 6654	.6064 5200	.6073 3620	.6082 1914	.6091 0084	.6099 8128	.39
0.40	0.6108 6049	0.6117 3846	0.6126 1519	0.6134 9069	0.6143 6497	0.6152 3803	0.6161 0987	0.6169 8049	0.6178 4991	0.6187 1812	0.40
.41	.6195 8513	.6204 5095	.6213 1557	.6221 7900	.6230 4125	.6239 0232	.6247 6221	.6256 2092	.6264 7847	.6273 3485	.41
.42	.6281 9007	.6290 4413	.6298 9703	.6307 4879	.6315 9940	.6324 4886	.6332 9719	.6341 4438	.6349 9044	.6358 3537	.42
.43	.6366 7918	.6375 2186	.6383 6343	.6392 0388	.6400 4322	.6408 8146	.6417 1859	.6425 5462	.6433 8955	.6442 2340	.43
.44	.6450 5615	.6458 8781	.6467 1840	.6475 4790	.6483 7633	.6492 0368	.6500 2997	.6508 5519	.6516 7934	.6525 0244	.44
0.45	0.6533 2448	0.6541 4547	0.6549 6540	0.6557 8429	0.6566 0214	0.6574 1895	0.6582 3472	0.6590 4945	0.6598 6316	0.6606 7584	0.45
.46	.6614 8749	.6622 9812	.6631 0773	.6639 1633	.6647 2391	.6655 3049	.6663 3606	.6671 4062	.6679 4419	.6687 4676	.46
.47	.6695 4833	.6703 4891	.6711 4850	.6719 4711	.6727 4474	.6735 4138	.6743 3705	.6751 3174	.6759 2547	.6767 1822	.47
.48	.6775 1001	.6783 0083	.6790 9070	.6798 7961	.6806 6756	.6814 5456	.6822 4061	.6830 2572	.6838 0988	.6845 9310	.48
.49	.6853 7528	.6861 5673	.6869 3714	.6877 1663	.6884 9518	.6892 7281	.6900 4952	.6908 2508	.6916 0018	.6923 7414	.49
0.50	0.6931 4718	0.6939 1931	0.6946 9054	0.6954 6086	0.6962 3029	0.6969 9881	0.6977 6643	0.6985 3316	0.6992 9900	0.7000 6394	0.50

THE FUNCTION $\phi(p) = -p \ln p/(1 - p)$

p	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009	p
0.50	0.6931 4718	0.6939 1931	0.6946 9054	0.6954 6086	0.6962 3029	0.6969 9881	0.6977 6643	0.6985 3316	0.6992 9900	0.7000 6394	0.50
.51	.7008 2800	.7015 9118	.7023 5347	.7031 1489	.7038 7542	.7046 3508	.7053 9387	.7061 5179	.7069 0884	.7076 6502	.51
.52	.7084 2034	.7091 7480	.7099 2840	.7106 8114	.7114 3303	.7121 8407	.7129 3426	.7136 8360	.7144 3210	.7151 7975	.52
.53	.7159 2656	.7166 7254	.7174 1768	.7181 6198	.7189 0544	.7196 4810	.7203 8991	.7211 3090	.7218 7107	.7225 9873	.53
.54	.7233 4895	.7240 8666	.7248 2356	.7255 5964	.7262 9492	.7270 2938	.7277 6304	.7284 9590	.7292 2796	.7299 5921	.54
0.55	0.7306 8967	0.7314 1933	0.7321 4820	0.7328 7627	0.7336 0356	0.7343 3006	0.7350 5577	0.7357 8070	0.7365 0485	0.7372 2822	0.55
.56	.7379 5081	.7386 7263	.7393 9367	.7401 1394	.7408 3344	.7415 5217	.7422 7014	.7429 8734	.7437 0378	.7444 1946	.56
.57	.7451 3438	.7458 4854	.7465 6195	.7472 7461	.7479 8652	.7486 9768	.7494 0809	.7501 1775	.7508 2667	.7515 3485	.57
.58	.7522 4229	.7529 4899	.7536 5496	.7543 6019	.7550 6469	.7557 6845	.7564 7149	.7571 7380	.7578 7539	.7585 7625	.58
.59	.7592 7638	.7599 7580	.7606 7450	.7613 7249	.7620 6975	.7627 6631	.7634 6215	.7641 5728	.7648 5171	.7655 4542	.59
0.60	0.7662 3844	0.7669 3074	0.7676 2235	0.7683 1326	0.7690 0347	0.7696 9298	0.7703 8180	0.7710 6992	0.7717 5735	0.7724 4409	0.60
.61	.7731 3014	.7738 1551	.7745 0019	.7751 8419	.7758 6750	.7765 5013	.7772 3209	.7779 1337	.7785 9397	.7792 7389	.61
.62	.7799 5315	.7806 3173	.7813 0965	.7819 8689	.7826 6347	.7833 3938	.7840 1463	.7846 8922	.7853 6315	.7860 3642	.62
.63	.7867 0903	.7873 8098	.7880 5228	.7887 2293	.7893 9292	.7900 6227	.7907 3096	.7913 9901	.7920 6624	.7927 3318	.63
.64	.7933 9929	.7940 6477	.7947 2960	.7953 9380	.7960 5736	.7967 2028	.7973 8257	.7980 4423	.7987 0525	.7993 6565	.64
0.65	0.8000 2542	0.8006 8455	0.8013 4307	0.8020 0096	0.8026 5822	0.8033 1487	0.8039 7089	0.8046 2629	0.8052 8108	0.8059 3525	0.65
.66	.8065 8880	.8072 4174	.8078 9407	.8085 4579	.8091 9690	.8098 4740	.8104 9729	.8111 4658	.8117 9526	.8124 4334	.66
.67	.8130 9082	.8137 3769	.8143 8397	.8150 2965	.8156 7473	.8163 1922	.8169 6311	.8176 0642	.8182 4912	.8188 9124	.67
.68	.8195 3277	.8201 7371	.8208 1407	.8214 5384	.8220 9302	.8227 3163	.8233 6965	.8240 0709	.8246 4395	.8252 8023	.68
.69	.8259 1594	.8265 5107	.8271 8562	.8278 1961	.8284 5302	.8290 8586	.8297 1813	.8303 4983	.8309 8096	.8316 1153	.69
0.70	0.8322 4154	0.8328 7098	0.8334 9985	0.8341 2817	0.8347 5592	0.8353 8312	0.8360 0976	0.8366 3584	0.8372 6137	0.8378 8634	0.70
.71	.8385 1076	.8391 3462	.8397 5794	.8403 8070	.8410 0292	.8416 2458	.8422 4570	.8428 6628	.8434 8631	.8441 0580	.71
.72	.8447 2474	.8453 4315	.8459 6101	.8465 7834	.8471 9512	.8478 1137	.8484 2709	.8490 4227	.8496 5691	.8502 7102	.72
.73	.8508 8461	.8514 9766	.8521 1018	.8527 2217	.8533 3364	.8539 4458	.8545 5499	.8551 6488	.8557 7425	.8563 8309	.73
.74	.8569 9142	.8575 9922	.8582 0651	.8588 1327	.8594 1952	.8600 2526	.8606 3047	.8612 3518	.8618 3937	.8624 4305	.74
0.75	0.8630 4622	0.8636 4888	0.8642 5103	0.8648 5267	0.8654 5380	0.8660 5443	0.8666 5455	0.8672 5417	0.8678 5329	0.8684 5190	0.75
.76	.8690 5001	.8696 4762	.8702 4474	.8708 4135	.8714 3747	.8720 3309	.8726 2821	.8732 2284	.8738 1698	.8744 1062	.76
.77	.8750 0378	.8755 9644	.8761 8861	.8767 8029	.8773 7149	.8779 6219	.8785 5241	.8791 4215	.8797 3140	.8803 2017	.77
.78	.8809 0845	.8814 9626	.8820 8358	.8826 7043	.8832 5679	.8838 4268	.8844 2809	.8850 1302	.8855 9748	.8861 8146	.78
.79	.8867 6497	.8873 4801	.8879 3057	.8885 1266	.8890 9429	.8896 7544	.8902 5613	.8908 3635	.8914 1610	.8919 9538	.79
0.80	0.8925 7420	0.8931 5256	0.8937 3046	0.8943 0789	0.8948 8486	0.8954 6137	0.8960 3741	0.8966 1300	0.8971 8814	0.8977 6281	0.80
.81	.8983 3703	.8989 1079	.8994 8410	.9000 5695	.9006 2935	.9012 0130	.9017 7279	.9023 4384	.9029 1443	.9034 8458	.81
.82	.9040 5428	.9046 2353	.9051 9233	.9057 6069	.9063 2860	.9068 9607	.9074 6309	.9080 2967	.9085 9581	.9091 6151	.82
.83	.9097 2676	.9102 9158	.9108 5596	.9114 1990	.9119 8340	.9125 4647	.9131 0910	.9136 7130	.9142 3306	.9147 9439	.83
.84	.9153 5528	.9159 1575	.9164 7578	.9170 3538	.9175 9455	.9181 5329	.9187 1161	.9192 6950	.9198 2696	.9203 8399	.84
0.85	0.9209 4060	0.9214 9679	0.9220 5255	0.9226 0788	0.9231 6280	0.9237 1729	0.9242 7137	0.9248 2502	0.9253 7825	0.9259 3107	0.85
.86	.9264 8347	.9270 3545	.9275 8701	.9281 3816	.9286 8889	.9292 3921	.9297 8911	.9303 3860	.9308 8768	.9314 3635	.86
.87	.9319 8460	.9325 3245	.9330 7989	.9336 2692	.9341 7354	.9347 1975	.9352 6555	.9358 1095	.9363 5595	.9369 0054	.87
.88	.9374 4473	.9379 8851	.9385 3189	.9390 7487	.9396 1744	.9401 5962	.9407 0139	.9411 5643	.9417 8375	.9423 2433	.88
.89	.9428 6451	.9434 0430	.9439 4369	.9444 8269	.9450 2129	.9455 5949	.9460 9731	.9466 3473	.9471 7176	.9477 0839	.89
0.90	0.9482 4464	0.9487 8050	0.9493 1596	0.9498 5104	0.9503 8573	0.9509 2004	0.9514 5395	0.9519 8748	0.9525 2063	0.9530 5338	0.90
.91	.9535 8576	.9541 1775	.9546 4936	.9551 8058	.9557 1143	.9562 4189	.9567 7197	.9573 0167	.9578 3099	.9583 5994	.91
.92	.9588 8850	.9594 1669	.9599 4450	.9604 7194	.9609 9900	.9615 2568	.9620 5199	.9625 7792	.9631 0348	.9636 2867	.92
.93	.9641 5349	.9646 7794	.9652 0201	.9657 2572	.9662 4905	.9967 7201	.9672 9461	.9678 1684	.9683 3870	.9688 6020	.93
.94	.9693 8133	.9699 0209	.9704 2249	.9709 4252	.9714 6219	.9719 8149	.9725 0044	.9730 1902	.9735 3724	.9740 5175	.94
0.95	0.9745 7259	0.9750 8973	0.9756 0651	0.9761 2293	0.9766 3899	0.9771 5469	0.9776 7004	0.9781 8503	0.9786 9967	0.9792 1394	0.95
.96	.9797 2788	.9802 4144	.9807 5465	.9812 6752	.9817 8003	.9822 9219	.9829 0399	.9833 1544	.9838 2655	.9843 3730	.96
.97	.9848 4769	.9853 5777	.9858 6747	.9863 7683	.9868 8585	.9873 9451	.9879 0283	.9884 1081	.9889 1844	.9894 2572	.97
.98	.9899 3264	.9904 3925	.9909 4551	.9914 5142	.9919 5699	.9924 6222	.9929 6710	.9934 7165	.9939 7586	.9944 7973	.98
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By

A. Clifford Cohen, Jr.

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NASA TM X-53372, "Estimation in the Negative Binomial Distribution," by A. Clifford Cohen, Jr., Marshall Space Flight Center, December 21, 1965.

Enclosed is an erratum for TM X-53372. This equation (the first part of equation (9) on page 5) was omitted in the reproduction process. All recipients of this report should remove the backing from the enclosed equation and stick the equation on the top of page 5 just above the first equation appearing on that page.

$$\frac{-p^{**} \ln p^{**}}{1 - p^{**}} = \left[\frac{\bar{x}}{s^2 + \bar{x}^2 - \bar{x}}\right] \ln (n\bar{x}/n_1).$$