NASA TECHNICAL
MEMORANDUM

NASA TM X-53464

May 25, 1966

NASA TM X-53464

FACILITY FORM 602

N66 36325	
(ACCESSION NUMBER)	(THRU)
39	1
(PAGES)	(CODE)
TMX-53464	30
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

By Lyle R. Dickey
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

GPO PRICE $ ______

CFSTI PRICE(S) $ ______

Hard copy (HC) 2.00

Microfiche (MF) .50

ff 653 July 65

TECHNICAL MEMORANDUM X-53464

TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

By

Lyle R. Dickey

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

The problem of trajectory optimization is considered from the standpoint of numerical analysis. A numerical solution is obtained for an assumed thrust angle history, and an explicit numerical solution is obtained for the linearized equations of motion for neighboring solutions. With the explicit solution available, it is not difficult to determine whether or not the assumed solution is optimum. The first and second variations together provide a straightforward iteration method of approaching the optimum solution. It is of particular interest that the procedure remains unaltered even by the introduction of discontinuities and intermediate constraints.

NASA - GEORGE C. MARSHALL SPACE FLIGHT CENTER

NASA - GEORGE C. MARSHALL SPACE FLIGHT CENTER

Technical Memorandum X-53464

May 25, 1966

TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

By

Lyle R. Dickey

TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY

DEFINITION OF SYMBOLS

Symbol	Definition
$X =$	$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$
$\dot{X}$	time derivative of X
f_1, f_2, g_1, g_2, h_1, h_2, m_1, m_2	impulse response functions defined by equations (4) and (12)
r	radial distance
r_c	required radial distance at cutoff time
r_n	actual radial distance at cutoff time
t	time
t_c	cutoff time
v	velocity
v_c	velocity at cutoff time
x,y	Cartesian coordinates of position in a space fixed system

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\dot{x}, \dot{y}$	time derivatives of x and y, respectively
z	an arbitrary function of X and t
z_1	the value of z at time t_1
z_c	required value of z at t_1
Δr	deviation in cutoff radial distance
Δt	deviation in cutoff time
Δv	deviation in cutoff velocity
$\Delta\lambda_1$, $\Delta\lambda_2$, $\Delta\lambda_3$	corrections for λ_1, λ_2 and λ_3, respectively
$\Delta\chi$	variation in thrust angle
$\Delta\bar{\chi}$	correction to $\bar{\chi}$
λ_1, λ_2, λ_3	coefficients determined such that $h_1 + \lambda_1 f_1 + \lambda_2 g_1 + \lambda_3 m_1 = 0$
$\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$	first approximations to λ_1, λ_2 and λ_3, respectively
χ	thrust angle measured from the y-axis
$\bar{\chi}$	first approximation for χ

TECHNICAL MEMORANDUM X-53464

TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

SUMMARY

The problem of trajectory optimization is considered from the standpoint of numerical analysis. A numerical solution is obtained for an assumed thrust angle history, and an explicit numerical solution is obtained for the linearized equations of motion for neighboring solutions. With the explicit solution available, it is not difficult to determine whether or not the assumed solution is optimum. The first and second variations together provide a straightforward iteration method of approaching the optimum solution. It is of particular interest that the procedure remains unaltered even by the introduction of discontinuities and intermediate constraints.

OPTIMIZATION WITH END POINT CONSTRAINTS

The problem of trajectory optimization can be considerably simplified in concept if advantage is taken of available numerical methods from the beginning. To illustrate the point, the problem of minimizing the burning time required to meet desired end conditions will be considered. The differential equations describing the motion of the vehicle are defined in the following vector equation:

$$\dot{X} = F(X, \chi, t). \tag{1}$$

The problem is to minimize the time required to fulfill the following conditions at cutoff:

$$\begin{aligned} r(t_c) &= r_c \\ \theta(t_c) &= \theta_c \\ v(t_c) &= v_c \end{aligned} \tag{2}$$

For some function $\bar{X}(t)$ with a velocity cutoff constraint, the following numerical solution is obtained:

$$\left.\begin{aligned} r(t_c) &= r_n \\ \theta(t_c) &= \theta_n \\ v(t_c) &= v_c \end{aligned}\right\} . \tag{3}$$

The linearized equations of motion can be solved explicitly for the same initial conditions and cutoff conditions to give a solution of the following form*:

$$\left.\begin{aligned} \Delta r &= r_n - r_c + \int_{t_o}^{t_n} \left[f_1 \Delta X + f_2 \Delta X^2 + \ldots \right] dt \\ \Delta \theta &= \theta_n - \theta_c + \int_{t_o}^{t_n} \left[g_1 \Delta X + g_2 \Delta X^2 + \ldots \right] dt \\ \Delta t &= \int_{t_o}^{t_n} \left[h_1 \Delta X + h_2 \Delta X^2 + \ldots \right] dt \end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned} \Delta r &= r - r_c \\ \Delta \theta &= \theta - \theta_c \\ \Delta t &= t - t_n \\ \Delta X &= X(t) - \bar{X}(t) \end{aligned}\right\} . \tag{5}$$

*The derivation of these equations is described in detail in Reference 2.

The above equations can be combined in the following expression:

$$\Delta t + \lambda_1 \Delta r + \lambda_2 \Delta\theta = \int_{t_o}^{t_n} \Big[(h_1 + \lambda_1 f_1 + \lambda_2 g_1)\ \Delta X$$

$$+ (h_2 + \lambda_1 f_2 + \lambda_2 g_2)\ \Delta X^2 + \dots \Big] dt + \lambda_1 (r_n - r_c)$$

$$+ \lambda_2 (\theta_n - \theta_c).$$

This expression is general at this point and imposes no constraint on r, θ or $\Delta X(t)$. Setting $r = r_c$ and $\theta = \theta_c$ gives the following expression for Δt:

$$\Delta t = \int_{t_o}^{t_n} \Big[(h_1 + \lambda_1 f_1 + \lambda_2 g_1)\ \Delta X + (h_2 + \lambda_1 f_2 + \lambda_2 g_2)\ \Delta X^2 + \dots \Big] dt$$

$$+ \lambda_1 (r_n - r_c) + \lambda_2 (\theta_n - \theta_c). \tag{6}$$

If $h_1 + \lambda_1 f_1 + \lambda_2 g_1 = 0$ and $(h_2 + \lambda_1 f_2 + \lambda_2 g_2) > 0$, any change in the function $\Delta X(t)$ over any interval will increase the value of Δt in a neighborhood. If this condition exists, the function $\bar{X}(t)$ yields a minimum value of t locally. If $h_1 + \lambda_1 f_1 + \lambda_2 g_1 \neq 0$, consider $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that

$$h_1 + \bar{\lambda}_1 f_1 + \bar{\lambda}_2 g_1 = \epsilon.$$

The values of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ can be obtained by the method of least squares or any other comparable method. Setting $\lambda_1 = \bar{\lambda}_1 + \Delta\lambda_1$, $\lambda_2 = \bar{\lambda}_2 + \Delta\lambda_2$ and $\Delta X = \Delta\bar{X} + \delta X$ gives the following expression for Δt.

$$\Delta t = \int_{t_o}^{t_n} \left\{ \left[\epsilon(t) + \Delta\lambda_1 f_1 + \Delta\lambda_2 g_1 + 2(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)\ \Delta\bar{X} \right] \delta X + \ldots \right\} dt$$

$$+ \lambda_1(r_n - r_c) + \lambda_2(\theta_n - \theta_c). \tag{7}$$

Setting the weighting function for δX equal to zero gives the following equation:

$$2(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2)\ \Delta\bar{X} = -\epsilon(t) - f_1\Delta\lambda_1 - g_1\Delta\lambda_2. \tag{8}$$

For simplification, the following definitions are employed:

$$\left.\begin{aligned} h_2^* &= (h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2) \\ \bar{\epsilon}(t) &= + \frac{\epsilon(t)}{2h_2^*} \\ \bar{f}_1 &= - \frac{f_1}{2h_2^*} \\ \bar{g}_1 &= - \frac{g_1}{2h_2^*} \end{aligned}\right\} . \tag{9}$$

The expression for $\Delta\bar{X}$ is the following:

$$\Delta\bar{X} = \bar{f}_1\Delta\lambda_1 + \bar{g}_1\Delta\lambda_2 - \bar{\epsilon}. \tag{10}$$

The terms $\Delta\lambda_1$ and $\Delta\lambda_2$ can be determined by setting $\Delta r = 0$ and $\Delta\theta = 0$ in equations (4) and keeping only linear terms. This results in the following system of equations:

$$\left.\begin{aligned} \Delta\lambda_1 \int_{t_o}^{t_n} f_1\bar{f}_1 \, dt + \Delta\lambda_2 \int_{t_o}^{t_n} f_1\bar{g}_1 \, dt &= \int_{t_o}^{t_n} f_1\bar{\epsilon} \, dt + r_c - r_n \\ \Delta\lambda_1 \int_{t_o}^{t_n} g_1\bar{f}_1 \, dt + \Delta\lambda_2 \int_{t_o}^{t_n} g_1\bar{g}_1 \, dt &= \int_{t_o}^{t_n} g_1\bar{\epsilon} \, dt + \theta_c - \theta_n \end{aligned}\right\} . \tag{11}$$

Solving this system for $\Delta\lambda_1$ and $\Delta\lambda_2$ provides the necessary quantities to determine ΔX from equation (10) at each time point along the trajectory. With the new function $X_1 = \bar{X} + \Delta X$, another iteration can be performed to determine X_2. This can be continued until $r_n - r_c$, $\theta_n - \theta_c$ and $\epsilon(t)$ all meet the required tolerances. Convergence of this iteration process ensures that the first order weighting function in equation (6) is zero. Similar operations can be applied to the second order term in the event that it is not everywhere positive.

INTERMEDIATE CONSTRAINTS

Assume that at some other time, t_1, which may or may not be a function of X, an additional constraint is imposed requiring that some other function $z(t_1) = z_c$ and that for $\bar{X}(t)$, $z(t_1) = z_1$. The expression for Δz can be written as

$$\Delta z = z_1 - z_c + \int_{t_o}^{t_1} \left[m_1 \Delta X + m_2 \Delta X^2 + \ldots \right] dt. \tag{12}$$

If $t_1 > t_n$, $m_1 = 0$ and $m_2 = 0$ for $t_n < t \leqq t_1$. Equation (12) can then be written as

$$\Delta z = z_1 - z_c + \int_{t_o}^{t_n} \left[m_1 \Delta X + m_2 \Delta X^2 + \ldots \right] dt. \tag{13}$$

If $t_o < t_1 \leqq t_n$, define $m_1 = m_2 = 0$ for $t_1 < t \leqq t_n$. Then Δz can also be represented by equation (13). This equation is identical in form to expressions for Δr and $\Delta\theta$ appearing in equations (4). The same analysis used before can be applied to the constraints that

$$r(t_c) = r_c$$

$$\theta(t_c) = \theta_c$$

$$v(t_c) = v_c$$

$$z(t_1) = z_c.$$

The resulting expression is obtained for $\Delta\bar{X}$:

$$\Delta\bar{X} = \bar{f}_1\Delta\lambda_1 + \bar{g}_1\Delta\lambda_2 + \bar{m}_1\Delta\lambda_3 + \bar{\epsilon} \tag{14}$$

where

$$\left.\begin{aligned} \bar{f}_1 &= -\frac{f_1}{2h_2^*}, \qquad h_2^* = h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2 + \bar{\lambda}_3 m_2 \\ \bar{g}_1 &= -\frac{g_1}{2h_2^*} \\ \bar{m}_1 &= -\frac{m_1}{2h_2^*} \\ \bar{\epsilon} &= \frac{\epsilon}{2h_2^*} \end{aligned}\right\} \tag{15}$$

and $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ have been determined as before. The values $\Delta\lambda_1$, $\Delta\lambda_2$ and $\Delta\lambda_3$ are solutions to the following system of equations:

$$\Delta\lambda_1 \int_{t_o}^{t_n} f_1\bar{f}_1\,dt + \Delta\lambda_2 \int_{t_o}^{t_n} f_1\bar{g}_1\,dt + \Delta\lambda_3 \int_{t_o}^{t_n} f_1\bar{m}_1\,dt = \int_{t_o}^{t_n} f_1\bar{\epsilon}\,dt + r_c - r_n$$

$$\Delta\lambda_1 \int_{t_o}^{t_n} g_1\bar{f}_1\, dt + \Delta\lambda_2 \int_{t_o}^{t_n} g_1\bar{g}_1\, dt + \Delta\lambda_3 \int_{t_o}^{t_n} g_1\bar{m}_1\, dt = \int_{t_o}^{t_n} g_1\bar{\epsilon}\, dt + \theta_c - \theta_n$$

$$\Delta\lambda_1 \int_{t_o}^{t_n} m_1\bar{f}_1\, dt + \Delta\lambda_2 \int_{t_o}^{t_n} m_1\bar{g}_1\, dt + \Delta\lambda_3 \int_{t_o}^{t_n} m_1\bar{m}_1\, dt = \int_{t_o}^{t_n} m_1\bar{\epsilon}\, dt + z_c - z_1.$$

The fact that the form of this system remains the same whether the constraints are applied to the end conditions or at some other point along the trajectory makes this method readily adaptable to computer operations and an extremely versatile multistage optimization program could easily be developed with a wide choice of constraints imposed at arbitrary points along the trajectory.

APPENDIX

An example to illustrate the method has been completed* from the following initial conditions:

$$x_o = 236.25546 \text{ km} \qquad \dot{x}_o = 2800.5746 \text{ m/sec}$$

$$y_o = 6456.66782 \text{ km} \qquad \dot{y}_o = 635.94317 \text{ m/sec}$$

$$t_o = 184.99481 \text{ sec.}$$

The motion is assumed to be described by the following two-dimensional system of differential equations:

$$\ddot{x} = \frac{F}{m} \sin \chi + \ddot{x}_g,$$

$$\ddot{y} = \frac{F}{m} \cos \chi + \ddot{y}_g,$$

$$\ddot{x}_g = \frac{x}{r} g, \quad \ddot{y} = \frac{y}{r} g,$$

$$g = -\frac{g_o r_o^2}{r^2}, \quad g_o = 9.81 \text{ m/sec}^2, \quad r_o = 6370 \text{ km}$$

$$\frac{F}{m} = \frac{10.787315}{1.2867664 - .261282644\tau_1}, \quad \tau_1 = \frac{t - 184.99481}{100}$$

$$184.99481 \leqq t < 412.85503$$

$$\frac{F}{m} = \frac{8.92405155}{.69140715 - .212616813\tau_2}, \quad \tau_2 = \frac{t - 412.85503}{100}$$

$$412.85503 \leqq t < 523.87973$$

$$\frac{F}{m} = 0, \quad 523.87973 \leqq t < 528.87973.$$

---

*The computer program used to solve this example was developed in cooperation with James Hilliard, R-COMP, MSFC, and is described in Reference 4.

$$\frac{F}{m} = \frac{1.96132994}{.36021199 - .046948359\tau_3}, \quad \tau_3 = \frac{t - 428.87973}{100}$$

$$t \geqq 428.87973.$$

F/m is in m/sec^2 and t is in seconds.

The problem is to determine the function χ which will minimize the burning time required to reach the following conditions:

$$r = r_c = 6555.2 \text{ km}$$

$$v = v_c = 7792.5746 \text{ m/sec}$$

$$\theta = \theta_c = 90°.$$

In addition to the above constraints, an intermediate constraint has been included which requires that the spent stage which separates at t = 412.85503 seconds must, moving only under the influence of gravity, pass through the prespecified point (x_1, y_1) where

$$x_1 = 2588.9871 \text{ km}$$

$$y_1 = 5971.13760 \text{ km}.$$

The following function was initially assumed for χ:

$$\chi = 75.111 + 1.1982 \left(\frac{t - t_o}{100}\right),$$

where χ is in degrees and t in seconds.

The equations of motion were solved numerically by fourth order Runge Kutta at five-second time steps until the stage dropping off at t = 412.85503 seconds fulfilled the condition that $x = x_1$ and the upper stage fulfilled the condition that $v = v_c$. The weighting functions, f_1, g_1, m_1 and k_1, were computed numerically by the method outlined in Chapter I of Reference 2. Corrections to the function χ were determined from equations (14) and (15), and another iteration was begun. After the seventh iteration, the following conditions were fulfilled:

$r(t_c) = r_c \pm 0.1m$

$v(t_c) = v_c \pm 0.01$ m/sec

$\theta(t_c) = \theta_c \pm 10^{-5}$ radians

$x(t_1) = x_1 \pm 0.1m$

$y(t_1) = y_1 \pm 0.1m$

$\epsilon(t_i) = 0 \pm 10^{-6}$/radian.

Figure 1 shows the resulting trajectory compared with the solution that resulted when the intermediate constraint was removed. The total burning time with the additional constraint required 768.60 seconds compared with 690.24 seconds without this constraint. The additional 78.36 seconds indicates that this is a rather expensive maneuver but illustrates the versatility of the method.

The guidance angle, X, is shown in Figure 2, where it is compared with the function used for the first iteration. A complete tabulation of X at every point for which it was computed is included in Table 1 at the end of the appendix. In performing the numerical integration, this table was used with second order interpolation employed to determine the midpoint value required by the Runge-Kutta method.

Figure 3 illustrates the first order weighting functions f_1, g_1, m_1, and f_1. These functions are tabulated for several time points in Table 1. For simplicity they have been rounded off to the fourth decimal place. It can be readily verified that they satisfy the equation $h_1 + \lambda_1 f_1 + \lambda_2 g_1 + \lambda_3 m_1 = 0$ within a tolerance of less than one unit in the last significant figure for the following values of λ_1, λ_2 and λ_3.

$\lambda_1 = .299026$ sec/km

$\lambda_2 = .702527\ 10^3$ sec/radian

$\lambda_3 = -.633224$ sec/km.

Figure 4 shows the functions $\bar{f}_1$, $\bar{g}_1$ and $\bar{h}_1$ which were used to determine the corrections to X. The second order weighting function, h_2, is also shown in this illustration, and it can be seen to be positive.

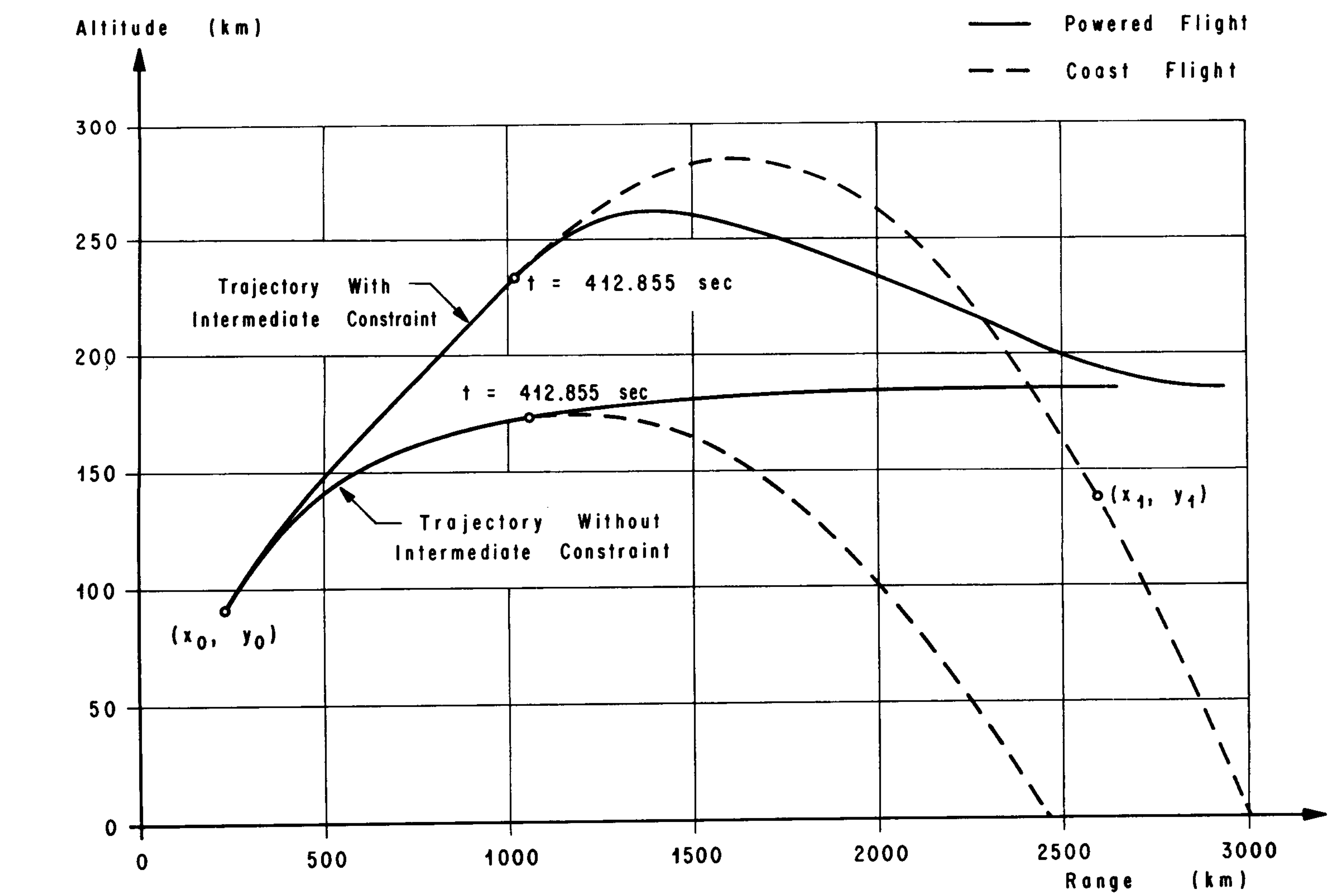

FIG. 1. OPTIMUM TRAJECTORIES WITH & WITHOUT INTERMEDIATE CONSTRAINTS

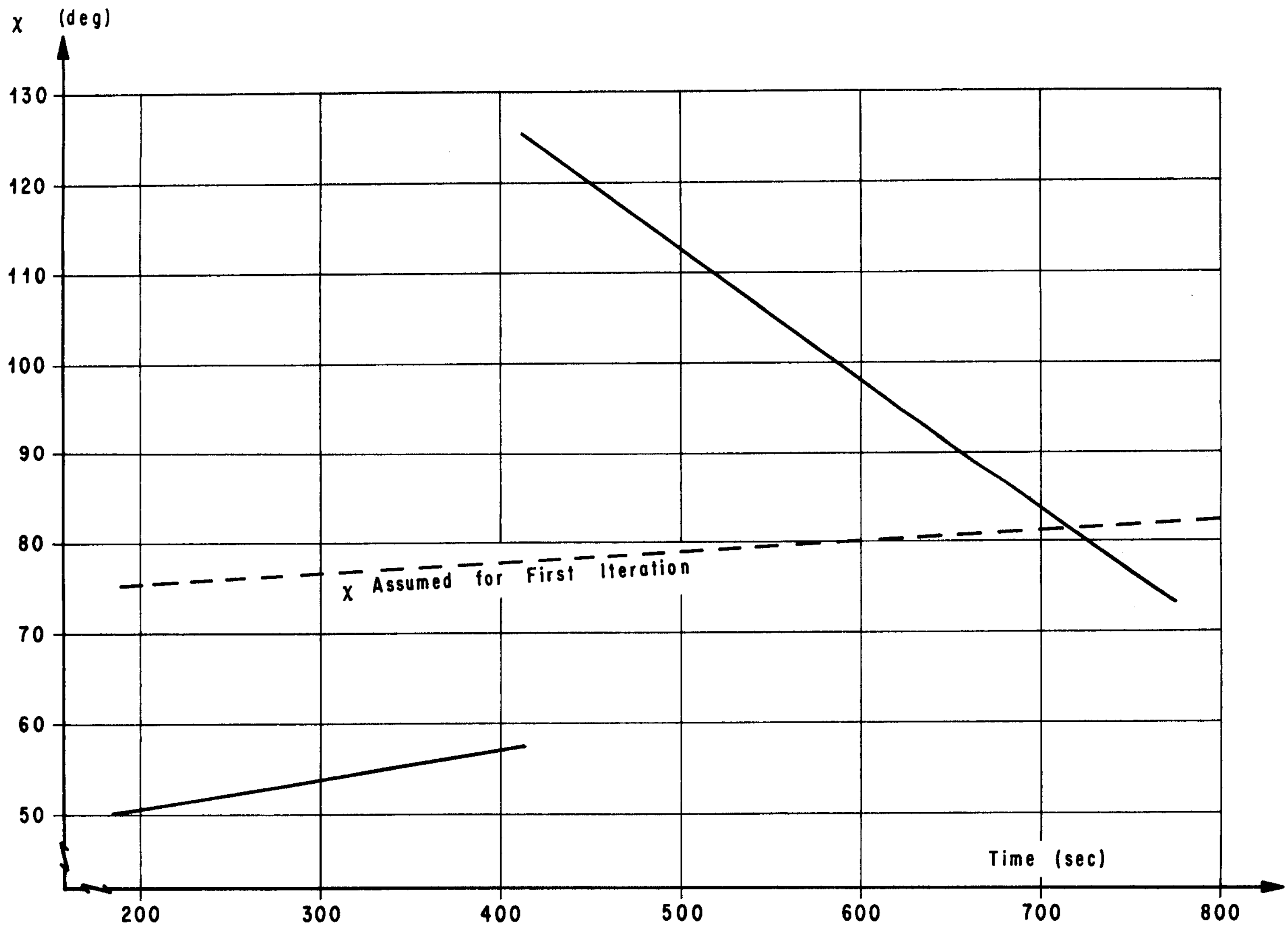

FIG. 2. χ FOR THE INTERMEDIATE CONSTRAINT TRAJECTORY

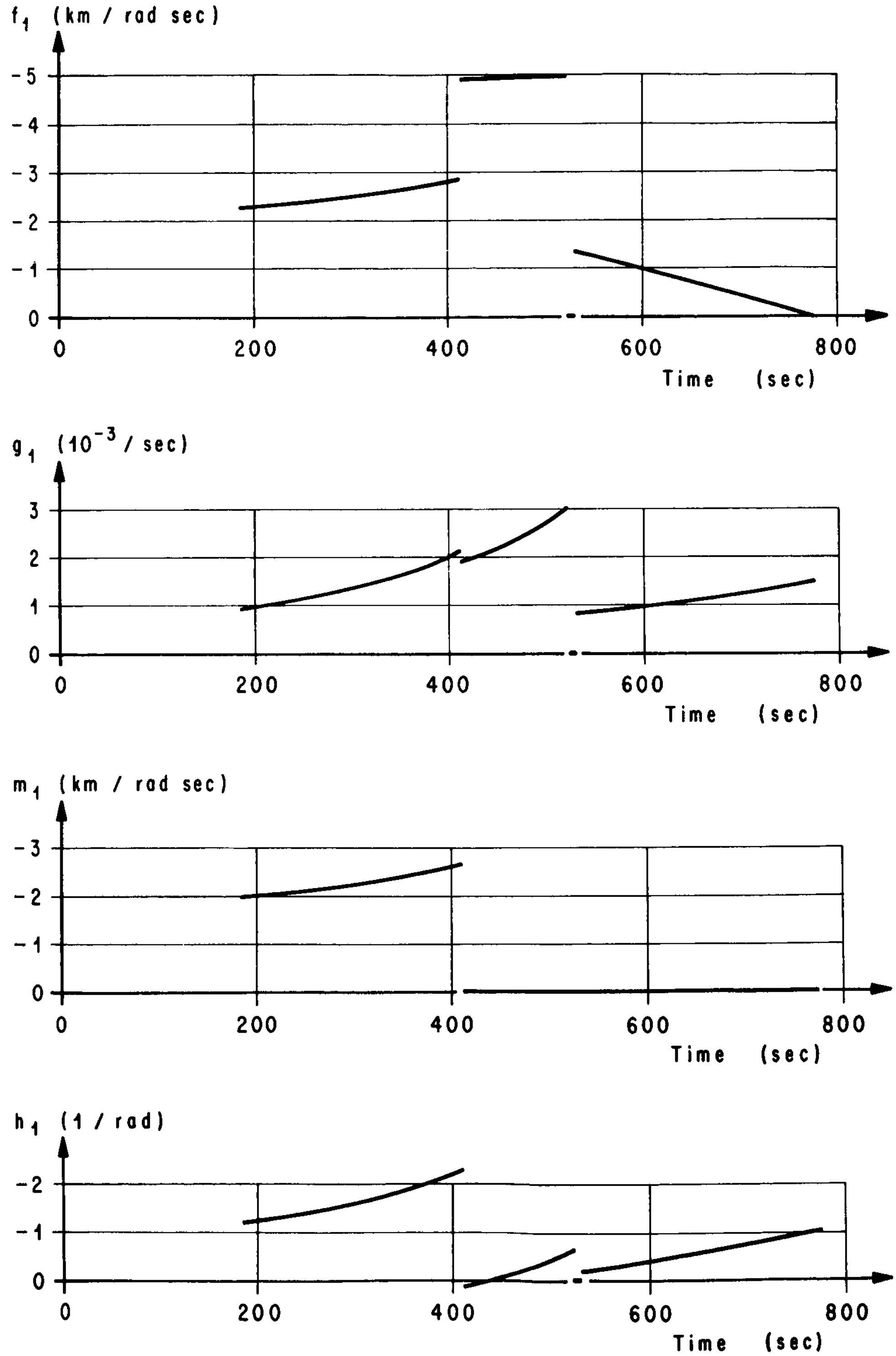

FIG. 3. FIRST ORDER WEIGHTING FUNCTIONS

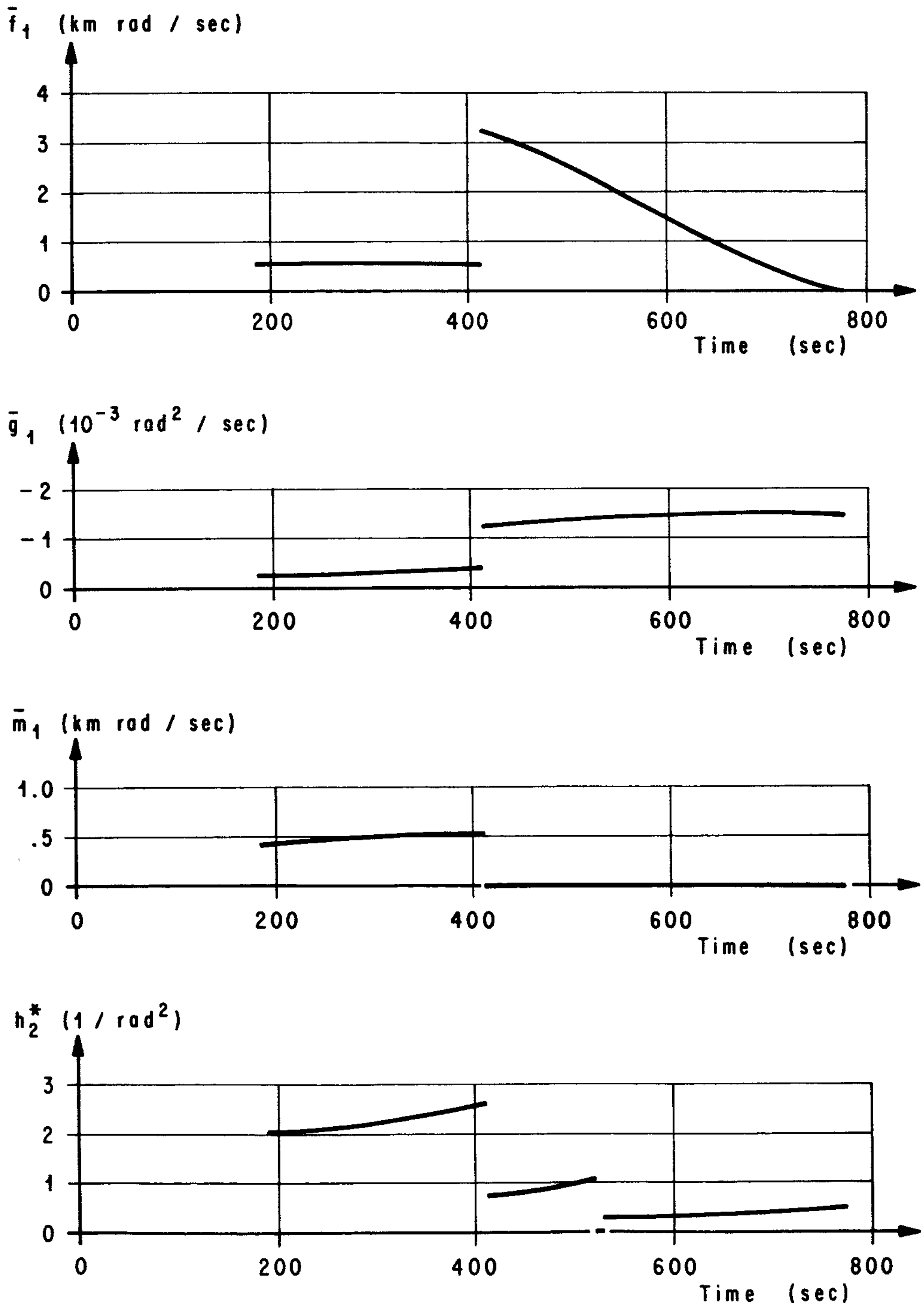

FIG. 4. FUNCTIONS USED TO DETERMINE $\Delta\chi$ AND THE SECOND ORDER WEIGHTING FUNCTION h_2^*

TABLE 1

t(sec)	f_1[km/rad sec]	$g_1[10^{-3}/sec]$	m_1[km/rad sec]	h_1[1/rad]
185	-2.2642	.9238	-1.9664	-1.2171
200	-2.2895	.9679	-1.9963	-1.2594
250	-2.3827	1.1380	-2.1054	-1.4201
300	-2.4932	1.3542	-2.2331	-1.6199
350	-2.6290	1.6382	-2.3880	-1.8769
400	-2.8032	2.0267	-2.5837	-2.2217
412.9	-2.8568	2.1507	-2.6433	-2.3305
412.9	-4.8807	1.8953	0	.1280
450	-4.9237	2.1708	0	- .0527
500	-4.9454	2.6950	0	- .4146
523.9	-4.9455	3.0395	0	- .6565
523.9	0	0	0	0
528.9	0	0	0	0
528.9	-1.3418	.8472	0	- .1939
550	-1.2401	.8836	0	- .2499
600	- .9809	.9823	0	- .3968
650	- .7032	1.1000	0	- .5625
700	- .4169	1.2384	0	- .7454
750	- .1317	1.3997	0	- .9440
773.6	.0000	1.4886	0	-1.0458

λ_1 = .299026 sec/km

λ_2 = .702527 10^3 sec/rad

λ_3 = -.633224 sec/km

TABLE 2

t(sec)	χ(degrees)	t(sec)	χ(degrees)
184.99481	50.0558152	325	54.7621191
185	50.0559990	330	54.9239170
190	50.2312133	335	55.0854054
195	50.4058228	340	55.2465994
200	50.5798353	345	55.4075148
205	50.7532584	350	55.5681676
210	50.9261002	355	55.7285747
215	51.0983688	360	55.8887534
220	51.2700725	365	56.0487214
225	51.4412196	370	56.2084973
230	51.6118189	375	56.3681001
235	51.7818790	380	56.5275498
240	51.9514090	385	56.6868666
245	52.1204181	390	56.8460719
250	52.2889156	395	57.0051875
255	52.4569111	400	57.1642363
260	52.6244145	405	57.3232420
265	52.7914357	410	57.4822289
270	52.9579851	412.855030	57.5733702
275	53.1240733	412.855030	125.234677
280	53.2897111	415	124.935432
285	53.4549096	420	124.236574
290	53.6196802	425	123.535040
295	53.7840346	430	122.830917
300	53.9479848	435	122.124289
305	54.1115432	440	121.415244
310	54.2747226	445	120.703869
315	54.4375359	450	119.990252
320	54.5999968	455	119.274482

TABLE 2 (Continued)

t(sec)	X(degrees)	t(sec)	X(degrees)
460	118.556646	585	100.202874
465	117.836835	590	99.4630176
470	117.115138	595	98.7235515
475	116.391644	600	97.9845716
480	115.666444	605	97.2461745
485	114.939627	610	96.5084576
490	114.211284	615	95.7715192
495	113.481505	620	95.0354585
500	112.750381	625	94.3003754
505	112.018000	630	93.5663709
510	111.284453	635	92.8335468
515	110.549831	640	92.1020062
520	109.814223	645	91.3718527
523.879730	109.242771	650	90.6431911
523.879730	109.242771	655	89.9161274
526.379730	108.874254	660	89.1907682
528.879730	108.505553	665	88.4672214
528.879730	108.505553	670	87.7455956
530	108.340267	675	87.0260004
535	107.602271	680	86.3085466
540	106.863648	685	85.5933453
545	106.124489	690	84.8805090
550	105.384885	695	84.1701506
555	104.644926	700	83.4623837
560	103.904703	705	82.7573229
565	103.164310	710	82.0550831
570	102.423837	715	81.3557796
575	101.683378	720	80.6595283
580	100.943026		

TABLE 2 (concluded)

t(sec)	χ(degrees)
725	79.9664454
730	79.2766473
735	78.5902504
740	77.9073712
745	77.2281261
750	76.5526312
755	75.8810022
760	75.2133544
765	74.5498023
770	73.8904599
773.599673	73.4188071

REFERENCES

1. Bellman, Richard, "Introduction to Matrix Analysis," Chapter 10, McGraw-Hill, New York, 1960.

2. Dickey, Lyle R., "Guidance Applications of Linear Analysis," NASA TM X-53166, MSFC, November 27, 1964.

3. Greenspan, Donald, "On Approximating Extremals for Boundary Value Problems," MRC Technical Summary Report #466, The University of Wisconsin, Madison Wisconsin, March 1964.

4. Hilliard, James, "Computer Program: Trajectory Optimization by Explicit Numerical Methods," to be published as a NASA TM X.

APPROVAL NASA TM X-53464

TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

By Lyle R. Dickey

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

E. D. Geissler
Director, Aero-Astrodynamics Laboratory

DISTRIBUTION

R-DIR
DEP-T

I-I-IB-DIR
Col. James

I-MO-MGR
Dr. Speer

I-MO-O
Mr. Kurtz

R-RP-U
Roy Bland

MS-IP
MS-IPL (8)
MS-H
CC-P
MS-T (6)
R-P&VE
Mr. Swanson, -J
Mr. Goerner, -A

R-ASTR
Mr. Moore
Mr. Richard
Mr. Gassaway
Mr. Taylor
Mr. Brooks
Mr. Hosenthien
Mr. Scofield
Mr. Woods
Mr. Digesu
Mr. R. Hill
Mrs. Neighbors
Dr. R. Decher

R-COMP
Dr. Arenstorf
Mr. Davidson
Mr. Harton
Mr. Scollard
Mr. Seely
Mr. Calhoun

R-AERO
Dr. Geissler
Mr. Jean
Mr. Dearman
Mr. Schwaniger
Mr. Ingram
Dr. H. Krause
Mr. Rheinfurth
Mr. Thomae
Mr. Baker
Mr. Burns
Mr. W. Vaughan
Mr. Hart
Mr. Lovingood
Mrs. Chandler
Mr. Lisle
Mr. Teague
Mr. Cummings
Dr. Sperling
Dr. Heybey
Mr. Mabry
Mr. Peasley
Mr. O. E. Smith
Mr. Dickey (20)

EXTERNAL DISTRIBUTION

Mr. Hans K. Hinz
Research Dept.
Grumman Aircraft Eng. Corp.
Bethpage, Long Island, N. Y.

Mr. J. S. Farrior
Lockheed
P. O. Box 1103
West Sta.
Huntsville, Alabama

Dr. George Nomicos
Applied Mathematics Sec.
Applied Research & Development
Republic Aviation Corp.
Farmingdale, Long Island, N. Y.

Mr. Theodore E. Edelbaum
Analytical Mechanics Assoc. Inc.
872 Mass. Ave.
Cambridge, Mass. 02139

Mr. Arthur C. Gilbert, Sc. D.
Chief, Space Systems Requirements
Corporate Systems Center
Div. United Aircraft Corp.
1690 New Britian Ave.
Farmington, Conn.

Mr. Robert A. Lerman
Sr. Analytical Engineer
Tech. Planning
Hamilton Standard Div.
United Aircraft Corp.
Windsor Locks, Conn.

Dr. M. G. Boyce
Dept. of Mathematics
Vanderbilt University
Nashville, Tenn.

Dr. Daniel Dupree
Dept. of Mathematics
Northeast La. State College
Monroe, La.

Dr. Steve Hu
Northrop Corp.
Box 1484
Huntsville, Ala.

Dr. William Nesline, Jr.
Analytical Research Dept.
Missile and Space Div.
Raytheon Company
Bedford, Mass.

Dept. of Mathematics
Southern Illinois University
Carbondale, Ill.
Attn: Dr. Robert W. Hunt
Mr. Robert Silber

Dr. H. Hermes
Mail #A127
Martin Company
P. O. Box 179
Denver 1, Colorado

Mr. R. W. Reck
Martin Company
3313 S. Memorial Parkway
Huntsville, Alabama

Dr. M. L. Anthony
Space Flight Technology
Mail No. A-153
The Martin Company
Denver 1, Colorado

Mr. Samuel Pines
Analytical Mechanics Associates, Inc.
941 Front St.
Uniondale, New York

Dr. W. A. Shaw
Mechanical Engineering Dept.
Auburn University
Auburn, Ala.

EXTERNAL DISTRIBUTION (Continued)

Mr. J. W. Hanson
Computation Center
University of North Carolina
Chapel Hill, North Carolina

Mr. Oliver C. Collins
Advanced Research
Flight Technology Dept.
Aero-Space Div.
Organization 2-5762
Mail Stop 15-12
Boeing Co.
P. O. Box 3707
Seattle, Wash.

Mr. Carl B. Cox, Chief
Flight Technology Dept.
Advanced Research
Aero-Space Div.
Organization 2-5762
Mail Stop 15-03
Boeing Co.
P. O. Box 3707
Seattle 24, Washington

Mr. Richard Hardy
Box AB-38
Boeing Co.
P. O. Box 1680
Huntsville, Ala.

Mr. Wes Morgan
Box AB-49
Boeing Co.
P. O. Box 1680
Huntsville, Alabama

Mr. Robert Glasson
Bendix Systems Div.
Bendix Corp.
3322 Memorial Parkway So.
Huntsville, Alabama

Chrysler Corp. Missile Div.
Sixteen Mile Rd. and Van Dyke
P. O. Box 2628
Detroit 31, Mich.
Attn: Mr. T. L. Campbell
Dept. 7162
Applied Mathematics

Mr. Harry Passmore
Hayes International Corp
P. O. Box 2287
Birmingham, Ala.

Astrodynamics Operation
Space Sciences Lab.
Missile and Space Vehicle Dept.
General Electric Company
Valley Forge Space Technology Center
P. O. Box 8555
Philadelphia 1, Pennsylvania
Attn: Mr. J. P. deVries
Mr. C. Cavoti

Dr. I. E. Perlin
Rich Computer Center
Ga. Institute of Technology
Atlanta, Ga.

Dr. Charles C. Conley
Dept. of Mathematics
University of Wisconsin
Madison, Wisconsin

Dr. O. R. Ainsworth
Dept. of Mathematics
University of Alabama

EXTERNAL DISTRIBUTION (Continued)

Dr. N. N.Puri
Electric Engineering Dept.
Drexel Institute of Technology
Philadelphia, Pennsylvania

University of Kentucky
College of Arts and Sciences
Dept. of Mathematics and Astronomy
Lexington, Kentucky
Attn: Dr. J. C. Eaves

Space Sciences Laboratory
Space and Info. Systems Div.
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.

Mr. J. W. Scheuch
P. O. Box 557
No. American Aviation, Inc.
Huntsville, Ala.

Mr. Joe Mason
Space & Info. Systems Div.
Dept. 41-595-720
No. American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.

Dr. D. M. Schrello, Director
Flight Sciences
Space & Info. Systems Div.
No. American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.

Mr. Myron Schall
Supervisor, Powered Trajectory
Space & Info. Div.
No. American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.

Mr. S. E. Cooper
Space & Info. Div.
Dept. 41-697-610
No. American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.

Space Sciences Lab.
Space & Info. Systems Div.
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, Calif.
Attn: Mr. Paul DesJardins
Mr. Harold Bell

Dr. E. R. Rang
Military Products Group
Aeronautical Div.
Minneapolis-Honeywell Regulator Co.
Mail Stop #340
Minneapolis, Minnesota

Dr. Dahlard Lukes
M.P.G. - Aeronautical Div.
Minneapolis-Honeywell Regulator Co.
2600 Ridgeway Road
Minneapolis, Minnesota

Dr. Donald Greenspan
Mathematics Research Center
U.S. Army, University of Wisconsin
Madison, Wisconsin

Mr. Ralph Q. Haumacher
A2-863: Space/Guidance & Control
Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California

Mr. F. A. Hewlett
Mgr. of Documentation
Federal Systems Div., IBM
6702 Gulf Freeway
Houston 17, Texas

Mr. Howard S. London
Rm. 910-E
Bellcomm, Inc.
1100 17th St., N. W.
Washington, D. C.

Mr. George Cherry
Mass. Institute of Technology
Cambridge, Mass.

Mr. William C. Marshall
Research Engineer
M. P. G. Research Dept.
Sta. #340
Minneapolis-Honeywell Regulator Co.
2600 Ridgeway Road
Minneapolis 3, Minnesota

Mr. W. G. Melbourne
Jet Propulsion Lab.
4800 Oak Grove Dr.
Pasadena 3, Calif.

Dr. Byron D. Tapley
Dept. of Aerospace Engineering
University of Texas
Austin, Texas

Dr. Hans G. Baussus
USA, GIMRADA
Fort Belvair,
Virginia

Dr. William A. Mersman
Chief,
Electronic Machine Computing Br.
Ames Research Center
Moffett Field, Calif.

Siegfried J. Gerathewohl
Code SB
NASA Headquarters
Washington, D. C.

Dr. Herman M. Dusek
A. C. Research & Development
A. C. Spark Plug
Electronics Div. of Gen. Motors
950 N. Sepulveda Blvd.
El Segundo, Calif.

Mr. Howard Haglund
Jet Propulsion Lab.
4800 Oak Grove Dr.
Pasadena 3, Calif.

Mr. Hewitt Phillips
Langley Research Center
Hampton, Va.

Mr. Frank J. Carroll
Systems Requirement Dept.
Equipment Div.
Raytheon Co.
40 Second Ave.
Waltham, Mass.

Mr. T. Perkins
Chrysler Corp.
HIC Bldg.
Huntsville, Ala.

Mr. Robert Allen
Mgr., Huntsville Sales Office
A. C. Spark Plug
Electronics Div. of Gen. Motors
Holiday Office Center
Huntsville, Ala.

Mr. Dale B. Ruhmel
Staff Associate
Aeronautical Research Assoc. of
Princeton, Inc.
50 Washington Rd.
Princeton, N. J.

Mr. Raymond Rishel
Mathematical Analysis Staff
Organization 2-5330
Mail Stop 89-75
Boeing Co.
P. O. Box 3707, Seattle, Wash.

EXTERNAL DISTRIBUTION (Continued)

Dr. Rudolf Hermann
Director
Univ. of Ala. Research Inst.
4701 University Ave., N. W.
Huntsville, Ala.

Dr. S. H. Lehnigk
Physical Sciences Lab.
Army Missile Command, Bldg. 5429
Redstone Arsenal, Ala.

Mr. R. J. Hayes
Code RE-TG
NASA Hdqs.
Washington, D. C.

Mr. Harold Chestnut
1 River Rd.
Schenectady, N. Y.

Space Sciences Lab.
Space & Info. Systems
No. American Aviation, Inc.
Downey, Calif.
Attn: Mr. Dave Engles

Dr. Lu Ting
Dept. of Applied Mechanics
Polytechnic Inst. of Brooklyn
333 Jay St.
Brooklyn 1, N. York

NASA, Langley Research Center
Hampton, Va.
Attn: Librarian

Dr. Peter Musen
Goddard Space Flt. Center
NASA
Greenbelt, Md.

Dr. V. Szebehely
Yale University Obs.
Box 2023, Yale Sta.
New Haven, Conn.

Dr. Rudolph Kalman
Division of Engineering Mechanics
Stanford University
Stanford, California 94305

Mr. Ken Kissel
Aeronautical Systems Div.
Applied Mathematics Research Br.
Wright-Patterson Air Force Base
Dayton, Ohio

Mr. Jack Funk
Manned Spacecraft Center
Flight Dynamics Branch
NASA
Houston, Texas

Dr. J. B. Rosser
Dept. of Mathematics
Cornell Univ.
Ithaca, N. Y.

Douglas Aircraft Corp.
3000 Ocean Park Blvd.
Santa Monica, Calif.
Attn: R. E. Holmen A2-263
Guidance & Control Section

Dr. Joseph F. Shea
Manned Spacecraft Center (MSC)
Houston, Texas

Mr. Charles F. Pontious
Guidance & Navigation Program
Office of Advanced Research & Tech.
Code: REG
NASA
Washington 25, D. C.

Mr. Jules Kanter
Guidance & Navigation Program
Office of Advanced Research & Tech.
Code: REG
NASA
Washington 25, D. C.

Dr. Ray Wilson
OART-Code RRA
Washington 25, D. C.

Dr. Joseph W. Siry
Theory & Analysis Office (547)
Data Systems Div.
Goddard Space Flt. Center
Greenbelt, Md.

Douglas Aircraft Corp.
3000 Ocean Park Blvd.
Santa Monica, Calif.
Attn: Mr. Joe Santa, A2-863

J. B. Cruz, Jr.
Research Associate Professor
Coordinated Science Lab.
Urbana, Ill.

Mr. Alan L. Friedlander
Research Engineer
Guidance and Control Section
IIT Research Center
10 W. 35th St.
Chicago 16, Ill.

Mr. Joseph V. Natrella
Manned Space Flight - Code MCR
NASA Hdqs., FOB 10B
Washington, D. C.

Mr. Donald Jezewski
Guidance Analysis Br.
Spacecraft Technology Div.
Manned Spacecraft Center
Houston, Texas

Mr. Robert M. Williams
Chief, Guidance Analysis
General Dynamics/Astronautics
Mail Zone 513-0
P. O. Box 166
San Diego 12, Calif.

Mr. M. D. Anderson, Jr.
General Dynamics Corp.
Suite 42 Holiday Office Center
South Memorial Parkway
Huntsville, Ala.

Mr. Y. L. Luke
Mathematics & Physics Div.
Midwest Research Institute
425 Volker Blvd.
Kansas City 10, Missouri

Mr. Ted Guinn
Advanced Space Technology
Engineering Research
Douglas Aircraft Corp.
Santa Monica, Calif.

Mr. Robert Chilton
NASA Manned Spacecraft Center - EG
P. O. Box 1537
Houston, Texas

Dr. S. E. Ross
Office of Manned Space Flt. - NASA
Washington, D. C.

Lt. Stephen J. Kahne, USAF
Applied Mathematics Branch
Data Sciences Lab.
Air Force Cambridge Research Labs.
Office of Aerospace Research
Lawrence G. Hanscom Field
Bedford, Mass.

Mr. Daniel B. Killeen
Computer Lab.
Norman Mayer Bldg.
Tulane University
New Orleans, La.

Dr. Bernard Friedland
Staff Scientist - Control
General Precision, Inc.
Little Falls, N. J.

EXTERNAL DISTRIBUTION (Continued)

Mr. C. H. Tross
Mgr., Aerospace Sciences
UNIVAC Div. of Sperry Rand Corp.
P. O. Box 6248, San Diego 6, Calif.

Mr. Walter L. Portugal
Mgr., Systems Sales
Aerospace Group
General Precision, Inc.
Little Falls, New Jersey

Mr. Ken Squires
Goddard Space Flt. Center,
Bldg #1, NASA
Greenbelt, Md.

Dr. Paul Degrarabedian
Astro Science Lab.
Bldg. G
Space Technology Lab., Inc.
One Space Park
Redondo Beach, Calif.

Dr. George Leitmann
Associate Professor, Engr. Science
University of California
Berkely, Calif.

Dr. R. P. Agnew
Dept. of Mathematics
Cornell University
Ithaca, N. Y.

Dr. Jurgen Moser
Professor of Mathematics
Graduate School of Arts And Science
New York University
New York, New York

Mr. Clint Pine
Dept. of Mathematics
Northwestern State College
Natchitoches, La.

Dr. John Gates
Jet Propulsion Lab.
4800 Oak Grove Drive
Pasadena 3, Calif.

Auburn Research Foundation
Auburn University
Auburn, Ala.

Grumman Library
Grumman Aircraft Engineering Corp.
Bethpage, L. I., New York

Jet Propulsion Lab.
Library
4800 Oak Grove Dr.
Pasadena 3, Calif.

Scientific and Tech. Info. Facility (25)
Attn: NASA Rep. (S-AK/RKT)
P. O. Box 5700
Bethesda, Md.

NASA Ames Research Center
Mountain View, Calif.
Attn: Librarian

NASA Flight Research Center
Edwards Air Force Base, Calif.
Attn: Librarian

NASA, Goddard Space Flight Center
Greenbelt, Md.
Attn: Librarian

EXTERNAL DISTRIBUTION (Continued)

NASA Launch Operations Directorate
Cape Kennedy, Fla.
Attn: Librarian

NASA Lewis Research Center
Cleveland, Ohio
Attn: Librarian

NASA Manned Spacecraft Center
Houston 1, Texas
Attn: Librarian

NASA Wallops Space Flt. Sta.
Wallops Island, Va.
Attn: Librarian

Space Flight Library
University of Kentucky
Lexington, Kentucky

University of Kentucky Library
University of Kentucky
Lexington, Kentucky

Office of Manned Space Flight
NASA Hdqs.
Federal Office Bldg. #6
Washington 25, D. C.
Attn: Mr. Eldon Hall
Dr. A. J. Kelly

Mr. Bryan F. Dorlin
Theoretical Guidance & Control Br.
NASA, Ames Research Center
Moffett Field, Calif.

Lt. Col. R. A. Newman
Air Force Space Systems Div.
M-SSVH
Bldg. 5250
Redstone Arsenal, Ala.

Roger Johnson
Honeywell Research
Roseville, Minnesota