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EXTENSION OF POPOV'S THEOREMS FOR STABILITY OF NONLINEAR CONTROL SYSTEMS

by Y. H. Ku and H. T. Chieh

Research Projects Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

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ABSTRACT

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In Part I of this paper, Popov's Theorem PI is first introduced and then a new Theorem I is formulated. The proof is given in Appendix I. An illustrative example shows that the result obtained from Theorem I agrees with that obtained from Lur'e's Theorem. In Part II the linear part transfer function may have poles along the imaginary axis with real positive residues. The nonlinear function f(e) is bounded as well as continuous. Popov's Theorem PII is extended to form a new Theorem II, which gives the condition for quasi-asymptotic stability. Two corollaries are also given. Corollary IIa gives the condition for asymptotic stability. The proof of Theorem II and its corollaries is given in Appendix II. Three examples check with the results of analog computer studies.

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FOREWORD

On April 19, 1966, at the Space Science Seminar at NASA's George C. Marshall Space Flight Center, Dr. Y. H. Ku of the University of Pennsylvania presented a lecture entitled Nonlinear Oscillations in High-Order Systems. This presentation was based largely on a paper published in the Journal of The Franklin Institute, June, 1965.

This Technical Memorandum is a reprint of the original Franklin Institute paper, Extensions of Popov's Theorems for Stability of Nonlinear Control Systems. The seminar report also included 30 or 40 additional figures of cyclic stable and unstable functions.

BIOGRAPHICAL NOTE

Dr. Y. H. Ku is Professor of Electrical Engineering, Moore School of Engineering, University of Pennsylvania. His interests include analysis and control of nonlinear systems and electrical energy conversion transient circuit analysis.

Born in China, Dr. Ku came to the United States for his college studies. He received his B. S., M. S., and Ph. D. degrees from the Massachusetts Institute of Technology. Upon completion of his studies, he returned to China as Professor of Electrical Engineering, National Ckekiang, University of Hangchow, 1929-1930. He was Dean of the Engineering College, National Central University, Nanking, 1931-1932. He spent the next five years as Dean, Engineering College, National Tsing Hua University, Peking. He was Vice Minister of Education, National Tsing Hua University, Peking, 1938-1944. The following year Dr. Ku was President of National Central University. He was Commissioner of Education at Shanghai 1945-1947. He then returned to Nanking as Professor at Central University.

Dr. Ku came back to MIT in 1950 as Visiting Professor. He became Professor of Electrical Engineering at Moore School of Engineering, his present position, in 1952. Concurrent with this academic past, Dr. Ku was an engineering consultant to the General Electric Company 1951-1956, and engineering consultant to the Radio Corporation of America 1960-1961. He is the author of numerous technical papers and several books on stability, nonlinear controls systems, and circuit analysis.

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EXTENSION OF POPOV'S THEOREMS FOR STABILITY OF NONLINEAR CONTROL SYSTEMS

INTRODUCTION

Popov has recently given two theorems by which we can determine the stability of nonlinear control systems. Theorem PI gives the asymptotic stability of the system involving a nonlinear function $f(e)$ by the condition $\text{Re}\,[(1 + j\omega q)G(j\omega)] \geq 0$, where $G(j\omega)$ denotes the linear part transfer function, and q is a non-negative constant. The new Theorem I is an extension of Theorem PI such that the restriction $\gamma > 0$ is now changed to $\gamma \geq 0$. Similar to Theorem PII, the new Theorem II is applicable to control systems with its linear part transfer function having pairs of imaginary poles. While Theorem PII gives the condition for hyperstability, Theorem II gives the condition for quasi-asymptotic stability. Corollary IIa then gives the condition for asymptotic stability whenever q can be set to zero.

EXTENSION OF POPOV'S FIRST THEOREM

Popov's Theorem PI (1) is first introduced in this section. Extension of this Theorem will be given as new Theorem I. (The proof of Theorem I is given in Appendix I.) An example shows that the result obtained from Theorem I agrees with that obtained from Lur'e's Theorem.

Popov's Theorem PI: If a non-negative quantity q exists such that for all real ω, the inequality

$$\text{Re}\,[(1 + j\omega q)G(j\omega)] \geq 0 \tag{1}$$

takes place, then the trivial solution of the system is asymptotically stable provided a given set of assumptions are valid.

Kalman (2) restated Popov's Theorem PI as follows:

"Assume that $\mathbf{A}$ is stable and that $\gamma > 0$. Then the system is g.a.s. if the condition

$$\text{Re}\,\{(2\alpha\gamma + j\omega q)[\mathbf{C}'(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \gamma/j\omega]\} \geq 0 \tag{2}$$

for all real ω holds for $2\alpha\gamma = 1$ and some $q \geq 0$."

In the above quotation, we have replaced the original symbol $\mathbf{F}$ by $\mathbf{A}$ for a real $n \times n$ matrix, $\mathbf{h}$ by $\mathbf{C}$ for a real n-vector, $\mathbf{g}$ by $\mathbf{b}$ for another real n-vector, and the constants ρ and β by γ and q, respectively. The prime denotes the transpose. Comparison of the two expressions gives the transfer function of the system linear part as

$$G(j\omega) = \mathbf{C}'(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \gamma/j\omega \qquad (\gamma > 0). \tag{3}$$

Note that on the righthand side of Eq. 3, the first term contains the poles with negative real parts while the second term denotes a pole at the origin.

The new Theorem I tries to remove the restriction that $\gamma > 0$ so as to include the case $\gamma = 0$ in its application to feedback control systems shown in Fig. 1 and introduces a new constant h which is non-negative. The system can be defined by the following differential equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}f(e), \tag{4}$$

$$e = -\mathbf{C}'\mathbf{x} - hf(e), \tag{5}$$

where $\mathbf{x}$ denotes a real n-vector, $\dot{\mathbf{x}} = d\mathbf{x}/dt$, $\mathbf{A}$ is a real $n \times n$ matrix whose eigenvalues may include zero, $\mathbf{b}$ and $\mathbf{C}$ are real n-vectors, and h is a non-negative constant. The nonlinear function $f(e)$ is a continuous function of e, with the conditions:

$$0 < ef(e) < ke^2(e \neq 0); \qquad f(0) = 0. \tag{6}$$

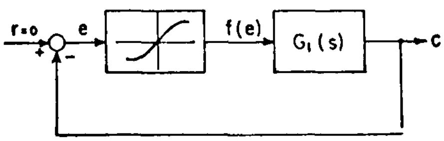

FIG. 1. Block diagram of a nonlinear control system.

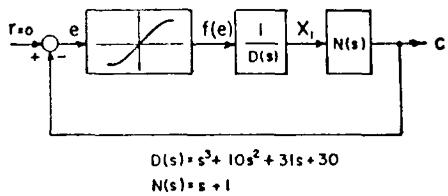

FIG. 2. Block diagram of example 1.

It is further assumed that $f'(e) = df(e)/de$ is continuous, and $1 + hf'(e) \neq 0$. Referring to Fig. 1, the linear part transfer function is defined by

$$G_1(s) = \mathbf{C}'(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + h. \tag{7}$$

For $s = j\omega$, Eq. 7 gives $G_1(j\omega)$, where γ is absorbed in the first term on the righthand side. The new Theorem I can be stated as follows:

Theorem I: If a non-negative quantity q exists such that for all real ω the inequality[1]

$$1/k + \mathrm{Re}\,[(1 + j\omega q)G_1(j\omega)] \geq 0 \tag{8}$$

takes place, then the trivial solution of the system Eqs. 4–5 is asymptotically stable provided the assumptions made are valid.

Substituting $G_1(j\omega)$ into Eq. 8 gives

$$1/k + \mathrm{Re}\,[(1 + j\omega q)\mathbf{C}'(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}] + h \geq 0. \tag{9}$$

[1] Note the factor $(1/k)$ appeared in Popov and Halanay (6) and in Aizerman and Gantmacher (7), p. 52). However the expression Eq. 8 in the present paper refers to $G_1(j\omega)$ which is more general than $G(j\omega)$ in (6), or $W(j\omega)$ in (7). Popov (4) did introduce the factor h (but not $1/k$). Maigarin (8) gave in (2.14) an additional term N which corresponds to h in Eq. 7, but the $(1/k)$ factor was not included.

When $h = 0$, the condition that $f'(e)$ is continuous can be removed and that the condition $1 + hf' \neq 0$ is automatically satisfied. The proof is given in Appendix I.

Example 1: Given a control system as shown in Fig. 1 with the linear part transfer function:

$$G_1(s) = \frac{s+1}{(s+2)(s+3)(s+5)}; \qquad h = 0; \qquad \gamma = 0. \tag{10}$$

The left hand side of inequality Eq. 8 gives

$$1/k + \mathrm{Re}\,[(1 + j\omega q)G_1(j\omega)] = (A_3\omega^6 + A_2\omega^4 + A_1\omega^2 + A_0)/B(\omega), \tag{11}$$

where $A_3 = 1/k$, $A_2 = 9q - 1 + 38/k$, $A_1 = q + 21 + 361/k$, $A_0 = 30(1 + 30/k)$, and $B(\omega) = 100(\omega^2 - 3)^2 + \omega^2(\omega^2 - 31)^2$. Expression Eq. 11 is greater than zero when $q \geq 1/9$ for any positive k and any ω. Hence the system is ***asymptotically stable***, according to Theorem I.

This result checks with the result by Lur'e's Theorem as shown in (3).

For this example, a set of state variables $\mathbf{x}$ can be chosen as x_1, x_2, and x_3 where x_1 is shown in Fig. 2. Referring to Eqs. 4 and 5, we get

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -31 & -10 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{12}$$

For $h = 0$, substituting Eqs. 12 into Eq. 7 gives $G_1(s)$ specified by Eq. 10. Note that from Fig. 2, we get the output c, which is the negative of e in an autonomous system (with input $r = 0$), as

$$c = x_1 + \dot{x}_1 = x_1 + x_2. \tag{13}$$

From Fig. 2, we also get the differential equation

$$f(e) = \dot{x}_3 + 10x_3 + 31x_2 + 30x_1 \tag{14}$$

where $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_3$ are defined in matrix **A**. The variables x_1, x_2, and x_3 are known as phase-space variables.

EXTENSION OF POPOV'S SECOND THEOREM

Popov's Theorem PII (4) is first introduced in this section. Extension of this Theorem will be given as new Theorem II. (The proof of Theorem II is given in Appendix II.) Three examples are given to illustrate the application of Theorem II. Analog computer results are presented to check the stability of the three control systems studied.

Popov's Theorem PII:

In order for the solution $\mathbf{x} = 0$ of the non-degenerate system Eqs. 15–16 to be hyperstable it is necessary and sufficient that the transfer function $G_1(s)$, defined in Eq. 18, be a real positive function.

The system considered in Theorem PII can be defined by the equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\phi(t) \tag{15}$$

$$\beta(t) = \mathbf{C}'\mathbf{x} + h\phi(t) \tag{16}$$

which satisfy the following inequality for any $T > 0$:

$$\int_0^T \phi(t)\beta(t)dt \leq \delta \sup_{0\leq t\leq T} |\mathbf{x}(t)|, \tag{17}$$

where δ is a non-negative constant. The transfer function $G_1(s)$ is defined by

$$G_1(s) = \sum_{i=1}^{m} \frac{p_i s}{s^2 + \omega_i^2} + G_0(s), \tag{18}$$

where $G_1(s)$ is defined in Eq. 7, $p_i > 0$ and ω_i are constants. The solution of the system Eqs. 15–16 is ***hyperstable*** if there exists a constant K such that any solution of the system satisfies the inequality

$$|\mathbf{x}(t) \leq K(|\mathbf{x}(0)| + \delta) \tag{19}$$

for all $\phi(t)$ that satisfies inequality 17.

For the sake of developing new Theorem II, let $\phi(t) = f[e(t)]$ and $\beta(t) = -e(t)$ with the conditions given in Eqs. 6. Substituting these into Eq. 17 gives

$$-\int_0^T ef(e)dt \leq \delta \sup_{0\leq t\leq T} |\mathbf{x}(t)|. \tag{20}$$

This is satisfied for any non-negative δ (including $\delta = 0$). The system Eqs. 15–16 now take the form of Eqs. 4–5, where the **A** matrix has eigenvalues on the imaginary axis as well as on the left half of the s-plane.

The new Theorem II can be stated as follows:

Theorem II: If $f(e)$ is bounded and a non-negative quantity q exists such that for all real ω the inequality

$$1/k + \mathrm{Re}\,[(1 + j\omega q)G_0(j\omega)] \geq 0 \tag{21}$$

takes place, then the trivial solution of the extended system Eqs. 4–5, with its transfer function $G_1(s)$ defined by Eq. 18, is quasi-asymptotically stable provided the above assumptions are valid.

Note that the definition of *quasi-asymptotic stability* follows that given in (5). The proof is given in Appendix II.

Corollary IIa: If q can be set to zero, then the trivial solution of the extended system Eqs. 4–5 is asymptotically stable provided the assumptions are valid.

Corollary IIb: If q can be set to zero, $f(e)$ is continuous, but the restrictions on $f'(e)$ are disregarded, then the trivial solution of the extended system Eqs. 4–5 is stable.

These corollaries are discussed in Appendix II.

Example 2: Given a control system shown in Fig. 1 with the linear part transfer function:

$$G_1(s) = \frac{3s^4 + 4s^3 + 5s^2 + s + 1}{s(3s + 1)(s^2 + 1)} \tag{22}$$

and $f(e) = \tanh e$, which is bounded and continuous. The derivative $f'(e) = \text{sech}^2\, e$ is continuous. Note that $f(0) = 0$ and $1 + hf'(e) = 1 + \text{sech}^2\, e > 1$ which is not equal to zero. Dividing the numerator of Eq. 22 by its denominator gives $h = 1$ and

$$G_1(s) = \frac{3s^3 + 2s^2 + 1}{s(3s + 1)(s^2 + 1)} + 1. \tag{23}$$

Expansion into partial fractions gives

$$G_1(s) = \frac{s}{s^2 + 1} + \frac{1}{s} - \frac{3}{3s + 1} + 1. \tag{24}$$

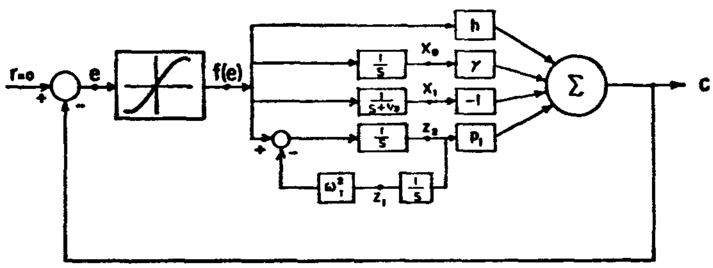

Fig. 3. Block diagram of example 2.

Comparison with Eq. 18 shows that $p_1 = 1$ and $\omega_1 = 1$. Equation 24 also shows that the coefficient of $1/s$ is $\gamma = 1$, and this term denotes a pole at the origin. Note that for this example, $G_0(s)$ in Eq. 18 is given by

$$G_0(s) = \frac{1}{s} - \frac{3}{3s + 1} + 1 = \frac{3s^2 + s + 1}{s(3s + 1)}. \tag{25}$$

Referring to inequality 6, we get

$$0 < e \tanh e < ke^2 (e \neq 0); \qquad \tanh 0 = 0 \tag{26}$$

and k can be chosen as 1. Then applying inequality 21 of Theorem II gives

$$1 + \text{Re}\left[(1 + j\omega q)\left(\frac{1}{j\omega} - \frac{3}{j3\omega + 1} + 1\right)\right] = \frac{(q - 1) + 18\omega^2}{1 + 9\omega^2} \geq 0 \quad \text{for} \quad q \geq 1. \tag{27}$$

According to Theorem II, the system is quasi-asymptotically stable. The system can be represented by the block diagram shown in Fig. 3. The variables chosen for analog computer study are: x_0, x_1, z_1, and z_2. The system equations are:

$$\left.\begin{aligned} \dot{x}_0 &= f(e) \\ \dot{x}_1 &= -\tfrac{1}{3}x_1 + f(e) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\omega_1^2 z_1 + f(e) \end{aligned}\right\} \tag{28}$$

and

$$-e = c = hf(e) + \gamma x_0 - x_1 + p_1 z_2, \tag{29}$$

where $h = 1, \gamma = 1, p_1 = 1, \omega_1 = 1$, and the coefficient of x_1 is -1. The result is checked by the analog computer study shown in Fig. 4.

Example 3: Given a control system shown in Fig. 1 with the linear part transfer function

$$G_1(s) = \frac{s^3 + s^2 + 2s}{(s+1)(s^2+1)} = \frac{s}{s^2+1} - \frac{1}{s+1} + 1 \tag{30}$$

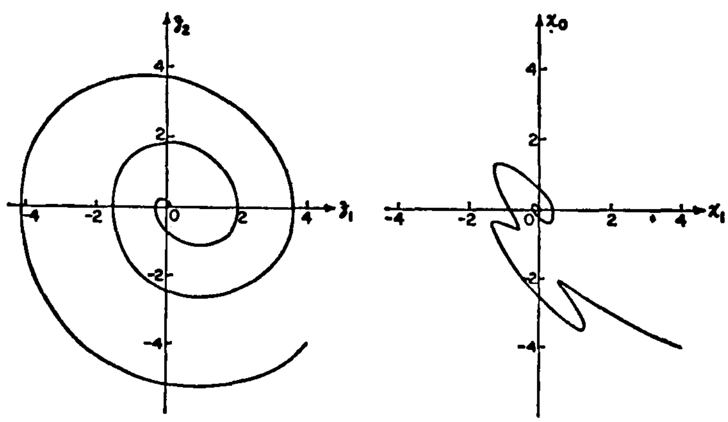

FIG. 4. Analog computer results—example 2.

and $f(e) = e + e^3$ which is continuous but not bounded. Note that $h = 1$ and the system has one pole at -1 and one pair of imaginary poles. The derivative is $f'(e) = 1 + 3e^2$ and $1 + hf'(e) = 2 + 3e^2 > 2$, and $f(0) = 0$. From condition Eq. 6, k has to be greater than $1 + e^2$. For e arbitrarily large, k has to be chosen as infinity. Then setting q to zero in inequality 21 gives, for $1/k = 0$,

$$\operatorname{Re} G_0(j\omega) = \operatorname{Re}\left[1 - \frac{1}{j\omega + 1}\right] = \frac{\omega^2}{1+\omega^2} \geq 0. \tag{31}$$

This system is asymptotically stable according to Corollary IIa. Referring to Fig. 3, $\gamma = 0$, $h = 1$, $p_1 = 1$, and $\omega_1 = 1$, but the $(s + 1/3)$ part is replaced by $(s + 1)$, with coefficient -1 for x_1. The system equations are:

$$\left.\begin{aligned}\dot{x}_1 &= -x_1 + f(e)\\ \dot{z}_1 &= z_2\\ \dot{z}_2 &= -z_1 + f(e)\end{aligned}\right\} \tag{32}$$

$$-e = c = f(e) - x_1 + z_2. \tag{33}$$

This result is checked by the analog computer study shown in Fig. 5.

Example 4: Given a control system shown in Fig. 1 with the linear part transfer function:

$$G_1(s) = \frac{s}{s^2+1} - \frac{1}{s+2} + 1 \tag{34}$$

and $f(e) = e + e^3$. This differs from Example 3 in having a pole at -2 instead of at -1. Setting $q = 0$ and $1/k = 0$ in inequality 21 gives

$$\text{Re}\, G_0(j\omega) = \text{Re}\left[1 - \frac{1}{j\omega + 2}\right] = \frac{2+\omega^2}{4+\omega^2} > 0. \tag{35}$$

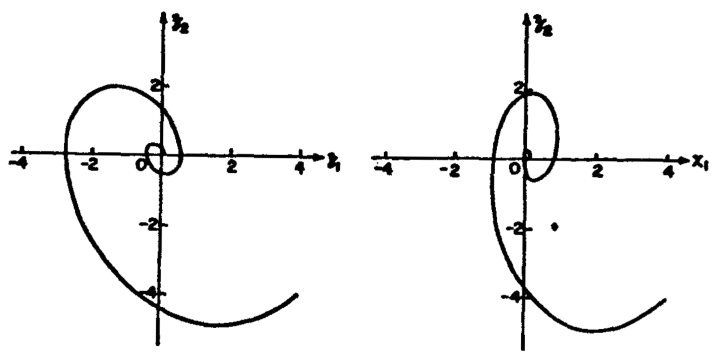

Fig. 5. Analog computer results—example 3.

This system is asymptotically stable according to Corollary IIa. This result is checked by the analog computer study shown in Fig. 6.

CONCLUSION

This paper has given two new theorems for the stability of control systems with one nonlinearity $f(e)$. These theorems are extensions of Popov's Theorems PI and PII. Theorem I removes the restriction that $\gamma > 0$ so that $\gamma \geq 0$ and a pole at the origin may not be necessary. It also introduces a non-negative constant h in the expression of the linear part transfer function $G_1(s)$, defined in Eq. 7. The inequality 8 in Theorem I includes a non-negative constant $(1/k)$. Example 1 demonstrates its application to an asymptotically stable control system. Theorem II is applicable to control systems with its linear part transfer function $G_1(s)$ having pairs of imaginary poles. If the nonlinear function $f(e)$ is bounded, it is shown that a similar test of inequality can be made for predicting the quasi-asymptotical stability of the system. Corollary IIa disregards the boundedness of $f(e)$ and predicts asymptotic stability whenever a non-negative quantity q (used in Theorems I and II) can be set to zero.

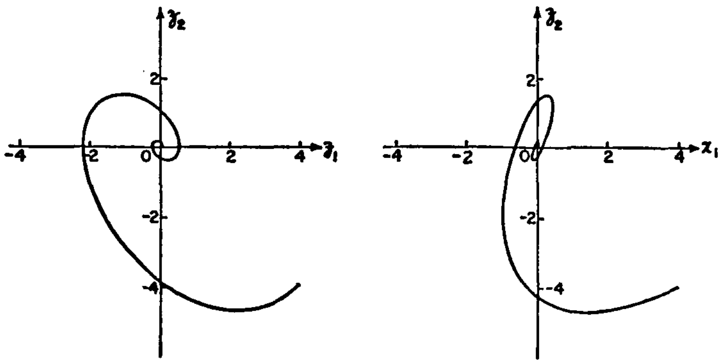

Fig. 6. Analog computer results—example 4.

Theorem II is applied to Example 2 and Corollary IIa is applied to Examples 3 and 4. The results check with those obtained from analog computer studies.

APPENDIX I

Lemmas

(1) If $G(s)$ is a real positive function of the form $N(s)/D(s)$ where $N(s) = d_{n+1}s^n + \cdots + d_2 s + d_1$ and $D(s) = s^n + \cdots + a_2 s + a_1$, then there exists a function $\epsilon(t)$ with the following properties:

$$\int_o^\infty \epsilon^2(t)dt \geq \alpha_1|\mathbf{y}(T)|^2 - \beta_1|\mathbf{y}(o)|\,|\mathbf{y}(T)|$$

$$\mathfrak{F}[\epsilon(t)] = H(j\omega)\Phi(j\omega) \qquad \text{and} \qquad |H(j\omega)|^2 = \mathrm{Re}\,[G(j\omega)],$$

where $\mathfrak{F}$ is the Fourier transform, α_1 and β_1 are positive constants. (See (4), Lemma 1 and p. 15.)

(2) If $\rho(t)$ satisfies the conditions $\mathrm{Sup}_{0\leq t\leq T}|\rho(t)| \leq \mu$, $\int_o^\infty |\rho(t)|dt \leq \mu$ and $\int_o^\infty |d\rho(t)/dt|dt \leq \mu$ then $\left|\int_o^\infty \rho(t)\phi(t)dt\right| \leq r_1\mu\ \mathrm{Sup}_{0\leq t\leq T}|\mathbf{y}(t)|$ where μ and r_1 are positive constants. (See (4), Lemma 3.)

(3) If $r^2(t) \leq \alpha\ \mathrm{Sup}_{0\leq t\leq T}|r(t)| + \beta$, then $|r(t)| \leq \alpha + \sqrt{\beta}$, $0 \leq t \leq T$ where α and β are positive constants. (See (4), Lemma 2.)

(4) If both $\int_0^T ef(e)dt$ and de/dt are bounded, then $\lim_{t\to\infty} e(t) = 0$. (See (1), Appendix 5.)

(5) If $\lim_{t\to\infty} e(t) = 0$ then $\lim_{t\to\infty} \mathbf{y}(t) = 0$. (See (1), Appendix 6.)

Proof of Theorem I

Let $G_1(s)$ be expanded into the form

$$G_1(s) = h + \gamma/s + (c_1 + c_2 s + \cdots + c_n s^{n-1})/D(s) \qquad \text{(I-1)}$$

where $D(s) = s^n + a_n s^{n-1} + \cdots + a_2 s + a_1$, a Hurwitz polynomial. This expansion is shown in Fig. 7. The system equations corresponding to Fig. 7

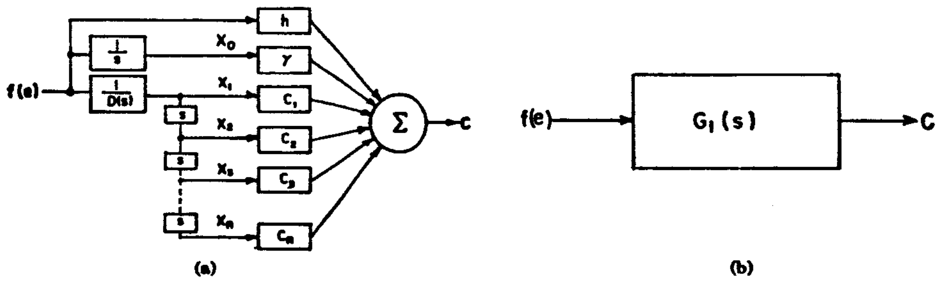

FIG. 7. Expanded block diagram of the linear part transfer function $G_1(s)$.

are given by Eqs. 4 and 5 where

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & 0 \\ \cdots & & & & & \cdots \\ 0 & -a_1 & & \cdots & & -a_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \gamma \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

For the convenience of proving the new Theorem I, let the matrix **A** be simplified to B by removing the first row and the first column. Similarly **x**, **b**, and **C** are simplified to get **y**, $\mathbf{b}_1$, and **g**. Thus, Eqs. 4 and 5 can be rewritten as

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{b}_1 f(e) \qquad \text{(I-2)}$$

$$\dot{x}_0 = f(e) \qquad \text{(I-3)}$$

$$e = -\gamma x_0 - \mathbf{g}'\mathbf{y} - hf(e) \qquad \text{(I-4)}$$

where

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & 0 \\ \cdots & & & & & \cdots \\ -a_1 & & \cdots & & & -a_n \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \mathbf{g} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and $\mathbf{g}'$ is the transpose of **g**. Let

$$\phi_T(t) = \phi(t) = f[e(t)] \quad \text{when} \quad 0 \le t \le T$$
$$= 0 \quad \text{when} \quad t > T$$

then the solution of Eq. (I-2) can be found in integral form as

$$\mathbf{y}(t) = \mathbf{W}(t)\mathbf{y}(0) + \int_0^t \mathbf{W}(t-u)\mathbf{b}_1\phi_T(u)du, \qquad \text{(I-5)}$$

where $\mathbf{W}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{B})^{-1}]$, the inverse Laplace transform is taken on every element of the square matrix.

A new function $j_T(t)$ is defined as

$$j_T(t) = -v(t) - q dv/dt - (h + 1/k + q\gamma)\phi_T(t) \qquad \text{(I-6)}$$

where

$$v(t) = \int_0^t \mathbf{g}'\mathbf{W}(t-u)\mathbf{b}_1\phi_T(u)du. \qquad \text{(I-7)}$$

The Fourier transform of Eq. (I-6) exists and is found to be

$$\mathfrak{F}[j_T(t)] = J_T(j\omega)$$
$$= -[(1 + j\omega q)\mathbf{g}'(j\omega\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}_1 + h + q\gamma + 1/k]\Phi(j\omega)$$
$$= -[(1 + j\omega q)G_1(j\omega) - \gamma/j\omega + 1/k]\Phi(j\omega) \qquad \text{(I-8)}$$

where $\Phi(j\omega)$ is the Fourier transform of $\phi_T(t)$.

Let

$$R(t) = \int_0^\infty j_T(t)\phi_T(t)dt + \int_0^\infty \epsilon^2(t)dt \qquad \text{(I-9)}$$

then it follows from Parseval's theorem that

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}[J_T(j\omega)\Phi^*(j\omega)]d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} [H(j\omega)\Phi(j\omega)][H(j\omega)\Phi(j\omega)]^*d\omega^{2}$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[1/k + (1 + j\omega q)G_1(j\omega)]|\Phi(j\omega)|^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[1/k + (1 + j\omega q)G_1(j\omega)]|\Phi(j\omega)|^2 d\omega = 0. \qquad \text{(I-10)}$$

This follows from the facts that $1/k + (1 + sq)G_1(s)$ is a real positive function, by using Lemma (1) and Eq. (I-8). Substituting Eq. (I-5) into Eq. (I-4) gives

$$e = -\mathbf{g}'\mathbf{W}(t)\mathbf{y}(0) - v(t) - h\phi_T(t) - \gamma x_o. \qquad \text{(I-11)}$$

Differentiating both sides of Eq. (I-11) with respect to t gives

[2] Asterisk denotes the complex conjugate.

$$\dot{e} = -\mathbf{g}'\dot{\mathbf{W}}(t)\mathbf{y}(0) - \dot{v}(t) - h\phi_T(t) - \gamma\dot{x}_o \quad \text{(I-12)}$$

subtracting both sides of Eq. (I-11) by $\phi_T(t)/k$, multiplying both sides of Eq. (I-12) by q and then adding them together yields

$$-v(t) - q\dot{v}(t) - (h + \gamma q + 1/k)\phi_T(t) = [e - \phi_T(t)/k] + (q\dot{e}) + \mathbf{g}'[\mathbf{W}(t) + q\dot{\mathbf{W}}(t)]\mathbf{y}(0) + \gamma x_o + qh\phi_T(t). \quad \text{(I-13)}$$

It follows from Eqs. (I-6), (I-9), and (I-13) that

$$R(T) = R_1(T) + R_2(T) + R_3(T) + R_4(T) + R_5(T) + R_6(T) \quad \text{(I-14)}$$

where

$$R_1(T) = 1/k \int_o^T [ke - \phi(t)]\phi(t)dt > 0$$

since the integrand is always positive according to the assumption.

$$R_2(T) = q\int_o^T \dot{e}\phi(t)dt = q\int_{e(o)}^{e(T)} f(e)de$$

$$= q\int_o^{e(T)} f(e)de - q\int_o^{e(o)} f(e)de \geq -qF(0)$$

where

$$F(0) = \int_o^{e(o)} f(e)de \geq 0.$$

$$R_3(T) = \int_o^T \mathbf{g}'[\mathbf{W}(t) + q\dot{\mathbf{W}}(t)]\mathbf{y}(0)\phi(t)dt \geq -r_1|\mathbf{y}(0)| \sup_{0\leq t\leq T} |\mathbf{y}(t)|$$

by Lemma (2) since **B** has eigenvalues with only negative real parts.

$$R_4(T) = \int_o^T \gamma x_0\phi(t)dt = \gamma\int_o^T x_0\dot{x}_0 dt = \gamma/2[x_0^2(T) - x_0^2(0)]$$

$$R_5(T) = \int_o^T qh\phi(t)\dot{\phi}(t)dt = qh/2[\phi^2(T) - \phi^2(0)] \geq -qh\phi^2(0)/2$$

$$R_6(T) = \int_o^\infty \epsilon^2(t)dt \geq \alpha_1|\mathbf{y}(T)|^2 - \beta_1|\mathbf{y}(0)|\,|\mathbf{y}(T)|, \qquad \text{by Lemma (1).}$$

Substituting all the R's into Eq. (I-10) gives the following inequality:

$$\alpha_1|\mathbf{y}(T)|^2 + \gamma x_o^2(T)/2 < |\mathbf{y}(0)|[\beta_1|\mathbf{y}(T)| + r_1 \sup_{0\leq t\leq T}|\mathbf{y}(t)|] + \gamma x_o^2(0)/2 + qF(0) + qh\phi^2(0)/2. \quad \text{(I-15)}$$

Now let

$$r_2 = \min.\,[\alpha_1, \gamma/2] \quad \text{when} \quad \gamma > 0$$
$$= \alpha_1 \quad \text{when} \quad \gamma = 0$$
$$r_3 = (\beta_1 + \gamma_1)/r_2$$
$$r_4^2 = 1/2r_2$$

then from the fact that $|\mathbf{y}(T) \leq \text{Sup}_{0\leq t\leq T}|\mathbf{y}(t) \leq \text{Sup}_{0\leq t\leq T}|\mathbf{x}(t)|$ and Lemma (3), the inequality (I-15) becomes

$$|\mathbf{x}(t)| < r_3|\mathbf{x}(0)| + r_4\sqrt{\gamma x_0^2(0) + 2qF(0) + qh\phi^2(0)}. \qquad \text{(I-16)}$$

This proves the stability of the system.

Now from Eq. 5, we have

$$e + hf(e) = -\mathbf{C}'\mathbf{x}. \qquad \text{(I-17)}$$

Since $\mathbf{x}$ is bounded and $f(e)$ has the same sign as e, both $f(e)$ and e have to be bounded. Differentiating both sides of Eq. (I-17) with respect to t gives

$$\dot{e} = -\mathbf{C}'\dot{\mathbf{x}}/[1 + hf'(e)]. \qquad \text{(I-18)}$$

Hence $\dot{e}$ is also bounded because $\mathbf{C}'\dot{\mathbf{x}}$ is bounded and $1 + hf'(e) \neq 0$.

From the assumption of inequality 6, there exists a positive quantity $N < k$ such that

$$R_1(t) = 1/k \int_0^T [ke - \Phi(t)]\Phi(t)dt > [(k - N)/k] \int_0^T c\phi(t)dt > 0. \qquad \text{(I-19)}$$

Since $R_1(t)$ is bounded from Eq. (I-10), $\int_0^T c\phi(t)dt$ is bounded and this bounded value is independent of T. It follows from Lemmas (4) and (5) that $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \mathbf{y} = 0$. Again from Eq. (I-14) we have $\lim_{t\to\infty} \dot{x}_o = 0$.

This proves the theorem.

APPENDIX II

Proof of Theorem II

The transfer function $G_1(s)$ is expanded as shown in Fig. 8. The system equations corresponding to Fig. 8 are given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}f(e), \tag{II-1}$$

$$\dot{z}_{2i-1} = z_{2i} \qquad (i = 1, 2, \cdots, m), \tag{II-2}$$

$$\dot{z}_{2i} = -\omega_i^2 z_{2i-1} + f(e), \tag{II-3}$$

$$e = -\mathbf{C}'\mathbf{x} - \mathbf{p}'\mathbf{z} - hf(e), \tag{II-4}$$

where

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2m} \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 0 \\ p_1 \\ \vdots \\ 0 \\ p_m \end{bmatrix}$$

and matrices **A** and **b** are the same as in Appendix I.

Following the same procedure as in proving Theorem I, we have in this case

$$R'(T) = R(T) + R_7(T) + R_8(T), \tag{II-5}$$

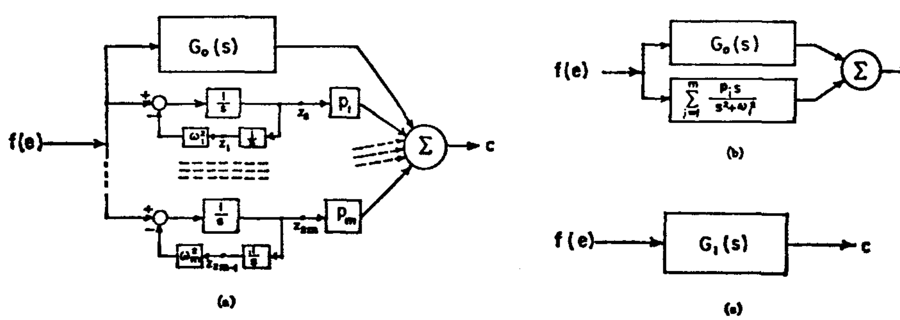

FIG. 8. Expanded block diagram of the linear part transfer function $G_1(s)$ with poles along imaginary axis.

where $R(T)$ is the same as given in Appendix I and $R_7(T)$ and $R_8(T)$ are

$$R_7(T) = \int_0^T \mathbf{p}'\mathbf{z}\phi(t)dt, \tag{II-6}$$

$$R_8(T) = q\int_0^T \mathbf{p}'\dot{\mathbf{z}}\phi(t)dt. \tag{II-7}$$

Using Eqs. (II-2) and (II-3), we can find $R_7(T)$ as

$$R_7(T) = \sum_{i=1}^{m} \int_0^T p_i z_{2i}\phi(t)dt = \sum_{i=1}^{m} \int_0^T p_i[z_{2i}\dot{z}_{2i} + \omega_i^2 z_{2i-1}\dot{z}_{2i-1}]dt$$

$$= 1/2 \sum_{i=1}^{m} p_i[z_{2i}^2(T) + \omega_i^2 z_{2i-1}^2(T) - z_{2i}^2(0) - \omega_i^2 z_{2i-1}^2(0)]$$

$$\geq \alpha_2|\mathbf{z}(T)|^2 - \beta_2|\mathbf{z}(0)|^2, \quad \text{(II-8)}$$

where $\alpha_2 = \text{min. } [p_i/2,\ p_i\omega_i^2/2]$ and $\beta_2 = \text{max. } [p_i/2,\ p_i\omega_i^2/2]$ for all i. Obviously, α_2 and β_2 are positive constants.

Assuming $\text{Sup}_{0\leq t\leq T}|\phi(t)| = M$, then $R_8(T)$ can be found as

$$R_8(T) = q\int_0^T \mathbf{p}'\dot{\mathbf{z}}\phi(t)dt = q\int \phi(t)d(\mathbf{p}'\mathbf{z})$$

$$\geq -qM|\mathbf{p}|[|\mathbf{z}(T)| + |\mathbf{z}(0)|]$$

$$\geq -2qM|\mathbf{p}| \sup_{0\leq t\leq T} |\mathbf{z}(t)|. \quad \text{(II-9)}$$

Substituting $R_7(T)$ and $R_8(T)$ into Eq. (II-5) gives

$$\alpha_1|\mathbf{y}(T)|^2 + \gamma x_0^2(T)/2 + \alpha_2|\mathbf{z}(T)|^2 < |\mathbf{y}(0)|[\beta_1|\mathbf{y}(T)| + r_1 \sup_{0\leq t\leq T} |\mathbf{y}(t)|]$$

$$+ 2qM|\mathbf{p}| \sup_{0\leq t\leq T} |\mathbf{z}(t)| + \gamma x_0^2(0)/2 + qF(0)$$

$$+ h\phi^2(0)/2 + \beta_2|\mathbf{z}(0)|^2. \quad \text{(II-10)}$$

Now let

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix},$$

$$r_5 = \text{min. } [\alpha_1, \gamma/2, \alpha_2] \quad \text{when} \quad \gamma > 0$$

$$= \text{min. } [\alpha_1, \alpha_2] \quad \text{when} \quad \gamma = 0$$

$$r_6 = \text{max. } [2\beta_1/r_5, 2r_1/r_5, 2|\mathbf{p}|/r_5]$$

$$r_7^2 = 1/2r_5 \quad \text{and} \quad r_8 = 2\beta_2.$$

Then by the same reason as in obtaining inequality (I-16) we have

$$|\boldsymbol{\eta}(T)|^2 < r_6[|\mathbf{y}(0)| + qM] \sup_{0\leq t\leq T} |\boldsymbol{\eta}(t)|$$

$$+ r_7^2[\gamma x_0^2(0) + 2qF(0) + qh\phi^2(0) + r_8|\mathbf{z}(0)|^2]. \quad \text{(II-11)}$$

It follows from Lemma (3) that

$$|\boldsymbol{\eta}(t)| < r_6[|\mathbf{y}(0)| + qM]$$

$$+ r_7\sqrt{\gamma x_0^2(0) + 2qF(0) + qh\phi^2(0) + r_8|\mathbf{z}(0)|^2}. \quad \text{(II-12)}$$

Inequality (II-12) shows that $\boldsymbol{\eta}$ is bounded. Hence both $\mathbf{x}$ and $\mathbf{z}$ are bounded.

In view of Eq. (II-4), e has to be bounded. Since $1 + hf'(e) \neq 0$, $\dot{e}$ is also bounded. For the same reason as in Appendix I, $\int_0^T c\phi(t)dt$ is bounded. It follows from Lemmas (4) and (5) that $\lim_{t\to\infty} e = 0$ and $\lim_{t\to\infty} \mathbf{y} = 0$.

Now rewriting Eq. (II-4) as

$$\gamma x_o + \mathbf{p}'\mathbf{z} = -e - hf(e) - \mathbf{g}'\mathbf{y} = \epsilon_o(t). \tag{II-13}$$

Differentiations of Eq. (II-13) with respect to t can be written as

$$\left.\begin{array}{l} (\gamma x_0) + (p_1 z_2) + \cdots + (p_m z_{2m}) = \epsilon_o(t) \\ 0 + \omega_1^2(p_1 z_2) + \cdots + \omega_m^2(p_m z_{2m}) = \epsilon_1(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 + \omega_1^{2m}(p_1 z_2) + \cdots + \omega_m^{2m}(p_m z_{2m}) = \epsilon_m(t) \end{array}\right\}. \tag{II-14}$$

The determinant of the simultaneous linear Eq. (II-14) is not equal to zero since $\omega_i \neq \omega_j \neq 0$ for $i \neq j$ by assumption. The quantities at the right side of Eq. (II-14) approach zero as t approaches infinity. This shows that Eq. 14 has a trivial solution as t approaches infinity. Thus,

$$\gamma x_0 = p_1 z_2 = p_2 z_4 = \cdots = p_m z_{2m} = 0, \quad \text{or} \quad x_0 = z_2 = z_4 = \cdots = z_{2m} = 0$$

as t approaches infinity. By Eqs. (II-2) and (II-3), $z_1 = z_3 = \cdots = z_{2m-1} = 0$ as t approaches infinity.

This completes the proof of Theorem II. Corollaries IIa and IIb follow quite obviously after the substitution of $q = 0$ into inequality (II-12).

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