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A GENERAL METHOD FOR THE COMPUTATION

OF CARTESIAN COORDINATES AND PARTIAL DERIVATIVES

OF THE TWO-BODY PROBLEM

By W. H. Goodyear

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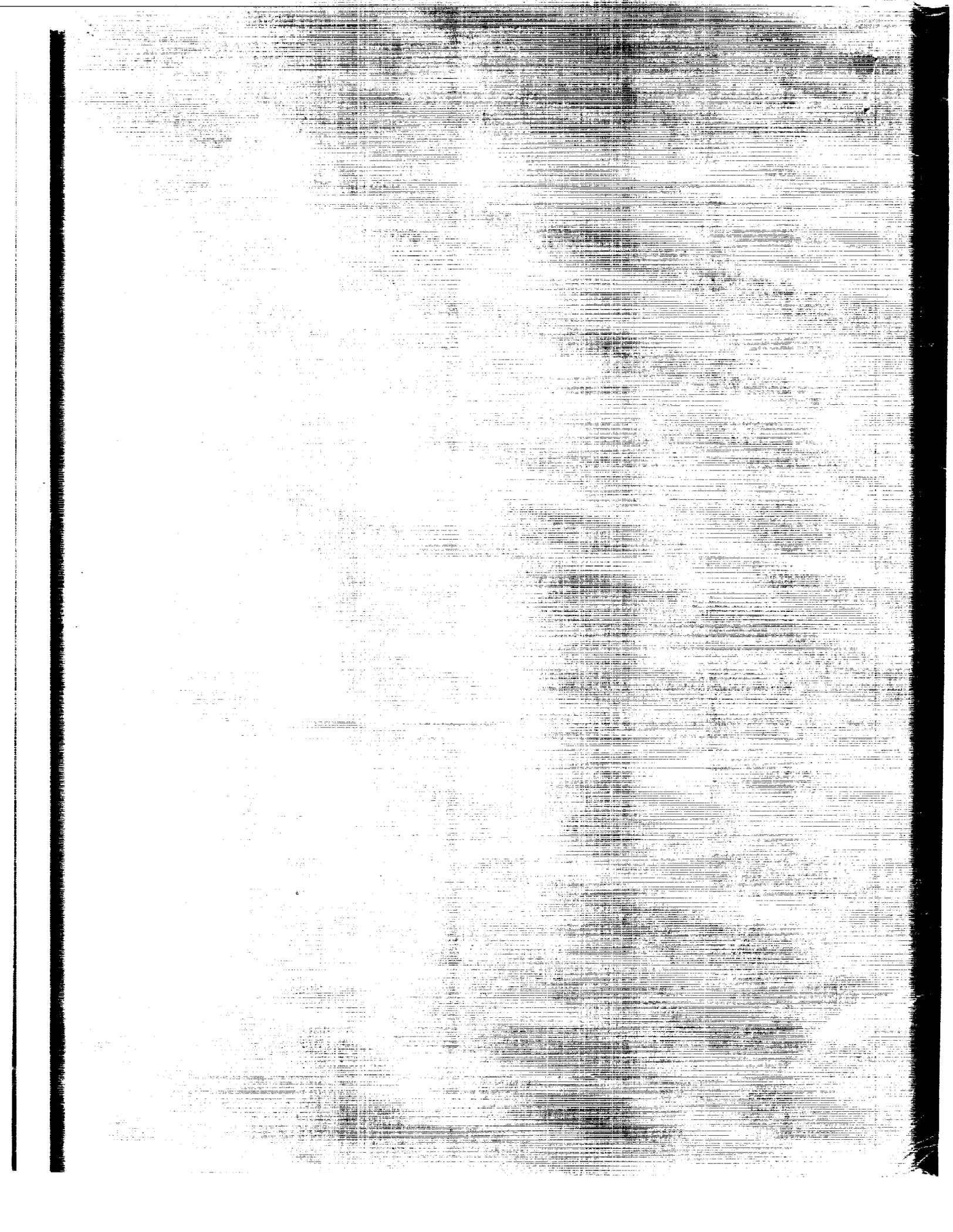
A General Method for the Computation of Cartesian Coordinates

And Partial Derivatives of the Two-Body Problem

A general solution for cartesian coordinates and partial derivatives of the two-body problem has been programmed as a double-precision Fortran 4 program for the IBM 7094. The compact subroutine provides an accurate and efficient computation of coordinates and partial derivatives for all cases of two-body motion. A derivation of all equations used by the subroutine is given for those interested in the formulation. A description of the subroutine is also given for those interested in the details of the computation.

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CONTENTS

INTRODUCTION

The classical two-body problem may be reduced to the solution of the differential equations

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$\frac{d\dot{\vec{r}}}{dt} = -\mu \frac{\vec{r}}{r^3}$$

where μ is a known constant and

$$r = \sqrt{\vec{r} \cdot \vec{r}}$$

in the magnitude of $\vec{r}$. A solution of the two-body problem gives the position vector $\vec{r}$ and velocity vector $\dot{\vec{r}}$ at any time t in terms of the position vector $\vec{r}_0$ and velocity vector $\dot{\vec{r}}_0$ at a reference time t_0. Many practical applications also require the partial derivatives of the components x, y, z of $\vec{r}$ and x, y, z of $\dot{\vec{r}}$ with respect to the components x_0, y_0, z_0 of $\vec{r}_0$ and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ of $\dot{\vec{r}}_0$. Classical formulas for the coordinates x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ and partial derivatives $\dfrac{\partial (x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)}$

have the disadvantage that several different formulations must be available and one must be selected depending on the particular values of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ and μ .

General formulas are available for computing the coordinates x, y, z, x, y, z and partial derivatives $\dfrac{\partial (x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)}$ as well as

$\dfrac{(x, y, z, \dot{x}, \dot{y}, \dot{z})}{\partial \mu}$ for all possible values of $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, t_0$, t and μ . The formulas have been programmed as a double precision Fortran 4 subroutine for the IBM 7094. The subroutine is advantageous in that it offers an efficient and compact program for the accurate computation of coordinates and partial derivatives for all possible cases of two-body motion. The purpose of this report is to give a derivation of all equations used by the subroutine and a detailed description of the computations. For those who are not interested in these details but only in using the subroutine, the inputs and outputs are described in Paragraph 4. 1 below and a Fortran 4 listing is given in Paragraph 4. 8.

The solution for the coordinates is derived in Section 1 by defining a new independent variable ψ by the differential equation

$$\frac{d\psi}{dt} = \frac{1}{r}$$

which Sundman applied to a theoretical investigation of the three-body problem. The solution is a modification of that given originally by Stumpff and differs primarily in that it is valid for all values of the constant μ . The relation of the general solution for coordinates to the classical formulas for elliptic two-body motion is also considered. It is shown that ψ is a generalization of the eccentric anomaly and that the relation between t and ψ is a generalization of Kepler's equation.

The partial derivatives are obtained in Section 2 by differentiating the solution for coordinates and manipulating the results algebraically. The approach is an extension of Sconzo's derivatives which are based on those of Kühnert. Although the derivation is tedious and involved, the final derivatives are compact and general. Some formulas used in the derivation of the coordinates and partial derivatives are discussed in Section 3 with formulas that are used for computation. A detailed description of the double-precision Fortran 4 subroutine for the IBM 7094 is given in Section 4. The description should be sufficient for those wishing to make modifications or additions to the subroutine.

1. THE SOLUTION FOR THE COORDINATES

1.1 THE EQUATIONS OF MOTION OF THE TWO-BODY PROBLEM

The equations of motion of the two-body problem are

$$m_1 \ddot{\vec{r}}_1 = -G\, m_1\, m_2\, \frac{\vec{r}_1 - \vec{r}_2}{r_{12}^3}$$

$$m_2 \ddot{\vec{r}}_2 = -G\, m_1 m_2\, \frac{\vec{r}_2 - \vec{r}_1}{r_{12}^3}$$

where

$$r_{12} = {}^+\sqrt{(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)}$$

is the distance between the two bodies and G is the universal constant of gravitation. At the time t the body of mass m_1 has the inertial cartesian position vector $\vec{r}_1$ and velocity vector $\dot{\vec{r}}_1$, and the body of mass m_2 has the inertial cartesian position vector $\vec{r}_2$ and velocity vector $\dot{\vec{r}}_2$. Each body is attracted toward the other with a force of magnitude $G\, m_1 m_2 / r_{12}^2$ where the acceleration of m_1 toward m_2 is $\ddot{\vec{r}}_1$ and the acceleration of m_2 toward m_1 is $\ddot{\vec{r}}_2$. Initial conditions are specified by the position $\vec{r}_{1,0}$ and velocity $\dot{\vec{r}}_{1,0}$ of m_1 and the position $\vec{r}_{2,0}$ and velocity $\dot{\vec{r}}_{2,0}$ of m_2 at a given reference time t_0.

A closed-form solution of the differential equations above gives $\vec{r}_1$, $\dot{\vec{r}}_1$ and $\vec{r}_2$, $\dot{\vec{r}}_2$ at any time t in terms of $\vec{r}_{1,0}$, $\dot{\vec{r}}_{1,0}$ and $\vec{r}_{2,0}$, $\dot{\vec{r}}_{2,0}$ at time t_0 as well as the constants m_1, m_2 and G. Such a solution is extremely important for many practical applications in astronomy and astronautics. However, it is convenient to transform the differential equations by defining the position $\vec{R}$ of the center of mass and the position $\vec{r}$ of m_2 relative to m_1 by

$$\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

so that

$$\dot{\vec{R}} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2}$$

$$\dot{\vec{r}} = \dot{\vec{r}}_2 - \dot{\vec{r}}_1$$

give the velocity $\dot{\vec{R}}$ of the center of mass and the velocity $\dot{\vec{r}}$ of m_2 relative to m_1. Thus the values of the four quantities above at the reference time t_0 are

$$\vec{R}_0 = \frac{m_1 \vec{r}_{1,0} + m_2 \vec{r}_{2,0}}{m_1 + m_2}$$

$$\dot{\vec{R}}_0 = \frac{m_1 \dot{\vec{r}}_{1,0} + m_2 \dot{\vec{r}}_{2,0}}{m_1 + m_2}$$

and

$$\vec{r}_0 = \vec{r}_{2,0} - \vec{r}_{1,0}$$

$$\dot{\vec{r}}_0 = \dot{\vec{r}}_{2,0} - \dot{\vec{r}}_{1,0} .$$

Time differentiation of $\dot{\vec{R}}$ and $\dot{\vec{r}}$ above and substitution from the differential equations show that

$$\ddot{\vec{R}} = 0$$

$$\ddot{\vec{r}} = - G (m_1 + m_2) \vec{r} / r^3$$

where

$$r = \sqrt[+]{\vec{r} \cdot \vec{r}}$$

An integration of these differential equations gives $\vec{R}$, $\dot{\vec{R}}$ and $\vec{r}$, $\dot{\vec{r}}$ at any time t. Simultaneous solution of the defining equations for $\vec{R}$ and $\vec{r}$ shows that

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2} \vec{r}$$

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and time differentiation gives

$$\dot{\vec{r}}_1 = \dot{\vec{R}} - \frac{m_2}{m_1 + m_2} \dot{\vec{r}}$$

$$\dot{\vec{r}}_2 = \dot{\vec{R}} + \frac{m_1}{m_1 + m_2} \dot{\vec{r}}$$

so that $\vec{r}_1$, $\dot{\vec{r}}_1$ and $\vec{r}_2$, $\dot{\vec{r}}_2$ may always be determined from $\vec{R}$, $\dot{\vec{R}}$ and $\vec{r}$, $\dot{\vec{r}}$.

Thus the integration of the differential equations for $\vec{r}_1$ and $\vec{r}_2$ has been transformed to an integration of differential equations for $\vec{R}$ and $\vec{r}$. Integration of the differential equation

$$\ddot{\vec{R}} = 0$$

for $\vec{R}$ shows that

$$\dot{\vec{R}} = \dot{\vec{R}}_0$$

and

$$\vec{R} = \vec{R}_0 + \dot{\vec{R}}_0 (t - t_0)$$

so that the center of mass moves along a straight line at constant velocity. However, the inertial coordinates $\vec{R}_0$, $\dot{\vec{R}}_0$ and thus $\vec{R}$, $\dot{\vec{R}}$ at any time t are usually unknown or the center of mass is chosen as the origin of inertial coordinates. In any case, the two-body problem reduces to the solution of the differential equation

$$\ddot{\vec{r}} = - \mu \vec{r}/r^3$$

where

$$r = \sqrt{\vec{r} \cdot \vec{r}}$$

is the magnitude of $\vec{r}$ and

$$\mu = G (m_1 + m_2)$$

is a postive constant. This differential equation is actually obtained for all inverse-square Newtonian forces between two bodies,

but the constant μ may be different. For example, the same differential equation describes the electrostatic repulsion of two electrons but μ is then negative and is a more complicated function of the electronic mass and charge.

1.2 DERIVATIVES WITH RESPECT TO A NEW VARIABLE

The differential equation

$$\ddot{\vec{r}} = -\mu\, \vec{r}/r^3$$

where

$$r = +\sqrt{\vec{r}\cdot\vec{r}}$$

is conveniently integrated by defining a new independent variable ψ by the differential equation of the Sundman transformation,

$$\dot{\psi} = 1/r$$

where ψ is zero when t is t_0. Using a prime to denote differentiation with respect to ψ,

$$t' = 1/\dot{\psi} = 1/(1/r)$$

$$= r$$

from the properties of derivatives. The chain rule for differentiation shows that

$$\vec{r}' = \dot{\vec{r}}\, t'$$

$$= \dot{\vec{r}}\, r.$$

The method of solution described below requires determination of the successive derivatives of t and $\vec{r}$ with respect to ψ. The second derivative of t is

$$t'' = r' = \frac{1}{2}(\vec{r}\cdot\vec{r})^{-\frac{1}{2}}\, 2\vec{r}\cdot\vec{r}' = \frac{\vec{r}}{r}\cdot\dot{\vec{r}}\, r$$

$$= \vec{r}\cdot\dot{\vec{r}}$$

where the quantity

$$\sigma = \vec{r}\cdot\dot{\vec{r}}$$

is defined so that

$$t'' = r' = \sigma .$$

The second derivative of $\vec{r}$ is

$$\vec{r}'' = \dot{\vec{r}}'\, r + \dot{\vec{r}}\, r'$$

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but

$$\ddot{\vec{r}}{}' = \ddot{\vec{r}} \, t' = (-\,\mu \vec{r}/r^3)\, r$$
$$= -\,\mu\,\vec{r}/r^2$$

so that

$$\vec{r}{}'' = -\,\mu\,\vec{r}/r + \sigma\,\dot{\vec{r}}.$$

The third derivative of t is

$$t''' = r'' = \sigma' = \vec{r}{}' \cdot \dot{\vec{r}} + \vec{r} \cdot \dot{\vec{r}}{}'$$
$$= r\dot{\vec{r}} \cdot \dot{\vec{r}} + \vec{r}\cdot(-\,\mu\,\vec{r}/r^2)$$
$$= r\,(\dot{\vec{r}} \cdot \dot{\vec{r}}) - \mu\ .$$

However, the quantity

$$\alpha = \dot{\vec{r}} \cdot \dot{\vec{r}} - 2\,\mu/r$$

is defined to obtain

$$t''' = r'' = \sigma' = r(\dot{\vec{r}} \cdot \dot{\vec{r}}) - 2\,\mu + \mu = r(\dot{\vec{r}} \cdot \dot{\vec{r}} - 2\,\mu/r) + \mu$$
$$= \alpha\,r + \mu.$$

The third derivative of $\vec{r}$ is then

$$\vec{r}{}''' = -\,\mu\,\vec{r}{}'/r + \mu\,\vec{r}\,r'/r^2 + \sigma\dot{\vec{r}} + \sigma\dot{\vec{r}}{}'$$
$$= -\,\mu\,\dot{\vec{r}} + \mu\,\vec{r}\,\sigma/r^2 + (\,\alpha\,r + \mu\,)\dot{\vec{r}} - \sigma\,.\,\mu\vec{r}/r^2$$
$$= \alpha\,r\,\dot{\vec{r}}.$$

The quanity α has the important property that

$$\alpha' = 2\dot{\vec{r}} \cdot \dot{\vec{r}}{}' + 2\,\mu\,r'/r^2$$
$$= 2\dot{\vec{r}}\cdot(-\,\mu\,\vec{r}/r^2) + 2\,\mu\,\sigma/r^2$$
$$= -2\,\mu\,\sigma/r^2 + 2\,\mu\,\sigma/r^2$$
$$= 0$$

so that α is constant for every value of ψ as well as t. This property gives all the derivatives of t and $\vec{r}$ as simple functions of the first two derivatives t', t'' and $\vec{r}{}'$, $\vec{r}{}''$. From the results above,

$$t''' = \alpha\,t' + \mu$$
$$\vec{r}{}''' = \alpha\,\vec{r}{}'$$

so that

$$t'''' = \alpha\, t''$$

$$\vec{r}'''' = \alpha\, \vec{r}''$$

$$t''''' = \alpha\, t''' = \alpha\, (\,\alpha\, t' + \mu\,)$$

$$= \alpha^2 t' + \alpha\,\mu$$

$$\vec{r}''''' = \alpha\, \vec{r}''' = \alpha\, (\,\alpha\, \vec{r}')$$

$$= \alpha^2\, \vec{r}'$$

$$t'''''' = \alpha^2\, t''$$

$$\vec{r}'''''' = \alpha^2\, \vec{r}''$$

$$t''''''' = \alpha^2\, t''' = \alpha^2\, (\,\alpha\, t' + \mu)$$

$$= \alpha^3\, t' + \alpha^2 \mu$$

$$\vec{r}''''''' = \alpha^2\, \vec{r}''' = \alpha^2\, (\,\alpha\, \vec{r}')$$

$$= \alpha^3\, \vec{r}'$$

etc.

The final result of the above derivation is that

$$t' = r$$

$$t'' = \sigma$$

$$t''' = \alpha\, r + \mu$$

$$t'''' = \alpha\, \sigma$$

$$t''''' = \alpha^2 r + \alpha\,\mu$$

$$t'''''' = \alpha^2 \sigma$$

$$t''''''' = \alpha^3 r + \alpha^2 \mu$$

$$t'''''''' = \alpha^3 \sigma$$

etc.

and that

$$\vec{r}' = r\,\dot{\vec{r}}$$

$$\vec{r}'' = -\mu\,\vec{r}/r + \sigma\,\dot{\vec{r}}$$

$$\vec{r}''' = \alpha\,(r\,\dot{\vec{r}})$$

$$\vec{r}'''' = \alpha\,(-\mu\,\vec{r}/r + \sigma\,\dot{\vec{r}})$$

$$\vec{r}''''' = \alpha^2\,(r\,\dot{\vec{r}})$$

$$\vec{r}'''''' = \alpha^2\,(-\mu\,\vec{r}/r + \sigma\,\dot{\vec{r}})$$

$$\vec{r}''''''' = \alpha^3\,(r\,\dot{\vec{r}})$$

$$\vec{r}'''''''' = \alpha^3\,(-\mu\,\vec{r}/r + \sigma\,\dot{\vec{r}})$$

etc.

where

$$\alpha = \dot{\vec{r}}_0 \cdot \dot{\vec{r}}_0 - 2\,\mu/r_0$$

since α' is zero and α is invariant for all values of ψ as well as t.

1.3 SERIES SOLUTION IN THE NEW VARIABLE

The Taylor's series for t in terms of ψ is

$$t = t_0 + t_0' \; \psi + t_0'' \; \psi^2/2: + t_0''' \; \psi^3/3! + t_0'''' \; \psi^4/4! \; \ldots\ldots$$

where t_0, t_0', t_0'', t_0''', t_0'''', $\ldots\ldots$ are the values of t, t', t'', t''', t'''', $\ldots$ when ψ is zero, i.e. when t is t_0. But

$$t_0' = r_0$$

$$t_0'' = \sigma_0$$

$$t_0''' = \alpha \; r_0 + \mu$$

$$t_0'''' = \alpha \; \sigma_0$$

$$t_0''''' = \alpha^2 \; r_0 + \alpha \; \mu$$

$$t_0'''''' = \alpha^2 \; \sigma_0$$

etc.

from the results above where

$$\alpha = \vec{r}_0 \cdot \vec{r}_0 - 2 \; \mu / r_0.$$

Thus substitution for t_0', t_0'', t_0''', t_0'''', $\ldots$ in the Taylor's series gives

$$t = t_0 + r_0 \; \psi + \sigma_0 \; \psi^2/2!$$
$$+ (\alpha \; r_0 + \mu) \; \psi^3/3! + \alpha \; \sigma_0 \; \psi^4/4!$$
$$+ (\alpha^2 r_0 + \alpha \mu) \; \psi^5/5! + \alpha^2 \; \sigma_0 \; \psi^6/6!$$
$$+ \; \ldots\ldots\ldots$$

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or, collecting coefficients of r_0, σ_0, and μ,

$$t = t_0 \quad + r_0 (\quad \psi + \alpha \quad \psi^3/3! + \alpha^2 \quad \psi^5/5! + \alpha^3 \quad \psi^7/7! + \ldots)$$

$$+ \sigma_0 (\quad \psi^2/2! + \alpha \quad \psi^4/4! + \alpha^2 \quad \psi^6/6! + \alpha^3 \quad \psi^8/8! + \ldots)$$

$$+ \mu (\quad \psi^3/3! + \alpha \quad \psi^5/5! + \alpha^2 \quad \psi^7/7! + \alpha^3 \quad \psi^9/9! + \ldots)$$

gives the value of t corresponding to any particular value of ψ.

Similarly, the Taylor's series for $\vec{r}$ in terms of ψ is

$$\vec{r} = \vec{r}_0 + \vec{r}_0' \psi + \vec{r}_0'' \ \psi^2/2! + \vec{r}_0''' \quad \psi^3/3! + \vec{r}_0'''' \quad \psi^4/4! + \ldots$$

where

$$\vec{r}_0' = r_0 \dot{\vec{r}}_0$$

$$\vec{r}_0'' = -\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0$$

$$\vec{r}_0''' = \alpha r_0 \dot{\vec{r}}_0$$

$$\vec{r}_0'''' = \alpha (-\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0)$$

$$\vec{r}_0''''' = \alpha^2 r_0 \dot{\vec{r}}_0$$

$$\vec{r}_0'''''' = \alpha^2 (-\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0)$$

etc.

from the results above. Thus substitution for $\vec{r}_0'$, $\vec{r}_0''$, $\vec{r}_0'''$, $\ldots$ in the Taylor's series gives

$$\vec{r} = \vec{r}_0 + r_0 \dot{\vec{r}}_0 \psi \quad + (-\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0) \ \psi^2/2!$$

$$+ \alpha r_0 \dot{\vec{r}}_0 \ \psi^3/3! + \alpha (-\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0) \ \psi^4/4!$$

$$+ \alpha^2 r_0 \dot{\vec{r}}_0 \ \psi^5/5! + \alpha^2 (-\mu \vec{r}_0/r_0 + \sigma_0 \dot{\vec{r}}_0) \ \psi^6/6!$$

$$+ \ \ldots\ldots\ldots$$

or, collecting coefficients of $\vec{r}_0$ and $\dot{\vec{r}}_0$,

$$\vec{r} = [1 - \frac{\mu}{r_0}(\psi^2/2! + \alpha\, \psi^4/4! + \alpha^2\, \psi^6/6! + \alpha^3\, \psi^8/8! + \ldots)]\vec{r}_0$$

$$+ \begin{bmatrix} r_0(\psi + \alpha\, \psi^3/3! + \alpha^2\, \psi^5/5! + \alpha^3\, \psi^7/7! + \ldots) \\ + \sigma_0(\psi^2/2! + \alpha\, \psi^4/4! + \alpha^2\, \psi^6/6! + \alpha^3\, \psi^8/8! + \ldots) \end{bmatrix} \dot{\vec{r}}_0$$

which gives the value of $\vec{r}$ corresponding to any value of ψ.

The results above are conveniently summarized by defining the transcendental functions

$$s_0 = 1 + \alpha\, \psi^2/2! + \alpha^2\, \psi^4/4! + \alpha^3\, \psi^6/6! + \ldots$$

$$s_1 = \psi + \alpha\, \psi^3/3! + \alpha^2\, \psi^5/5! + \alpha^3\, \psi^7/7! + \ldots$$

$$s_2 = \psi^2/2! + \alpha\, \psi^4/4! + \alpha^2\, \psi^6/6! + \alpha^3\, \psi^8/8! + \ldots$$

$$s_3 = \psi^3/3! + \alpha\, \psi^5/5! + \alpha^2\, \psi^7/7! + \alpha^3\, \psi^9/9! + \ldots$$

which have the derivatives

$$s_0' = \alpha\, s_1$$

$$s_1' = s_0$$

$$s_2' = s_1$$

$$s_3' = s_2$$

with respect to ψ. In this notation

$$t = t_0 + r_0\, s_1 + \sigma_0\, s_2 + \mu\, s_3$$

gives t for any value of ψ. Also

$$r = t' = r_0 s_1' + \sigma_0\, s_2' + \mu\, s_3'$$

$$= r_0 s_0 + \sigma_0\, s_1 + \mu\, s_2$$

and

$$\sigma = r' = r_0 s_0' + \sigma_0\, s_1' + \mu\, s_2' = r_0\, \alpha s_1 + \sigma_0\, s_0 + \mu\, s_1$$

$$= \sigma_0 s_0 + (\alpha r_0 + \mu)\, s_1$$

gives r and σ corresponding to any value of ψ.

It is also convenient to define the functions

$$f = 1 - \mu s_2/r_0$$

$$g = r_0 s_1 + \sigma_0 s_2$$

which have the derivatives

$$f' = - \mu s_2'/r_0$$

$$= - \mu s_1/r_0$$

$$g' = r_0 s_1' + \sigma_0 s_2'$$

$$= r_0 s_c + \sigma_0 s_1.$$

However, the equation for t as a function of ψ shows that

$$g = t - t_0 - \mu s_3$$

is an alternate expression for g. Differentiation of this equation shows that

$$g' = t' - \mu s_3'$$

$$= r - \mu s_2$$

is an alternate expression for g'.

In terms of f and g,

$$\vec{r} = f \vec{r}_0 + g \dot{\vec{r}}_0$$

gives $\vec{r}$ corresponding to any value of ψ. Differentiation of this equation with respect to ψ gives

$$\vec{r}' = f' \vec{r}_0 + g' \dot{\vec{r}}_0$$

and differentiation of $\vec{r}$ with respect to t gives

$$\dot{\vec{r}} = \dot{f} \vec{r}_0 + \dot{g} \dot{\vec{r}}_0$$

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where

$$\dot{f} = f' \dot{\psi} = (- \mu\, s_1/r_0)\,(1/r)$$

$$= -\mu\, s_1/(r_0 r)$$

and

$$\dot{g} = g' \dot{\psi} = (r - \mu s_2)\,(1/r)$$

$$= 1 - \mu\, s_2/r$$

or

$$\dot{g} = g' \dot{\psi} = (r_0 s_0 + \sigma_0 s_1)\,(1/r)$$

$$= (r_0 s_0 + \sigma_0 s_1)/r.$$

Another form of the equation relating t and ψ above is obtained by defining

$$\gamma = r(\vec{\dot{r}} \cdot \vec{\dot{r}}) - \mu$$

which satisfies the identity

$$\gamma = r\,(\vec{\dot{r}} \cdot \vec{\dot{r}} - 2\mu/r) + \mu$$

$$= \alpha r + \mu .$$

Then

$$t = t_0 + r_0\, s_1 + \sigma_0\, s_2 + \mu s_3$$

$$= t_0 + r_0\,(\psi + \alpha s_3) + \sigma_0\, s_2 + \mu s_3$$

$$= t_0 + r_0\, \psi + \sigma_0\, s_2 + (\alpha r_0 + \mu)\, s_3$$

so that

$$t = t_0 + r_0\, \psi + \sigma_0\, s_2 + \gamma_0 s_3.$$

Also, differentiation of this equation gives alternate expressions for r and σ.

$$t' = r = r_0 + \sigma_0 s_1 + \gamma_0 s_2$$

$$t'' = r' = \sigma = \sigma_0 s_0 + \gamma_0 s_1.$$

In addition

$$\sigma' = (\vec{r} \cdot \dot{\vec{r}})' = \vec{r}' \cdot \dot{\vec{r}} + \vec{r} \cdot \dot{\vec{r}}' = r\,\dot{\vec{r}} \cdot \dot{\vec{r}} + \vec{r} \cdot (-\mu\,\vec{r}/r^2) = r(\dot{\vec{r}} \cdot \dot{\vec{r}}) - \mu$$

$$= \gamma$$

so that

$$t''' = r'' = \sigma' = \gamma = \gamma_0\, s_0 + \sigma_0\,\alpha s_1.$$

1.4 GENERAL SOLUTION FOR COORDINATES

The formulas above may be used to calculate $\vec{r}$, $\dot{\vec{r}}$ from given values of $\vec{r}_0$, $\dot{\vec{r}}_0$ and μ, t_0,t. First

$$r_0 = \sqrt{\vec{r}_0 \cdot \vec{r}_0}$$

$$\sigma_0 = \vec{r}_0 \cdot \dot{\vec{r}}_0$$

$$\alpha = \dot{\vec{r}}_0 \cdot \dot{\vec{r}}_0 - 2\mu/r_0$$

are determined. Then the parameter ψ and its transcendental functions

$$s_0 = 1 + \alpha \psi^2/2! + \alpha^2 \psi^4/4! + \alpha^3 \psi^6/6! + \dots$$

$$s_1 = \psi + \alpha \psi^3/3! + \alpha^2 \psi^5/5! + \alpha^3 \psi^7/7! + \dots$$

$$s_2 = \psi^2/2! + \alpha \psi^4/4! + \alpha^2 \psi^6/6! + \alpha^3 \psi^8/8! + \dots$$

$$s_3 = \psi^3/3! + \alpha \psi^5/5! + \alpha^2 \psi^7/7! + \alpha^3 \psi^9/9! + \dots$$

are obtained by solving the equation

$$t = t_0 + r_0 s_1 + \sigma_0 s_2 + \mu s_3$$

for ψ. Then

$$r = r_0 s_0 + \sigma_0 s_1 + \mu s_2$$

and

$$f = 1 - \mu s_2/r_0 \qquad\qquad g = (t - t_0) - \mu s_3$$

$$\dot{f} = -\mu s_1/(r\, r_0) \qquad\qquad \dot{g} = 1 - \mu s_2/r$$

give the final solution for the coordinates.

$$\vec{r} = f\, \vec{r}_0 + g\, \dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f}\, \vec{r}_0 + \dot{g}\, \dot{\vec{r}}_0$$

These are the equations for coordinates given previously by the author (1965).

The great advantage of these formulas is their complete generality. They are a modification of equations which were derived by Karl Stumpff (1947). Stumpff's derivation differs somewhat from that above and he normalizes his equations for computational convenience and simplicity. Similar formulas have been obtained independently by Herrick (1960) and Sperling (1961), but Stumpff was apparently the first to recognize that a generalized solution for coordinates of the two-body problem could be obtained by utilizing new transcendental functions similar to s_0, s_1, s_2, s_3 above. Stumpff's work (1947) is given in his textbook (1959) and is also available in English (1962). The formulas above differ primarily in that the equations are not normalized but are left in a form that is continuous through the trival cases where μ or $(t - t_0)$ is zero. The formulas are thus equally valid for negative values of μ or $(t - t_0)$.

The transformation from the variable t to the variable ψ is often treated as a regularizing transformation since the solution for $\vec{r}$ above is continuous through a collision which can occur if $\mu > 0$ and $\dot{\vec{r}}_0$ is parallel to $\vec{r}_0$. For such a collision $\dot{\vec{r}}$ is indeterminate at the origin but $\vec{r}'$ is zero at the origin and continuous through the collision. Also, r and σ equal zero at the origin and are continuous through the collision. The continuation of the solution through a collision is of course of little physical importance. However, it is practically advantageous because numerical problems are eliminated in the computation of position coordinates for near-collisions.

The regularizing transformation from t to ψ was used by Sundman (1912) in an investigation of the three-body problem. Stumpff realized that the transformation was of computational

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value in solving the two-body problem, and that the resulting formulas could be placed in a form that was equally valid for elliptic, parabolic, and hyperbolic orbits including the cases of circular and rectilinear motion. He also realized the relation of the general formulas to classic formulas for elliptic, parabolic, and hyperbolic motion. This provides an alternate way of deriving the general solution, i.e. it may be obtained by manipulating classical formulas. Herrick (1960) used this approach and his formulas were modified to obtain the general solution above before Sconzo informed the author of Stumpff's prior work.

1.5 CORRESPONDENCE TO CLASSICAL FORMULAS

In the case where α is negative, let

$$\frac{\mu}{a} = -\alpha$$

define the length a and let

$$\Delta E = \sqrt[+]{\mu/a}\ \psi$$

define the angle ΔE. Then the functions s_0, s_1, s_2, s_3 may be expressed in terms of $\sqrt{\mu/a}$ and trigonometric functions of ΔE. For example,

$$s_3 = \psi^3/3! + \alpha\ \psi^5/5! + \alpha^2\ \psi^7/7! + \alpha^3\ \psi^9/9! + \ldots$$

$$= \psi^3/3! - (\mu/a)\ \psi^5/5! + (\mu/a)^2\ \psi^7/7! - (\mu/a)^3 \psi^9/9! + \ldots$$

$$= \frac{\sqrt{\mu/a}^3\ \psi^3/3! - \sqrt{\mu/a}^5\ \psi^5/5! + \sqrt{\mu/a}^7\ \psi^7/7! - \ldots}{\sqrt{\mu/a}^3}$$

$$= \frac{(\sqrt{\mu/a}\ \psi)^3/3! - (\sqrt{\mu/a}\ \psi)^5/5! + (\sqrt{\mu/a}\ \psi)^7/7! - \ldots}{\sqrt{\mu/a}^3}$$

$$= \frac{\Delta^3E/3! - \Delta^5E/5! + \Delta^7E/7! - \ldots}{\sqrt{\mu/a}^3}$$

$$= \frac{\Delta E - (\Delta E - \Delta^3E/3! + \Delta^5E/5! - \Delta^7E/7! - \ldots)}{\sqrt{\mu/a}^3}$$

$$= \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3}$$

with similar derivations for s_0, s_1, s_2.

The result is that

$$s_0 = \cos \Delta E$$

$$s_1 = \frac{\sin \Delta E}{\sqrt{\mu/a}}$$

$$s_2 = \frac{1 - \cos \Delta E}{(\mu/a)}$$

$$s_3 = \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3}$$

These formulas give the following alternate solution when α is negative (which can occur only when μ is positive). First

$$r_0 = \sqrt[+]{\vec{r}_0 \cdot \vec{r}_0}$$

$$\sigma_0 = \vec{r}_0 \cdot \dot{\vec{r}}_0$$

$$(\mu/a) = 2\mu/r_0 - \dot{\vec{r}}_0 \cdot \dot{\vec{r}}_0$$

are determined. Then solution of the equation

$$t = t_0 + r_0 \frac{\sin \Delta E}{\sqrt{\mu/a}} + \sigma_0 \frac{1 - \cos \Delta E}{(\mu/a)} + \mu \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3}$$

gives ΔE. Then

$$r = r_0 \cos \Delta E + \sigma_0 \frac{\sin \Delta E}{\sqrt{\mu/a}} + \mu \frac{1 - \cos \Delta E}{(\mu/a)}$$

and

$$f = 1 - \frac{\mu}{r_0} \frac{1 - \cos \Delta E}{\mu/a} \qquad g = t - t_0 - \mu \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3}$$

$$\dot{f} = - \frac{\mu}{r \, r_0} \frac{\sin \Delta E}{\sqrt{\mu/a}} \qquad \dot{g} = 1 - \frac{\mu}{r} \frac{1 - \cos \Delta E}{(\mu/a)}$$

give

$$\vec{r} = f \vec{r}_0 + g \dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f} \vec{r}_0 + \dot{g} \dot{\vec{r}}_0$$

This solution is also valid when α is positive although ΔE and $\sqrt{\mu/a}$ are then imaginary quantities. However, the solution is not valid when α is zero and its numerical accuracy decreases as α approaches zero.

The formulas for t, r, f, g, $\dot{f}$, $\dot{g}$ above all reduce to classical forms such as those given by Herget (1948) if

$$\Delta E = E - E_0$$

where E and E_0 are, respectively, the eccentric anomalies at time t and at time t_0. The well-known relations

$$r_0 = a (1 - e\cos E_0)$$

$$\sigma_0 = \sqrt{\mu \, a} \; e \, \sin E_0$$

are used to obtain the results. For example,

$$t = t_0 + r_0 \; \frac{\sin \Delta E}{\sqrt{\mu/a}} \; + \; \sigma_0 \; \frac{1 - \cos \Delta E}{\mu/a} \; + \mu \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3}$$

$$= t_0 + a \, (1 - e \cos E_0) \; \sin \Delta E \, / \sqrt{\mu/a}$$

$$+ \sqrt{\mu a} \; e \, \sin E_0 \, (1 - \cos \Delta E) \, / \, (\mu/a)$$

$$+ \mu \, (\Delta E - \sin \Delta E) \, / \sqrt{\mu/a}^{\,3}$$

so that

$$\sqrt{\mu/a^3}\,(t - t_0) = (1 - e \cos E_0) \sin \Delta E$$

$$+\, e \sin E_0 \, (1 - \cos \Delta E)$$

$$+\, \Delta E - \sin \Delta E$$

$$= \sin \Delta E - e \cos E_0 \sin \Delta E + e \sin E_0$$

$$-\, e \sin E_0 \cos \Delta E$$

$$+\, \Delta E - \sin \Delta E$$

$$= \Delta E + e \sin E_0 - e\,(\sin E_0 \cos \Delta E + \cos E_0 \sin \Delta E)$$

$$= \Delta E + e \sin E_0 - e \sin (E_0 + \Delta E)$$

$$= (E - E_0) + e \sin E_0 - e \sin [E_0 + (E - E_0)]$$

reduces to the classic form of Kepler's equation

$$E - e \sin E = E_0 - e \sin E_0 + \sqrt{\mu/a^3}\,(t - t_0).$$

Since all the steps above are reversible, an alternate derivation is to define

$$\psi = \frac{E - E_0}{\sqrt{\mu/a}}$$

and express Kepler's equation and the classic relations

$$\dot{r} = a\,(1 - e \cos E)$$

$$f = 1 - \frac{a}{r_0}\,[1 - \cos (E - E_0)]$$

$$g = t - t_0 - \frac{(E - E_0) - \sin (E \quad E_0)}{\sqrt{\mu/a^3}}$$

$$\dot{f} = - \frac{\sqrt{\mu a}}{r_0 \, r}\,\sin (E - E_0)$$

$$\dot{g} = 1 - \frac{a}{r}\,[1 - \cos (E - E_0)]$$

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in terms of ψ and α .

In the case where α is positive, let

$$\Delta F = \sqrt{\alpha}\ \psi$$

define the angle ΔF. An equivalent definition

$$\Delta F = i \quad \Delta E$$

where $i = \sqrt{-1}$ and

$$\alpha = -\mu/a .$$

Either definition gives

$$s_0 = \cosh \Delta F$$

$$s_1 = \frac{\sinh \quad F}{\sqrt{\alpha}}$$

$$s_2 = \frac{\cosh \Delta F - 1}{\alpha}$$

$$s_3 = \frac{\sinh \Delta F - \Delta F}{\sqrt{\alpha^3}}$$

which may be reduced as above to obtain classic formulas for hyperbolic orbits. The solutions in terms of ΔE and ΔF are both equally valid for all cases where α is not zero, but imaginary quantities are eliminated by utilizing ΔE for negative α and ΔF for positive α . However, both solutions decrease in accuracy as α approaches zero.

In the case where α is zero, the functions s_0, s_1, s_2, s_3 reduce to

$$s_0 = 1$$

$$s_1 = \psi$$

$$s_2 = \psi^2/2$$

$$s_3 = \psi^3/3\ !$$

since only the leading term in these series is then non-zero. The resulting formulas may also be reduced to classical formulas for parabolic orbits. This suggests the possibility of using ΔE for negative α, ψ for zero α, and ΔF for positive α, which corresponds to the classical approach. However, use of ψ in the general solution for all values of α is advantageous for several reasons. Computer storage requirements are reduced since only one formulation is required. The general soltuion is also continuous through zero α so that small values of α do no require a separate formulation for nearly-parabolic orbits. These advantages can become very important in practical applications where different types of orbits may be encountered.

2. THE PARTIAL DERIVATIVES OF THE SOLUTION

2.1 DIFFERENTIATION OF THE SOLUTION FOR COORDINATES

The solutions $\vec{r}$ and $\dot{\vec{r}}$ are functions of the initial conditions $\vec{r}_0$ and $\dot{\vec{r}}_0$ as well as μ and the times t and t_0. The differential relationships giving $d\vec{r}$ and $d\dot{\vec{r}}$ in terms of $d\vec{r}_0$ and $d\dot{\vec{r}}_0$ as well as $d\mu$ and the time differentials dt and dt_0 are very important. The differential relation between $d\vec{r}$ and $d\vec{r}_0$, $d\dot{\vec{r}}_0$ is given by Herget in a form due to Bower (1932) which is similar to the earlier work of Kühnert (1879). Sconzo (1963) has extended this approach to give the differential relation between $d\dot{\vec{r}}$ and $d\vec{r}_0$, $d\dot{\vec{r}}_0$ as well. The approach given here is similar but concise expressions are obtained which are valid for all cases of two-body motion. The differential relationships are obtained by first differentiating the equations of the solution for coordinates and then combining the results to eliminate all differentials other than $d\vec{r}$, $d\dot{\vec{r}}$ and $d\vec{r}_0$, $d\dot{\vec{r}}_0$, $d\mu$, dt, dt_0.

The differentiations are as follows.

Differentiation of

$$\vec{r} = f \, \vec{r}_0 + g \, \dot{\vec{r}}_0$$
$$\dot{\vec{r}} = \dot{f} \, \vec{r}_0 + \dot{g} \, \dot{\vec{r}}_0$$

gives

$$d\vec{r} = f \, d\vec{r}_0 + g \, d\dot{\vec{r}}_0$$
$$+ \vec{r}_0 \, d f + \dot{\vec{r}}_0 \, d g$$
$$d\dot{\vec{r}} = \dot{f} \, d\vec{r}_0 + \dot{g} \, d\dot{\vec{r}}_0$$
$$+ \vec{r}_0 \, d \dot{f} + \dot{\vec{r}}_0 \, d \dot{g}.$$

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Then

$$f = 1 - \mu s_2 / r_0$$

gives

$$d f = -d\mu \ s_2 / r_0 - \mu \ d s_2 / r_0 + \mu \ s_2 \ d \ r_0 / r_0^2 \qquad ,$$

$$= [-(f - 1) \ d \ r_0 - \mu \ d \ s_2 - s_2 \ d \mu] / \ r_0$$

and

$$g = t - t_0 - \mu \ s_3$$

gives

$$dg = dt - d \ t_0 - d \mu \ s_3 - \mu \ d \ s_3$$

$$= - \mu \ d \ s_3 - s_3 \ d \mu + dt - dt_0 .$$

But the alternate expression

$$g = r_0 \ s_1 + \sigma_0 \ s_2$$

for g gives

$$dg = r_0 \ d \ s_1 + s_1 \ d \ r_0 + \sigma_0 \ d \ s_2 + s_2 \ d\sigma_0$$

$$= s_1 \ d \ r_0 + r_0 \ d \ s_1 + \sigma_0 \ d \ s_2 + s_2 \ d \ \sigma_0$$

Similarly,

$$\dot{f} = - \mu \ s_1 / (r \ r_0)$$

gives

$$d \dot{f} = - d \mu \ s_1 / (r \ r_0) - \mu \ d \ s_1 / (r \ r_0) + \mu \ s_1 \ d \ r_0 / (r \ r_0^2) + \mu s_1 \ d \ r / (r^2 \ r_0)$$

$$= - \dot{f} \ dr_0 / r_0 - \dot{f} \ dr / r - \mu \ d \ s_1 / (r \ r_0) - d \ \mu \ s_1 / (r \ r_0)$$

and

$$\dot{g} = 1 - \mu s_2 / r$$

gives

$$d \dot{g} = - d \mu s_2 / r - \mu \ d \ s_2 / r + \mu s_2 dr / r^2$$

$$= [-\mu \ d \ s_2 - (\dot{g} - 1) \ d \ r - s_2 d\mu] \ / \ r .$$

But the alternate expression

$$\dot{g} = (r_0 s_0 + \sigma_0 s_1)/r$$

gives

$$dg = (r_0\ ds_0 + s_0\ dr_0 + \sigma_0\ ds_1 + s_1\ d\ \sigma_0)/r$$
$$- (r_0 s_0 + \sigma_0 s_1)\ dr/r^2$$

$$= [s_0\ dr_0 - \dot{g}\ dr + r_0\ ds_0 + \sigma_0\ ds_1 + s_1\ d\sigma_0]/r.$$

The differentiation of the equation

$$t = t_0 + r_0\ s_1 + \sigma_0\ s_2 + \mu\ s_3$$

is equivalent to equating the two different expressions above for dg.
This gives

$$-\mu\ ds_3 - s_3 d\mu + dt - dt_0 = s_1\ dr_0 + r_0\ ds_1 + \sigma_0\ ds_2 + s_2\ d\sigma_0$$

$$r_0\ ds_1 + \sigma_0\ ds_2 + \mu\ ds_3 = -s_1\ dr_0 - s_2\ d\ \sigma_0 - s_3\ d\mu$$
$$+ dt - dt_0$$

Similarly, differentiation of

$$r = r_0 s_0 + \sigma_0 s_1 + \mu\ s_2$$

is equivalent to equating the two expressions above for $\dot{g}$. This gives

$$[-\mu\ ds_2 - (\dot{g}-1)\ dr - s_2 d\mu]\ /r$$

$$= [s_0 dr_0 - \dot{g}\ dr + r_0\ ds_0 + \sigma_0\ ds_1 + s_1\ d\sigma_0]\ /r$$

$$dr = s_0\ dr_0 + s_1\ d\sigma_0 + s_2\ d\mu$$

$$+ r_0\ ds_0 + \sigma_0\ ds_1 + \mu\ ds_2.$$

The differentiation of

$$s_0 = 1 + \alpha \, \psi^2/2! + \alpha^2 \, \psi^4/4! + \alpha^3 \, \psi^6/6! + \ldots.$$

$$s_1 = \psi + \alpha \, \psi^3/3! + \alpha^2 \, \psi^5/5! + \alpha^3 \, \psi^7/7! + \ldots$$

$$s_2 = \psi^2/2! + \alpha \, \psi^4/4! + \alpha^2 \, \psi^6/6! + \alpha^3 \, \psi^8/8! + \ldots$$

$$s_3 = \psi^3/3! + \alpha \, \psi^5/5! + \alpha^2 \, \psi^7/7! + \alpha^3 \, \psi^9/9! + \ldots.$$

is conveniently expressed by utilizing

$$\frac{\partial s_0}{\partial \alpha} = \psi^2/2! + 2 \alpha \, \psi^4/4! + 3 \alpha^2 \, \psi^6/6! + \ldots.$$

$$\frac{\partial s_1}{\partial \alpha} = \psi^3/3! + 2 \alpha \, \psi^5/5! + 3 \alpha^2 \, \psi^7/7! + \ldots.$$

$$\frac{\partial s_2}{\partial \alpha} = \psi^4/4! + 2 \alpha \, \psi^6/6! + 3 \alpha^2 \, \psi^8/8! + \ldots.$$

$$\frac{\partial s_3}{\partial \alpha} = \psi^5/5! + 2 \alpha \, \psi^7/7! + 3 \alpha^2 \, \psi^9/9! + \ldots.$$

Term by term differentiation of the series and combination of the results gives

$$ds_0 = \alpha \, s_1 \, d\psi + \frac{\partial s_0}{\partial \alpha} \, d\alpha$$

$$ds_1 = s_0 \, d\psi + \frac{\partial s_1}{\partial \alpha} \, d\alpha$$

$$ds_2 = s_1 \, d\psi + \frac{\partial s_2}{\partial \alpha} \, d\alpha$$

$$ds_3 = s_2 \, d\psi + \frac{\partial s_3}{\partial \alpha} \, d\alpha \ .$$

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Finally, differentiation of

$$r_0 = (\vec{r}_0 \cdot \vec{r}_0)^{1/2}$$

$$\sigma_0 = \vec{r}_0 \cdot \dot{\vec{r}}_0$$

$$\alpha = \dot{\vec{r}}_0 \cdot \dot{\vec{r}}_0 - 2 \mu / r_0$$

gives

$$dr_0 = \frac{1}{2} (\vec{r}_0 \cdot \vec{r}_0)^{-\frac{1}{2}} 2\vec{r}_0 \, d\vec{r}_0$$

$$= d\vec{r}_0 \cdot \vec{r}_0 / r_0$$

$$d\sigma_0 = \dot{\vec{r}}_0 \cdot d\vec{r}_0 + \vec{r}_0 \cdot d\dot{\vec{r}}_0$$

$$d\alpha = 2 \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 - 2d\mu / r_0 + 2\mu \, dr_0 / r_0^2$$

$$= 2 (\mu \, dr_0 / r_0^2 + \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 - d\mu / r_0).$$

2.2 THE COMBINATION OF THE DIFFERENTIALS

The combination of the differentials above to give $d\vec{r}$ and $d\dot{\vec{r}}$ as a function of $d\vec{r}_0$, $d\dot{\vec{r}}_0$, $d\mu$, dt, and dt_0 is simplified by using the relations.

$$\vec{r}_0 = \dot{g}\,\vec{r} - g\,\dot{\vec{r}}$$

$$\dot{\vec{r}}_0 = -\dot{f}\,\vec{r} + f\,\dot{\vec{r}}$$

and

$$f\dot{g} - \dot{f}g = 1$$

which are derived in Section 2.4 below.

Substituting the two expressions for $\vec{r}_0$ and $\dot{\vec{r}}_0$ into the formulas above for $d\vec{r}$ and $d\dot{\vec{r}}$ gives

$$d\vec{r} = f\,d\vec{r}_0 + g\,d\dot{\vec{r}}_0 + \vec{r}_0\,df + \dot{\vec{r}}_0\,dg$$

$$= f\,d\vec{r}_0 + g\,d\dot{\vec{r}}_0 + (\dot{g}\,\vec{r} - g\,\dot{\vec{r}})\,df$$

$$+ (-\dot{f}\,\vec{r} + f\,\dot{\vec{r}})\,dg$$

so that

$$d\vec{r} = f\,d\vec{r}_0 + g\,d\dot{\vec{r}}_0 + \vec{r}\,(\dot{g}\,df - \dot{f}\,dg)$$

$$+ \dot{\vec{r}}\,(-g\,df + f\,dg)$$

and, similarly,

$$d\dot{\vec{r}} = \dot{f}\,d\vec{r}_0 + \dot{g}\,d\dot{\vec{r}}_0 + \vec{r}_0\,d\dot{f} + \dot{\vec{r}}_0\,d\dot{g}$$

$$= \dot{f}\,d\vec{r}_0 + \dot{g}\,d\dot{\vec{r}}_0 + (\dot{g}\,\vec{r} - g\,\dot{\vec{r}})\,d\dot{f}$$

$$+ (-\dot{f}\,\vec{r} + f\,\dot{\vec{r}})\,d\dot{g}$$

so that

$$\dot{d\vec{r}} = \dot{f}\, d\vec{r}_0 + \dot{g}\, d\dot{\vec{r}}_0 + \vec{r}\,(\dot{g}\, d\,\dot{f} - \dot{f}\, d\,\dot{g})$$

$$+ \dot{\vec{r}}\,(-g\, d\,\dot{f} + f\, d\,\dot{g}).$$

The expressions $(\dot{g}\, d\, f - \dot{f}\, d\, g)$, $(-g\, d\, f + f\, d\, g)$ and $(\dot{g}\, d\,\dot{f} - \dot{f}\, d\,\dot{g})$, $(-g\, d\,\dot{f} + f\, d\,\dot{g})$ are evaluated in order as follows.

Firstly,

$$(\dot{g}\, d\, f - \dot{f}\, d\, g) = \frac{\dot{g}}{3}[-(f-1)\, dr_0 - \mu\, ds_2 - s_2\, d\mu\,]\, /r_0$$

$$-\dot{f}\,[\, s_1\, dr_0 + r_0\, ds_1 + \sigma_0\, ds_2 + s_2\, d\,\sigma_0\,]$$

$$= [-\dot{f}s_1 - (f-1)\,\dot{g}/r_0]\, dr_0 - \dot{f}\, r_0\, ds_1 - (\dot{g}\,\mu\,/r_0 + \dot{f}\,\sigma_0)\, ds_2$$

$$-\dot{f}\, s_2\, d\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}s_1 - (f-1)\,\dot{g}/r_0]\, dr_0 - \dot{f}r_0\,(s_0\, d\,\psi + \frac{\partial s_1}{\partial\alpha}\, d\alpha\,)$$

$$-(\dot{g}\mu\,/r_0 + \dot{f}\,\sigma_0)\,(s_1\, d\psi + \frac{\partial s_2}{\partial\alpha}\, d\alpha)$$

$$-\dot{f}\, s_2\, d\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}s_1 - (f-1)\,\dot{g}/r_0]\, dr_0 - [\dot{f}\, r_0 s_0 + (\dot{g}\,\mu\,/r_0 + \dot{f}\,\sigma_0)s_1]\, d\psi$$

$$-[\dot{f}\, r_0\, \frac{\partial s_1}{\partial\alpha} + (\dot{g}\,\mu\,/r_0 + \dot{f}\,\sigma_0)\, \frac{\partial s_2}{\partial\alpha}]\, d\alpha$$

$$-\dot{f}\, s_2\, d\,\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)\,\dot{g}\,/r_0]\, dr_0 - [\dot{f}\, \frac{r_0 s_0 + \sigma_0 s_1}{r} - \frac{\mu s_1}{r\, r_0}\,\dot{g}]\, r\, d\,\psi$$

$$- [- \frac{\mu s_1}{r\, r_0}\, r_0\, \frac{\partial s_1}{\partial\alpha}$$

$$+ (\frac{r_0 s_0 + \sigma_0 s_1}{r}\, \frac{\mu}{r_0} - \frac{\mu s_1}{r\, r_0}\,\sigma_0)\, \frac{\partial s_2}{\partial\alpha}]\, d\,\alpha$$

$$- \dot{f}\, s_2\, d\,\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)\, \dot{g}/r_0]\, dr_0 - (\dot{f}\,\dot{g} - \dot{f}\,\dot{g})\, r\, d\psi$$

$$+ \frac{\mu}{r}\, (s_1\, \frac{\partial s_1}{\partial \alpha} - s_0\, \frac{\partial s_2}{\partial \alpha})\, d\alpha\ -\dot{f}\, s_2\, d\, \sigma_0$$

$$- (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)\, \dot{g}/r_0]\, dr_0 + \frac{\mu}{r}\, 2(s_1\, \frac{\partial s_1}{\partial \alpha} - s_0\, \frac{\partial s_2}{\partial \alpha})\, \frac{d\alpha}{2}$$

$$-\dot{f}\, s_2\quad d\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

But the identity

$$s_2^{\,2} = 2\, (s_1\, \frac{\partial s_1}{\partial \alpha} - s_0\, \frac{\partial s_2}{\partial \alpha})$$

is treated in Section 3.1.

This gives

$$(\dot{g}\, d\, f - \dot{f}\, d\, g) = [-\dot{f}\, s_1 - (f-1)\, \dot{g}/r_0]\, dr_0$$

$$- \frac{\mu\, s_2}{r}\, s_2\, 2(\mu\, dr_0/r_0^{\,2} + \vec{r}_0\cdot d\vec{r}_0 - d\mu/r_0)/2$$

$$- \dot{f}\, s_2\, d\,\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)\, \dot{g}/r_0]\, dr_0 - (\dot{g}-1)\, s_2\, (\mu\, dr_0/r_0^{\,2} + \vec{r}_0\cdot d\vec{r}_0 - d\mu/r_0)$$

$$- \dot{f}\, s_2\, d\,\sigma_0 - (\dot{g}\, s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)\dot{g}/r_0 + [(f-1)\, (\dot{g}-1)/r_0]\, dr_0$$

$$-(\dot{g}-1)\, s_2\, \vec{r}_0\cdot d\vec{r}_0 - \dot{f}\, s_2\, d\sigma_0 - (s_2/r_0)\, d\mu$$

$$= [-\dot{f}\, s_1 - (f-1)/r_0]\, d\vec{r}_0\cdot\vec{r}_0/r_0 - (\dot{g}-1)\, s_2\, \vec{r}_0\cdot d\vec{r}_0$$

$$- \dot{f}\, s_2\, (\vec{r}_0\cdot d\vec{r}_0 + \vec{r}_0\cdot d\vec{r}_0) - (s_2/r_0)\, d\mu$$

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or, finally,

$$(\dot{g}\, d\, f - \dot{f}\, d\, g) = -\frac{\dot{f}\, s_1 + (f-1)/r_0}{r_0}\ \vec{r}_0 \cdot d\vec{r}_0 - \dot{f}\, s_2\ \dot{\vec{r}}_0 \cdot d\vec{r}_0$$

$$- \dot{f}\, s_2\ \vec{r}_0 \cdot d\dot{\vec{r}}_0 - (\dot{g}-1)\, s_2\ \dot{\vec{r}}_0 \cdot d\vec{r}_0 - (s_2/r_0)\, d\mu \quad .$$

Secondly,

$$(-g\, d\, f + f\, d\, g) = -g\ [-(f-1)\, dr_0 - \mu\, ds_2 - s_2 d\mu\ ]\ /r_0$$

$$+ f\ [\ s_1 dr_0 + r_0\, ds_1 + {}^\tau_0\, ds_2 + s_2\ d\ {}^\tau_0]\ \cdot$$

$$= [fs_1 + (f-1)\, g/r_0\ ]\, dr_0 + f\, r_0\, ds_1 + (f\ \sigma_0 + g\, \mu/r_0)\, ds_2$$

$$+ f\, s_2\, d\ \sigma_0 + (g\, s_2/r_0)\, d\mu$$

$$= [f\, s_1 + (f-1)\, g/r_0]\, dr_0 + f\, r_0\, (s_0\, d\psi + \frac{\partial s_1}{\partial\alpha}\, d\alpha\ )$$

$$+ (f\ \sigma_0 + g\ \mu\ /r_0)\, (s_1\, d\psi + \frac{\partial s_2}{\partial\alpha}\, d\alpha) + f\, s_2\ d\sigma_0$$

$$+ (g\, s_2/r_0)\, d\mu$$

$$= [f\, s_1 + (f-1)\, g/r_0]\, dr_0 + [f\, (r_0 s_0 + \sigma_0\, s_1) + g\mu\, s_1/r_0]\ d\psi$$

$$+ [f\, r_0\ \frac{\partial s_1}{\partial\alpha} + (f\ \sigma_0 + g\mu\ /r_0)\frac{\partial s_2}{\partial\alpha}]\, d\alpha$$

$$+ f\, s_2\ d\sigma_0 + (g\, s_2/r_0)\, d\mu$$

$$= [f\, s_1 + (f-1)\, g/r_0]\, dr_0 + (f\ \frac{r_0 s_0 + \sigma_0 s_1}{r} - \frac{-\mu\, s_1}{r\, r_0}\ \dot{g})\, r\, d\psi$$

$$+ [f\, r_0\ \frac{\partial s_1}{\partial\alpha} + (f\ \sigma_0 + g\, \mu\ /r_0)\ \frac{\partial s_2}{\partial\alpha}]\, d\alpha$$

$$+ f\, s_2\ d\ \sigma_0 + (g\, s_2/r_0)\, d\mu$$

$$= [f\, s_1 + (f-1)\, g/r_0\,]dr_0 + (f\, \dot{g} - \dot{f}\, g)\, r\, d\psi$$

$$+ [f\, r_0\, \frac{\partial s_1}{\partial \alpha} + (f\, \sigma_0 + g\mu/r_0)\frac{\partial s_2}{\partial \alpha}]\, d\alpha + f\, s_2\, d\,\sigma_0 + (g\, s_2/r_0)d\mu$$

$$= [\, f s_1 + (f-1)\, g/r_0]\, dr_0 + r\, d\psi + \left[f\, r_0\, \frac{\partial s_1}{\partial \alpha} \right.$$

$$\left. + (f\, \sigma_0 + g\mu/r_0)\, \frac{\partial s_2}{\partial \alpha} \right] d\alpha + f\, s_2\, d\,\sigma_0 + (g\, s_2/r_0)\, d\mu$$

But

$$r_0\, ds_1 + \sigma_0\, ds_2 + \mu\, ds_3 = -\, s_1\, dr_0 - s_2\, d\,\sigma_0 - s_3\, d\mu + dt - dt_0$$

from above so that

$$r_0\, (s_0\, d\psi + \frac{\partial s_1}{\partial \alpha}\, d\alpha) + \sigma_0\, (s_1 d\psi + \frac{\partial s_2}{\partial \alpha}\, \alpha) + \mu\, (s_2\, d\psi + \frac{\partial s_3}{\partial \alpha}\, d\alpha)$$

$$= -\, s_1\, dr_0 - s_2\, d\sigma_0 - s_3\, d\mu + dt - dt_0$$

$$(r_0 s_0 + \sigma_0 s_1 + \mu s_2)\, d\psi = -(r_0\, \frac{\partial s_1}{\partial \alpha} + \sigma_0\, \frac{\partial s_2}{\partial \alpha} + \mu\, \frac{\partial s_3}{\partial \alpha})\, d\alpha$$

$$-\, s_1\, dr_0 - s_2\, d\,\sigma_0 - s_3 d\,\mu + dt - dt_0$$

$$r\, d\psi = -(r_0\, \frac{\partial s_1}{\partial \alpha} + \sigma_0\frac{\partial s_2}{\partial \alpha} + \mu\,\frac{\partial s_3}{\partial \alpha})\, d\alpha$$

$$-\, s_1\, dr_0 - s_2\, d\,\sigma_0 - s_3\, d\mu + dt - dt_0 \quad .$$

This gives

$$(-g\, d\, f + f\, d\, g) = [\, f\, s_1 + (f-1)\, g/r_0]\, dr_0 - (r_0\, \frac{\partial s_1}{\partial \alpha} + \sigma_0\, \frac{\partial s_2}{\partial \alpha} + \mu\frac{\partial s_3}{\partial \alpha})\, d\alpha$$

$$-\, s_1\, dr_0 - s_2\, d\,\sigma_0 - s_3\, d\mu + dt - dt_0$$

$$+ [f\, r_0\, \frac{\partial s_1}{\partial \alpha} + (f\, \sigma_0 + g\mu/r_0)\, \frac{\partial s_2}{\partial \alpha}]\, d\alpha$$

$$+ f\, s_2\, d\,\sigma_0 + (g\, s_2/r_0)\, d\mu$$

$$= [\,(f-1)\ s_1 + (f-1)\ g/r_0\,]\ dr_0$$

$$+ \left[-r_0\ \frac{\partial s_1}{\partial \alpha} - \sigma_0\ \frac{\partial s_2}{\partial \alpha} - \mu\ \frac{\partial s_3}{\partial \alpha} + f\ r_0\ \frac{\partial s_1}{\partial \alpha} \right.$$

$$\left. + (f\ \sigma_0 + g\mu\,/r_0)\ \frac{\partial s_2}{\partial \alpha} \right]\ d\alpha + (f-1)\ s_2\ d\sigma_0 + (g\ s_2/r_0 - s_3)\ d\mu$$

$$+ dt - dt_0.$$

But the coefficient of $d\alpha$ is

$$(f-1)\ r_0\ \frac{\partial s_1}{\partial \alpha} + [\,(f-1)\ \sigma_0 + g\mu\,/r_0]\ \frac{\partial s_2}{\partial \alpha} - \mu\ \frac{\partial s_3}{\partial \alpha}$$

$$= \frac{-\mu\ s_2}{r_0}\ r_0\ \frac{\partial s_1}{\partial \alpha} + [\,\frac{-\mu\ s_2}{r_0}\ \sigma_0 + (r_0 s_1 + \sigma_0 s_2)\ \mu\,/r_0]\frac{\partial s_2}{\partial \alpha} - \mu\ \frac{\partial s_3}{\partial \alpha}$$

$$= -\mu \left[(s_2\ \frac{\partial s_1}{\partial \alpha} - s_1\ \frac{\partial s_2}{\partial \alpha}) + \frac{\partial s_3}{\partial \alpha} \right]$$

and the identity

$$s_2 s_3 = 2\ (s_2\ \frac{\partial s_1}{\partial \alpha} - s_1 \frac{\partial s_2}{\partial \alpha})$$

is treated in Section 3.1. This gives the following coefficient for $d\alpha$.

$$-\mu\ (s_2 s_3/2 + \frac{\partial s_3}{\partial \alpha}) = (g\ s_2 - g\ s_2 - \mu s_2 s_3 - \mu\ 2\ \frac{\partial s_3}{\partial \alpha})/2$$

$$= [\,g\ s_2 - s_2\ (r_0 s_1 + \sigma_0 s_2 + \mu s_3) - \mu\ 2\ \frac{\partial s_3}{\partial \alpha}]\ 2$$

$$= \left(g\ s_2 - [\,s_2\ (t-t_0) + \mu\ 2\ \frac{\partial s_3}{\partial \alpha}\,] \right)/2$$

Thus

$$(-g\ d\ f + f\ d\ g) = [\,(f-1)\ s_1 + (f-1)g/r_0\,]dr_0 + \left(g\ s_2 \right.$$

$$\left. - [\,s_2\ (t-t_0) + \mu\ 2\frac{\partial s_3}{\partial \alpha}\,] \right)\ d\alpha\ /\ 2 + (f-1)\ s_2\ d\sigma_0$$

$$+ (g\ s_2/r_0 - s_3)\ d\mu + dt - dt_0.$$

Setting

$$U = s_2 (t-t_0) + \mu \cdot 2 \; \frac{\partial \cdot s_3}{\partial \alpha}$$

gives

$$(-g\,d\,f + f\,d\,g) = [\,(f-1)\,s_1 + (f-1)\,g/r_0]\;dr_0$$

$$+ (g\,s_2 - U)\,(\mu\,dr_0/r_0^2 + \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 - d\,\mu/r_0)$$

$$+ (f-1)\,s_2\,d\,\sigma_0 + (g\,s_2/r_0 - s_3)\;d\mu + dt - dt_0$$

$$= [\,(f-1)\,s_1 + (f-1)g/r_0 - (-\mu\,s_2/r_0)g/r_0]\,dr_0 + U\;(-\mu/r_0^2)\,dr_0$$

$$+ (g\,s_2 - U)\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + (f-1)\,s_2\,d\,\sigma_0$$

$$+ (-g\,s_2/r_0 + U/r_0 + g\,s_2/r_0 - s_3)\,d\mu + dt - dt_0$$

$$= [\,(f-1)s_1 + (f-1)g/r_0 - (f-1)g/r_0]\,d\vec{r}_0 \cdot \vec{r}_0/r_0$$

$$+ U\,(-\mu/r_0^2)\,d\vec{r}_0 \cdot \vec{r}_0/r_0 + (g\,s_2 - U)\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0$$

$$+ (f-1)\,s_2\,d\,\sigma_0 + (U/r_0 - s_3)\,d\mu + dt - dt_0$$

$$= [(f-1)\,s_1/r_0]\,\vec{r}_0 \cdot d\vec{r}_0 + U\,(-\mu\,\vec{r}_0/r_0^3) \cdot d\vec{r}_0 - U\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0$$

$$+ g\,s_2\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + (f-1)\,s_2\,(\dot{\vec{r}}_0 \cdot d\vec{r}_0 + \vec{r}_0 \cdot d\dot{\vec{r}}_0)$$

$$+ (U/r_0 - s_3)\,d\mu + dt - dt_0$$

and, finally

$$(-g\,d\,f + f\,d\,g) = \frac{(f-1)\,s_1}{r_0}\;\vec{r}_0 \cdot d\vec{r}_0 + (f-1)\,s_2\,\dot{\vec{r}}_0 \cdot d\vec{r}_0 + (f-1)\,s_2\,\vec{r}_0 \cdot d\dot{\vec{r}}_0$$

$$+ g\,s_2\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + U\,\ddot{\vec{r}}_0 \cdot d\vec{r}_0 - U\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + (U/r_0 - s_3)\,d\mu$$

$$+ dt - dt_0 \; .$$

The quantity U may be placed in the form

$$U = s_2 (t-t_0) + \mu (\psi s_4 - 3s_5)$$

where

$$s_4 = \psi^4/4! + \alpha \psi^6/6! + \alpha^2 \psi^8/8! + \alpha^3 \psi^{10}/10! + \ldots$$

$$s_5 = \psi^5/5! + \alpha \psi^7/7! + \alpha^2 \psi^9/9! + \alpha^3 \psi^{11}/11! + \ldots$$

by using the identity

$$2 \frac{\partial s_3}{\partial \alpha} = \psi s_4 - 3s_5$$

treated in Section 3.1.

Thirdly,

$$(\dot{g} d \dot{f} - \dot{f} d \dot{g}) = \dot{g} (-\dot{f} dr_0/r_0 - \dot{f} dr/r - \mu ds_1/r r_0 - d\mu s_1/ r r_0)$$

$$-\dot{f} (s_0 dr_0 - \dot{g} dr + r_0 ds_0 + \sigma_0 ds_1 + s_1 d \sigma_0)/r$$

$$= (-\dot{f} \dot{g}/r_0 - \dot{f} s_0/r) dr_0 - (\dot{f} r_0/r) ds_0 - (\dot{g}\mu/(r r_0) + \dot{f}\sigma_0/r) ds_1$$

$$- (\dot{f} s_1/r) d \sigma_0 - (\dot{g} s_1/ r r_0) d\mu$$

$$= (-\dot{f} \dot{g}/r_0 - \dot{f} s_0/r) dr_0 - \frac{\dot{f} r_0}{r} (\alpha s_1 d\psi + \frac{\partial s_0}{\partial \alpha} d\alpha)$$

$$-(\dot{g}\mu /(r r_0) + \dot{f} \sigma_0 /r) (s_0 d\psi + \frac{\partial s_1}{\partial \alpha} d\alpha)$$

$$-(\dot{f} s_1/r) d \sigma_0 - (\dot{g} s_1/(r r_0) d\mu$$

$$= (-\dot{f} \dot{g}/r_0 - \dot{f} s_0/r) dr_0 + [- \dot{f} r_0 \alpha s_1/r^2$$

$$- (\dot{g}\mu/r r_0 + \dot{f} \sigma_0/r) s_0/r] r d\psi - [\frac{\dot{f} r_0}{r} \frac{\partial s_0}{\partial \alpha}$$

$$+ (\dot{g} \mu /(r r_0) + \dot{f} \sigma_0/r) \frac{\partial s_1}{\partial \alpha}] d\alpha - (\dot{f} s_1/r) d \sigma_0 - (\dot{g} s_1/r r_0) d\mu .$$

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But the coefficient of $r\,d\psi$ is

$$-\dot{f}\,r_0\,\alpha s_1/r^2 - (\dot{g}\mu/r\,r_0 + \dot{f}\,\sigma_0/r)s_0/r = \frac{\mu s_1}{r\,r_0}\frac{r_0\,\alpha s_1}{r^2}$$

$$-\left(\frac{r_0 s_0 + \sigma_0 s_1}{r}\frac{\mu}{r\,r_0} - \frac{\mu s_1}{r\,r_0}\frac{\sigma_0}{r}\right)\frac{s_0}{r}$$

$$= -\frac{\mu}{r^3}(s_0^2 - \alpha s_1^2)$$

and the identity

$$s_0^2 - \alpha s_1^2 = 1$$

is treated in Section 3.1.

Thus

$$(\dot{g}\,d\dot{f} - \dot{f}\,d\dot{g}) = (-\dot{f}\,\dot{g}/r_0 - \dot{f}\,s_0/r)\,dr_0$$

$$- \frac{\mu}{r^3}\left[-\left(r_0\frac{\partial s_1}{\partial\alpha} + \sigma_0\frac{\partial s_2}{\partial\alpha} + \mu\frac{\partial s}{\partial\alpha}\right)d\alpha\right.$$

$$\left. - s_1\,dr_0 - s_2\,d\sigma_0 - s_3\,d\mu + dt - dt_0\right]$$

$$-\left[\frac{\dot{f}r_0}{r}\frac{\partial s_0}{\partial\alpha} + \frac{\dot{g}\mu/r_0 + \dot{f}\,\sigma_0}{r}\frac{\partial s_1}{\partial\alpha}\right]d\alpha$$

$$-(\dot{f}\,s_1/r)\,d\sigma_0 - (\dot{g}\,s_1/r\,r_0)\,d\mu$$

$$= \left[-\dot{f}\,\dot{g}/r_0 - \dot{f}\,s_0/r + \frac{\mu}{r^3}\,s_1\right]dr_0$$

$$+ \frac{\mu}{r^3}\left[\left(r_0\frac{\partial s_1}{\partial\alpha} + \sigma_0\frac{\partial s_2}{\partial\alpha} + \mu\frac{\partial s_3}{\partial\alpha}\right)\right.$$

$$\left. - \frac{\dot{f}r_0}{r}\frac{\partial s_0}{\partial\alpha} - \frac{\dot{g}\mu/r_0 + \dot{f}\,\sigma_0}{r}\frac{\partial s_1}{\partial\alpha}\right]d\alpha$$

$$+ \ (\quad \frac{\mu}{r^3} \quad s_2 \ - \ \frac{\dot{f} \, s_1}{r}) \ d \ \sigma_0 + (\quad \frac{\mu}{r^3} \quad s_3 \ - \ \frac{\dot{g} \, s_1}{r \, r_0}) \ d\mu$$

$$- \ \frac{\mu}{r^3} \ (dt \ - \ dt_0).$$

But the coefficient of $d\alpha$ is

$$\frac{\mu}{r^3} \ (r_0 \ \frac{\partial s_1}{\partial \alpha} \ + \ \sigma_0 \ \frac{\partial s_2}{\partial \alpha} + \mu \frac{\partial s_3}{\partial \alpha}) \ + \ \frac{\mu \, s_1}{r \, r_0} \ \frac{r_0}{r} \ \frac{\partial s_0}{\partial \alpha}$$

$$- \ \frac{1}{r} \ (\frac{r_0 s_0 + \sigma_0 s_1}{r} \ \frac{\mu}{r_0} \ - \ \frac{\mu \, s_1}{r \, r_0} \ \sigma_0 \ ) \ \frac{\partial s_1}{\partial \alpha}$$

$$= \quad \frac{\mu}{r^3} \ [\, r_0 \ \frac{\partial s_1}{\partial \alpha} \ + \ \sigma_0 \ \frac{\partial s_2}{\partial \alpha} \ + \ \mu \ \frac{\partial s_3}{\partial \alpha} \ + \ r \ (s_1 \ \frac{\partial s_0}{\partial \alpha} \ - \ s_0 \frac{\partial s_1}{\partial \alpha}) \,]$$

and the identity
$$2 \ (s_1 \ \frac{\partial s_0}{\partial \alpha} \ - \ s_0 \ \frac{\partial s_1}{\partial \alpha}) \ = \ s_1 s_2 + s_3$$

treated in Section 3.1 below gives

$$\frac{\mu}{r^3} \left[r_0 \ \frac{\partial s_1}{\partial \alpha} \ + \ \sigma_0 \ \frac{\partial s_2}{\partial \alpha} \ + \ \mu \ \frac{\partial s_3}{\partial \alpha} \ + \ r \ \frac{s_1 s_2 + s_3}{2} \right]$$

$$= \ \frac{\mu}{r^3} \ [r_0 \ \frac{\partial s_1}{\partial \alpha} \ + \ \sigma_0 \ \frac{\partial s_2}{\partial \alpha} + \mu \frac{\partial s_3}{\partial \alpha} \ + r \ \frac{s_1 s_2}{2} \ + (r_0 s_0 + \sigma_0 \ s_1 + \mu s_2) \frac{s_3}{2}]$$

$$= \ \frac{\mu}{r^3} \left[r \ \frac{s_1 s_2}{2} + r_0 \ (\frac{\partial s_1}{\partial \alpha} \ + \ \frac{s_0 s_3}{2}) \ + \ \sigma_0 \ (\frac{\partial s_2}{\partial \alpha} \ + \ \frac{s_1 s_3}{2}) \right.$$

$$\left. + \ \mu \ \frac{s_2 s_3}{2} \ + \ \mu \ \frac{\partial s_3}{\partial \alpha} \right]$$

for the coefficient of $d\alpha$. Also, the identities

$$2 \ \frac{\partial s_1}{\partial \alpha} \ + \ s_0 s_3 = s_1 s_2$$

$$2 \ \frac{\partial s_2}{\partial \alpha} \ + \ s_1 s_3 \ = \ s_2^2$$

treated in Section 3.1 give

$$\frac{\mu}{r^3} \left[r \, s_1 s_2 + r_0 \, s_1 s_2 + \sigma_0 \, s_2^2 + \mu \, s_2 s_3 + \mu \cdot 2 \, \frac{\partial s_3}{\partial \alpha} \right] \ / 2$$

$$= \frac{\mu \, s_2}{r} \, \frac{s_1}{r} + \frac{\mu}{r^3} \left[s_2 \, (r_0 s_1 + \sigma_0 s_2 \cdot + \mu \, s_3) + \mu \cdot 2 \, \frac{\partial s_3}{\partial \alpha} \right] \ / 2$$

$$= - (\dot{g} - 1) \, s_1 / r + \frac{\mu}{r^3} \left[s_2 \, (t - t_0) + \mu \cdot 2 \, \frac{\partial s_3}{\partial \alpha} \right] / 2$$

$$= \left[-(\dot{g} - 1) s_1 / r + \frac{\mu}{r^3} \, U \right] / 2$$

for the coefficient of $d\alpha$. Thus

$$(\dot{g} \, d \, \dot{f} - \dot{f} \, d \, \dot{g}) = \left[-\dot{f} \, \dot{g} / r_0 - \dot{f} \, s_0 / r + \frac{\mu}{r^3} \, s_1 \right] dr_0$$

$$+ \left[-(\dot{g} - 1) \, s_1 / r + \frac{\mu}{r^3} \, U \right] \left(\frac{\mu \, dr_0}{r_0^2} + \vec{r_0} \cdot \dot{\vec{dr_0}} - \frac{d\mu}{r_0} \right)$$

$$+ \left(\frac{\mu}{r^3} \, s_2 - \frac{\dot{f} s_1}{r} \right) d \, \sigma_0 + \left(\frac{\mu}{r^3} \, s_3 - \frac{\dot{g} s_1}{r \, r_0} \right) d\mu$$

$$- \frac{\mu}{r^3} \, (dt - dt_0)$$

$$= \left[-\dot{f} \, \dot{g} / r_0 - \dot{f} \, s_0 / r + \frac{\mu}{r^3} \, s_1 - (\dot{g} - 1) \frac{\mu s_1}{r \, r_0^2} + \frac{\mu}{r^3} \, \frac{\mu}{r_0^2} \, U \right] dr_0$$

$$+ \left[-(\dot{g} - 1) s_1 / r + \frac{\mu}{r^3} U \right] \dot{\vec{r_0}} \cdot \dot{\vec{dr_0}} + \left(\frac{\mu \, s_2}{r^3} - \frac{\dot{f} s_1}{r} \right) d \, \sigma_0$$

$$+ \left[+(\dot{g} - 1) \frac{s_1}{r \, r_0} - \frac{\mu}{r^3} \, \frac{U}{r_0} + \frac{\mu}{r^3} \, s_3 - \frac{\dot{g} s_1}{r \, r_0} \right] d\mu$$

$$- \frac{\mu}{r^3} \, (dt - dt_0)$$

$$= [-\dot{f}\,\dot{g}/r_0 - \dot{f}\,s_0/r + \frac{\mu}{r^3}\,s_1 + (\dot{g}-1)\,\dot{f}/r_0 + \frac{\mu}{r^3}\,\frac{\mu}{r_0^2}\,U]d\vec{r}_0 \cdot \vec{r}_0/r_0$$

$$+ [-\frac{(\dot{g}-1)\,s_1}{r} + \frac{\mu}{r^3}\,U]\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + \left(-\frac{(\dot{g}-1)}{r^2} - \frac{\dot{f}\,s_1}{r}\right)\,d\,\sigma_0$$

$$+ [-\frac{s_1}{r\,r_0} - \frac{\mu}{r^3}\,(\frac{U}{r_0} - s_3)]d\mu - \frac{\mu}{r^3}\,(dt - dt_0)$$

$$= [-\dot{f}\,s_0/r\,r_0 + \frac{\mu\,s_1}{r\,r_0}\,\frac{1}{r^2} - \frac{\dot{f}}{r_0^2}]\,\vec{r}_0 \cdot d\vec{r}_0 - \frac{\mu}{r^3}\,U\,(-\mu\,\vec{r}_0/r_0^3) \cdot d\vec{r}_0$$

$$+ [-\frac{(\dot{g}-1)\,s_1}{r} + \frac{\mu}{r^3}\,U\,\dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 - \frac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r}\,(\dot{\vec{r}}_0 \cdot d\vec{r}_0 + \vec{r}_0 \cdot d\dot{\vec{r}}_0)$$

$$+ [-\frac{s_1}{r\,r_0} - \frac{\mu}{r^3}\,(\frac{U}{r_0} - s_3)]\,d\mu - \frac{\mu}{r^3}\,(dt - dt_0)$$

or, finally,

$$(\dot{g}\,d\,\dot{f} - \dot{f}\,d\,\dot{g}) = -\dot{f}\,(\frac{s_0}{r\,r_0} + \frac{1}{r^2} + \frac{1}{r_0^2})\,\vec{r}_0 \cdot d\vec{r}_0 - \frac{\mu}{r^3}\,U\,\ddot{\vec{r}}_0 \cdot d\vec{r}_0$$

$$+ -\frac{(\dot{g}-1)\,s_1}{r} + \frac{\mu}{r^3}\,U]\,\dot{\vec{r}}_0 \cdot d\,\dot{\vec{r}}_0 - \frac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r}\,(\dot{\vec{r}}_0 \cdot d\vec{r}_0 + \vec{r}_0 \cdot d\dot{\vec{r}}_0)$$

$$+ [-\frac{s_1}{r\,r_0} - \frac{\mu}{r^3}\,(\frac{U}{r_0} - s_3)]\,d\mu - \frac{\mu}{r^3}\,(dt - dt_0)\ .$$

Lastly, since

$$f\,\dot{g} - \dot{f}\,g = 1$$

it follows that

$$d\,f \cdot \dot{g} + f\,d\,\dot{g} - d\dot{f} \cdot g - \dot{f}\,dg = 0$$

$$(-g\,d\,\dot{f} + f\,d\,\dot{g}) = -(\dot{g}\,d\,f - \dot{f}\,d\,g)$$

$$= \frac{\dot{f}\,s_1 + (f-1)/r_0}{r_0}\,\vec{r}_0 \cdot d\,\vec{r}_0 + \dot{f}\,s_2\,\dot{\vec{r}}_0 \cdot d\,\vec{r}_0 + \dot{f}\,s_2\,\vec{r}_0 \cdot d\,\dot{\vec{r}}_0$$

$$+ (\dot{g}-1)\,s_2\,\dot{\vec{r}}_0 \cdot d\,\dot{\vec{r}}_0 + (s_2/r_0)d\mu\ \ .$$

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Then substitution of the final expressions for $(\dot{g}\, d\, f - \dot{f}\, d\, g)$, $(-g\, d\, f + f\, d\, g)$, $(\dot{g}\, d\, \dot{f} - \dot{f}\, d\, \dot{g})$, $(-g\, d\, \dot{f} + f\, d\, \dot{g})$ into the expressions above for $d\vec{r}$ and $d\dot{\vec{r}}$ gives the final expressions for the differentials. Firstly,

$$d\dot{\vec{r}} = f\, d\dot{\vec{r}}_0 + g\, d\ddot{\vec{r}}_0 + \vec{r}\left[-\frac{\dot{f}\, s_1 + (f-1)/r_0}{r_0}\, \vec{r}_0 \cdot d\,\dot{\vec{r}}_0 - \dot{f}\, s_2\, \dot{\vec{r}}_0 \cdot d\,\vec{r}_0 - \dot{f}\, s_2\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 \right.$$

$$\left. - (\dot{g}-1)\, s_2\, \ddot{\vec{r}}_0 \cdot d\vec{r}_0 - (s_2/r_0)\, d\mu \right]$$

$$+ \dot{\vec{r}}\left[+\frac{(f-1)\, s_1}{r_0}\, \vec{r}_0 \cdot d\vec{r}_0 + (f-1)\, s_2\, \dot{\vec{r}}_0 \cdot d\, \vec{r}_0 + (f-1)\, s_2\, \vec{r}_0 \cdot d\, \dot{\vec{r}}_0 \right.$$

$$+ g\, s_2\, \dot{\vec{r}}_0 \cdot d\, \dot{\vec{r}}_0 + U\dot{\vec{r}}_0 \cdot d\vec{r}_0 - U\dot{\vec{r}}_0 \cdot d\vec{r}_0 + (U/r_0 - s_3)\, d\mu$$

$$\left. + dt - dt_0 \right]$$

or, as the final differential $d\vec{r}$,

$$d\vec{r} = f\, d\vec{r}_0 + U\, \dot{\vec{r}}\, \ddot{\vec{r}}_0 \cdot d\vec{r}_0$$

$$-\frac{\dot{f}\, s_1 + (f-1)/r_0}{r_0}\, \vec{r}\, \vec{r}_0 \cdot d\vec{r}_0 - \dot{f}\, s_2\, \vec{r}\, \dot{\vec{r}}_0 \cdot d\vec{r}_0 + \frac{(f-1)}{r_0}\, s_1\, \dot{\vec{r}}\, \vec{r}_0 \cdot d\vec{r}_0$$

$$+ (f-1)\, s_2\, \dot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\vec{r}_0 + g\, d\dot{\vec{r}}_0 - U\, \dot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0$$

$$-\dot{f}\, s_2\, \vec{r}\, \vec{r}_0 \cdot d\dot{\vec{r}}_0 - (\dot{g}-1)\, s_2\, \vec{r}\, \ddot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 + (f-1)\, s_2\, \dot{\vec{r}}\, \vec{r}_0 \cdot d\dot{\vec{r}}_0 + g\, s_2\, \dot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0$$

$$+ [\vec{r}\,(-s_2/r_0) + \dot{\vec{r}}\,(U/r_0 - s_3)]\, d\mu + \dot{\vec{r}}\, dt - \dot{\vec{r}}\, dt_0.$$

Secondly,

$$d\dot{\vec{r}} = \dot{f}\,d\vec{r}_0 + \dot{g}\,d\dot{\vec{r}}_0 + \dot{\vec{r}}\left[-\dot{f}\left(\frac{s_0}{r r_0} + \frac{1}{r^2} + \frac{1}{r_0^2}\right)\vec{r}_0\cdot d\vec{r}_0 - \frac{\mu}{r^3}U\ddot{\vec{r}}_0\cdot d\vec{r}_0\right.$$

$$- \frac{(\dot{g}-1)\,s_1}{r}\dot{\vec{r}}_0\cdot d\dot{\vec{r}}_0 + \frac{\mu}{r^3}\;U\,\dot{\vec{r}}_0\cdot d\dot{\vec{r}}_0$$

$$- \frac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r}(\dot{\vec{r}}_0\cdot d\vec{r}_0 + \vec{r}_0\cdot d\dot{\vec{r}}_0)$$

$$\left.- \frac{s_1}{r r_0}\,d\mu - \frac{\mu}{r^3}\left(\frac{U}{r_0} - s_3\right)d\mu - \frac{\mu}{r^3}(dt - dt_0)\right]$$

$$+ \dot{\vec{r}}\left[\frac{\dot{f}\,s_1 + (f-1)/r_0}{r_0}\,\vec{r}_0\cdot d\vec{r}_0 + \dot{f}\,s_2\,\dot{\vec{r}}_0\cdot d\vec{r}_0 + \dot{f}s_2\,\vec{r}_0\cdot d\dot{\vec{r}}_0\right.$$

$$\left.+ (\dot{g}-1)\,s_2\,\dot{\vec{r}}_0\cdot d\dot{\vec{r}}_0 + (s_2/r_0)\,d\mu\right]$$

$$= \dot{f}\,d\vec{r}_0 + U\left(-\mu\frac{\vec{r}}{r^3}\right)\ddot{\vec{r}}_0\cdot d\vec{r}_0 - \dot{f}\left(\frac{s_0}{r r_0} + \frac{1}{r^2} + \frac{1}{r_0^2}\right)\vec{r}\,\vec{r}_0\cdot d\vec{r}_0$$

$$- \frac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r}\,\vec{r}\,\dot{\vec{r}}_0\cdot d\vec{r}_0 + \frac{\dot{f}\,s_1 + (f-1)/r_0}{r_0}\,\dot{\vec{r}}\,\vec{r}_0\cdot d\vec{r}_0$$

$$+ \dot{f}\,s_2\,\dot{\vec{r}}\,\dot{\vec{r}}_0\cdot d\vec{r}_0 + \dot{g}\,d\dot{\vec{r}}_0 - U\left(-\mu\frac{\vec{r}}{r^3}\right)\ddot{\vec{r}}_0\cdot d\dot{\vec{r}}_0$$

$$- \frac{(\dot{g}-1)\,s_1}{r}\,\vec{r}\,\dot{\vec{r}}_0\cdot d\dot{\vec{r}}_0 - \frac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r}\,\vec{r}\,\vec{r}_0\cdot d\dot{\vec{r}}_0$$

$$+ \dot{f}\,s_2\,\dot{\vec{r}}\,\vec{r}_0\cdot d\dot{\vec{r}}_0 + (\dot{g}-1)\,s_2\,\dot{\vec{r}}\,\dot{\vec{r}}_0\cdot d\dot{\vec{r}}_0 + \left[\vec{r}\left(-\frac{s_1}{r r_0}\right)\right.$$

$$\left.+ \dot{\vec{r}}\;\frac{s_2}{r_0} - \mu\frac{\vec{r}}{r^3}\left(\frac{U}{r_0} - s_3\right)\right]\,d\mu - \mu\frac{\vec{r}}{r^3}(dt - dt_0)$$

or, as the final differential $d\dot{\vec{r}}$,

$$d\dot{\vec{r}} = \dot{f}\, d\vec{r}_0 + U\, \ddot{\vec{r}}\, \ddot{\vec{r}}_0 \cdot d\vec{r}_0$$

$$-\dot{f}\left(\frac{s_0}{r\, r_0} + \frac{1}{r^2} + \frac{1}{r_0^2}\right) \vec{r}\, \vec{r}_0 \cdot d\vec{r}_0 \; - \; \frac{\dot{f}\, s_1 + (\dot{g}-1)/r}{r}\, \ddot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\vec{r}_0$$

$$+ \; \frac{\dot{f}\, s_1 + (f-1)/r_0}{r_0}\, \dot{\vec{r}}\, \vec{r}_0 \cdot d\vec{r}_0 \; + \; \dot{f}\, s_2\, \dot{\vec{r}}\, \vec{r}_0 \cdot d\vec{r}_0 \; + \; \dot{g}\, d\dot{\vec{r}}_0 \; - \; U\, \ddot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0$$

$$- \; \frac{\dot{f}\, s_1 + (\dot{g}-1)/r}{r}\, \ddot{\vec{r}}\, \vec{r}_0 \cdot d\dot{\vec{r}}_0 \; - \; \frac{(\dot{g}-1)s_1}{r}\, \ddot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 \; + \; \dot{f}\, s_2\, \dot{\vec{r}}\, \vec{r}_0 \cdot d\dot{\vec{r}}_0$$

$$+ \, (\dot{g}-1)\, s_2\, \dot{\vec{r}}\, \dot{\vec{r}}_0 \cdot d\dot{\vec{r}}_0 \; + [\, \vec{r}(-s_1/r\, r_0 + \dot{\vec{r}}\,(s_2/r_0) + \ddot{\vec{r}}\,(U/r_0 - s_3)\,]\, d\mu$$

$$+ \ddot{\vec{r}}\, dt \; - \; \ddot{\vec{r}}\, dt_0.$$

2.3 THE PARTIAL DERIVATIVES

The differential relations obtained above may be expressed in terms of a set of partial drivatives as follows. Let x, y, z be the components of $\vec{r}$ and $\dot{x}$, $\dot{y}$, $\dot{z}$ be the components of $\dot{\vec{r}}$ so that x_0, y_0, z_0 are the components of $\vec{r}_0$ and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are the components of $\dot{\vec{r}}_0$. Then the differential relations above may be expressed in the following matrix notation.

$$
\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} =
\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ \upsilon & 0 & f \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
+ U \cdot
\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}
\begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
$$

$$
- \frac{\dot{f}\, s_1 + (f-1)/r_0}{r_0}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
\begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
$$

$$
- \dot{f}\, s_2 \cdot
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
\begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
$$

$$
+ \frac{(f-1)\, s_1}{r_0} \cdot
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}
\begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
$$

$$
+ (f-1)\, s_2 \cdot
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}
\begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}
$$

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$$+ \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} - U \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$-\dot{f} s_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} - (\dot{g}-1)s_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+(f-1)s_2 \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} + g\, s_2 \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -s_2/r_0 \\ U/r_0 - s_3 \end{bmatrix} \cdot d\mu + \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cdot dt - \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cdot dt_0$$

and,

$$\begin{bmatrix} d\dot{x} \\ d\dot{y} \\ d\dot{z} \end{bmatrix} = \begin{bmatrix} \dot{f} & 0 & 0 \\ 0 & \dot{f} & 0 \\ 0 & 0 & \dot{f} \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix} + U \cdot \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$-\dot{f}\left(\frac{s_0}{r\,r_0} + \frac{1}{r^2} + \frac{1}{r_0^2}\right) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$- \frac{\dot{f} s_1 + (\dot{g}-1)/r}{r} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$+ \frac{\dot{f} s_1 + (f-1)/r_0}{r_0} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix} + \dot{f} s_2 \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$+ \begin{bmatrix} \dot{g} & 0 & 0 \\ 0 & \dot{g} & 0 \\ 0 & 0 & \dot{g} \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} \qquad - U \cdot \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$- \frac{\dot{f} s_1 + (\dot{g}-1)/r}{r} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} - \frac{(\dot{g}-1)s_1}{r} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+ \dot{f} s_2 \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} + (\dot{g}-1) s_2 \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} & \ddot{x} \\ y & \dot{y} & \ddot{y} \\ z & \dot{z} & \ddot{z} \end{bmatrix} \begin{bmatrix} -s_1/r \; r_0 \\ s_2/r_0 \\ U/r_0 - s_3 \end{bmatrix} \cdot d\mu$$

$$+ \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \cdot dt - \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \cdot dt_0$$

The coefficient of dt in the above equations is a check on the solution of the differential equation. The coefficient of dt_0 is the negative of that for dt which indicates that an increase in dt_0 is equivalent to a decrease in dt and vice-versa. In most applications of the differential relations, t and t_0 are treated as fixed quantities and x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ are considered as functions of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ and sometimes μ as well.

The differential relations are usually expressed in terms of a set of partial derivatives as follows.

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$+ \begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} \partial x/\partial \mu \\ \partial y/\partial \mu \\ \partial z/\partial \mu \end{bmatrix} \cdot d\mu + \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cdot dt - \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cdot dt_0$$

$$\begin{bmatrix} d\dot{x} \\ d\dot{y} \\ d\dot{z} \end{bmatrix} = \begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix}$$

$$+ \begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix} \begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} \partial \dot{x}/\partial \mu \\ \partial \dot{y}/\partial \mu \\ \partial \dot{z}/\partial \mu \end{bmatrix} \cdot d\mu + \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} dt - \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} dt_0$$

The partial derivatives of x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to μ are merely the coefficients of $d\mu$ in the explicit differential expressions above.

$$\begin{bmatrix} \partial x/\partial\mu \\ \partial y/\partial\mu \\ \partial z/\partial\mu \end{bmatrix} = \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -s_2/r_0 \\ U/r_0 - s_3 \end{bmatrix}$$

$$\begin{bmatrix} \partial\dot{x}/\partial\mu \\ \partial\dot{y}/\partial\mu \\ \partial\dot{z}/\partial\mu \end{bmatrix} = \begin{bmatrix} x & \dot{x} & \ddot{x} \\ y & \dot{y} & \ddot{y} \\ z & \dot{z} & \ddot{z} \end{bmatrix} \begin{bmatrix} -s_1/(r\,r_0) \\ s_2/r_0 \\ U/r_0 - s_3 \end{bmatrix}$$

The partial derivatives of x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to x_0, y_0, z_0 and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are given by collecting the coefficients of dx_0, dy_0, dz_0 and $d\dot{x}_0$, $d\dot{y}_0$, $d\dot{z}_0$ in the explicit differential expressions above. A more compact matrix notation is also used to express the results as follows.

$$\begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix} + U \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -\dfrac{\dot{f}\,s_1 + (f-1)/r_0}{r_0} & -\dot{f}\,s_2 \\[3mm] \dfrac{(f-1)\,s_1}{r_0} & (f-1)s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$\begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix} = \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} - U \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -\dot{f}\,s_2 & -(\dot{g}-1)s_2 \\ (f-1)\,s_2 & g\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix} = \begin{bmatrix} \dot{f} & 0 & 0 \\ 0 & \dot{f} & 0 \\ 0 & 0 & \dot{f} \end{bmatrix} + U \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -\dot{f}(\dfrac{s_0}{r\,r_0} + \dfrac{1}{r^2} + \dfrac{1}{r_0^2}) & -\dfrac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r} \\[3mm] \dfrac{\dot{f}\,s_1 + (f-1)/r_0}{r_0} & \dot{f}\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix} = \begin{bmatrix} \dot{g} & 0 & 0 \\ 0 & \dot{g} & 0 \\ 0 & 0 & \dot{g} \end{bmatrix} - U \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$+ \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} -\dfrac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r} & -\dfrac{(\dot{g}-1)\,s_1}{r} \\[3mm] \dot{f}\,s_2 & (\dot{g}-1)\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ \ddot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

In the expressions above for the partial derivatives, the parameter

$$U = s_2 (t-t_0) + \mu (\psi s_4 - 3 s_5)$$

is a monotonically increasing and unbounded function of the time t. In the case of elliptic motion where α is negative, all other quantities in the partial derivatives are periodic with the period

$$T = 2 \pi \; \mu \sqrt{-\alpha}^{\,3}$$

of the elliptic orbit. In general, the partial derivatives of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ and μ then have an unbounded oscillation in time since the velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$ and acceleration components $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ which are multiplied by U have an oscillation about zero with the period T. However, this effect can be nullified for some derivatives if some of the other components $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ and $\ddot{x}_0$, $\ddot{y}_0$, $\ddot{z}_0$ multiplying U are identically zero. In particular, the coordinate reference frame may be chosen so that only x_0 and $\dot{y}_0$ are not identically zero, and the unbounded oscillation will then occur in only eight of the thirty-six partial derivatives of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

In the case of elliptic motion, it is shown in Section 3.1 that

$$s_4 = \frac{\Delta^2 E/2! - (1-\cos \Delta E)}{\sqrt{\mu/a}^{\,4}}$$

$$s_5 = \frac{\Delta^3 E/3! - (\Delta E - \sin \Delta E)}{\sqrt{\mu/a}^{\,5}}$$

which, with

$$\psi = \frac{\Delta E}{\sqrt{\mu/a}}$$

$$s_2 = \frac{1 - \cos \Delta E}{\mu/a}$$

$$(t-t_0) = r_0 \frac{\sin \Delta E}{\sqrt{\mu/a}} + \sigma_0 \frac{1-\cos \Delta E}{\mu/a} + \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^{\,3}}$$

$$g = r_0 \frac{\sin \Delta E}{\sqrt{\mu/a}} + \sigma_0 \frac{1-\cos \Delta E}{\mu/a}$$

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from above, gives

$$U = 3\mu \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^5} + \frac{1 - \cos \Delta E}{\mu/a} \left(g - \mu \frac{\sin \Delta E}{\sqrt{\mu/a}^3}\right) .$$

Of course, the partial derivatives may be computed for elliptic motion by using ΔE rather than ψ to solve for the coordinates and determine all the quantities in the partial derivatives. Then the only secular term in all the derivatives is that due to ΔE in the expression above for U .

2.4 THE INVERSE PARTIAL DERIVATIVES

The solution of the two-body problem is still of course valid if t is treated as the reference time and $\vec{r}$ and $\dot{\vec{r}}$ as initial conditions. Then $\vec{r}_0$ and $\dot{\vec{r}}_0$ are obtained as solutions of the differential equation at the time t_0. For this inverse solution,

$$r = \sqrt[+]{\vec{r} \cdot \vec{r}}$$

$$\sigma = \vec{r} \cdot \dot{\vec{r}}$$

$$\alpha = \dot{\vec{r}} \cdot \dot{\vec{r}} - 2\mu/r$$

where α is constant in time and therefore equal to the value above computed from $\vec{r}_0$ and $\dot{\vec{r}}_0$. As in the solution above, $\bar{\psi}$ and its transcendental functions.

$$\bar{s}_0 = 1 + \alpha \bar{\psi}^2/2! + \alpha^2 \bar{\psi}^4/4! + \alpha^3 \bar{\psi}^6/6! + \ldots$$

$$\bar{s}_1 = \bar{\psi} + \alpha \bar{\psi}^3/3! + \alpha^2 \bar{\psi}^5/5! + \alpha^3 \bar{\psi}^7/7! + \ldots$$

$$\bar{s}_2 = \bar{\psi}^2/2! + \alpha \bar{\psi}^4/4! + \alpha^2 \bar{\psi}^6/6! + \alpha^3 \bar{\psi}^8/8! + \ldots.$$

$$\bar{s}_3 = \bar{\psi}^3/3 + \alpha \bar{\psi}^5/5! + \alpha^2 \bar{\psi}^7/7! + \alpha^3 \bar{\psi}^9/9! + \ldots$$

satisfy the equation

$$t_0 = t + r\bar{s}_1 + \sigma\bar{s}_2 + \mu\bar{s}_3.$$

The bar is used to distinguish parameters of the inverse solution from those of the solution above. Also

$$r_0 = r\bar{s}_0 + \sigma\bar{s}_1 + \mu\bar{s}_2$$

and

$$\bar{f} = 1 - \mu\bar{s}_2/r \qquad\qquad \bar{g} = t_0 - t - \mu\bar{s}_3$$

$$\dot{\bar{f}} = -\mu\bar{s}_1/r_0 r \qquad\qquad \dot{\bar{g}} = 1 - \mu\bar{s}_2/r_0$$

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which give

$$\vec{r}_0 = \bar{f}\,\vec{r} + \bar{g}\,\dot{\vec{r}}$$

$$\dot{\vec{r}}_0 = \bar{\dot{f}}\,\vec{r} + \bar{\dot{g}}\,\dot{\vec{r}}$$

as the inverse solutions of the differential equations.

However,

$$\psi = \int_{t_0}^{t} \frac{dt^*}{r^*}$$

from the definition of ψ above where r^* is the value of the radius at the times t^* between t_0 and t. But

$$\bar{\psi} = \int_{t}^{t_0} \frac{dt^*}{r^*} = - \int_{t_0}^{t} \frac{dt^*}{r^*} = -\psi$$

so that $\bar{\psi}$ is merely the negative of ψ. Thus

$$\bar{s}_0 = s_0$$

$$\bar{s}_1 = -s_1$$

$$\bar{s}_2 = s_2$$

$$\bar{s}_3 = -s_3$$

so that ψ satisfies the equation

$$t_0 = t - r\,s_1 + \sigma\,s_2 - \mu\,s_3.$$

Also

$$r_0 = r\,s_0 - \sigma s_1 + \mu\,s_2$$

and

$$\bar{f} = 1 - \mu\,s_2/r \qquad\qquad \bar{g} = t_0 - t + \mu\,s_3 = -(t - t_0 - \mu s_3)$$

$$= \dot{g} \qquad\qquad\qquad\qquad = -g$$

$$\bar{\dot{f}} = \mu\,s_1/r\,r_0 \qquad\qquad \bar{\dot{g}} = 1 - \mu\,s_2/r_0$$

$$= -\dot{f} \qquad\qquad\qquad\qquad = f.$$

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This gives the formulas

$$\vec{r}_0 = \dot{g}\,\vec{r} - g\,\dot{\vec{r}}$$

$$\dot{\vec{r}}_0 = -\dot{f}\,\vec{r} + f\,\dot{\vec{r}}$$

which were used above to obtain the partial derivatives. In addition, substitution of

$$\vec{r} = f\,\vec{r}_0 + g\,\dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f}\,\vec{r}_0 + \dot{g}\,\dot{\vec{r}}_0$$

into the first equation gives

$$\vec{r}_0 = (f\,\dot{g} - \dot{f}\,g)\,\vec{r}_0$$

so that

$$f\,\dot{g} - \dot{f}\,g = 1$$

which was also used to obtain the partial derivatives. Another property of the inverse solution is that

$$\bar{U} = \bar{s}_2\,(t_0 - t) + \mu\,(\,\bar{\psi}\,\bar{s}_4 - 3\bar{s}_5)$$

where

$$\bar{s}_4 = s_4$$

$$\bar{s}_5 = -\,s_5$$

which gives

$$\bar{U} = -\,U\,.$$

The relations above may be used to express the partial derivatives in the inverse differential relations

$$
\begin{bmatrix} dx_0 \\ dy_0 \\ dz_0 \end{bmatrix} = \begin{bmatrix} \partial x_0/\partial x & \partial x_0/\partial y & \partial x_0/\partial z \\ \partial y_0/\partial x & \partial y_0/\partial y & \partial y_0/\partial z \\ \partial z_0/\partial x & \partial z_0/\partial y & \partial z_0/\partial z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} + \begin{bmatrix} \partial x_0/\partial \dot{x} & \partial x_0/\partial \dot{y} & \partial x_0/\partial \dot{z} \\ \partial y_0/\partial \dot{x} & \partial y_0/\partial \dot{y} & \partial y_0/\partial \dot{z} \\ \partial z_0/\partial \dot{x} & \partial z_0/\partial \dot{y} & \partial z_0/\partial \dot{z} \end{bmatrix} \begin{bmatrix} d\dot{x} \\ d\dot{y} \\ d\dot{z} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \partial x_0/\partial \mu \\ \partial y_0/\partial \mu \\ \partial z_0/\partial \mu \end{bmatrix} \cdot d\mu \; + \; \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \cdot dt_0 \; - \; \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \cdot dt
$$

$$
\begin{bmatrix} d\dot{x}_0 \\ d\dot{y}_0 \\ d\dot{z}_0 \end{bmatrix} = \begin{bmatrix} \partial \dot{x}_0/\partial x & \partial \dot{x}_0/\partial y & \partial \dot{x}_0/\partial z \\ \partial \dot{y}_0/\partial x & \partial \dot{y}_0/\partial y & \partial \dot{y}_0/\partial z \\ \partial \dot{z}_0/\partial x & \partial \dot{z}_0/\partial y & \partial \dot{z}_0/\partial z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} + \begin{bmatrix} \partial \dot{x}_0/\partial \dot{x} & \partial \dot{x}_0/\partial \dot{y} & \partial \dot{x}_0/\partial \dot{z} \\ \partial \dot{y}_0/\partial \dot{x} & \partial \dot{y}_0/\partial \dot{y} & \partial \dot{y}_0/\partial \dot{z} \\ \partial \dot{z}_0/\partial \dot{x} & \partial \dot{z}_0/\partial \dot{y} & \partial \dot{z}_0/\partial \dot{z} \end{bmatrix} \begin{bmatrix} d\dot{x} \\ d\dot{y} \\ d\dot{z} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \partial \dot{x}_0/\partial \mu \\ \partial \dot{y}_0/\partial \mu \\ \partial \dot{z}_0/\partial \mu \end{bmatrix} \cdot d\mu \; + \; \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{bmatrix} \cdot dt_0 \; - \; \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{bmatrix} \cdot dt
$$

In particular, since

$$
\begin{bmatrix} \partial x_0/\partial \mu \\ \partial y_0/\partial \mu \\ \partial z_0/\partial \mu \end{bmatrix} = \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} -\bar{s}_2/r \\ \bar{U}/r & -\bar{s}_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \partial \dot{x}_0/\partial \mu \\ \partial \dot{y}_0/\partial \mu \\ \partial \dot{z}_0/\partial \mu \end{bmatrix} = \begin{bmatrix} x_0 & \dot{x}_0 & \ddot{x}_0 \\ y_0 & \dot{y}_0 & \ddot{y}_0 \\ z_0 & \dot{z}_0 & \ddot{z}_0 \end{bmatrix} \begin{bmatrix} -\bar{s}_1/(r_0 r) \\ \bar{s}_2/r \\ \bar{U}/r - \bar{s}_3 \end{bmatrix}
$$

it follows that

$$
\begin{bmatrix} \partial x_0/\partial \mu \\ \partial y_0/\partial \mu \\ \partial z_0/\partial \mu \end{bmatrix} = \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} -s_2/r \\ -U/r + s_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \partial \dot{x}_0/\partial \mu \\ \partial \dot{y}_0/\partial \mu \\ \partial \dot{z}_0/\partial \mu \end{bmatrix} = \begin{bmatrix} x_0 & \dot{x}_0 & \ddot{x}_0 \\ y_0 & \dot{y}_0 & \ddot{y}_0 \\ z_0 & \dot{z}_0 & \ddot{z}_0 \end{bmatrix} \begin{bmatrix} s_1/(r\,r_0) \\ s_2/r \\ -U/r + s_3 \end{bmatrix}
$$

A similar method of expressing the partials of x_0, y_0, z_0 and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ with respect to x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ by first using the functions of $\mathcal{T}$ and then converting to functions of $\mathcal{U}$ gives

$$\begin{bmatrix} \partial x_0/\partial x & \partial x_0/\partial y & \partial x_0/\partial z \\ \partial y_0/\partial x & \partial y_0/\partial y & \partial y_0/\partial z \\ \partial z_0/\partial x & \partial z_0/\partial y & \partial z_0/\partial z \end{bmatrix} = \begin{bmatrix} \dot{g} & 0 & 0 \\ 0 & \dot{g} & 0 \\ 0 & 0 & \dot{g} \end{bmatrix} - U \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{z} \end{bmatrix}$$

$$+ \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} \dfrac{-\dot{f} s_1 + (\dot{g}-1)/r}{r} & \dot{f} s_2 \\ -\dfrac{(\dot{g}-1) s_1}{r} & (\dot{g}-1)s_2 \end{bmatrix} \begin{bmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

$$\begin{bmatrix} \partial x_0/\partial \dot{x} & \partial x_0/\partial \dot{y} & \partial x_0/\partial \dot{z} \\ \partial y_0/\partial \dot{x} & \partial y_0/\partial \dot{y} & \partial y_0/\partial \dot{z} \\ \partial z_0/\partial \dot{x} & \partial z_0/\partial \dot{y} & \partial z_0/\partial \dot{z} \end{bmatrix} = - \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} + U \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

$$- \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} -\dot{f} s_2 & (f-1) s_2 \\ -(\dot{g}-1)s_2 & g s_2 \end{bmatrix} \begin{bmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial x & \partial \dot{x}_0/\partial y & \partial \dot{x}_0/\partial z \\ \partial \dot{y}_0/\partial x & \partial \dot{y}_0/\partial y & \partial \dot{y}_0/\partial z \\ \partial \dot{z}_0/\partial x & \partial \dot{z}_0/\partial y & \partial \dot{z}_0/\partial z \end{bmatrix} = - \begin{bmatrix} \dot{f} & 0 & 0 \\ 0 & \dot{f} & 0 \\ 0 & 0 & \dot{f} \end{bmatrix} - U \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{bmatrix} \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{z} \end{bmatrix}$$

$$- \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} -\dot{f}\left(\dfrac{s_0}{rr_0} + \dfrac{1}{r^2} + \dfrac{1}{r_0^2}\right) & \dfrac{\dot{f} s_1 + (f-1)/r_0}{r_0} \\ -\dfrac{f s_1 + (\dot{g}-1)/r}{r} & \dot{f} s_2 \end{bmatrix} \begin{bmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial \dot{x} & \partial \dot{x}_0/\partial \dot{y} & \partial \dot{x}_0/\partial \dot{z} \\ \partial \dot{y}_0/\partial \dot{x} & \partial \dot{y}_0/\partial \dot{y} & \partial \dot{y}_0/\partial \dot{z} \\ \partial \dot{z}_0/\partial \dot{x} & \partial \dot{z}_0/\partial \dot{y} & \partial \dot{z}_0/\partial \dot{z} \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix} + U \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

$$+ \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix} \begin{bmatrix} -\dfrac{\dot{f} s_1 + (f-1)/r_0}{r_0} & \dfrac{(f-1)\, s_1}{r_0} \\ -\dot{f} s_2 & (f-1)\, s_2 \end{bmatrix} \begin{bmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix}$$

These inverse partial derivatives are often utilized for computation. However, they satisfy identically the well-known formulas

$$\begin{bmatrix} \partial x_0/\partial x & \partial x_0/\partial y & \partial x_0/\partial z \\ \partial y_0/\partial x & \partial y_0/\partial y & \partial y_0/\partial z \\ \partial z_0/\partial x & \partial z_0/\partial y & \partial z_0/\partial z \end{bmatrix} = \begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial x_0/\partial \dot{x} & \partial x_0/\partial \dot{y} & \partial x_0/\partial \dot{z} \\ \partial y_0/\partial \dot{x} & \partial y_0/\partial \dot{y} & \partial y_0/\partial \dot{z} \\ \partial z_0/\partial \dot{x} & \partial z_0/\partial \dot{y} & \partial z_0/\partial \dot{z} \end{bmatrix} = - \begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial x & \partial \dot{x}_0/\partial y & \partial \dot{x}_0/\partial z \\ \partial \dot{y}_0/\partial x & \partial \dot{y}_0/\partial y & \partial \dot{y}_0/\partial z \\ \partial \dot{z}_0/\partial x & \partial \dot{z}_0/\partial y & \partial \dot{z}_0/\partial z \end{bmatrix} = - \begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial \dot{x} & \partial \dot{x}_0/\partial \dot{y} & \partial \dot{x}_0/\partial \dot{z} \\ \partial \dot{y}_0/\partial \dot{x} & \partial \dot{y}_0/\partial \dot{y} & \partial \dot{y}_0/\partial \dot{z} \\ \partial \dot{z}_0/\partial \dot{x} & \partial \dot{z}_0/\partial \dot{y} & \partial \dot{z}_0/\partial \dot{z} \end{bmatrix} = \begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{bmatrix}^T$$

where the T indicates matrix transposition. Thus the inverse partials of x_0, y_0, z_0 and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ with respect to x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ may be obtained directly from the partials of x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to x_0, y_0, z_0 and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

3. SOME USEFUL FORMULAS FOR DERIVATION AND COMPUTATION

3.1 THE PROOF OF SERIES IDENTITIES

Several identities were used in the derivation above for the partial derivatives.

As an example, consider

$$s_0^2 - \alpha s_1^2 = 1.$$

This may be proven by collecting coefficients of the same powers of ψ in the products of the series.

$$(1 + \alpha \psi^2/2! + \alpha^2 \psi^4/4! + \dots) \cdot (1 + \alpha \psi^2/2! + \alpha^2 \psi^4/4! + \dots)$$
$$- \alpha (\psi + \alpha \psi^3/3! + \alpha^2 \psi^5/5! + \dots)(\psi + \alpha \psi^3/3! + \alpha^2 \psi^5/5! + \dots)$$
$$= 1 + \alpha \psi^2(1/2! + 1/2!) + \alpha^2 \psi^4(1/4! + 1/(2! 2!) + 1/4!) + \dots$$
$$\quad - \alpha \psi^2 \qquad\qquad - \alpha^2 \psi^4(1/3! + 1/3!) \qquad\qquad - \dots$$
$$= 1 + \alpha \psi^2(1-1) + \alpha^2 \psi^4(1+6+1-4-4)/4! + \dots$$
$$= 1$$

The product is valid since the series s_0, s_1, s_2, s_3, s_4, s_5 above are always absolutely convergent. However, an alternate proof is to first observe that the identity is true for zero α where each series reduces to its leading term. If α is negative, then

$$\alpha = - \mu/a$$
$$s_0 = \cos \Delta E$$
$$s_1 = \frac{\sin \Delta E}{\sqrt{\mu/a}}$$

From above where

$$\Delta E = \sqrt{\mu/a}\ \psi$$

so that

$$s_0^2 - \alpha s_1^2 = \cos^2 \Delta E + \frac{\mu}{a} \left(\frac{\sin \Delta E}{\sqrt{\mu/a}} \right)^2 = \cos^2 \Delta E + \sin^2 \Delta E$$
$$= 1.$$

Similarly, if α is positive, then

$$s_0 = \cosh \Delta F$$

$$s_1 = \frac{\sinh \Delta F}{\sqrt{\alpha}}$$

from above where

$$\Delta F = \sqrt{\alpha}\, \psi$$

so that

$$s_0^2 - \alpha\, s_1^2 = \cosh^2 \Delta F - \alpha \left(\frac{\sinh \Delta F}{\sqrt{\alpha}} \right)^2 = \cosh^2 \Delta F - \sinh^2 \Delta F$$

$$= 1.$$

Thus the identity is true for all α and all ψ.

The other series identities used in the partial derivatives involve $\frac{\partial s_0}{\partial \alpha}, \frac{\partial s_1}{\partial \alpha}, \frac{\partial s_2}{\partial \alpha}, \frac{\partial s_3}{\partial \alpha}$. They may also be proven by collecting coefficients of the same powers of ψ in the products of the series. However, the relations

$$2\,\frac{\partial s_0}{\partial \alpha} = \psi\, s_1$$

$$2\,\frac{\partial s_1}{\partial \alpha} = \psi\, s_2 - s_3$$

$$2\,\frac{\partial s_2}{\partial \alpha} = \psi\, s_3 - 2 s_4$$

$$2\,\frac{\partial s_3}{\partial \alpha} = \psi\, s_4 - 3 s_5$$

may be proven by a term-by-term comparison of the two sides of each equation. If these formulas are substituted in the identities, each identity is valid for zero α. But

$$s_0 = 1 + \alpha\, s_2$$
$$s_1 = \psi + \alpha\, s_3$$
$$s_2 = \psi^2/2! + \alpha\, s_4$$
$$s_3 = \psi^3/3! + \alpha\, s_5$$

similarly, so that, for non-zero α,

$$s_2 = \frac{s_0 - 1}{\alpha} = \frac{1 - s_0}{-\alpha} =$$

$$= \frac{\cos h\ \Delta F - 1}{\alpha} = \frac{1 - \cos\ \Delta E}{\mu/a}$$

$$s_3 = \frac{s_1 - \psi}{\alpha} = \frac{\psi - s_1}{-\alpha}$$

$$= \frac{\sin h\ \Delta F/\sqrt{\alpha} - \Delta F/\sqrt{\alpha}}{\alpha} = \frac{\Delta E/\sqrt{\mu/a} - \sin\ \Delta E/\sqrt{\mu/a}}{\mu/a}$$

$$= \frac{\sinh \Delta F - \Delta F}{\sqrt{\alpha}^3} = \frac{\Delta E - \sin\ \Delta E}{\sqrt{\mu/a}^3}$$

$$s_4 = \frac{s_2 - \psi^2/2!}{\alpha} = \frac{\psi^2/2! - s_2}{-\alpha}$$

$$= \frac{(\cos h\ \Delta F - 1)/\alpha - (\Delta F/\sqrt{\alpha})^2/2!}{\alpha} = \frac{(\Delta E/\sqrt{\mu/a})^2/2! - (1-\cos\Delta E)/(\mu/a)}{\mu/a}$$

$$= \frac{(\cos h\ \Delta F - 1) - \Delta^2 F/2!}{\alpha^2} = \frac{\Delta^2 E/2! - (1 - \cos \Delta E)}{(\mu/a)^2}$$

$$s_5 = \frac{s_3 - \psi^3/3!}{\alpha} = \frac{\psi^3/3! - s_3}{-\alpha}$$

$$= \frac{(\sin h\ \Delta F - \Delta F)/\sqrt{\alpha}^3 - (\Delta F/\sqrt{\alpha})^3/3!}{\alpha} = \frac{(\Delta E\sqrt{\mu/\alpha})^3/3 - (\Delta E - \sin\Delta E)\sqrt{\mu/\alpha}^3}{\mu/\alpha}$$

$$= \frac{(\sin h\ \Delta F - \Delta F) - \Delta^3 F/3!}{\sqrt{\alpha}^5} = \frac{\Delta^3 E/3! - (\Delta E - \sin\ \Delta E)}{\sqrt{\mu/a}^5}$$

Use of these relations proves the identities for both negative α where ΔE is real or positive α where ΔF is real. For example,

$$s_2 \, s_3 = 2 \, (s_2 \frac{\partial s_1}{\partial \alpha} - s_1 \frac{\partial s_2}{\partial \alpha})$$

$$= s_2 \, (2\frac{\partial s_1}{\partial \alpha}) - s_1 \, (2 \frac{\partial s_2}{\partial \alpha})$$

$$= s_2 \, (\psi s_2 - s_3) - s_1 \, (\psi s_3 - 2s_4)$$

is true when α is zero since

$$(\psi^2/2!) \, (\psi^3/3!) = (\psi^2/2!)(\psi \, \psi^2/2! - \psi^3/3!) - (\psi)(\psi \, \psi^3/3! - 2\psi^4/4!)$$

$$\psi^5/12 = \psi^5 (1/4 - 1/12 - 1/3 + 1/12)$$

$$= \psi^5/12.$$

For negative α ,

$$\frac{1 - \cos \Delta E}{\sqrt{\mu/a}} \quad \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3} = \frac{1-\cos \Delta E}{\mu/a} \left(\frac{\Delta E}{\sqrt{\mu/a}} \quad \frac{1-\cos \Delta E}{\mu/a} - \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3} \right)$$

$$- \frac{\sin \Delta E}{\sqrt{\mu/a}} \left(\frac{\Delta E}{\sqrt{\mu/a}} \quad \frac{\Delta E - \sin \Delta E}{\sqrt{\mu/a}^3} \right.$$

$$\left. - 2 \, \frac{\Delta^2 E/2! - (1 -\cos \Delta E)}{(\mu/a)^2} \right)$$

$$(1-\cos \Delta E) \, (\Delta E - \sin \Delta E) = (1-\cos \Delta E) \, (\Delta E -\Delta E \cos \Delta E - \Delta E + \sin \Delta E)$$

$$- \Delta E \sin \Delta E(\Delta E - \sin \Delta E) + 2 \sin \Delta E \, \Delta^2 E/2!$$

$$-2 \sin \Delta E \, (1 - \cos \Delta E)$$

$$= (1 - \cos \Delta E) \, (\sin \Delta E - \Delta E \cos \Delta E)$$

$$+ \Delta E \, (1 - \cos^2 \Delta E) - 2 \sin \Delta E \, (1 - \cos \Delta E)$$

$$= (1 - \cos \Delta E) \, (\sin \Delta E - \Delta E \cos \Delta E + \Delta E + \Delta E \cos \Delta E$$

$$- 2 \sin \Delta E)$$

$$= (1 - \cos \Delta E) \, (\Delta E - \sin \Delta E).$$

A similar result follows by using ΔF for positive α, but the use of ΔE is actually sufficient since the trigonometric formulas used are valid for imaginary arguments.

3.2 THE USE OF DIMENSIONLESS SERIES

For many purposes it is convenient to define the dimensionless argument

$$\lambda = \alpha \psi^2$$

and its transcendental functions

$$c_0 = 1 + \lambda/2! + \lambda^2/4! + \lambda^3/6! + \ldots$$
$$\dot{c}_1 = 1/1! + \lambda/3! + \lambda^2/5! + \lambda^3/7! + \ldots$$
$$c_2 = 1/2! + \lambda/4! + \lambda^2/6! + \lambda^3/8! + \ldots$$
$$c_3 = 1/3! + \lambda/5! + \lambda^2/7! + \lambda^3/9! + \ldots$$
$$c_4 = 1/4! + \lambda/6! + \lambda^2/8! + \lambda^3/10! + \ldots$$
$$c_5 = 1/5! + \lambda/7! + \lambda^2/9! + \lambda^3/11! + \ldots$$

which are related to s_0, s_1, s_2, s_3, s_4, s_5 by

$$s_0 = c_0$$
$$s_1 = \psi c_1$$
$$s_2 = \psi^2 c_2$$
$$s_3 = \psi^3 c_3$$
$$s_4 = \psi^4 c_4$$
$$s_5 = \psi^5 c_5$$

The dimensionless series have the property that

$$c_0 = 1 + \lambda c_2$$

$$c_1 = 1 + \lambda c_3$$

$$c_2 = 1/2! + \lambda c_4$$

$$c_3 = 1/3! + \lambda c_5$$

or, conversely, if $\lambda \neq 0$, that

$$c_5 = (c_3 - 1/3!)/\lambda$$
$$c_4 = (c_2 - 1/2!)/\lambda$$
$$c_3 = (c_1 - 1)/\lambda$$
$$c_2 = (c_0 - 1)/\lambda$$

In terms of trigonometric or hyperbolic functions, for $\lambda \neq 0$,

$$c_0 = 1 + \lambda/2! + \lambda^2/4! + \lambda^3/6! + \ldots.$$

$$= 1 + \sqrt{\lambda}^2/2! + \sqrt{\lambda}^4/4! + \sqrt{\lambda}^6/6! + \ldots.$$

$$= \cos h \sqrt{\lambda} = \cos h(i\sqrt{-\lambda})$$

$$= \cos \sqrt{-\lambda}$$

$$c_1 = 1 + \lambda/3! + \lambda^2/5! + \lambda^3/7! + \ldots.$$

$$= (\sqrt{\lambda} + \sqrt{\lambda}^3/3! + \sqrt{\lambda}^5/5! + \ldots.) /\!/\sqrt{\lambda}$$

$$= \sin h\sqrt{\lambda} \;/\!/\sqrt{\lambda} = \sin h (i\sqrt{-\lambda}) / (i\sqrt{-\lambda}) = i \sin \sqrt{-\lambda}/(i\sqrt{-\lambda})$$

$$= \sin \sqrt{-\lambda} / \sqrt{-\lambda}.$$

Let $c_0\{4\lambda\}$ and $c_1\{4\lambda\}$ be, respectively, the values of c_0 and c_1 for the argument (4λ) rather than λ. Then, for $\lambda > 0$ and $\lambda < 0$, respectively,

$$c_0 \{4\lambda\} = \cos h \sqrt{4\lambda} = \cos \sqrt{-4\lambda}$$

$$= \cos h (2\sqrt{\lambda}) = \cos (2\sqrt{-\lambda})$$

$$= 2 \cos h^2\sqrt{\lambda} - 1 = 2 \cos^2 \sqrt{-\lambda} - 1$$

$$c_1 \{4\lambda\} = \sin h\sqrt{4\lambda} / \sqrt{4\lambda} = \sin \sqrt{-4\lambda} / \sqrt{-4\lambda}$$

$$= \sin h (2\sqrt{\lambda}) / (2\sqrt{\lambda}) = \sin (2\sqrt{-\lambda}) / (2\sqrt{-\lambda})$$

$$= 2 \sin h \sqrt{\lambda} \cos h \sqrt{\lambda} / (2\sqrt{\lambda}) = 2 \sin \sqrt{-\lambda} \cos \sqrt{-\lambda} / (2\sqrt{-\lambda})$$

$$= \cos h\sqrt{\lambda} \cdot (\sin h\sqrt{\lambda} /\!/\sqrt{\lambda}) = \cos \sqrt{-\lambda} \cdot (\sin \sqrt{-\lambda} / \sqrt{-\lambda})$$

so that

$$c_0 \{ 4\lambda \} = 2 \, c_0^2 \{ \lambda \} - 1$$

$$c_1 \{ 4\lambda \} = c_0 \{ \lambda \} \cdot c_1 \{ \lambda \} .$$

These two relations are also true for zero λ, and are thus valid for all values of λ. It is worth noting that the argument

$$\lambda = \alpha \, \psi^2 .$$

and α are simultaneously positive, zero, or negative except in the trivial case where ψ and thus λ are both zero.

3.3 FORMULAS FOR THE SOLUTION OF KEPLER'S EQUATION

It was shown above that the general equation

$$t = t_0 + r_0 \, s_1 + \sigma_0 \, s_2 + \mu \, s_3$$

relating t and ψ is a generalization of Kepler's equation for elliptic motion. Stumpff (1962) calls it the main equation but prefers an equivalent of the alternate form

$$t = t_0 + r_0 \, \psi + \sigma_0 \, s_2 + \gamma_0 \, s_3.$$

The first, second, and third derivatives of this alternate form with respect to ψ give the formulas

$$r = r_0 + \sigma_0 \, s_1 + \gamma_0 \, s_2$$

for the radius,

$$\sigma = \sigma_0 \, s_0 + \gamma_0 \, s_1$$

for the scalar product of the position and velocity vectors, and

$$\gamma = \gamma_0 \, s_0 + \alpha \, \sigma_0 \, s_1$$

for the parameter γ which equals μ for parabolic motion, i.e. for $\alpha = 0$. When σ_0 and γ_0 are both zero, r is always equal to r_0 whereas σ and γ are both always zero. Then the motion is circular and ψ is a linear function of t.

In computing coordinates with or without partial derivatives for a particular time t, it is necessary to solve for that value of ψ which satisfies the main equation above. With a slight change in terminology, let

$$(t - t_0) = r_0 \, s_1 + \sigma_0 \, s_2 + \mu \, s_3$$

be the time interval computed directly from any value of ψ, and let τ be the desired time interval. Then the change $\Delta\psi$ in ψ required to change $(t - t_0)$ to τ is given by

$$\tau \doteq (t - t_0) + \frac{d(t - t_0)}{d\psi} \Delta\psi$$

to first order accuracy. Thus

$$\tau - (t - t_0) \doteq r \Delta\psi$$

where r is the radius for the particular value of ψ. The value

$$\psi* = \psi + \Delta\psi$$

is then a better approximation for the desired value of ψ if the first order approximation is accurate. Explicitly, if

$$\Delta\tau = r_0 s_1 + \sigma_0 s_2 + \mu s_3 - \tau$$

$$r = r_0 s_0 + \sigma_0 s_1 + \mu s_2$$

are first computed from ψ, then

$$\psi^* = \psi - \Delta\tau / r .$$

This is Newton's method for the solution of the main equation.

Using ψ^* as a new value for ψ then gives a new approximation, etc. However, a method of stopping the iterations is necessary. Also, Newton's method is not always convergent and should be backed up by alternate methods for obtaining a better approximation ψ^* from ψ. The iterations may be controlled by always bounding the solution by the maximum known value ψ_- and minimum known value ψ_+ for which the respective residuals $\Delta\tau_-$ and $\Delta\tau_+$ are, respectively, negative and positive. Then any approximation ψ^* is accepted at a new value of ψ only if

$$\psi_- < \psi^* < \psi_+$$

i.e. if ψ^* is between ψ_- and ψ_+ but not equal to ψ_- or ψ_+.

When the value ψ^* computed by Newton's method fails to lie between ψ_- and ψ_+, alternate methods may be tried to obtain a

satisfactory value of ψ^*. One such formula is

$$\psi^* = \psi \ (1 - \Delta\tau/\tau)$$

which gives smaller corrections to ψ for smaller values of $\left|\Delta\tau\right|$.

Another is to define ψ^* as the value satisfying the interpolation

formula

$$\frac{\psi_- - \psi^*}{\Delta\tau_-} = \frac{\psi_+ - \psi^*}{\Delta\tau_+} \ .$$

Explicit solution of this equation gives

$$\psi^* = \psi_- \ - \ \frac{\Delta\tau_-}{\Delta\tau_+ - \Delta\tau_-} \ (\psi_+ - \psi_-)$$

The use of these formulas is treated in the description of the

Fortran subroutine in Section 4.1.

3.4 FORMULAS FOR SERIES COMPUTATION

Each iteration in the solution of the general form of Kepler's equation requires the evaluation of s_0, s_1, s_2, and s_3 from a given value of ψ . The evaluation of s_4 and s_5 from the final value of ψ is also required for the determination of partial derivatives. However, it is convenient to compute the dimensionless series c_0, c_1, c_2, c_3, c_4, c_5 of the argument

$$\lambda = \alpha \, \psi^2$$

rather than s_0, s_1, s_2, s_3, s_4, s_5 directly. Then

$$s_0 = c_0$$
$$s_1 = \psi \, c_1$$
$$s_2 = \psi^2 c_2$$
$$s_3 = \psi^3 c_3$$

give the values required for each iteration of Kepler's equation. The final value of c_4 and c_5 may be used to obtain

$$U = s_2 (t-t_0) + \mu (c_4 - 3c_5) \; \psi^5$$

so that s_4 and s_5 are never required explicitly.

The series c_0, c_1, c_2, c_3, c_4, and c_5 must be computed from arbitrarily large values of $|\lambda|$, i.e. for arbitrarily large values of $|\alpha|$ and $|\psi|$, in order to determine coordinates and partial derivatives for arbitrarily large values of $|\tau|$, i.e. for any value of t . This may be done by repeatedly dividing λ by 4 when $|\lambda| > 1$ until the mth division reduces the argument to one or less in magnitude. Then c_0 and c_1 are computed as described below with the reduced argument. Then the formulas

$$c_0\{4\lambda\} = 2c_0^2\{\lambda\} - 1$$

$$c_1\{4\lambda\} = c_0\{\lambda\} \cdot c_1\{\lambda\}$$

are applied repeatedly m times to obain their true values for the original value of λ . Then the formulas

$$c_2 = (c_0 - 1) / \lambda$$

$$c_3 = (c_1 - 1)/\lambda$$

$$c_4 = (c_2 - 1/2) / \lambda$$

$$c_5 = (c_5 - 1/6) / \lambda$$

above give the true values of the four remaining series. This approach seems to be quite satisfactory, but an alternate method is to determine

$$c_0 = \cos \sqrt{-\lambda}$$

$$c_1 = \sin \sqrt{-\lambda} / \sqrt{-\lambda}$$

directly when $\lambda < -1$ or

$$c_0 = \cosh \sqrt{\lambda}$$

$$c_1 = \sinh \sqrt{\lambda} / \sqrt{\lambda}$$

directly when $\lambda > 1$. Then the same four formulas above give c_2, c_3, c_4 and c_5.

The series c_0, c_1, c_2, c_3, c_4, c_5 may all be determined from a given value of λ by first using the nesting formulas

$$c_5 = \left\{ 1 + \left[1 + \left(1 + \cdots \left[1 + \frac{\lambda}{(2n+3)(2n+2)} \right] \frac{\lambda}{(2n+1)(2n)} \cdots \right) \frac{\lambda}{9\cdot 8} \right] \frac{\lambda}{7\cdot 6} \right\} / 5!$$

$$c_4 = \left\{ 1 + \left[1 + \left(1 + \cdots \left[1 + \frac{\lambda}{(2n+2)(2n+1)} \right] \frac{\lambda}{(2n)(2n-1)} \cdots \right) \frac{\lambda}{8\cdot 7} \right] \frac{\lambda}{6\cdot 5} \right\} / 4!$$

to determine c_5 and c_4. The number of terms n required for the computation is considered below. Eight terms, i.e. seven leading 1's, are sufficient for accurate floating-point computation to sixteen decimal

digits when $|\lambda| \leq 1$ which is always true since if $|\lambda| > 1$ initially it is either reduced as described above or the alternate method is utilized. Then the resulting values of c_5 and c_4 give

$$c_3 = 1/6 + \lambda c_5$$

$$c_2 = 1/2 + \lambda c_4$$

$$c_1 = 1 + \lambda c_3$$

$$c_0 = 1 + \lambda c_2$$

for the remaining four series. If the argument has been reduced, c_0 and c_1 are then used as described above to obtain the true values of c_0, c_1, c_2, c_3, c_4, and c_5.

3.5 THE ACCURACY OF THE SERIES COMPUTATION

The limiting cases in the nesting computation above occur when $|\lambda| = 1$. If $\lambda = -1$, then

$$c_0 = \cos \sqrt{-\lambda} = \cos (1)$$
$$= .54030$$
$$c_2 = (c_0 - 1) / \lambda = (.54030 - 1)/(-1)$$
$$= .45970$$
$$c_4 = (c_2 - 1/2) / \lambda = (.45970 - .50000) / (-1)$$
$$= .04030$$

and, similarly,

$$c_1 = \sin \sqrt{-\lambda} / \sqrt{-\lambda} = \sin (1)/(1)$$
$$= .84147$$
$$c_3 = (c_1 - 1) / \lambda = (.84147 - 1) / (-1)$$
$$= .15853$$
$$c_5 = (c_3 - 1/6) / \lambda = (.15853 - .16666) / (-1)$$
$$= .00813$$

Also, if $\lambda = 1$, then

$$c_0 = \cosh \sqrt{\lambda} = \cosh (1)$$
$$= 1.54308$$
$$c_2 = (c_0 - 1)/\lambda = (1.54308 - 1)/(1)$$
$$= .54308$$
$$c_4 = (c_2 - 1/2)/\lambda = (.54308 - .50000)/(1)$$
$$= .04308$$

and, similarly,

$$c_1 = \sinh \sqrt{\lambda} / \sqrt{\lambda} = \sinh (1)/(1)$$
$$= 1.17520$$
$$c_3 = (c_1 - 1)/\lambda = (1.17520 - 1)/(1)$$
$$= .17520$$
$$c_5 = (c_3 - 1/6)/\lambda = (.17520 - .16666)/(1)$$
$$= .00754$$

Suppose that eight terms are chosen to compute c_4 and c_5 for $|\lambda| = 1$. Then if $\lambda = -1$, the truncation error Δc_4 introduced in computing c_4 is

$$\Delta c_4 = \lambda^8/20! + \lambda^9/22! + \lambda^{10}/24! + \ldots.$$

$$= + 1/20! - 1/22! + 1/24! + \ldots.$$

so that

$$\Delta c_4 < 1/20! = .41103 \times 10^{-18}$$

from the properties of alternating series.

Thus

$$\frac{\Delta c_4}{c_4} < \frac{.41103 \times 10^{-18}}{.04030} = 10.19 \times 10^{-18}$$

$$\left| \frac{\Delta c_4}{c_4} \right| < .1019 \times 10^{-16} .$$

Similarly, the truncation error Δc_5 introduced in computing c_5 is

$$\Delta c_5 = \lambda^8/21! + \lambda^9/23! + \lambda^{10}/25! + \ldots.$$

$$= 1/21! - 1/23! + 1/25! - \ldots.$$

so that

$$\Delta c_5 < 1/21! = .01957 \times 10^{-18}$$

and

$$\frac{\Delta c_5}{c_5} < \frac{.01957 \times 10^{-18}}{.00813} = 2.407 \times 10^{-18}$$

$$\left| \frac{\Delta c_5}{c_5} \right| < .2407 \times 10^{-17}$$

If $\lambda = + 1$ with eight terms, the truncation error Δc_4 introduced in computing c_4 is

$$\Delta c_4 = \lambda^8/20! + \lambda^9/22! + \lambda^{10}/24! + \ldots.$$

$$= 1/20! + 1/22! + 1/24! + \ldots.$$

$$< 1/20! + 1/(20^2 \cdot 20!) + 1/(20^4 \cdot 20!) + \ldots.$$

$$= (1 + 1/20^2 + 1/20^4 + \ldots)/20!$$

$$= [1/(1 - 1/20)] \ /20! = (20/19)/20! = (1/19!)/19$$

$$= .04326 \times 10^{-17}$$

so that

$$\frac{\Delta c_4}{c_4} < \frac{.04326 \times 10^{-17}}{.04308} = 1.004 \times 10^{-17}$$

$$\left| \frac{\Delta c_4}{c_4} \right| < .1004 \times 10^{-16} \ .$$

Similarly, the truncation error Δc_5 introduced in computing c_5 is

$$\Delta c_5 = \lambda^8/21! + \lambda^9/23! + \lambda^{10}/25! + \ldots.$$

$$= 1/21! + 1/23! + 1/25! + \ldots.$$

$$< 1/21! + 1/(21^2 \cdot 21!) + 1/(21^4 \cdot 21!) + \ldots.$$

$$= (1 + 1/21^2 + 1/21^4 + \ldots)/21!$$

$$= [1/(1-1/21)] \ /21! = (21/20)/21! = (1/20!)/20$$

$$= .02055 \times 10^{-18}$$

so that

$$\frac{\Delta c_5}{c_5} < \frac{.02055 \times 10^{-18}}{.00754} = 2.72 \times 10^{-18}$$

$$\left|\frac{\Delta c_5}{c_5}\right| < .272 \times 10^{-17}$$

Thus eight terms are sufficient for sixteen significant figures in the nesting computation above for c_5 and c_4 with $|\lambda| \leq 1$. A similar derivation will determine a sufficient number of terms if a different number of significant figures is required. Ordinarily this is determined by the particular electronic computer utilized, e.g. the IBM 7094 has 54 binary bits in a double-precision fraction which gives slightly more than 16 significant decimal digits for floating-point computation.

4. THE DOUBLE-PRECISION FORTRAN 4 IBM 7094 SUBROUTINE

4.1 DESCRIPTION OF PROGRAM INPUTS AND OUTPUTS

The Fortran 4 IBM 7094 double-precision subroutine TWOBDY has the following nine inputs. The six initial position and velocity components at the reference time t_0 are input as a state vector

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix}$$

The seventh input is the time interval

$$\tau = t - t_0$$

between the reference time t_0 and the time t at which a solution is desired. The eighth input is the constant μ in the differential equation where

$$\mu = G(m_1 + m_2)$$

if relative coordinates between two gravitationally attracting masses m_1 and m_2 are being obtained. The ninth input is an approximate value for the final solution ψ of Kepler's equation where

$$\psi = 0$$

is used if no approximation is available.

The input value of ψ is changed by the program to the actual solution of Kepler's equation for the input time interval τ. Thus the ninth input to the program is also an output quantity. The number

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of iterations required to solve the general form of Kepler's equation can be considerably reduced if a good input approximation for ψ is available. In particular, solutions may be required for a whole series of values of τ which differ by given increments. Then the use of the previous value of ψ as an input approximation for ψ at each new incremented value of τ is very efficient. It is usually better to use an input approximation for ψ even if it is only known very roughly. If not, it is more efficient to use an input value of zero although the program will work with any input value for the parameter ψ .

The program outputs the six position and velocity components in the form of a state vector

$$\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

for the time t. The 36 partial derivatives of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are output in a 6 x 6 matrix

$$\begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 & \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 & \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 & \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \\ \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 & \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 & \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 & \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix}$$

Similarly, the inverse partial derivatives are output in a 6x6 matrix

$$\begin{bmatrix} \partial x_0/\partial x & \partial x_0/\partial y & \partial x_0/\partial z & \partial x_0/\partial \dot{x} & \partial x_0/\partial \dot{y} & \partial x_0/\partial \dot{z} \\ \partial y_0/\partial x & \partial y_0/\partial y & \partial y_0/\partial z & \partial y_0/\partial \dot{x} & \partial y_0/\partial \dot{y} & \partial y_0/\partial \dot{z} \\ \partial z_0/\partial x & \partial z_0/\partial y & \partial z_0/\partial z & \partial z_0/\partial \dot{x} & \partial z_0/\partial \dot{y} & \partial z_0/\partial \dot{z} \\ \partial \dot{x}_0/\partial x & \partial \dot{x}_0/\partial y & \partial \dot{x}_0/\partial z & \partial \dot{x}_0/\partial \dot{x} & \partial \dot{x}_0/\partial \dot{y} & \partial \dot{x}_0/\partial \dot{z} \\ \partial \dot{y}_0/\partial x & \partial \dot{y}_0/\partial y & \partial \dot{y}_0/\partial z & \partial \dot{y}_0/\partial \dot{x} & \partial \dot{y}_0/\partial \dot{y} & \partial \dot{y}_0/\partial \dot{z} \\ \partial \dot{z}_0/\partial x & \partial \dot{z}_0/\partial y & \partial \dot{z}_0/\partial z & \partial \dot{z}_0/\partial \dot{x} & \partial \dot{z}_0/\partial \dot{y} & \partial \dot{z}_0/\partial \dot{z} \end{bmatrix}$$

The two 6x6 matrices above are also matrix inverses. That is, their product should give a 6x6 identity matrix.

The six partial derivatives of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ with respect to μ are output as a vector

$$\begin{bmatrix} \partial x/\partial \mu \\ \partial y/\partial \mu \\ \partial z/\partial \mu \\ \partial \dot{x}/\partial \mu \\ \partial \dot{y}/\partial \mu \\ \partial \dot{z}/\partial \mu \end{bmatrix}$$

Similarly, the six inverse partial derivatives of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ with respect to μ are output as a vector

$$\begin{bmatrix} \partial x_0/\partial \mu \\ \partial y_0/\partial \mu \\ \partial z_0/\partial \mu \\ \partial \dot{x}_0/\partial \mu \\ \partial \dot{y}_0/\partial \mu \\ \partial \dot{z}_0/\partial \mu \end{bmatrix}$$

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Additional outputs are the acceleration vector

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}$$

at time t and the acceleration vector

$$\begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{bmatrix}$$

at time t_0, as well as the radius

$$r = \sqrt{x^2 + y^2 + z^2}$$

at time t and the radius

$$r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

at time t_0.

The Fortran symbols for the inputs and outputs above are listed in comment statements at the beginning of the subroutine. The program may be used as a "black box" by those not interested in the formulation or computation. However, a knowledge of the operation of the subroutine is very important if modifications are desired for particular applications. For example, it may be desired to eliminate the partial derivatives or at least make their computation optional since they are not required in many cases. However, such modifications will be worthwhile only where the particular application is encountered repeatedly. Ordinarily it is better to use the program as it stands without modification although this may involve some redundant computation. In any case, the operation of the program is described below for those who are interested.

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4.2 THE INITIAL COMPUTATIONS

The following initial computations are performed only once before starting the series computation below that is repeated for each iteration in the solution of Kepler's equation. The following technique for computing the radius r_0 at time t_0 is used to avoid utilizing a square root subroutine. First

$$b = \text{maximum } (|x_0|, \ |y_0|, \ |z_0|)$$

$$c = (x_0/b)^2 + (y_0/b)^2 + (z_0/b)^2$$

are computed and the initial value of d is set equal to two. Then a better approximation d* for the square root of c is obtained from

$$d* = (d + c/d)/2$$

and d* is used as a new value of d to iterate the formula. The iterations are continued until d is no longer greater than d* which is then accepted as the square root of c. Then

$$r_0 = b \cdot d*$$

gives the radius r_0 at the reference time. The quantities

$$\sigma_0 = x_0 \dot{x}_0 + y_0 \dot{y}_0 + z_0 \dot{z}_0$$

$$\alpha = \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 - 2\mu/r_0$$

are then computed. It is also necessary to initialized a counter m to zero in order to compute the series described below for large arguments.

In the trivial case where τ is zero, ψ is set equal to zero and the series computation below is performed. The known value zero for ψ satisfies Kepler's equation identically so that all the outputs will then be computed by the program. If τ is negative, it is known that ψ must also be negative. Thus the right bound ψ_+ for ψ is set equal to zero and its residual $\Delta\tau_+$ is set equal

to the known value $-\tau$, whereas the left bound ψ_- and its residual $\Delta\tau_-$ are both set to $-\infty$, i.e. the negative of the largest number available for computation. If τ is positive, it is known that ψ must be positive. Thus the left bound ψ_- for ψ is set equal to zero and its residual $\Delta\tau_-$ is set equal to the known value $-\tau$, whereas the right bound ψ_+ and its residual $\Delta\tau_+$ are both set to $+\infty$, i.e. the largest number available for computation.

If the input approximation for ψ then lies between ψ_- and ψ_+, it is used in the series computation below. If not, Newton's method using an initial ψ of zero gives the approximation

$$\psi = \tau / r_0$$

which is then tried. If this value is not numerically between ψ_- and ψ_+,

$$\psi = \tau$$

is used to start the series computation. Thus an initial approximation for ψ is always obtained which is between ψ_- and ψ_+ but not equal to either.

4.3   THE SERIES COMPUTATION

The computation of the series is begun by computing the argument

$$\lambda = \alpha \, \psi^2$$

of the simensionless series. If $|\lambda| > 1$, the value is saved as λ_p but λ itself is repeatedly divided by 4 until $|\lambda| \leq 1$, and the number m of required divisions is also counted. Then $3c_5$ and c_4 are computed by the nesting formulas

$$3c_5 = (1+(1+(1+(1+(1+(1+(1+\frac{\lambda}{19 \cdot 18})\frac{\lambda}{17 \cdot 16})\frac{\lambda}{15 \cdot 14})\frac{\lambda}{13 \cdot 12})\frac{\lambda}{11 \cdot 10})\frac{\lambda}{9 \cdot 8})\frac{\lambda}{7 \cdot 6})/\ 40$$

$$c_4 = (1+(1+(1+(1+(1+(1+(1+\frac{\lambda}{18 \cdot 17})\frac{\lambda}{16 \cdot 15})\frac{\lambda}{14 \cdot 13})\frac{\lambda}{12 \cdot 11})\frac{\lambda}{10 \cdot 9})\frac{\lambda}{8 \cdot 7})\frac{\lambda}{6 \cdot 5})/\ 24$$

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which in turn give

$$c_3 = [1/2 + \lambda (3c_5)] / 3$$

$$c_2 = 1/2 + \lambda c_4$$

$$c_1 = 1 + \lambda c_3$$

$$c_0 = 1 + \lambda c_2$$

where each series is correct to sixteen significant figures.

If the original value of λ has not been reduced, the series s_3, s_2, s_1 are computed as described below. If the original λ has been reduced, i.e. if $m > 0$, the formulas

$$c_0^* = 2 c_0^2 - 1$$

$$c_1^* = c_1 c_0$$

are applied m times with c_0^* and c_1^* as new values for c_0 and c_1. The final values of c_0^* and c_1^* are accepted as the true values of c_0 and c_1 for the original value of λ, i.e. for λ_p. Also

$$c_2 = (c_0 - 1)/ \lambda_p$$

$$c_3 = (c_1 - 1)/ \lambda_p$$

$$c_4 = (c_2 - 1/2)/ \lambda_p$$

$$3c_5 = (3c_3 - 1/2)/ \lambda_p$$

are computed as the true value of the dimensionless series.

The dimensionless series c_3, c_2, c_1 give the series s_3, s_2, s_1 by the formulas

$$s_3 = \psi^3 c_3$$

$$s_2 = \psi^2 c_2$$

$$s_1 = \psi c_1$$

which are used to compute the residual in Kepler's equation below. The value of c_0 is equal to s_0 and will be used in Newton's method for solving Kepler's equation. If this particular series computation should be the last of the iterations for solving Kepler's equation, c_4 and $3c_5$ will be used below to compute partial derivatives. The series computation described here seems to be quite satisfactory for obtaining solutions for two-body motion. The absolute error in the solutions will increase as τ and thus λ increase, but this is an inherent error since the absolute error in τ itself increases as it becomes larger if it is represented as a floating point number.

4.4 THE SOLUTION OF KEPLER'S EQUATION

The residual $\Delta \tau$ for solving Kepler's equation is found by first computing

$$g = r_0 s_1 + \sigma_0 s_2$$

for the current value of ψ . Then

$$\Delta \tau = g + \mu s_3 - \tau$$

gives the residual. The radius

$$r = r_0 c_0 + \sigma_0 s_1 + \mu s_2$$

for the current value of ψ is also determined. If $\Delta \tau$ is zero, the coordinates are then computed as described below since Kepler's equation has been solved. If $\Delta \tau$ is negative, it replaces the old value of $\Delta \tau_-$ and ψ replaces the left bound ψ_- . If $\Delta \tau$ is positive, it replaces the old value of $\Delta \tau_+$ and ψ replaces the right bound ψ_+. Then Newton's method

$$\psi^* = \psi - \Delta \tau / r$$

is used in an attempt to obtain a better approximation for ψ .

If $\psi^* > \psi_-$ and $\psi^* < \psi_+$, ψ^* is accepted as a new value of ψ and the series computation above is repeated. Otherwise, the following

methods are used succesively in an attempt to obtain an approximation between ψ_- and ψ_+. First

$$\psi^* = \psi_- \, (1 - 4\Delta\tau_- / \tau_-)$$

is tried if $\left|\Delta\tau_-\right| < \left|\Delta\tau_+\right|$ but

$$\psi^* = \psi_+ \, (1 - 4\Delta\tau_+ / \tau_+)$$

is tried if $\left|\Delta\tau_-\right| > \left|\Delta\tau_+\right|$. The factor 4 helps to give significant corrections to ψ in floating-point computation when $\left|\Delta\tau_-\right|$ or $\left|\Delta\tau_+\right|$ is small. If $\left|\Delta\tau_-\right| = \left|\Delta\tau_+\right|$ or ψ^* fails to lie between ψ_- and ψ_+, the value

$$\psi^* = 2\psi_-$$

is tried if $\tau > 0$, but

$$\psi^* = 2\psi_+$$

is tried if $\tau < 0$. This insures that the solution will always be isolated between finite bounds. If this method fails, the interpolation formula

$$\psi^* = \psi_- - (\psi_+ - \psi_-) \, \Delta\tau / (\Delta\tau_+ - \Delta\tau_-)$$

is tried. If this fails, the mid-point

$$\psi^* = \psi_- + (\psi_+ - \psi_-)/2$$

between ψ_- and ψ_+ is tried. If this fails, the iterations are terminated and the coordinates are computed since a better value for ψ is not obtainable.

4.5 THE COMPUTATION OF COORDINATES AND ACCELERATIONS

The final values of the radius r and the function g have been obtained in the last iteration of Kepler's equation above. The functions s_1 and s_2 from the last iteration are used to determine

$$(f-1) = -\mu s_2 / r_0$$

$$\dot{f} = -\mu s_1 / (r \, r_0)$$

$$(\dot{g}-1) = -\mu s_2 / r \quad .$$

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Then the coordinates are obtained from the relations

$$x = x_0 + [(f-1) x_0 + g \dot{x}_0]$$

$$y = y_0 + [(f-1) y_0 + g \dot{y}_0]$$

$$z = z_0 + [(f - 1) z_0 + g \dot{z}_0]$$

$$\dot{x} = [\dot{f} x_0 + (\dot{g}-1) \dot{x}_0] + \dot{x}_0$$

$$\dot{y} = [\dot{f} y_0 + (\dot{g}-1) \dot{y}_0] + \dot{y}_0$$

$$\dot{z} = [\dot{f} z_0 + (\dot{g}-1) \dot{z}_0] + \dot{z}_0$$

which give maximum accuracy with small values of τ for which $(f-1)$, g, $\dot{f}$, $(\dot{g}-1)$ are each close to zero. Then

$$\ddot{x} = - \mu x / r^3$$

$$\ddot{y} = - \mu y / r^3$$

$$\ddot{z} = - \mu z / r^3$$

give the acceleration components at time t, and the acceleration components

$$\ddot{x}_0 = - \mu x_0 / r_0^3$$

$$\ddot{y}_0 = - \mu y_0 / r_0^3$$

$$\ddot{z}_0 = - \mu z_0 / r_0^3$$

at time t_0 are also computed.

4.6 THE COMPUTATION OF PARTIAL DERIVATIVES

The partial derivatives are now computed by determining

$$U = s_2 (t-t_0) + \mu (c_4 - 3c_5) \psi^5$$

and using locations in the matrix of partial derivatives to store elements of the matrices on the right side of the formulas

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$$
\begin{bmatrix} \partial x/\partial\mu \\ \partial y/\partial\mu \\ \partial z/\partial\mu \end{bmatrix}
=
\begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix}
\begin{bmatrix} -\,s_2/r_0 \\[4pt] U/r_0 - s_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \partial \dot{x}/\partial\mu \\ \partial \dot{y}/\partial\mu \\ \partial \dot{z}/\partial\mu \end{bmatrix}
=
\begin{bmatrix} x & \dot{x} & \ddot{x} \\ y & \dot{y} & \ddot{y} \\ z & \dot{z} & \ddot{z} \end{bmatrix}
\begin{bmatrix} -s_1/(r\,r_0) \\[4pt] s_2/r_0 \\[4pt] U/r_0 - s_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \partial x_0/\partial\mu \\ \partial y_0/\partial\mu \\ \partial z_0/\partial\mu \end{bmatrix}
=
\begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \\ z_0 & \dot{z}_0 \end{bmatrix}
\begin{bmatrix} -\,s_2/r \\[4pt] -U/r + s_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \partial \dot{x}_0/\partial\mu \\ \partial \dot{y}_0/\partial\mu \\ \partial \dot{z}_0/\partial\mu \end{bmatrix}
=
\begin{bmatrix} x_0 & \dot{x}_0 & \ddot{x}_0 \\ y_0 & \dot{y}_0 & \ddot{y}_0 \\ z_0 & \dot{z}_0 & \ddot{z}_0 \end{bmatrix}
\begin{bmatrix} s_1/(r\,r_0) \\[4pt] s_2/r \\[4pt] -U/r + s_3 \end{bmatrix}
$$

which give the partial derivatives with respect to μ . Similarly, locations in the matrix of partial derivatives are used to store elements of the first matrix on the right side of the formulas for the 2x3 matrices

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$$[P_1] = \begin{bmatrix} -\dfrac{\dot{f}\,s_1 + (f-1)/r_0}{r_0} & -\dot{f}\,s_2 \\[2ex] \dfrac{(f-1)\,s_1}{r_0} & (f-1)\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\[1ex] \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$[P_2] = \begin{bmatrix} -\dot{f}\,s_2 & -(\dot{g}-1)s_2 \\[2ex] (f-1)\,s_2 & g\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\[1ex] \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$[P_3] = \begin{bmatrix} -\dot{f}\left(\dfrac{s_0}{r\,r_0} + \dfrac{1}{r^2} + \dfrac{1}{r_0^2}\right) & -\dfrac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r} \\[3ex] \dfrac{\dot{f}\,s_1 + (f-1)/r_0}{r_0} & \dot{f}\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\[1ex] \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

$$[P_4] = \begin{bmatrix} -\dfrac{\dot{f}\,s_1 + (\dot{g}-1)/r}{r} & \dfrac{-(\dot{g}-1)\,s_1}{r} \\[3ex] \dot{f}\,s_2 & (\dot{g}-1)\,s_2 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\[1ex] \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}$$

which are stored in locations of the inverse partial derivative matrix.

Then the computations

$$\begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial y_0 \end{bmatrix} = \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} P_1 \end{bmatrix} + U \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix} + \begin{bmatrix} (f-1)+1 & 0 & 0 \\ 0 & (f-1)+1 & 0 \\ 0 & 0 & (f-1)+1 \end{bmatrix}$$

$$\begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\ \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\ \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix} = \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} P_2 \end{bmatrix} - U \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} + \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\ \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\ \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix} = \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} P_3 \end{bmatrix} + U \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{bmatrix} + \begin{bmatrix} \dot{f} & 0 & 0 \\ 0 & \dot{f} & 0 \\ 0 & 0 & \dot{f} \end{bmatrix}$$

$$\begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\ \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\ \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix} = \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} \begin{bmatrix} P_4 \end{bmatrix} - U \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} + \begin{bmatrix} (\dot{g}-1)+1 & 0 & 0 \\ 0 & (\dot{g}-1)+1 & 0 \\ 0 & 0 & (\dot{g}-1)+1 \end{bmatrix}$$

give the 36 partial derivatives of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, with respect to x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. Finally, the formulas

$$\begin{bmatrix} \partial x_0/\partial x & \partial x_0/\partial y & \partial x_0/\partial z \\[6pt] \partial y_0/\partial x & \partial y_0/\partial y & \partial y_0/\partial z \\[6pt] \partial z_0/\partial x & \partial z_0/\partial y & \partial z_0/\partial z \end{bmatrix} = \begin{bmatrix} \partial \dot{x}/\partial \dot{x}_0 & \partial \dot{x}/\partial \dot{y}_0 & \partial \dot{x}/\partial \dot{z}_0 \\[6pt] \partial \dot{y}/\partial \dot{x}_0 & \partial \dot{y}/\partial \dot{y}_0 & \partial \dot{y}/\partial \dot{z}_0 \\[6pt] \partial \dot{z}/\partial \dot{x}_0 & \partial \dot{z}/\partial \dot{y}_0 & \partial \dot{z}/\partial \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial x_0/\partial \dot{x} & \partial x_0/\partial \dot{y} & \partial x_0/\partial \dot{z} \\[6pt] \partial y_0/\partial \dot{x} & \partial y_0/\partial \dot{y} & \partial y_0/\partial \dot{z} \\[6pt] \partial z_0/\partial \dot{x} & \partial z_0/\partial \dot{y} & \partial z_0/\partial \dot{z} \end{bmatrix} = - \begin{bmatrix} \partial x/\partial \dot{x}_0 & \partial x/\partial \dot{y}_0 & \partial x/\partial \dot{z}_0 \\[6pt] \partial y/\partial \dot{x}_0 & \partial y/\partial \dot{y}_0 & \partial y/\partial \dot{z}_0 \\[6pt] \partial z/\partial \dot{x}_0 & \partial z/\partial \dot{y}_0 & \partial z/\partial \dot{z}_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial x & \partial \dot{x}_0/\partial y & \partial \dot{x}_0/\partial z \\[6pt] \partial \dot{y}_0/\partial x & \partial \dot{y}_0/\partial y & \partial \dot{y}_0/\partial z \\[6pt] \partial \dot{z}_0/\partial x & \partial \dot{z}_0/\partial y & \partial \dot{z}_0/\partial z \end{bmatrix} = - \begin{bmatrix} \partial \dot{x}/\partial x_0 & \partial \dot{x}/\partial y_0 & \partial \dot{x}/\partial z_0 \\[6pt] \partial \dot{y}/\partial x_0 & \partial \dot{y}/\partial y_0 & \partial \dot{y}/\partial z_0 \\[6pt] \partial \dot{z}/\partial x_0 & \partial \dot{z}/\partial y_0 & \partial \dot{z}/\partial z_0 \end{bmatrix}^T$$

$$\begin{bmatrix} \partial \dot{x}_0/\partial \dot{x} & \partial \dot{x}_0/\partial \dot{y} & \partial \dot{x}_0/\partial \dot{z} \\[6pt] \partial \dot{y}_0/\partial \dot{x} & \partial \dot{y}_0/\partial \dot{y} & \partial \dot{y}_0/\partial \dot{z} \\[6pt] \partial \dot{z}_0/\partial \dot{x} & \partial \dot{z}_0/\partial \dot{y} & \partial \dot{z}_0/\partial \dot{z} \end{bmatrix} = \begin{bmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\[6pt] \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\[6pt] \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{bmatrix}^T$$

give the 36 inverse partial derivatives of x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ with respect to x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$. This concludes the computation of all quantities which are output by the subroutine.

4.7 SOME PROPERTIES OF THE SUBROUTINE

A Fortran 4 listing of the double-precision subroutine TWOBDY is given below. The compiled subroutine occupies less than 1500 single-precision storage locations on the IBM 7094. The minimum running time is a little over 10 milliseconds and occurs for the trivial case where the input value of τ is zero. The program will almost always run in less than 100 milliseconds for a non-zero value of τ where no input approximation for ψ is available. A good input approximation for ψ can reduce the running time to as low as 15 milliseconds. Extremely unusual cases with no input approximation for ψ can take as much as 1000 milliseconds, but the solution will always be obtained.

Accuracy checks indicate that errors in the outputs are always close to a lower bound below which accuracy improvements are unwarranted. The lower bound occurs because of uncertainties in the inputs due to the floating-point representation of numbers. For example, the absolute error in x should not be reduced below about

$$\delta_x = \left|\frac{\partial x}{\partial x_0}\right| \cdot \left|10^{-16}x_0\right| + \left|\frac{\partial x}{\partial y_0}\right| \cdot \left|10^{-16}y_0\right| + \left|\frac{\partial x}{\partial z_0}\right| \cdot \left|10^{-16}z_0\right|$$

$$+ \left|\frac{\partial x}{\partial \dot{x}_0}\right| \cdot \left|10^{-16}\dot{x}_0\right| + \left|\frac{\partial x}{\partial \dot{y}_0}\right| \cdot \left|10^{-16}\dot{y}_0\right| + \left|\frac{\partial x}{\partial \dot{z}_0}\right| \cdot \left|10^{-16}\dot{z}_0\right|$$

$$+ \left|\frac{\partial x}{\partial \mu}\right| \cdot \left|10^{-16}\mu\right| + \left|\dot{x}\right| \left|10^{-16}\tau\right|$$

because each of the eight inputs x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, μ, and τ is known to only about sixteen significant figures. Partial derivatives have been checked by multiplying the 6x6 inverse partial derivative matrix. The difference between the product and the unit matrix is then easily explainable in terms of errors caused by the floating-point representation

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of the partial derivatives.

Results obtained in some limited applications of the subroutine have been very satisfactory. The author will sincerely appreciate any suggestions, criticisms, or comments from those using the Fortran program. A Fortran version for the IBM 360 should be available shortly. There are several programs available which solve the equivalent problem accurately and efficiently for particular cases of two-body motion. The main advantage of the program described here is that it offers equal accuracy and efficiency with complete generality as well. This can become extremely important in applications where a compact program is desired to handle many different types of two-body motion. In any case, the generality is provided without any sacrifice of accuracy, efficiency, or compactness.

```
$IBFTC TWOBDY
       SUBROUTINE TWOBDY(S0,TAU,MU,PSI,S,P,PI,PMU,POMU,ACC,ACC0,R,R0)
C
C GENERAL SOLUTION OF TWO BODY PROBLEM WITH PARTIAL DERIVATIVES
C FORTRAN 4 DOUBLE PRECISION SUBROUTINE FOR IBM 7094 WITH IBSYS SYSTEM
C SEE APRIL 1965 ASTRONOMICAL JOURNAL FOR FORMULATION BY W. H. GOODYEAR
C
C CALLING SEQUENCE IS AS FOLLOWS
C      CALL TWOBDY(S0,TAU,MU,PSI,S,P,PI,PMU,POMU,ACC,ACC0,R,R0)
C
C DOUBLE PRECISION QUANTITIES IN CALLING SEQUENCE ARE AS FOLLOWS
C      DOUBLE PRECISION S0(6),TAU,MU,PSI
C     1,S(6),P(6,6),PI(6,6),PMU(6),POMU(6),ACC(3),ACC0(3),R,R0
C
C                 INPUTS
C S0(1),S0(2),S0(3)=X0,Y0,Z0=POSITION COMPONENTS AT REFERENCE TIME T0
C S0(4),S0(5),S0(6)=XD0,YD0,ZD0=VELOCITY COMPONENTS AT REFERENCE TIME T0
C TAU=TIME INTERVAL (T-T0) FROM REFERENCE TIME T0 TO SOLUTION TIME T
C MU=CONSTANT IN DIFFERENTIAL EQUATIONS (XDD,YDD,ZDD)=-MU*(X,Y,Z)/(R**3)
C PSI=APPROXIMATION FOR FINAL SOLUTION PSI OF KEPLER'S EQUATION
C
C                 OUTPUTS
C PSI=GENERALIZED ECCENTRIC ANOMALY=SOLUTION OF KEPLERS EQUATION
C S(1),S(2),S(3)=X,Y,Z=POSITION COMPONENTS AT SOLUTION TIME T=T0+TAU
C S(4),S(5),S(6)=XD,YD,ZD=VELOCITY COMPONENTS AT SOLUTION TIME T=T0+TAU
C P(I,J)=PARTIAL DERIVATIVE DS(I)/DS0(J) OF S(I) WITH RESPECT TO S0(J)
C PI(I,J)=PARTIAL DS0(I)/DS(J) WITH ROLES OF T0 AND T REVERSED
C PMU(I)=PARTIAL DS(I)/DMU OF S(I) WITH RESPECT TO MU
C POMU(I)=PARTIAL DS0(I)/DMU WITH ROLES OF T0 AND T REVERSED
C ACC(I)=-MU*S(I)/(R**3)=ACCELERATION COMPONENT AT SOLUTION TIME T
C ACC0(I)=-MU*S0(I)/(R0**3)=ACCELERATION COMPONENT AT REFERENCE TIME T0
C R=RADIUS AT TIME T=SQUARE ROOT OF(X**2+Y**2+Z**2)
C R0=RADIUS AT TIME T0=SQUARE ROOT OF(X0**2+Y0**2+Z0**2)
C
C ADDITIONAL DOUBLE PRECISION QUANTITIES FOR COMPUTATION
C     2,SIG0,ALPHA,PSIN,PSIP,A,AP,C0,C1,C2,C3,C4,C5X3,S1,S2,S3,DTAU,DTAUN
C     3,DTAUP,U,FM1,G,FD,GDM1
C
C START OF INITIAL COMPUTATIONS
C COMPUTE RADIUS R0=SQUARE ROOT OF(X0**2+Y0**2+Z0**2)
       S1=DMAX1(DABS(S0(1)),DABS(S0(2)),DABS(S0(3)))
       S2=(S0(1)/S1)**2+(S0(2)/S1)**2+(S0(3)/S1)**2
       R0=2.
    10 R=R0
```

```
            RO=(R+S2/R)*.5
            IF(RO.LT.R) GO TO 10
            RO=RO*S1
C COMPUTE OTHER PARAMETERS
            SIGO=SO(1)*SO(4)+SO(2)*SO(5)+SO(3)*SO(6)
            ALPHA=SO(4)**2+SO(5)**2+SO(6)**2-2.*MU/RO
C INITIALIZE SERIES MOD COUNT M TO ZERO
            M=0
C INITIALIZE BOUNDS PSIN AND PSIP FOR PSI OR SET PSI=0 IF TAU=0
            IF(TAU) 20,30,40
      20 PSIN=-1.0+38
            PSIP=0.
            DTAUN=PSIN
            DTAUP=-TAU
            GO TO 50
      30 PSI=0.
            GO TO 100
      40 PSIN=0.
            PSIP=+1.0+38
            DTAUN=-TAU
            DTAUP=PSIP
C USE APPROXIMATION FOR PSI IF IT IS BETWEEN BOUNDS PSIN AND PSIP
      50 IF(PSI.GT.PSIN.AND.PSI.LT.PSIP) GO TO 100
C TRY NEWTON'S METHOD FOR INITIAL PSI SET EQUAL TO ZERO
            PSI=TAU/RO
C SET PSI=TAU IF NEWTON'S METHOD FAILS
            IF(PSI.LE.PSIN.OR.PSI.GE.PSIP) PSI=TAU
C END OF INITIAL COMPUTATIONS
C
C BEGINNING OF LOOP FOR SOLVING KEPLER'S EQUATION
C BEGINNING OF SERIES SUMMATION
C COMPUTE ARGUMENT A IN REDUCED SERIES OBTAINED BY FACTORING OUT PSI'S
      100 A=ALPHA*PSI*PSI
            IF(DABS(A).LE.1.) GO TO 120
C SAVE A IN AP AND MOD A IF IT EXCEEDS UNITY IN MAGNITUDE
            AP=A
      110 M=M+1
            A=A*.25
            IF(DABS(A).GT.1.) GO TO 110
C SUM SERIES C5X3=3*S5/PSI**5 AND C4=S4/PSI**4
      120 C5X3=(1.+(1.+(1.+(1.+(1.+(1.+(1.+A/342.)*A/272.)*A/210.)*A/156.)
          1       *A/110.)*A/72.)*A/42.)/40.
            C4  =(1.+(1.+(1.+(1.+(1.+(1.+(1.+A/306.)*A/240.)*A/182.)*A/132.)
          1       *A/90.)*A/56.)*A/30.)/24.
C COMPUTE SERIES C3=S3/PSI**3,C2=S2/PSI**2,C1=S1/PSI,C0=S0
            C3=(.5+A*C5X3)/3.
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        C2= .5+A*C4
        C1= 1.+A*C3
        C0= 1.+A*C2
        IF(M.LE.0) GO TO 140
C DEMOD SERIES C0 AND C1 IF NECESSARY WITH DOUBLE ANGLE FORMULAS
    130 C1=C1*C0
        C0=2.*C0*C0-1.
        M=M-1
        IF(M.GT.0) GO TO 130
C DETERMINE C2,C3,C4,C5X3 FROM C0,C1,AP IF DEMOD REQUIRED
        C2=(C0-1.)/AP
        C3=(C1-1.)/AP
        C4=(C2-.5)/AP
        C5X3=(3.*C3-.5)/AP
C COMPUTE SERIES S1,S2,S3 FROM C1,C2,C3
    140 S1=C1*PSI
        S2=C2*PSI*PSI
        S3=C3*PSI*PSI*PSI
C END OF SERIES SUMMATION
C COMPUTE RESIDUAL DTAU AND SLOPE R FOR KEPLER'S EQUATION
        G=R0*S1+SIG0*S2
        DTAU=(G+MU*S3)-TAU
        R=DABS(R0*C0+(SIG0*S1+MU*S2))
        IF(DTAU) 200,300,210
C RESET BOUND
    200 PSIN=PSI
        DTAUN=DTAU
        GO TO 220
    210 PSIP=PSI
        DTAUP=DTAU
C TRY NEWTON'S METHOD AND INITIALIZE SELECTOR N
    220 PSI=PSI-DTAU/R
        N=0
C ACCEPT PSI IF IT IS BETWEEN BOUNDS PSIN AND PSIP
    230 IF(PSI.GT.PSIN.AND.PSI.LT.PSIP) GO TO 100
C SELECT ALTERNATE METHOD OF COMPUTING PSI OR STOP ITERATIONS
        N=N+1
        GO TO (1,2,3,4,300),N
C TRY INCREMENTING BOUND WITH DTAU NEAREST ZERO BY THE RATIO 4*DTAU/TAU
      1 IF(DABS(DTAUN).LT.DABS(DTAUP)) PSI=PSIN*(1.-(4.*DTAUN)/TAU)
        IF(DABS(DTAUP).LT.DABS(DTAUN)) PSI=PSIP*(1.-(4.*DTAUP)/TAU)
        GO TO 230
C TRY DOUBLING BOUND CLOSEST TO ZERO
      2 IF(TAU.GT.0.) PSI=PSIN+PSIN
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          IF(TAU.LT.0.) PSI=PSIP+PSIP
          GO TO 230
C TRY INTERPOLATION BETWEEN BOUNDS
      3 PSI=PSIN+(PSIP-PSIN)*(-DTAUN/(DTAUP-DTAUN))
          GO TO 230
C TRY HALVING BETWEEN BOUNDS
      4 PSI=PSIN+(PSIP-PSIN)*.5
          GO TO 230
C END OF LOOP FOR SOLVING KEPLER'S EQUATION
C
C COMPUTE REMAINING THREE OF FOUR FUNCTIONS FM1,G,FD,GDM1
    300 FM1=-MU*S2/R0
          FD=-MU*S1/R0/R
          GDM1=-MU*S2/R
C COMPUTE COORDINATES AT SOLUTION TIME T=T0+TAU
          DO 310 I=1,3
          S(I)=S0(I)+(FM1*S0(I)+G*S0(I+3))
          S(I+3)=(FD*S0(I)+GDM1*S0(I+3))+S0(I+3)
C COMPUTE ACCELERATIONS
          ACC(I)=-MU*S(I)/R/R/R
    310 ACC0(I)=-MU*S0(I)/R0/R0/R0
C END OF COMPUTATION FOR COORDINATES AND ACCELERATIONS
C
C COMPUTATION OF PARTIAL DERIVATIVES
C COMPUTE COEFFICIENTS FOR STATE PARTIALS
          U= S2*TAU+MU*(C4-C5X3)*PSI*PSI*PSI*PSI*PSI
          P(1,1)=-(FD*S1+FM1/R0)/R0
          P(1,2)=-FD*S2
          P(2,1)= FM1*S1/R0
          P(2,2)= FM1*S2
          P(1,3)= P(1,2)
          P(1,4)=-GDM1*S2
          P(2,3)= P(2,2)
          P(2,4)= G*S2
          P(3,1)=-FD*(C0/R0/R+1./R/R+1./R0/R0)
          P(3,2)=-(FD*S1+GDM1/R)/R
          P(4,1)=-P(1,1)
          P(4,2)=-P(1,2)
          P(3,3)= P(3,2)
          P(3,4)=-GDM1*S1/R
          P(4,3)=-P(1,2)
          P(4,4)=-P(1,4)
C COMPUTE COEFFICIENTS FOR MU PARTIALS
          P(1,5)=-S1/R0/R
          P(2,5)= S2/R0
          P(3,5)= U/R0-S3
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      P(1,6)=-P(1,5)
      P(2,6)= S2/R
      P(3,6)=-U/R+S3
      DO 400 I=1,3
C COMPUTE MU PARTIALS
      PMU(I)=-S(I)*P(2,5)+S(I+3)*P(3,5)
      PMU(I+3)= S(I)*P(1,5)+S(I+3)*P(2,5)+ACC(I)*P(3,5)
      POMU(I)=-SO(I)*P(2,6)+SO(I+3)*P(3,6)
      POMU(I+3)= SO(I)*P(1,6)+SO(I+3)*P(2,6)+ACCO(I)*P(3,6)
C MATRIX ACCUMULATIONS FOR STATE PARTIALS
      DO 400 J=1,4
      PI(J,I)= P(J,1)*SO(I)+P(J,2)*SO(I+3)
  400 PI(J,I+3)= P(J,3)*SO(I)+P(J,4)*SO(I+3)
      DO 410 I=1,3
      DO 420 J=1,3
      P(I,J)     =S(I)*PI(1,J)  +S(I+3)*PI(2,J)  +U*S(I+3)*ACCO(J)
      P(I,J+3)   =S(I)*PI(1,J+3)+S(I+3)*PI(2,J+3)-U*S(I+3)*SO(J+3)
      P(I+3,J)   =S(I)*PI(3,J)  +S(I+3)*PI(4,J)  +U*ACC(I)*ACCO(J)
  420 P(I+3,J+3)=S(I)*PI(3,J+3)+S(I+3)*PI(4,J+3)-U*ACC(I)*SO(J+3)
      P(I,I)     =P(I,I)     +FM1+1.
      P(I,I+3)   =P(I,I+3)   +G
      P(I+3,I)   =P(I+3,I)   +FD
  410 P(I+3,I+3)=P(I+3,I+3)+GDM1+1.
C TRANSPOSITIONS FOR INVERSE STATE PARTIALS
      DO 430 I=1,3
      DO 430 J=1,3
      PI(J+3,I+3)= P(I,J)
      PI(J+3,I)  =-P(I+3,J)
      PI(J,I+3)  =-P(I,J+3)
  430 PI(J,I)    = P(I+3,J+3)
C END OF COMPUTATION FOR PARTIAL DERIVATIVES
C
C END OF PROGRAM - ALL OUTPUTS HAVE BEEN COMPUTED
      RETURN
      END
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