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SUMMARY

The transfer function for a constant-thickness, thin-wall cylindrical tank containing a fluid is derived. The fluid in the tank is assumed to have a constant ullage gas pressure and a pressure distribution initially linear with longitudinal position. The treatment is one-dimensional, and the effects of damping are considered. The transfer function developed is the response of pressure at any longitudinal position to velocity at the tank base. General expressions for the resonant and nodal frequencies are given, and the time response to a sinusoidal driving velocity for an undamped liquid is derived. A numerical example which shows the general form of the transfer function solution is given, and a typical liquid-oxygen tank is analyzed.

INTRODUCTION

The occurrence of longitudinal oscillations (known as pogo) of significant amplitude in launch vehicles has produced efforts to define the problem analytically. This dynamic instability, which is usually due to the coupling of the vehicle structure (in a longitudinal mode) and the vehicle propulsion system, is known as the longitudinal structural dynamics problem. Analytical definition of the problem requires dynamic descriptions of the vehicle structure, the tankage characteristics, the propellant lines, the pumps, and the engines and their intercoupling.

Studies of the overall problem (refs. 1 to 4) have placed emphasis on the modeling of pump-inlet and feed-line characteristics and other important areas, but have assumed relatively simple tank models for predicting pressure at the feed-line inlets. Generally, these studies have represented the propellant-tank characteristics as an equivalent single lumped mass or some minor modification of this concept. This study was conducted to determine the transfer function of the tank with the fluid mass considered to be distributed. Other studies (e.g., refs. 5 and 6) have determined the modal characteristics of the propellant tank.

In this investigation, the propellant tank was isolated as a single component in the longitudinal system, and transfer functions for the tank (i.e., response of pressure to driving point velocity) were derived in order to be useful either in a computer simulation of the entire system or in a direct calculation.

The basic model considers the longitudinal wave propagation effects of a compressible liquid in a thin-wall, constant-diameter tank. For this model, the transfer function for the pressure at each longitudinal position to the velocity at the tank base is derived. The model used is one-dimensional and includes the effect of damping. In general, the problem is analogous to a modified form of the shorted load transmission line.

SYMBOLS

A	area, sq ft
a	dummy variable defined by eq. (41)
B	fluid compressibility, sq ft/lb
b	dummy variable defined by eq. (42)
c	reciprocal of effective wave velocity, sec/ft
D	amplitude of sinusoidal input velocity
d	index for inverse transform, $\left[n - (1/2)\right]\pi/\alpha$
E	modulus of elasticity, lb/sq ft
f	frequency, cps
G	dummy variable
g	gravitational constant, ft/sec^2
H	dummy variable
K	proportionality constant per unit length for fluid damping, lb-sec/ft^2
K^*	$(K/A\ dx)\left[B + (2R/E\tau)\right]$, sec/sq ft
k	constant pressure above liquid, lb/sq ft
$\mathscr{L}$	Laplace transform operator
ℓ	height of fluid in tank, ft

M	constant, lb/cu ft
m, n	index numbers
P	pressure, lb/sq ft
R	radius, ft
s	Laplace variable
t	time, sec
u	dummy variable for hyperbolic indentities
V	velocity, ft/sec
$\mathcal{V}$	volume, cu ft
v	dummy variable for hyperbolic identities
W	weight, lb
x	distance coordinate, ft
y	dummy variable for inverse transform computation
α	constant
ϵ	strain
$\theta(x,t)$	$P(x,t) - P(x,0)$, lb/sq ft
λ	phase angle, deg
ρ	fluid weight density, lb/cu ft
σ	stress, lb/sq ft
τ	wall thickness, ft
$\varphi(x,s)$	$\mathscr{L}\left[\theta(x,t)\right]$
ω	angular frequency, rad/sec

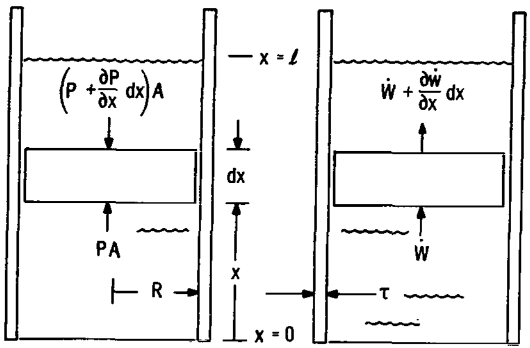
Subscripts:

FC	fluid compression
N	nodal
OV	original volume
R	resonant
S	stored
U	undamped

ΔV wall volume change

Superscript:

$\dot{}$ differentiations with respect to time



(a) Forces. (b) Flows.

Figure 1. - Coordinate system.

DERIVATION OF BASIC EQUATIONS

If the one-dimensional longitudinal motion of liquid in a cylindrical tank is considered, the coordinate system shown in figure 1 can be established.

Considering the conservation of mass for a cylindrical element with area A and height dx gives

$$\frac{\partial W_S}{\partial t} = \dot{W} - \left(\dot{W} + \frac{\partial \dot{W}}{\partial x} dx\right) = - \frac{\partial \dot{W}}{\partial x} dx \tag{1}$$

The weight stored W_S in the elementary volume at time t is the sum of the weight in the original volume, the weight stored in the change of volume of the walls, and the weight stored by fluid compression; hence,

$$W_S = W_{S,OV} + W_{S,\Delta V} + W_{S,FC} \tag{2}$$

where

$$W_{S,OV} = \rho A\, dx \tag{3}$$

and

$$W_{S,FC} = \rho ABP\, dx \tag{4}$$

The weight stored due to a small volume change is

$$W_{S,\Delta V} = \rho\, \Delta V \tag{5}$$

For a circular element of radius R,

$$\Delta V = 2\pi R\, dx\, \Delta R$$

If stresses remain within the elastic limit, the tank walls will obey Hooke's law, or

$$\epsilon = \frac{\sigma}{E} = \frac{\Delta R}{R}$$

The hoop stress in the wall is $\sigma = PR/\tau$; then, if the induced strain due to longitudinal stress (through the Poisson's ratio effect) is neglected, the radius change is given by

$$\Delta R = \frac{PR^2}{E\tau}$$

Hence, the stored weight due to volume change is

$$W_{S,\Delta V} = \left(\frac{2\pi\rho R^3}{E\tau}\right) P \, dx \tag{6}$$

Therefore,

$$W_S = \left(\frac{2RP}{E\tau} + BP + 1\right)\rho A \, dx \tag{7}$$

For constant-thickness walls and a constant-radius cylinder, this result can be substituted into equation (1) to give

$$\rho A\left(B + \frac{2R}{E\tau}\right)\frac{\partial P}{\partial t} = -\frac{\partial \dot{W}}{\partial x} \tag{8}$$

If, instead, velocities are considered,

$$\left(B + \frac{2R}{E\tau}\right)\frac{\partial P}{\partial t} = -\frac{\partial V}{\partial x} \tag{9}$$

In order to derive the equation of motion, Newton's law is applied to an element acted upon by forces PA and $-\left[P + (\partial P/\partial x)dx\right]A$ and a damping force. The tank base velocity $V(0, t)$ is used as an approximation to the wall velocity (assumes longitudinal tank rigidity), and damping is assumed to be a linear function of the relative velocity. Further, since the tank wall is "thin" ($\tau/R \ll 1$), the associated mass can be neglected yielding the following equation of motion

$$PA - \left(P + \frac{\partial P}{\partial x} dx\right) A - W - K\left[V(x, t) - V(0, t)\right] dx = \frac{W}{g} \frac{\partial V}{\partial t} \tag{10}$$

or

$$-\frac{\partial P}{\partial x} - \rho - \left(\frac{K}{A}\right) \left[V(x, t) - V(0, t)\right] = \frac{\rho}{g} \frac{\partial V}{\partial t} \tag{11}$$

Differentiating equation (9) with respect to time and equation (11) with respect to x and eliminating the cross derivative terms gives

$$\frac{\partial^2 P}{\partial x^2} = c^2 \frac{\partial^2 P}{\partial t^2} + K^* \frac{\partial P}{\partial t} \tag{12}$$

where

$$\left.\begin{aligned} c^2 &= \frac{\rho}{g}\left(B + \frac{2R}{E\tau}\right) \\ &\text{and} \\ K^* &= \frac{K}{A}\left(B + \frac{2R}{E\tau}\right) \end{aligned}\right\} \tag{13}$$

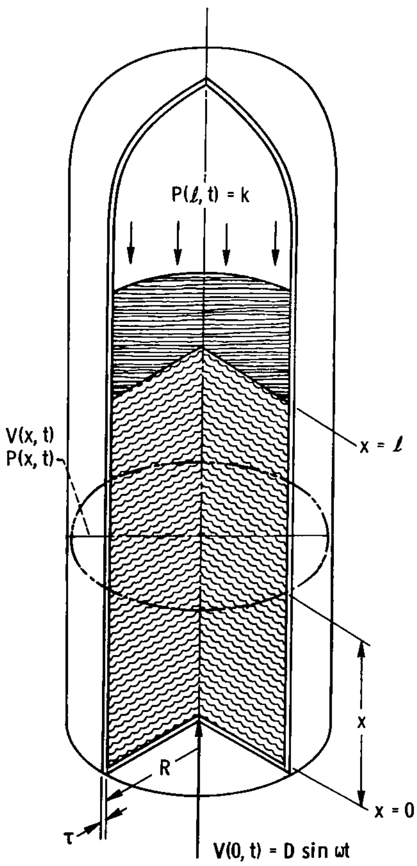


Figure 2. - Typical tank under consideration.

When $K^* = 0$, equation (12) is the undamped wave equation, and c is the reciprocal of the wave velocity. The term $K^* \partial P/\partial t$ accounts for fluid dissipation losses.

BOUNDARY CONDITIONS

The system of interest and the boundary conditions are illustrated in figure 2. The ullage gas pressure above the liquid is assumed to remain constant. (The effect of acoustic variations in the ullage gas is given in ref. 7.) Thus,

$$P(\ell, t) = k \tag{14}$$

The response of the tank due to the sinusoidal motion of a rigid tank bottom is desired; therefore,

$$V(0,t) = D \sin \omega t \tag{15}$$

Equation (15) is transformed into a pressure relation by considering equation (11) at $x = 0$:

$$\frac{\partial P(0,t)}{\partial x} + \rho = -\frac{\rho}{g} \frac{\partial V(0,t)}{\partial t} \tag{16}$$

or

$$\frac{\partial P(0,t)}{\partial x} = -\rho - \frac{\rho D \omega}{g} \cos \omega t \tag{17}$$

If a linear initial pressure distribution is assumed,

$$P(x,0) = k - M(x - \ell) \tag{18}$$

With the fluid at rest initially,

$$\frac{\partial P(x,0)}{\partial t} = 0 \tag{19}$$

Under normal conditions, $M = \rho$; however, in certain special cases (i. e., bladder pressurization systems for tanks under zero-gravity conditions ($m = 0$)), the added generality is desirable. The transformation

$$\theta(x,t) = P(x,t) - P(x,0) \tag{20}$$

is introduced into equations (12), (14), (17), (18), and (19), respectively, to simplify the initial conditions; then,

$$\frac{\partial^2 \theta}{\partial x^2} = c^2 \frac{\partial^2 \theta}{\partial t^2} + K^* \frac{\partial \theta}{\partial t} \tag{21}$$

$$\theta(\ell, t) = 0 \tag{22}$$

$$\frac{\partial \theta(0,t)}{\partial x} = M - \rho - \frac{\rho D \omega}{g} \cos \omega t \tag{23}$$

$$\theta(x, 0) = 0 \tag{24}$$

$$\frac{\partial\theta(x, 0)}{\partial t} = 0 \tag{25}$$

TRANSFER FUNCTION SOLUTION

The Laplace transformation is applied to the preceding equations and boundary conditions. The transform of $\theta(x, t)$ is defined by

$$\varphi(x, s) = \mathscr{L}\left[\theta(x, t)\right] = \int_0^\infty \theta(x, t)e^{-st}\, dt \tag{26}$$

where s is the Laplace operator. Since the initial conditions are zero (eqs. (24) and (25)), the transform of equation (21) is

$$\frac{\partial^2\varphi(x, s)}{\partial x^2} - (c^2s^2 + K^*s)\varphi(x, s) = 0 \tag{27}$$

Equation (22) becomes

$$\varphi(\ell, s) = 0 \tag{28}$$

and equation (23) becomes

$$\frac{\partial\varphi(0, s)}{\partial x} = \frac{M - \rho}{s} - \frac{\rho D\omega}{g} \frac{s}{s^2 + \omega^2} \tag{29}$$

Since s is a parameter, equations (27) and (29) can be considered as ordinary differential equations.

The solution of the following equations and conditions is thus required:

$$\frac{d^2\varphi(x, s)}{dx^2} - (c^2s^2 + K^*s)\varphi(x, s) = 0 \tag{30}$$

$$\varphi(\ell, s) = 0 \tag{28}$$

$$\frac{d\varphi(0, s)}{dx} = \frac{M - \rho}{s} - \frac{\rho D\omega}{g} \frac{s}{s^2 + \omega^2} \tag{31}$$

The solution of the ordinary differential equation (30) can be written in the form

$$\varphi(x, s) = G \sinh\left(\sqrt{c^2 s^2 + K^* s}\; x\right) + H \cosh\left(\sqrt{c^2 s^2 + K^* s}\; x\right) \tag{32}$$

Then, applying condition (28) yields

$$G \sinh\left(\sqrt{c^2 s^2 + K^* s}\; \ell\right) + H \cosh\left(\sqrt{c^2 s^2 + K^* s}\; \ell\right) = 0 \tag{33}$$

and applying equation (31) gives

$$\frac{d\varphi(0, s)}{dx} = G\sqrt{c^2 s^2 + K^* s} = \frac{M - \rho}{s} - \frac{\rho D\omega}{g} \frac{s}{s^2 + \omega^2} \tag{34}$$

from which

$$G = \frac{M - \rho}{\sqrt{c^2 s^2 + K^* s}\; s} - \frac{\rho D\omega}{g} \frac{s}{\sqrt{c^2 s^2 + K^* s}\,(s^2 + \omega^2)} \tag{35}$$

Equations (33) and (35) combine to give

$$H = -G \tanh\left(\sqrt{c^2 s^2 + K^* s}\; \ell\right) \tag{36}$$

The solution of equation (30) is thus equation (32), where G and H are given by equations (35) and (36), respectively. After some manipulation, equation (32) can be written as follows:

$$\varphi(x, s) = -\left[\frac{M - \rho}{s\sqrt{c^2 s^2 + K^* s}} - \frac{\rho D\omega}{g} \frac{s}{\sqrt{c^2 s^2 + K^* s}\,(s^2 + \omega^2)}\right]\left\{\frac{\sinh\left[\sqrt{c^2 s^2 + K^* s}\,(\ell - x)\right]}{\cosh\left(\sqrt{c^2 s^2 + K^* s}\; \ell\right)}\right\} \tag{37}$$

In order to evaluate the transfer function, that is, the ratio of the Laplace transform of the system response to the Laplace transform of the driving function with quiescent initial conditions, M is set equal to ρ. Since $V(0, s) = D\omega/(s^2 + \omega^2)$,

$$\frac{\varphi(x, s)}{V(0, s)} = \frac{\rho}{g} \frac{s}{\sqrt{c^2 s^2 + K^* s}} \left\{ \frac{\sinh\left[\sqrt{c^2 s^2 + K^* s}\,(\ell - x)\right]}{\cosh\left(\sqrt{c^2 s^2 + K^* s}\,\ell\right)} \right\} \tag{38}$$

This function is the transfer function sought in terms of the Laplace variable s and the system parameters. The form is such that it can be substituted into many existing simulations that derive tank pressure to driving point velocity by using a single lump approximation.

In some situations, having the time response of this system is desirable. Hence, the time response is determined in the appendix for the case with zero damping and general linear initial conditions on pressure distribution. The pressure response in time due to a sinusoidal input in tank base velocity is given by

$$\theta(x, t) = \mathscr{L}^{-1}\left[\varphi(x, s)\right] = -(M - \rho)\left[(\ell - x) + \frac{2}{c^2 \ell} \sum_{n=1}^{\infty} \frac{(-1)^n}{\delta^2} \sin \delta c(\ell - x) \cos \delta t\right]$$

$$+ \left(\frac{\rho D \omega}{g}\right) \left\{(\ell - x) \cos \omega t - \frac{2}{c^2 \ell} \sum_{n=1}^{\infty} \frac{(-1)^n}{\delta^2} \frac{1}{\omega^2 - \delta^2} \sin\left[\delta c(\ell - x)\right]\left(\delta^2 \cos \delta t - \omega^2 \cos \omega t\right)\right\}$$

(A8)

where

$$\delta = \left(n - \frac{1}{2}\right) \frac{\pi}{c\ell} \tag{A9}$$

The amplitude ratio and phase relations corresponding to equation (38) are important for experimental testing and for various other analyses. Substituting $i\omega$ for s to obtain the magnitude and phase relations associated with steady-state vibration gives

$$\frac{\varphi(x, i\omega)}{V(0, i\omega)} = \frac{\rho}{g} \frac{i\omega}{\sqrt{-c^2 \omega^2 + K^* \omega i}} \left\{ \frac{\sinh\left[\sqrt{-c^2 \omega^2 + K^* \omega i}\,(\ell - x)\right]}{\cosh\left(\sqrt{-c^2 \omega^2 + K^* \omega i}\,\ell\right)} \right\} \tag{39}$$

The quantity $\sqrt{-c^2\omega^2 + K^*\omega i}$ can be considered a complex number; hence, if $\sqrt{-c^2\omega^2 + K^*\omega i}$ is replaced by $a + ib$, equation (38) can be written

$$\frac{\varphi(x, i\omega)}{V(0, i\omega)} = \frac{\rho}{g} \frac{i\omega}{a + ib} \left\{ \frac{\sinh\left[(a + ib)(\ell - x)\right]}{\cosh\left[(a + ib)\ell\right]} \right\} \tag{40}$$

where

$$a = \sqrt{\frac{1}{2}\left[\sqrt{c^4\omega^4 + (K^*\omega)^2} - c^2\omega^2\right]} \tag{41}$$

and

$$b = \sqrt{\frac{1}{2}\left[\sqrt{c^4\omega^4 + (K^*\omega)^2} + c^2\omega^2\right]} \tag{42}$$

Equation (40) and its parameters a and b are similar in form to the shorted load transmission line equations. They differ basically in the form of dissipation considered; that is, for transmission line equations, damping depends on absolute velocity (current) instead of the relative velocity considered here (i.e., eq. (10)).

The amplitude ratio can be obtained from equation (42) as

$$\left|\frac{\varphi(x, i\omega)}{V(0, i\omega)}\right| = \left|\frac{\rho\omega i}{g(a + ib)}\right| \frac{\left|\sinh\left[(a + ib)(\ell - x)\right]\right|}{\left|\cosh\left[(a + ib)\ell\right]\right|} \tag{43}$$

The following relations are substituted into equation (43):

$$\left|\frac{\rho\omega i}{g(a + ib)}\right| = \frac{\rho\omega}{g} \frac{1}{\sqrt{a^2 + b^2}} \tag{44}$$

$$\left|\sinh(u + iv)\right| = \sqrt{\sinh^2 u + \sin^2 v} \tag{45}$$

$$\left|\cosh(u + iv)\right| = \sqrt{\sinh^2 u + \cos^2 v} \tag{46}$$

$$a^2 + b^2 = \sqrt{c^4\omega^4 + (K^*\omega)^2} \tag{47}$$

which yield as the expression for the amplitude ratio

$$\left|\frac{\varphi(x, i\omega)}{V(0, i\omega)}\right| = \frac{\rho\omega}{g\left[c^4\omega^4 + (K^*\omega)^2\right]^{1/4}} \sqrt{\frac{\sinh^2[a(\ell - x)] + \sin^2[b(\ell - x)]}{\sinh^2(a\ell) + \cos^2(b\ell)}} \tag{48}$$

Thus, since the amplitude ratio has been determined, the phase relation is determined from equation (40) and from the relations

$$\sinh(u + iv) = \sinh u \cos v + i \cosh u \sin v \tag{49}$$

$$\cosh(u + iv) = \cosh u \cos v + i \sinh u \sin v \tag{50}$$

Then

$$\frac{\varphi(x, i\omega)}{V(0, i\omega)} = \frac{\rho\omega i}{g(a + ib)}\left(\frac{\sinh[a(\ell - x)]\cos[b(\ell - x)] + i\{\cosh[a(\ell - x)]\sin[b(\ell - x)]\}}{\cosh(a\ell)\cos(b\ell) + i[\sinh(a\ell)\sin(b\ell)]}\right) \tag{51}$$

Since $\arg(z_1/z_2) = \arg z_1 - \arg z_2$, the general phase angle solution λ (in deg) is given by

$$\lambda = \frac{180}{\pi}\left(\tan^{-1}\{\coth[a(\ell - x)]\cot[b(\ell - x)]\} - \tan^{-1}[\tanh(a\ell)\tan(b\ell)] - \tan^{-1}\left(\frac{-a}{b}\right)\right) \tag{52}$$

For the special case of zero damping ($K^* = 0$), the amplitude ratio (eq. (48)) reduces to

$$\left|\frac{\varphi(x, i\omega)}{V(0, i\omega)}\right|_U = \frac{\rho}{gc}\left|\frac{\sin[\omega c(\ell - x)]}{\cos(\omega c\ell)}\right| \tag{53}$$

or

$$\left|\frac{\varphi(x, i\omega)}{V(0, i\omega)}\right|_U = \frac{\rho}{gc}\left|\tan(\omega c\ell)\cos(\omega cx) - \sin(\omega cx)\right| \tag{54}$$

The phase angle for the undamped case is determined from equation (40) for $K^* = 0$:

$$\frac{\varphi(x, i\omega)}{V(0, i\omega)} = \frac{\rho}{gc}\left\{\frac{\sin\left[\omega c(\ell - x)\right]}{\cos(\omega c\ell)}\right\}i \tag{55}$$

From equation (55) it is clear that

$$\lambda_U = \left\{\begin{matrix} +90^{\circ} & \text{for } \dfrac{\sin\left[\omega c(\ell - x)\right]}{\cos(\omega c\ell)} > 0 \\ \\ -90^{\circ} & \text{for } \dfrac{\sin\left[\omega c(\ell - x)\right]}{\cos(\omega c\ell)} < 0 \end{matrix}\right\} \tag{56}$$

The driving point impedance relations can be determined in a similar manner by letting $x = 0$ in equations (48) and (52). The resonant and nodal frequencies can be calculated for the undamped case by considering the expression for the amplitude ratio (eq. (54)).

Resonant Frequencies

For the undamped resonant frequencies, the amplitude ratio becomes infinite. If the bottom of the tank ($x = 0$) is considered, from equation (53)

$$\left|\tan(\omega c\ell)\right| = \infty$$

which requires

$$\omega_R = \frac{\left(n - \frac{1}{2}\right)\pi}{c\ell} \text{ (rad/sec)} \qquad n = 1, 2, 3, \cdots \tag{57}$$

and

$$f_R = \frac{\left(n - \frac{1}{2}\right)}{2c\ell} \text{ (cps)} \qquad n = 1, 2, 3, \cdots \tag{58}$$

This result can also be shown by a consideration of the time response (eq. (A8)). This expression for the resonant frequency has the same form as that for a closed organ pipe, as might be expected, since the undamped case was considered.

Equations (57) and (58) are not strictly applicable for all tank stations since certain x values produce the indeterminate form zero over zero, which when evaluated can become a finite amplitude ratio. This fact is, however, of no practical consequence.

Nodal Characteristics

The location of the nodes (the condition under which the amplitude ratio becomes zero) for the undamped case, is determined from equation (53), hence

$$\sin \omega c(\ell - x) = 0$$

which requires

$$x = \ell - \frac{n\pi}{\omega c} \qquad \text{for } 0 \leq x \leq \ell \qquad n = 0, 1, 2, \cdot \cdot \cdot$$

For a given x-position the frequency at which nodes will occur is given by

$$\omega_N = \frac{n\pi}{c(\ell - x)} \qquad \text{for } 0 \leq x \leq \ell \qquad n = 0, 1, 2, \cdot \cdot \cdot \tag{59}$$

At resonance, the nodes will be determined by the formula

$$\frac{x}{\ell} = \frac{n - \frac{1}{2} - m}{n - \frac{1}{2}} \qquad \text{for } 0 \leq x \leq \ell \qquad n = 0, 1, 2, \cdot \cdot \cdot \qquad m = 0, 1, 2, \cdot \cdot \cdot, n - 1 \tag{60}$$

NUMERICAL EXAMPLE

Because of the relative complexity of forms of the transfer function (amplitude ratio and phase), a special case was evaluated to indicate the form of the solution. For simplicity, the following tank and fluid characteristics were assumed: $\ell = 1.0$, $\rho/g = 1.0$, and $c = 1.0$. Under these conditions, the magnitude and the phase angle were evaluated from equations (48) and (52), respectively.

For the undamped case ($K^* = 0$), the effect of the x position in the tank is given in figure 3. The bottom of the tank ($x = 0$) shows nodes at $\omega = n\pi$ (for $n = 0, 1, 2, \cdot \cdot \cdot$), while for $x = 0.25$ and $x = 0.75$ the nodes appear at $4n\pi/3$ and $4n\pi$, respectively.

The amplitude ratio plot has a period of π radians per second for the $x = 0$ position and a period of 4π radians per second for $x = 0.25$ and $x = 0.75$. This periodicity of the magnitude with ω (or, in general with $\omega c\ell$) occurs when an integral number of half-cycles of the numerator and an integral number of half-cycles of the denominator (of eq. (53)) occur at a common value of $\omega c\ell$. For $x = 1/3$, for example,

$$\left|\frac{\varphi(x, i\omega)}{V(0, i\omega)}\right| = \frac{\rho}{gc}\left|\frac{\sin\left(\frac{2}{3}\,\omega c\ell\right)}{\cos(\omega c\ell)}\right| \tag{61}$$

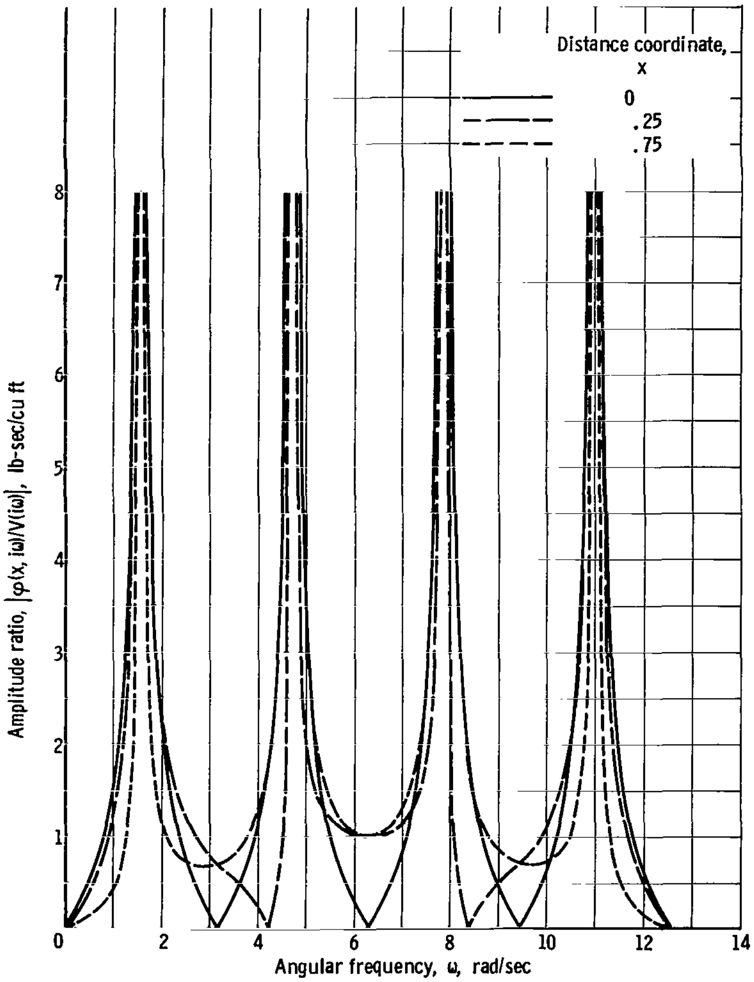


Figure 3. - Effect of position on response of pressure to velocity for undamped case (magnitude).

When $\omega c \ell = 3\pi$, two half-cycles of the numerator and three half-cycles of the denominator will have occurred and this sequence will be repeated for each successive 3π interval of $\omega c \ell$. In general, the amplitude ratio for zero damping is periodic with $\omega c \ell$ and has a period of $m\pi$, where m is the denominator of the least common fraction $(\ell - x)/\ell$ for a given x position.

The resonant frequencies for all the locations shown occur at $[n - (1/2)]\pi$ (for $n = 1, 2, 3, \cdot \cdot \cdot$), as predicted by equation (57); however, this is not the case for all locations, as discussed in the section Resonant Frequencies. Since the undamped case was considered, the phase angle is either 90^o or -90^o, as determined by equation (56), and is therefore not plotted.

The effect of damping was studied, and typical results are shown in figure 4. The position $x = 0$ (line inlet) was chosen because of its special importance in many cases. The amplitude ratio plot (fig. 4(a)) indicates that increasing damping K^* decreases the

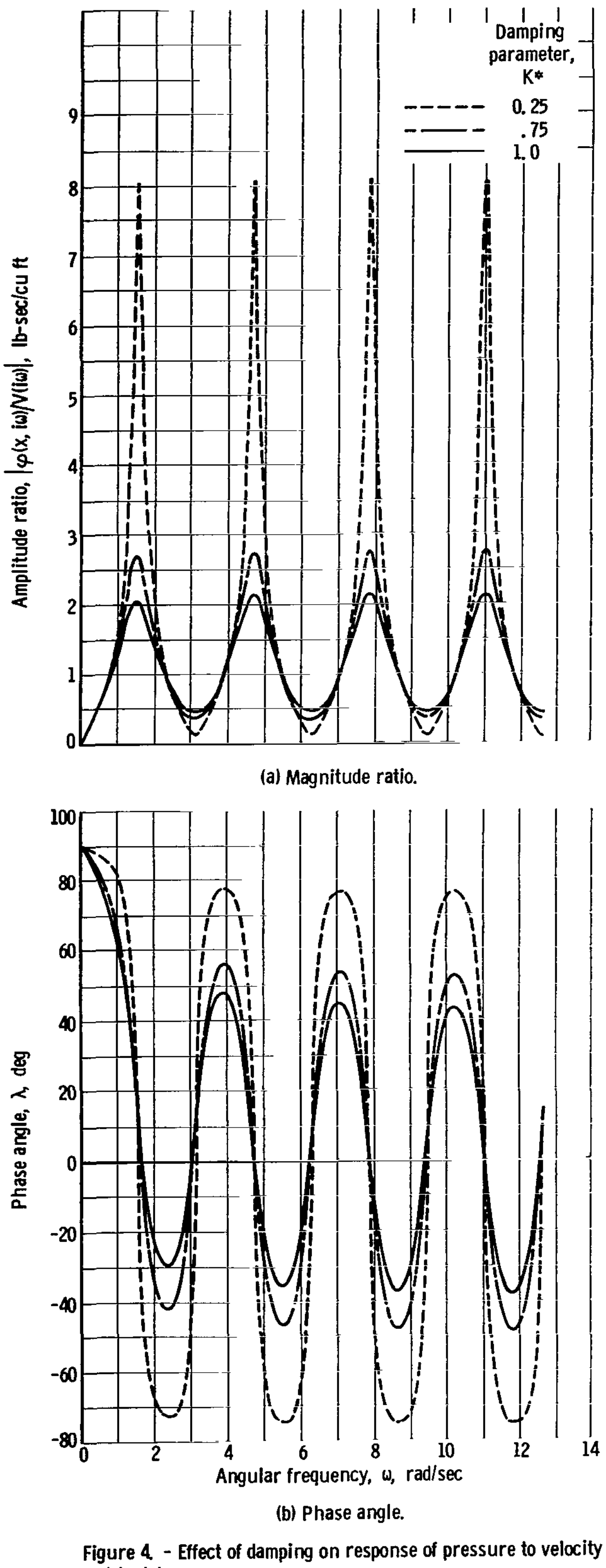


Figure 4. - Effect of damping on response of pressure to velocity at tank base.

height of the resonant peaks as would be expected. The shifting of the resonant frequency, however, appears to be negligible. The effect of damping on phase angle (fig. 4(b)) also shows negligible shifting of the zero crossing at the resonant frequencies for the cases considered. The phase angle at zero frequency approaches 90° for the $x = 0$ position instead of zero that might be expected, because of the form of the damping assumed in equation (10).

LIQUID-OXYGEN TANK ANALYSIS

In order to interpret the results of this study in a quantitative manner, the dynamics of a tank having missile dimensions is considered. The assumed tank has a 5-foot radius and a 0.0173-inch average wall thickness and is filled to a depth of 20 feet with liquid oxygen. For stainless-steel walls, these dimensions imply (eq. (13)) an effective acoustic velocity $1/c$ of 487 feet per second, or a value of c of 0.00205 second per foot. The amplitude of the resonant peaks is, of course, dependent on the damping magnitude. Since experimental values of K or K^* are unknown, a plot of the effect of K^* on peak amplitude is presented in figure 5. Over the range shown, peak amplitude is nearly linear with K^* on the log-log plot with a slope approximately equal to 1. For an assumed value of $K^* = 2\times10^{-5}$, the model indicates that the amplitude ratio $|\varphi/V|$ at the first resonant peak is 1.14×10^{4} at the tank base.

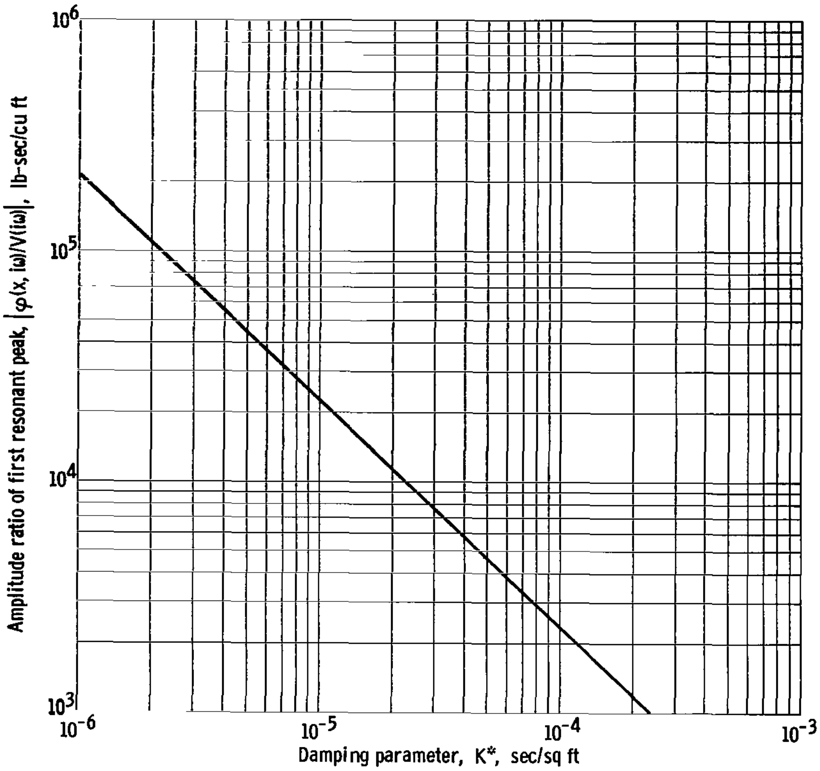


Figure 5. - Effect of damping on amplitude ratio of first resonant peak of liquid-oxygen tank. Effective wave velocity, 487 feet per second; fluid weight density, 71.14 pounds per cubic foot; height of fluid in tank, 20 feet.

Hence, for an input velocity amplitude of 0.2 feet per second, the corresponding pressure amplitude is 15.8 pounds per square inch. Pressure amplitudes of this magnitude have been observed in unpublished data.

SUMMARY OF RESULTS

The transfer function of a fluid contained in a longitudinally excited tank was developed, based on the following assumptions:

(a) Constant ullage pressure

(b) Constant-thickness, thin, cylindrical walls

(c) Negligible Poisson's ratio effect in the tank walls

(d) Initial pressure distribution linear with position

(e) One-dimensional motion (no geysering or three dimensional effects)

(f) Linear dissipative losses

The following determinations resulted from the analysis:

1. The transfer function (pressure to tank base velocity) was derived in terms of the Laplace variable and in terms of the amplitude ratio and the phase angle. The amplitude ratio and phase angle for undamped motion were also determined. The special case of driving point impedance is readily determined in the preceding cases by allowing $x = 0$, where x is the distance from the tank base.

2. The time response in pressure to a sinusoidal input velocity at the tank base was determined for the undamped case.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 9, 1966.

APPENDIX - TIME RESPONSE

The time response of the system under study is of some interest in understanding the physics of the problem. The undamped response of the system is considered. In this case, from equation (37), the inverse Laplace transform of the following equation is required:

$$\varphi(x, s) = -\left(\frac{M - \rho}{cs^2} - \frac{\rho D\omega}{gc} \frac{1}{s^2 + \omega^2}\right)\left\{\frac{\sinh[cs(\ell - x)]}{\cosh(cs\ell)}\right\} \tag{A1}$$

By reference 8,

$$\mathscr{L}^{-1}\left(\frac{\sinh ys}{s^2 \cosh \alpha s}\right) = y + \frac{2\alpha}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n - \frac{1}{2}\right)^2} \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi y}{\alpha}\right] \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi t}{\alpha}\right] \quad \text{for } 0 \leq y \leq \alpha \tag{A2}$$

also,

$$\mathscr{L}^{-1}\left(\frac{1}{s^2 + \omega^2}\right) = \frac{1}{\omega} \sin \omega t \tag{A3}$$

The convolution theorem (ref. 9) is applied to these functions to obtain

$$\mathscr{L}^{-1}\left(\frac{1}{s^2 + \omega^2} \frac{\sinh ys}{s^2 \cosh \alpha s}\right) = \int_0^t \frac{1}{\omega} \sin \omega(t - T)$$

$$\times\left\{y + \frac{2\alpha}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n - \frac{1}{2}\right)^2} \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi y}{\alpha}\right] \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi T}{\alpha}\right]\right\} d\tau \tag{A4}$$

Performing these integrations and manipulating trigonometric forms produces

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+\omega^2}\cdot\frac{\sinh ys}{s^2\cosh\alpha s}\right)=\frac{y}{\omega^2}(1-\cos\omega t)$$

$$+\frac{2}{\alpha\omega}\sum_{n=1}^{\infty}\frac{(-1)^n}{d^2}\left(\frac{\omega}{\omega^2-d^2}\right)\sin yd(\cos td+\cos\omega t) \qquad \text{(A5)}$$

where

$$d=\left(n-\frac{1}{2}\right)\frac{\pi}{\alpha} \qquad \text{(A6)}$$

Suppressing the s^2 term in the denominator by differentiating the time result twice with respect to t (ref. 10) determines the inverse transform:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+\omega^2}\,\frac{\sinh ys}{\cosh\alpha s}\right)=y\cos\omega t$$

$$-\frac{2}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{d^2}\left(\frac{1}{\omega^2-d^2}\right)\sin yd\left(d^2\cos td-\omega^2\cos\omega t\right) \qquad \text{(A7)}$$

By application of the inverse transforms (eqs. (A2) and (A7)) to equation (A1), the time solution is determined as

$$\theta(x,t)=\mathscr{L}^{-1}\left[\varphi(x,s)\right]=-(M-\rho)\left[(\ell-x)+\frac{2}{c^2\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\delta^2}\sin\delta c(\ell-x)\cos\delta t\right]$$

$$+\left(\frac{\rho D\omega}{g}\right)\left\{(\ell-x)\cos\omega t-\frac{2}{c^2\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\delta^2}\,\frac{1}{\omega^2-\delta^2}\sin\left[\delta c(\ell-x)\right]\left(\delta^2\cos\delta t-\omega^2\cos\omega t\right)\right\}$$

(A8)

where

$$\delta = \left(n - \frac{1}{2}\right)\frac{\pi}{c\ell} \tag{A9}$$

Equation (A8) can be verified as the solution of the undamped boundary-value problem by direct differentiation of equation (A8) and recognition of the validity of the relation

$$\ell - x = \frac{-2}{c^2\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\delta^2}\sin\left[\delta c(\ell - x)\right] \qquad 0 \leq x \leq \ell \tag{A10}$$

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