AFCRL-66-352

A TECHNIQUE FOR SOLVING CERTAIN WIENER-HOPF TYPE BOUNDARY VALUE PROBLEMS

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C. P. BATES

R. MITTRA

May 1966

Contract No. AF19(628)-3819
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Task No. 563502
Technical Report No. 9

Prepared For
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

ANTENNA LABORATORY
DEPARTMENT OF ELECTRICAL ENGINEERING
ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

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AFCRL-66-352

Antenna Laboratory Report No. 66-4

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* This work was supported in part by Air Force Cambridge Research Center under Contract AF19(628)-3819 and in part by the National Aeronautics and Space Administration under Grant NSG-395.

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ABSTRACT

The time-harmonic analysis of three boundary value problems con-
aining semi-infinite boundaries is presented. The first problem considered
s a parallel plate waveguide with one plate truncated and radiating into free
pace. The excitation of a dielectric slab and the excitation of an isotropic,
ncompressible, plasma slab by means of a parallel plate waveguide with
ne plate truncated are the second and third problems analyzed, respectively.
Both TE and TEM polarizations are considered in these open-region problems.

A function of a complex variable is factored in each of these Wiener-
Hopf type boundary value problems. The function is analytic in a strip and is
actored into a product of two functions. One of these functions is analytic in
a half-plane while the other is analytic in the adjacent half-plane with an over-
ap in the regions of analyticity coinciding with the strip. This factorization is
obtained by a technique developed in this work.

The technique obtains the factorization for the open-region problem
from a function and its factorization that occurs in a related closed-region
problem. A closed-region problem is one whose transverse dimensions are
finite. The chosen closed-region boundary value problem yields a function of
a complex variable which can be factored. The factorization of the function
for the open-region boundary value problem is obtained by taking the limit, as
a parameter approaches infinity, of the function and factorization appropriate
to the closed-region structure. By this means the factorization and hence the
solution to the open-region boundary value problem is obtained.

It is also found that the limiting procedure may be used to obtain more han just the open-region factorization. It is shown that the limit of the comolete closed-region solution becomes the open-region solution. Hence, this rields one possible method for the solution of problems of this type.

The results of the numerical computations are presented. These in-:lude the average power reflected in the waveguide, the average power radiated n the space wave, the average power transmitted by the surface waves, and he radiation pattern of the space wave.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Prof. Raj ttra, his advisor, for the advice and guidance so generously given during course of this research work. Thanks are also expressed to Prof. G. A. schamps for his interest and advice. Appreciation for the many helpful scussions is extended to Dr. S. W. Lee and Mr. A. Neureuther.

Financial support was provided by the Air Force Cambridge Research boratories, Office of Aerospace Research, under contract AF 19 (628) - 19 and supported in part by the National Aeronautics and Space Adminis- ation under Grant NSG 395. The IBM 7094 computation facilities were op- ated by the Department of Computer Science, University of Illinois, and re partially supported by a National Science Foundation Grant, NSF GP700.

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1. INTRODUCTION

The time-harmonic analysis of certain radiation and diffraction problems with semi-infinite boundaries requires solutions of the steady-state wave equation satisfying various boundary conditions. This class of problems is conventionally formulated in terms of the Wiener-Hopf technique, that is, at some point in the analysis a complex variable equation is solved by analytic continuation. An exhaustive discussion and numerous illustrations of this technique may be found in Noble [1958].

Difficulty with the Wiener-Hopf technique is encountered because a factorization of a function of a complex variable must be made. This function of a complex variable, which is analytic in a strip, must be factored into a product of two functions. One function of the product is analytic in a half-plane while the other is analytic in the adjacent half-plane, with an overlap in the regions of analyticity coinciding with the strip.

In this work a C-R boundary value problem will refer to a closed-region boundary value problem (one whose transverse dimensions are finite; see Fig. 2, for example). An O-R boundary value problem means an open region boundary value problem (one in which radiation may occur; see Fig. 1, for example).

The factorization in the case of a C-R boundary value problem may be obtained by using the infinite product expansion of an integral (entire) function; see for example Titchmarsh [1932]. This results from the fact that the function is a ratio of two integral functions. The function to be factored in the case of an

-R boundary value problem contains branch line singularities and recourse to a ɪrmal factorization procedure, for example Noble [1958], may be made. How- ver, as usually occurs with such procedures, specific results are difficult to btain except in a few simple cases.

The possibility of attacking an O-R boundary value problem through a elated C-R boundary value problem, which by its nature is easier to analyze, as been suggested and attempted to a limited extent by various authors. Noble 1958] expressed interest in knowing how far the results for a parallel plate uct (semi-infinite parallel plate waveguide) enclosed in a larger parallel plate 'aveguide, with a finite (but large) spacing between the plates, could be used to pproximate the results near the mouth of the parallel plate duct when radiating a unbounded space. Talanov [1959], desiring the analysis of surfave wave aunching in a dielectric slab backed by a perfect conductor by means of a semi- afinite parallel plate waveguide, enclosed the O-R structure in a larger parallel late waveguide. This procedure reduced the O-R structure to a C-R structure. Ie then analyzed this C-R structure and calculated the desired field quantities or increasing values of the spacing between the plates of the parallel plate wave- uide. He suggested that the results obtained for the C-R structure are in the imit, as the spacing between the plates of the parallel plate waveguide becomes arge, the results of the O-R problem. Mittra and Karjala [1964] showed that he expression for the reflection coefficient of a parallel plate duct enclosed in a arger parallel plate waveguide yields, in the limit of the waveguide walls approaching infinity, the expression for the reflection coefficient in the duct when radiating into free space. Mittra and VanBlaricum [1965] numerically calculated the reflection coefficient in the duct enclosed in the larger parallel plate waveguide and showed that the numerical values approached, as the spacing of the plates of the waveguide became large, the known numerical value of the reflection coefficient for the duct radiating into free space. Mittra and Bates [1965] used a limiting procedure to obtain an extension of the function-theoretic technique introduced by Whitehead [1951]. The limit, as a dimension became infinitely large, of a certain function that occurs in a related C-R problem gave the desired unknown function necessary in the O-R problem. The mode matching technique was used in that analysis.

The extension of a C-R boundary value problem solution to yield the solution of an O-R boundary value problem is expected if one takes into account the physical phenomenon occurring. For example, consider a source in a parallel plate waveguide where the medium has a slight loss. At any location A within the waveguide the field is made up of two components: a direct wave from the source and reflected waves from the boundary. The magnitude of the reflected waves at A, as the spacing of the waveguide walls approaches infinity, would approach zero due to the loss in the medium. Therefore, point A would see only the incident field in the limit. That is, we are left with a source radiating in an unbounded region. The idea of a slight loss in the medium is not restrictive. When the analysis is completed the loss is permitted to be as small desired, in fact, zero. The inclusion of the loss is usually used regardless :he method of solution of these problems.

This work determines a method by which the O-R factorization may be ained from a related C-R problem. The method involves a limit, as a pa- neter approaches infinity, as suggested by the physics of these problems. ; chosen C-R boundary value problem yields a function of a complex variable .ch can be factored. The factorization of the function for the O-R boundary ue problem is obtained by taking the limit, as the transverse dimension ap- aches infinity, of the function and factorization appropriate to the chosen C-R ucture. By this means the factorization and hence the solution to the O-R ndary value problem is obtained.

It is found that the limiting procedure may be used to obtain more than t the O-R factorization. It is shown that the limit of the complete C-R solu- ı becomes the O-R solution. Hence, this yields one possible method for the ution of problems of this type. This is a rather useful method as the C-R so- .on is usually readily obtained.

Obviously there is more than one possibility for the choice of a C-R ucture. However, the results obtained for the O-R structure are unique since O-R solution is a limit point of the C-R solutions. This result is expected m the physics of the problem which implies that the field reflected from a ndary that is receding to infinity in a lossy medium will be zero in the vicin- of the source. Hence, the boundary condition satisfied by the boundary that

hat recedes to infinity in the C-R structure is immaterial.

The first problem discussed is the analysis of the fields associated with a parallel plate waveguide having the top plate terminated (semi-infinite). The solution is obtained in closed form and thus the method is clearly demonstrated. The factorization is verified by reference to the solution for the fields of a parallel plate duct obtained by Noble [1958], since the function to be factored is the same in each problem. The launching of surface waves on a dielectric slab with a relative dielectric constant greater than one by means of a semi-infinite parallel plate waveguide is analyzed. Solutions for both TE and TEM excitations are obtained. Numerical results for the power reflected in the waveguide, power trapped in the surface waves, power radiated by the space wave, and also the radiation pattern of the space wave are obtained for various parameters. The power results for the TEM excitation are compared with those obtained by Angulo and Chang [1959] who worked with the formal factorization procedure and the differences in the results are noted. The final problem analyzed is the case where the dielectric slab is replaced by an incompressible, isotropic, plasma slab. This gives the possibility of a relative dielectric constant less than one. Again, numerical results for the power reflected in the waveguide, power radiated in the space wave, and the radiation pattern of the space wave are presented for various parameters. No trapped waves can occur in this case.

The boundary value problems investigated here are formulated by a method used by Jones [1950] as opposed to an integral equation approach. Fourier transforms are applied directly to the partial differential equation and the complex variable equation is obtained without the use of an integral equation. The integral equation approach would lead to equivalent results as the integral equation would be of the Wiener-Hopf type; see for example Morse and Feshbach [1953]. Jones' method also has the advantage that the application of the edge condition, Meixner [1954], which is necessary in this type of problem, may be clearly applied.

The usual method of calculating the far field is by means of saddle point integration. However, the problems considered here are such that the far field pattern may be obtained more directly by using an equivalent Huygen source in the aperture. The far field pattern is then related to the Fourier transform of this aperture distribution and in these problems becomes an evaluation of a function on an interval.

2. RADIATION FROM A TRUNCATED PARALLEL PLATE WAVEGUIDE

2.1 Formulation of the Problem

The first problem considered is a parallel plate waveguide with one plate truncated (semi-infinite) and radiating into free space. The structure is shown in Fig. 1. The incident field, in the parallel plate waveguide section of the structure, is taken to be the TEM mode with the magnetic field intensity parallel to the walls of the structure. The case of a TE incident field is not discussed in detail as the function to be factored turns out to be the same as in the TEM case. However, the TE case can be obtained from section 3.1 by setting the relative dielectric constant (κ) to one.

The incident field is therefore the lowest order TM mode.

$$H_y^i = e^{-ikz} \; ^{**} \quad ; \qquad 0 \leq x \leq b \tag{2.1}$$

We wish to find the electromagnetic fields which satisfy Maxwell's equation and the necessary boundary conditions pertaining to this structure with a source as given by (2.1).

Maxwell's equations for a medium with loss and the time convention chosen are

$$\nabla \times \bar{H} = (-i\omega\epsilon_0 + \sigma_1)\bar{E} \quad , \quad \nabla \times \bar{E} = i\omega\mu_0 \bar{H} \tag{2.2}$$

The loss is due to the conductivity (σ_1) of the medium. From (2.2) it can be

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** $e^{-i\omega t}$ time convention

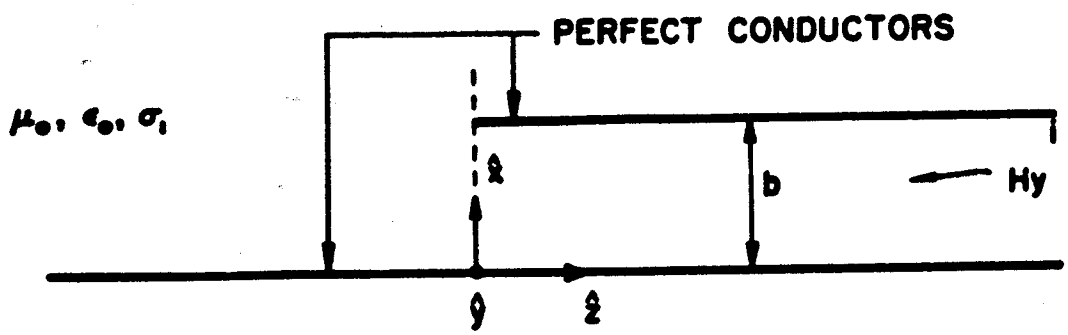


Fig. 1 Parallel plate waveguide with one plate terminated.

shown that the magnetic field intensity must satisfy

$$\nabla^2 \overline{H} + k^2 \overline{H} = 0 \tag{2.3}$$

where

$$k = (\omega^2 \mu_0 \epsilon_0 + i\omega\mu_0\sigma_1)^{1/2} = k_1 + i k_2 \tag{2.4}$$

with $k_1 > 0$ and $k_2 > 0$. Since the incident field is independent of the y-coordinate and the entire structure is uniform with respect to the y-direction, the total field will also be independent of y. Therefore the solution of (2.3) is equivalent to solving the two-dimensional wave equation for the scalar potential ϕ_t.

$$\frac{\partial^2 \phi_t}{\partial x^2} + \frac{\partial^2 \phi_t}{\partial z^2} + k^2 \phi_t = 0 \tag{2.5}$$

All the field quantities are derivable from ϕ_t by letting

$$H_y = \phi_t \tag{2.6}$$

$$E_x = \frac{1}{(i\omega\epsilon_0 - \sigma_1)} \frac{\partial \phi_t}{\partial z} \tag{2.7}$$

$$E_z = -\frac{1}{(i\omega\epsilon_0 - \sigma_1)} \frac{\partial \phi_t}{\partial x} \tag{2.8}$$

Let

$$\phi_t = \phi_i + \phi \tag{2.9}$$

where ϕ_i is the incident field and ϕ is the scattered field. Obviously

$$\phi_i = e^{-ikz} \; ; \quad 0 \le x \le b \; , \; \forall z \tag{2.10}$$

$$\phi_i = 0 \quad ; \quad b \le x \quad , \; \forall z \tag{2.10.a}$$

and ϕ must satisfy.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \tag{2.11}$$

The following boundary conditions on ϕ_t and hence on ϕ may be obtained by recalling (2.6) to (2.9) and the fact that the walls of the structure are perfect electric conductors:

a) $\frac{\partial \phi_t}{\partial x} = 0$ at $x = 0, \forall z \Rightarrow \frac{\partial \phi}{\partial x} = 0$ at $x = 0$, $\forall z$

b) $\frac{\partial \phi_t}{\partial x} = 0$ at $x = b$, $z > 0 \Rightarrow \frac{\partial \phi}{\partial x} = 0$ at $x = b$, $z > 0$

c) ϕ_t continuous at $x = b$, $z < 0$ $\Rightarrow$

$$\phi(b+0, z) - \phi(b-0, z) = e^{-ikz} \; , \quad z < 0$$

where

$$\phi(b+0, z) = \left. \phi(x, z) \right|_{x = b+0}$$

$x = b + 0 = b + \delta$ where δ is arbitrarily small

d) $\frac{\partial \phi_t}{\partial x}$ continuous at $x = b, \forall z \Rightarrow \frac{\partial \phi(b+0, z)}{\partial x} = \frac{\partial \phi(b-0, z)}{\partial x}$

where
$$\frac{\partial \phi(b+0, z)}{\partial x} = \left. \frac{\partial \phi(x, z)}{\partial x} \right|_{x = b+0}$$

e) ϕ_t and hence ϕ must be decaying waves since a loss has been included as $r = (x^2+z^2)^{1/2} \to \infty$ for $x \geq b$ and for $z < 0$, $0 \leq x \leq b$.

f) ϕ and $|\nabla\phi|$ satisfies the edge condition, Meixner [1954], as the edge of the plate is approached, that is, ϕ goes to zero as $z^{1/2}$ and $|\nabla\phi|$ as $z^{-1/2}$ for $x = b$, $z = 0 - \zeta$, $\zeta \to 0$.

Define

$$\alpha = \sigma + i\,\tau \tag{2.12}$$

$$\Phi(x,\alpha) = \Phi_+(x,\alpha) + \Phi_-(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x,z)\, e^{i\alpha z}\, dz \tag{2.13}$$

$$\Phi_+(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \phi(x,z)\, e^{i\alpha z}\, dz \tag{2.14}$$

$$\Phi_-(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \phi(x,z)\, e^{i\alpha z}\, dz \tag{2.15}$$

$$\Phi'(x,\alpha) = \Phi'_+(x,\alpha) + \Phi'_-(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial\phi(x,z)}{\partial x}\, e^{i\alpha z}\, dz \tag{2.16}$$

From the behavior of $\phi(x, z)$ for any given x as $z \to \infty$ it can be deduced that $\Phi_+(x,\alpha)$ is analytic for $\tau > -k_2$ and $\Phi_-(x,\alpha)$ is analytic for $\tau < k_2$. Hence $\Phi(x,\alpha)$ for any given x, is analytic in the strip $-k_2 < \tau < k_2$.

The boundary conditions (except e), in view of definitions (2.12) to (2.16), may now be written in terms of transforms as

a. 1) $\Phi'(0,\alpha) = 0$

b. 1) $\Phi'_+(b+0,\alpha) = \Phi'_+(b-0,\alpha) = 0$

c. 1) $\Phi_-(b+0,\alpha) - \Phi_-(b-0,\alpha) = -\dfrac{i}{\sqrt{2\pi}\,(\alpha-k)}$

d. 1) $\Phi'(b+0,\alpha) = \Phi'(b-0,\alpha)$

f. 1) $\Phi_+(b,\alpha) \sim \alpha^{-1}$ as $\alpha \to \infty$ for $\tau > -k_2$

and $\Phi'_-(b,\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau < k_2$

(Abelian theorem given in Noble [1958]).

Multiplying the wave equation (2. 11) by $(2\pi)^{-1} e^{i\alpha z}$ and integrating from $-\infty$ to ∞ with respect to z gives

$$\frac{d^2\Phi(x,\alpha)}{dx^2} - \gamma^2\,\Phi(x,\alpha) = 0 \tag{2. 17}$$

with

$$\gamma = (\alpha^2 - k^2)^{1/2} = -i\,(k^2-\alpha^2)^{1/2} \tag{2. 18}$$

The branch cuts used in (2. 18) and shown in Fig. 2 have been chosen so that γ has a positive real part when $-k_2 < \tau < k_2$. General solutions for $\Phi(x,\alpha)$ satisfying (2. 17) and in a form convenient for applying the boundary conditions are

$$\Phi(x,\alpha) = A(\alpha)\,\cosh(\gamma x) + C(\alpha)\,\sinh(\gamma x)\;; \quad 0 \le x \le b \tag{2. 19}$$

$$\Phi(x,\alpha) = B(\alpha)\,e^{-\gamma x} + D(\alpha)\,e^{\gamma x} \;; \quad b \le x \tag{2. 20}$$

The solution for $\phi(x, y)$ is obtained, once the unknown functions A, B, C, and D are found, by using the Fourier inversion integral

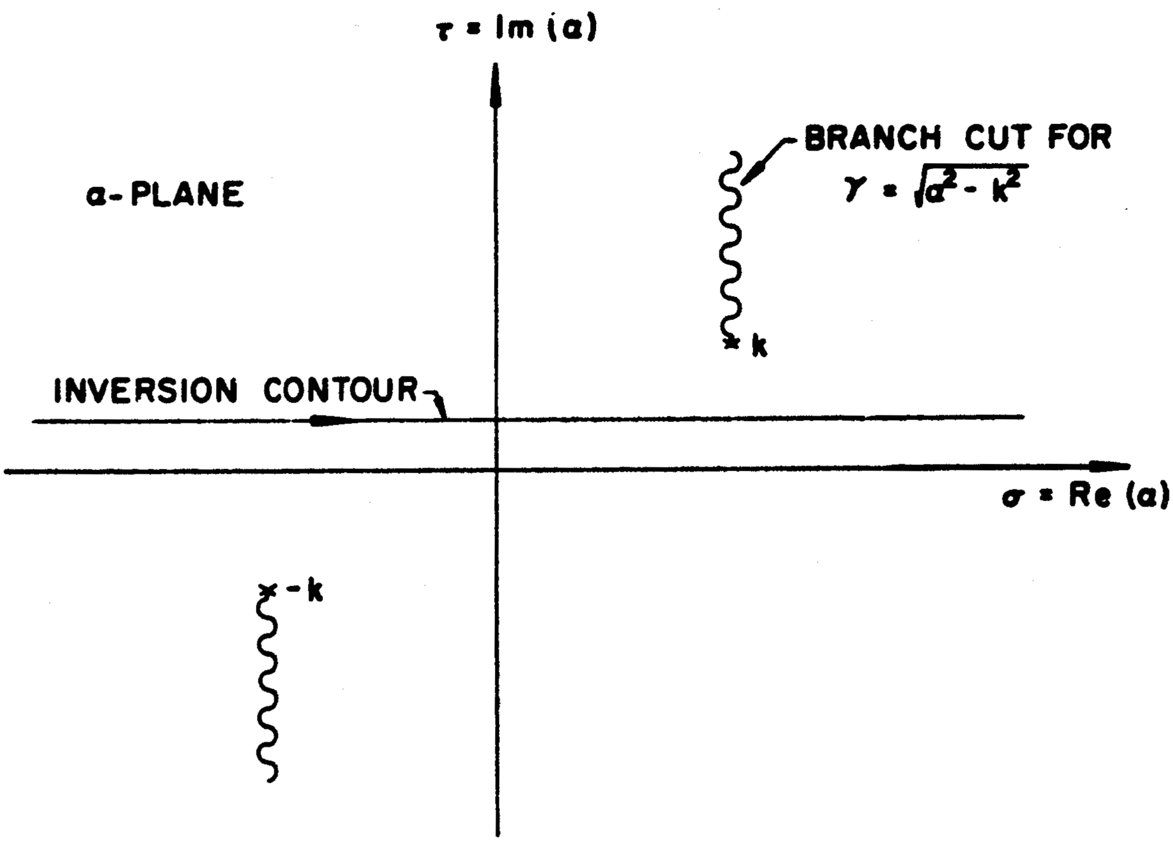


Fig. 2 Choice of branch cuts for $\gamma = (\alpha^2 - k^2)^{1/2}$ and the contour used in the Fourier inversion integral.

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \bar{\Phi}(x,\alpha)\, e^{-i\alpha z}\, d\alpha \tag{2.21}$$

fact that $\bar{\Phi}(x,\alpha)$ is analytic in the strip requires that the path of integra- be in the strip, that is, in (2.21) $-k_2 < \tau < k_2$.

The boundary condition (e) and the fact that γ has a positive real part $-k_2 < \tau < k_2$ requires $D(\alpha)$ to be set to zero. Likewise boundary condition) requires $C(\alpha)$ to be zero. Now at x = b we may write

$$\bar{\Phi}_-(b-o) + \bar{\Phi}_+(b-o) = A \cosh(\gamma b) \tag{2.22}$$

$$\bar{\Phi}_-(b+o) + \bar{\Phi}_+(b+o) = B e^{-\gamma b} \tag{2.23}$$

$$\bar{\Phi}_-'(b-o) + \bar{\Phi}_+'(b-o) = \gamma A \sinh(\gamma b) \tag{2.24}$$

$$\bar{\Phi}_-'(b+o) + \bar{\Phi}_+'(b+o) = -\gamma B e^{-\gamma b} \tag{2.25}$$

that the α in the argument of the functions in the above equations has not written for convenience and will not be if it does not lead to confusion. ever, $\bar{\Phi}$, A, B, and γ are still functions of α. Boundary conditions) and (d. 1) show that $\bar{\Phi}_-'(b+o) = \bar{\Phi}_-'(b-o)$ and is now defined as $\bar{\Phi}_-'(b)$.

t is,

$$\bar{\Phi}_-'(b+o) = \bar{\Phi}_-'(b-o) = \bar{\Phi}_-'(b) \tag{2.26}$$

and also

$$\Phi'_+(b+o) = \Phi'_+(b-o) = o \tag{2.26.a}$$

Using these facts in (2.24) and (2.25) yields

$$\Phi'_-(b) = \gamma A \ \sinh(\gamma b) = -\gamma B e^{-\gamma b} \tag{2.27}$$

The two unknown functions A and B are obtained by solving for the unknown $\Phi'_-(b)$.

Define

$$D_+ = \Phi_+(b+o) - \Phi_+(b-o) \tag{2.28}$$

which is obviously analytic for $\tau > -k_2$. Subtracting (2.22) and (2.23) and using (2.27), (2.28), and the boundary condition (c.1) yields

$$D_+ = \frac{i}{\sqrt{2\pi}\,(\alpha-k)} - \frac{\Phi'_-(b)}{b\,(\alpha+k)(\alpha-k)\,L(\alpha)} \tag{2.29}$$

where

$$L(\alpha) = \frac{e^{-\gamma b} \ \sinh(\gamma b)}{\gamma b} \tag{2.30}$$

The function L has branch points at k and -k and (refer to the definition of γ) L is analytic for $-k_2 < \tau < k_2$ Therefore, L may be factored into the product $L_+ L_-$ which holds in the strip. The function L_+ is analytic for $\tau > -k_2$ and L_- is analytic for $\tau < k_2$. This factorization is the d·ificult step in this class of problems and it is obtained by a limiting procedure as discussed in section 2.3.

Let

$$E(\alpha) = E_+(\alpha) + E_-(\alpha) = \frac{i(\alpha+k)\, L_+(\alpha)}{\sqrt{2\pi}\,(\alpha-k)} \tag{2.31}$$

th

$$E_-(\alpha) = \frac{i\, 2k\, L_+(k)}{\sqrt{2\pi}\,(\alpha-k)} \quad ; \text{ analytic for } \tau < k_2 \tag{2.32}$$

$$) = \frac{i}{\sqrt{2\pi}\,(\alpha-k)}\left[(\alpha+k)\,L_+(\alpha) - 2k\,L_+(k)\right]; \text{ analytic for } \tau > -k_2 \tag{2.33}$$

ultiplying (2.29) by ($L_+(\alpha)$). (α + k) and using (2.31), (2.32), and (2.33) ves

$$D_+\,(\alpha+k)\,L_+(\alpha) - E_+(\alpha) = E_-(\alpha) - \frac{\Phi'_-(b)}{b\,(\alpha-k)\,L_-(\alpha)} \tag{2.34}$$

hich holds in the strip $-k_2 < \tau < k_2$. The left side of (2.34) is analytic for $\tau > -k_2$ and the right side is analytic for $\tau < k_2$. Therefore, one side is ie analytic continuation of the other and they may both be equated to a function (α) which is analytic over the whole α - plane.

$$D_+\,(\alpha+k)\,L_+(\alpha) - E_+(\alpha) = E_-(\alpha) - \frac{\Phi'_-(b)}{b\,(\alpha-k)\,L_-(\alpha)} = J(\alpha) \tag{2.35}$$

he application of the edge condition gives the unknown function J (α). The fac- orization obtained in section 2.3 shows that $L_+(\alpha)$ behaves as $\alpha^{-1/2}$ for $\alpha \to \infty$, $\tau > -k_2$ and $L_-(\alpha)$ behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$, $\tau < k_2$. The edge ondition (f. 1) shows that D_+ behaves as α^{-1} as $\alpha \to \infty$, $\tau > -k_2$ and $\Phi'_-(b)$ ehaves as $\alpha^{-1/2}$ as $\alpha \to \infty$, $\tau < k_2$. The definitions of E_- and E_+ given in (2.32) and (2.33), respectively, show that E_- behaves as α^{-1} as $\alpha \to \infty$, $\tau < k_2$ and E_+ behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$, $\tau > -k_2$. Therefore $J(\alpha)$, which is analytic in the whole of the α-plane, behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$, $\tau < k_2$ and as α^{-1} as $\alpha \to \infty$, $\tau > -k_2$. Hence from the extended form of Louiville's theorem, for example Hille [1959], $J(\alpha)$ must be identically zero, that is

$$\Phi'_-(b) = b\, E_-(\alpha)\ (\alpha - k)\ L_-(\alpha) \tag{2.36}$$

Use of (2.32) and (2.27) gives

$$A_1(x,\alpha) = A(\alpha)\,\mathrm{Cosh}(\gamma x) = \frac{i\, b\, 2k\, L_+(k)\ L_-(\alpha)\ \mathrm{Cosh}(\gamma x)}{\sqrt{2\pi}\ \ \gamma\ \ \mathrm{Sinh}(\gamma b)} \tag{2.37}$$

$$B_1(x,\alpha) = B(\alpha)\, e^{-\gamma x} = -\frac{i\, b\, 2k\, L_+(k)\, L_-(\alpha)\, e^{\gamma b - \gamma x}}{\sqrt{2\pi}\ \ \gamma} \tag{2.38}$$

The formal solution is obtained by inverting (2.19) and (2.20) and may be written as

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} A_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\ ; \qquad 0 \le x \le b \tag{2.39}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} B_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\ ; \qquad b \le x \tag{2.40}$$

with $-k_2 < \tau < k_2$ in (2.39) and (2.40).

2.2 Choice of a Closed-Region Structure

The factorization of $L(\alpha)$, given by (2.30), is obtained in section 2.3 by taking the limit, as a parameter approaches infinity, of the function and factorization which occurs in a related C-R problem. The C-R structure should yield a function whose factorization is obtainable and the limit of the function, as the transverse dimension approaches infinity, should be the O-R function (2.30).

It should be pointed out that it is not necessary to resort to a C-R structure to accomplish the factorization. It is certainly possible to intuitively choose a function which may be factored and be such that some sort of mathematical limiting process on the function will yield the O-R function. The choice of a C-R structure seems to be the easier choice to make and has the added advantage that all mathematical results must be consistent with the physical phenomenon.

The chosen C-R structure for this problem is shown in Fig. 3. The choice of a related C-R structure is not unique. The one chosen here is considered to be the obvious one, that is, the simplest way to convert the O-R structure to a C-R structure. The O-R structure is obtained from the C-R structure by the limit of a, $c \rightarrow \infty$ while maintaining $a - c = b$.

The solution for the electromagnetic fields inside the C-R structure is formulated in the same manner as for the O-R structure in section 2.1. The only major change is in the boundary condition (e) which must now be

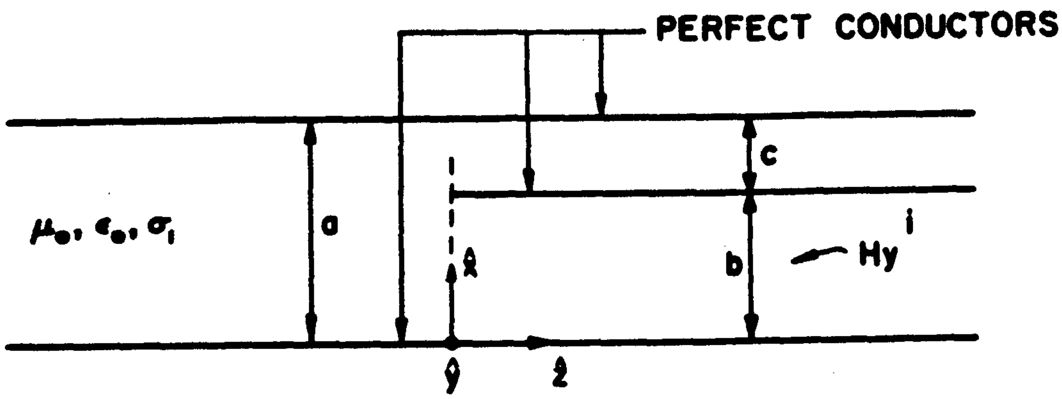


Fig. 3 Chosen C-R structure corresponding to Fig. 1.

e) $\frac{\partial \phi_t}{\partial x} = 0$ at $x = a, \forall z \Rightarrow \frac{\partial \phi}{\partial x} = 0$ at $x = a, \forall z$

or in terms of transforms

e. 1) $\Phi'(a, \alpha) = 0$

This change will be reflected in the general solution for $\Phi(x, \alpha)$ in $x \geq b$, that is, (2. 20) will now become

$$\Phi(x,\alpha) = B(\alpha) \operatorname{Cosh}[\gamma(a-x)] + D(\alpha) \operatorname{Sinh}[\gamma(a-x)]; \quad b \leq x \leq a \qquad (2.41)$$

Duplicating the arguments used in section 2. 1 will then give the following results:

$$C(\alpha) = D(\alpha) = 0 \qquad (2.42)$$

$$C_1(x,\alpha) = A(\alpha) \operatorname{Cosh}(\gamma x) = \frac{i2k}{\sqrt{2\pi}} \frac{K_+(k)}{\gamma} \frac{K_-(\alpha) \operatorname{Cosh}(\gamma x)}{\operatorname{Sinh}(\gamma b)} \qquad (2.43)$$

$$Q(x,\alpha) = B(\alpha) \operatorname{Cosh}[\gamma(a-x)] = -\frac{i2k}{\sqrt{2\pi}} \frac{K_+(k)}{\gamma} \frac{K_-(\alpha) \operatorname{Cosh}[\gamma(a-x)]}{\operatorname{Sinh}(\gamma c)} \qquad (2.44)$$

$$K(\alpha) = K_+(\alpha) K_-(\alpha) = \frac{\operatorname{Sinh}(\gamma b)}{\gamma} \frac{\operatorname{Sinh}(\gamma c)}{\operatorname{Sinh}(\gamma a)} \qquad (2.45)$$

where $K(\alpha)$ is a ratio of integral function. Note that $K(\alpha)$ is actually meromorphic with poles at

$$\alpha = \pm \left(k^2 - \left(\frac{n\pi}{a}\right)^2\right)^{1/2} = \pm i \left(\left(\frac{n\pi}{a}\right)^2 - k^2\right)^{1/2} \qquad (2.46)$$

and hence is analytic in the strip $-k_2 < \tau < k_2$. $K(\alpha)$ is factored into $K_+(\alpha)$, analytic for $\tau > -k_2$ and $K_-(\alpha)$, analytic for $\tau < k_2$ in section 2. 3. The asymptotic behavior for $K_+(\alpha)$ is chosen such that $K_+(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau > -k_2$ and $K_-(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau < k_2$. With these conditions the formal solution for the C-R structure is given by

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} C_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha \, ; \quad 0 \le x \le b \tag{2.47}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} D_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha \, ; \quad b \le x \le a \tag{2.48}$$

with $-k_2 < \tau < k_2$ in (2.46) and (2.47).

2.3 Factorization Obtained by a Limiting Procedure on the Function Appropriate to the Closed-Region Problem

A formal solution for the EM-fields of the structure shown in Fig. 1 is obtained once L_+ and L_- of (2.30) are found. This factorization will be obtained by talking the limit of $K(\alpha)$, given in (2.45), and its factorization. Recall that we are interested in a factorization using functions whose common region of analyticity is the strip $-k_2 < \tau\ (\mathrm{Im}\,\alpha) < k_2$ and whose product is $L(\alpha)$ for values of α within the strip. This strip of analyticity for $L(\alpha)$ is also the strip of analyticity for $K(\alpha)$. Also recall that Υ as defined in (2.18) has a positive real part for α within the strip. Hence, for any α of interest (within the strip), the limit, as suggested by the physics of the problem, gives

$$\lim_{\substack{a,c\to\infty\\ a-c=b}} K(\alpha) = \lim_{\substack{a,c\to\infty\\ a-c=b}} \left\{ \frac{\sinh(\gamma b)\ \sinh(\gamma c)}{\gamma\ \sinh(\gamma a)} \right\}$$

$$\lim_{\substack{a,c\to\infty\\ a-c=b}} K(\alpha) = \frac{\sinh(\gamma b)}{\gamma} e^{-\gamma b} = b\,L(\alpha) \qquad (2.49)$$

:e

$$\frac{\sinh(\gamma c)}{\sinh(\gamma a)} = \lim_{\substack{a,c\to\infty\\ a-c=b}} \left\{ \frac{e^{(\sigma_2+i\tau_2)c} - e^{-(\sigma_2+i\tau_2)c}}{e^{(\sigma_2+i\tau_2)a} - e^{-(\sigma_2+i\tau_2)a}} \right\} = e^{-\gamma b} \qquad (2.50)$$

ause $\gamma = \sigma_2 + i\tau_2$ with $\sigma_2 > 0$ for any α within the strip. Thus, the ction to be factored for the O-R problem is the limit, as given by (2.49), of function to be factored for the C-R problem. This fact enables us to obtain factorization of $L(\alpha)$ by this limiting procedure.

The factorization of $K(\alpha)$, given in (2.45), may be realized by using fact that $K(\alpha)$ is a ratio of integral functions. Hence, $K(\alpha)$ may be ex-ssed in terms of infinite products from which $K_+(\alpha)$ and $K_-(\alpha)$ may be con-iently extracted. The results are

$$)= \left[\frac{\sin(kb)\ \sin(kc)}{k\ \sin(ka)}\right]^{1/2} \frac{\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{n\pi}}\right]\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\eta_n}\right)e^{-\frac{\alpha}{\eta_n}}\right] e^{-\chi(\alpha)}}{\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_n}\right)e^{-\frac{\alpha}{\alpha_n}}\right]} \qquad (2.51)$$

$$K_-(\alpha) = K_+(-\alpha) \qquad (2.52)$$

ere:

$$\beta_n = \left(k^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2} = i\left(\left(\frac{n\pi}{b}\right)^2 - k^2\right)^{1/2} \tag{2.53}$$

$$a_n = \left(k^2 - \left(\frac{n\pi}{a}\right)^2\right)^{1/2}, \quad \eta_n = \left(k^2 - \left(\frac{n\pi}{c}\right)^2\right)^{1/2} \tag{2.54}$$

he exponential factors are inserted in the infinite products in order to make em converge uniformly. The function $\chi(\alpha)$ is chosen to be analytic and to ake $K_+(\alpha)$ behave as $\alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau > -k_2$ as required in order to able to solve the Wiener-Hopf equation for the C-R problem. The function $\chi(\alpha)$ is obtained from a knowledge of the asymptotic form of $K_+(\alpha)$. The symptotic form of the infinite products is obtained by comparing them with e infinite product

$$\prod_{n=1}^{\infty}\left(1 + \frac{\alpha}{an+b}\right)e^{-\frac{\alpha}{an}} = e^{-\frac{\alpha\Gamma}{a}} \frac{\Gamma\left(\frac{b}{a}+1\right)}{\Gamma\left(\frac{\alpha}{a}+\frac{b}{a}+1\right)} \tag{2.54.a}$$

here $\Gamma(\alpha)$ is the Gamma function and Γ is Euler's constant. The asymptotic form of (2.54.a) is obtained by the use of Sterling's formula. Since each finite product in equation (2.52) behaves in the manner of

$$\beta_n = \frac{in\pi}{b} + O(n^{-1}) \tag{2.54.b}$$

r large n, their asymptotic form may be obtained from (2.54.a) to yield

$$\chi(\alpha) = -\frac{i\alpha}{\pi}\left[c\ln\left(\frac{a}{c}\right) + b\ln\left(\frac{a}{b}\right)\right] + \alpha\sum_{n=1}^{\infty}\left(-\frac{b}{in\pi} + \frac{1}{a_n} - \frac{1}{\eta_n}\right) \tag{2.54.c}$$

The function $\chi(\alpha)$ results in $K_+(\alpha)$ behaving algebraically for the olution of the C-R problem. However, when merely using $K(\alpha)$ to obtain the

O-R factorization it is not necessary to include this term. Its inclusion does lead to the possibility of obtaining the O-R solution from the C-R solution as will be discussed in section 2.5.

$K_+(\alpha)$ and hence $K_-(\alpha)$, without including the $\chi(\alpha)$ term, may be expressed in an alternate form. This will prove to be advantageous when limits are to be taken. In particular, one can write

$$K_+(\alpha) = \left[\frac{\sin(kb)\,\sin(kc)}{k\,\sin(ka)}\right]^{1/2} \cdot \prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{in\pi}} \quad \cdot\; e^{\overset{1}{H}(\alpha)} \qquad (2.55)$$

with

$$\overset{1}{H}(\alpha) = \lim_{N\to\infty} \frac{1}{2\pi i}\int_{\Sigma} f(\alpha,\omega)\; g(\omega)\; d\omega \qquad (2.56)$$

where:

$$f(\alpha,\omega) = \ln\left(1+\frac{\alpha}{(k^2-\omega^2)^{1/2}}\right) - \frac{\alpha}{(k^2-\omega^2)^{1/2}} \qquad (2.57)$$

$$g(\omega) = \frac{F_1'(\omega)}{F_1(\omega)} - \frac{F_2'(\omega)}{F_2(\omega)} \qquad (2.58)$$

and

a) $F_1(\omega)$ has simple zeros at $\frac{n\pi}{c}$; $n = 1, 2, \cdots$

b) $F_2(\omega)$ has simple zeros at $\frac{n\pi}{a}$; $n = 1, 2, \cdots$

c) $F_1'(\omega) = \dfrac{d\,F_1(\omega)}{d\,\omega}$, $F_2'(\omega) = \dfrac{d\,F_2(\omega)}{d\,\omega}$

d) Σ in the contour shown in Fig. 4 with $0 < r < \pi/a$, $r < \epsilon < k_2$,

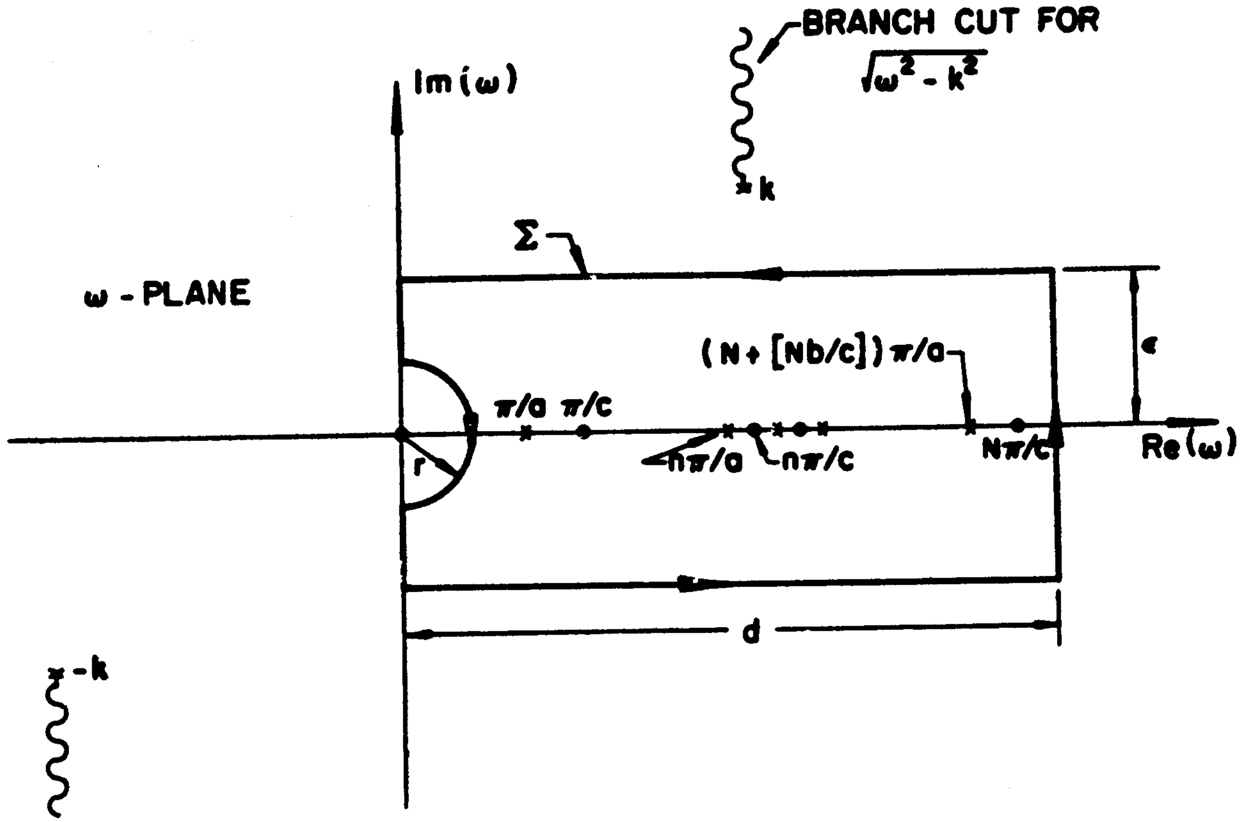


Fig. 4 Contour used in the representation of $K_+(\alpha)$ given by (2. 55).

$$\frac{N\pi}{c} < d < \left(N + \left[\frac{Nb}{c}\right] + 1\right)\frac{\pi}{a} \;{}^{**}$$

The functions, $F_1(\omega)$ and $F_2(\omega)$, may be identified with the functions hich, when set to zero, result in the characteristic equations which give the ansverse wave members for the regions $b \leq x \leq a$, $z > o$ and $o \leq x \leq a$, $< o$, respectively, in the C-R structure, Fig. 3. In this particular prob- n:

$$F_1(\omega) = \text{Sin}\,(\omega c) \tag{2.59}$$

$$F_2(\omega) = \text{Sin}\,(\omega a) \tag{2.60}$$

That $K_+(\alpha)$, given by (2.55), is the same as $K_+(\alpha)$, given by (2.51) ay be verified by integrating (2.56). For any α with $\tau > -k_2$ the integrand (2.56) is analytic with respect to ω in $-k_2 < \text{Im}\,\omega < k_2$ except for the simple oles at the zeros of $F_1(\omega)$ and $F_2(\omega)$. In this case $F_1'(\omega)$ and $F_2'(\omega)$ do not ave any poles within Σ. Using the calculus of residues one obtains equal- y between (2.51) and (2.55).

The expression for $H^1(\alpha)$ may be expanded further by integrating ong the path of integration. That is

$$\int_\Sigma = \int_{d-i\epsilon}^{d+i\epsilon} + \int_{d+i\epsilon}^{o+i\epsilon} + \int_{o+i\epsilon}^{c+i(\epsilon-\gamma)} + \int_{\gamma} + \int_{o-i(\epsilon-\gamma)}^{o-i\epsilon} + \int_{o-i\epsilon}^{1-i\epsilon} \tag{2.61}$$

ow

$^{*}\left[\frac{Nb}{c}\right]$ means the largest integer in $\frac{Nb}{c}$.

$$\int_{0+i\epsilon}^{0+i(\epsilon-\gamma)} + \int_{0-i(\epsilon-\gamma)}^{0-i\epsilon} = 0 \tag{2.62}$$

because the integrand is an odd function of ω.

Also

$$\int = -\pi i \ \mathrm{Res}\Big|_{\omega=0} = \left[\ln\left(1+\frac{\alpha}{k}\right) - \frac{\alpha}{k}\right]\cdot(1-1) = 0 \tag{2.63}$$

and

$$\int_{d-i\epsilon}^{d+i\epsilon} \sim \frac{1}{N} \tag{2.64}$$

because the integrand behaves as ω^{-2} for large ω.

Therefore

$$H^{1}(\alpha) = \frac{1}{2\pi i} \int_{\xi=0}^{\infty} \left[f(\alpha,\xi-i\epsilon)\, g(\xi-i\epsilon) - f(\alpha,\xi+i\epsilon)\, g(\xi+i\epsilon)\right] d\xi \tag{2.65}$$

where ϵ is any positive number such that $0 < \epsilon < k_2$, that is, $H^1(\alpha)$ is independent of ϵ . A useful representation for $K(\alpha)$ is now available and may be written as

$$K(\alpha) = \frac{\mathrm{Sinh}(\gamma b)}{\gamma} \frac{\mathrm{Sin}(kc)}{\mathrm{Sin}(ka)} e^{\left[H^1(\alpha) + H^1(-\alpha)\right]} \tag{2.66}$$

with $H^1(\alpha)$ given by (2.65).

A representation for $L(\alpha)$ for any α within the strip, $-k_2 < \tau < k_2$, and from which L_+ and L_- may be chosen, is obtained by taking the limit of

66) as a, $c \to \infty$ while maintaining a-c=b. The limit of the left side of 66) for values of α within the strip is given by (2.49), that is, $bL(\alpha)$.

$$\lim_{\substack{a,c\to\infty \\ a-c=b}} \frac{\sin(kc)}{\sin(ka)} = e^{ikb} \tag{2.67}$$

l let

$$H(\alpha) = \lim_{\substack{a,c\to\infty \\ a-c=b}} \overset{1}{H}(\alpha) \tag{2.68}$$

e function $f(\alpha, \omega)$ is independent of a, c and the function $g(\xi \mp i\epsilon)$, for s problem, is

$$g(\xi \mp i\epsilon) = \frac{c\cos[c(\xi \mp i\epsilon)]}{\sin[c(\xi \mp i\epsilon)]} - \frac{a\cos[a(\xi \mp i\epsilon)]}{\sin[a(\xi \mp i\epsilon)]} \tag{2.69}$$

nce

$$\lim_{\substack{a,c\to\infty \\ a-c=b}} g(\xi \mp i\epsilon) = \mp ib \tag{2.70}$$

fine

$$\overset{1}{G}(\xi) = \frac{1}{2\pi i}\left\{\lim_{\substack{a,c\to\infty \\ a-c=b}} g(\xi - i\epsilon) - \lim_{\substack{a,c\to\infty \\ a-c=b}} g(\xi + i\epsilon)\right\}\Bigg|_{\epsilon\to 0} \tag{2.70. a}$$

en

$$H(\alpha) = \int_{\omega=0}^{\infty} f(\alpha,\omega)\, \overset{1}{G}(\omega)\, d\omega \tag{2.71}$$

here

$$G^1(\omega) = -b/\pi \tag{2.71.a}$$

It is interesting to observe the physical significance of (2.70.a). $^1(\omega)$ given by (2.70.a) is equal to the difference of Δ_1 and Δ_2. Δ_1 and Δ_2 re the inverses of the spacing of the zeros of $F_1(\omega)$ and $F_2(\omega)$. This is ex- ected if one had worked with the infinite product form of $K(\alpha)$. The natural garithm of $K(\alpha)$ would convert products over n to sums of logarithms over n. ne could then change the summation to a sum over $n\pi/a$ and $n\pi/c$, that is, the eros of $F_1(\omega)$ and $F_2(\omega)$. As a and c approach infinity, $\omega_n = \frac{n\pi}{a}$ and $\omega_n^1 = \frac{n\pi}{c}$ ould approach continuous variables ω and ω^1. Also, $\Delta\omega_n = \pi/a$ and $\Delta\omega^1{}_n =$ $/c$ would approach $d\omega$ and $d\omega^1$. In the limit one would obtain

$$\int_{\omega=0}^{\infty} f(\alpha,\omega)\left[\frac{1}{\Delta\omega_n^1} - \frac{1}{\Delta\omega_n}\right] d\omega \tag{2.72}$$

ut

$$\frac{1}{\Delta\omega_n^1} - \frac{1}{\Delta\omega_n} = -\frac{b}{\pi} \tag{2.73}$$

omparing (2.72) with (2.71) justifies the interpretation of $G^1(\omega)$.

This method could have been used but it is not as convenient or as suf- ciently general as the representation given by equation (2.55) when analyzing ore difficult problems as in sections 3 and 4. However, it does provide a heck on the limit of $g(\xi_+ i\epsilon)$. The difference of the inverse of the spacing of e zeros of $F_1(\omega)$ and $F_2(\omega)$ can be calculated for increasing values of a, c, a-c=b, and the result should approach the expression obtained for $G^1(\omega)$, (2.70.a). The check in this case is trivial as the zeros of $F_1(\omega)$ and $F_2(\omega)$ are obvious.

The limit of (2.66) and the results given in (2.49), (2.67), and (2.71) show that $L(\alpha)$ is

$$L(\alpha) = \frac{\sinh(\gamma b)}{\gamma b} \cdot e^{ikb} \cdot e^{[H(\alpha) + H(-\alpha)]} \tag{2.74}$$

with $H(\alpha)$ given by (2.71). The choice of $L_+(\alpha)$ may now be made as

$$L_+(\alpha) = \left[\frac{\sin(kb)}{kb}\right]^{1/2} \cdot \left[\prod_{n=1}^{\infty}\left(1 + \frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{i n \pi}}\right] \cdot e^{\frac{ikb}{2} + H(\alpha)} \tag{2.75}$$

and

$$L_-(\alpha) = L_+(-\alpha) \tag{2.76}$$

The range of analyticity of $L_+(\alpha)$ can be deduced from the regions of analyticity of the infinite product and $H(\alpha)$. The imaginary part of βn, (2.53), is greater than k_2 and hence the infinite product in (2.75) is analytic for $\tau > -k_2$. The function $f(\alpha, \omega)$, (2.57), satisfies the conditions of theorem A, page 11, Noble [1958] and also restated in the Appendix. Hence $H(\alpha)$ is analytic for $\tau > -k_2$. Therefore, $L_+(\alpha)$ is analytic for $\tau > -k_2$ and $L_-(\alpha)$ is analytic for $\tau < k_2$, the desired range of analyticity. The product of $L_+(\alpha)$ and $L_-(\alpha)$ is obviously $L(\alpha)$.

The functions $L_+(\alpha)$ and $L_-(\alpha)$ can be arranged in a slightly different form which is preferable for numerical calculations.

$$L_+(\alpha) = \left[\frac{L_+(\alpha)\,L(\alpha)}{L_-(\alpha)}\right]^{1/2} = \left[\frac{\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{i n\pi}}}{\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\beta_n}\right)e^{\frac{\alpha b}{i n\pi}}}\right]^{1/2} \cdot \left[\frac{\operatorname{Sinh}(\gamma b)}{\gamma}\,e^{-\gamma b}\right]^{1/2} \cdot e^{\left[\frac{H(\alpha)-H(-\alpha)}{2}\right]} \tag{2.77}$$

In this section equation (2. 77) will not be used but its form will be used in section 3.

Equation (2. 71) may be integrated to obtain $H(\alpha)$ and hence $L_+(\alpha)$ in closed form. The nature of $H(\alpha)$ in equation (2. 71) may be obtained once the following integration is performed.

$$\int_{\omega=0}^{\infty}\left[\ln\left(1+\frac{\alpha}{(k^2-\omega^2)^{1/2}}\right)-\frac{\alpha}{(k^2-\omega^2)^{1/2}}\right]d\omega \tag{2.77.a}$$

Consider the function $Q(\alpha, \omega)$

$$Q(\alpha,\omega) = \omega\ln\left(1+\frac{\alpha}{(k^2-\omega^2)^{1/2}}\right)-\frac{k}{2}\ln\left(\frac{k+\omega}{k-\omega}\right) -$$

$$-(k^2-\alpha^2)^{1/2}\ln\left\{\alpha+\frac{\sqrt{k^2-\alpha^2}\left(\sqrt{k^2-\alpha^2}-\omega\right)}{\sqrt{k^2-\omega^2}+\alpha}\right\} \tag{2.77.b}$$

The branch of $\ln(k+\omega)$ is chosen to be in the lower half plane and the branch of $\ln(k-\omega)$ is chosen to be in the upper half plane. Hence for any α, such that $\tau > -k_2$, $Q(\alpha, \omega)$ is an analytic function of ω for the $\operatorname{Im}(\omega) = 0$. The derivative of $Q(\alpha, \omega)$ with respect to ω for any ω in this range is

$$\frac{\partial Q(\alpha,\omega)}{\partial\omega} = \ln\left(1+\frac{\alpha}{\sqrt{k^2-\omega^2}}\right)-\frac{\alpha}{\sqrt{k^2-\omega^2}}$$

:efore, the integral (2. 77. a) becomes

$$Q(k,\omega)\Big|_{\omega=0}^{\infty} = -i\alpha + \frac{i\pi k}{2} - i\gamma \ln\left(\frac{\alpha-\gamma}{k}\right) \tag{2.77.c}$$

hence

$$H(\alpha) = \frac{i\,b\gamma}{\pi} \ln\left(\frac{\alpha-\gamma}{k}\right) + \frac{i\alpha b}{\pi} - \frac{ibk}{2} \tag{2.78}$$

$$) = \left[\frac{\sin(kb)}{kb}\right]^{1/2} \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{n\pi}}\right] \cdot e^{\frac{i\alpha b}{\pi} + \frac{ib}{\pi}\gamma \ln\left(\frac{\alpha-\gamma}{k}\right) - \chi(\alpha)} \tag{2.79}$$

$L_-(\alpha) = L_+(-\alpha)$ and $\chi(-\alpha) = -\chi(\alpha)$. The principal determination of logarithm is used, that is, $\ln(1) = 0$. The unknown factor, exp. $[-\chi(\alpha)]$, :h must be analytic in $\tau > -k_2$, is added so that the asymptotic form of α) is algebraic. The algebraic behavior of $L_+(\alpha)$, and hence $L_-(\alpha)$, en- :s algebraic behavior of $J(\alpha)$ and hence its determination in equation (2. 35) he extended form of Liouville's theorem.

For large α in $\tau > -k_2$, we have

$$i\frac{b}{\pi}\gamma \ln\left(\frac{\alpha-\gamma}{k}\right) \sim i\frac{b\alpha}{\pi}\ln\left(\frac{2\alpha}{k}\right) \tag{2.80}$$

asymptotic form of the infinite product may be found by the use of equa- (2. 54. a) and gives

$$\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{n\pi}} \sim \alpha^{-1/2}\, e^{i\frac{\Gamma\alpha b}{\pi} + i\frac{\alpha b}{\pi}\left[\ln\left(\frac{\alpha b}{\pi}\right) - i\frac{\pi}{2} - 1\right]} \tag{2.81}$$

where Γ is Euler's constant. Therefore if $\chi(\alpha)$ is chosen as

$$\chi(\alpha) = i\frac{\alpha b}{\pi}\left[\Gamma - i\frac{\pi}{2} + \ln\left(\frac{kb}{2\pi}\right)\right] \tag{2.82}$$

then

$$L_+(\alpha) = \left[\frac{\sin(kb)}{kb}\right]^{1/2} \cdot \prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{i n\pi}} \cdot$$

$$\cdot \mathrm{Exp}\left\{ i\frac{b}{\pi}\gamma \ln\left(\frac{\alpha-\gamma}{k}\right) - \frac{\alpha b}{\pi} + i\frac{\alpha b}{\pi}\left[1-\Gamma+\ln\left(\frac{2\pi}{kb}\right)\right]\right\} \tag{2.83}$$

with $L_+(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau > -k_2$. The function $L_-(\alpha) = L_+(-\alpha)$ with $L_-(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau < k_2$.

2.4 Solution of the Problem

The factorization given by equation (2.83) and the transformed field quantities given by equations (2.37), (2.38), (2.39), and (2.40) permit the determination of the field quantities of interest. The scattered fields within the waveguide are given by

$$\phi(x,z) = \frac{ibk L_+(k)}{\pi}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{L_+(\alpha)\cosh(\gamma x)}{\gamma \sinh(\gamma b)}\, e^{-i\alpha z}\, d\alpha\,; \quad 0 \le x \le b,\ z > 0 \tag{2.84}$$

with $-k_2 < \tau < k_2$. Since $z > 0$ and the integrand has no branch cuts in the lower half plane, the path of integration may be closed in the lower half plane. The solution for ϕ is then obtained by using the calculus of residues, for example, the reflection coefficient, R, in the waveguide is given by the residue at $\alpha = -k$.

$$R = -L_+^2(k) = -L_+^2(k_o) \tag{2.85}$$

Higher order reflected modes are available by the same procedure. Note that once a result as given in equation (2.85), or those following in this section, are reached k may be taken as real and equal to k_o. This may be done because k_2 is infitesimally small compared to k_1 and the evaluation of a function at k is equivalent to its evaluation at $k_1 = \omega\sqrt{\mu_o\epsilon_o} = k_o$.

The fields for $x \geq b$ and hence the radiated field is given by equation (2.40) with $B_1(x, \alpha)$ given by equation (2.38). The function $B_1(x, \alpha)$ has a branch cut in both half planes and hence little is gained by closing the contour by an infinite semicircle. The radiated fields, asymptotic behavior of ϕ given by equation (2.40), is usually obtained by an integration procedure known as the saddle-point method, for example, Morse and Feshback [1953]. However, in this class of problems the use of equivalence theorems for fields, for example, Deschamps [1962], will prove to be more direct and convenient. The field in the aperture is given by

$$\phi(b+o,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} B_1(b,\alpha)\, e^{-i\alpha z}\, d\alpha\ ; \quad \forall z \tag{2.86}$$

Recall that $H_y(b + o, z) = \phi_t(b + o, z) = \phi(b + o, z)$. Therefore, H_y in the aperture is given by equation (2.86). The equivalent electric surface current in the aperture is given by

$$\bar{J} = \hat{n} \times \bar{H} = \hat{x} \times H_y \hat{y} = \hat{z}\, \phi(b+0, z)\, \delta(x-b)^{**} \tag{2.87}$$

The actual far field is equal to the far field due to the equivalent electric current 2J. The vector potential due to the current 2J is obtained by knowing the asymptotic form of the Hankel function $H_0^{(2)}(k_0\rho)$ as described in Harrington [1961] and gives

$$\bar{A} = \frac{e^{ik_0\rho}}{\sqrt{-i2\pi k_0\rho}}\, \hat{z} \int_{z_1=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} \phi(b+0, z_1)\, \delta(x_1-b)\, e^{-i\bar{k}\cdot\bar{r}_1}\, dz_1\, dx_1 \tag{2.88}$$

where:

$$\bar{k} = k_0 \cos(\theta)\hat{z} + k_0 \sin(\theta)\hat{x} = k_z\hat{z} + k_x\hat{x} \tag{2.89}$$

$$\bar{r}_1 = z_1\hat{z} + x_1\hat{x} \tag{2.90}$$

$$\rho = (x^2 + z^2)^{1/2} \tag{2.91}$$

Integrating (2.88) with respect to x_1 gives

$$\bar{A} = \frac{e^{i\frac{\pi}{4} + ik_0\rho - ik_0 b \sin(\theta)}}{\sqrt{k_0\rho}\; 2\pi} \int_{z_1=-\infty}^{\infty} e^{-ik_z z_1} \left(\int_{-\infty+i\tau}^{\infty+i\tau} B_1(b,\alpha)\, e^{-i\alpha z_1}\, d\alpha \right) dz_1 \tag{2.92}$$

and hence

** $\delta(x)$ is the delta function.

$$\bar{A} = \hat{z}\,\frac{e^{i\frac{\pi}{4} + ik_o\rho - ik_o b\,\mathrm{Sin}(\theta)}}{\sqrt{k_o\rho}} \cdot B_1(b, -k_o\,\mathrm{Cos}\,\theta) \tag{2.93}$$

n Maxwell's equations we obtain

$$H_y = -ik_o\,A\,\mathrm{Sin}(\theta)$$

$$H_y = \sqrt{\frac{k_o}{\rho}} \cdot e^{ik_o\rho - i\frac{\pi}{4} - ik_o b\,\mathrm{Sin}(\theta)} \cdot B_1(b, -k_o\,\mathrm{Cos}\,\theta) \cdot \mathrm{Sin}(\theta) \tag{2.94}$$

In this problem H_y in the far field is

$$= e^{ik_o\rho - i\frac{\pi}{4} - ik_o b\,\mathrm{Sin}(\theta)} \cdot \sqrt{\frac{2k_o}{\pi\rho}} \cdot b\,L_+(k_o) \cdot L_+(k_o\,\mathrm{Cos}\,\theta) \tag{2.95}$$

$.0 < \theta < \pi.$

The real power reflected in the waveguide and the real power radiated ie space wave are obtainable once the number of modes propagating in the eguide are known. If the guide dimension and frequency of operation are sen so that only the TEM mode can propagate, the results are:

malized Reflected Power $= W_{ref.} = |R|^2 = |L_+(k_o)|^4$ (2.96)

normalized power radiated is given by the Poynting vector.

malized Radiated Power $= W_{rad} = \dfrac{\omega\epsilon_o}{k_o b}\displaystyle\int_{\theta=0}^{\pi} \bar{E} \times \bar{H}^* \cdot \rho\hat{\rho}\, d\theta$ (2.97)

e the incident power is $k_o b/\omega\,\epsilon_o$. Now

$$E_\theta = \sqrt{\frac{\mu_0}{\epsilon_0}}\, H_y.$$

)

$$W_{rad} = \frac{1}{b}\int_{\theta=0}^{\pi}\left| B_1(b, -k_0 \cos\theta)\cdot \sin\theta\right|^2 d\theta \qquad (2.98)$$

ı this problem (2.98) becomes

$$W_{rad} = \frac{2b}{\pi}\left|L_+(k_0)\right|^2 \int_{\theta=0}^{\pi}\left|L_+(k_0 \cos\theta)\right|^2 d\theta \qquad (2.99)$$

The radiation pattern as a function of θ is given by

$$|H_y| = \left|B_1(b, -k_0 \cos\theta)\cdot \sin\theta\right| \qquad (2.100)$$

nd in this problem is proportional to

$$|H_y| \doteq \left|L_+(k_0 \cos\theta)\right| \qquad (2.101)$$

The numerical value for $W_{ref.}$, $W_{rad.}$, and the far field patterns are iven in section 4. Some of the equations in this section have been obtained, rith sufficient generality so that they will apply in the analysis of the problems n section 3.

In this section $L_+(\alpha)$ and hence $L_-(\alpha)$ were available in closed form. lowever, this is not always feasible as the integration usually cannot be carried ut. Once $L_+(\alpha)$ is obtained in closed form, for example (2.83), the loss may e reduced to zero. Setting σ_1 to zero results in $k_2 = 0$, $k_1 = k_o = \omega\sqrt{\mu_o \epsilon_o}$. Therefore, when $\sigma_1 = o$, $k = k_o$ in (2.83). The path of the Fourier invern integral used in (2.39) and (2.40) when σ_1 is greater than zero is shown Fig. 2. This path must be indented when the loss approaches zero, that is, n $k \rightarrow k_o$ the contour is shown in Fig. 5.

The equation for $L_+(\alpha)$ must be interpreted in a similar manner as Fourier inversion integral when the loss is reduced to zero. For example, α) used in (2.75) and defined by (2.71) becomes, when $k \rightarrow k_o'$,

$$H(\alpha) = -\frac{b}{\pi} \int_{\Sigma_1} \left\{ \ln\left(1 + \frac{\alpha}{\sqrt{k_o^2 - \omega^2}}\right) - \frac{\alpha}{\sqrt{k_o^2 - \omega^2}} \right\} d\omega \tag{2.102}$$

h the contour Σ_1 given by Fig. 6. However, no use is made of the inte- l form of $H(\alpha)$ given by (2.71) and (2.102) as the closed form is available. he more difficult problems, discussed in the following sections, the integral m is used and the contours of integration must be interpreted as discussed e.

Comments on the Method

The factorization obtained in section 2.3 for $L(\alpha)$ can be verified by erence to the problem of the parallel plate duct discussed by Noble [1958]. function to be factored is the same in each problem. A comparison of the torization for L, given by equation (2.83), with Noble's solution, shows t they are the same. He obtained the factorization not on $L(\alpha)$ directly but a function related to $L(\alpha)$ and by use of a formal factorization procedure.

The form of L_+ obtained by means of a limiting procedure on K,

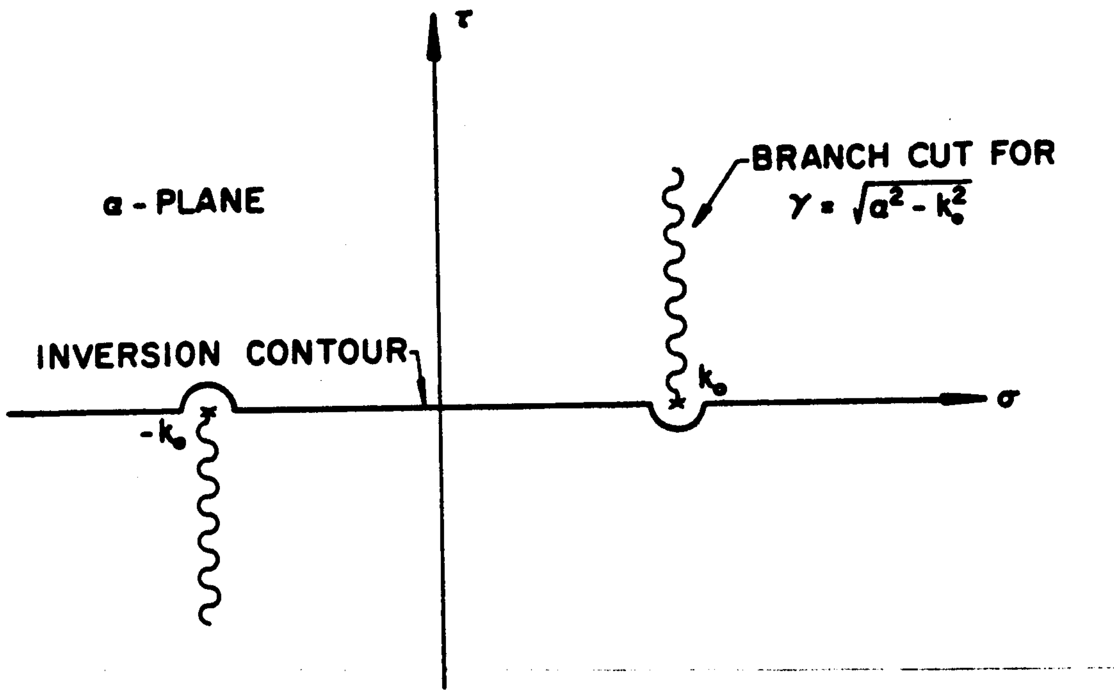
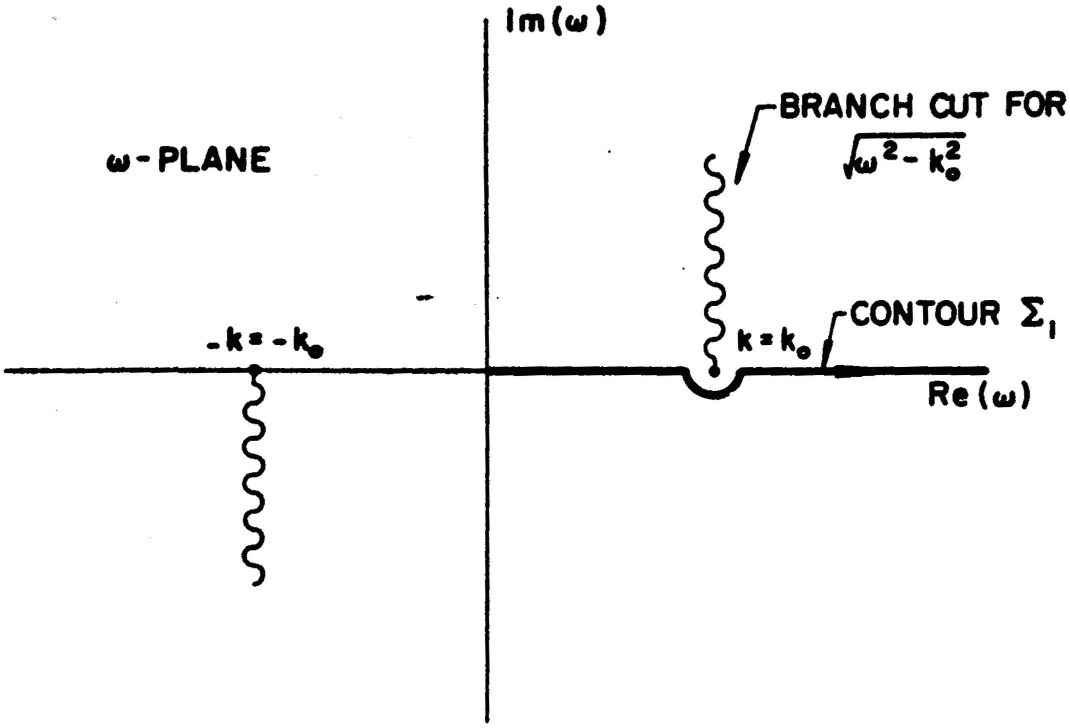


Fig. 5 Branch cuts for γ and the Fourier inversion contour when $k \rightarrow k_o$.



6 Contour used in the integral representation of H(α), (2. 102), when $k \rightarrow k_o$.

2.66), proved advantageous because of the generality of $H^1(\alpha)$ given by equation (2.56). In the problems discussed in section 3 it will be seen that $g(\omega)$ changes for the various problems and not $f(\alpha, \omega)$. As already mentioned, a check on the limit of $g(\xi \pm i\epsilon)$ is available because it is equal to the difference of the inverse of the spacing of the zeros of the characteristic equations $F_1(\omega)$ and $F_2(\omega)$. The form of $H^1(\alpha)$, equation (2.56), also applies to certain other geometries involving coupled waveguides. It is found that in these cases the form of (2.56) remains unchanged. One needs only to substitute the characteristic expressions for $F_1(\omega)$ and $F_2(\omega)$ appropriate for the particular geometry under consideration. For example, this method of solution should be applicable for analyzing radiation from a semi-infinite circular waveguide. The above discussion brings out a strong similarity between cylindrical and parallel plate waveguide problems that may not be apparent without a close look at these problems.

The form of L_+ given by equation (2.77) will also prove convenient when numerical calculations are attempted in section 4.

The C-R problem has been used primarily to generate the O-R factorization. However, the relationship between these two problems may even be made more general. If the asymptotic form of $K_+(\alpha)$, namely $\chi(\alpha)$, is included in (2.55) and hence in the limiting process, we obtain

$$\lim_{\substack{a, c \to \infty \\ a-c=b}} K_+(\alpha) = \sqrt{b}\, L_+(\alpha) \qquad (2.103)$$

This follows from the fact that the limit of the asymptotic form of $K_+(\alpha)$ becomes the asymptotic form of $L_+(\alpha)$. That is, for $K_+(\alpha)$, the $\chi(\alpha)$ is given by (2.54.c), namely

$$K(\alpha) = -\frac{i\alpha}{\pi}\left[c\ln\left(\frac{a}{c}\right) + b\ln\left(\frac{a}{b}\right)\right] + \lim_{M\to\infty}\left\{\alpha\sum_{n=1}^{M}\left(-\frac{b}{in\pi} + \frac{1}{\alpha_n} - \frac{1}{\eta_n}\right)\right\} \tag{2.104}$$

The series involving the terms α_n and η_n may be expressed as

$$\sum_{n=1}^{M}\left(\frac{1}{\alpha_n} - \frac{1}{\eta_n}\right) = \frac{1}{2\pi i}\left\{\int_{\Sigma_2}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_2'}{F_2}\,d\omega - \int_{\Sigma_3}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_1'(\omega)}{F_1(\omega)}\,d\omega\right\} \tag{2.105}$$

where F_1 and F_2 are given by equation (2.59) and (2.60), respectively. The contour Σ_2 is the same as the contour Σ shown in Fig. 4, except that now $\frac{M\pi}{a} < d < (M+1)\frac{\pi}{a}$. Likewise Σ_3 is given by Fig. 4 with $\frac{M\pi}{c} < d < (M+1)\frac{\pi}{c}$. Integrating along the contours gives (ϵ is set equal to r)

$$\sum_{n=1}^{M}\left(\frac{1}{\alpha_n} - \frac{1}{\eta_n}\right) = \frac{1}{2\sqrt{k^2-\left(\frac{M\pi}{a}\right)^2}} - \frac{1}{2\sqrt{k^2-\left(\frac{M\pi}{c}\right)^2}} +$$

$$+\frac{1}{2\pi i}\left\{\int_{\omega=0-i\epsilon}^{\frac{M\pi}{a}-i\epsilon}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_2'}{F_2}\,d\omega - \int_{\omega=0+i\epsilon}^{\frac{M\pi}{a}+i\epsilon}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_2'}{F_2}\,d\omega\right\} -$$

$$-\frac{1}{2\pi i}\left\{\int_{\omega=0-i\epsilon}^{\frac{M\pi}{c}-i\epsilon}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_1'}{F_1}\,d\omega + \int_{\omega=0+i\epsilon}^{\frac{M\pi}{c}+i\epsilon}\frac{1}{\sqrt{k^2-\omega^2}}\frac{F_1'}{F_1}\,d\omega\right\} \tag{2.106}$$

w a and c $\to\infty$ but so does M such that $M/a \to \infty$ and $M/c \to \infty$. Hence

may write as a and c become large

$$\sum_{n=1}^{M}\left(\frac{1}{a_n}-\frac{1}{\eta_n}\right) \doteq \int_{\omega=0}^{M\pi/a}\frac{a}{\pi\sqrt{k^2-\omega^2}}\,d\omega - \int_{\omega=0}^{M\pi/c}\frac{c}{\pi\sqrt{k^2-\omega^2}}\,d\omega \qquad (2.107)$$

nce

$$_{1}\left(\frac{1}{a_n}-\frac{1}{\eta_n}\right) \doteq \frac{1}{i\pi}\left[a\ln\left(\frac{2M\pi}{a}\right)-c\ln\left(\frac{2M\pi}{c}\right)\right]+i\frac{b}{\pi}\ln(-ik) \qquad (2.108)$$

erefore, (2.104) becomes as a, c $\to\infty$

$$\lim_{\substack{,c\to\infty\\ c=b}} \chi(\alpha) = -i\frac{\alpha}{\pi}\left[-b\ln\left(-\frac{ibk}{2\pi}\right)+b\left\{\lim_{M\to\infty}\left[\ln(M)-\sum_{n=1}^{M}\frac{1}{n}\right]\right\}\right] =$$

$$= i\frac{\alpha b}{\pi}\left[\Gamma - i\frac{\pi}{2}+\ln\left(\frac{bk}{2\pi}\right)\right] \qquad (2.109)$$

ich is $\chi(\alpha)$ for the O-R structure as can be seen by comparing (2.109) th (2.82).

The fields for $0 \leq x \leq b$ involves the Fourier inversion of $A_1(x, \alpha)$, ven by equation (2.37) for the O-R structure, and of $C_1(x, \alpha)$, given by equa-n (2.43) for the C-R structure. For any α within the strip and any $0 \leq x \leq b$, e limit as a, c $\to\infty$, while a-c=b, of $C_1(x, \alpha)$ is $A_1(x, \alpha)$ provided (2.103) true. That is, $K_+(\alpha)$ includes the term $\chi(\alpha)$ which makes it algebraic the proper half-plane. Hence the fields in $0 \leq x \leq b$ for the C-R structure approach, in the limit, the actual fields of the O-R structure.

This establishes, for example, that the reflection coefficient for this chosen C-R structure converges to the value of the reflection coefficient for the O-R structure. Indeed, this is what Mittra and VanBlaricum [1965] reported.

Likewise, the limit of $D_1(x, \alpha)$, given in (2.44), for $b \leq x < a$ and α within the strip is equal to $B_1(x, \alpha)$, given in (2.38). Therefore, the fields for $b \leq x < a$ in the chosen C-R structure approach, in the limit, the fields for the O-R structure for $b \leq x$ (a goes to infinity on the limit). This establishes that the O-R solution is a limit point of the C-R solution in all space. Therefore, O-R field quantities may be obtained from the corresponding C-R field quantities by a limiting process as suggested by Talanov [1959] and Mittra, et.al. [1966].

3. EXCITATION OF A DIELECTRIC SLAB BY MEANS OF A TRUNCATED PARALLEL PLATE WAVEGUIDE

3.1 TE Excitation of a Surface Wave Structure

The excitation of a dielectric slab by means of a parallel plate waveguide with one plate truncated is analyzed in this section. The surface wave structure is shown in Fig. 7. The relative dielectric constant of the slab (κ) is assumed to be greater than one and hence the possibility of the structure supporting surface waves. The incident field in the waveguide section is taken to be the lowest order TE mode with the electric field intensity parallel to the walls of the guide. A slight loss due to finite conductivities (σ_1, σ_2) is assumed in each region. However, when the final solution is obtained this loss is permitted to approach zero.

The TE polarization is described in detail in sections 3.1.1 to 3.1.4 as opposed to the TEM excitation. The details of the formulation of the problem for the TEM excitation follow closely those in section 2 and only the pertinent differences and results are given in section 3.2.

3.1.1 Formulation of the Problem

The total electromagnetic fields are obtained by solving the scalar wave equations for the scattered scalar potential ϕ. Define

$$\phi_t = \phi_i + \phi \tag{3.1}$$

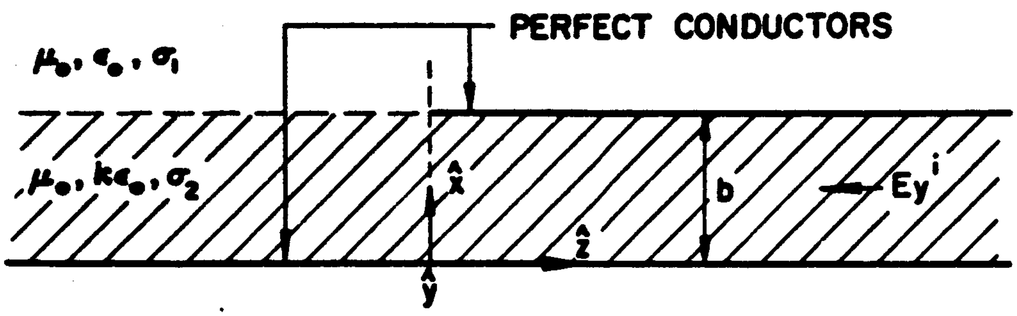


Fig. 7 **Surface wave structure excited by means of a parallel plate waveguide.**

$$\phi_i = \sin\left(\frac{\pi x}{b}\right) e^{-i\beta_1 z} \; ; \quad 0 \le x \le b \, , \forall z \tag{3.2}$$

$$\beta_1 = \left(k_d^2 - \left(\frac{\pi}{b}\right)^2\right)^{1/2} = i\left(\left(\frac{\pi}{b}\right)^2 - k_d^2\right)^{1/2} \tag{3.3}$$

$$k_d = \kappa^{1/2}\left(\omega^2 \mu_0 \epsilon_0 + i \frac{\omega \mu_0 \sigma_2}{\kappa}\right)^{1/2} = k_3 + i k_4 \tag{3.4}$$

with $k_3 > 0$, $k_4 > 0$. Note that the parameters are chosen so that $(2\pi/b)^2 > \text{Re}\, k_d^2 > (\pi/b)^2$. This ensures that the lowest order mode propagates in the waveguide and that the dielectric slab is excited. The scattered scalar potential ϕ is given by the solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k_d^2 \phi = 0 \quad ; \quad 0 \le x \le b \tag{3.5}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \quad ; \quad b \le x \tag{3.6}$$

The constant k is given by equation (2.4). The electromagnetic fields are obtained from

$$E_y = \phi_t \tag{3.7}$$

$$H_x = -\frac{1}{i\omega\mu_0} \frac{\partial \phi_t}{\partial z} \tag{3.8}$$

$$H_z = \frac{1}{i\omega\mu_0}\frac{\partial\phi_t}{\partial x} \tag{3.9}$$

Defining Fourier transforms of ϕ by equations (2.12) to (2.16) reduces the problem to the solution of

$$\frac{d^2\Phi(x,\alpha)}{dx^2} - \gamma_1^2\,\Phi(x,\alpha) = c\,; \quad 0 \le x \le b \tag{3.10}$$

$$\frac{d^2\Phi(x,\alpha)}{dx^2} - \gamma^2\,\Phi(x,\alpha) = 0\,; \quad b \le x \tag{3.11}$$

where:

$$\gamma_1 = (\alpha^2 - k_d^2)^{1/2} \tag{3.12}$$

and γ is given by equation (2.18). The asymptotic behavior of $\phi(x, z)$ results in $\Phi_+(x, \alpha)$ being analytic for $\tau > -\mathrm{Min}(k_2, k_4)$** and $\Phi_-(x,\alpha)$ being analytic for $\tau < \mathrm{Min}(k_2, k_4)$.

Reference to Fig. 7, equations (3.7) to (3.9), and equations (2.12) to (2.16) gives as the boundary conditions on $\Phi(x, \alpha)$

a) $\Phi(c,\alpha) = c$

b) $\Phi_+(b+0,\alpha) = \Phi_+(b-0,\alpha) = \Phi_+(b,\alpha) = 0$

c) $\Phi_-(b+0,\alpha) = \Phi_-(b-0,\alpha) = \Phi_-(b)$

d) $\Phi_-'(b+0,\alpha) - \Phi_-'(b-0,\alpha) = \sqrt{\frac{\pi}{2}}\;\frac{1}{b}\;\frac{i}{(\alpha-\beta_1)}$

** Min = Minimum value of

e) $\Phi_-(b,\alpha) \sim \alpha^{-3/2}$ as $\alpha \to \infty$ for $\tau < \mathrm{Min}(k_2, k_4)$

$$\Phi_+'(b,\alpha) \sim \alpha^{-1/2} \text{ as } \alpha \to \infty \text{ for } \tau > -\mathrm{Min}(k_2, k_4)$$

Boundary condition (e) reflects the fact that $E_y = \phi$ and $E_y \sim z^{+1/2}$ at $x = b$, $z \longrightarrow -0$.

A solution of (3.10) and (3.11) in a form suitable for the application of the boundary conditions is

$$\Phi(x,\alpha) = A(\alpha)\,\mathrm{Sinh}(\gamma_1 x) + C(\alpha)\,\mathrm{Cosh}(\gamma_1 x); \quad 0 \le x \le b \tag{3.13}$$

$$\Phi(x,\alpha) = B(\alpha)\,e^{-\gamma x} + D(\alpha)\,e^{\gamma x}; \quad b \le x \tag{3.14}$$

Recalling that γ has a positive real part for any α for $-k_2 < \tau < k_2$ and the fact that we are seeking decaying waves at infinity requires that $D(\alpha)$ be zero. Also boundary condition (a) requires $C(\alpha)$ to be zero. We may now write at $x = b$, using boundary conditions (b) and (c),

$$\Phi_-(b) = A(\alpha)\,\mathrm{Sinh}(\gamma_1 b) = B(\alpha)\,e^{-\gamma b} \tag{3.15}$$

$$\Phi_-'(b-0) + \Phi_+'(b+0) = A(\alpha)\,\gamma_1\,\mathrm{Cosh}(\gamma_1 b) \tag{3.16}$$

$$\Phi_-'(b+0) + \Phi_+'(b+0) = -B(\alpha)\,\gamma\,e^{-\gamma b} \tag{3.17}$$

A solution for $\Phi_-(b)$ will yield $A(\alpha)$ and $B(\alpha)$ and hence a solution for $\Phi(x, \alpha)$ and a formal solution for $\phi(x, z)$.

Define

$$D_+' = \Phi_+'(b+0) - \Phi_+'(b-0) \qquad (3.18)$$

and therefore D_+' is analytic in $\tau > -\mathrm{Min}(k_2, k_4)$ and behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$ in $\tau > -\mathrm{Min}(k_2, k_4)$. Subtracting equations (3.16) and (3.17), using the boundary condition (d), and the results in (3.15) gives

$$D_+' = -\frac{\Phi_-(b)}{L(\alpha)} - \sqrt{\frac{\pi}{2}}\,\frac{i}{b(\alpha-\beta_1)} \qquad (3.19)$$

where:

$$L(\alpha) = \frac{\mathrm{Sinh}(\gamma_1 b)}{\gamma\,\mathrm{Sinh}(\gamma_1 b) + \gamma_1\,\mathrm{Cosh}(\gamma_1 b)} \qquad (3.20)$$

The function $L(\alpha)$ has branch points at k and -k, the branch points of γ. Choosing the branch cuts for γ as was done in section 2, (2.18), and shown in Fig. 2, results in $L(\alpha)$ again being analytic in $-k_2 < \tau < k_2$. This function must be factored into a product $L_+ L_-$. The function L_+ is analytic in $\tau > -k_2$ and L_- is analytic in $\tau < k_2$. Again, this factorization is the difficult step. The facterization is obtained in section 3.1.3 by a limiting procedure. The method is similar to the one used in section 2.3 and gives the factorization in a form convenient for numerical work. Multiplying (3.19) by $L_+(\alpha)$ and rearranging yields

$$D_+' L_+(\alpha) - E_+(\alpha) = -\frac{\Phi_-(b)}{L_-(\alpha)} + E_-(\alpha) \qquad (3.21)$$

where:

$$E(\alpha) = -\sqrt{\frac{\pi}{2}}\,\frac{i\,L_+(\alpha)}{b(\alpha-\beta_1)} \tag{3.22}$$

$$E_-(\alpha) = -\sqrt{\frac{\pi}{2}}\,\frac{i\,L_+(\beta_1)}{b(\alpha-\beta_1)} \tag{3.23}$$

$$E_+(\alpha) = -\sqrt{\frac{\pi}{2}}\,\frac{i}{b(\alpha-\beta_1)}\left[L_+(\alpha) - L_+(\beta_1)\right] \tag{3.24}$$

Obviously $E_-(\alpha)$ is analytic for $\tau < -k_4$ and $E_+(\alpha)$ is analytic for $\tau >$ $-\mathrm{Min}(k_2, k_4)$. Therefore equation (3.21) holds for $-\mathrm{Min}(k_2, k_4) < \tau <$ $\mathrm{Min}(k_2, k_4)$. The left side of (3.21) is analytic for $\tau > -\mathrm{Min}(k_2, k_4)$ and the right side is analytic for $\tau < \mathrm{Min}(k_2, k_4)$. Therefore, one side is the analytic continuation of the other and both sides of (3.21) may be set equal to $J(\alpha)$, which is analytic in the whole α-plane.

In section 3.1.3 we will find that L_+ and L_- behave as $\alpha^{-1/2}$ in $\tau > -k_2$ and $\tau < k_2$, respectively. Therefore, $E_-(\alpha)$ behaves as α^{-1} for $\tau < k_4$ and $E_+(\alpha)$ behaves as α^{-1} for $\tau > -\mathrm{Min}(k_2, k_4)$. Using these results in 3.21 shows that $J(\alpha)$ behaves as α^{-1} for $\tau > -\mathrm{Min}(k_2, k_4)$ and also for $\tau < \mathrm{Min}(k_2, k_4)$. The extended form of Liouville's theorem proves that $J(\alpha)$ is zero. Hence, we obtain

$$A_1(x,\alpha) = A(\alpha)\,\mathrm{Sinh}(\gamma_1 x) = -\sqrt{\frac{\pi}{2}}\,\frac{i\,L_+(\beta_1)\,L_-(\alpha)\ \mathrm{Sinh}(\gamma_1 x)}{b\,(\alpha-\beta_1)\ \mathrm{Sinh}(\gamma_1 b)} \tag{3.25}$$

$$B_1(x,\alpha) = B(\alpha)\, e^{-\gamma x} = -\sqrt{\frac{\pi}{2}}\; \frac{i\, L_+(\beta_1)\, L_-(\alpha)\; e^{\gamma b - \gamma x}}{b\,(\alpha - \beta_1)} \tag{3.26}$$

nd the formal solution as

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty + i\tau}^{\infty + i\tau} A_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\, ; \quad 0 \le x \le b \tag{3.27}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty + i\tau}^{\infty + i\tau} B_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\, ; \quad b \le x \tag{3.28}$$

with $-\mathrm{Min}(k_2, k_4) < \tau < \mathrm{Min}(k_2, k_4)$ in (3.27) and (3.28).

3.1.2 Choice of a Closed-Region Structure

The chosen C-R structure is shown in Fig. 8. The formulation of this problem is identical to that of the O-R problem in section 3.1.1 except for one important change. The boundary condition which required decaying waves at infinity now becomes

f) $\Phi(a, \alpha) = 0$

This results in the following equation, corresponding to (3.14), as a solution of the wave equation (3.11).

$$\Phi(x,\alpha) = B(\alpha)\, \mathrm{Sinh}\left[\gamma(a-x)\right] + D(\alpha)\, \mathrm{Cosh}\left[\gamma(a-x)\right] \; ; \; b \le x \le a \tag{3.29}$$

Following the method of solution as outlined in section 3.1.1 gives

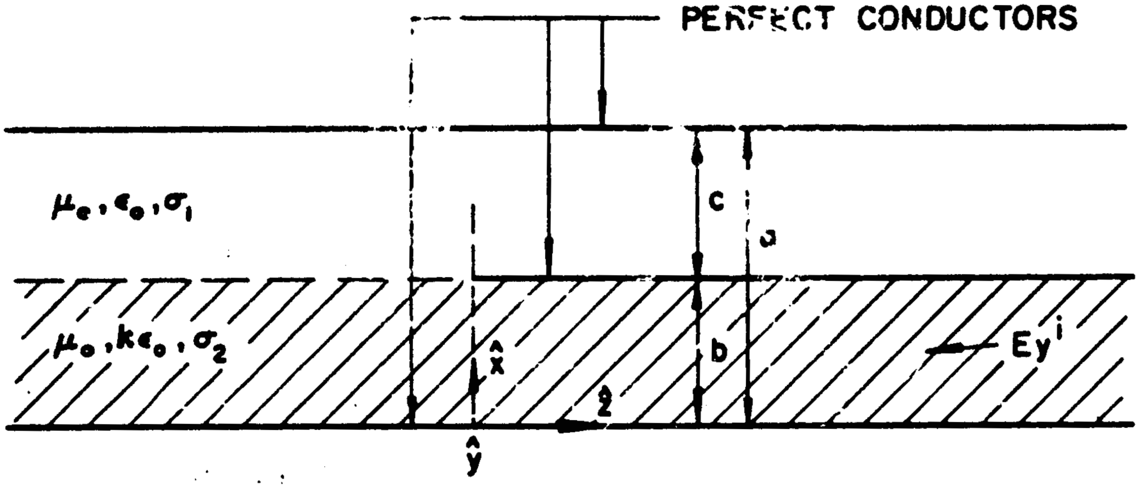


Fig. 8 Chosen closed-region structure corresponding to Fig. 7.

$$C_1(x,\alpha) = A(\alpha)\,\mathrm{Sinh}(\gamma_1 x) = -\sqrt{\frac{\pi}{2}}\;\frac{i\,K_+(\beta_1)\,K_-(\alpha)\;\mathrm{Sinh}(\gamma_1 x)}{b\,(\alpha-\beta_1)\;\mathrm{Sinh}(\gamma_1 b)} \tag{3.30}$$

$$D_1(x,\alpha) = B(\alpha)\,\mathrm{Sinh}[\gamma(a-x)] = -\sqrt{\frac{\pi}{2}}\;\frac{i\,K_+(\beta_1)\,K_-(\alpha)\;\mathrm{Sinh}[\gamma(a-x)]}{b\,(\alpha-\beta_1)\;\mathrm{Sinh}(\gamma c)} \tag{3.31}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\tau}^{\infty+i\tau} C_1(x,\alpha)\,e^{-i\alpha z}\,d\alpha\,; \quad 0 \le x \le b \tag{3.32}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\tau}^{\infty+i\tau} D_1(x,\alpha)\,e^{-i\alpha z}\,d\alpha\,; \quad b \le x \le a \tag{3.33}$$

with $-\mathrm{Min}(k_2, k_4) < \tau < \mathrm{Min}(k_2, k_4)$ in (3.32) and (3.33). The function $K_+(\alpha)$ is analytic for $\tau > -\mathrm{Min}(k_2, k_4)$ and behaves as $\alpha^{-1/2}$ with $\alpha \to \infty$ for $\tau > -\mathrm{Min}(k_2, k_4)$. $K_-(\alpha)$ is analytic for $\tau < \mathrm{Min}(k_2, k_4)$ and behaves as $\alpha^{-1/2}$ with $\alpha \to \infty$ for $\tau < \mathrm{Min}(k_2, k_4)$. The product $K_+(\alpha)\,K_-(\alpha)$ is equal to $K(\alpha)$ in the strip.

$$K(\alpha) = \frac{\mathrm{Sinh}(\gamma_1 b)\;\mathrm{Sinh}(\gamma c)}{\gamma\,\mathrm{Sinh}(\gamma_1 b)\,\mathrm{Cosh}(\gamma c) + \gamma_1\,\mathrm{Cosh}(\gamma_1 b)\,\mathrm{Sinh}(\gamma c)} \tag{3.34}$$

The function $K(\alpha)$ is a ratio of integral functions. It is actually meromorphic with poles at

$$\alpha_n = \pm\,(k^2 - l_n^2)^{1/2} = \pm\, i\,(l_n^2 - k^2)^{1/2} \tag{3.35}$$

with l_n being the zeros of the characteristic equation (transverse wave numbers of the waveguide) for the inhomogeneously filled waveguide.

$$\mathrm{Cos}(\ell c)\,\frac{\mathrm{Sin}(\ell_1 b)}{\ell_1} + \mathrm{Cos}(\ell_1 b)\,\frac{\mathrm{Sin}(\ell c)}{\ell} \tag{3.36}$$

where:

$$\ell_1 = \left(\ell^2 + k_d^2 - k^2\right)^{1/2} \tag{3.37}$$

Note that the zeros approach ($n\pi/a$) as $l \to \infty$.

The functions $K_+(\alpha)$ and $K_-(\alpha)$ are now readily obtained by using the infinite product expansions of the integral functions.

$$K_+(\alpha) = f_0^{1/2} \cdot \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{i n\pi}}\right] \cdot \frac{\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\eta_n}\right)e^{-\frac{\alpha}{\eta_n}}\right]}{\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_n}\right)e^{-\frac{\alpha}{\alpha_n}}\right]} \cdot e^{-\chi(\alpha)} \tag{3.38}$$

where:

$$f_0 = \frac{\mathrm{Sin}(k_d b)\,\mathrm{Sin}(kc)}{k\,\mathrm{Sin}(k_d b)\,\mathrm{Cos}(kc) + k_d\,\mathrm{Cos}(k_d b)\,\mathrm{Sin}(kc)} \tag{3.39}$$

$$\beta_n = \left(k_d^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2} = i\left(\left(\frac{n\pi}{b}\right)^2 - k_d^2\right)^{1/2} \tag{3.40}$$

$$\eta_n = \left(k^2 - \left(\frac{n\pi}{c}\right)^2\right)^{1/2} \tag{3.40.a}$$

$$\alpha_n = \left(k^2 - \ell_n^2\right)^{1/2} \tag{3.40.b}$$

$$\chi(\alpha) = -i\frac{\alpha}{\pi}\left[c\ln\left(\frac{a}{c}\right) + b\ln\left(\frac{a}{b}\right)\right] + \alpha\sum_{n=1}^{\infty}\left(-\frac{b}{i n\pi} + \frac{1}{\alpha_n} - \frac{1}{\eta_n}\right) \tag{3.41}$$

The $\chi(\alpha)$ given by (3.41) ensures that $K_+(\alpha)$ behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau > -\mathrm{Min}(k_2, k_4)$. It is obtained from a knowledge of the asymptotic form of the infinite products in (3.38) by means of equation (2.54.a). The function $K_-(\alpha)$ is equal to $K_+(-\alpha)$. This function, $K(\alpha)$, and its factorization will yield the factorization of $L(\alpha)$, equation (3.20), via limit in a form convenient for numerical calculations.

3.1.3 Factorization for the Open-Structure

The factorization of $L(\alpha)$, equation (3.20), will be done by a limiting procedure analogous to the method discussed in section 2.3. The only difference here is that $L(\alpha)$ is more difficult (compare (3.20) with (2.30)) and a closed form for the answer is not obtainable. However, the form of the factorization obtained readily lends itself to numerical processing and hence numerical results for the electromagnetic field quantities of interest.

The relationship between the O-R structure and the C-R structure is seen by comparing Fig. 7 and 8. The O-R structure is obtained by letting a, $c \to \infty$ while maintaining $a-c=b$. Using this limit on $K(\alpha)$, equation (3.34), for any α such that $-k_2 < \tau < k_2$, gives

$$\lim_{\substack{a,c\to\infty \\ a-c=b}} K(\alpha) = \lim_{c\to\infty} \left\{ \frac{\sinh(\gamma_1 b)\,\sinh(\gamma c)}{\gamma \sinh(\gamma_1 b)\cosh(\gamma c) + \gamma_1 \cosh(\gamma_1 b)\sinh(\gamma c)} \right\} =$$

$$= \frac{\sinh(\gamma_1 b)}{\gamma \sinh(\gamma_1 b) + \gamma_1 \cosh(\gamma_1 b)} = L(\alpha) \qquad (3.42)$$

This results from the fact that, for any α within the strip, γ has a positive, non zero, real part. Therefore,

$$\lim_{c \to \infty} \frac{\cosh(\gamma c)}{\sinh(\gamma c)} = 1 \tag{3.43}$$

The function $K(\alpha)$ can be expressed in a convenient (for taking limits) form by first expressing $K_+(\alpha)$ and $K_-(\alpha)$ in a form like the one used in section 2.3. In particular, we have

$$K_+(\alpha) = f_0^{1/2} \cdot \left[\prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{i n \pi}} \right] \cdot e^{H^1(\alpha)} \tag{3.44}$$

with $H^1(\alpha)$ defined by (2.56), (2.57), and (2.58). Again $\chi(\alpha)$ is not included in (3.44). In this case the functions $F_1(\omega)$ and $F_2(\omega)$ are the characteristic equations of the structure shown in Fig. 8, namely,

$$F_1(\omega) = \sin(\omega c) \tag{3.45}$$

$$F_2(\omega) = \omega \cos(\omega c) \sin(\omega_1 b) + \omega_1 \cos(\omega_1 b) \sin(\omega c) \tag{3.46}$$

with

$$\omega_1 = (\omega^2 + k_d^2 - k^2)^{1/2} \tag{3.46.a}$$

The contour used in (2.56) must enclose the proper zeros of equations (3.45) and (3.46), that is, half the total number as the zeros occur in pairs $\pm\omega_n$. Equation (3.46) now has the possibility of zeros that give rise to surface waves and again zeros whose spacing approaches a continuum as a, c, approach infinity. This is clearly demonstrated by considering the zeros of (3.46) when the loss σ_1 and σ_2 are reduced to zero. Under this condition there are two possibilities, real roots and imaginary roots. The imaginary roots will exist if κ is greater than one, which is the case being considered here. The possibility of κ less than one is discussed in section 3.3.

The real zeros of equation (3.46) approach $\pm n\pi/a$ for large, real, ω and also the spacing between the zeros becomes infinitesimally small as a approaches infinity. For imaginary roots let $\omega = i\,p$ (p a real number) and equation (3.46) becomes

$$F_2(ip) = i\left[p \cosh(pc)\sin(p_1 b) + p_1 \cos(p_1 b)\sinh(pc)\right] \qquad (3.47)$$

with $p_1 = \left((\kappa-1)k_o^2 - p^2\right)^{1/2}$, Equation (3.47) can be shown only to have zeros for $0 < |p| < \sqrt{\kappa - 1}\,k_o$. The number of zeros is discrete and there may actually be none. How many real zeros equation (3.47) may have is determined by the parameters b, κ , k_o. When c approaches infinity and p is positive (3.47) reduces to

$$p \sin(p_1 b) + p_1 \cos(p_1 b) \qquad (3.48)$$

which is the characteristic equation for determining the surface roots of a dielectric slab backed by a perfect conductor with a TE-polarization. (See Collin [1960], p. 474). Therefore, the imaginary roots of (3.46) which are in the upper half of the ω-plane go into the surface wave roots and the positive real roots go into the continuous eigenvalue spectrum as a and c approach infinity for the open structure shown in Fig. 7. Note that equation (3.46) is slightly different from equation (3.36). However, they have the same zeros because $l = 0$ or $l_1 = 0$ are not roots of (3.36). A zero at $l = 0$ is present only when the parameters b, κ , k_o are such that the transition point of a new surface wave occurs. We will pick the parameters such that the transition point is not obtained. When the loss σ_1 and σ_2 are reinserted the zeros will become slightly complex. The previous real roots will now contain a small imaginary component and the imaginary roots will contain a small real component.

The contour used in (3.44) is the one shown in Fig. 4 with the addition of contours to pick up the surface wave roots in the upper half-plane, if any. The zeros $(n\pi/a)$ are now replaced by the ω_n, the zeros of (3.46). The radius r, in Fig. 4, must now be $0 < r < \mathrm{Re}(\omega_o)$, $\mathrm{Im}(\omega_o) < \epsilon < k_2$, where ω_o is the smallest root of equation (3.46) belonging to the set which goes into a continuous spectrum as a and c approach infinity.

That $K_+(\alpha)$ given by (3.44) is the same as (3.38) can be verified by using the calculus of residues. The only singularities within the contour are the zeros of $F_1(\omega)$ and $F_2(\omega)$. $F_1'(\omega)$ and $F_2'(\omega)$ do not have any poles within Σ.

The contour integral for $H^1(\alpha)$ can now be written along each path and gives for the surface waves

$$\sum_{n=1}^{M} \frac{1}{2\pi i} \int_{\rho} f(\alpha,\omega)\, g(\omega)\, d\omega = -\sum_{n=1}^{M} \left\{ \ln\left(1 + \frac{\alpha}{\sqrt{k^2+p_n^2}}\right) - \frac{\alpha}{\sqrt{k^2+p_n^2}} \right\} \tag{3.49}$$

where p_n is a zero of (3.47) and M is the number of zeros. The integrals along the other paths give the same results as in section 2.3. Therefore,

$$K_+(\alpha) = f_o^{1/2} \cdot \left[\prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{2n\pi}} \right] \cdot \left[\prod_{n=1}^{M} \left(1 + \frac{\alpha}{\sqrt{k^2+p_n^2}}\right) e^{-\frac{\alpha}{\sqrt{k^2+p_n^2}}} \right]^{-1} \cdot e^{H^1(\alpha)} \tag{3.50}$$

with $H^1(\alpha)$ given by (2.65). The function $K_-(\alpha) = K_+(-\alpha)$. Hence,

$$K(\alpha) = f_0^{1/2} \cdot \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{2n\pi}}\right] \cdot \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\sqrt{k^2+p_n^2}}\right) e^{-\frac{\alpha}{\sqrt{k^2+p_n^2}}}\right]^{-1} \cdot e^{H^1(\alpha)} \cdot$$

$$\cdot f_0^{1/2} \cdot \left[\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\beta_n}\right) e^{\frac{\alpha b}{2n\pi}}\right] \cdot \left[\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\sqrt{k^2+p_n^2}}\right) e^{\frac{\alpha}{\sqrt{k^2+p_n^2}}}\right]^{-1} \cdot e^{H^1(-\alpha)} \tag{3.51}$$

Taking the limit of (3.51) and using (3.42), (3.39) yields

$$\lim_{\substack{a,c\to\infty \\ a-c=b}} K(\alpha) = L(\alpha) = \frac{\sin(k_d b) \cdot \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right) e^{-\frac{\alpha b}{2n\pi}}\right] \cdot \left[\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\beta_n}\right) e^{\frac{\alpha b}{2n\pi}}\right]}{-ik \sin(k_d b) + k_d \cos(k_d b)} \cdot$$

$$\cdot \left[\prod_{n=1}^{M}\left(1+\frac{\alpha}{\sqrt{k^2+p_n^2}}\right) e^{-\frac{\alpha}{\sqrt{k^2+p_n^2}}}\right]^{-1} \cdot \left[\prod_{n=1}^{M}\left(1-\frac{\alpha}{\sqrt{k^2+p_n^2}}\right) e^{\frac{\alpha}{\sqrt{k^2+p_n^2}}}\right]^{-1} \cdot e^{H(\alpha)+H(-\alpha)} \tag{3.52}$$

where p_n is now a surface wave root (positive root) of (3.48), M the number of roots, and

$$H(\alpha) = \lim_{\substack{a,c\to\infty \\ a-c=b}} H^1(\alpha) \tag{3.53}$$

with $H^1(\alpha)$ given by (2.65).

Define ($\epsilon > 0$)

$$-\frac{b}{2\pi} + G_1(\xi - i\epsilon) = \lim_{\substack{a,c\to\infty \\ a-c=b}} \frac{g(\xi - i\epsilon)}{2\pi i} = \frac{1}{2\pi i} \lim_{\substack{a,c\to\infty \\ a-c=b}} \left\{ \frac{F_1'(\xi - i\epsilon)}{F_1(\xi - i\epsilon)} - \frac{F_2'(\xi - i\epsilon)}{F_2(\xi - i\epsilon)} \right\} =$$

$$= -\left\{ \frac{\mathrm{Sin}(\omega_1 b) + \omega b \left[\frac{\omega}{\omega_1}\mathrm{Cos}(\omega_1 b) + i\,\mathrm{Sin}(\omega_1 b)\right] - i\frac{\omega}{\omega_1}\mathrm{Cos}(\omega_1 b)}{2\pi i\left[\omega\,\mathrm{Sin}(\omega_1 b) - i\,\omega_1 \mathrm{Cos}(\omega_1 b)\right]} \right\}\Bigg|_{\omega = \xi - i\epsilon} \tag{3.54}$$

and

$$\frac{b}{2\pi} + G_2(\xi + i\epsilon) = \lim_{\substack{a,c\to\infty \\ a-c=b}} \frac{g(\xi + i\epsilon)}{2\pi i} =$$

$$= -\left\{ \frac{\mathrm{Sin}(\omega_1 b) + \omega b \left[\frac{\omega}{\omega_1}\mathrm{Cos}(\omega_1 b) - i\,\mathrm{Sin}(\omega_1 b)\right] + i\frac{\omega}{\omega_1}\mathrm{Cos}(\omega_1 b)}{2\pi i\left[\omega\,\mathrm{Sin}(\omega_1 b) + i\,\omega_1 \mathrm{Cos}(\omega_1 b)\right]} \right\}\Bigg|_{\omega = \xi + i\epsilon} \tag{3.55}$$

so

$$H(\alpha) = -\frac{b}{\pi}\int_{\omega=0}^{\infty} f(\alpha,\omega)\,d\omega + \int_{\omega=0-i\epsilon}^{\infty-i\epsilon} f(\alpha,\omega)G_1(\omega)\,d\omega - \int_{\omega=0+i\epsilon}^{\infty+i\epsilon} f(\alpha,\omega)\,G_2(\omega)\,d\omega \tag{3.56}$$

with $\mathrm{Im}\,\omega_0 < \epsilon < k_2$. In this form the terms due to G_1 and G_2 can be considered the perturbation, due to κ varying from 1. The equations (3.52) and (3.56) give $L(\alpha)$ in a suitable form for obtaining $L_+(\alpha)$ and $L_-(\alpha)$.

The choice for $L_+(\alpha)$ may be made as

$$L_+(\alpha) = \left[\frac{\sin(k_d b)}{-ik\sin(k_d b) + k_d\cos(k_d b)}\right]^{1/2} \cdot \left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{i n\pi}}\right] \cdot$$

$$\cdot \left[\prod_{n=1}^{M}\left(1+\frac{\alpha}{\sqrt{k^2+\beta_n^2}}\right)e^{-\frac{\alpha}{\sqrt{k^2+\beta_n^2}}}\right]^{-1} \cdot e^{H(\alpha) - \chi(\alpha)} \tag{3.57}$$

and

$$L_-(\alpha) = L_+(-\alpha) \tag{3.58}$$

The function $L_+(\alpha)$ is analytic for $\tau > -\mathrm{Min}(k_2, k_4)$. This results from the fact that β_n, equation (2.40), has a positive imaginary part greater than k_4, and therefore the infinite product is analytic for $\tau > -k_4$. The function $H(\alpha)$, equation (3.56), is analytic for $\tau > -k_2$ as $f(\alpha, \omega)$ and $G_1(\omega)$ and $G_2(\omega)$ satisfy the conditions of theorem A, page 11, Noble [1958] and restated in the Appendix.

The solution of the Wiener-Hopf type equation, (3.21), requires that $L_+(\alpha)$ and hence $L_-(\alpha)$ have algebraic behavior for large α. Therefore, the term $e^{-\chi(\alpha)}$ must be included in (3.57) to ensure this behavior. The asymptotic form of (3.57) will dictate the choice of $\chi(\alpha)$.

The infinite product behaves as

$$\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{i n\pi}} \sim \alpha^{-1/2}\, e^{i\frac{\alpha b}{\pi}\left[\Gamma + \ln\left(\frac{\alpha b}{\pi}\right) - 1\right] + \frac{\alpha b}{2}} \tag{3.59}$$

for $\tau > -\mathrm{Min}(k_2, k_4)$. The finite product behaves as

$$\prod_{n=1}^{M}\left(1+\frac{\alpha}{\sqrt{k^2+p_n^2}}\right)e^{-\frac{\alpha}{\sqrt{k^2+p_n^2}}} \sim \alpha^M\, e^{-\sum_{n=1}^{M}\frac{\alpha}{\sqrt{k^2+p_n^2}}} \tag{3.60}$$

Also

$$-\frac{b}{\pi}\int_{\omega=0}^{\infty} f(\alpha,\omega)\,d\omega \sim i\,\frac{\alpha b}{\pi} - i\,\frac{\alpha b}{\pi}\ln\left(\frac{2\alpha}{k}\right) \tag{3.61}$$

which is obtained from the work in section 2. 3.

This leaves the term

$$\int_{\omega=0-i\epsilon}^{\infty-i\epsilon} f(\alpha,\omega)\,G_1(\omega)\,d\omega - \int_{\omega=0+i\epsilon}^{\infty+i\epsilon} f(\alpha,\omega)\,G_2(\omega)\,d\omega \tag{3.62}$$

Consider it in two parts by referring to the definition of $f(\alpha, \omega)$ given by (2. 57).

One term is

$$I(\alpha) = -\alpha\left\{\int_{\omega=0-i\epsilon}^{\infty-i\epsilon}(k^2-\omega^2)^{-1/2}\,G_1(\omega)\,d\omega - \int_{\omega=0+i\epsilon}^{\infty+i\epsilon}(k^2-\omega^2)^{-1/2}\,G_2(\omega)\,d\omega\right\} \tag{3.63}$$

which behaves as $(\alpha)\cdot$ (constant). These integrals converge because $G_1(\omega)$ and $G_2(\omega)$ go to zero for large ω on the contours indicated in (3. 63). The remaining terms of (3. 62) are

$$I_1(\alpha) = \int_{\omega=0-i\epsilon}^{\infty-i\epsilon}\ln\left(1+\frac{\alpha}{\sqrt{k^2-\omega^2}}\right)G_1(\omega)\,d\omega - \int_{\omega=0+i\epsilon}^{\infty+i\epsilon}\ln\left(1+\frac{\alpha}{\sqrt{k^2-\omega^2}}\right)G_2(\omega)\,d\omega \tag{3.64}$$

Changing the variable in the second integral of (3.64) to $-\omega$ and using the definitions of $G_1(\omega)$ and $G_2(\omega)$ given by equations (3.54) and (3.55), respectively, gives for $I_1(\alpha)$

$$I_1(\alpha) = \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \ln\left(1 + \frac{\alpha}{\sqrt{k^2 - \omega^2}}\right) G_1(\omega)\, d\omega \tag{3.65}$$

The path of integration may be closed in the lower half plane since $G_1(\omega)$ goes to zero on the infinite semicircle. The integrand of (3.65), however, may have poles at $\omega = i\,p$ where p is the surface wave pole at equation (3.48), and also the branch of $\sqrt{k + \omega}$ is in the lower half plane. Therefore,

$$I_1(\alpha) = \sum_{n=1}^{M} \ln\left(1 + \frac{\alpha}{\sqrt{k^2 + p_n^2}}\right) + \int_B \ln\left(1 + \frac{\alpha}{\sqrt{k^2 - \omega^2}}\right) G_1(\omega)\, d\omega \tag{3.66}$$

The branch line integral of (3.66) is equal to

$$\int_{-k}^{-i\infty} \left[\ln\left(1 + \frac{i\alpha}{\sqrt{\omega^2 - k^2}}\right) - \ln\left(1 - \frac{i\alpha}{\sqrt{\omega^2 - k^2}}\right)\right] G_1(\omega)\, d\omega \sim \text{Constant} \tag{3.67}$$

as $\alpha \to \infty$. Therefore

$$I_1(\alpha) \sim \ln(\alpha^M) \tag{3.68}$$

for $\tau > -\mathrm{Min}(k_2, k_4)$. Combining these results gives

$$L_+(\alpha) \sim \alpha^{-1/2} \operatorname{Exp.}\left\{\frac{i\alpha b}{\pi}\left[\Gamma + \ln\left(\frac{kb}{2\pi}\right)\right] + \frac{\alpha b}{2} + I(\alpha) + \sum_{n=1}^{M} \frac{\alpha}{\sqrt{k^2+p_n^2}}\right\} \tag{3.69}$$

Hence, the choice of $\chi(\alpha)$ as

$$\chi(\alpha) = \operatorname{Exp}\left\{\frac{i\alpha b}{\pi}\left[\Gamma + \ln\left(\frac{kb}{2\pi}\right)\right] + \frac{\alpha b}{2} + I(\alpha) + \alpha \sum_{n=1}^{M} \frac{1}{\sqrt{k^2+p_n^2}}\right\} \tag{3.70}$$

will result in $L_+(\alpha)$ remaining analytic for $\tau > -\mathrm{Min}(k_2, k_4)$ but now behaving algebraically, namely, $\alpha^{-1/2}$, as $\alpha \to \infty$ in this region. The function $L_-(\alpha)$ is given by $L_+(-\alpha)$ and hence $L_-(\alpha)$ behaves as $\alpha^{-1/2}$ as $\alpha \to \infty$ for $\tau < \mathrm{Min}(k_2, k_4)$. The product $L_+(\alpha)\, L_-(\alpha)$ is still $L(\alpha)$ in the strip because of the fact that $\chi(-\alpha) = -\chi(\alpha)$ (refer to the definition of $\chi(\alpha)$ given by equation (3.70)).

Reducing losses, σ_1 and σ_2, to zero once $L_+(\alpha)$ is obtained, yields a simplification in the equations. Setting σ_1 and σ_2 to zero results in $k = k_o$, $k_d = \sqrt{\kappa}\, k_o$. Also $\epsilon \to o$ in equations (3.56) and (3.63). Now $G_2(\omega) = -G_1^*(\omega)$ with ω real (refer to equations (3.54) and (3.55) defining $G_1(\omega)$ and $G_2(\omega)$, respectively) and the loss reduced to zero. Therefore, we obtain for (3.56) and (3.63), under the condition that the loss is zero,

$$H(\alpha) = \int_{\Sigma_1} f(\alpha, \omega)\left(-\frac{b}{\pi} + G(\omega)\right) d\omega \tag{3.71}$$

$$I(\alpha) = -\alpha \int_{\Sigma_1} (k^2 - \omega^2)^{-1/2}\, G(\omega)\, d\omega \tag{3.72}$$

where:

$$G(\omega) = 2\,\mathrm{Re}\; G_1(\omega) = \frac{b}{\pi} - \frac{\omega^2 b - \mathrm{Cos}(\omega_1 b)\,\mathrm{Sin}(\omega_1 b)\left[\frac{\omega^2}{\omega_1} - \omega_1\right]}{\pi\left[\omega^2\,\mathrm{Sin}^2(\omega_1 b) + \omega_1^2\,\mathrm{Cos}^2(\omega_1 b)\right]} \tag{3.73}$$

$$\omega_1 = \left(\omega^2 + (K-1)k_o^2\right)^{1/2} \tag{3.73.a}$$

and $\sum_1$ is the contour shown in Fig. 6. Convergence of the integral in the definition of $I(\alpha)$, (3.72), is ensured as $G(\omega)$, (3.73), goes to zero as $\omega \to \infty$.

A form of $L_+(\alpha)$ that is extremely convenient for numerical work when the loss is reduced to zero is given by

$$L_+(\alpha) = \left[\frac{L_+(\alpha)}{L_-(\alpha)}\,L(\alpha)\right]^{1/2} = \left[\frac{\prod_{n=1}^{\infty}(\beta_n+\alpha)\,e^{-\frac{\alpha b}{i n\pi}}}{\prod_{n=1}^{\infty}(\beta_n-\alpha)\,e^{\frac{\alpha b}{i n\pi}}}\right]^{1/2} \cdot \left[\frac{\mathrm{Sinh}(Y_1 b)}{Y_1}\right]^{1/2} \cdot$$

$$\cdot\left[\frac{Y_1}{Y\,\mathrm{Sinh}(Y_1 b) + Y_1\,\mathrm{Cosh}(Y_1 b)}\right]^{1/2} \cdot \left[\frac{\prod_{n=1}^{M}\left(\sqrt{k_o^2+p_n^2}-\alpha\right)e^{\frac{\alpha}{\sqrt{k_o^2+p_n^2}}}}{\prod_{n=1}^{M}\left(\sqrt{k_o^2+p_n^2}+\alpha\right)e^{-\frac{\alpha}{\sqrt{k_o^2+p_n^2}}}}\right]^{1/2} \cdot e^{H_1(\alpha) - \chi(\alpha)} \tag{3.74}$$

where:

$$H_1(\alpha) = \frac{1}{2}\int_{\Sigma_1}\left[f(\alpha,\omega) - f(-\alpha,\omega)\right]\left(-\frac{b}{\pi} + G(\omega)\right)d\omega \tag{3.75}$$

with $\chi(\alpha)$ defined by equation (3.70) with $k = k_o$, $I(\alpha)$ by equation (3.72), $G(\omega)$ by equation (3.73), β_n by equation (3.40), Y_1 by equation (3.12), p_n the real positive zeros of (3.48), and by setting $k_d = \sqrt{K}\,k_o$, $k = k_o$ wherever they occur.

3.1.4 Solution of the Problem:

The evaluation of the field quantities of interest is obtained by the Fourier inversion integrals given in (3.27) and (3.28). The function $L_+(\alpha)$ is given by (3.74) and $L_-(\alpha) = L_+(-\alpha)$. Within the waveguide section of the structure, $z > 0$ and $0 \leq x \leq b$, the modes are obtained from equation (3.27). The contour may be closed in the lower half plane enclosing only pole-type singularities. The calculus of residues gives the modes directly, in particular, the lowest order reflected mode (only reflected mode carrying average real power) is

$$E_y = -\frac{\pi^2}{2b^3\beta_1^2} L_+(\beta_1) L_-(-\beta_1) \operatorname{Sin}\left(\frac{\pi x}{b}\right) e^{i\beta_1 z} \tag{3.76}$$

Hence, the reflection coefficient is

$$R = -\frac{\pi^2}{2b^3\beta_1^2} L_+^2(\beta_1) \tag{3.77}$$

The far field is again obtained by the use of the Huygen equivalent source in the aperture $x = b + 0$ in a manner analogous to that described in section 2.4. The only difference is now we have an equivalent magnetic current in the aperture. The results are

$$E_y = \left(\frac{\pi k_0}{2\rho}\right)^{1/2} \cdot \frac{e^{ik_0\rho + i\frac{\pi}{4} - ik_0 b \operatorname{Sin}(\theta)}}{b} \cdot L_+(\beta_1) \cdot \left[\frac{\operatorname{Sin}(\theta) \cdot L_+(k_0 \operatorname{Cos}\theta)}{k_0 \operatorname{Cos}\theta + \beta_1}\right] \tag{3.78}$$

with $\rho = \sqrt{x^2 + z^2}$, $0 < \theta < \pi$, and $k_o = \omega\sqrt{\mu_o \epsilon_o}$.

In the results given in (3.77) and (3.78) the loss has been made negligible, in fact, zero. The loss set at zero results in $k = k_o$ and $k_d = \sqrt{\kappa}\, k_o$ and hence equation (3.78). The loss will also be set at zero in the results that follow.

The average real power reflected in the waveguide and the average real power radiated in the space wave are

Normalized Reflected Power = $W_{ref} = |R|^2$ (3.79)

Normalized Radiated Power = W_{rad} =

$$= \frac{\pi k_o^2 |L_+(\beta_1)|^2}{\beta_1 b^3} \int_{\theta=0}^{\pi} \left| \frac{\sin(\theta) \cdot L_+(k_o \cos\theta)}{k_o \cos(\theta) + \beta_1} \right|^2 d\theta \tag{3.80}$$

The normalization consists in the incident power being set to one. Recall that the parameters are chosen so that only the lowest order mode propagates in the waveguide. Therefore, the higher order reflected modes carry zero average real power.

The structure of Fig. 7 has the possibility of the existence of surface waves. The surface wave modes for $b \le x$ are obtained from

$$E_y(x,z) = -\frac{i L_+(\beta_1)}{2b} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{\sinh(\gamma_1 b)\, e^{\gamma b - \gamma x - i\alpha z}\, d\alpha}{(\alpha-\beta_1)\left[\gamma \sinh(\gamma_1 b) + \gamma_1 \cosh(\gamma_1 b)\right] L_+(\alpha)};\ b \le x,\ z < 0 \tag{3.81}$$

Use of $L_-(\alpha) = L(\alpha)/L_+(\alpha)$ is made in (3.81). The surface waves are given by α_n, the positive zeros of

$$\gamma \sinh(\gamma_1 b) + \gamma_1 \cosh(\gamma_1 b) \tag{3.81.a}$$

with α_n real and $k_o < \alpha_n < \sqrt{K}\, k_o$. The possibility of surface waves exists because $K > 1$ is being considered here. The residues of the integrand of (3.81) at the surface wave poles give the surface wave modes as

$$E_y = \frac{\pi}{b} L_+(\beta_1) \sum_{n=1}^{M} Resn \cdot e^{-p_n x - i\alpha_n z} \; ; \; b \leq x, \; z < o \tag{3.82}$$

where:

$$Resn = \frac{e^{p_n b}\, h_n p_n \sin^2(h_n b)}{L_+(\alpha_n)(\alpha_n - \beta_1)\left[\alpha_n h_n \sin^2(h_n b) - \alpha_n p_n \cos(h_n b)\sin(h_n b) + b\alpha_n p_n h_n\right]} \tag{3.83}$$

p_n $(0 < p_n < \sqrt{K - 1}\, k_o)$ is a positive root of (3.48) with the loss set to zero.

M = number of surface waves (number of positive real zeros of (3.48))

$$\alpha_n = \sqrt{k_o^2 + p_n^2} \tag{3.84}$$

$$h_n = \sqrt{(K-1)k_o^2 - p_n^2} > o \tag{3.85}$$

The surface wave modes for $0 \leq x \leq b$ are obtained from

$$E_y = -\frac{i L_+(\beta_1)}{2b} \int_{-\infty + i\tau}^{\infty + i\tau} \frac{\sinh(\gamma x)\, e^{-i\alpha z}}{L_+(\alpha)(\alpha - \beta_1)\left[\gamma \sinh(\gamma_1 b) + \gamma_1 \cosh(\gamma_1 b)\right]} d\alpha \; ; \; o \leq x \leq b, \; z < o \tag{3.86}$$

The residue of the integrand of (3.86) at the surface wave poles again gives for the surface wave modes.

$$E_y = \frac{\pi}{b} L_+(\beta_1) \sum_{n=1}^{M} \frac{Resn \cdot e^{-p_n b - i\alpha_n z} \cdot \sin(h_n x)}{\sin(h_n b)} \; ; 0 \le x \le b, \; z < 0 \qquad (3.87)$$

The surface wave modes are orthogonal in the following sense:

$$\int_0^\infty E_y^m \, E_y^n \, dz = 0 \, ; \; n \neq m \qquad (3.88)$$

Hence, the total real average power carried by the surface wave modes is the sum of the power in each mode. This gives the normalized real average power carried in the surface waves as

$$W_{sw} = \frac{2\pi^2}{\beta_1 b^3} \left| L_+(\beta_1) \right|^2 \sum_{n=1}^{M} \left\{ \frac{\alpha_n \left| Resn \right|^2 e^{-2p_n b}}{\sin^2(h_n b)} \cdot \right.$$

$$\left. \cdot \left[\frac{\sin^2(h_n b)}{2 p_n} + \frac{b}{2} - \frac{\sin(2h_n b)}{4 h_n} \right] \right\} \qquad (3.89)$$

An inspection of the results given in this section shows that once $L_+(\sigma)$ is known (σ real), the radiation pattern of the space wave, the power reflected in the waveguide, the power carried in the surface waves, and the power carried in the space wave can be determined. These numerical calculations are discussed and the results presented in section 4.

The radiation of a truncated parallel plate waveguide in free space, Fig. 1, with an E_y incident may be obtained from the results of section 3.1 by setting the relative dielectric constant to one. This value of the dielectric constant removes any surface wave phenomenon and hence any equations for surface wave quantities are set to zero.

3.2 TEM Excitation of a Surface Wave Structure

In this section we will obtain the excitation of the surface wave structure, shown in Fig. 7, for a TEM mode incident in the waveguide. The incident field is

$$H_y^i = e^{-i k_d z}; \; 0 \leq x \leq b, \; \forall z$$

with k_d given by equation (3.4). The formulation of the problem follows exactly the one given in section 2 except that now the presence of the dielectric must be taken into account. The results of the analysis are

$$H_y = \phi_t \tag{3.90}$$

$$E_x = \frac{1}{Y} \frac{\partial \phi_t}{\partial z} \tag{3.91}$$

$$E_z = -\frac{1}{Y} \frac{\partial \phi_t}{\partial x} \tag{3.92}$$

$$\phi_t = e^{-i k_d z} + \phi \; ; \quad 0 \leq x \leq b, \forall z \tag{3.93}$$

$$\phi_t = \phi \; ; \quad b \leq x, \forall z \tag{3.93. a}$$

where:

$$Y = Y_2 = i \omega \kappa \epsilon_0 - \sigma_2 \; ; \quad 0 \leq x \leq b \tag{3.94}$$

$$Y = Y_1 = i \omega \epsilon_0 - \sigma_1 \; ; \quad b \leq x \tag{3.94. a}$$

and

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} A_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\,; \quad 0 \le x \le b \tag{3.95}$$

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} B_1(x,\alpha)\, e^{-i\alpha z}\, d\alpha\,; \quad b \le x \tag{3.96}$$

$$A_1(x,\alpha) = \frac{i\kappa_1 L_+(k_d)\, L_-(\alpha)\, \mathrm{Cosh}(\gamma_1 x)}{\sqrt{2\pi}\, \gamma_1 \,\mathrm{Sinh}(\gamma_1 b)\, (\alpha - k_d)} \tag{3.97}$$

$$B_1(x,\alpha) = -\frac{i L_+(k_d)\, L_-(\alpha)\, e^{\gamma b}}{\sqrt{2\pi}\, \gamma\, (\alpha - k_d)} \tag{3.98}$$

with $\kappa_1 = Y_2/Y_1$ and γ_1 defined by equation (3. 12) and γ by (2. 18).

The function $L(\alpha)$ for this polarization of the incident field is

$$L(\alpha) = \frac{\gamma \gamma_1 \,\mathrm{Sinh}(\gamma_1 b)}{\gamma_1 \,\mathrm{Sinh}(\gamma_1 b) + \gamma \kappa_1 \,\mathrm{Cosh}(\gamma_1 b)} \tag{3.99}$$

This function is factored by the method of section 3. 1. 3 by using the related closed-region function

$$K(\alpha) = \frac{\gamma \gamma_1 \,\mathrm{Sinh}(\gamma c)\, \mathrm{Sinh}(\gamma_1 b)}{\gamma_1 \,\mathrm{Sinh}(\gamma_1 b)\, \mathrm{Cosh}(\gamma c) + \kappa_1 \gamma \,\mathrm{Cosh}(\gamma_1 b)\, \mathrm{Sinh}(\gamma c)} \tag{3.100}$$

with the result

$$L_+(\alpha) = (\alpha + k)^{1/2}\, (\alpha + k_d)\, M_+(\alpha) \tag{3.101}$$

$$M_{+}(\alpha)=\frac{\left[\mathrm{Sin}(k_d b)\right]^{1/2}\cdot\left[\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\beta_n}\right)e^{-\frac{\alpha b}{i n\pi}}\right]\cdot e^{H(\alpha)-\chi(\alpha)}}{\left[-k_d^2\,\mathrm{Sin}(k_d b)-i\,K_1 k\,\mathrm{Cos}(k_d b)\right]\cdot\left[\prod_{n=1}^{M}\left(1+\frac{\alpha}{\sqrt{k^2+p_n^2}}\right)e^{-\frac{\alpha}{\sqrt{k^2+p_n^2}}}\right]} \tag{3.102}$$

$$L_{-}(\alpha)=(\alpha-k)(\alpha-k_d)\;M_{+}(-\alpha) \tag{3.102.a}$$

and k is defined by equation (2.4) and β_n by (3.40). The numbers p_n are the surface wave zeros given by the zeros of the characteristic equation.

$$K_1\,p\,\mathrm{Cos}(b p_1)=p_1\,\mathrm{Sin}(b p_1) \tag{3.103}$$

$$p_1=(k_d^2-k^2-p^2)^{1/2} \tag{3.104}$$

The function $H(\alpha)$ is still given by (3.56), $\chi(\alpha)$ by (3.70), and $I(\alpha)$ by (3.63), but now the functions $G_1(\omega)$ and $G_2(\omega)$ become for this problem

$$G_1(\omega)=\frac{b}{2\pi}-\frac{-iK_1\mathrm{Cos}(\omega_1 b)+\frac{\omega}{\omega_1}\mathrm{Sin}(\omega_1 b)+\omega b\left[\mathrm{Cos}(\omega_1 b)+iK_1\frac{\omega}{\omega_1}\mathrm{Sin}(\omega_1 b)\right]}{2\pi i\left[-i\omega K_1\mathrm{Cos}(\omega_1 b)+\omega_1\,\mathrm{Sin}(\omega_1 b)\right]} \tag{3.105}$$

$$\omega_1=(\omega^2+k_d^2-k^2)^{1/2} \tag{3.106}$$

$$G_2(\omega)=-\,G_1(-\omega) \tag{3.107}$$

This results in $L_+(\alpha)$ being analytic and behaving as $\alpha^{1/2}$ as $\alpha\to\infty$ for $\tau>-\mathrm{Min}(k_2,k_4)$. Likewise $L_-(\alpha)$ is analytic and behaves as $\alpha^{1/2}$ as $\alpha\to\infty$ for $\tau<\mathrm{Min}(k_2,k_4)$.

The form of $L_{\pm}(\alpha)$ that is convenient for numerical calculations when the loss σ_1 and σ_2 are reduced to zero is given by

$$L_+(\alpha) = (\alpha + k_o)^{1/2} \cdot (\alpha + K^{1/2} k_o) \cdot \left[\frac{\prod_{n=1}^{\infty} (\beta_n + \alpha) e^{-\frac{\alpha b}{2n\pi}}}{\prod_{n=1}^{\infty} (\beta_n - \alpha) e^{\frac{\alpha b}{2n\pi}}} \cdot \frac{\sinh(Y_1 b)}{Y_1} \right]^{1/2} \cdot$$

$$\cdot \left[\frac{\prod_{n=1}^{M} \left(\sqrt{k_o^2 + p_n^2} - \alpha\right) e^{\frac{\alpha}{\sqrt{k_o^2 + p_n^2}}}}{\prod_{n=1}^{M} \left(\sqrt{k_o^2 + p_n^2} + \alpha\right) e^{-\frac{\alpha}{\sqrt{k_o^2 + p_n^2}}}} \right]^{1/2} \cdot \left[\frac{1}{Y_1 \sinh(Y_1 b) + K Y \cosh(Y_1 b)} \right]^{1/2} \cdot e^{H_1(\alpha) - \chi(\alpha)} \qquad (3.108)$$

where:

$$H_1(\alpha) = \frac{1}{2} \int_{\Sigma_1} \left[f(\alpha, \omega) - f(-\alpha, \omega) \right] \cdot \left[-\frac{b}{\pi} + G(\omega) \right] d\omega \qquad (3.109)$$

with Σ_1 the contour shown in Fig. 6 and $G(\omega)$ given by

$$G(\omega) = \frac{b}{\pi} - \frac{K \sin(\omega_1 b) \cos(\omega_1 b) \left[\frac{\omega^2}{\omega_1} - \omega_1 \right] + K \omega^2 b}{\pi \left[K^2 \omega^2 \cos^2(\omega_1 b) + \omega_1^2 \sin^2(\omega_1 b) \right]} \qquad (3.110)$$

$$\omega_1 = \left(\omega^2 + (K-1) k_o^2 \right)^{1/2} \qquad (3.111)$$

The $\chi(\alpha)$ is given by (3.70) with $k_d = \sqrt{K}\, k_o$, $k = k_o$ replaced throughout. The $I(\alpha)$ used in $\chi(\alpha)$ now becomes

$$I(\alpha) = -\alpha \int_{\Sigma_1} (k_o^2 - \omega^2)^{-1/2} G(\omega)\, d\omega \qquad (3.112)$$

with $G(\omega)$ given in (3.110). This integral converges as $G(\omega)$ behaves as $\sin(\omega)$ for large ω. The p_n become the positive real zeros of (3.103) with $k_d = \sqrt{K}\, k_o$, $k = k_o$, $K_1 = K$.

The results of interest for this problem are obtained as was done in section 3. 1. 4 and are merely stated here. The normalized (incident energy set at one) reflected power

$$W_{ref} = |R|^2 = \left| \frac{L_+^2(\kappa^{1/2} k_o)}{4 k_o^2 b} \right|^2 \tag{3.113}$$

This follows since the parameters are restricted to values such that only the lowest order mode can propagate in the waveguide. That is, $\kappa^{1/2} k_o < \pi/b$. The normalized power radiated in the space wave is

$$W_{rad} = \frac{\kappa^{1/2}}{2\pi b k_o} \left| L_+(\kappa^{1/2} k_o) \right|^2 \int_{\theta=0}^{\pi} \left| \frac{L_+(k_o \cos\theta)}{k_o \cos\theta + \kappa^{1/2} k_o} \right|^2 d\theta \tag{3.114}$$

with the radiation pattern given by $(0 < \theta < \pi)$

$$|H_y| = \left| \frac{L_+(k_o \cos\theta)}{k_o \cos\theta + \kappa^{1/2} k_o} \right| \tag{3.115}$$

The normalized energy in the surface waves becomes

$$W_{sw} = \frac{\kappa^{1/2} \left| L_+(\kappa^{1/2} k_o) \right|^2}{k_o b} \sum_{n=1}^{M} a_n \cdot | Dresn |^2 \cdot \left\{ \frac{\kappa b}{2} + \frac{\kappa \sin(2 h_n b)}{4 h_n} + \frac{h_n^2 \sin^2(h_n b)}{2 p_n^3} \right\} \tag{3.116}$$

where M is the number of surface waves and

$$Dresn = - \frac{p_n^2 h_n}{(a_n - \kappa^{1/2} k_o) \cdot L_+(a_n) \cdot D} \tag{3.117}$$

$$D = a_n p_n \sin(h_n b) + b a_n p_n h_n \cos(h_n b) + a_n h_n \kappa \cos(h_n b) + b a_n p_n^2 \kappa \sin(h_n b) \tag{3.118}$$

$$a_n = (k_0^2 + p_n^2)^{1/2} \tag{3.119}$$

$$h_n = ((\kappa - 1) k_0^2 - p_n^2)^{1/2} \geq 0 \tag{3.120}$$

These numerical calculations are discussed and the results presented in section 4 along with those for the TE-polarization.

3.3 Excitation of an Incompressible, Isotropic, Plasma Slab

The structure in question is still given by Fig. 7 but with one change. The dielectric between $0 \leq x \leq b$ is replaced by an incompressible, isotropic, plasma medium. This plasma medium behaves as a dielectric with a relative dielectric constant less than 1. This results from the fact that the relative dielectric constant for this medium is

$$\kappa = (1 - X) \tag{3.121}$$

$$X = \frac{\omega_N^2}{\omega^2} \tag{3.122}$$

with ω_N the plasma frequency and ω the wave frequency, for example Budden [1964].

Values of the relative dielectric constant less than zero are not of any interest as it is impossible to have propagating waves in the waveguide. However, there are propagating modes for the relative dielectric constant between zero and one. It can be shown, for this range of κ, that the structure will not support surface waves.

The analysis of this problem may be extracted from sections 3.1 and 3.2 by considering the effect of $0 < \kappa < 1$ on the analysis. It is found that the only change occurs in $G(\omega)$, given by (3.73) for the TE case and by (3.110) for the TEM case. Recall that ω_1 in these equations is

$$\omega_1 = \left((\kappa - 1)\, k_o^{\,2} + \omega^2\right)^{1/2} \tag{3.123}$$

with ω varying from zero to infinity. The quantity $(\kappa - 1)\, k_o^{\,2}$ is now negative. Hence, for values of $\omega < \sqrt{(1-\kappa)}\,k_o$, ω_1 becomes imaginary. The choice of the sign of the imaginary number is immaterial as $G(\omega)$ is not a multivalued function of ω. Therefore, the results of sections 3.1 and 3.2 stand unaltered for the case of $0 < \kappa < 1$. However, all surface wave phenomena is non-existent for this range of κ and the equations in those sections must be interpreted accordingly.

3.4 Discussion of the Method for the Dielectric Slab Structure

Basically the related C-R structure was used only to obtain the O-R factorization. However, the O-R and C-R solutions may be related as was done in section 2.5 in the following way. To be specific, we will discuss the TE case.

The function $L_+(\alpha)$ given by (3.57) is the limit, $a, c \to \infty$ while $a-c=b$, of $K_+(\alpha)$ given by (3.38) if we include the $\chi(\alpha)$, (3.41), in $K_+(\alpha)$ (recall that $\chi($ makes $K_+(\alpha)$ behave algebraically). This is true if the limit of (3.41) is (3.70). We may write (3.41) as

$$\chi(\alpha) = Q(\alpha) + a\sum_{n=1}^{\infty}\left[\left(k^2 - l_n^2\right)^{-1/2} - \left(k^2 - \left(\frac{n\pi}{a}\right)^2\right)^{-1/2}\right] \tag{3.124}$$

Convergence of the series is ensured because $l_n = n\pi/a + O(1/n)$ for large n. The function $Q(\alpha)$ is equal to $\chi(\alpha)$ in section 2.5 and is given by (2.104). Recall that the l_n are the roots of equation (3.36). Taking the limit of $\chi(\alpha)$ given by (3.124) yields

$$\lim_{\substack{a,c\to\infty \\ a-c=b}} \chi(\alpha) = i\frac{\Gamma a b}{\pi} + i\frac{\alpha b}{\pi}\ln\left(\frac{kb}{2\pi}\right) + \frac{\alpha b}{2} + I(\alpha) + \alpha\sum_{n=1}^{M}\frac{1}{\sqrt{k^2+p_n^2}} \tag{3.125}$$

with $I(\alpha)$ given by (3.63). The limit of $Q(\alpha)$ had already been obtained in section 2.5; refer to equation (2.109). Now equation (3.125) is $\chi(\alpha)$, (3.70) for the O-R problem. Therefore, the limit of $K_+(\alpha)$ is $L_+(\alpha)$. Hence, one may now show that the limit of $C_1(x, \alpha)$ and $D_1(x, \alpha)$, given by equations (3.30) and (3.31), respectively, for the C-R structure, become $A_1(x, \alpha)$ and $B_1(x, \alpha)$, given by equations (3.25) and (3.26), respectively, for the O-R structure. Therefore, the complete C-R solution becomes, in the limit, the O-R solution. This means that instead of formulating the O-R problem one could obtain the solution by formulating a related C-R problem and taking the limit of its solution as dictated by the physics. This phenomena is what Talanov [1959] suggested would happen. In fact, he solved the C-R problem for a TEM excitation and calculated the energy in the slow waves of the inhomogeneously filled guide and claimed that their value as $a \to \infty$ is the energy in the surface waves.

In section 2 we obtained the factorization for the C-R problem in a particular form. The advantage of doing this should now be obvious. As a and c become large, the series which occurred could be recast as integrals providing the spacing between adjacent zeros of the characteristic equations, $F_1(\omega)$ and $F_2(\omega)$, was known. This representation yielded the information as the function obtained, $-\frac{b}{\pi} + G(\omega)$, by taking the limit of the integral representation, is the desired information. In particular, $-\frac{b}{\pi} + G(\omega)$, should be the difference of the inverse of the spacing of the zeros of the characteristic equations. That this is so is verified in section 4 with numerical calculations.

4. NUMERICAL RESULTS

The usual field quantities of interest in problems of this type are obtainable once $L_+(\sigma)$ is known (σ real). That is, the average power radiated in the space wave, reflected in the waveguide, and trapped in the surface waves (if any) and the radiation pattern of the space wave are given as functions of $|L_+(\sigma)|$. This can be seen by referring to equations (3.79), (3.80), (3.89), and (3.78) for the TE polarization and (3.113), (3.114), (3.116), and (3.115) for the TEM excitation.

The function $|L_+(\sigma)|$ is given in convenient form for numerical processing by (3.74) and (3.108) for the TE and TEM polarizations respectively. To indicate what is involved in the evaluation of $|L_+(\sigma)|$, a close look at the TE case is made. The evaluation of $|L_+(\sigma)|$ for the TEM polarization will be similar. Taking the absolute value of (3.74) for real arguments gives

$$|L_+(\sigma)| = \left|\frac{\prod_{n=1}^{\infty}(\beta_n+\sigma)}{\prod_{n=1}^{\infty}(\beta_n-\sigma)} \frac{\operatorname{Sinh}(Y_1 b)}{Y_1}\right|^{1/2} \cdot \left|\frac{\prod_{n=1}^{M}\left(\sqrt{k_0^2+p_n^2}-\sigma\right)}{\prod_{n=1}^{M}\left(\sqrt{k_0^2+p_n^2}+\sigma\right)}\right|^{1/2} \cdot$$

$$\cdot \left|\frac{Y_1}{Y \operatorname{Sinh}(Y_1 b)+Y_1 \operatorname{Cosh}(Y_1 b)}\right|^{1/2} \cdot e^{\operatorname{Re} H_1(\sigma)-\frac{\sigma b}{2}-\operatorname{Re} I(\sigma)} \qquad (4.\ 1)$$

The real part of $H_1(\sigma)$ is given by

$$\mathrm{Re}\left[\frac{1}{2}\int_{\Sigma_1}\left\{\ln\left(\frac{\sqrt{k_o^2-\omega^2}+\sigma}{\sqrt{k_o^2-\omega^2}-\sigma}\right)-\frac{2\sigma}{\sqrt{k_o^2-\omega^2}}\right\}\cdot\left(-\frac{b}{\pi}+G(\omega)\right)\,d\omega\right]=$$

$$=\frac{1}{2}\int_{\omega=0}^{k_o}\left\{\ln\left|\frac{\sqrt{k_o^2-\omega^2}+\sigma}{\sqrt{k_o^2-\omega^2}-\sigma}\right|\right\}\cdot\left(-\frac{b}{\pi}+G(\omega)\right)\,d\omega+\frac{\sigma b}{2}+\mathrm{Re}\,I(\sigma) \tag{4.2}$$

This result is obtained since the square root term is purely imaginary for $\omega > k_o$. The definition of β_n, (3.40), and the fact that we are considering the case where only one mode propagates in the waveguide $\left(\frac{\pi}{b}<\sqrt{K}\,k_o<\frac{2\pi}{b}\right)$, gives

$$\left|\frac{\prod_{n=1}^{\infty}(\beta_n+\sigma)}{\prod_{n=1}^{\infty}(\beta_n-\sigma)}\right| = \left|\frac{\beta_1+\sigma}{\beta_1-\sigma}\right| \tag{4.3}$$

Therefore, equation (4.1) becomes

$$\left|L_+(\sigma)\right| = \left|\frac{(\beta_1+\sigma)}{(\beta_1-\sigma)}\;\frac{\mathrm{Sinh}(Y_1 b)}{Y_1}\right|^{1/2}\cdot\left|\frac{\prod_{n=1}^{M}\left(\sqrt{k_o^2+p_n^2}-\sigma\right)}{\prod_{n=1}^{M}\left(\sqrt{k_o^2+p_n^2}+\sigma\right)}\right|^{1/2}\cdot$$

$$\cdot\left|\frac{Y_1}{Y\,\mathrm{Sinh}\,(Y_1 b)+Y_1\,\mathrm{Cosh}\,(Y_1 b)}\right|^{1/2}\cdot e^{H_2(\sigma)} \tag{4.4}$$

with $H_2(\sigma)$ given by

$$H_2(\sigma) = \frac{1}{2}\int_{\omega=0}^{k_0} \left\{ \ln \left| \frac{\sqrt{k_0^2-\omega^2}+\sigma}{\sqrt{k_0^2-\omega^2}-\sigma} \right| \right\} \cdot \left(-\frac{b}{\pi} + G(\omega)\right) d\omega \tag{4.5}$$

and $G(\omega)$ given by (3.73). The convenience of $L_+(\sigma)$ as given in (4.4) is now obvious. We are left with only a finite integral to evaluate.

When $|\sigma|$ is less than k_o, the integrand of equation (4.5) becomes singular at $\omega = \omega_o = \left(k_o^2 - \sigma^2\right)^{1/2}$. This results from the fact that we have reduced the loss to zero. However, this is an integrable singularity, as one expects, and may be handled in the following way. Rewriting equation (4.5) for $0 \leq \sigma \leq k_o$ gives

$$\begin{aligned} H_2(\sigma) = \frac{1}{2}\int_{\omega=0}^{k_0} \Bigg\{ &\left(\ln \left| \frac{\sqrt{k_0^2-\omega^2}+\sigma}{\sqrt{k_0^2-\omega^2}-\sigma} \right| \right) \cdot \left(-\frac{b}{\pi} + G(\omega)\right) - \left(\ln \left| \frac{2\sigma^2}{\omega_0(\omega_0-\omega)} \right| \right) \cdot \left(-\frac{b}{\pi} + G(\omega_0)\right) \Bigg\} \cdot d\omega \\ &+ \frac{1}{2}\left(-\frac{b}{\pi} + G(\omega_0)\right) \int_{\omega=0}^{k_0} \left(\ln \left| \frac{2\sigma^2}{\omega_0(\omega_0-\omega)} \right| \right) d\omega \end{aligned} \tag{4.6}$$

The first integral in equation (4.6) is no longer singular at ω_o and is conveniently handled by a digital computer. The second integral in (4.6) can be obtained in closed form. For negative values of σ we have, $H_2(\sigma) = -H_2(|\sigma|)$. Therefore, $H_2(\sigma)$ is obtained from (4.6) for all σ and hence $|L_+(\sigma)|$ is conveniently calculated. The characteristic response of the structures for both TE and TEM excitations will be given in graphical form in what follows.

In going from a series to an integral, which occurs when obtaining

$L_+(\alpha)$ and $L_-(\alpha)$ by letting a and c approach infinity in the representation of $K(\alpha)$, we needed knowledge of the spacing of the zeros of the characteristic equations. We concluded that the function $b/\pi - G(\omega)$ is the difference of the inverse of the spacing of the zeros of the characteristic equations $F_1(\omega)$ and $F_2(\omega)$ as a, $c \rightarrow \infty$. For example, the characteristic equations are given by (3.45) and (3.46) and $G(\omega)$ by (3.73) for the TE polarization. This conclusion may be verified by actually finding the zeros of $F_1(\omega)$ and $F_2(\omega)$ and seeing if indeed the zeros behave, as a and c approach infinity, in a manner given by the function $\frac{b}{\pi} - G(\omega)$. The result of this is shown in Fig. 9 for the TE excitation. One can see that, for c/b equal to 80, the curve obtained from a knowledge of the zeros is identical to the theoretical curve given by $b/\pi - G(\omega)$. For $c/b = 8$ there is some difference, as expected. That is, only in the limit as $a \rightarrow \infty$ does $G(\omega)$ hold.

The response of the structure to a TE polarized source is shown in Figs. 10 to 15. The relative dielectric constant (κ) has three distinct ranges: a) the plasma phenomena with $0 < \kappa < 1.0$; b) free space radiation with $\kappa = 1.0$; c) the surface wave phenomena with $\kappa > 1.0$. The response of the structure for values of κ in each of these ranges is given in Figs. 10, 11, 12, and 13. Recall that these are normalized values with the incident energy set at 1 or 100 per cent and the maximum far field value set at one.

An inspection of Figs. 10 and 11 shows that it is possible to have all the incident power radiated in the space wave. This is also true for the sur-

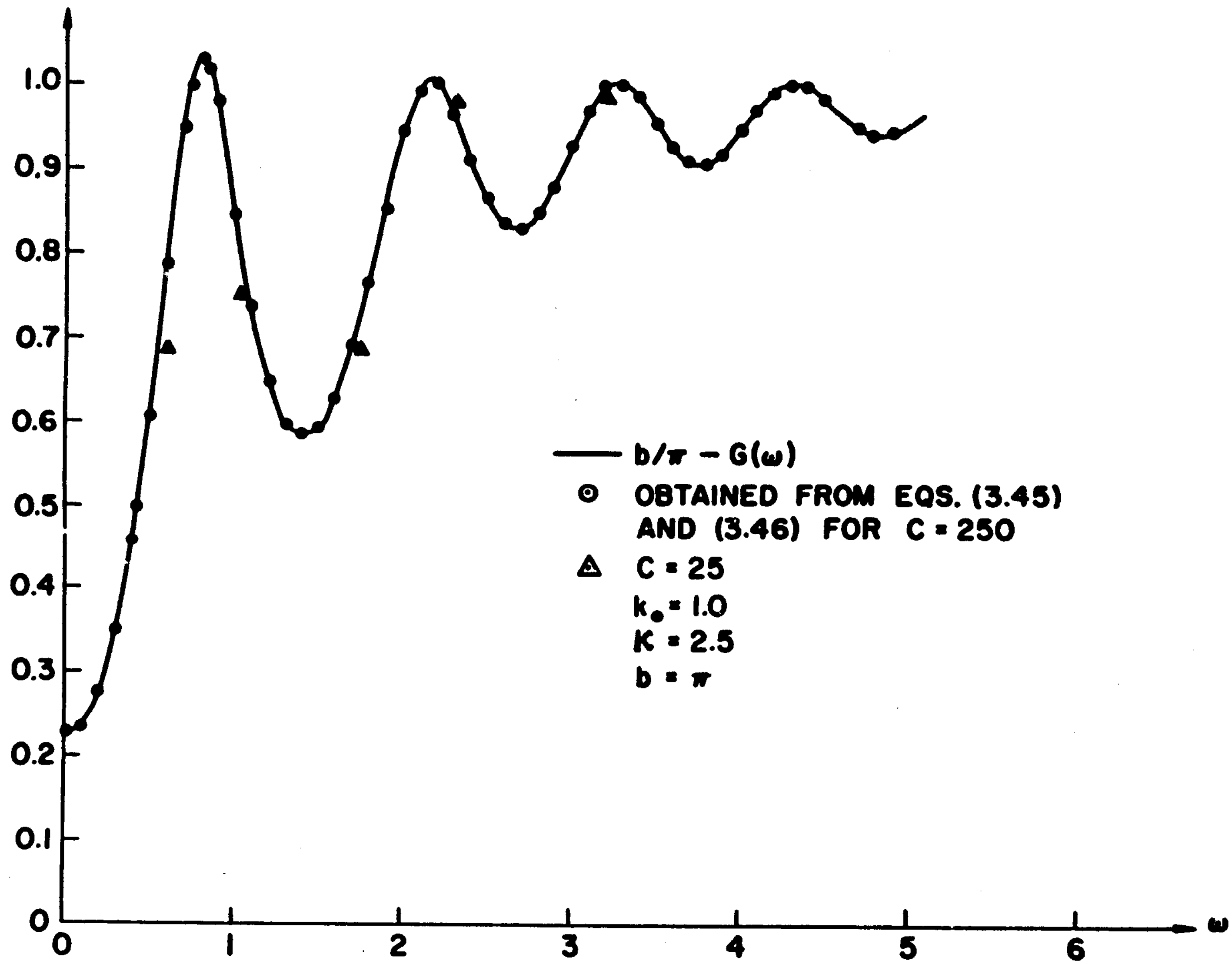


Fig. 9 Comparison of $b/\pi - G(\omega)$ given by (3. 73) with the results obtained from a knowledge of the zeros of (3. 45) and (3. 46).

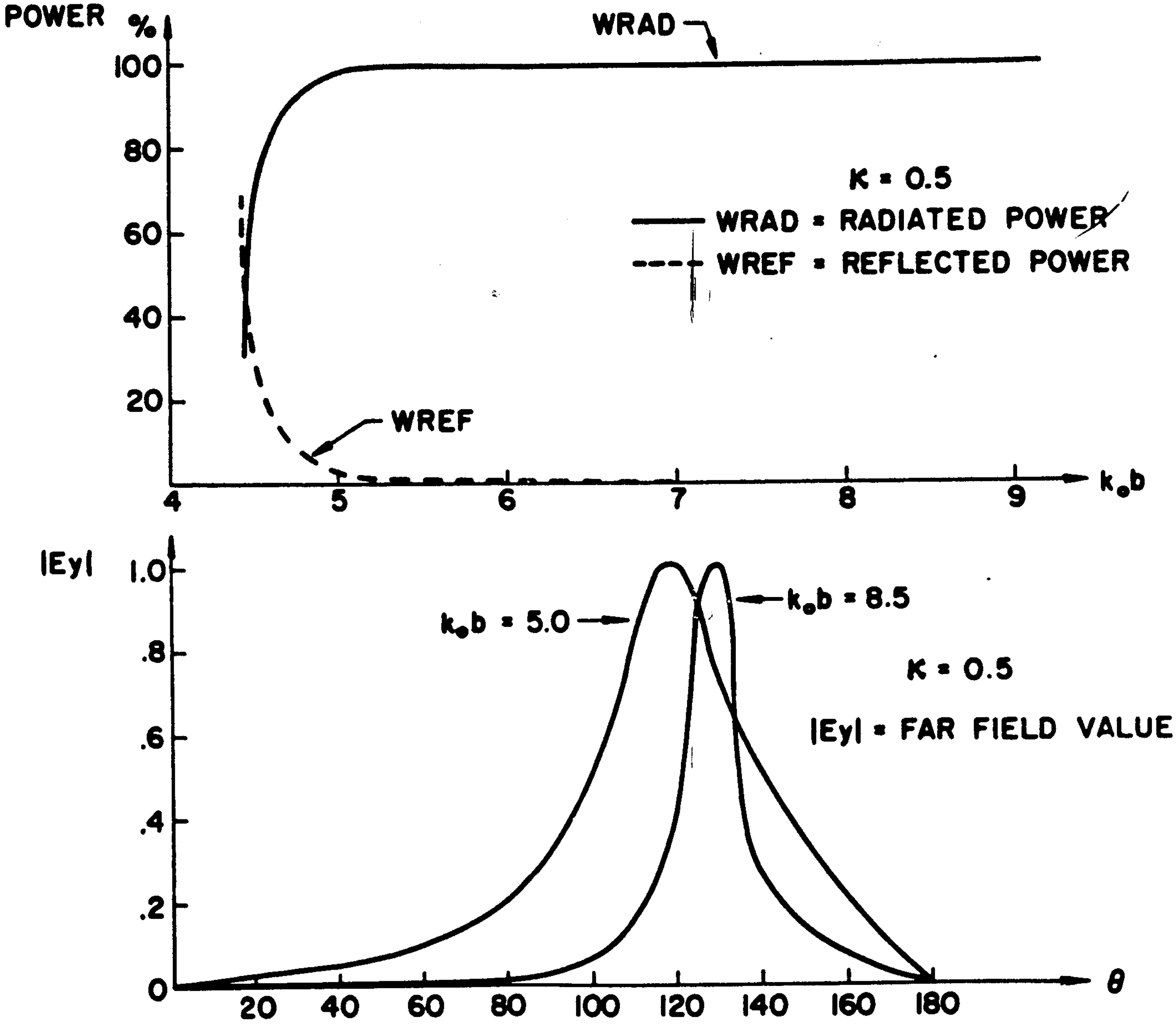
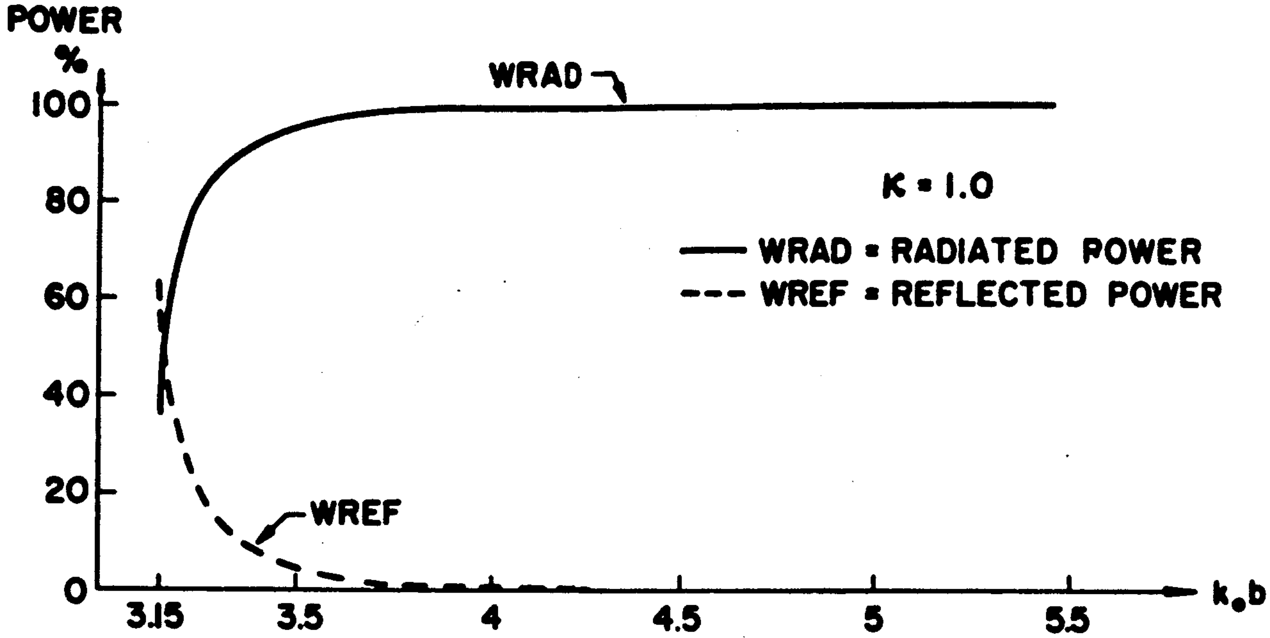


Fig. 10 Power distribution and far field patterns for an isotropic, incompressible, TE excited, plasma slab.



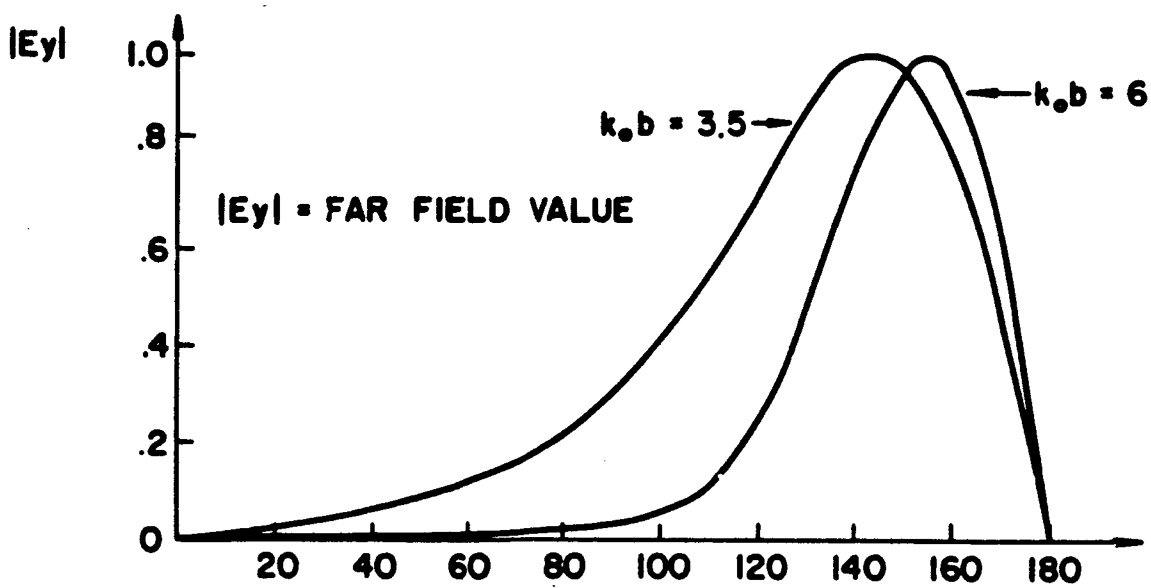


Fig. 11 Power distribution and far field patterns for a TE excited, parallel plate waveguide radiating in free space.

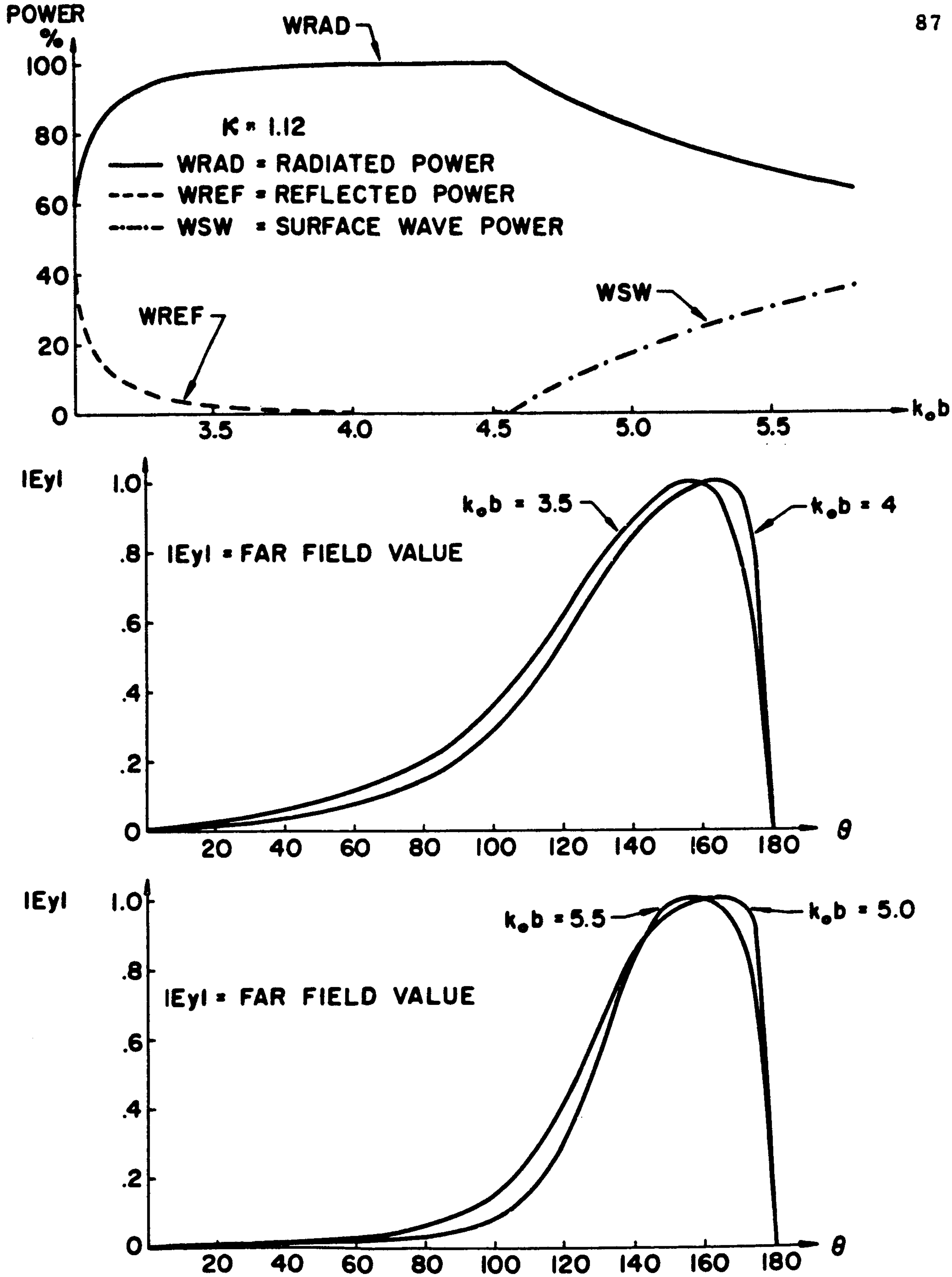


Fig. 12 Power distribution and far field patterns for a TE excited surface wave structure.

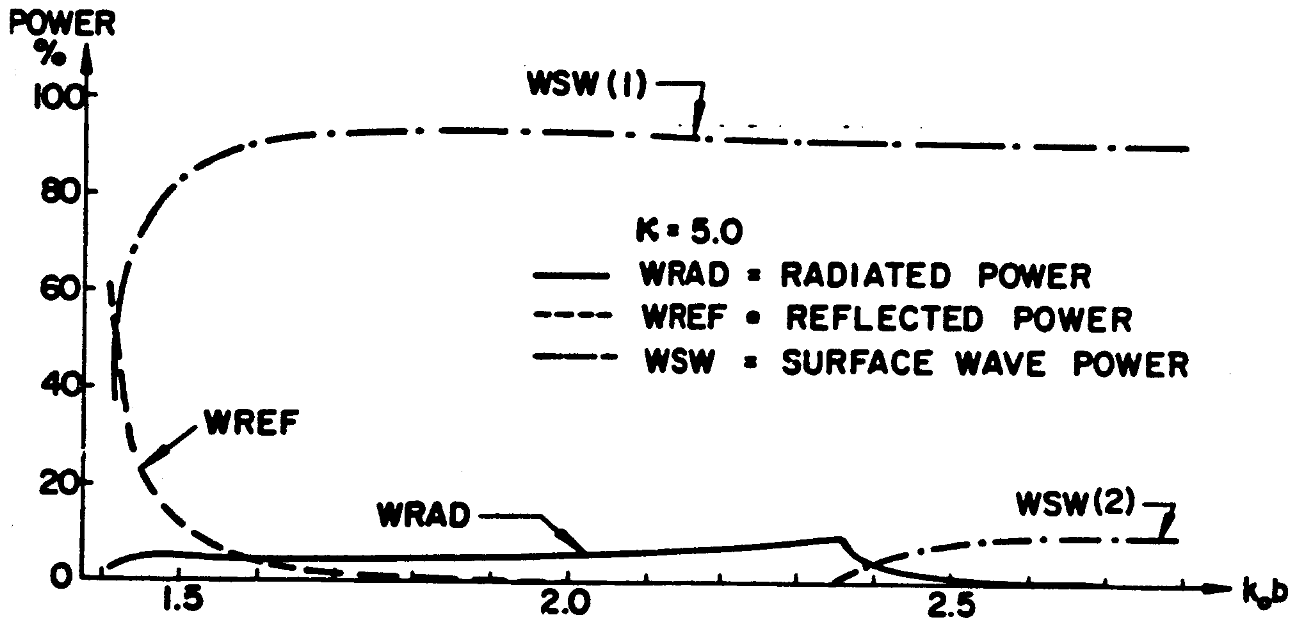


Fig. 13 Power distribution for a TE excited surface wave structure.

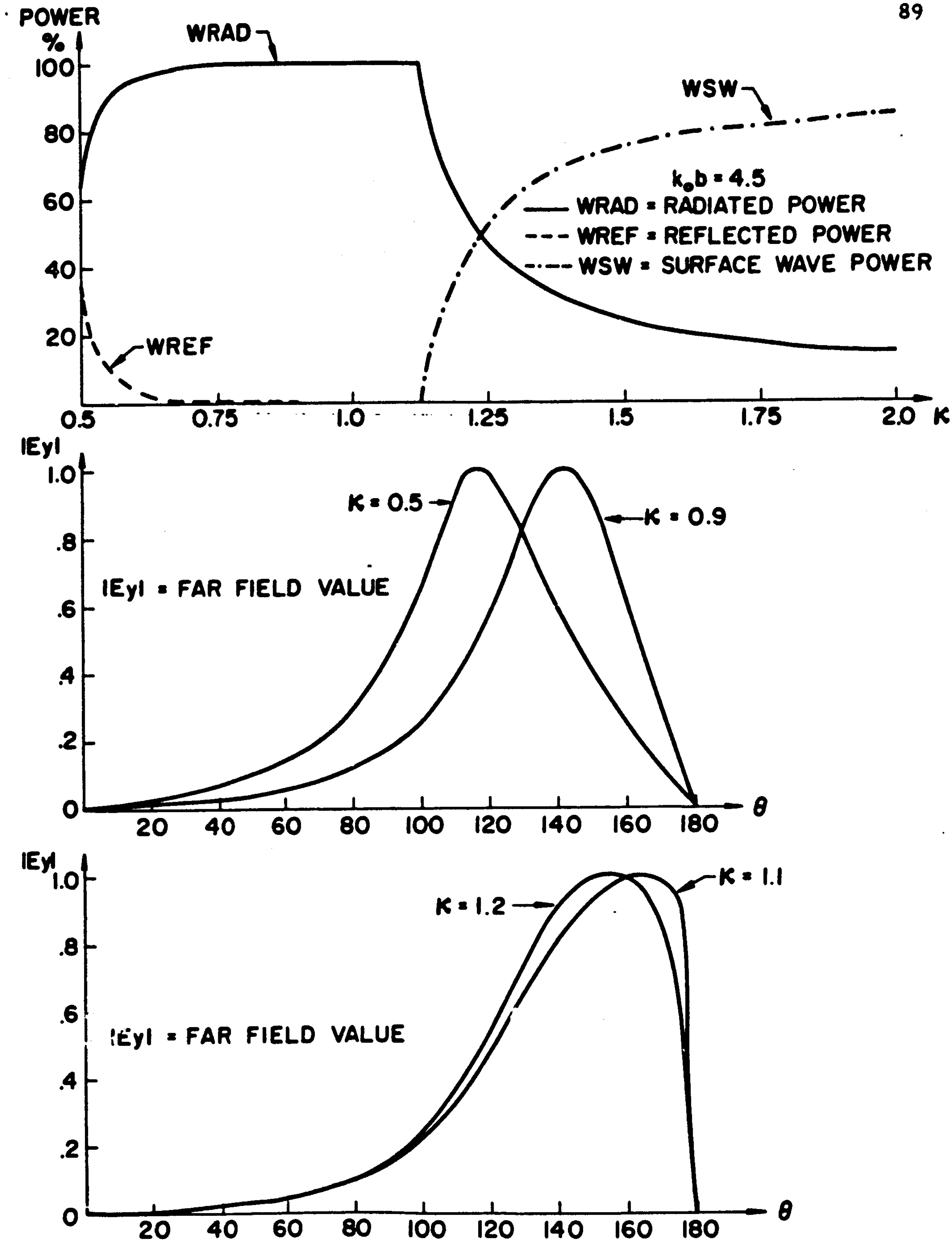


Fig. 14 Power distribution and far field patterns for a TE excited structure as a function of K.

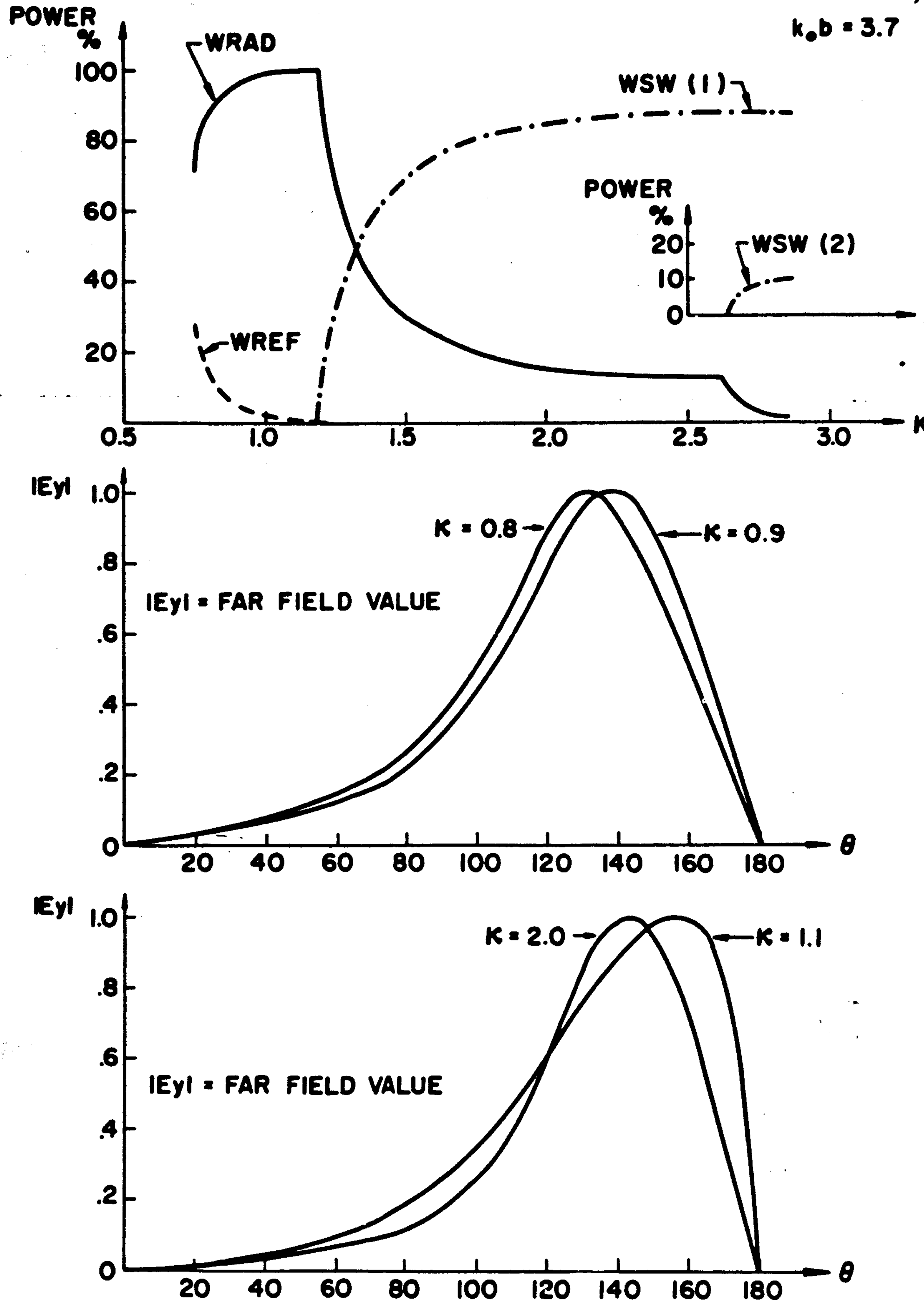


Fig. 15 Power distribution and far field patterns for a TE excited structure as a function of K.

face wave structure, shown in Fig. 12, until the structure starts to support surface waves. Once surface waves can be supported, a transfer of power from the space wave to the surface wave occurs. The structure will support surface waves only when $\sqrt{\kappa-1}\,k_o b > \pi/2$. The incident power may all be transferred to the surface waves by a suitable choice of parameters as seen in Fig. 13. There are no far field patterns plotted in Fig. 13 as the power radiated in this case is negligible compared to the power carried in the surface waves. Fig. 13 also shows the effect of two surface waves existing on the structure. This is the maximum number that the structure can support since we assumed that the parameters are such that only the lowest order mode can propagate in the waveguide. The effect of holding $k_o b$ constant and varying κ is shown in Figs. 14 and 15.

The response of the structure to a TEM polarized source is given in Figs. 16 to 19. The structure is again quite efficient as the reflected power may be made negligible with all the incident power radiated in the space wave (refer to Figs. 16 and 17). When κ is greater than one the structure supports a surface wave and again the power is transferred from the space wave to the surface wave as seen in Fig. 18. Only one surface wave is supported with the values of parameters that permit only the lowest order mode to propagate in the waveguide. The effect of varying κ is shown in Fig. 19.

A check on the algebra and computer results is possible by using the conservation of energy principle. The sum of the power radiated in the space wave, the power reflected in the waveguide, and the power carried by the surface waves (if any) must equal the incident power. This equality was obtained to within 0.5 per cent. This also gave a check on the far field patterns as the radiated power is equal to a constant times the integral of the square of the far field function. A verification of the numerical work for the radiated power is then an indirect verification of the far field pattern.

A comparison of the numerical results for the TEM excitation of the surface wave structure, given in Fig. 18, with those published by Angulo and Chang [1959] shows that they are not in agreement. The results given here should compare with their results with "h" set to zero. Their results show that the reflected power has a maximum at $k_o b$ approximately 1.25 with a corresponding minimum in the power radiated. Our results do not display these phenomena. Another paper by Angulo and Chang [1958] gives the results for a cylindrical geometry. The results published there have the functional form of our results given in Fig. 18. One would not expect the change in the geometry from cylindrical to rectangular to cause the change in response as found in their two papers.

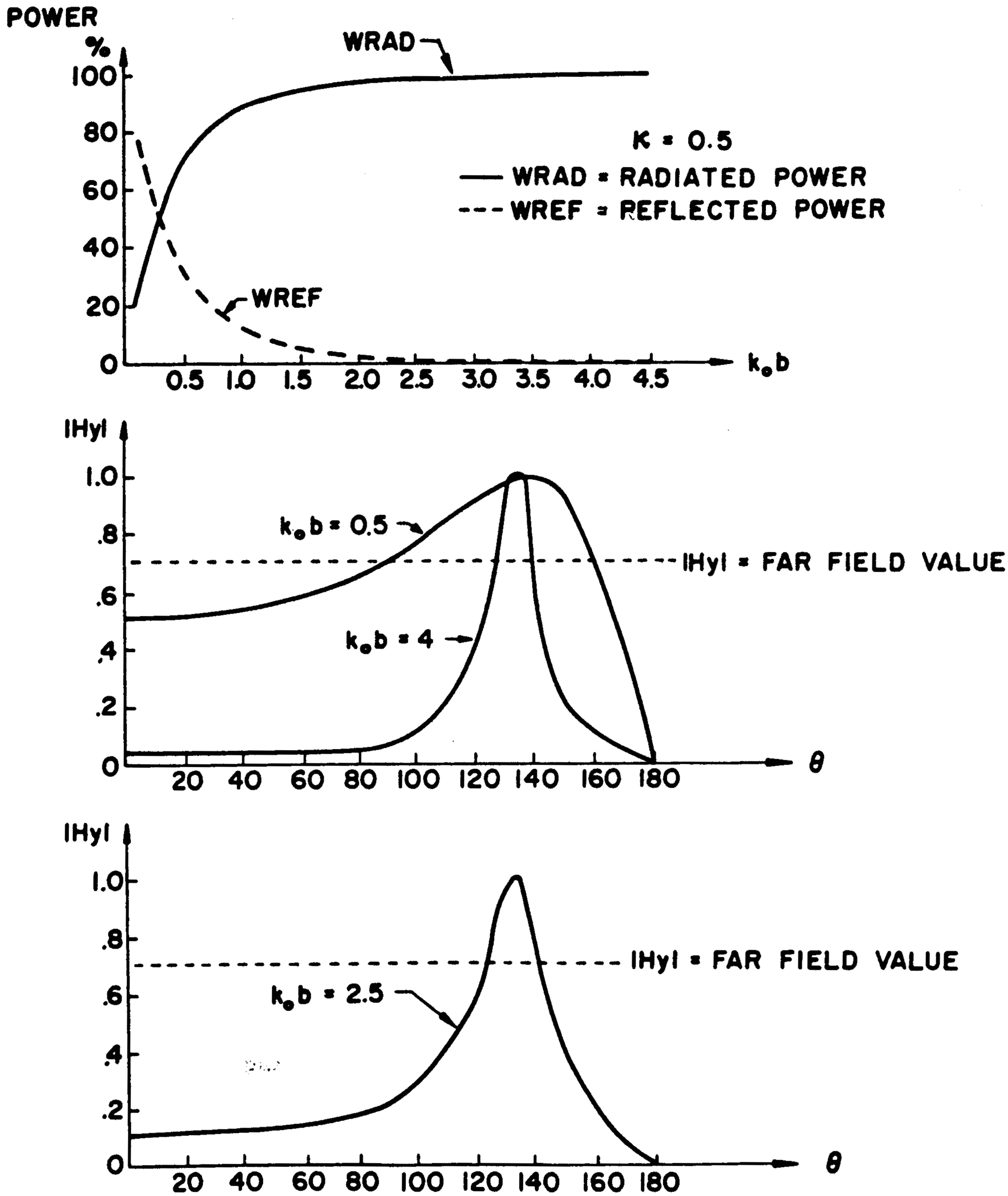


Fig. 16 Power distribution and far field patterns for an isotropic, incompressible, TEM excited, plasma slab.

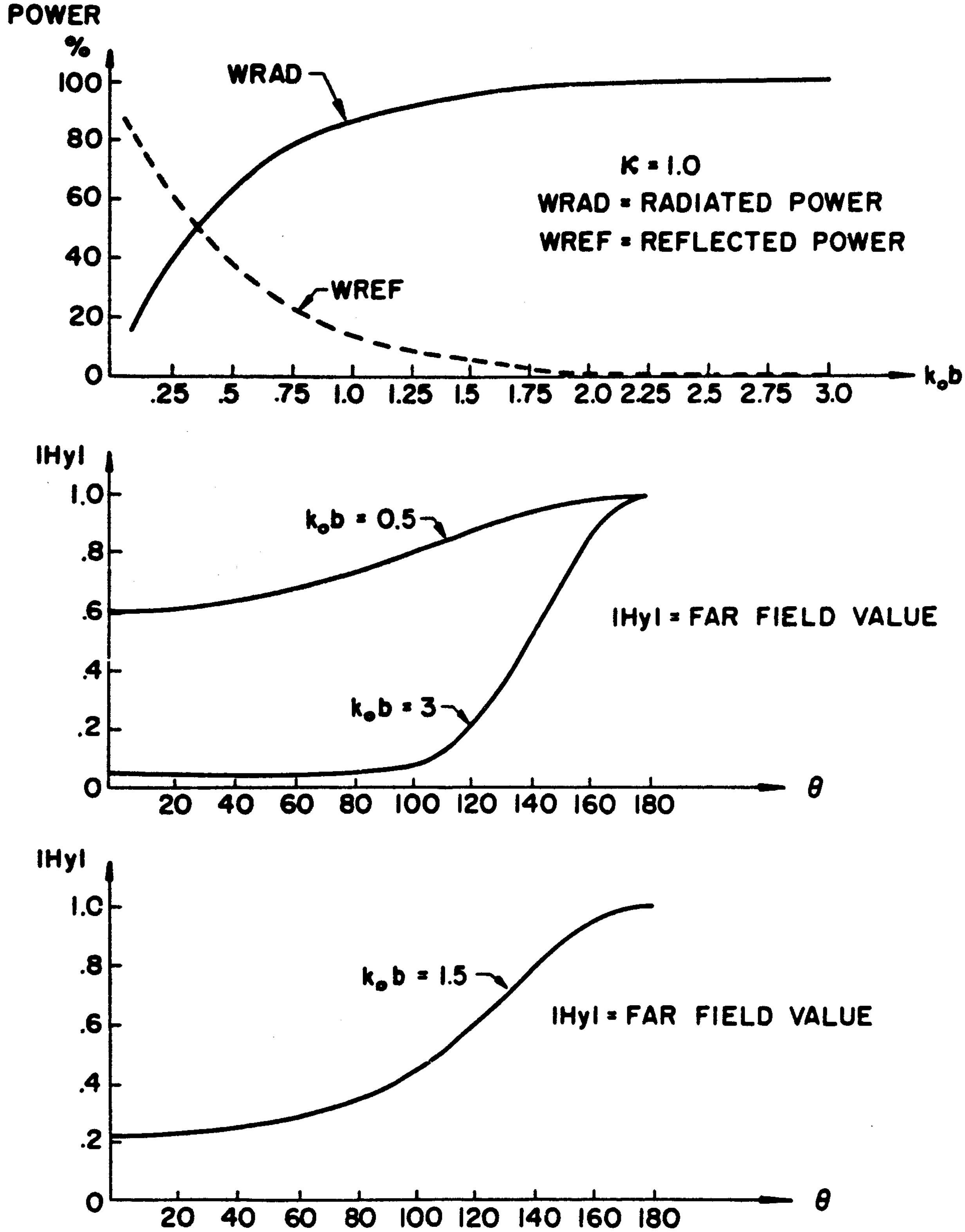


Fig. 17 Power distribution and far field patterns for a TEM excited parallel plate waveguide radiating in free space.

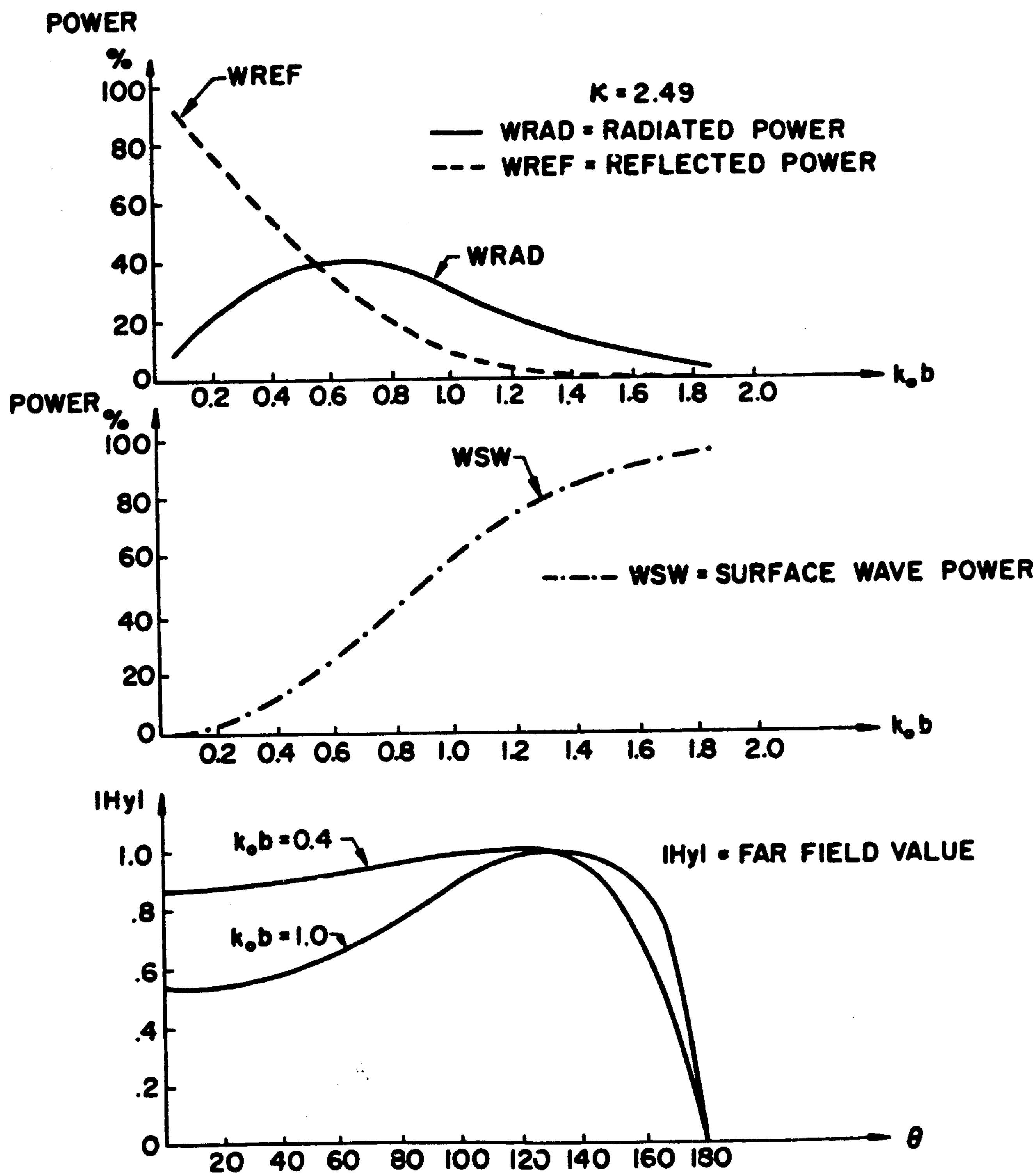


Fig. 18 Power distribution and far field patterns for a TEM excited surface wave structure.

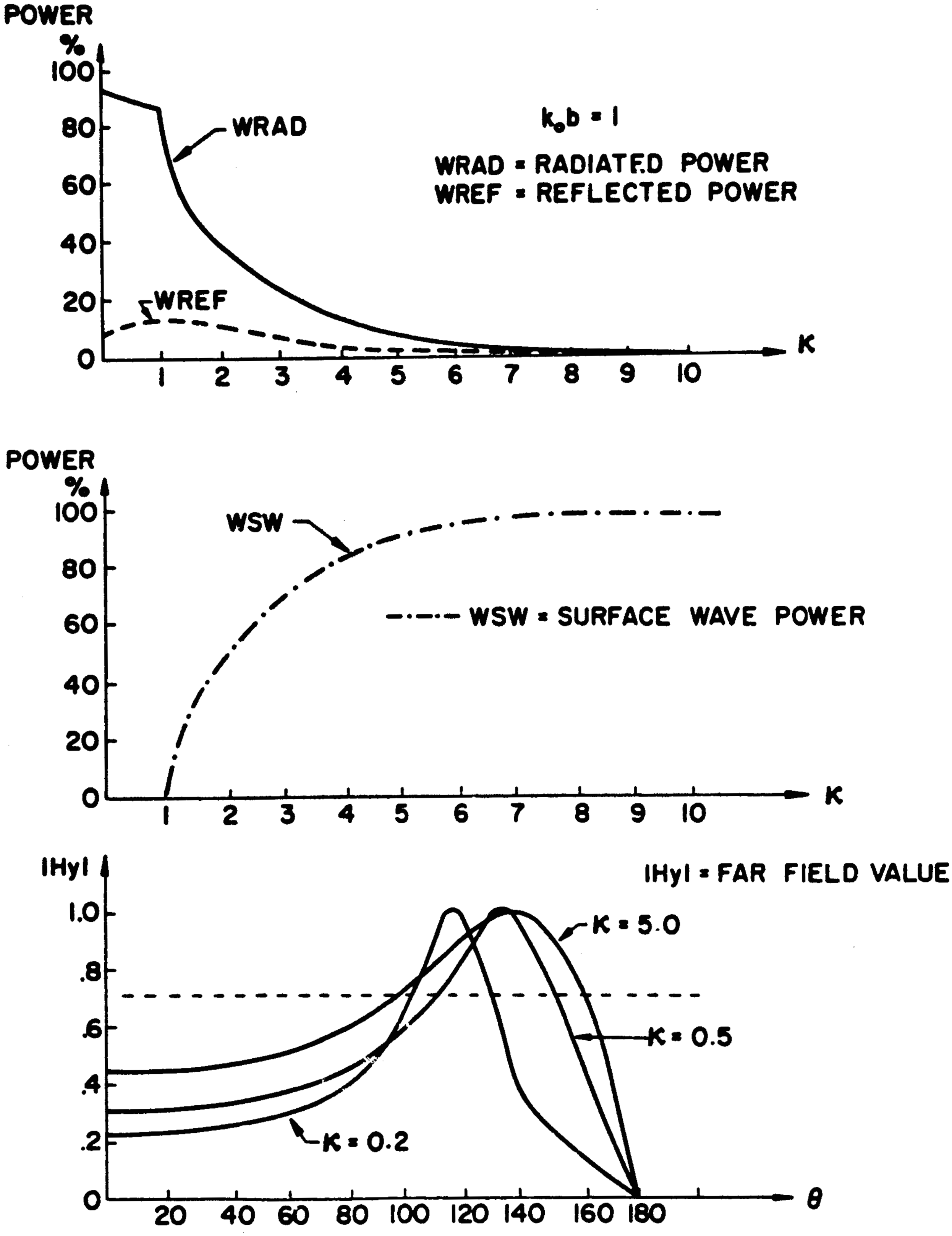


Fig. 19 Power distribution and far field patterns for a TEM excited structure as a function of *K*.

5. CONCLUSION

The solutions of three Wiener-Hopf type boundary value problems have been given in this work. The problem of a parallel plate waveguide with one plate truncated and radiating in free space was solved in section 2. A verification of the factorization was obtained by reference to the solution of a semi-infinite parallel plate waveguide radiating in free space given in Noble [1958]. The function to be factored is the same for both problems.

The excitation of a dielectric slab (surface wave structure) and the excitation of an incompressible, isotropic, plasma slab by means of a truncated parallel plate waveguide were given in section 3. Both TE and TEM polarizations of the exciting field were considered. The results for the TEM excitation of the surface wave structure were compared with those obtained by Angulo and Chang [1959] and the differences noted. They used a formal factorization procedure in their paper. The graphical results for the TE excitation of the surface wave structure and the graphical results for both TE and TEM excitations of the incompressible, isotropic, plasma slab presented in section 4 have not, to the best of the author's knowledge, been given elsewhere.

The factorization, one of the key steps, was obtained by a technique described in this work. The factorization was obtained by taking the limit, as the transverse dimension approaches infinity, of the function and factorization appropriate to the related closed-region structure. A closed form of the factorization was obtained only in section 2. In the more difficult problems discussed in section 3, the factorization was in a form convenient for numerical processing.

This technique for obtaining the factorization is certainly applicable to other open-region problems as discussed in section 2. 5. For example, this technique should prove useful for finding the electromagnetic fields associated with an incompressible, anisotropic, plasma slab when excited by a truncated parallel plate waveguide.

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APPENDIX

Region of Analyticity of a Function Given by an Integral Representation

Let $G(\alpha)$ be defined by equation (A-1)

$$G(\alpha) = \int_C g(\alpha, \zeta)\, d\zeta \tag{A-1}$$

The conditions under which $G(\alpha)$ is analytic are given in Noble [1958], page 11, and are presented here for convenience.

Theorem: Let $g(\alpha, \zeta) = f(\zeta) h(\alpha, \zeta)$ satisfy the conditions

(i) $h(\alpha, \zeta)$ is a continuous function of the complex variables α and ζ where α lies inside a region R and ζ lies on a contour C.

(ii) $h(\alpha, \zeta)$ is a regular function of α in R for every ζ on C.

(iii) $f(\zeta)$ has only a finite number of finite discontinuities on C and a finite number of maxima and minima on any finite part of C.

(iv) $f(\zeta)$ is bounded except at a finite number of points. If ζ_0 is such a point, so that $g(\alpha, \zeta) \to \infty$ as $\zeta \to \zeta_0$, then

$$\int_C g(\alpha, \zeta)\, d\zeta = \lim_{\delta \to 0} \int_{C-\delta} g(\alpha, \zeta)\, d\zeta$$

exists where the notation $(C - \delta)$ denotes the contour C apart from a small length δ surrounding ζ_0, and lim $(\delta \to 0)$ denotes the limit as this excluded length tends to zero. The limit must be approached uniformly when α lies in any closed domain R' within R.

(v) If C goes to infinity then any bounded part of C must be smooth and conditions (i) and (ii) must be satisfied for any bounded part of C. The infinite-integral defining $G(\alpha)$ must be uniformly convergent when α lies in any closed domain R' within R.

Then $G(\alpha)$ defined by (A-1) is a regular function of α in R.

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Unclassified
SECURITY CLASSIFICATION

DOCUMENT CONTROL DATA - R & D

1. ORIGINATING ACTIVITY University of Illinois Urbana, Illinois	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
A TECHNIQUE FOR SOLVING CERTAIN WIENER-HOPF TYPE BOUNDARY VALUE PROBLEMS

4. DESCRIPTIVE NOTES
Technical Report No. 9 - Interim

5. AUTHOR(S)
Bates, C. P. and Mittra, R.

6. REPORT DATE May 1966	7a. TOTAL NO. OF PAGES 104	7b. NO. OF REFS. 28

8a. CONTRACT OR GRANT NO. AF19(628)-3819 b. PROJECT AND TASK NUMBERS 5635, 02 c. DOD ELEMENT 61445014 d. DOD SUBELEMENT 681305	9a. ORIGINATOR'S REPORT NUMBER(S) Antenna Laboratory Report No. 66-4 9b. OTHER REPORT NO(S). AFCRL - 66-352

10. AVAILABILITY/LIMITATION NOTICES
"Distribution of this document is unlimited."

11. SUPPLEMENTARY NOTES This research was supported in part by the National Aeronautics and Space Administration.	12. SPONSORING MILITARY ACTIVITY HQ, Air Force Cambridge Research Laboratories, OAR(CRD) U. S. Air Force L. G. Hanscom Field, Bedford, Mass.

13. ABSTRACT

The time-harmonic analysis of three boundary value problems containing semi-infinite boundaries is presented. The first problem considered is a parallel plate waveguide with one plate truncated and radiating into free space. The excitation of a dielectric slab and the excitation of an isotropic, incompressible, plasma slab by means of a parallel plate waveguide with one plate truncated are the second and third problems analyzed, respectively.

A function of a complex variable is factored in each of these Wiener-Hopf type boundary value problems. The function is analytic in a strip and is factored into a product of two functions. One of these functions is analytic in a half-plane while the other is analytic in the adjacent half-plane with an overlap in the regions of analyticity coinciding with the strip. This factorization is obtained by a technique developed in this work.

The technique obtains the factorization for the open-region problem from a function and its factorization that occurs in a related closed-region problem. A closed-region problem is one whose transverse dimensions are finite. The chosen closed-region boundary value problem yields a function of a complex variable which can be factored. The factorization of the function for the open-region boundary value problem is obtained by taking the limit, as a parameter approaches infinity,

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SECURITY CLASSIFICATION

Unclassified
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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Wiener-Hopf Type Boundary Value Problems						
Truncated Parallel Plate Waveguide						
Surface Wave Structure						
Dielectric Slab						
Incompressible, Isotropic, Plasma Slab						
Open-Region Structure						
Closed-Region Structure						
Factorization Procedure						
Surface Waves						
Far Field Patterns						
Entire Functions						
Analytic Continuation						

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13. ABSTRACT (continued)

of the function and factorization appropriate to the closed-region structure. By this means the factorization and hence the solution to the open-region boundary value problem is obtained.

It is also found that the limiting procedure may be used to obtain more than just the open-region factorization. It is shown that the limit of the complete closed-region solution becomes the open-region solution. Hence, this yeilds one possible method for the solution of problems of this type.

The results of the numerical computations are presented. These include the average power reflected in the waveguide, the average power radiated in the space wave, the average power transmitted by the surface waves, and the radiation pattern of the space wave.