THE PLASMA CAPACITOR IN A MAGNETIC FIELD

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ABSTRACT

This paper extends previous work on the impedance of a parallel plate capacitor, filled with warm plasma, to take account of a static magnetic field. Computations are presented for parallel plate and coaxial cylinder geometries, and indicate series of resonances occurring at, and between, the electron cyclotron frequency harmonics. It is suggested that if the theoretical results were confirmed experimentally, plasma admittance measurements would form the basis of a simple and powerful diagnostic technique for measuring plasma parameters such as electron density and temperature in the laboratory and in the ionosphere.

Author

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CONTENTS

LIST OF FIGURES

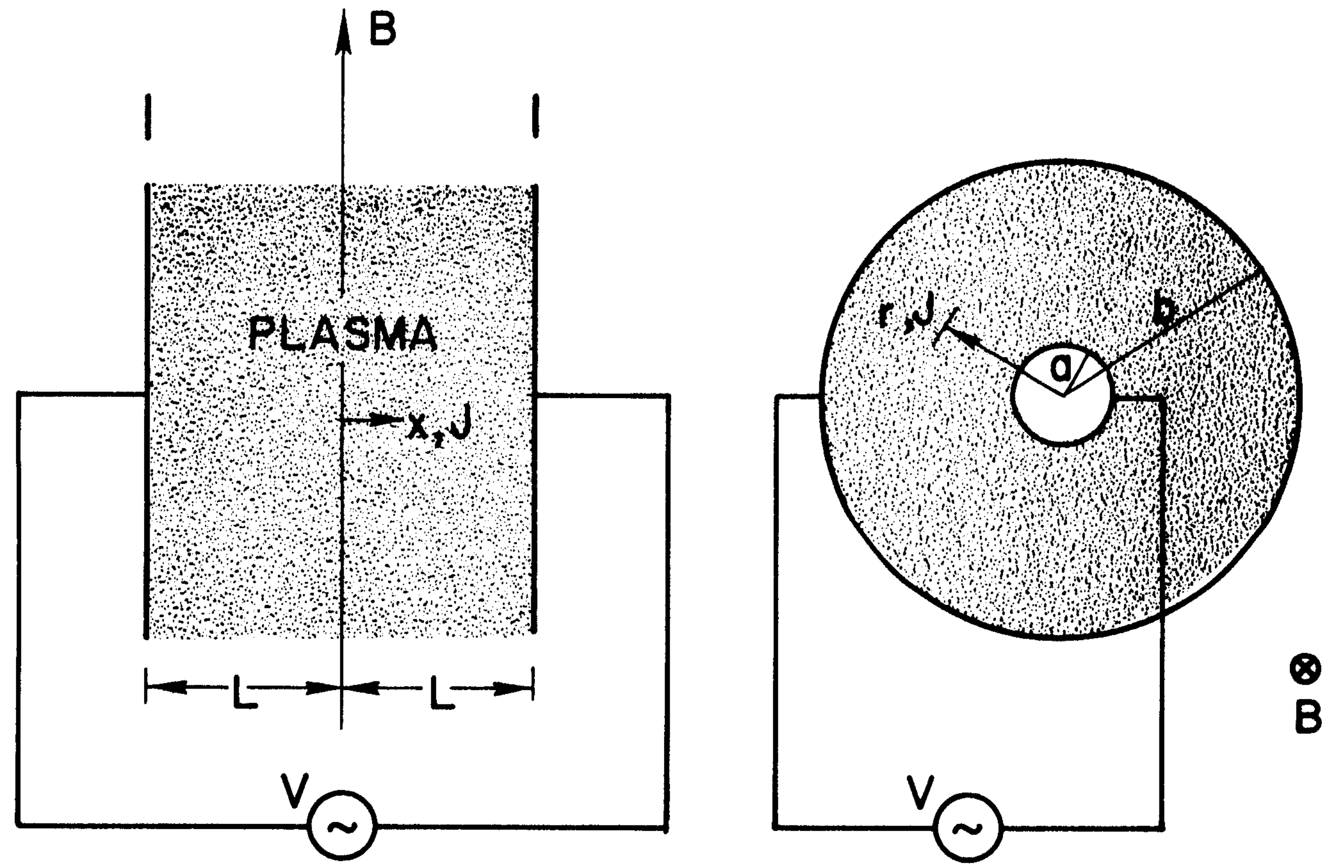

Fig. 1. Parallel plate and coaxial cylinder plasma capacitor geometries.

§ 1 INTRODUCTION

Several previous analyses have concerned the impedance of a parallel plate capacitor containing plasma, in the absence of a magnetic field (Wolff 1956, 1958; Vandenplas and Gould 1961; Weissglas 1962; Hall 1963; Shure 1964). The object of this paper is to extend the theory to situations where a static magnetic field is present, parallel to the metallic boundaries. The studies are restricted to geometries in which the wave equation is separable. This implies a simple set of normal modes, and the occurrence of series of resonance frequencies. Specifically, the two cases to be examined are parallel plates, and infinitely long coaxial cylinders, as shown in fig. 1.

In these problems, equivalent plasma permittivity concepts play an important role. For example, for a capacitor of arbitrary geometry filled with homogeneous cold plasma, the impedance, Z_p , at frequency ω , is simply (Z_0/ϵ_p) , where Z_0 is the impedance with free space as the dielectric, the equivalent plasma permittivity, ϵ_p is given by $\left[1 - (\omega_p^2/\omega^2)\right]$, ω_p is the electron plasma frequency, and the static magnetic field strength, B , is zero. Of far more practical and theoretical interest are cases in which the electron temperature, T_e , is non-zero. The equivalent plasma permittivity is then dispersive, i.e. its variation depends not only on ω , but also on the wave-number, $\underset{\sim}{k}$ (Stix 1962).

The case $T_e \neq 0$, $B = 0$ has been approached from several different directions, and an excellent tutorial paper by Hall (1963) describes and contrasts these. Basically, the treatment may be either microscopic, in which case it is founded directly on the Boltzmann equation, or it may employ macroscopic equations derived by taking moments of the Boltzmann equation. To take a particular case, the latter approach yields the following mathematically equivalent solutions for the normalized unit area impedance, Z , of a parallel plate plasma capacitor,

$$Z(\omega) = \frac{Z_p(\omega)}{Z_0(\omega)} = \left[\frac{1 - \frac{\omega_p^2}{\omega^2}\left(\frac{\tan kL}{kL}\right)}{1 - (\omega_p^2/\omega^2)}\right] = \frac{8}{\pi^2} \sum_{n\ \mathrm{odd}}^{\infty} \frac{1}{n^2\ \epsilon_p(k_n, \omega)} , \quad (1)$$

$$\epsilon_p\ (k_n, \omega) = 1 - \frac{\omega_p^2}{\omega^2 - k_n^2 v_T^2} , \quad (2)$$

Where $Z_0(\omega) = (1/i\omega C_0)$; $C_0 = (\epsilon_0/2L)$; 2L is the plate separation; $v_T = (\gamma\kappa T_e/m_e)^{1/2}$; γ is the adiabatic compression constant for the electron gas (=3, 2 and 5/3 in 1,2, and 3 dimensions); $k_n = (n\pi/2L)$, and k is obtained from,

$$0 = 1 - \frac{\omega_p^2}{\omega^2 - k^2 v_T^2} . \quad (3)$$

It is clear that resonances occur when $kL = [(2r + 1)\ \pi/2]$ $(r = 0, 1, \ldots)$, i.e. when $(2L/\lambda_D)\ [(\omega^2/\omega_p^2) - 1]^{1/2} = (2r + 1)\pi$, where $\lambda_D = (v_T/\omega_p)$ is approximately the electronic Debye length, $[(\kappa T_e/m_e)^{1/2}/\omega_p]$. Since, in practice, the quantity $(2L/\lambda_D)$ must be a large number so as to satisfy the definition of a plasma, the implication is that a very closely spaced series of resonances should be observed for $\omega \gtrsim \omega_p$.

Equation (1) has been criticized on the grounds that it ignores collisionless damping, an effect which may be caused by electrons in the unperturbed velocity distribution, $f_0(v)$, whose transit time between the plates is an odd number of half-periods of the applied signal, and which are consequently continuously accelerated by the electric field (Weissglas 1962; Hall 1963; Shure 1964). This phenomenon can only be described correctly by the microscopic theory. A summation similar to that of eqn. (1) is again obtained but with,

$$\epsilon_p(k_n,\omega) = 1 + \omega_p^2 \int_{-\infty}^{\infty} \frac{v(\partial f_0/\partial v)}{\omega^2 - k_n^2 v^2} dv \quad . \tag{4}$$

The residues at the poles in the integrand introduce loss analogous to Landau damping (Stix 1962), and would certainly make observation of the resonances difficult, even if a sufficiently homogeneous plasma could be realized. In the laboratory, however, inhomogeneities change the phenomena considerably: The separation between the resonances increases, and they are observable as the well-known "Tonks-Dattner" series of peaks. Several detailed numerical analysis of these have been carried out successfully using both macroscopic and microscopic theories (Parker *et al*. 1964; Crawford 1964; Leavens 1965; Harker 1965).

The foregoing discussion serves to explain the motivation for extending the microscopic theory to the case with $T_e \neq 0$, $B \neq 0$. In this situation, the resonant modes of the system are longitudinal cyclotron harmonic waves which do not suffer collisionless damping (Bernstein 1958). They have the additional feature that their propagation characteristics tend to be influenced less by inhomogeneity of the plasma than by that in B , which can easily be controlled experimentally. The implication is that a more readily realizable experimental test of the theory may be possible for the warm magnetoplasma dielectric. The relevant theory, and some computations, will be presented in § 2, and discussed briefly in § 3. It should be emphasized at the outset that an exact description of the plasma behavior, particularly near boundaries, would be extremely complicated. Many assumptions will be made, and although the final results are plausible physically, their degree of validity requires to be established by experiment.

§ 2 THEORY

2.1 Parallel Plate Capacitor

The total conventional current density, J_0 , is continuous and consists of the plasma convection current density, J_p , and the displacement current density, J_d . In the plasma region, $-L < x < L$, we have

$$- J_o(\omega) = J_p(x,\omega) + i\omega\epsilon_0 E(x,\omega) \quad . \tag{5}$$

Use of the equivalent permittivity forms derived for an infinite plasma is usually justified by imaging, i.e. setting up an infinite series of parallel plates with period $2L$. Successive plates are driven in antiphase so that the total current density has a rectangular waveform in space (Hall,1963). This procedure is formally equivalent to using the finite sine transform (Irving and Mullineux 1959),

$$f(k_n) = \int_{-L}^{+L} f(x) \cos k_n x \, dx \quad , \quad f(x) = \frac{1}{L} \sum_{n=1}^{\infty} f(k_n) \cos k_n x \ . \tag{6}$$

Application of this to eqn (5) yields,

$$- J_0(k_n,\omega) = J_p(k_n,\omega) + i\omega \epsilon_0 E(k_n,\omega) = i\omega \epsilon_0 \epsilon_\perp(k_n,\omega)E(k_n,\omega) \ , \tag{7}$$

where we have introduced the equivalent plasma permittivity component perpendicular to the magnetic field, $\epsilon_\perp(k_n,\omega)$ (Stix 1962). Taking collisions into account approximately (Tataronis and Crawford 1965), this is given by

$$\epsilon_\perp(k_n,\omega) = 1 - \left(1 - \frac{i\nu}{\omega}\right)\frac{\omega_p^2}{\omega_c^2} \sum_{m=1}^{\infty} \frac{\exp(-\lambda_n)I_m(\lambda_n)}{\frac{\lambda_n}{2}\left[\left(\frac{\omega - i\nu}{m\omega_c}\right)^2 - 1\right]} \quad . \tag{8}$$

Here ν is the electron/neutral momentum transfer collision frequency; $\lambda_n = (k_n R)^2$; ω_c is the electron cyclotron frequency, and $R = \left[(\kappa T_e/m_e)^{1/2}/\omega_c\right]$ is the Larmor radius of an electron with thermal energy. Equation (8) is appropriate to a Maxwellian electron velocity distribution. Ion motions have been neglected. For the expression to be valid, it is required that $R \ll L$.

Rearranging eqn (7), and inverting the transform, yields

$$E(x,\omega) = -\frac{1}{i\omega\epsilon_0 L}\sum_{n=1}^{\infty}\frac{J_0(k_n,\omega)\cos k_n x}{\epsilon_\perp(k_n,\omega)} , \tag{9}$$

which may be integrated to give the potential,

$$V(\omega) = \frac{4}{i\omega\epsilon_0\pi}\sum_{n=1}^{\infty}\frac{J_0(k_n,\omega)\sin\left(\frac{n\pi}{2}\right)}{n\,\epsilon_\perp(k_n,\omega)} , \tag{10}$$

Applying eqn. (6) we have,

$$J_0(k_n,\omega) = \frac{2}{k_n}\sin\left(\frac{n\pi}{2}\right) , \tag{11}$$

which may be substituted in eqn. (10) to yield, finally,

$$Z(\omega) = \frac{8}{\pi^2}\sum_{n\ \mathrm{odd}}\frac{1}{n^2\,\epsilon_\perp(k_n,\omega)} , \tag{12}$$

As in the previous work described in § 1, it is implicit in the analysis leading to this result that a boundary condition of specular reflection of the electrons has been assumed.

Two points may be observed directly from eqn. (12). First, there will be a series of plasma resonances occurring at the cyclotron harmonics, where the normalized impedance goes to zero and, second, that wherever the condition $\epsilon_{\perp}(k_n, \omega) = 0$ is satisfied, there are geometric resonances and the impedance will be infinite. These are just the frequencies for which cyclotron harmonic waves propagate. The resonances occur due to a coupling between the forced field and standing cyclotron harmonic waves. For frequencies and wave numbers which do not satisfy the dispersion relation, the fields established within the plasma may be understood as the superposition of an infinite number of longitudinal waves. As the operating frequency is made more nearly equal to a resonance frequency, a single wave component will dominate the summation. Precisely at resonance, the coupling is optimum between the forcing field and a particular wave component.

It will be remarked that eqn. (12) shows no collisionless damping. This is so because the electrons gyrate in the magnetic field, returning to the same position every cyclotron period. There is no group traveling backwards and forwards between the plates, as was the case for zero magnetic field. For propagation at angles other than exact perpendicularity to the magnetic field, very strong cyclotron and Landau damping effects are to be expected.

The impedance described by eqn. (12) can be expressed as an equivalent circuit. It consists of an infinite set of impedances in series, a typical member of which is

$$Z_{pn} = \frac{1}{i\,\omega C_0} \left[\frac{8}{n^2\,\pi^2\,\epsilon_{\perp}(k_n, \omega)}\right] . \tag{13}$$

This can be further broken down as shown in fig. 2.

2.2 Coaxial Cylinder Capacitor

The analysis follows similar lines to that of the previous section except that the appropriate transform is now the finite Hankel transform

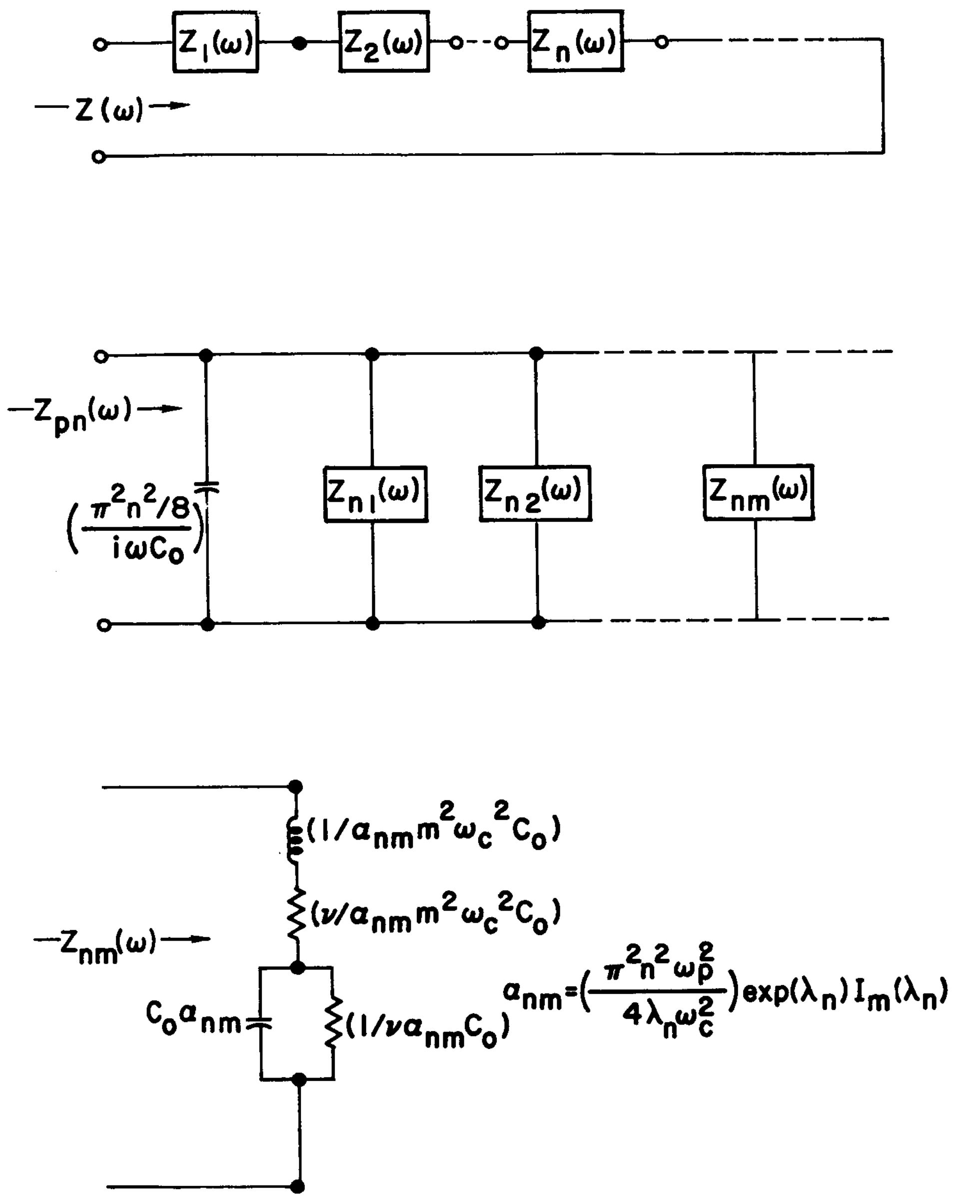

Fig. 2. Equivalent circuit representation of normalized, unit area impedance, $Z(\omega)$, of a parallel plate capacitor filled with a warm magnetoplasma.

(Sneddon 1951) defined by,

$$f(k_n) = \int_a^b f(r)\, B_0(k_n r)\, r\, dr \quad , \quad f(r) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{k_n^2 f(k_n)\, B_0(k_n r)}{\left\{ \dfrac{J_0^2(k_n a)}{J_0^2(k_n b)} - 1 \right\}} \quad , \tag{14}$$

where the function $B_0(k_n r)$ is given by,

$$B_0(k_n r) = J_0(k_n r)\, N_0(k_n a) - J_0(k_n a)\, N_0(k_n r) \quad . \tag{15}$$

J_0 and N_0 are the zero-order Bessel functions of the first and second kind, respectively. In this section, k_n is defined by the roots of $B_0(k_n, b) = 0$.

In place of eqn. (5), we have for the current/unit length, I_0 , the expression,

$$- \frac{I_0(\omega)}{2\pi r} = J_p(r,\omega) + i\omega\, \epsilon_0 E(r,\omega) \quad . \tag{16}$$

Applying the transformation; solving for $E(k_n, \omega)$; inverting the transform, and integrating to obtain the potential, exactly as before, yields the following expression for the normalized impedance,

$$Z = \frac{\pi^2}{2 \ln(b/a)} \sum_{n=1}^{\infty} \frac{\left\{ \int_a^b B_0(k_n r)\, d(k_n r) \right\}^2}{\left\{ \dfrac{J_0^2(k_n a)}{J_0^2(k_n b)} - 1 \right\} \epsilon_\perp(k_n, \omega)} \tag{17}$$

where for this geometry $C_0 = \left[2\pi\epsilon_0/\ln(b/a)\right]$, and $\epsilon_\perp(k_n,\omega)$ is again given by eqn. (8). It may easily be shown that if $a \to \infty$, $b \to \infty$, and $(b - a) = 2L$, eqn. (17) reduces correctly to the parallel plate case of eqn. (12).

2.3 Computations

For the computations, it is convenient to put the permittivity into the integral form (Crawford 1965),

$$\epsilon_\perp(k_n,\omega) = 1 + \left(1 - \frac{i\nu}{\omega}\right)\frac{\omega_p^2}{\omega_c^2}\int_0^\pi \frac{\sin\left[(\omega - i\nu)\phi/\omega_c\right]\sin\phi\,\exp\left[-2\lambda_n\cos^2(\phi/2)\right]d\phi}{\sin\left[(\omega - i\nu)\pi/\omega_c\right]} \tag{18}$$

This has been used in conjunction with eqns. (2) and (17) to obtain the normalized admittance plots for parallel plate geometry shown in figs. 3a and b. In the figures, the shaded zones mark regions where the density of resonance peaks was very high and detailed computations were not carried out. Admittance has been chosen for this representation, rather than impedance since an experiment would probably be carried out with V constant. The admittance plots then indicate the rf current variation to be expected.

Figure 3(a) shows clearly the effect of increasing (ω_p^2/ω_c^2) . The number of resonances in each passband increases rapidly with this parameter. It will be noted that the resonances are extremely sharp. The implication is that effects such as magnetic field inhomogeneity or collisions would tend to damp the resonances strongly. Figure 3(b) demonstrates this clearly for a relatively low collision frequency. Only the resonances in the first passband remain well marked.

In any experimental verification of the theory, it would be more usual to maintain ω constant while the discharge current and magnetic field were varied. This implies that plots of the admittance variation with (ω_c/ω) for different values of (ω_p^2/ω_c^2) would be obtained. The computed forms of these are shown in figs. 4 and 5, and again indicate the powerful influence of collisions in damping the resonances.

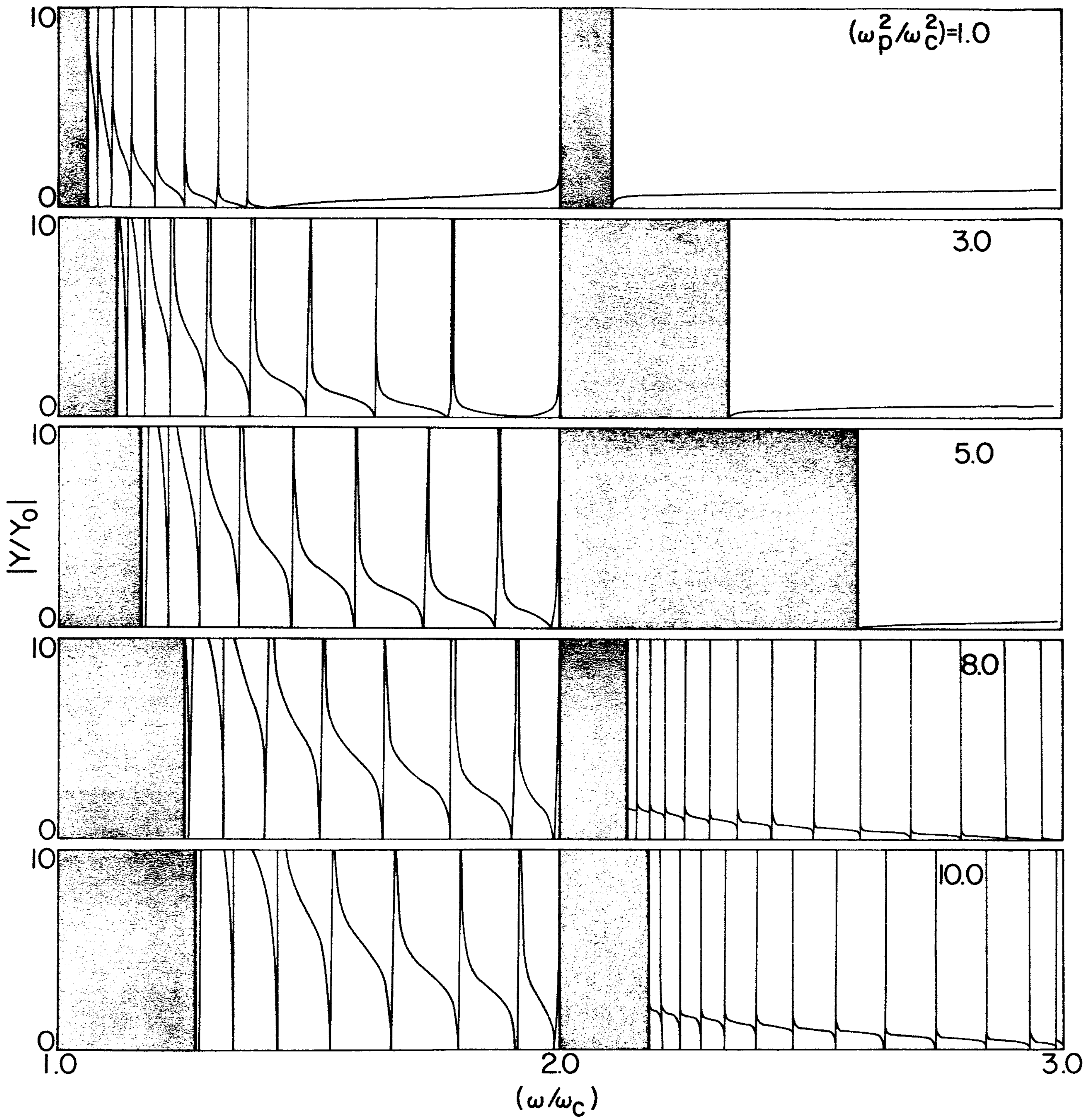

Fig. 3(a). Parallel plate capacitor - Normalized admittance variation with (ω_c/ω) and $(\omega_p^{\ 2}/\omega_c^{\ 2})$ for $(\pi v_T/2L\omega_c) = 0.1$,

(a) $(\nu/f_c) = 0$

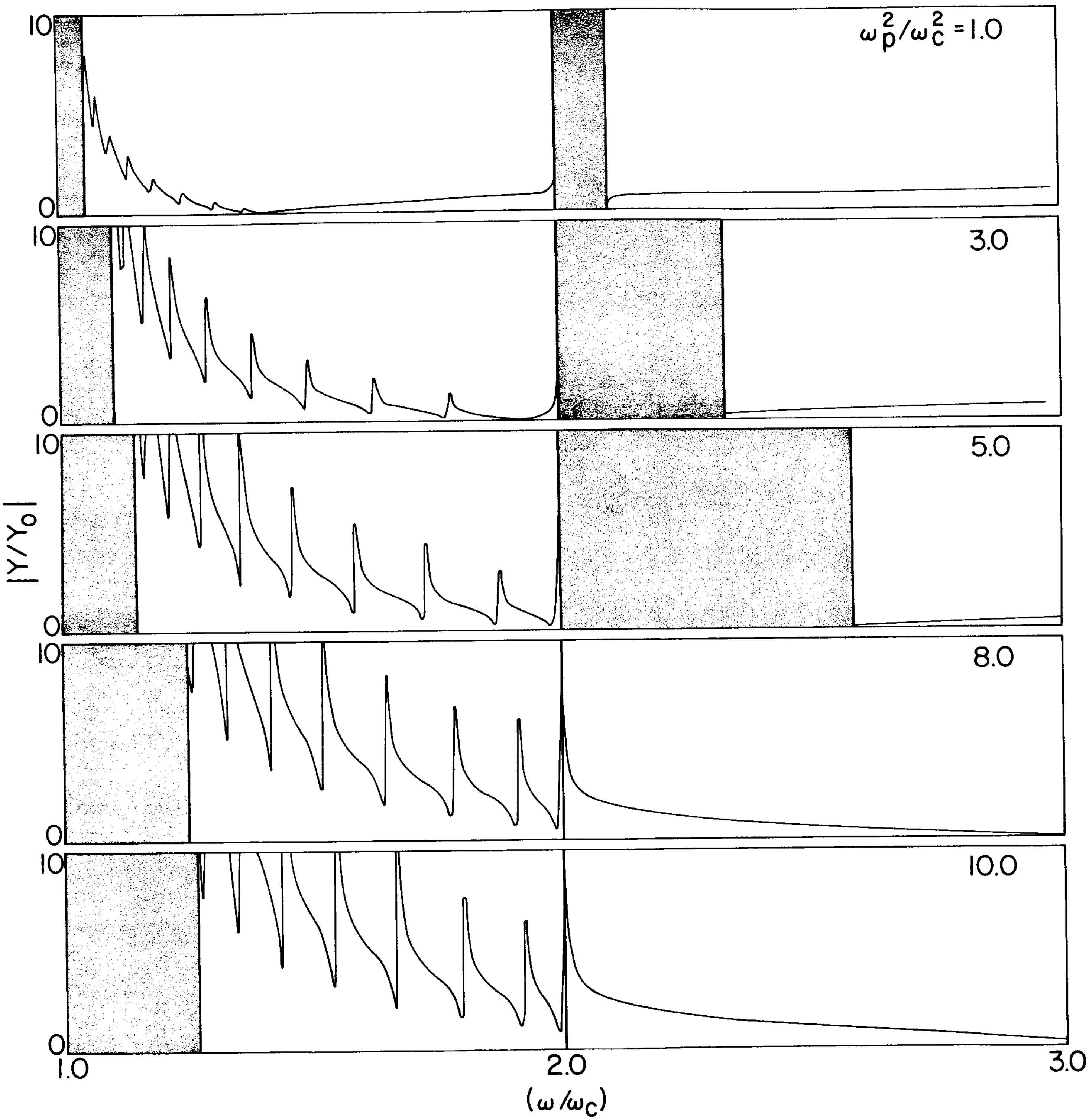

Fig. 3(b). Parallel plate capacitor - Normalized admittance variation with (ω_c/ω) and $(\omega_p{}^2/\omega_c{}^2)$ for $(\pi v_T/2L\omega_c) = 0.1$,

(b) $(\nu/f_c) = 0.003$

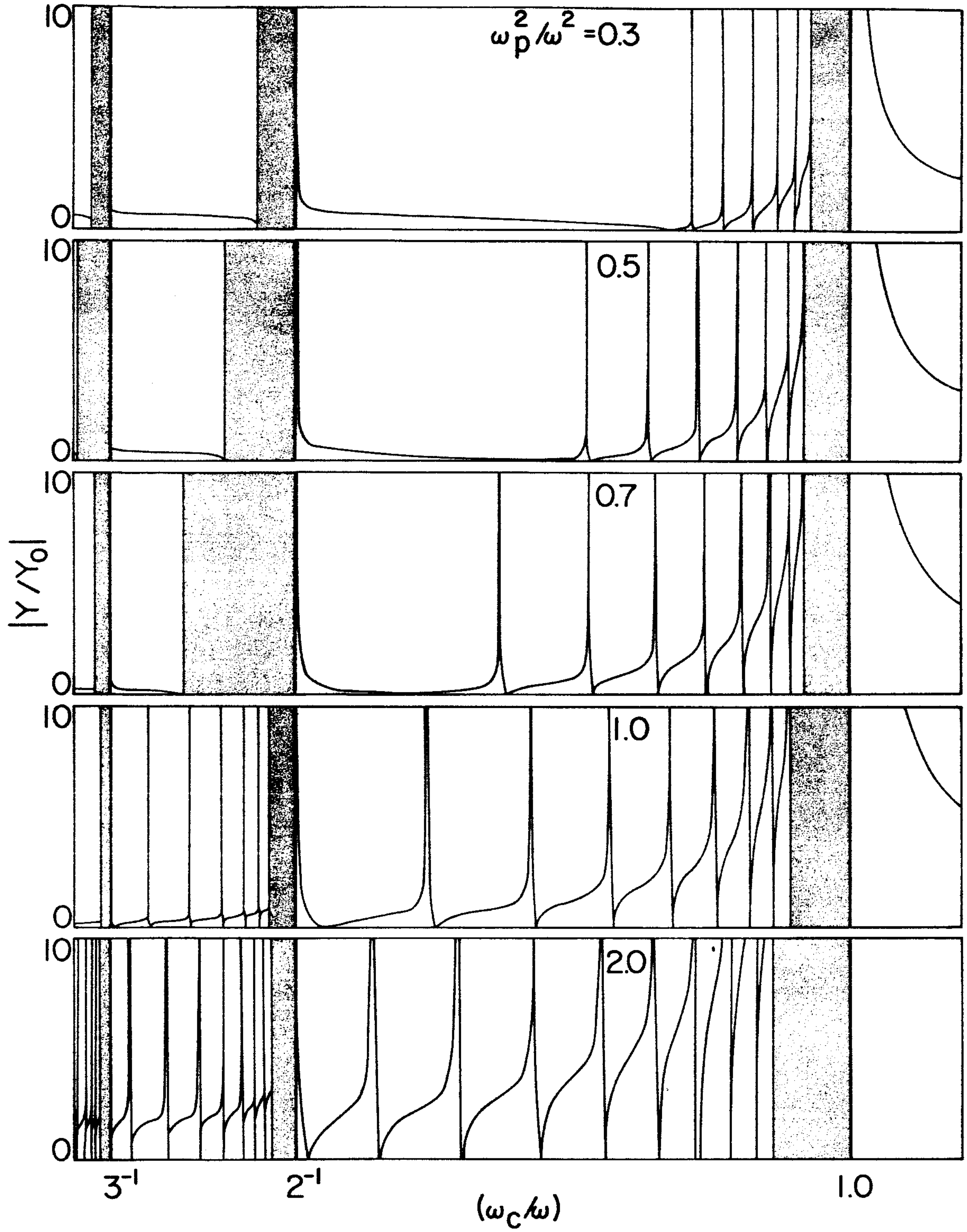

Fig. 4(a). Parallel plate capacitor - Normalized admittance variation with (ω_c/ω) and (ω_p^2/ω^2) for $(\pi v_T/2L\omega) = 0.1$, (a) $(\nu/f_c) = 0$

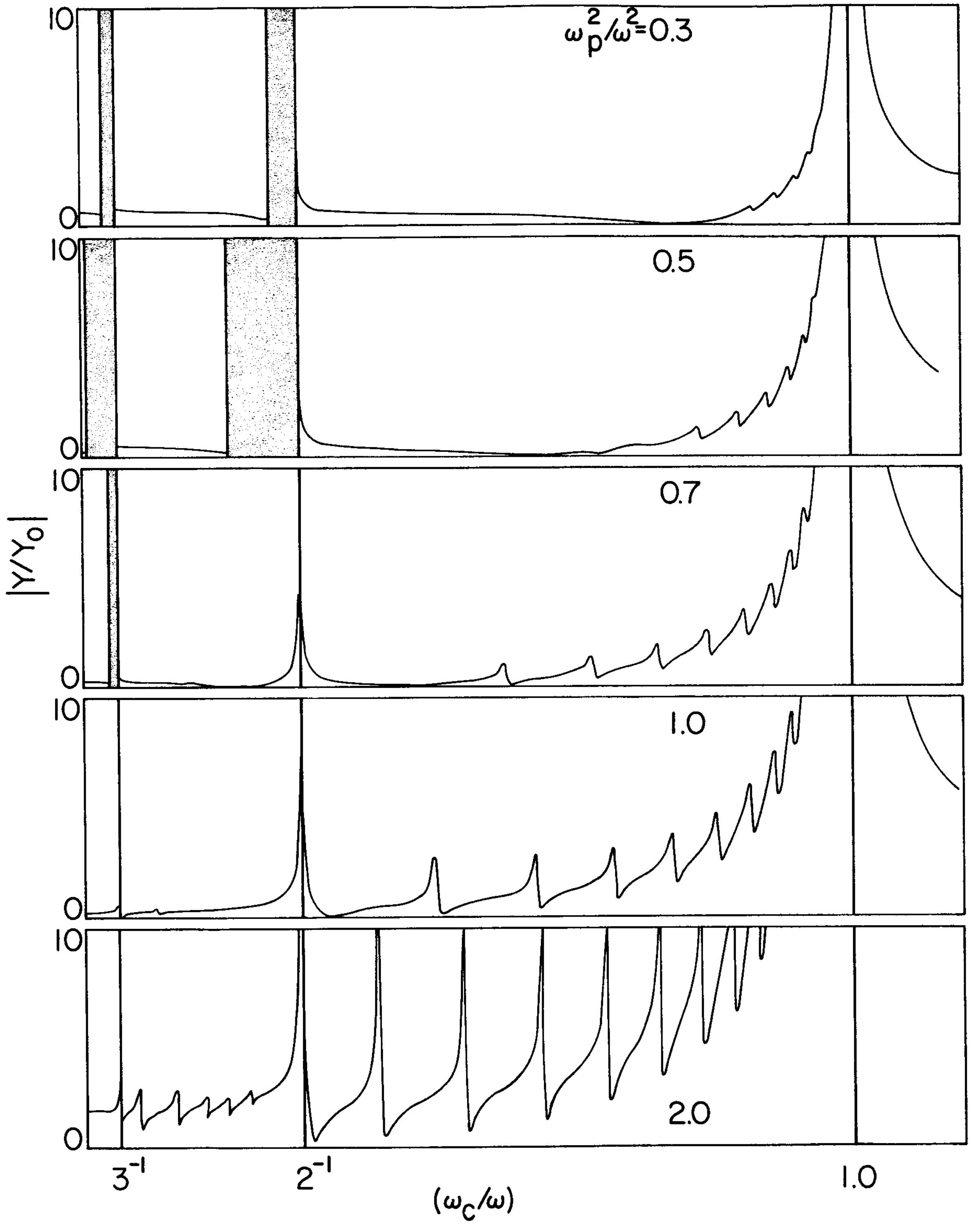

Fig. 4(b). Parallel plate capacitor - Normalized admittance variation with (ω_c/ω) and $(\omega_p{}^2/\omega^2)$ for $(\pi_T/2L\omega) = 0.1$,

(b) $(\nu/f_c) = 0.003$

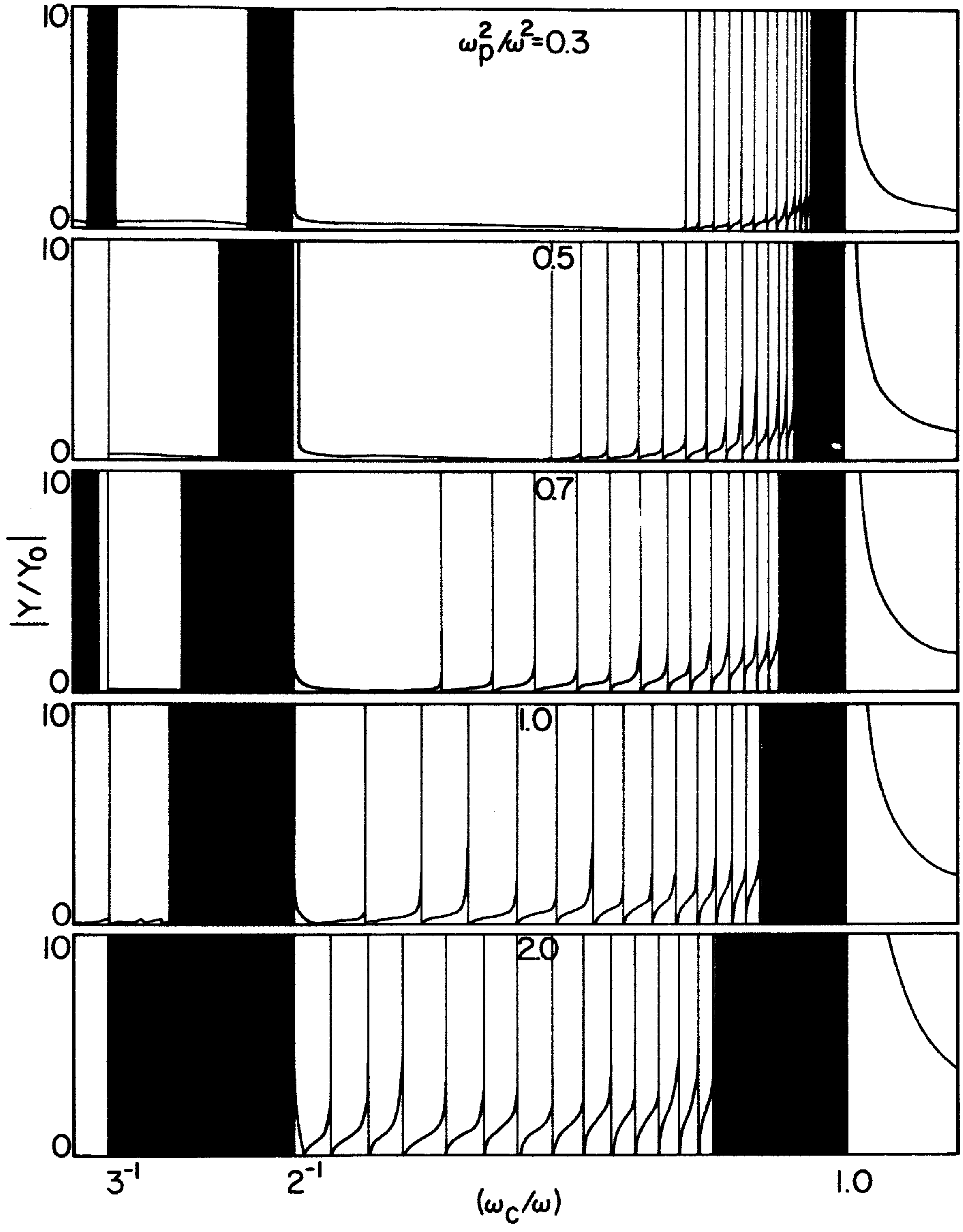

Fig. 5(a). Coaxial cylinder capacitor - Normalized admittance variation with (ω_c/ω) and (ω_p^2/ω^2) for $(v_T/b\omega) = 0.02$, $(b/a) = 3.0$, (a) $(\nu/f_c) = 0$

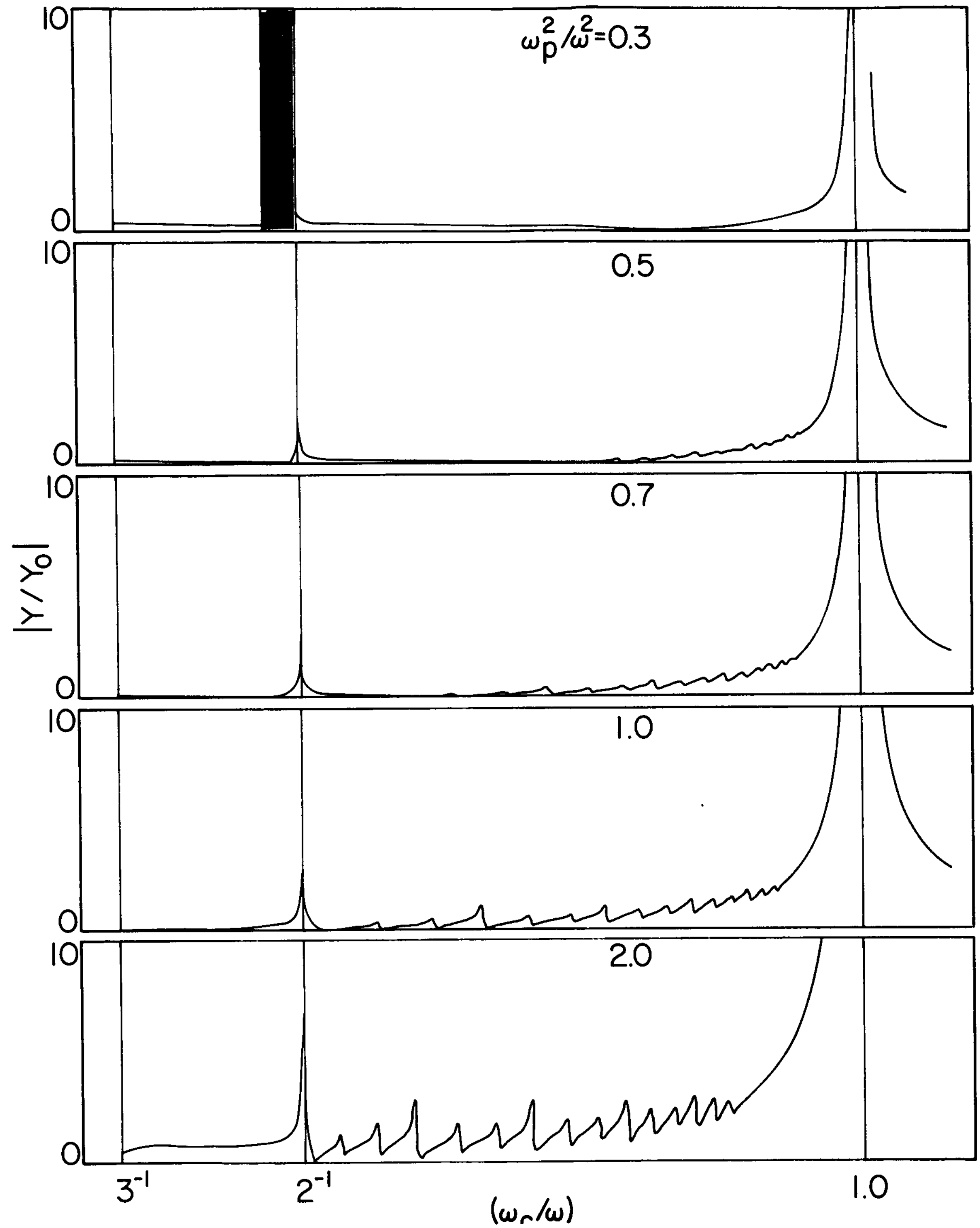

Fig. 5(b). Coaxial cylinder capacitor - Normalized admittance variation with (ω_c/ω) and $(\omega_p{}^2/\omega^2)$ for $(v_T/b\omega) = 0.02$, $(b/a) = 3.0$,

(b) $(\nu/f_c) = 0.003$

§ 3 DISCUSSION

The most serious objection to the analysis in this paper, and to all previous plasma capacitance theories, is the assumption of a specularly reflecting boundary condition. This assumption is as easy to criticize as it is difficult to improve upon, but is frequently used because of the tractability it lends to the equations, and because it appeals strongly to physical intuition. It is clear, however, that near the boundary plasma density gradients and sheath effects, finite Larmor radius effects, etc., would all have to be taken into account in any exact treatment. It is probably better to approach the problem empirically, and indeed one of the strongest motives for making experimental measurements on plasma capacitors to check the theory in this paper is to verify whether its use of the simple specular reflection condition leads to valid predictions or not.

Assuming that our analysis is adequate, we note that the theoretical impedance solutions indicate that the warm plasma capacitor in a magnetic field should be rich in resonances. Since these can be well spaced, and are not subject to collisionless damping, there is a somewhat greater hope of success in attempting to find them experimentally than there is in the absence of a magnetic field. Difficulties in carrying out such an experiment would result first from any misalignment of the electrodes, since the wavelengths are very short. Second, in view of the high "Q" of the resonances, magnetic field inhomogeneities and collisions would exert smoothing effects. It is not difficult experimentally to obtain values of $(\nu/\omega) \sim 10^{-4}$, so that the latter should not be a deciding factor. Now that space probes have made the ionosphere readily accessible to experiment, it might be possible to carry out measurements on large volumes of uniform, almost collisionless, warm plasma, in a highly uniform magnetic field.

It should be noted that the standing wave resonances in the two geometries discussed in this paper differ from the plasma resonances occurring at the upper hybrid frequency and harmonics of the cyclotron frequency which have already been observed in laboratory plasma experiments using parallel wire probes (Crawford et al 1964; Harp 1965 a, b; Crawford and Weiss 1966). Mathematically, this geometry is non-resonant. The driven probe simply excites a propagating cyclotron harmonic wave which spreads with a cylindrical wave front. The receiving probe measures the total potential resulting from interference between the propagating wave and the directly coupled capacitive signal. There are interference peaks, but no resonance effects. These results do establish the validity of eqn. (8), however, for the infinite plasma permittivity description employed in our analysis. Since the geometric resonances occurring for $\epsilon_{\perp}(k_n,\omega) = 0$ effectively give discrete points on the cyclotron harmonic wave dispersion curves, it should be possible to trace out these curves from observations of the resonances for various capacitor dimensions.

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