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ENHANCEMENT OF THE TRANSVERSE PROPERTIES OF FIBROUS COMPOSITES

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SUMMARY

A method of analysis is developed for the determination of the elastic constants of a variety of configurations of three-dimensional filamentary reinforcement for plastics. This analysis is employed in preliminary evaluations of the effectiveness of several approaches to the enhancement of properties transverse to the filaments. For glass-reinforced epoxies, binders of increased stiffness, elliptical filaments, and triangular filaments are all shown capable of effecting improvements in transverse stiffnesses. For epoxies reinforced by advanced filaments like boron, however, nearly this degree of improvement was found possible through proper filament orientation. Further studies of other configurations and combinations are suggested.

KEY WORDS

Composites, Filaments, Fibers, Materials

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MECHANICS SECTION

ENHANCEMENT OF THE TRANSVERSE PROPERTIES
OF FIBROUS COMPOSITES*

By

Norris F. Dow

*Based upon final report for NASA under Contract NASw-1144. Prepared for presentation at 21st Annual Meeting of the Reinforced Plastics Divisions, Society of the Plastics Industry. Chicago, Feb. 8-10, 1966.

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ABSTRACT

A method of analysis is developed for the determination of the elastic constants of a variety of configurations of three-dimensional filamentary reinforcement for plastics. This analysis is employed in preliminary evaluations of the effectiveness of several approaches to the enhancement of properties transverse to the filaments. For glass-reinforced epoxies, binders of increased stiffness, elliptical filaments, and triangular filaments are all shown capable of effecting improvements in transverse stiffnesses. For epoxies reinforced by advanced filaments like boron, however, nearly this degree of improvement was found possible through proper filament orientation. Further studies of other configurations and combinations are suggested.

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INTRODUCTION

Filamentary reinforcements for composites while exhibiting potentials for advanced properties along the filaments provide less attractive characteristics transversely. Indeed effective utilization of the high strength and stiffness of advanced filaments has been shown (for example, Ref. 1) to be hampered by the usual poor properties transverse to the filament orientation. Possibilities of enhancement of transverse properties need to be considered simultaneously from two standpoints. That is, one needs to know both what the magnitudes of the possible improvements are as well as how they can be effected. In this paper both these problems will be considered. First a method of analysis will be developed for the evaluation of the transverse effectiveness of various types of reinforcement. Second this analysis will be applied to the use of shaped filaments and multi-directional reinforcement to assess the improvements effected and to determine directions for further advances. The use of particulate fillers in combination with filaments will also be evaluated.

The investigations reported herein were supported by the NASA on Contract NASw-1144.

ANALYTICAL APPROACH

The analytical approach used relates to that followed in Reference 2 to determine the properties of integrally stiffened plates. Therein the reinforcement provided by integral stiffening is evaluated as fully effective in the direction of the stiffening but reduced in stretching effectiveness transverse to the stiffening by a factor β. Similarly the transverse shearing effectiveness is evaluated as reduced, - in this case by a different factor β'. With the longitudinal and transverse effectivenesses established, the remainder of the analysis is a straight forward elasticity problem of trigonometric resolution and summation of stiffnesses to yield the desired elastic constants.

The basis for the extension of the integral stiffening analysis to filamentary composites is illustrated schematically in Figure 1. In this figure the portion of the binder material between filaments in a uni-directionally reinforced filamentary composite is shown to be similar to the skin and integral ribbing of integrally stiffened plates. Neglecting the filaments (i.e., treating them for the moment as holes) the stiffness of such a two-dimensional array may be written according to the analysis of Reference 2 either as

$$E_{T_o} = E_b \left[\frac{t_s}{1-\nu^2} + \beta_\perp \left(\frac{A_{W_\perp}}{b_x} \right) - \frac{\left(\frac{\nu}{1-\nu^2} \right)^2 t_s^2}{\frac{t_s}{1-\nu^2} + \frac{A_{W_\perp}}{b_x}} \right] \tag{1}$$

or converting to the nomenclature for composites

$$E_{T_o} = E_b \left[\frac{1}{1-\nu^2} - \beta_{-o} (1 - v_b) - \frac{\left(\frac{\nu}{1-\nu^2} \right)^2}{\frac{1}{1-\nu^2} - (1 - v_b)} \right] \tag{2}$$

In these equations

E_{T_o} = stiffness transverse to round holes in binder

E_b = Young's modulus of binder material

t_s = "skin thickness" - thickness of straight elements of binder (if any) between holes

ν = Poisson's ratio

$\beta_\perp$ = transverse effectiveness of $\perp$ - shaped material between holes in binder

$A_{W_\perp}$ = cross-sectional area of $\perp$ - shaped material between holes

b_x = hole spacing

β_{-o} = transverse effectiveness lost by making holes in binder

v_b = volume fraction of binder material

The next step is evidently to fill the holes with filaments to yield an equation for the two-dimension composite as follows

$$E_{T_\bullet} = E_b \left[\frac{1}{1-\nu^2} - \beta_{-o}(1-v_b) + \beta_\bullet(1-v_b)\frac{E_f}{E_b} - \frac{\left(\dfrac{\nu}{1-\nu^2}\right)^2}{\dfrac{1}{1-\nu^2} - (1-v_b) + (1-v_b)\dfrac{E_f}{E_b}} \right] \tag{3}$$

where for simplicity the Poisson's ratios of filaments and binder have been assumed equal; $\beta_\bullet$ represents the transverse effectiveness of the filament, and E_f is the Young's modulus of the filamentary material.

Extension of this type of analysis to three dimension and for application to filaments and binders of different Poisson's ratio is discussed in the following sections. The derivation of the equations is given in the Appendix.

EQUATIONS FOR ELASTIC CONSTANTS

General Equations - The elastic constants evaluated in the Appendix are those applicable to an orthotropic composite having reinforcements symmetrically disposed about the three principal axes at plus and minus the angles indicated in Figure 2. For such a composite there are nine elastic constants defined by the following equations

$$
\begin{aligned}
\sigma_1 &= A_1 \epsilon_1 + A_2 \epsilon_2 + A_3 \epsilon_3 \\
\sigma_2 &= A_2 \epsilon_1 + A_4 \epsilon_2 + A_5 \epsilon_3 \\
\sigma_3 &= A_3 \epsilon_1 + A_5 \epsilon_2 + A_6 \epsilon_3 \\
\tau_{12} &= A_7 \gamma_{12} \\
\tau_{23} &= A_8 \gamma_{23} \\
\tau_{13} &= A_9 \gamma_{13}
\end{aligned}
\tag{4}
$$

where

$\sigma_1, \sigma_2, \sigma_3$ direct stresses in the 1-, 2-, and 3 - directions

$\epsilon_1, \epsilon_2, \epsilon_3$ direct strains in the 1-, 2-, and 3 - directions

$\tau_{12}, \tau_{23}, \tau_{13}$ shear stresses in the 1-2, 2-3, and 1-3 planes

$\gamma_{12}, \gamma_{23}, \gamma_{13}$ shear strains in the 1-2, 2-3, and 1-3 planes

and the A's are the elastic constants given by the equation in the Appendix. These constants are related to the conventional stretching and shearing stiffnesses E and G and Poisson's ratio ν by the following equations.

$$E_1 = A_1 - \frac{A_2^2 A_6 - A_2 A_3 A_5}{A_4 A_6 - A_5^2} - \frac{A_3^2 A_4 - A_2 A_3 A_5}{A_4 A_6 - A_5^2}$$

$$E_2 = A_4 - \frac{A_2^2 A_6 - A_2 A_3 A_5}{A_1 A_6 - A_3^2} - \frac{A_1 A_5^2 - A_2 A_3 A_5}{A_1 A_6 - A_3^2}$$

$$
\begin{aligned}
E_3 &= A_6 - \frac{A_3^2 A_4 - A_2 A_3 A_5}{A_1 A_4 - A_2^2} - \frac{A_1 A_5^2 - A_2 A_3 A_5}{A_1 A_4 - A_2^2} \\
\nu_{12} &= \frac{A_2 A_6 - A_3 A_5}{A_1 A_6 - A_3^2} \\
\nu_{13} &= \frac{A_3 A_4 - A_2 A_5}{A_1 A_4 - A_2^2} \\
\nu_{23} &= \frac{A_1 A_5 - A_2 A_3}{A_1 A_4 - A_2^2} \\
G_{12} &= A_7 \\
G_{23} &= A_8 \\
G_{13} &= A_9
\end{aligned}
\qquad (5)
$$

General equations like (4) and (5) may be found in the literature of three-dimensional elasticity for application to any orthotropic solid with the specified symmetries (so that couplings among shears and displacements are avoided). The evaluation of the constants employed in these equations for filamentary composites, however, involves less standard procedures. These special characteristics are discussed in the Appendix and illustrated for specific cases in the following sections.

Equations for specific reinforcement configurations - In order to illustrate the application of the equations derived in the Appendix to specific configurations of fibrous reinforcement, four sample cases will be considered:

(1) Uni-directional triangular filaments in the 1-direction

(2) Orthogonal elliptical filaments in the 1- and 2 - directions with minor axes of the ellipses oriented in the 3 - direction

(3) Round filaments oriented in the 1- and 2- directions and at angles thereto.

(4) In this case the effects of variations in the properties of filaments and binder are considered for the orthogonal filaments of Cases (2) and (3).

The equations for the elastic constants for these four cases are given in Tables 1, 2, 3, and 4 respectively, and their application to the evaluation of transverse effectiveness is discussed in the following section.

EVALUATIONS OF TRANSVERSE EFFECTIVENESSES

In the section ANALYTICAL APPROACH the concept of the use of transverse effectiveness factors β was introduced. Let us now consider the evaluation of these factors, and their application to the determination of the effectivenesses of the several example cases given above.

Evaluations of β's

Equations relating the stiffnesses of uni-directionally reinforced composites to the effectiveness coefficients β, β', and β'' may be readily derived as special cases of the general equations developed in the Appendix. For the three-dimensional case, with the Poisson's ratio of the filaments ν_f not equal to those of the binder, these equations are:

$$(1-\beta_{-\bullet} v_f) = \frac{1}{4}\left\{\left[\left(\frac{1-\nu_b}{\nu_b^2}\right) v_b + (1+\nu_b)\frac{E_{T_o}}{E_b}\right] - \sqrt{\left[\left(\frac{1-\nu_b}{\nu_b^2}\right) v_b + (1+\nu_b)\frac{E_{T_o}}{E_b}\right]^2 - 8\left(\frac{1-\nu_b}{\nu_b}\right)(v_b)(1+\nu_b)\frac{E_{T_o}}{E_b}}\right\} \tag{6}$$

$$\begin{aligned}
(\beta_{\bullet} v_f)^3 &+ \left\{(T+2T')(1-\beta_{-\bullet} v_f) - \frac{1}{2}\left(\frac{1-\nu_f}{\nu_f^2}\right) V - \left(\frac{1+\nu_f}{2}\right)\frac{E_{T_\bullet}}{E_f}\right\}(\beta_{\bullet} v_f)^2 \\
&+ \left\{(T'+2T)\,T'(1-\beta_{-\bullet} v_f)^2 - \frac{1}{2}\left(\frac{1-\nu_f}{\nu_f}\right) V\left[T' + \frac{1}{\nu_f}\,T + \left(\frac{1-\nu_f}{\nu_f}\right)T''\right](1-\beta_{-\bullet} v_f) \right. \\
&\qquad \left. + \left(\frac{1+\nu_f}{2}\right)\frac{E_{T_\bullet}}{E_f}\left[\left(\frac{1-\nu_f}{\nu_f}\right)^2 V - 2T'(1-\beta_{-\bullet} v_f)\right]\right\}(\beta_{\bullet} v_f) \\
&+ \left\{(T')^2(1-\beta_{-\bullet} v_f) - \frac{1}{2}\left(\frac{1-\nu_f}{\nu_f}\right) V\left[T' + \left(\frac{1-\nu_f}{\nu_f}\right)T''\right]\right\} T(1-\beta_{-\bullet} v_f)^2 \\
&\qquad + \left\{\left(\frac{1+\nu_f}{2}\right)\frac{E_{T_\bullet}}{E_f}\left[\left(\frac{1-\nu_f}{\nu_f}\right)^2 T''\, V - (T')^2(1-\beta_{-\bullet} v_f)\right]\right\}(1-\beta_{-\bullet} v_f) = 0
\end{aligned} \tag{7}$$

$$(1-\beta'_{-\bullet}\, v_f) = \frac{G_{12_o}}{G_b} \tag{8}$$

$$\beta'_{\bullet}\, v_f = \frac{G_{12_\bullet}}{G_f} - \frac{G_b}{G_f}(1-\beta'_{-\bullet}\, v_f) = \frac{G_{12_\bullet}}{G_f} - \frac{G_{12_o}}{G_f} \tag{9}$$

$$(1-\beta''_{-\bullet}\, v_f) = \frac{G_{23_o}}{G_b} \tag{10}$$

$$\beta''\, v_f = \frac{G_{23_\bullet}}{G_f} - \frac{G_{23_o}}{G_f} \tag{11}$$

where

$$T = \frac{\dfrac{E_b}{1+\nu_b}}{\dfrac{E_f}{1+\nu_f}}$$

$$T' = \frac{\dfrac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}}{\dfrac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}}$$

$$T'' = \frac{\dfrac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}}{\dfrac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}}$$

$$V = T'' v_b + v_f$$

v_f = volume fraction of filaments

ν_f = Poisson's ratio of filaments

G_{12_o} = shear stiffness of binder having unidirectional round holes, in the plane of the holes

G_{23_o} = shear stiffness of binder having unidirectional round holes, transverse to the holes

Other symbols as before.

In essence equations (6) - (11) define factors (β) for transverse effectiveness-for use in multi-directional reinforcement patterns-in terms of unidirectional reinforcement. Accordingly, any available data on the transverse effectiveness of unidirectional filamentary reinforcement may be employed via these equations, and those of the Appendix for multi-directional configurations.

In order to obtain values of the β's for use in the present evaluations of approaches to improvements in properties, the upper bounds of the elastic constant analysis of Reference 3 were used to yield values of β. Typical results are plotted in Figure 3 showing that, as for the integral stiffening of Reference 2, the transverse effectivenesses as represented by the values are slightly - but not substantially - different for stretching and shearing. The values calculated using the upper bounds of Reference 3 will be employed in the following sections to measure the merits of triangular and elliptical filaments and multi-directional reinforcement patterns.

Evaluations of Triangular Filaments - The special case of filaments in the shape of equilateral triangles is of interest because, with ideal packing, it permits straight-line binder elements among the filaments (Figure 4).

In the limit the configuration shown in Figure 4 is akin to those for which the waffle-type analysis of the appendix applies. That is, the material between the triangular filaments is effective along the straight-line elements as well as transverse to them; the components of stiffnesses both along and transverse to each element thus contribute to the overall stiffness, as described by the equations of Table 1.

The ratio of transverse stiffnesses produced by the straight-line continuous binder elements of Figure 4 (as represented by the equations of Table 1) to the transverse stiffnesses of the usual discontinuous binder are plotted in Figure 5 for triangular glass filaments in epoxy. This material combination was chosen as a typical one to use as a basic reference. The same materials will also be used for other evaluations; effects of changes in materials will be considered later. Not surprisingly the gains shown are relatively greater for the higher binder contents, but they never approach the factors of two or more improvements that would be desirable, and that, as will be shown, appear accessible by other approaches.

One characteristic of the triangular filament which may deserve further consideration, however, is the fact that the transverse stiffness is enhanced in both transverse directions ($E_2 = E_3$). This isotropy through the thickness may be of value for applications in which multi-axial stresses are encountered.

The fact that the straight-line binder elements apparently become more effective as the binder contributes a greater percentage of the overall properties also suggests that for relatively stiffer binders these straight-line elements may be more effective.

<u>Evaluations of Elliptical Filaments</u> - For most shell structures, stiffness properties in the thickness direction of the shell wall have little influence upon the overall response. For such applications, let us consider configurations like ellipses to increase properties in the plane of the shell. Experimental data (Reference 1), for example, have shown that 4 to 1 aspect ratio glass ellipses properly packed in 50% by volume epoxy enhance the transverse stiffness in the major axis direction of the ellipses by very nearly 100%. (Subsequent (unpublished) data have shown that the accompanying reduction in stiffness properties through the thickness - i.e. in the major-axis direction - is negligible.) At first glance, then, ellipses appear to have a promising potential. To find just how valuable such an improvement is, however, requires some examination of the importance of the transverse stiffness.

One measure of the importance of this property is obtained by comparison with other approaches which achieve the same property. As a first example, then, compare the properties of round and elliptical-filament reinforced composites having comparable amounts of orthogonal reinforcement to provide biaxial stiffness. The comparison is made in Figure 6.

In Figure 6 the stretching stiffness in the 1-direction (E_1) is plotted against the percentage of reinforcement oriented transversely. In all cases the total amount of reinforcement (i.e. the sum of the reinforcements in the two directions) is held constant at 50% by volume of the composite. The values plotted were calculated from the equations of Table 2 and 3 for

$$E_f = 10{,}500{,}000 \text{ psi}, \quad \nu_f = 0.2$$

$$E_b = 500{,}000 \text{ psi}, \quad \nu_b = 0.35$$

that is for properties representative of E-glass filaments in epoxy binder.

Two curves are given for both the round and elliptical filaments, representing (1) the β - values applicable to 50% volume fraction laminates with each lamina having unidirectional reinforcement (the upper curve) and (2) the β - values for random mixtures of filaments in the two-directions (the lower curve). Differences between the upper and lower curves are small, as can be seen. The β - values for the round filaments are those plotted in Figure 3. Those for the ellipses were calculated to make $E_1 = 2E_1$ for 100% of the reinforcement in the 2-direction.

With the curves of Figure 6, it is possible to compare directly the relative effectiveness of rounds and ellipses for providing a given transverse stiffness. For example, suppose that transverse stiffnesses ranging upward from that for the unidirectional ellipses is to be obtained by the orthogonal rounds. To achieve these stiffnesses some of the longitudinal (1-direction) round filaments must be oriented in the 2-direction; the stiffness in the 1-direction is thus reduced, and the reduction is substantial, - as shown by the "equivalent rounds" curve on Figure 6.

Each point on the "equivalent rounds" curve of Figure 6 has the same transverse (2-direction) stiffness as the elliptical filaments at the same value of the abscissa. Thus, for example, with 20% transverse reinforcement the ellipses provide an E_1 of 5100 ksi approximately, whereas the <u>equivalent</u> rounds (i.e. the rounds giving the same E_2(= 3400 ksi) as this configuration of ellipses) would provide only the E_1 given by the "equiv" curve at this abscissa (20%) or 3900 ksi.

<u>Evaluations of Bi-Axial Stiffness of Filaments at Angles to the 1- and 2-Directions</u> - Filaments making small angles (of $\pm\,\theta$ degrees) to the 1- or 2-directions are less effective in providing transverse stiffness than orthogonal

filaments. The same type of comparison made for orthogonal rounds and ellipses in Figure 6 is made for rounds at $\pm\theta^{o}$ and orthogonal ellipses in Figure 7. The "equivalent rounds" curve in Figure 7 is substantially below that in Figure 6.

Values shown for the stiffnesses with various angles of reinforcement were calculated from the equations of Table 4 with the β - values corresponding to those for the lower curves of Figure 6.

Effects of Material Properties on the Importance of Transverse Effectiveness - Comparisons like those of Figure 6 suggest that for glass-reinforced epoxy, if the application requires transverse stiffness of one-half or more of the axial stiffness, shaped filaments like 4 to 1 ellipses may provide substantial structural improvement. If advanced filaments like boron are considered, however, a different result is obtained.

In Figure 8, the curves of Figure 6 are replotted for boron instead of glass reinforcement. With the high ratio of longitudinal to transverse stiffness provided by the boron, the factor 2 improvement associated with the elliptical geometry for unidirectional reinforcement is nearly as readily attained with a few transverse round filaments. Hence, the "equivalent round" curve is only slightly below the curve for the ellipses.

A similar result is obtainable for changes in binder properties. Thus the use of a hypothetical filled binder (properties like those of the alumina-filled epoxy of Reference 1 were used for calculation) can raise the overall stiffness level (i.e. the longitudinal as well as the transverse stiffnesses) of a glass-reinforced plastic as shown in Figure 9. Thus the binder improvement is more effective than the filamentary ellipses, for example, for they enhance only the transverse properties. If, however, the reinforcement were boron, the improvement arising from the stiffer binder would be a much smaller percentage of the overall stiffnesses. With boron, then, once again transverse stiffness properties could be attained nearly as readily with a few transverse filaments as with a binder twice as stiff as epoxy.

CONCLUDING REMARKS

A method of analysis has been developed for the elastic constants of plastics having reinforcing filaments in various directions in a three-dimensional array. This analysis has been applied to make preliminary evaluations of the effectiveness of several approaches to the enhancement of the transverse stiffnesses of composites. For glass-reinforced plastics, binders of increased stiffness, elliptical filaments, and triangular filaments were all found to offer potentials of improvements - the binder improvement appearing most effective and the triangles least. With advanced filaments like boron, however, nearly the same magnitude improvement in transverse properties could be obtained by proper filament orientation as by filament shaping - or as could be obtained from binders only about twice as stiff as epoxy. Further studies of other configurations and combinations are desirable and may readily be carried out using the elastic constant analysis developed. For example, diamond-shape filaments, which approximate the 4 to 1 aspect ratio ellipse and also permit straight-line matrix elements among the filaments, would be a logical extension of the present studies. Such filaments combined with a high modulus binder should provide attractive elastic properties in all axis directions.

APPENDIX - DERIVATION OF EQUATIONS

The elastic constants for the three-dimensionally reinforced composites are derived by partial differentiations of the general expression for the strain energy of a repeating rectangular element b_1 by b_2 by b_3 of the composite. This derivation is analogous to that in Reference 2 for integrally stiffened plates with the following differences.

(1) It is a three-dimensional rather than a two-dimensional analysis.

(2) Properties of binder and filaments are different, whereas ribs and skin in Reference 2 were of the same material.

(3) Only extension and shearing are considered. Reference 2 also evaluated bending and twisting stiffnesses.

The general expressions for the strain energy of stretching of a composite subjected to the strains ϵ_1, ϵ_2, and ϵ_3 may be written as follows:

$$V = \frac{1}{2}\int_o^{b_1}\int_o^{b_2}\int_o^{b_3} \frac{E_b}{(1+\nu_b)(1-2\nu_b)}\left[(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2)(1-\nu_b)+(\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_1\epsilon_3)(2\nu_b)\right]d(1)d(2)d(3)$$

$$+\frac{1}{2}\int_o^{b_1}\left[\frac{E_{f_1}(1-\nu_{f_1})}{(1+\nu_{f_1})(1-2\nu_{f_1})}-\frac{E_b(1-\nu_b)}{(1+\nu_b((1-2\nu_b)}\right]\left[\epsilon_1^2(v_{f_1})\right]d(1) \qquad \text{(A-1)}$$

$$+\frac{1}{2}\int_o^{b_1}\left[\frac{\nu_{f_1}E_{f_1}\beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})}-\frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_1\epsilon_3)(2v_{f_1})\right]d(1)$$

$$+\frac{1}{2}\int_o^{b_1}\left[\frac{E_{f_1}(1-\nu_{f_1})\beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_2^2+\epsilon_3^2)(v_{f_1})\right]d(1)$$

$$+\frac{1}{2}\int_o^{b_2}\left[\frac{E_{f_2}(1-\nu_{f_2})}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\left[\epsilon_2^2(v_{f_2})\right]d(2)$$

$$+\frac{1}{2}\int_{o}^{b_2}\left[\frac{\nu_{f_2}E_{f_2}\beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_1\epsilon_3)(2v_{f_2})\right]d(2)$$

$$+\frac{1}{2}\int_{o}^{b_2}\left[\frac{E_{f_2}(1-\nu_{f_2})\beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_1^{\;2}+\epsilon_3^{\;2})(v_{f_2})\right]d(2)$$

$$+\frac{1}{2}\int_{o}^{b_3}\left[\frac{E_{f_3}(1-\nu_{f_3})}{(1+\nu_{f_3})(1-2\nu_{f_3})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\left[\epsilon_3^{\;2}(v_{f_3})\right]d(3) \qquad \text{(A-1, cont.)}$$

$$+\frac{1}{2}\int_{o}^{b_3}\left[\frac{\nu_{f_3}E_{f_3}\beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_1\epsilon_3)(2v_{f_3})\right]d(3)$$

$$+\frac{1}{2}\int_{o}^{b_3}\left[\frac{E_{f_3}(1-\nu_{f_3})\beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_1^{\;2}+\epsilon_2^{\;2})(v_{f_3})\right]d(3)$$

$$+\frac{1}{2}\int_{o}^{b_s}\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\epsilon_{S_L}^{\;2}(v_{f_s})\right]d(3)$$

$$+\frac{1}{2}\int_{o}^{b_s}\left[\frac{\nu_{f_s}E_{f_s}\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\left(\epsilon_{S_L}\epsilon_{S_{T_1}}+\epsilon_{S_{T_1}}\epsilon_{S_{T_2}}+\right.\right.$$

$$\left.\left.+\epsilon_{S_L}\epsilon_{S_{T_2}}\right)\left(2v_{f_s}\right)\right]d(s)$$

$$+\frac{1}{2}\int_{o}^{b_s}\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\left(\epsilon_{S_{T_1}}^{2}+\epsilon_{S_{T_2}}^{2}\right)\left(v_{f_s}\right)\right]d(s)$$

$$+\frac{1}{2}\int_{o}^{b_s}\left[G_{f_s}\beta'_{\bullet_s}-G_b\beta'_{-\bullet}\right]\left[\left(\gamma^2_{S_{L_{T_1}}}+\gamma^2_{S_{L_{T_2}}}\right)\left(v_{f_s}\right)\right]d(s) \qquad \text{(A-1, cont.)}$$

$$+\frac{1}{2}\int_{o}^{b_s}\left[G_{f_s}\beta''_{\bullet_s}-G_b\beta''_{-\bullet}\right]\left[\gamma^2_{S_T}\left(v_{f_s}\right)\right]d(s)$$

where

- V = strain energy of distortion
- E = Young's modulus
- G = shear modulus
- ν = Possion's ratio
- ϵ = extensional strain
- γ = shear strain
- v = volume fraction

Subscripts

- f = filament
- b = binder
- 1, 2, 3, s = 1-, 2-, 3-, skew directions

Evident in the foregoing expression are the various β's representative of the transverse effectivenesses of the filaments and binder elements among filaments as discussed in the body of the paper. For simple extension, such that the energy is measured by an expression of the form

$$\frac{1}{2}\int_{o}^{b}\frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left(\beta_{\bullet}v_f\right)\left(\epsilon^2\right)d(n) \qquad \text{(A-2)}$$

for example, the analogy between the β's of Reference 2 and those used herein is complete. For Poisson extensions of the form

$$\frac{1}{2}\int_{o}^{b} \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)} (\beta_{\bullet} \nu_f)(2\epsilon_1\epsilon_2)\, d(n) \tag{A-3}$$

however, the physical model of reduced effectiveness is somewhat different, and strictly speaking a different effectiveness factor, as $(\beta_{\bullet} + \delta)$ should perhaps be employed. For simplicity herein such a refinement is not considered. In consequence slight errors are introduced which show up primarily as slightly high calculated values of E_1 for uni-directionally reinforced (in the 1-direction) composites. Inasmuch as this E_1 is the most easily calculated of all the constants, via the rule of mixtures, so it can be readily corrected, if desired, and since the other values of stiffnesses appear accurately calculated ($\pm$ 5%) with one β for direct and for Poisson strains, only one β is used in the following development. (The use of one β like this also affects the calculation of the Poisson's ratios for the composite, as ν_{21}. The magnitude of this effect has not been evaluated.)

In order to evaluate the strain energy, we require expressions for $\epsilon_{f_{S_L}}$, $\gamma_{f_{S_{L_1}}}$, etc. in terms of ϵ_1, ϵ_2, and ϵ_3. These expressions are

(1) The strain along a skew filament

$$\epsilon_{f_{S_L}} = \epsilon_1 \cos^2\phi + \epsilon_2 \cos^2\psi + \epsilon_3 \cos^2\Omega \tag{A-4}$$

(2) The strain perpendicular to a skew filament and in the plane of the filament and the 1-axis

$$\epsilon_{f_{S_{T_1}}} = \epsilon_1 \sin^2\phi + \epsilon_2 \cos^2\psi \cot^2\phi + \epsilon_3 \cos^2\Omega \cot^2\phi \tag{A-5}$$

(3) The strain perpendicular to $\epsilon_{f_{S_L}}$ and $\epsilon_{f_{S_{T_1}}}$

$$\epsilon_{f_{S_{T_2}}} = \epsilon_2 \frac{\cos^2 \Omega}{\sin^2 \phi} + \epsilon_3 \frac{\cos^2 \psi}{\sin^2 \phi} \qquad \text{(A-6)}$$

Similar expressions can be written for the orthogonal shearing strains as

$$\gamma_{f_{S_{L_{T_1}}}} = 2\left[\epsilon_1 \sin\phi \cos\phi - \epsilon_2 \cot\phi \cos^2\psi - \epsilon_3 \cot\phi \cos^2\Omega\right] \qquad \text{(A-7)}$$

$$\gamma_{f_{S_{L_{T_2}}}} = 2\left[\epsilon_2 \frac{\cos\Omega}{\sin\phi} - \epsilon_3 \frac{\cos\psi \cos\Omega}{\sin\phi}\right] \qquad \text{(A-8)}$$

$$\gamma_{f_{S_T}} = 2 \; \epsilon_2 \cot\phi \frac{\cos\psi \cos\Omega}{\sin\phi} - \epsilon_3 \cot\phi \frac{\cos\psi \cos\Omega}{\sin\phi} \qquad \text{(A-9)}$$

Substituting equations (A-4) - (A-9) in equation (A-1), integrating and simplifying, yields -

$$\frac{V}{b_1 b_2 b_3} = \epsilon_1^2 \left\{ \frac{E_b (1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)} + \left[\frac{E_{f_1}(1-\nu_{f_1})}{2(1+\nu_f)(1-2\nu_{f_1})} - \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})\right.$$

$$+\left[\frac{E_{f_2}(1-\nu_{f_2})\beta_{\bullet 2}}{2(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_2}) + \left[\frac{E_{f_3}(1-\nu_{f_s})\beta_{\bullet 3}}{2(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right]\left[(\cos^4\phi)(v_{f_s})\right] +$$

$$\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right]\left[(\sin^4\phi)(v_{f_s})\right]$$

$$
+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\sin^2\phi \ \cos^2\phi)\ (v_{f_s})\right] +
$$

$$
2\left[G_{f_s} \beta'_{\bullet_s} - G_b \beta'_{-\bullet}\right]\left[(\sin^2\phi \cos^2\phi)(v_{f_s})\right]\Bigg\}
$$

$$
+ \ \epsilon_2^{\,2}\left\{\frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)} + \left[\frac{E_{f_1}(1-\nu_{f_1})\beta_{\bullet_1}}{2(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_1}) \right. \qquad \text{(A-10)}
$$

$$
+\left[\frac{E_{f_2}(1-\nu_{f_2})}{2(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_2}) + \left[\frac{E_{f_3}(1-\nu_{f_3})\beta_{\bullet_3}}{2(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b}\right](v_{f_3})
$$

$$
+\left[\frac{E_{f_s}(1-\nu_{f_s})}{2(1+\nu_{f_s})(1-2\nu_{f_s})} \quad \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right]\left[(\cos^4\psi)(v_{f_s})\right] +
$$

$$
+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right]\left[\frac{\cos^4\phi \cos^4\psi + \cos^4\Omega}{\sin^4\phi} v_{f_s}\right]
$$

$$
+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right]\left[\left(2\cot^2\phi \cos^4\psi \ + \right.\right.
$$

$$
\left.\left.\frac{\cot^2\phi \cos^2\psi \cos^2\Omega}{\sin^4\phi}\right)(v_{f_s})\right]
$$

$$
+ \ 2\left[G_{f_s} \beta'_{\bullet} - G_b \beta'_{-\bullet}\right]\left[\left(\cot^2\phi \cos^4\psi + \frac{\cos^2\psi \cos^2\Omega}{\sin^2\psi}\right) v_{f_s}\right] +
$$

$$
2\left[G_{f_s} \beta''_{\bullet} - G_b \beta''_{-\bullet}\right]\left[\left(\frac{\cot^2\phi \cos^2\psi \cos^2\Omega}{\sin^4\phi}\right) v_{f_s}\right]\Bigg\}
$$

$$+ \epsilon_2 \epsilon_3 \left\{ \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)} + \left[\frac{\nu_{f_1} E_{f_1} \beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_1}) \right.$$

$$+ \left[\frac{\nu_{f_2} E_{f_2} \beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_2})$$

$$+ \left[\frac{\nu_{f_3} E_{f_3} \beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_3})$$

$$+ \left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] \left[\left(\cos^2\psi \cos^2\Omega\right)\left(1 + \frac{1+\cos^4\phi}{\sin^4\phi}\right) \right] (v_{f_s})$$

$$+ \left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] \left[\left(\cot^2\phi \cos^2\psi \cos^2\Omega + \frac{\sec^2\phi\{\cos^4\psi + \cos^4\Omega\}}{\sin^2\phi} \right) \right] (v_{f_s})$$

$$\left. + 4\left[G_{f_s} \beta'_{\bullet s} - G_b \beta'_{-\bullet} \right] \left[\left(\cos^2\psi \cos^2\Omega\right) \right] (v_{f_s}) - 4\left[G_{f_s} \beta''_{\bullet s} - G_b \beta''_{-\bullet} \right] \left[\frac{\cos^2\phi \cos^2\psi \cos^2\Omega}{\sin^4\phi} \right] (v_{f_s}) \right\}$$

(A-10 cont.)

$$+ \epsilon_1 \epsilon_3 \left\{ \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)} + \left[\frac{\nu_{f_1} E_{f_1} \beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_1}) + \right.$$

$$+ \left[\frac{\nu_{f_2} E_{f_2} \beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_2})$$

$$+ \left[\frac{\nu_{f_3} E_{f_3} \beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] (v_{f_3})$$

$$+ \left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] \left[2\cos^2\phi \cos^2\Omega \right] (v_{f_s})$$

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\sin^2\phi \cos^2\Omega + \cot^2\phi\{\cos^2\phi\cos^2\Omega + \cos^2\psi\} + \right.$$

$$\left. + \cos^2\psi)\right](v_{f_s}) - 4\left[G_{f_s} \beta'_{\bullet_s} - G_b \beta'_{-\bullet}\right]\left[(\cos^2\phi \cos^2\Omega)(v_{f_s})\right]\Bigg\}$$

$$+ \epsilon_3^{\,2}\Bigg\{\frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)} + \left[\frac{E_{f_1}(1-\nu_f)\beta_{\bullet_1}}{2(1+\nu_{f_1})(1-2\nu_f)} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_1}) +$$

$$+ \left[\frac{E_{f_2}(1-\nu_{f_2})\beta_{\bullet_2}}{2(1+\nu_{f_2})(1-2\nu_b)} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})$$

$$+ \left[\frac{E_{f_3}(1-\nu_{f_3})}{2(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_3}) +$$

$$+ \left[\frac{E_{f_s}(1-\nu_{f_s})}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)}{2(1+\nu_b)(1-2\nu_b)}\right](v_{f_s})(\cos^4\Omega)$$

$$+ \left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right]\left[\left(\frac{\cos^4\phi \cos^4\Omega + \cos^4\psi}{\sin^4\phi}\right) v_{f_s}\right]$$

$$+ \left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{2(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{2(1+\nu_b)(1-2\nu_b)}\right]\left[\left(2\cot^2\phi \cos^4\Omega + \frac{\cot^2\phi \cos^2\psi \cos^2\Omega}{\sin^4\phi}\right) v_{f_s}\right]$$

$$+ 2\left[G_{f_s} \beta'_{\bullet_s} - G_b \beta'_{-\bullet}\right]\left[\left(\cot^2\phi \cos^4\Omega + \frac{\cos^2\psi \cos^2\Omega}{\sin^2\phi}\right) v_{f_s}\right] +$$

$$+ 2\left[G_{f_s} \beta''_{\bullet_s} - G_b \beta''_{-\bullet}\right]\left[\frac{\cos^2\phi \cos^2\psi \cos^2\Omega}{\sin^4\phi}\right](v_{f_s})\Bigg\}$$

(A-10 cont.)

$$+\ \epsilon_1\epsilon_2\left\{\frac{\nu_b E_b}{(1+\nu_b(1-2\nu_b)} + \left[\frac{\nu_{f_1} E_{f_1} \beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_{f_1})(1-2\nu_{f_1})}\right](v_{f_1}) +\right.$$

$$+\left[\frac{\nu_{f_2} E_{f_2} \beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{\nu_{f_2} E_{f_2} \beta_{\bullet 2}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})$$

$$+\left[\frac{\nu_{f_3} E_{f_3} \beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](2\cos^2\phi\ \cos^2\psi)(v_{f_s})$$

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\sin^2\phi\cos^2\psi + \cot^2\phi\{\cos^2\phi\cos^2\nu + \cos^2\Omega\} +\right.$$

$$\left.\left.\cos^2\Omega)(v_{f_s})\right] - 4\left[G_{f_s}\beta'_{\bullet s} - G_b\beta'_{-\bullet}\right]\left[(\cos^2\phi\ \cos^2\psi)(v_{f_s})\right]\right\}$$

Differentiating successively with respect to ϵ_1, ϵ_2, and ϵ_3, and collecting the factors of each of these strains for each partial derivative yields the desired elastic constants, as follows:

$$\frac{\dfrac{\partial\left(\dfrac{V}{b_1 b_2 b_3}\right)}{\partial\epsilon_1}}{\epsilon_1} = \frac{\sigma_1}{\epsilon_1}$$

$$= A_1 + A_2\left(\frac{\epsilon_2}{\epsilon_1}\right) + A_3\left(\frac{\epsilon_3}{\epsilon_1}\right)$$

$$A_1 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b} + \left[\frac{E_{f_1}(1-\nu_{f_1})}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})$$

$$
\begin{aligned}
&+\left[\frac{E_{f_2}(1-\nu_{f_2})\beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})\\
&+\left[\frac{E_{f_s}(1-\nu_{f_3})\beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})\\
&+\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_3})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](\cos^4\phi)(v_{f_s})\\
&+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](\sin^4\phi)(v_{f_s})\\
&+\left[\frac{\nu_{f_s}E_{f_s}\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](2\sin^2\phi\cos^2\phi)(v_{f_s})\\
&+\left[G_{f_s}\beta'_{\bullet s}-G_b\beta'_{-\bullet}\right]\left[(4\sin^2\phi\cos^2\phi)(v_{f_s})\right]
\end{aligned}
\tag{A-11}
$$

Similarly

$$
\begin{aligned}
A_2 = {} & \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}+\left[\frac{\nu_{f_1}E_{f_1}\beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})\\
&+\left[\frac{\nu_{f_2}E_{f_2}\beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})+\left[\frac{\nu_{f_3}E_{f_3}\beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})}\right.\\
&\left.-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})+\left\{\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\right.\\
&\left.+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\right\}(\cos^2\phi\cos^2\psi)(v_{f_s})
\end{aligned}
\tag{A-12}
$$

(A-12 cont.)

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{o_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\sin^2\phi \cos^2\psi + \cot^2\phi(\cos^2\phi\cos^2\psi + \cos^2\Omega)\right.$$

$$\left. + \cos^2\Omega\right](v_{f_s}) - \left[G_{f_s}\beta'_{o_s} - G_b \beta'_{-o}\right]\left[(4\cos^2\phi \cos^2\psi)(v_{f_s})\right]$$

and

$$A_3 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)} + \left[\frac{\nu_{f_1} E_{f_1} \beta_{o_1}}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{\nu_b E_b \beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})$$

$$+\left[\frac{\nu_{f_2} E_{f_2} \beta_{o_2}}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{\nu_b E_b \beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2}) + \left[\frac{\nu_{f_3} E_{f_3} \beta_{o_3}}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{\nu_b E_b \beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

(A-13)

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})(1+\beta_{o_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)(1+\beta_{-o})}{(1+\nu_b)(1-2\nu_b)}\right](\cos^2\phi \cos^2\Omega)(v_{f_s})$$

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{o_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\sin^2\phi\cos^2\Omega + \cot^2\phi(\cos^2\phi\cos^2\Omega + \cos^2\psi)\right.$$

$$\left. + \cos^2\psi\right](v_{f_s}) - \left[G_{f_s}\beta'_{o_s} - G_b\beta'_{-o}\right]\left[(4\cos^2\phi\cos^2\Omega)(v_{f_s})\right]$$

likewise

$$A_4 = \left[\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)} + \frac{E_{f_1}(1-\nu_{f_1})\beta_{o_1}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})$$

$$+\left[\frac{E_{f_2}(1-\nu_{f_2})}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})$$

$$+\left[\frac{E_{f_3}(1-\nu_{f_3})\beta_{o_3}}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{E_b(1-\nu_b)\beta_{-o}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](\cos^4\psi)(v_{f_s}) \qquad \text{(A-14)}$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\frac{\cos^4\phi\cos^4\psi+\cos^4\Omega}{\sin^4\phi}\right](v_{f_s})$$

$$+\left[\frac{\nu_{f_s}E_{f_s}\beta_{\bullet s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[2\cot^2\phi\cos^4\psi+\cos^2\psi\cos^2\Omega\left(\frac{1}{\sin^2\phi}+\frac{1}{\sin^4\phi}\right)\right](v_{f_s})$$

$$+\left[G_{f_s}\beta'_{\bullet s}-G_b\beta'_{-\bullet}\right]\left[4\left(\cot^2\phi\cos^4\psi+\frac{\cos^2\psi\cos^2\Omega}{\sin^2\phi}\right)\right](v_{f_s})$$

$$+\left[G_{f_s}\beta''_{\bullet s}-G_b\beta''_{-\bullet}\right]\left[4\left(\frac{\cos^2\psi\cos^2\Omega\cot^2\phi}{\sin^2\phi}\right)\right](v_{f_s})$$

also

$$A_5=\frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}+\left[\frac{\nu_{f_1}E_{f_1}\beta_{\bullet 1}}{(1+\nu_{f_1})(1-2\nu_{f_1})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})$$

$$+\left[\frac{\nu_{f_2}E_{f_2}\beta_{\bullet 2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})$$

$$+\left[\frac{\nu_{f_3}E_{f_3}\beta_{\bullet 3}}{(1+\nu_{f_3})(1-2\nu_{f_3})}-\frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\left[(\cos^2\psi\cos^2\Omega)(v_{f_s})\right] \qquad \text{(A-15)}$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\cot^4\phi\cos^2\psi\cos^2\Omega + \frac{\cos^2\psi\cos^2\Omega}{\sin^4\phi}\right](v_{f_s})$$

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[2\cot^2\phi\cos^2\psi\cos^2\Omega + \frac{\cos^4\psi + \cos^4\Omega}{\sin^4\phi}\right](v_{f_s})$$

$$+\left[G_{f_s}\beta'_{\bullet_s} - G_b\beta'_{-\bullet}\right]\left[4\left(\cot^2\phi\cos^2\psi\cos^2\Omega - \frac{\cos^2\psi\cos^2\Omega}{\sin^2\phi}\right)\right](v_{f_s})$$

$$+\left[G_{f_s}\beta''_{\bullet_s} - G_b\beta''_{-\bullet}\right]\left[4\left(\frac{\cos^2\phi\cos^2\psi\cos^2\Omega}{\sin^4\phi}\right)\right](v_{f_s})$$

and finally,

$$A_6 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)} + \left[\frac{E_{f_1}(1-\nu_{f_1})\beta_{\bullet_1}}{(1+\nu_{f_1})(1-2\nu_{f_1})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_1})$$

$$+\left[\frac{E_{f_2}(1-\nu_{f_2})\beta_{\bullet_2}}{(1+\nu_{f_2})(1-2\nu_{f_2})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_2})$$

$$+\left[\frac{E_{f_3}(1-\nu_{f_3})}{(1+\nu_{f_3})(1-2\nu_{f_3})} - \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_3})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](\cos^4\Omega)(v_{f_s}) \qquad \text{(A-16)}$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\frac{\cos^4\phi\cos^4\Omega + \cos^4\psi}{\sin^4\phi}\right](v_{f_s})$$

$$+\left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[2\cot^2\phi\cos^4\Omega + \cos^2\psi\cos^2\Omega\left(\frac{1}{\sin^2\phi} + \frac{1}{\sin^4\phi}\right)\right](v_{f_s})$$

$$+\left[G_{f_s}\beta'_{\bullet_s} - G_b\beta'_{-\bullet}\right]\left[4\left(\frac{\cos^2\Omega\,\{\cos^2\phi + \cos^2\psi\}}{\sin^2\phi}\right)\right](v_{f_s})$$

$$+\left[G_{f_s}\beta''_{\bullet_s} - G_b\beta''_{-\bullet}\right]\left[\frac{4\cos^2\phi\cos^2\psi\cos^2\Omega}{\sin^4\phi}\right](v_{f_s})$$

The elastic constants for shearing are found in a similar fashion to those for stretching. Shears γ_{12}, γ_{23}, and γ_{13} are imposed and the strain energy is evaluated as

$$\begin{aligned}
V' = {} & \frac{1}{2}\int_o^{b_1}\int_o^{b_2}\int_o^{b_3} G_b\left(\gamma_{12}^2 + \gamma_{23}^2 + \gamma_{13}^2\right) d(1)\,d(2)\,d(3) \\
& + \frac{1}{2}\int_o^{b_n}\left[G_{f_n}\beta'_{\bullet_n} - G_b\beta'_{-\bullet}\right](v_{f_n})\left(\gamma^2_{f_{n_{L_{T_1}}}} + \gamma^2_{f_{n_{L_{T_2}}}}\right) d(n) \\
& + \frac{1}{2}\int_o^{b_n}\left[G_{f_n}\beta''_{\bullet_n} - G_b\beta''_{-\bullet}\right](v_{f_n})\,\gamma^2_{f_{n_T}}\, d(n) \\
& + \frac{1}{2}\int_o^{b_s}\left[G_{f_s}\beta'_{\bullet_s} - G_b\beta'_{-\bullet}\right](v_{f_s})\left(\gamma^2_{f_{s_{L_{T_1}}}} + \gamma^2_{f_{s_{L_{T_2}}}}\right) d(s) \\
& + \frac{1}{2}\int_o^{b_s}\left[G_{f_s}\beta''_{\bullet_s} - G_b\beta''_{-\bullet}\right](v_{f_s})\left(\gamma^2_{f_{s_T}}\right) d(s)
\end{aligned} \qquad \text{(A-17)}$$

$$+\frac{1}{2}\int_0^{b_s}\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_s})(\epsilon^2_{f_{s_L}})\,d(s)$$

$$+\frac{1}{2}\int_0^{b_s}\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_s})\left(\epsilon^2_{f_{s_{T_1}}}+\epsilon^2_{f_{s_{T_2}}}\right)d(s)$$

$$+\frac{1}{2}\int_0^{b_s}\left[\frac{2\nu_{f_s}E_{f_s}\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})}-\frac{2\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right](v_{f_s})\left(\epsilon_{f_{s_L}}\epsilon_{f_{s_{T_1}}}+\epsilon_{f_{s_{T_1}}}\epsilon_{f_{s_{T_2}}}+\epsilon_{f_{s_L}}\epsilon_{f_{s_{T_2}}}\right)d(s)$$

In (A-17) n = 1, 2, 3; and

$$\gamma_{f_{s_{L_{T_1}}}}=\gamma_{12}\left[\frac{\cos\psi}{\sin\phi}\left(\cos^2\phi-\sin^2\phi\right)\right]+\gamma_{23}\left[\frac{2\cos\phi\cos\psi\cos\Omega}{\sin\phi}\right]+\gamma_{13}\left[\frac{\cos\Omega}{\sin\phi}\left(\cos^2\phi-\sin^2\phi\right)\right] \tag{A-18}$$

$$\gamma_{f_{s_{L_{T_2}}}}=\gamma_{12}\left[\frac{\cos\phi\cos\Omega}{\sin\phi}\right]+\gamma_{23}\left[\frac{\cos^2\psi-\cos^2\Omega}{\sin\phi}\right]+\gamma_{13}\left[\frac{\cos\phi\cos\psi}{\sin\phi}\right] \tag{A-19}$$

$$\gamma_{f_{s_T}}=-\gamma_{12}\cos\Omega+\gamma_{23}\left[\frac{\cos\phi\left(\cos^2\psi-\cos^2\Omega\right)}{\sin^2\phi}\right]-\gamma_{13}\cos\psi \tag{A-20}$$

$$\epsilon_{f_{s_L}}=\pm\gamma_{12}\cos\phi\cos\psi\pm\gamma_{23}\cos\psi\cos\Omega\pm\gamma_{13}\cos\phi\cos\Omega \tag{A-21}$$

$$\epsilon_{f_{s_{T_1}}} = \mp \gamma_{12} \cos\phi \cos\psi \pm \gamma_{23} \cot^2\phi \cos\psi \cos\Omega \mp \gamma_{13} \cos\phi \cos\Omega \tag{A-22}$$

$$\epsilon_{f_{s_{T_2}}} = \pm \gamma_{23} \left[\frac{\cos\psi \cos\Omega}{\sin^2\phi} \right] \tag{A-23}$$

Substituting (A-18) - (A-23) in (A-17), carrying out the integrations and differentiations, etc. yields

(A-24)

$$A_7 = G_b + \left[G_{f_1} \beta'_{\bullet} - G_b \beta'_{-\bullet} \right] (v_{f_1}) + \left[G_{f_2} \beta'_{\bullet_2} - G_b \beta'_{-\bullet} \right] (v_{f_2})$$

$$+ \left[G_{f_3} \beta''_{\bullet_3} - G_b \beta''_{-\bullet} \right] (v_{f_3}) + \left[G_{f_s} \beta'_{\bullet_s} - G_b \beta'_{-\bullet} \right] \left[\frac{\cos^2\psi(\cos^2\phi - \sin^2\phi)^2 + \cos^2\phi \cos^2\Omega}{\sin^2\phi} \right] (v_{f_s})$$

$$+ \left[G_{f_s} \beta''_{\bullet_s} - G_b \beta''_{-\bullet} \right] \left[\cos^2\Omega \right] (v_{f_s})$$

$$+ \left[\frac{E_{f_s} (1-\nu_{f_s})(1+\beta_{\bullet_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b (1-\nu_b)(1+\beta_{-\bullet})}{(1+\nu_b)(1-2\nu_b)} \right] \left[\cos^2\phi \cos^2\psi \right] (v_{f_s})$$

$$- \left[\frac{\nu_{f_s} E_{f_s} \beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b \beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)} \right] \left[2 \cos^2\phi \cos^2\psi \right] (v_{f_s})$$

$$A_8 = G_b + \left[G_{f_1} \beta''_{\bullet_1} - G_b \beta''_{-\bullet} \right] (v_{f_1}) + \left[G_{f_2} \beta'_{\bullet_2} - G_b \beta'_{-\bullet} \right] (v_{f_2})$$

$$+ \left[G_{f_3} \beta'_{\bullet_3} - G_b \beta'_{-\bullet} \right] (v_{f_3})$$

(A-25)

$$+ \left[G_{f_s} \beta'_{\bullet_s} - G_b \beta'_{-\bullet} \right] \left[\frac{\cos^2\phi \cos^2\psi \cos^2\Omega + (\cos^2\psi - \cos^2\Omega)^2}{\sin^2\phi} \right] (v_{f_s})$$

$$+\left[G_{f_s}\beta''_{\bullet_s} - G_b\beta''_{-\bullet}\right]\left[\cos^2\phi\left(\frac{\cos^2\psi - \cos^2\Omega}{\sin^2\phi}\right)^2\right](v_{f_s}) \qquad \text{(A-25 cont.)}$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\right]\left[\cos^2\psi\cos^2\Omega\right](v_{f_s})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[\cos^2\psi\cos^2\Omega\left(\frac{1+\cos^4\phi}{\sin^4\phi}\right)\right](v_{f_s})$$

$$+\left[\frac{\nu_{f_s}E_{f_s}\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[2\cos^2\psi\cos^2\Omega\left(\cos^2\phi + \frac{1}{\sin^2\phi}\right)\right](v_{f_s})$$

$$A_9 = G_b + \left[G_{f_1}\beta'_{\bullet_1} - G_b\beta'_{-\bullet}\right](v_{f_1}) + \left[G_{f_2}\beta''_{\bullet_2} - G_b\beta''_{-\bullet}\right](v_{f_2})$$

$$+\left[G_{f_3}\beta'_{\bullet_3} - G_b\beta'_{-\bullet}\right](v_{f_3}) \qquad \text{(A-26)}$$

$$+\left[G_{f_s}\beta'_{\bullet_s} - G_b\beta'_{-\bullet}\right]\left[\frac{\cos^2\Omega(\cos^2\phi - \sin^2\phi) + \cos^2\phi\cos^2\psi}{\sin^2\phi}\right](v_{f_s})$$

$$+\left[G_{f_s}\beta''_{\bullet_s} - G_b\beta''_{-\bullet}\right]\left[\cos^2\psi\right](v_{f_s})$$

$$+\left[\frac{E_{f_s}(1-\nu_{f_s})(1+\beta_{\bullet_s})}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{E_b(1-\nu_b)(1+\beta_{-\bullet})}{(1+\nu_b)(1-2\nu_b)}\right]\left[\cos^2\phi\cos^2\Omega\right](v_{f_s})$$

$$-\left[\frac{\nu_{f_s}E_{f_s}\beta_{\bullet_s}}{(1+\nu_{f_s})(1-2\nu_{f_s})} - \frac{\nu_b E_b\beta_{-\bullet}}{(1+\nu_b)(1-2\nu_b)}\right]\left[2\cos^2\phi\cos^2\Omega\right](v_{f_s})$$

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TABLE 1. - EQUATIONS FOR ELASTIC CONSTANTS FOR TRIANGULAR GLASS FILAMENTS IN THE 1-DIRECTION IN EPOXY BINDER ($\beta_{\bullet}$ for $E_f/E_b = 0.05$; $\beta_{\blacktriangle}$ for $E_f/E_b = 21$)

$$A_1 = \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\,(v_f) + \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\,(v_b)$$

$$A_2 = \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}\,(1-\beta_{-\bullet} v_b) + \frac{\nu_b E_b}{(1+\nu_b)((1-2\nu_b)}\,(\beta_{\bullet} v_b)$$

$$A_3 = A_2$$

$$A_4 = \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left\{1 - \frac{3}{4} v_b\left[\frac{1+\beta_{-\bullet}}{2} + \frac{1}{3}\left(\nu_f\left\{\frac{1+\beta_{-\bullet}}{2}\right\} + \left\{1-2\nu_f\right\}\beta'_{-\bullet}\right)\left(\frac{1}{1-\nu_f}\right)\right]\right\}$$

$$+ \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left\{\frac{3}{4} v_b\left[\frac{1+\beta_{\bullet}}{2} + \frac{1}{3}\left(\nu_b\left\{\frac{1+\beta_{\bullet}}{2}\right\} + \left\{1-2\nu_b\right\}\beta'_{\bullet}\right)\left(\frac{1}{1-\nu_b}\right)\right]\right\}$$

$$A_5 = \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}\left\{1 - \frac{3}{4} v_b\left[\beta_{-\bullet} + \frac{1}{3}\left(\left\{1-\nu_f\right\}\beta_{-\bullet} - \left\{1-2\nu_f\right\}\beta'_{-\bullet}\right)\left(\frac{1}{\nu_f}\right)\right]\right\}$$

$$+ \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left\{\frac{3}{4} v_b\left[\beta_{\bullet} + \frac{1}{3}\left(\left\{1-\nu_b\right\}\beta_{\bullet} - \left\{1-2\nu_b\right\}\beta'_{\bullet}\right)\left(\frac{1}{\nu_b}\right)\right]\right\}$$

$$A_6 = A_4$$

$$A_7 = G_f(\beta'_{\blacktriangle} v_f) + G_b(1-\beta'_{-\blacktriangle} v_f)$$

$$A_8 = G_f(\beta''_{\blacktriangle} v_f) + G_b v_b\left\{\frac{1}{4}\left[\frac{1-\nu_b}{1-2\nu_b}\right] + \frac{1}{4}\left[\frac{1-3\nu_b}{1-2\nu_b}\right]\beta_{\bullet} + \frac{1}{2}\beta'_{\bullet}\right\}$$

$$A_9 = A_7$$

TABLE 2. - EQUATIONS FOR ELASTIC CONSTANTS FOR STRETCHING FOR ORTHOGONAL, ELLIPTICAL FILAMENTS IN 1- and 2- DIRECTIONS WITH MINOR AXES IN 3-DIRECTION

$$A_1 = \frac{E_b(1-\nu_b)}{(1-\nu_b)(1-2\nu_b)}\left[1 - v_{f_1} - \beta_{-\bullet} v_{f_2}\right] + \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left[v_{f_1} + \beta_{\bullet} v_{f_2}\right]$$

$$A_2 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left[1 - \beta_{-\bullet} v_{f_1} - \beta_{-\bullet} v_{f_2}\right] + \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}\left[\beta_{\bullet} v_{f_1} + \beta_{\bullet} v_{f_2}\right]$$

$$A_3 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left[1 - \frac{\beta_{-\bullet} + \beta_{-\bullet}}{2}(v_{f_1}) - \frac{\beta_{-\bullet} + \beta_{-\bullet}}{2}(v_{f_2})\right]$$

$$+ \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}\left[\frac{\beta_{\bullet} + \beta_{\bullet}}{2}(v_{f_1}) + \frac{\beta_{\bullet} + \beta_{\bullet}}{2}(v_{f_2})\right]$$

$$A_4 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left[1 - \beta_{-\bullet} v_{f_1} - v_{f_2}\right] + \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left[\beta_{\bullet} v_{f_1} + v_{f_2}\right]$$

$$A_5 = A_3$$

$$A_6 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left[1 - \beta_{-\bullet} v_{f_1} - \beta_{-\bullet} v_{f_2}\right] + \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left[\beta_{\bullet} v_{f_1} + \beta_{\bullet} v_{f_2}\right]$$

TABLE 3. - EQUATIONS FOR ELASTIC CONSTANTS FOR ORTHOGONAL, ROUND FILAMENTS IN 1- and 2-DIRECTIONS

$$A_1 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left[1-\nu_{f_1}-\beta_{-\bullet_2} v_{f_2}\right] + \frac{E_{f_1}(1-\nu_{f_1})}{(1+\nu_{f_1})(1-2\nu_{f_1})}\left[v_{f_1}\right] + \frac{E_{f_2}(1-\nu_{f_2})}{(1+\nu_{f_2})(1-2\nu_{f_2})}\left[\beta_{\bullet_2} v_{f_2}\right]$$

$$A_2 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left[1-\beta_{-\bullet_1} v_{f_1}-\beta_{-\bullet_2} v_{f_2}\right] + \frac{\nu_{f_1} E_{f_1}}{(1+\nu_{f_1})(1-2\nu_{f_1})}\left[\beta_{\bullet_1} \nu_{f_1}\right] + \frac{\nu_{f_2} E_{f_2}}{(1+\nu_{f_2})(1-2\nu_{f_2})}\left[\beta_{\bullet_2} v_{f_2}\right]$$

$$A_3 = A_2$$

$$A_4 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left[1-\beta_{-\bullet_1} v_{f_1}-v_{f_2}\right] + \frac{E_{f_1}(1-\nu_{f_1})}{(1+\nu_{f_1})(1-2\nu_{f_1})}\left[\beta_{\bullet_1} v_{f_1}\right] + \frac{E_{f_2}(1-\nu_{f_2})}{(1+\nu_{f_2})(1-2\nu_{f_2})}\left[v_{f_2}\right]$$

$$A_5 = A_2$$

$$A_6 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left[1-\beta_{-\bullet_1} v_{f_1}-\beta_{-\bullet_2} v_{f_2}\right] + \frac{E_{f_1}(1-\nu_{f_1})}{(1+\nu_{f_1})(1-2\nu_{f_1})}\left[\beta_{\bullet_1} v_{f_1}\right] + \frac{E_{f_2}(1-\nu_{f_2})}{(1+\nu_{f_2})(1-2\nu_{f_2})}\left[\beta_{\bullet_2} v_{f_2}\right]$$

$$A_7 = G_b\left[1-\beta'_{-\bullet_1} v_{f_1}-\beta'_{-o_2} v_{f_2}\right] + G_{f_1}\left[\beta'_{\bullet_1} v_{f_1}\right] + G_{f_2}\left[\beta'_{\bullet_2} v_{f_2}\right]$$

$$A_8 = G_b\left[1-\beta''_{-\bullet_1} v_{f_1}-\beta'_{-o_2} v_{f_2}\right] + G_{f_1}\left[\beta''_{\bullet_1} v_{f_1}\right] + G_{f_2}\left[\beta'_{\bullet_2} v_{f_2}\right]$$

$$A_9 = G_b\left[1-\beta'_{-\bullet_1} v_{f_1}-\beta''_{-\bullet_2} v_{f_2}\right] + G_{f_1}\left[\beta'_{\bullet_1} v_{f_1}\right] + G_{f_2}\left[\beta''_{\bullet_2} v_{f_2}\right]$$

TABLE 4. - EQUATIONS FOR ELASTIC CONSTANTS FOR ROUND FILAMENTS MAKING ANGLES OF $\pm\theta$ DEGREES TO THE 1-DIRECTION

$$A_1 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left\{1-v_f\left[\cos^4\theta + \beta_{-\bullet}\sin^4\theta + \left(\nu_b\beta_{-\bullet} + \left\{1-2\nu_b\right\}\beta'_{-\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{1-\nu_b}\right)\right]\right\}$$

$$+\frac{E_f(1-\nu_f)(v_f)}{(1+\nu_f)(1-2\nu_f)}\left\{\cos^4\theta + \beta_{\bullet}\sin^4\theta + \left(\nu_f\beta_{\bullet} + \left\{1-2\nu_f\right\}\beta'_{\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{1-\nu_f}\right)\right\}$$

$$A_2 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left\{1-v_f\left[\beta_{-\bullet}(\sin^4\theta+\cos^4\theta)+\left(\left\{1-\nu_b\right\}\left\{\frac{1+\beta_{-\bullet}}{2}\right\} - \left\{1-2\nu_b\right\}\beta'_{-\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{\nu_b}\right)\right]\right\}$$

$$+\frac{\nu_f E_f(v_f)}{(1+\nu_f)(1-2\nu_f)}\left\{\beta_{\bullet}(\sin^4\theta+\cos^4\theta)+\left(\left\{1-\nu_f\right\}\left\{\frac{1+\beta_{\bullet}}{2}\right\} - \left\{1-2\nu_f\right\}\beta'_{\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{\nu_f}\right)\right\}$$

$$A_3 = \frac{\nu_b E_b}{(1+\nu_b)(1-2\nu_b)}\left\{1-\beta_{-\bullet}v_f\right\} + \frac{\nu_f E_f}{(1+\nu_f)(1-2\nu_f)}\left\{\beta_{\bullet}v_f\right\}$$

$$A_4 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left\{1-\nu_f\left[\sin^4\theta + \beta_{-\bullet}\cos^4\theta + \left(\nu_b\beta_{-\bullet} + \left\{1-2\nu_b\right\}\beta'_{-\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{1-\nu_b}\right)\right]\right\}$$

$$+\frac{E_f(1-\nu_f)(v_f)}{(1+\nu_f)(1-2\nu_f)}\left\{\sin^4\theta + \beta_{\bullet}\cos^4\theta + \left(\nu_f\beta_{\bullet} + \left\{1-2\nu_f\right\}\beta'_{\bullet}\right)\left(\frac{2\sin^2\theta\cos^2\theta}{1-\nu_f}\right)\right\}$$

$$A_5 = A_3$$

$$A_6 = \frac{E_b(1-\nu_b)}{(1+\nu_b)(1-2\nu_b)}\left\{1-\beta_{-\bullet}v_f\right\} + \frac{E_f(1-\nu_f)}{(1+\nu_f)(1-2\nu_f)}\left\{\beta_{\bullet}v_f\right\}$$

$$A_7 = G_b\left\{1-v_f\left[\left[\left\{(1-\nu_b)+(1-3\nu_b)\beta_{-\bullet}\right\}\left\{\frac{2\sin^2\theta\cos^2\theta}{1-2\nu_b}\right\} + \beta'_{-\bullet}\cos^2 2\theta\right]\right]\right\}$$

TABLE 4. - Continued

$$+ \; G_f v_f \left\{ \left[(1-\nu_f) + (1-3\nu_f)\beta_{\bullet} \right] \left[\frac{2 \sin^2\theta \cos^2\theta}{1-2\nu_f} \right] + \beta'_{\bullet} \cos^2 2\theta \right\}$$

$$A_8 = G_b \left\{ 1 - v_f \left[\beta'_{-\bullet} \sin^2\theta + \beta''_{-\bullet} \cos^2\theta \right] \right\} + G_f v_f \left\{ \beta'_{\bullet} \sin^2\theta + \beta''_{\bullet} \cos^2\theta \right\}$$

$$A_9 = G_b \left\{ 1 - v_f \left[\beta'_{-\bullet} \cos^2\theta + \beta''_{-\bullet} \sin^2\theta \right] \right\} + G_f v_f \left\{ \beta'_{\bullet} \cos^2\theta + \beta''_{\bullet} \sin^2\theta \right\}$$

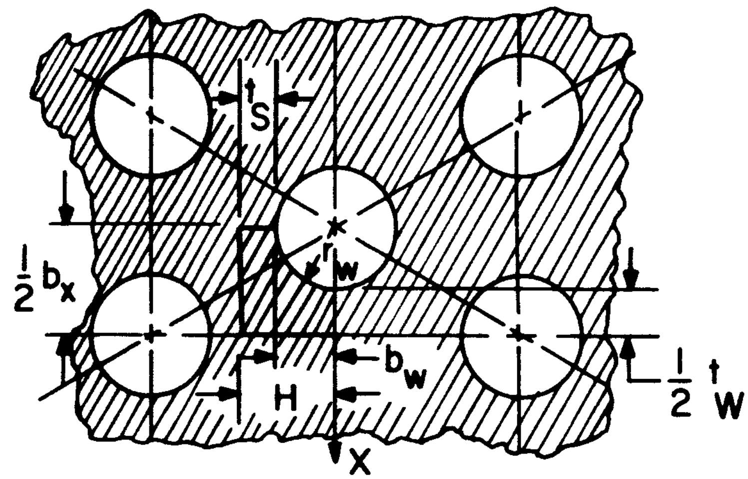

Figure 1. Repeating Element of Unidirectionally Reinforced Composite Corresponding to That for the Integrally Stiffened Plate of Reference 2.

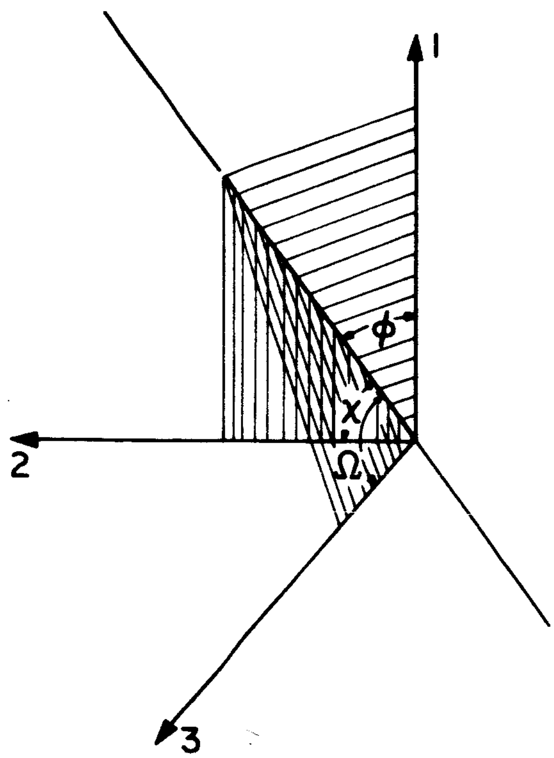

Figure 2. Three-Dimensional Angle Notation Used in Analysis.

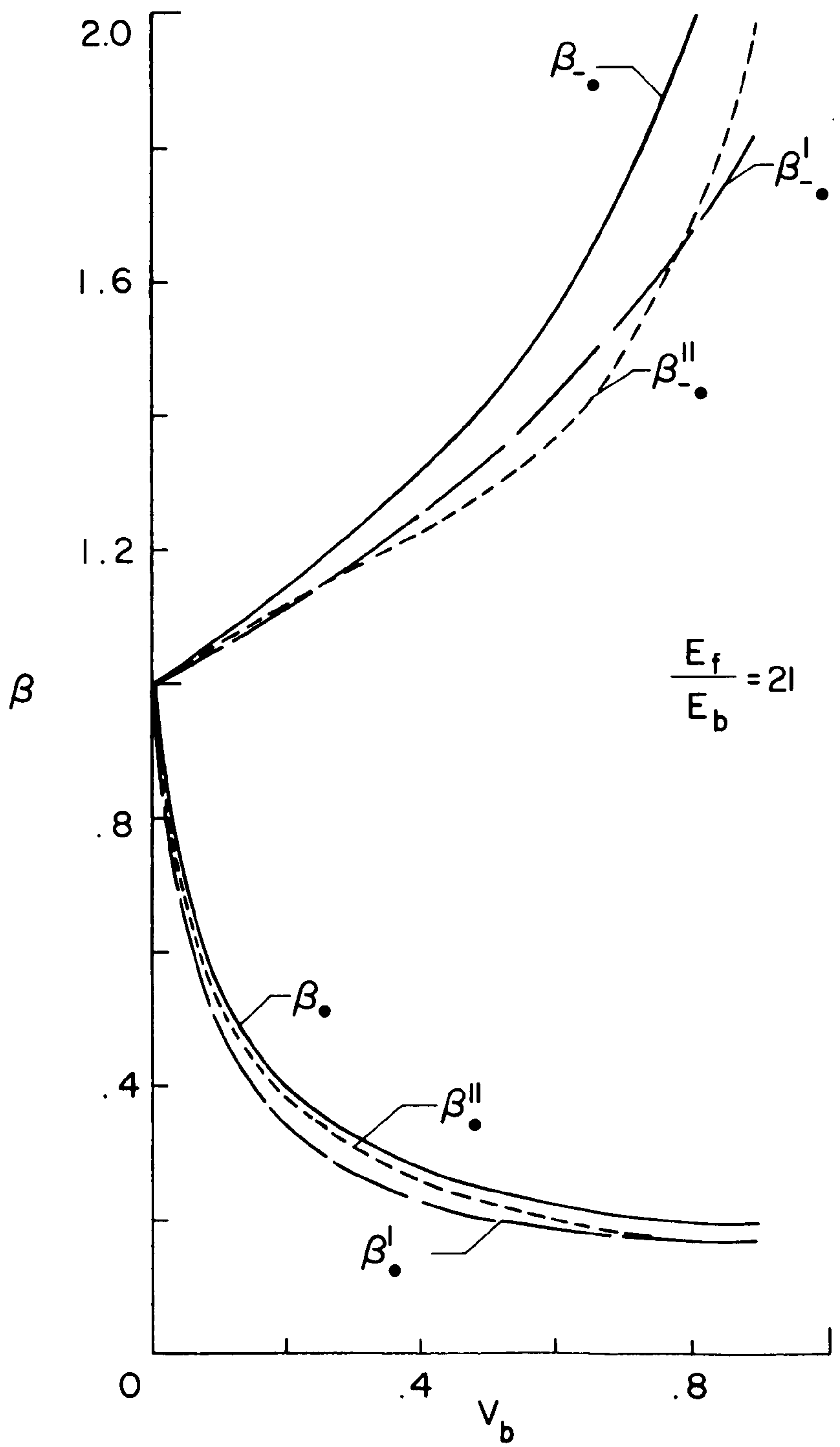

Figure 3. Values of Transverse Effectiveness Coefficients β for Glass Filaments in Epoxy Binder, Based on Upper Bounds of Reference 3.

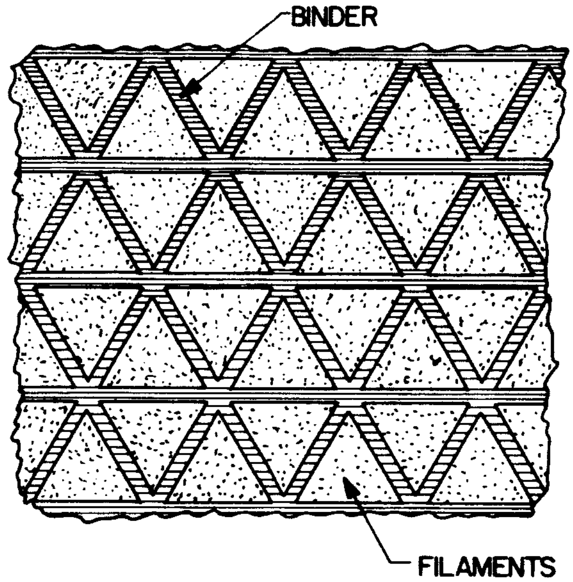

Figure 4. Schematic Representation of Straight-Line Binder Elements Possible Among Perfectly-Packed, Equilateral Triangular Cross-Section Filaments.

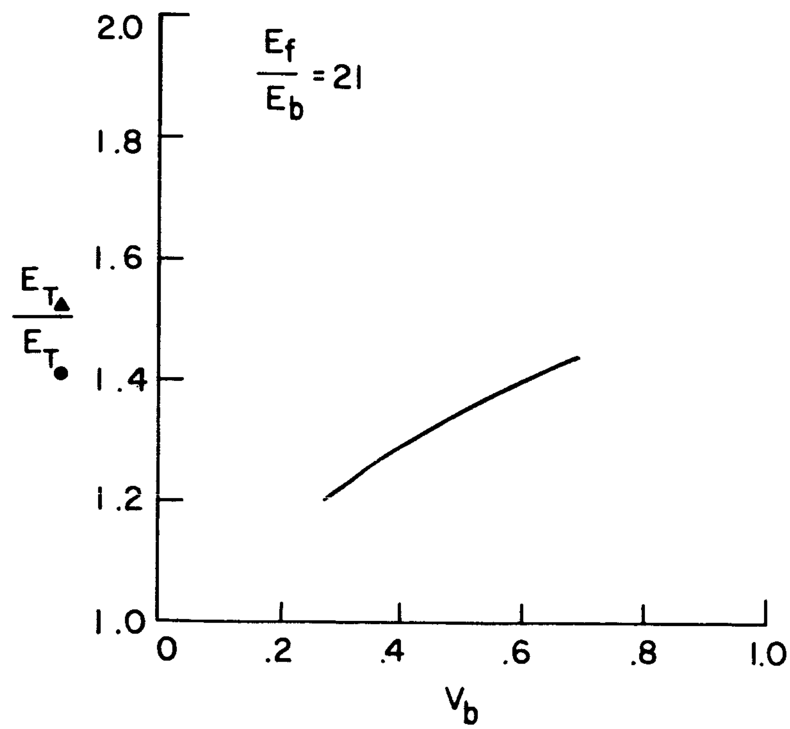

Figure 5. Ratio of Transverse Stiffnesses of Epoxy Composites Having Unidirectional Filamentary Glass Reinforcement of Triangular and Circular Cross-Section.

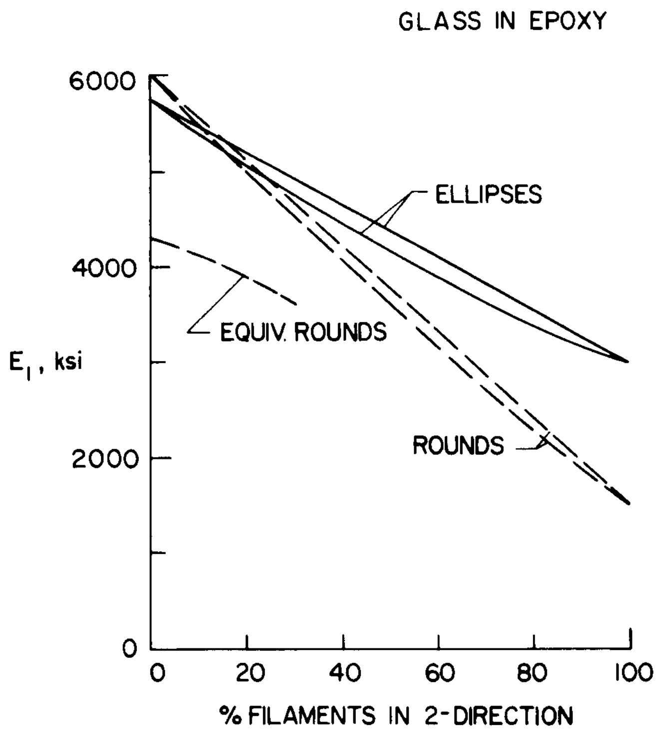

Figure 6. Evaluation of Enhancement of Transverse Stiffness Provided by 4 to 1 Aspect Ratio Elliptical Cross-Sections of Glass in Epoxy.

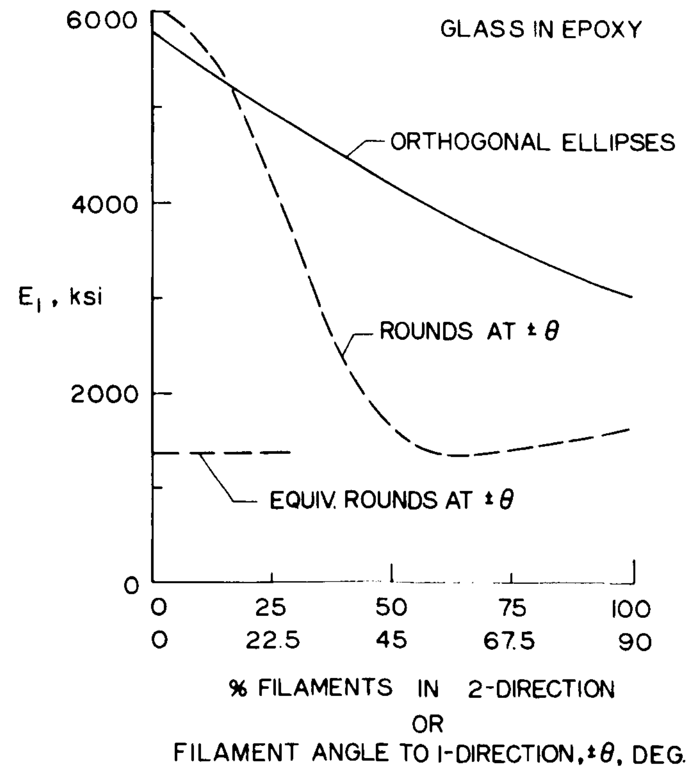

Figure 7. Evaluation of Enhancement of Transverse Stiffness by Angular Filament Orientation.

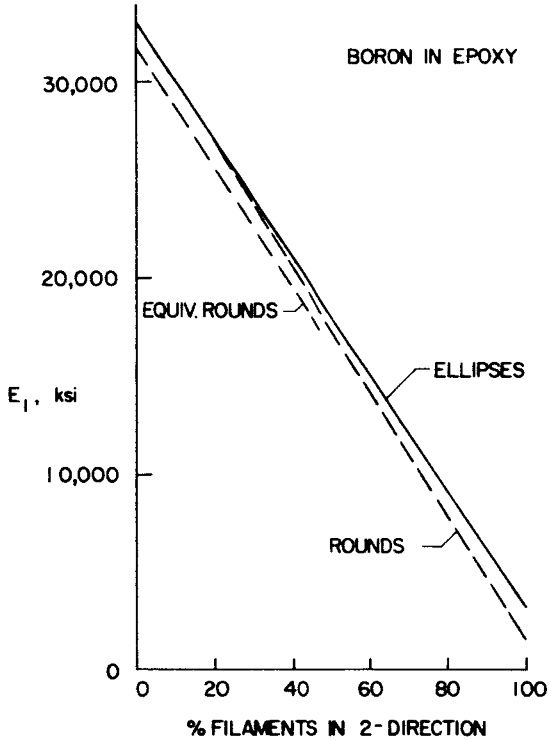

Figure 8. Evaluation of Enhancement of Transverse Stiffness Provided by 4 to 1 Aspect Ratio Elliptical Cross-Sections of Boron in Epoxy.

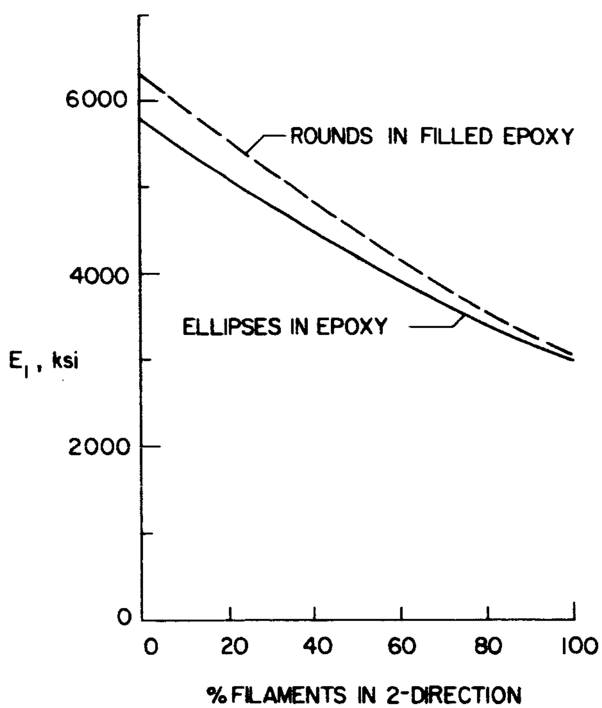

Figure 9. Evaluation of Enhancement of Transverse Stiffness Provided to Glass-Filament Reinforced Composites by a Filled Binder Having a Stiffness Increase of 160%.