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A STUDY OF DAMPING IN NONLINEAR OSCILLATIONS

by Maurice L. Rasmussen
Stanford University

and

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Ames Research Center

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A STUDY OF DAMPING IN NONLINEAR OSCILLATIONS

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SUMMARY

An investigation is made of nonlinear oscillations in which the damping and static moments are represented by arbitrary polynomial functions of the dependent variable. When the nonlinear damping is small but the static nonlinearities arbitrarily large, an approximate solution is established which leads to expressions for the damping decrement involving elliptic integrals and gamma functions in special cases. An "effective linear damping" is defined and a generalized formula for this parameter is obtained that is valid for a wide range of nonlinearities in both the damping and static moments. This formula is useful, for instance, in deducing the dynamic-stability parameters of missiles observed in nearly planar motion in free flight.

INTRODUCTION

Many physical systems are describable only in terms of nonlinear ordinary differential equations. One example is the large-amplitude pitching motion of hypersonic aerodynamic configurations for which the frequency can be strongly dependent on amplitude.[1] In the analysis of ballistic-range data, it is desired to deduce the aerodynamic properties, linear and nonlinear, from observations of a given set of oscillations. Toward this end, a knowledge of the effects of the nonlinearities involved in the differential equations governing the motion is very important.

In this study, we will be concerned chiefly with the damping of oscillations of nonlinear systems. Although we will be interested in solutions to the nonlinear equations for their own sake, the eventual goal will be to derive formulas allowing the extraction of both linear and nonlinear damping parameters from a set of data. From the standpoint of aerodynamics, this goal is equivalent to determining from data the dynamic-stability parameters of a missile oscillating in a plane. A theoretical study indicating the form of the

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[1]Although the present analysis is applicable to a wide variety of oscillatory phenomena, the text is cast in terms of pitching oscillations of bodies having nonlinear damping and static moments in flight in a medium of constant density.

nonlinearities to be expected in the dynamic-stability parameters can be found in reference 1. Aspects of obtaining free-flight damping in a conventional wind tunnel are discussed in reference 2.

Since we will be interested in problems in which some of the nonlinearities may be large, conventional methods of analysis in which all nonlinearities are considered small (see, e.g., refs. 3 and 4) are not always suitable. Two methods that treat large nonlinearities in the static moment have been devised by Murphy: a perturbation technique (refs. 5 and 6) and a quasi-linear technique (ref. 7) that yields results essentially the same as those of Coakley, Laitone, and Maas (ref. 8). Although these methods are quite general in that they are applicable for combined pitching and yawing motions, only cubic static moments are discussed, and explicit damping formulas for only linear damping in planar and near-circular oscillations are derived and plotted. Redd, Olsen, and Barton (ref. 9) derived formulas for the effective damping of nonlinear oscillations, but considered the static moment to be linear.

Many aerodynamic systems are not describable by cubic static moments, and quintic or higher order polynomials are often essential for proper data reduction. There is reason to expect that if the static moment requires description by higher order polynomials, then the damping moment will also. It is desirable, therefore, to understand the influence of arbitrary nonlinear damping and static moments on oscillatory motions. The intent of the present paper is to contribute to this understanding.

The analysis proceeds along two distinct lines after appropriate transformations are used to render the original second-order differential equation into an equivalent integral equation on the one hand and an equivalent first-order differential equation on the other. By means of the integral equation, for which a special exact solution exists, an approximate solution is found that is valid for large damping nonlinearities and small effective static nonlinearities. For small damping but arbitrarily large nonlinearities in the static moment, an approximate solution is then found by utilizing the first-order differential equation. For this latter line of analysis, the damping decrement can be conveniently represented by an "effective linear damping," which can be evaluated in terms of quadratures leading to elliptic integrals and gamma functions in special cases. Based on limiting values of the quadrature expressions, a simplified approximate formula for the effective linear damping is devised that is extremely accurate for a wide range of nonlinear parameters and, further, is amenable to data-reduction techniques.

NOMENCLATURE

A,B constants of integration

a amplitude variable defined by equation (28b)

b defined by equation (81b)

D_o defined by equation (57c)

E	complete elliptic integral of second kind
f	arbitrary function occurring in equation (27)
H	$\sum_{s \geq 0} H_s y^s$
H_{o_e}	effective linear damping defined by equation (66)
H_s	damping-moment coefficients
$\bar{H}_s$	defined by equation (11)
h	H_o/ν
h_s	$\frac{1}{2+s} \frac{H_s}{H_o} y_a{}^s$ (eq. (54))
I	defined by equation (81c)
K	complete elliptic integral of first kind
k	modulus of elliptic integrals
M	$\sum_{s \geq 0} M_s y^{s+1}$
$\bar{M}_o$	$M_o - \frac{H_o{}^2}{4}$
M_s	static-moment coefficients
$\bar{M}_s$	$M_s - \bar{H}_s$
m_s	$\bar{M}_s y_m{}^s/\nu^2$ or $\bar{M}_s y_a{}^s/\nu^2$ (eqs. (42), (54))
P	defined by equation (18)
Q	defined by equation (30a)
R	defined by equation (34)
s	summation index
t	independent variable in equation (1)
t_o	initial value of t
u,z	functions defined by equations (3), (7), and (8)

Y $\quad y^2/y_a^2$

Y_o $\quad$ value of Y for zero damping

Y_m $\quad$ mth order expansion term in Y (eq. (56))

y $\quad$ dependent variable and amplitude of nonlinear oscillation

y_a $\quad \left[\frac{1}{2}\left(y_n^2 + y_{n+1}^2\right)\right]^{1/2}$

y_m $\quad$ a particular maximum amplitude

y_n $\quad$ nth maximum amplitude

y_{n+1} $\quad$ (n+1)th maximum amplitude

y_o $\quad$ initial amplitude of y

$y_{(o)}$ $\quad$ zeroth order approximation (eq. (50))

γ $\quad$ defined by equation (79b)

φ $\quad \tan^{-1}(H_o/2\nu)$ or $\sin^{-1}(H_o/2\sqrt{M_o})$

μ $\quad m_2(2m_o + m_2)$

ν $\quad |\overline{M}_o|^{1/2}$ if $\overline{M}_o \neq 0$, $|\overline{M}_n y^n|^{1/2}$ if $\overline{M}_o = 0$

τ $\quad$ frequency variable defined by equation (28c)

ω $\quad$ angular frequency

BASIC ANALYSIS

The Basic Equation

In this study we wish to consider oscillations that are influenced by nonlinearities in both the damping and static forces or moments. We shall assume that the nonlinearities are functions of the dependent variable only.

Consider a nonlinear oscillation to be governed by the following differential equation for $y = y(t)$:

$$\frac{d^2y}{dt^2} + \left(\sum_{s \geq o} H_s y^s\right)\frac{dy}{dt} + \left(\sum_{s \geq o} M_s y^s\right) y = 0 \qquad (1)$$

The middle term represents the damping moment, the last term, the static restoring moment. We shall refer to the term represented by $s = 0$ in both the damping and static moments as the linear moment. Similarly, we shall refer to polynomials terminated by $s = 1$, $s = 2$, and $s = 4$ as quadratic, cubic, and quintic moments, respectively. When H_s and M_s are nonzero only for even values of s, the motion is called symmetric; when H_s and M_s exist for odd values of s, the motion is called unsymmetric. Symmetric motions are of the greater interest, but unsymmetric motions, such as oscillations about a trim angle in aerodynamics, do occur.

Equation (1) includes as special cases several well-known equations. If $H_0 \neq 0$, $M_0 \neq 0$, but $H_s = M_s = 0$ for $s \neq 0$, equation (1) is linear and has a well-known solution. If $H_0 = -H_2 = -\epsilon$ and $H_s = 0$, $s \neq 0, 2$; $M_0 \neq 0$ and $M_s = 0$, $s \neq 0$, then equation (1) reduces to the classical Van der Pol equation

$$\frac{d^2y}{dt^2} + \epsilon(y^2 - 1)\frac{dy}{dt} + M_0 y = 0 \tag{2}$$

If all the damping terms vanish, leaving only a nonlinear restoring moment, then the motion is periodic. If, in addition, the restoring moment, the last term in equation (1), is a quintic or less in y, then exact solutions may be found in terms of elliptic functions for symmetric oscillations.

Equivalent Pair of Equations

It is convenient to seek a transformation that will express equation (1) in such a form that the first derivative of the dependent variable does not appear.

Let us introduce two functions $u(t)$ and $z(t)$ so that

$$y(t) = u(t)z(t) \tag{3}$$

Since we have introduced two functions to replace one, we have an arbitrary condition at our disposal. We will select $u(t)$ so that the differential equation for $z(t)$ does not explicitly involve its first derivative.

If the transformation given by equation (3) is substituted into equation (1), then

$$\frac{d^2z}{dt^2} + \left(\frac{2}{u}\frac{du}{dt} + \sum_{s \geq 0} H_s u^s z^s\right)\frac{dz}{dt} + \left[\frac{1}{u}\frac{d^2u}{dt^2} + \sum_{s \geq 0}\left(M_s + \frac{H_s}{u}\frac{du}{dt}\right)u^s z^s\right]z = 0 \tag{4}$$

It does not suffice to set the coefficient of dz/dt in equation (4) equal to zero in order to specify the needed equation for u. If this were done, a term involving dz/dt would appear in the coefficient of z owing to the presence of d^2u/dt^2. The form of the coefficient of dz/dt, however, suggests the following relation for u:

$$\frac{1}{u}\frac{du}{dt} = -\frac{1}{2}\sum_{s\geq o} a_s H_s u^s z^s \tag{5}$$

Here the a_s are constants to be determined so that the dz/dt terms vanish in equation (4).

Taking the derivative of equation (5) to determine d^2u/dt^2 and substituting into equation (4), we find

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o}\left(-a_s + 1 - \frac{s}{2}a_s\right)H_s u^s z^s\right]\frac{dz}{dt} + \left\{\sum_{s\geq o}\left[M_s + \left(-\frac{s}{2}a_s - \frac{a_s}{2} + 1\right)\frac{H_s}{u}\frac{du}{dt}\right]u^s z^s\right\} z = 0 \tag{6}$$

We now eliminate the derivative dz/dt from equation (6) by requiring that its coefficient vanish. This can be done term by term if

$$a_s = \frac{2}{2 + s}$$

Hence,

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o}\left(M_s + \frac{H_s}{2 + s}\frac{1}{u}\frac{du}{dt}\right)u^s z^s\right] z = 0 \tag{7}$$

$$\frac{1}{u}\frac{du}{dt} = -\sum_{s\geq o}\frac{H_s}{2 + s}u^s z^s \tag{8}$$

Equations (7) and (8) are the ones we have sought. These equations together with equation (3) replace the single equation (1) for y by two simultaneous equations in the new variables u and z. It may appear that the solution of the two simultaneous equations for u and z would be more complicated than the single equation for y. We will find, however, that the equations become uncoupled for special values of M_s and H_s, and exact solutions may then be found. Moreover, transformations of these equations will yield a single integral equation for y and a single first-order equation for y that are appropriate for establishing approximate solutions.

To see more clearly the nature of equations (7) and (8), let us write them in a slightly different form. Substituting equation (8) into equation (7) gives presence of d^2u/dt^2. The form of the coefficient of dz/dt, however, suggests the following relation for u:

$$\frac{1}{u}\frac{du}{dt} = -\frac{1}{2}\sum_{s\geq o} a_s H_s u^s z^s \tag{5}$$

Here the a_s are constants to be determined so that the dz/dt terms vanish in equation (4).

Taking the derivative of equation (5) to determine d^2u/dt^2 and substituting into equation (4), we find

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o}\left(-a_s + 1 - \frac{s}{2}a_s\right)H_s u^s z^s\right]\frac{dz}{dt} + \left\{\sum_{s\geq o}\left[M_s + \left(-\frac{s}{2}a_s - \frac{a_s}{2} + 1\right)\frac{H_s}{u}\frac{du}{dt}\right]u^s z^s\right\} z = 0 \tag{6}$$

We now eliminate the derivative dz/dt from equation (6) by requiring that its coefficient vanish. This can be done term by term if

$$a_s = \frac{2}{2 + s}$$

Hence,

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o}\left(M_s + \frac{H_s}{2 + s}\frac{1}{u}\frac{du}{dt}\right)u^s z^s\right] z = 0 \tag{7}$$

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To see more clearly the nature of equations (7) and (8), let us write them in a slightly different form. Substituting equation (8) into equation (7) gives

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o} M_s u^s z^s - \left(\sum_{s\geq o} \frac{H_s}{2+s} u^s z^s\right)^2\right] z = 0 \tag{9}$$

Equation (9) may be simplified by introducing new parameters, $\overline{H}_s$, defined by

$$\sum_{s\geq o} \overline{H}_s u^s z^s = \left(\sum_{s\geq o} \frac{H_s}{2+s} u^s z^s\right)^2 \tag{10}$$

The square of the series in equation (10) may be arranged so that $\overline{H}_s$ may be identified as

$$\overline{H}_s \equiv \sum_{m=o}^{s} \frac{H_m H_{s-m}}{(2+m)(2+s-m)} \tag{11}$$

Equation (9) becomes

$$\frac{d^2z}{dt^2} + \left[\sum_{s\geq o} (M_s - \overline{H}_s) u^s z^s\right] z = 0 \tag{12}$$

With equation (7) written in this form, special exact solutions suggest themselves.

Special Exact Solutions

It can be seen that when

$$M_s = \overline{H}_s, \qquad s \neq 0 \tag{13}$$

equation (12) reduces to

$$\frac{d^2z}{dt^2} + \left(M_o - \frac{H_o^2}{4}\right) z = 0 \tag{14}$$

which has the general solution

$$z = \cos \tau \tag{15}$$

where

$$\tau = \nu(t - t_o), \qquad \nu^2 = M_o - \frac{H_o^2}{4}$$

and t_o is an arbitrary constant. The other arbitrary constant which would normally appear has been suppressed since it will appear in the solution of the equation for $u(t)$, equation (8).

Consider now a two-term damping moment given by

$$\left(\sum_{s\geq o} H_s y^s\right)\frac{dy}{dt} = \left(H_o + H_n y^n\right)\frac{dy}{dt} \tag{16}$$

Here we have kept the linear term H_o and one nonlinear term of arbitrary power n, where n is an integer. With this damping moment, equation (8) takes the form

$$\frac{1}{u}\frac{du}{dt} = -\frac{H_o}{2} - \frac{H_n}{2+n} u^n z^n \tag{17}$$

A solution to equation (17) may be found by introducing the transformation

$$U^2 = P\, e^{-H_o t} \tag{18}$$

so that

$$P^{-1-\frac{n}{2}}\frac{dP}{dt} = -\frac{2H_n}{2+n} e^{-\frac{n}{2}H_o t} z^n \tag{19}$$

which has the solution

$$P \equiv U^2 e^{H_o t} = \left(A + \frac{nH_n}{2+n}\int_o^t z^n e^{-\frac{n}{2}H_o t'}\, dt'\right)^{-\frac{2}{n}} \tag{20}$$

where A is a constant of integration.

From equations (10) and (13) we find that for this special case the only nonzero values of M_s and $\overline{H}_s$ are given by

$$\left.\begin{aligned} M_n &= \overline{H}_n = \frac{H_o H_n}{2+n} \\ M_{2n} &= \overline{H}_{2n} = \frac{H_n^{\,2}}{(2+n)^2} \end{aligned}\right\} \tag{21}$$

Finally, by combining the solutions for $u(t)$ and $z(t)$, we find that the function $y(t)$ can be expressed as a function of $\tau = \nu(t - t_o)$ by

$$y(t) = \frac{e^{-\frac{H_o\tau}{2\nu}} \cos \tau}{\left[\frac{1}{y_o^{\,n}} + \frac{nH_n}{(2 + n)\nu}\int_0^{\tau} e^{-\frac{n}{2}\frac{H_o\tau'}{\nu}} \cos^n \tau' \, d\tau'\right]^{\frac{1}{n}}} \tag{22}$$

where $y_o = y(t = t_o)$ and t_o are arbitrary constants, and $y(t)$ is a general solution to the following equation:

$$\frac{d^2y}{dt^2} + (H_o + H_n y^n)\frac{dy}{dt} + \left[M_o - \frac{H_o^{\,2}}{4} + \frac{H_oH_n}{2 + n}\, y^n + \frac{H_n^{\,2}}{(2 + n)^2}\, y^{2n}\right] y = 0 \tag{23}$$

where n is an integer.

Solution (22) to equation (23) was first given by Smith (ref. 10), although by an entirely different derivation. Subsequently, we shall use this exact solution as a basis of an approximate solution that does not restrict the nonlinear restoring terms M_n and M_{2n} to the values given by equation (21).

It is interesting to see the explicit forms that solution (22) takes for specific values of n. Here we present the solutions for $n = 2$ and $n = 4$, which are most likely to be of practical interest.

<u>n = 2</u>:

$$y = \frac{e^{-\frac{H_o\tau}{2\nu}} \cos \tau}{\left\{B - \frac{H_2}{4H_o}\, e^{-\frac{H_o\tau}{\nu}}\left[1 - \sin \varphi \sin(2\tau - \varphi)\right]\right\}^{\frac{1}{2}}} \tag{24}$$

$$\nu^2 = M_o - \frac{H_o^{\,2}}{4}, \qquad \tau = \nu(t - t_o)$$

$$\sin \varphi = \frac{H_o}{2\sqrt{M_o}}$$

<u>n = 4</u>:

$$y = \frac{e^{-\frac{H_o\tau}{2\nu}} \cos \tau}{\left\{B - \frac{H_4}{8H_o}\, e^{-\frac{2H_o\tau}{\nu}}\left[1 - \frac{4}{3} \sin \psi \sin(2\tau - \psi) - \frac{1}{3} \sin \varphi \sin(4\tau - \varphi)\right]\right\}^{\frac{1}{4}}} \tag{25}$$

$$\sin \psi = \frac{H_o}{\sqrt{\nu^2 + H_o^{\,2}}}, \qquad \sin \varphi = \frac{H_o}{2\sqrt{M_o}}$$

where B is a constant. Formulas (24) and (25) are useful because they indicate explicitly and analytically how the motion may be affected by arbitrarily large damping nonlinearities.

The special exact solution (22) has the shortcoming that the nonlinear static-moment terms M_n are not open constants, but are determined by the damping-moment terms H_o and H_n as given by equation (21). We would like to lift this restriction and find approximate solutions for arbitrary values of M_n. To do this, we must take into account that the frequency may be variable and recast the equations into more appropriate forms.

Equivalent Integral Equation

The two simultaneous equations that we shall deal with are (12) and (8), which we rewrite here as

$$\frac{d^2z}{dt^2} + \left(\sum_{s\geq o} \overline{M}_s u^s z^s\right) z = 0 \tag{26a}$$

$$\frac{1}{u^2}\frac{du^2}{dt} = -2\sum_{s\geq o} \frac{H_s}{2 + s} u^s z^s \tag{26b}$$

where $\overline{M}_s = M_s - \overline{H}_s'$. The parameters $\overline{M}_s$ may be viewed as the "effective" static-moment parameters.

We introduce a modified version of the Kryloff-Bogoliuboff technique, which is outlined in appendix A. Consider a nonlinear differential equation having the form

$$\frac{d^2x}{dt^2} + \nu^2 x + f\left(x, \frac{dx}{dt}\right) = 0 \tag{27}$$

where f is an arbitrary function of x and dx/dt, and ν^2 is a positive quantity. If we assume that equation (27) has a solution of the form

$$x = a(\tau)\cos \tau \ , \qquad \tau = \tau(t) \tag{28a}$$

then, as shown in appendix A, $a(\tau)$ and $\tau(t)$ are determined by

$$\frac{da}{d\tau} = \frac{\frac{1}{\nu^2} f(a \cos \tau, -a\nu \sin \tau)\sin \tau}{1 + \frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau} \tag{28b}$$

$$\frac{d\tau}{dt} = \nu\left[1 + \frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau\right] \tag{28c}$$

The amplitude variable, a, is determined as a function of τ, the frequency variable, by solving equation (28b). Using this solution, $a = a(\tau)$, one may then find the relationship between τ and the original independent variable t by solving equation (28c) in the form of a quadrature

$$t - t_o = \frac{1}{\nu}\int_0^{\tau} \frac{d\tau'}{1 + \frac{f}{a\nu^2}\cos \tau'} \tag{28d}$$

Consider now the application of the modified Kryloff-Bogoliuboff equations to our particular problem. To account for oscillations for which $\overline{M}_o \leq 0$, we shall select special values for the arbitrary parameter ν^2. If $\overline{M}_o \neq 0$, then we shall choose $\nu^2 = |\overline{M}_o|$. If $\overline{M}_o = 0$, then we shall choose $\nu^2 = |\overline{M}_n y_m{}^n|$, where y_m is a suitable characteristic amplitude of the motion. Noting that $z = a \cos \tau$, we may identify the function $f(a \cos \tau, -a\nu \sin \tau)$ from equation (26a) as follows:

$$f = \left(-1 + \sum_{s \geq o} \frac{\overline{M}_s}{\nu^2} a^s u^s \cos^s \tau\right) \nu^2 a \cos \tau \tag{29}$$

Expression (29), in this case, is not in convenient form to solve equation (28b) because the variable u is present, and u must be found as a function of $a(\tau)$ and τ by combining equations (26b) and (28c) to eliminate the variable t. Because this step cannot be easily accomplished, we instead introduce a new variable, Q, defined as follows:

$$Q \equiv a^2u^2 e^{\frac{H_o\tau}{\nu}} \tag{30a}$$

It now can be established that y is determined as a function of τ by

$$y^2(\tau) = Q(\tau)e^{-\frac{H_o\tau}{\nu}} \cos^2 \tau \tag{30b}$$

An equation for $Q(\tau)$ now remains to be found. From equation (30a) it follows that

$$u^2 e^{\frac{H_o\tau}{\nu}} \frac{da^2}{d\tau} = \frac{dQ}{d\tau} - \frac{Q}{u^2}\frac{du^2}{dt}\frac{dt}{d\tau} - \frac{H_oQ}{\nu} \tag{30c}$$

Substituting equations (26b), (28b), and (28c) into equation (30c), we obtain

$$\frac{1}{Q^2}\frac{dQ}{d\tau} = \frac{-\frac{2}{\nu}\sum\limits_{s\geq 1}\frac{H_s}{2+s} Q^{\frac{s-2}{2}} e^{-\frac{sH_o\tau}{2\nu}} \cos^s \tau + \left(\frac{H_o}{\nu}\cos\tau + 2\sin\tau\right)\frac{f}{a\nu^2 Q}}{1 + \frac{f}{a\nu^2}\cos\tau} \tag{31a}$$

This equation is the more general counterpart of equation (19). The appropriate expression for f in terms of Q and τ is found from equation (29) to be

$$\frac{f}{a\nu^2} = \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2} Q^{\frac{s}{2}} e^{-\frac{sH_o\tau}{2\nu}} \cos^s\tau\right)\cos\tau \tag{31b}$$

Thus, when equation (31b) is used for f in equation (31a), a first-order nonlinear equation for $Q(\tau)$ results.

It is, in general, too difficult to integrate equation (31a) for $Q = Q(\tau)$. Nevertheless, it is convenient to go a step further and find an integral equation for y that will serve as a basis for approximate solutions. We first note that f can be expressed in terms of y^2 by using equation (30b):

$$\frac{f}{a\nu^2} = Q\, e^{-\frac{H_o\tau}{\nu}} y^{-2}\left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2} y^s\right)\cos^3\tau \tag{32a}$$

or

$$\frac{f}{a\nu^2} = \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2} y^s\right)\cos\tau \tag{32b}$$

Equation (31a) may now be expressed as

$$\frac{1}{Q^2}\frac{dQ}{d\tau} = \frac{e^{-\frac{H_o\tau}{\nu}} y^{-2}\left[-\frac{2}{\nu}\cos^2\tau\sum\limits_{s\geq 1}\frac{H_s}{2+s} y^s + \left(-1 + \sum\limits_{s\geq o}\frac{\overline{M}_s}{\nu^2} y^s\right)R(\tau)\right]}{1 + \left(-1 + \sum\limits_{s\geq o}\frac{\overline{M}_s}{\nu^2} y^s\right)\cos^2\tau} \tag{33}$$

where

$$R(\tau) = \frac{H_o}{\nu}\cos^4\tau + 2\sin\tau\cos^3\tau \tag{34}$$

Direct integration of equation (33) gives

$$Q = \left\{ B + \int_0^{\tau} \frac{e^{-\frac{H_o\tau'}{\nu}}\, y^{-2}\left[\frac{2\cos^2\tau'}{\nu}\sum_{s\geq 1}\frac{H_s}{2+s}y^s - \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}y^s\right)R(\tau')\right]d\tau'}{1 + \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}y^s\right)\cos^2\tau'} \right\}^{-1} \tag{35}$$

Substitution of equation (35) into equation (30b) now yields an integral equation for $y(\tau)$:

$$y^2 = \frac{e^{-\frac{H_o\tau}{\nu}}\cos^2\tau}{\frac{1}{{y_o}^2} + \displaystyle\int_0^{\tau} \frac{e^{-\frac{H_o\tau'}{\nu}}\, y^{-2}\left[\frac{2\cos^2\tau'}{\nu}\sum_{s\geq 1}\frac{H_s}{2+s}y^s + \left(1 - \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}y^s\right)R(\tau')\right]d\tau'}{1 + \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}y^s\right)\cos^2\tau'}} \tag{36}$$

Here we have identified the constant of integration B as $1/{y_o}^2$, where $y_o = y(\tau = 0)$.

If it is assumed that $y = y(\tau)$ can be found, then, at least formally, the corresponding relation for $t(\tau)$ can be determined from equation (28d):

$$t - t_o = \frac{1}{\nu}\int_0^{\tau}\frac{d\tau'}{1 + \left(-1 + \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}y^s\right)\cos^2\tau'} \tag{37}$$

The integral equation (36) is interesting because it reflects roughly the nature of a solution that one might expect for a nonlinear oscillation.

Furthermore, when $\nu^2 = \overline{M}_o > 0$, $\overline{M}_s = 0$ for $s \geq 1$, and $H_s = 0$ for $s \neq 0,2$, the exact solution (22) for $n = 2$ is recovered immediately. Equation (36) will thus be useful for a perturbation of this exact solution.

Equivalent First-Order Equation

We shall now derive the most useful and most important equation of this analysis. Since $y = uz$, the first derivative of y can be written in terms of u and z as

$$\frac{dy}{dt} = y\left(\frac{1}{z}\frac{dz}{dt} + \frac{1}{u}\frac{du}{dt}\right) \tag{38}$$

We now again introduce the modified Kryloff-Bogoliuboff equations, derived in appendix A, so that $z = a \cos \tau$ and $dz/dt = -a\nu \sin \tau$. In addition, we make use of equations (26b), (28c), and (32b), and write equation (38) as follows:

$$\frac{dy}{d\tau} = \frac{-y\left(\tan \tau + \frac{1}{\nu}\sum_{s\geq o}\frac{H_s}{2 + s}\, y^s\right)}{\sin^2 \tau + \cos^2 \tau \sum_{s\geq o}\frac{\overline{M}_s}{\nu^2}\, y^s} \tag{39}$$

The general second-order differential equation (1) with t as the independent variable has now been reduced to a single first-order equation with τ as the independent variable. The relation between t and τ is again provided by equation (37). Although equation (39) is equivalent to the integral equation (36) because it describes the same nonlinear oscillation, it has a very different appearance.

Distortion Due to Nonlinear Damping

Equation (39) can immediately be used to determine where the maximum amplitudes occur in a nonlinear oscillation. When there is no damping in an oscillation, the motion is periodic, and it is easy to deduce that the maximum amplitudes occur at $\tau_m = n\pi$, where n is an integer. For a general oscillation, the maximum amplitude occurs when $dy/d\tau$ vanishes, and thus from equation (39) we deduce that

$$\tan \tau_m = -\frac{1}{\nu}\sum_{s\geq o}\frac{H_s}{2 + s}\, {y_m}^s \tag{40}$$

where $y_m = y(\tau_m)$ is the maximum amplitude.

Formula (40) indicates how much the maximum amplitude is shifted from the point of symmetry, $\tau_m = n\pi$. Consider a special simple example, namely, the limit cycle for the exact solution (24). The limit cycle occurs when $H_o < 0$, $H_2 > 0$, and $\tau \to \infty$ and, in this case, equation (24) yields

$$(y)_{lim} = \frac{\cos \tau}{\sqrt{-\frac{H_2}{4H_o}[1 - \sin \varphi \sin(2\tau - \varphi)]}} \tag{41}$$

where

$$\sin \varphi = \frac{H_o}{2\sqrt{M_o}}$$

or

$$\tan \varphi = \frac{H_o}{2\nu}$$

The value of y_m from equation (41) is

$$y_m = \sqrt{-\frac{4H_o}{H_2}}$$

and equation (40) now yields for this example

$$\tan \tau_m = -\frac{H_o}{2\nu}\left(1 + \frac{H_2}{2H_o} y_m^2\right) = \tan \varphi$$

Hence,

$$\tau_m = n\pi + \varphi$$

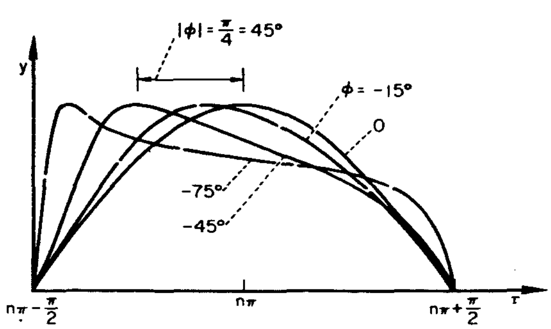

Figure 1.- Distortion of motion due to nonlinear damping.

and the limit motion is distorted from the symmetry position by an amount $\varphi = \tan^{-1}(H_o/2\nu)$. This motion is shown in figure 1 for values of $\varphi = 0°$, $-15°$, $-45°$, and $-75°$. When $|\varphi|$ is large, a substantial distortion of the nearly cosine oscillation characteristic of small nonlinearities is found, and the oscillation is no longer symmetric about $\tau_m = n\pi$.

It is interesting and instructive to see how exact solutions for zero damping appear in the present formulation of nonlinear oscillations. Moreover, these exact solutions will be useful later in approximations for nonzero damping.

General Solution

Let us now consider only symmetric motions, so that s is even, and all the damping terms are zero ($H_s \equiv 0$). The motion will be periodic and have a maximum amplitude that we denote by y_m. We introduce a normalized form of y by the transformation

$$Y_o = \left(\frac{y}{y_m}\right)^2$$

In addition, we introduce the coefficients m_s defined by:

$$m_s = \frac{\overline{M}_s y_m{}^s}{\nu^2}, \qquad s \geq 0 \tag{42}$$

Note that if $M_o \neq 0$, m_s is a ratio of the magnitude of the sth nonlinear term of the static moment to the magnitude of the linear term. With this notation, one may express equation (39) for zero damping as follows:

$$\frac{dY_o}{d\tau} = \frac{-2Y_o \tan \tau}{\sin^2 \tau + \cos^2 \tau \sum_{s \geq o} m_s Y_o^{s/2}} \tag{43}$$

Equation (43) can be immediately integrated if a transformation to a new independent variable $x = \tan \tau$ is made. If the boundary condition is imposed that $Y_o = 1$ when $\tau = n\pi$ (n is an integer), then

$$\tan^2 \tau = Y_o^{-1} \sum_{s \geq o} \frac{2m_s}{2 + s}\left(1 - Y_o^{\frac{s+2}{2}}\right) \tag{44}$$

The zero-damping solution becomes complete when a relationship between τ and t is established. This is done by means of equation (37), which now becomes

$$t - t_o = \frac{1}{\nu} \int_o^{\tau} \frac{d\tau'}{\sin^2 \tau' + \cos^2 \tau' \sum_{s \geq o} m_s Y_o^{s/2}} \tag{45}$$

Formula (45) is not always a convenient expression to use because Y_o is generally an implicit function of τ. Consequently, an alternate expression in which t is determined as a function of Y_o is often more useful. The alternate formula may be found by changing variables according to equations (43) and (44), and thus obtaining

$$t - t_{n\pi} = \frac{1}{2\nu} \int_{Y_o}^{1} \frac{dY_o'}{\sqrt{Y_o' \sum_{s \geq o} \frac{2m_s}{2 + s}\left(1 - Y_o'^{\frac{s+2}{2}}\right)}} \tag{46a}$$

for

$$n\pi \leq \tau \leq n\pi + \frac{\pi}{2} \qquad \text{and} \qquad 0 \leq Y_o \leq 1$$

Expression (46a) has been obtained by more conventional methods. If $m_s = 0$ when $s \geq 6$, then equation (46a) can be evaluated by elliptic integrals (see refs. 11 and 12). The angular frequency ω of the oscillation is defined by

$$\frac{\nu}{\omega} = \frac{1}{\pi} \int_o^{1} \frac{dY_o}{\sqrt{Y_o \sum_{s \geq o} \frac{2m_s}{2 + s}\left(1 - Y_o^{\frac{s+2}{2}}\right)}} \tag{46b}$$

An approximate formula for the frequency with arbitrary values of m_s has been derived in reference 13.

Special Case of a Cubic Static Moment

For the case of a linear-plus-cubic moment, $m_s = 0$ when $s \geq 4$, it is easy to find an explicit function for $Y_o(\tau)$ from equation (44); namely,

$$Y_o = \frac{(\mu/m_2)\cos^2 \tau}{\sin^2 \tau + m_o \cos^2 \tau + \sqrt{(\sin^2 \tau + m_o \cos^2 \tau)^2 + \mu \cos^4 \tau}} \tag{47}$$

where $\mu = m_2(2m_o + m_2)$.

With this explicit representation, equation (37) yields the following relation between τ and t

$$t - t_o = \frac{1}{\nu} \int_0^{\tau} \frac{d\tau'}{\sqrt{(\sin^2 \tau' + m_o \cos^2 \tau')^2 + \mu \cos^4 \tau'}} \tag{48a}$$

It follows that the frequency is given by

$$\frac{\nu}{\omega} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\tau}{\sqrt{(\sin^2 \tau + m_o \cos^2 \tau)^2 + \mu \cos^4 \tau}} \tag{48b}$$

It is interesting to compare equation (48b) for the frequency with the corresponding formula (46b), which may be evaluated in terms of elliptic integrals. The quadrature (48b) is a proper integral and may be evaluated numerically without difficulty.

Formulas (48) contain four possible cases of interest, depending on the value of the moment terms $\overline{M}_o$ and $\overline{M}_2$. To discuss them, we adopt the following convention. If $\overline{M}_n$ is positive, it is called stable; if it is negative, it is called unstable. The various possibilities are stated by describing the term $\overline{M}_o$ and then $\overline{M}_2$. For a cubic static moment we thus have the following cases:[2]

<u>Stable-stable and stable-unstable static moment</u>

$\overline{M}_o > 0$, $\nu^2 = \overline{M}_o$, $m_o = 1$ $\qquad$ $0 < m_2 < \infty$ (stable-stable); $-1 < m_2 < 0$ (stable-unstable)

<u>Unstable-stable static moment</u>

$\overline{M}_o < 0$, $\nu^2 = |\overline{M}_o|$, $m_o = -1$ $\qquad$ $2 < m_2 < \infty$

<u>Pure cubic static moment</u>

$\overline{M}_o = 0$, $\nu^2 = \overline{M}_2 y_m^2 > 0$, $m_o = 0$, $m_2 = 1$ $\qquad$ (49)

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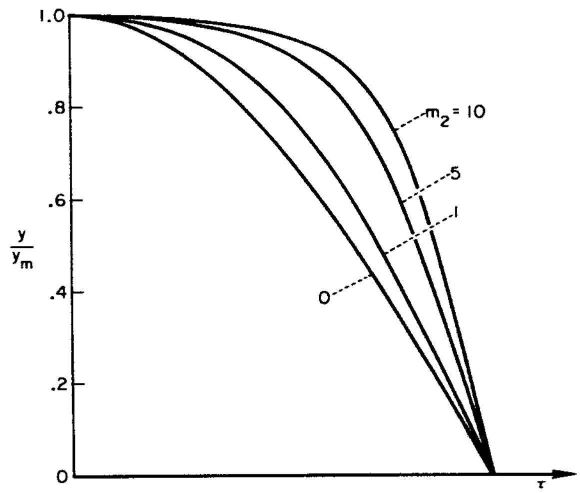
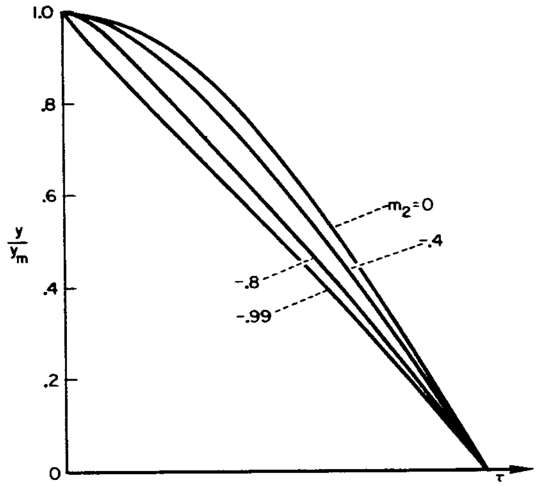
[2]Note that if $m_2 < -1$ for the stable-unstable case, tumbling motion results; also, if $1 < m_2 < 2$ for the unstable-stable case, unsymmetric motion about a trim angle occurs. Since neither of these cases is covered in the present analysis, initial conditions must be such that the restrictions on m_2 are observed.

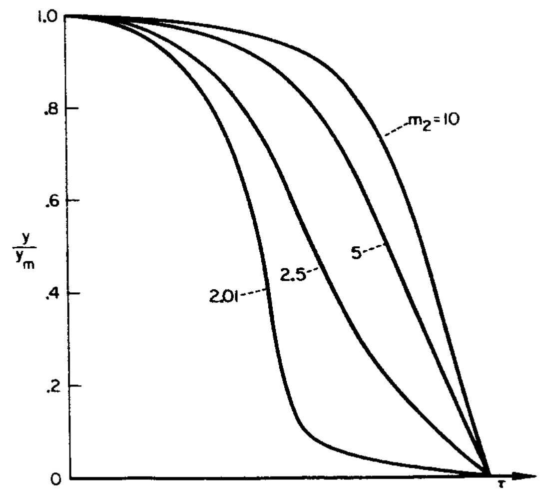
The oscillations (eq. (47)) for the cubic moments are plotted in figures 2(a), (b), and (c) to show the effect of m_2 for the various cases. In these figures, $\sqrt{Y_o} \equiv y/y_m$ is plotted versus the transformed independent variable, τ, over a quarter cycle of oscillation. As indicated by equation (42), m_2 is a ratio of the magnitude of the nonlinear part of the static moment to the magnitude of the linear part. The value $m_2 = 0$ represents the pure linear case, $m_2 = \infty$ the pure cubic case. In figure 2(d), y/y_m is plotted versus the original independent variable, t, normalized in such a way that the curves have the same frequency. This is equivalent to comparing each nonlinear moment with a different linear moment, in each case the linear moment being chosen to give the same frequency of oscillation as did the particular nonlinear moment. With



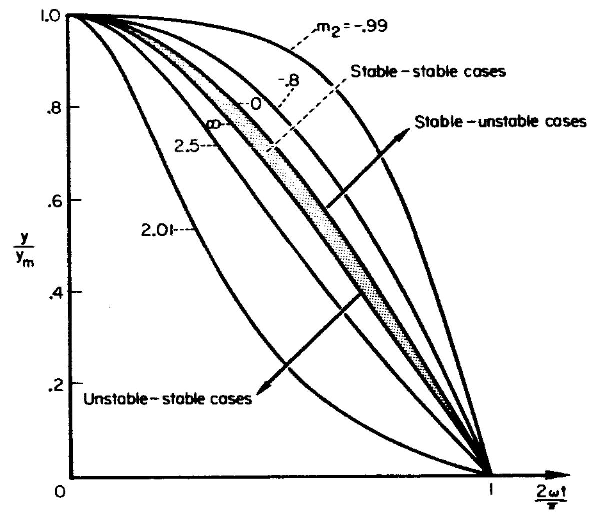
(a) Variation with τ; stable-stable 1-3 static moment.



(b) Variation with τ; stable-unstable 1-3 static moment.



(c) Variation with τ; unstable-stable 1-3 static moment.



(d) Variation with t.

Figure 2.- Effect of m_2 on oscillatory behavior.

this kind of plot, it can be seen that most nonlinearities in the moment have no drastic effect on the appearance of the motion (although the frequency is greatly affected). For instance, all stable-stable cases would fall within the shaded region of figure 2(d). Furthermore, the vast majority of stable-unstable cases ever encountered would be bracketed by the curves labeled 0.0 and -0.8; most unstable-stable cases would be bracketed by the curves labeled 2.5 and ∞.

APPROXIMATE SOLUTIONS

We now wish to find solutions for arbitrary values of H_s and $\overline{M}_s$. With the use of the special exact solutions already discussed, it is possible to find approximate solutions valid when the parameters H_s and $\overline{M}_s$ are small in a certain sense. Essentially, we shall find first-order perturbation solutions to the exact solutions that are special cases of equations (36) and (39). We shall consider only symmetric motion and hence make the restriction $H_s = \overline{M}_s = 0$ when s is odd.

Perturbation Solution Using the Integral Equation

Because the integral equation (36) has an exact solution for a special case, that is, when $\overline{M}_o > 0$ and $\overline{M}_s = 0$, $s > 0$, and when $H_s = 0$, $s \geq 4$, it is natural to ask if there exists a small perturbation of this exact solution in which certain parameters are small in an appropriate sense. The exact solution can be expressed in the form of equation (22), with $n = 2$,

$$y^2_{(o)}(\tau) = \frac{e^{-\frac{H_o\tau}{\nu}} \cos^2 \tau}{\frac{1}{{y_o}^2} + \frac{H_2}{2\nu}\int_o^\tau e^{-\frac{H_o\tau'}{\nu}} \cos^2 \tau' \, d\tau'} \tag{50}$$

Expression (50) can be considered as a zeroth order approximation. Since equation (36) is an integral equation, it is straightforward to use equation (50) and determine a first-order approximation to equation (36) as a first step in a scheme of successive approximations.

Consistent with the zeroth approximation, we require that $\overline{M}_o > 0$, and that

$$\left| \sum_{s \geq 2} \frac{\overline{M}_s}{\nu^2} y^s \right| \ll 1 \tag{51a}$$

and

$$\left| \sum_{s \geq 4} \frac{4H_s}{(2 + s)H_2} y^{s-2} \right| \ll 1 \tag{51b}$$

With these conditions, we can expand the denominator of the integrand of the integral in equation (36) in a Taylor series and keep the first term. The first-order approximation can now be obtained by replacing the y^2 that appears under the integral by the zeroth-order approximation $y^2_{(o)}$ and retaining terms consistent with equations (51). The first-order approximation thus can be written as

$$y^2_{(1)} = \frac{e^{-\frac{H_o\tau}{\nu}}\cos^2\tau}{\frac{1}{y_o^2} + \int_0^\tau e^{-\frac{H_o\tau'}{\nu}}\left\{\frac{2\cos^2\tau'}{\nu}\sum_{s\geq 2}\frac{H_s}{2+s}y^{s-2}_{(o)} - \sum_{s\geq 2}\frac{\overline{M}_s}{\nu^2}y^{s-2}_{(o)}\left[R(\tau') + \frac{H_2}{2\nu}y^2_{(o)}\cos^4\tau'\right]\right\}d\tau'} \tag{52}$$

Formula (52) for the first approximation is expressed in terms of quadratures involving the zeroth-order approximation. In general, these quadratures are difficult to evaluate in terms of tabulated functions. For the special case when the nonlinear damping terms are zero, however, the quadratures can be evaluated with the use of elementary functions. We now write the corresponding formal solution for the frequency function τ as a function of t. Starting with equation (37) and using the approximations consistent with the first-order solution, we obtain

$$\nu(t - t_o) = \tau - \int_0^\tau \left(\sum_{s\geq 2}\frac{\overline{M}_s}{\nu^2}y^s_{(o)}\right)\cos^2\tau'\,d\tau' \tag{53a}$$

where, as before, $y^2_{(o)}$ is given by equation (50). Since higher order products of $\overline{M}_s$ are neglected in the first approximation, we can also express τ explicitly in terms of t by rearranging equation (53a) to read

$$\tau = \nu(t - t_o) + \int_0^{\nu(t-t_o)}\left(\sum_{s\geq 2}\frac{\overline{M}_s}{\nu^2}y^s_{(o)}\right)\cos^2\tau\,d\tau \tag{53b}$$

In many practical problems of interest in aerodynamics, the nonlinear restoring-moment terms are not small, and thus the above first-order approximations are not particularly useful. The damping, however, is usually small; consequently, results having wide application can be obtained from a perturbation solution of the first-order differential equation (39).

Perturbation Solution Using the First-Order Differential Equation

Let us write equation (39) in a normalized form that naturally suggests a perturbation procedure. We introduce a characteristic amplitude y_a that we shall identify later and define the following symbols:

$$\left.\begin{aligned} Y &= \left(\frac{y}{y_a}\right)^2 , \qquad h = \frac{H_o}{\nu} \\ m_s &= \frac{\overline{M}_s {y_a}^s}{\nu^2} , \qquad h_s = \frac{1}{2 + s} \frac{H_s}{H_o} {y_a}^s , \qquad s \quad \text{even} \end{aligned}\right\} \tag{54}$$

Since $\overline{M}_s \approx M_s$, m_s is essentially a ratio of the magnitude of the sth nonlinear term of the static moment to the magnitude of the linear term; h_s is a similar ratio for the damping moment but multiplied by a constant, $1/(2 + s)$. With this notation, equation (39) becomes

$$\frac{dY}{d\tau} = \frac{-2Y\left(\tan \tau + h \sum_{s \geq o} h_s Y^{s/2}\right)}{\sin^2 \tau + \cos^2 \tau \sum_{s \geq o} m_s Y^{s/2}} \tag{55}$$

Since an exact solution exists for equation (55) when $h \equiv 0$, namely, equation (44), we seek a perturbation solution of equation (55) valid when $h \ll 1$. Therefore, we consider that $h \ll 1$ and $h_s = O(1)$ and assume over a half cycle, at least, that $Y(\tau)$ can be expanded in a series of the following form:

$$Y(\tau) = \sum_{m=o}^{\infty} h^m Y_m(\tau) \tag{56}$$

Substituting equation (56) into (55), collecting terms, and requiring that the coefficient of each power of h vanish identically yields an infinite set of equations for the functions Y_m. We shall confine our interest to the first-order perturbation and thus to only the functions Y_o and Y_1. The equations for Y_o and Y_1 are found to be

$$\frac{dY_o}{d\tau} = - \frac{2Y_o \tan \tau}{D_o} \tag{57a}$$

$$D_o \frac{dY_1}{d\tau} + \frac{2 \tan \tau}{D_o}\left[\sin^2 \tau + \cos^2 \tau \sum_{s \geq 0} \left(1 - \frac{s}{2}\right) m_s Y_o^{s/2}\right] Y_1 = -2Y_o \sum_{s \geq 0} h_s Y_o^{s/2} \tag{57b}$$

where

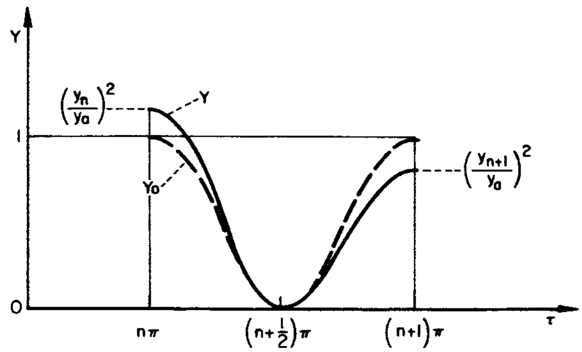
$$D_o \equiv \sin^2 \tau + \cos^2 \tau \sum_{s \geq 0} m_s Y_o^{s/2} \tag{57c}$$

With the use of equations (57a) and (57c), it is not difficult to verify that an integrating factor for equation (57b) is merely $\sec^2 \tau$, so that equation (57b) can be expressed as

$$\frac{d}{d\tau}(D_o \sec^2 \tau \, Y_1) = -2 \sec^2 \tau \, Y_o \sum_{s \geq 0} h_s Y_o^{s/2} \tag{58}$$

With the above equations we will be interested in the solution between two consecutive maximum amplitudes, say the nth maximum amplitude, y_n, and the (n+1)th, y_{n+1}. The zeroth order equation (57a) yields a periodic solution with a given maximum amplitude. We now select y_a so that the maximum amplitude squared of the zeroth order solution is the mean of the squares of the nth and (n+1)th maximum amplitudes

$$y_a^2 = \frac{1}{2}(y_n^2 + y_{n+1}^2)$$



Sketch (a)

We can now take equation (44) to be the solution of (57a) and identify y_m with y_a. The situation is shown in sketch (a).

Correct within the first-order approximation, y_n occurs at $\tau = n\pi$ and y_{n+1} occurs at $\tau = (n + 1)\pi$. We integrate equation (58) so that $Y_1(n\pi) = -Y_1(n\pi + \pi)$

$$Y_1(\tau) = -\frac{2 \cos^2 \tau}{D_o} \int_{(n+\frac{1}{2})\pi}^{\tau} \sec^2 \tau' \, Y_o \sum_{s \geq 0} h_s Y_o^{s/2} \, d\tau' \tag{59}$$

The function $Y_1(\tau)$ is expressed by a quadrature. Since $Y_o(\tau)$ can be written explicitly as a function of τ only for special cases, such as the cubic restoring moment (eq. (47)), a general explicit form for the integrand cannot be found. With a change in variable, however, the quadrature can be expressed in the form of elliptic or hyperelliptic integrals.

Consider a change of variable in the integral of equation (59) to Y_o, as defined by equations (57a) and (44). Then, in terms of Y_o, the function Y_1 can be written as

$$Y_1(\tau) = \frac{-Y_o G(Y_o)}{\sum\limits_{s\geq o} \frac{m_s}{2 + s}\left(2 + sY_o^{\frac{s+2}{2}}\right)} \tag{60a}$$

where

$$G(Y_o) = \pm \int_o^{Y_o} \frac{\sum\limits_{s\geq o} \frac{m_s}{2 + s}\left(2 + sY_o'^{\frac{s+2}{2}}\right)\sum\limits_{s\geq o} h_s Y_o'^{s/2}\, dY_o'}{\sqrt{Y_o' \sum\limits_{s\geq o} \frac{2m_s}{2 + s}\left(1 - Y_o'^{\frac{s+2}{2}}\right)}} \tag{60b}$$

for $n\pi \leq \tau \leq (n + 1)\pi$. The plus sign is used when $[n + (1/2)]\pi \leq \tau \leq (n+1)\pi$ and the minus sign when $n\pi \leq \tau \leq [n + (1/2)]\pi$. It is clear that $Y_1(\tau)$ and $G(Y_o)$ are odd functions of $\{\tau - [n + (1/2)]\pi\}$. When the polynomial in the radical of equation (60b) is a quartic or less, then the integral can be evaluated in terms of elliptic integrals and elliptic functions. We shall evaluate explicitly the complete integral later for special cases of the damping decrement.

The corresponding first-order form of the frequency equation (37) can now be determined. Inserting equation (56) into (37) and keeping only first-order terms in h yields

$$\nu(t - t_o) = \int_{(n+\frac{1}{2})\pi}^{\tau} \frac{d\tau'}{\sin^2 \tau' + \cos^2 \tau' \sum\limits_{s\geq o} m_s Y_o^{s/2}} - \frac{h}{2}\int_{(n+\frac{1}{2})\pi}^{\tau} \frac{\cos^2 \tau'\, Y_1 \sum\limits_{s\geq o} s m_s Y_o^{\frac{s-2}{2}}\, d\tau'}{\left(\sin^2 \tau' + \cos^2 \tau' \sum\limits_{s\geq o} m_s Y_o^{s/2}\right)^2} + O(h^2) \tag{61}$$

where here $t = t_o$ when $\tau = [n + (1/2)]\pi$. This completes the formal first-order solution. Because the first-order integral in equation (61) is very complicated, we shall not investigate it further here, but merely note that it can be integrated numerically without great difficulty when Y_o and Y_1 are known functions of τ.

THE DAMPING DECREMENT

We now wish to derive formulas that portray the distinguishing features of the nonlinear oscillation. Of an oscillation that is sufficiently well-defined, two distinguishing features are the maximum amplitudes and the half period. (By half period, we mean the time interval between two successive maximum amplitudes of the motion. In this sense, it is a pseudo-period, since the motion is not necessarily periodic. Given the half period, we can likewise define an angular frequency.) A pertinent parameter to consider is the ratio of two successive maximum amplitudes, called the damping decrement. A slightly more significant parameter is the logarithmic decrement because it is a constant for a strictly linear oscillation and thus forms a good standard for comparison.

Decrement for Small Damping

Consider now the first-order solution that neglects terms of order h^2 in equation (56); that is,

$$Y(\tau) = Y_o(\tau) + hY_1(\tau) + O(h^2) \tag{62a}$$

From this expression we find y_n and y_{n+1} to be given by

$$\left(\frac{y_n}{y_a}\right)^2 = 1 + hY_1(n\pi) + O(h^2) \tag{62b}$$

$$\left(\frac{y_{n+1}}{y_a}\right)^2 = 1 + hY_1(n\pi + \pi) + O(h^2) \tag{62c}$$

Noting that $Y_1(n\pi) = -Y_1(n\pi + \pi)$, dividing one equation by the other, and expanding for small h, we find the damping decrement to be

$$\left(\frac{y_n}{y_{n+1}}\right)^2 = 1 - 2hY_1(n\pi + \pi) + O(h^2) \tag{63}$$

Taking the logarithm of this expression and again expanding for small h yields the following formula for the logarithmic decrement:

$$\ln\left|\frac{y_n}{y_{n+1}}\right| = -hY_1(n\pi + \pi) + O(h^2) \tag{64}$$

Here $Y_1(n\pi + \pi)$ can be evaluated by equation (59) or (60). Before proceeding, however, it is convenient to modify equation (64) by defining an effective linear damping.

Effective Linear Damping

In a strictly linear oscillation it is known that the logarithmic decrement is a constant with the value

$$l n\left|\frac{y_n}{y_{n+1}}\right| = \frac{H_o \pi}{2\nu} \tag{65}$$

where ν is the frequency defined by $\nu^2 = |\overline{M}_o|$.[3] Analogously, we can define an effective linear damping H_{o_e} by

$$l n\left|\frac{y_n}{y_{n+1}}\right| \equiv \frac{H_{o_e}\pi}{2\omega} = h\,\frac{\pi}{2}\,\frac{\nu}{\omega}\,\frac{H_{o_e}}{H_o} \tag{66}$$

where now, consistent with the first-order analysis, we define the frequency ω by

$$\frac{\nu}{\omega} = \frac{1}{\pi}\int_{n\pi}^{n\pi+\pi} \frac{d\tau}{\sin^2\tau + \cos^2\tau \sum\limits_{s\geq o} m_s Y_o^{s/2}} \tag{67a}$$

or

$$\frac{\nu}{\omega} = \frac{1}{\pi}\int_{o}^{1} \frac{dY_o}{\sqrt{Y_o \sum\limits_{s\geq o} \frac{2m_s}{2+s}\left(1 - Y_o^{\frac{s+2}{2}}\right)}} \tag{67b}$$

---

[3]The logarithmic decrement can be written in more conventional form if we introduce the so-called damping ratio δ as $\delta = H_o/2M_o$. Then, realizing that $\nu = \sqrt{\overline{M}_o} = \sqrt{M_o - (H_o^2/4)}$ for a linear oscillator, we have

$$l n\left|\frac{y_n}{y_{n+1}}\right| = \frac{\pi\delta}{\sqrt{1 - \delta^2}}$$

For a linear system, the logarithmic decrement is independent of amplitude, and, therefore, we can also write

$$l n\left|\frac{y_n}{y_{n+2}}\right| = \frac{2\pi\delta}{\sqrt{1 - \delta^2}}$$

We now combine equations (64) and (66) to get

$$\frac{H_{O_e}}{H_O} = -\frac{2}{\pi}\frac{\omega}{\nu} Y_1(n\pi + \pi) + O(h) \tag{68}$$

Expression (68) is an important formula, and we shall investigate it in some detail.

General Formula for Effective Linear Damping

The formal first-order value for the effective linear damping can be found by evaluating $Y_1(n\pi + \pi)$ in formula (68) by (60) (noting that $Y_O(n\pi + \pi) = 1$)

$$\frac{H_{O_e}}{H_O} = \frac{\frac{2}{\pi}\frac{\omega}{\nu}}{\sum\limits_{s \geq o} m_s} \int_O^1 \frac{\sum\limits_{s \geq o} \frac{m_s}{2+s}\left(2 + sY_O^{\frac{s+2}{2}}\right) \sum\limits_{s \geq o} h_s Y_O^{s/2}\, dY_O}{\sqrt{Y_O \sum\limits_{s \geq o} \frac{2m_s}{2+s}\left(1 - Y_O^{\frac{s+2}{2}}\right)}} \tag{69}$$

where ω/ν is given by equation (67b). Formula (69) can be explicitly evaluated in terms of tabulated functions for restricted combinations of m_s. We shall investigate these special solutions in order to gain an appreciation for the influence of the nonlinearities on the damping, and to ascertain the asymptotic values of the general formula in various limits. On the basis of this information, an approximate, but quite general, formula can be established that is valid for almost the whole range of possible values of the nonlinear parameters.

Cubic Damping and Cubic Static Moment

Formula (69) can be evaluated in terms of elliptic integrals for a cubic static moment; that is, $m_s = 0$ when $s \geq 4$. For the sake of simplicity, we also consider only a cubic damping moment; that is, $H_s = 0$ when $s \geq 4$. For this case, formula (69) appears as

$$\frac{H_{O_e}}{H_O} = \frac{\frac{1}{\pi}\frac{\omega}{\nu}}{m_O + m_2} \int_O^1 \frac{\left[m_O + \frac{m_2}{2}\left(1 + Y_O^{\,2}\right)\right](1 + 2h_2Y_O)\, dY_O}{\sqrt{Y_O(1 - Y_O)\left[m_O + \frac{m_2}{2}(1 + Y_O)\right]}} \tag{70a}$$

where

$$\frac{\nu}{\omega} = \frac{1}{\pi} \int_0^1 \frac{dY_o}{\sqrt{Y_o(1 - Y_o)\left[m_o + \frac{m_2}{2}(1 + Y_o)\right]}} \tag{70b}$$

Two representations are required for the various static moments defined by equation (49) to express (70a) in terms of elliptic integrals. (No such distinction is needed for the various cubic damping moments that could exist.) These are as follows:

$$\frac{H_{o_e}}{H_o} = \frac{2m_o + m_2}{m_o + m_2}\left[\frac{1}{2}\left(1 + \frac{m_2}{2m_o + m_2}\frac{\gamma_2}{K}\right) + h_2\left(\frac{\gamma_1}{K} + \frac{m_2}{2m_o + m_2}\frac{\gamma_3}{K}\right)\right] \tag{71a}$$

$$K = K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad \text{Complete elliptic integral of first kind} \tag{71b}$$

$$E = E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \quad \text{Complete elliptic integral of second kind} \tag{71c}$$

Stable-stable and unstable-stable static moment: (formula 236.16, ref. 12)

$$\left.\begin{aligned}
\gamma_1 &\equiv \frac{1}{k^2}\left[(k^2 - 1)K + E\right] \\
\gamma_2 &\equiv \frac{1}{3k^4}\left[(3k^4 - 5k^2 + 2)K + 2(2k^2 - 1)E\right] \\
\gamma_3 &\equiv \frac{1}{5k^2}\left[4(2k^2 - 1)\gamma_2 + 3(1 - k^2)\gamma_1\right] \\
k^2 &= \frac{1}{2}\frac{m_2}{m_o + m_2}
\end{aligned}\right\} \tag{72}$$

Stable-unstable static moment: (formula 234.16, ref. 12)

$$\left.\begin{aligned}
\gamma_1 &\equiv \frac{1}{k^2}(K - E) \\
\gamma_2 &\equiv \frac{1}{3k^4}\left[(2 + k^2)K - 2(1 + k^2)E\right] \\
\gamma_3 &\equiv \frac{1}{5k^2}\left[4(k^2 + 1)\gamma_2 - 3\gamma_1\right] \\
k^2 &= \frac{-m_2}{2 + m_2}
\end{aligned}\right\} \tag{73}$$

An equivalent expression for H_{o_e}/H_o for a cubic static moment, but a linear damping moment, that is, $h_2 = 0$, has been given by Murphy (ref. 5) and Murphy and Hodes (ref. 6). Murphy obtained his results in a different manner. He assumed that the damping was small and then used the exact elliptic functions obtained for zero damping in an averaging process over a cycle of motion. The perturbation method of Murphy thus yields the same value for the effective damping as the present analysis for a cubic static moment, but it is difficult to extend the method of Murphy to higher order static moments owing to the complexities in representing more complicated static-moment solutions by elliptic functions. An approximate formula for H_{o_e}/H_o for a cubic static moment and linear damping has also been derived by Murphy (ref. 7) by a "quasi-linear" method, and we will discuss this later in the section entitled, "Comparison for a Cubic Moment."

Expression (71a) for H_{o_e}/H_o in terms of elliptic integrals is actually a fairly involved formula. Also, the quadratures given by equation (70) are difficult to evaluate numerically because of their singular nature at both limits. It is interesting that the effective linear damping can be expressed by another set of quadratures that possibly may be easier to evaluate numerically (especially with electronic computers) than either equation (70a) or (71a). We obtain this alternate formula by transforming equation (70a) from the Y_o variables to the τ variables with the use of equations (43), (44), and (47). The alternate result is

$$\frac{H_{o_e}}{H_o} = \frac{2\omega}{\pi\nu}\frac{2m_o+m_2}{m_o+m_2}\left\{\int_0^{\pi/2}\frac{d\tau}{(\sin^2\tau+m_o\cos^2\tau)+\sqrt{(\sin^2\tau+m_o\cos^2\tau)^2+\mu\cos^4\tau}}\right.$$
$$\left.+\,2h_2(2m_o+m_2)\int_0^{\pi/2}\frac{\cos^2\tau\,d\tau}{\left[(\sin^2\tau+m_o\cos^2\tau)+\sqrt{(\sin^2\tau+m_o\cos^2\tau)^2+\mu\cos^4\tau}\right]^2}\right\} \tag{74a}$$

where

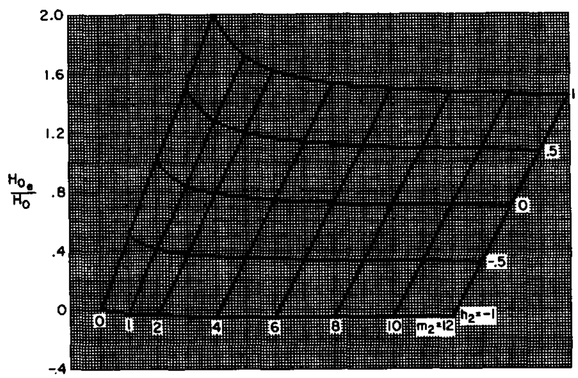
$$\frac{\nu}{\omega} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\tau}{\sqrt{\left(\sin^2 \tau + m_o \cos^2 \tau\right)^2 + \mu \cos^4 \tau}} \tag{74b}$$

and

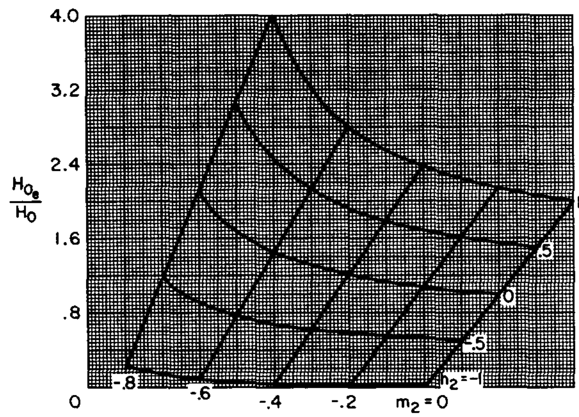
$$\mu = m_2(2m_o + m_2)$$

The above quadratures are proper integrals and can be evaluated numerically without difficulty. It is worth mentioning that equation (74a) can easily be extended to higher order damping terms.

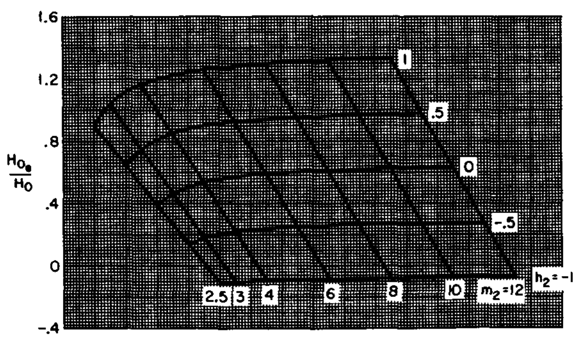
There are several ways in which formulas (71a) and (74a) can be plotted. If one is interested in knowing H_{O_e}/H_O for given values of h_2 and m_2, carpet plots are useful. Such carpet plots are shown in figures 3(a), (b), and (c). Because of the ease of making linear interpolations, these plots are convenient to use.



(a) Stable-stable 1-3 static moment.



(b) Stable-unstable 1-3 static moment.



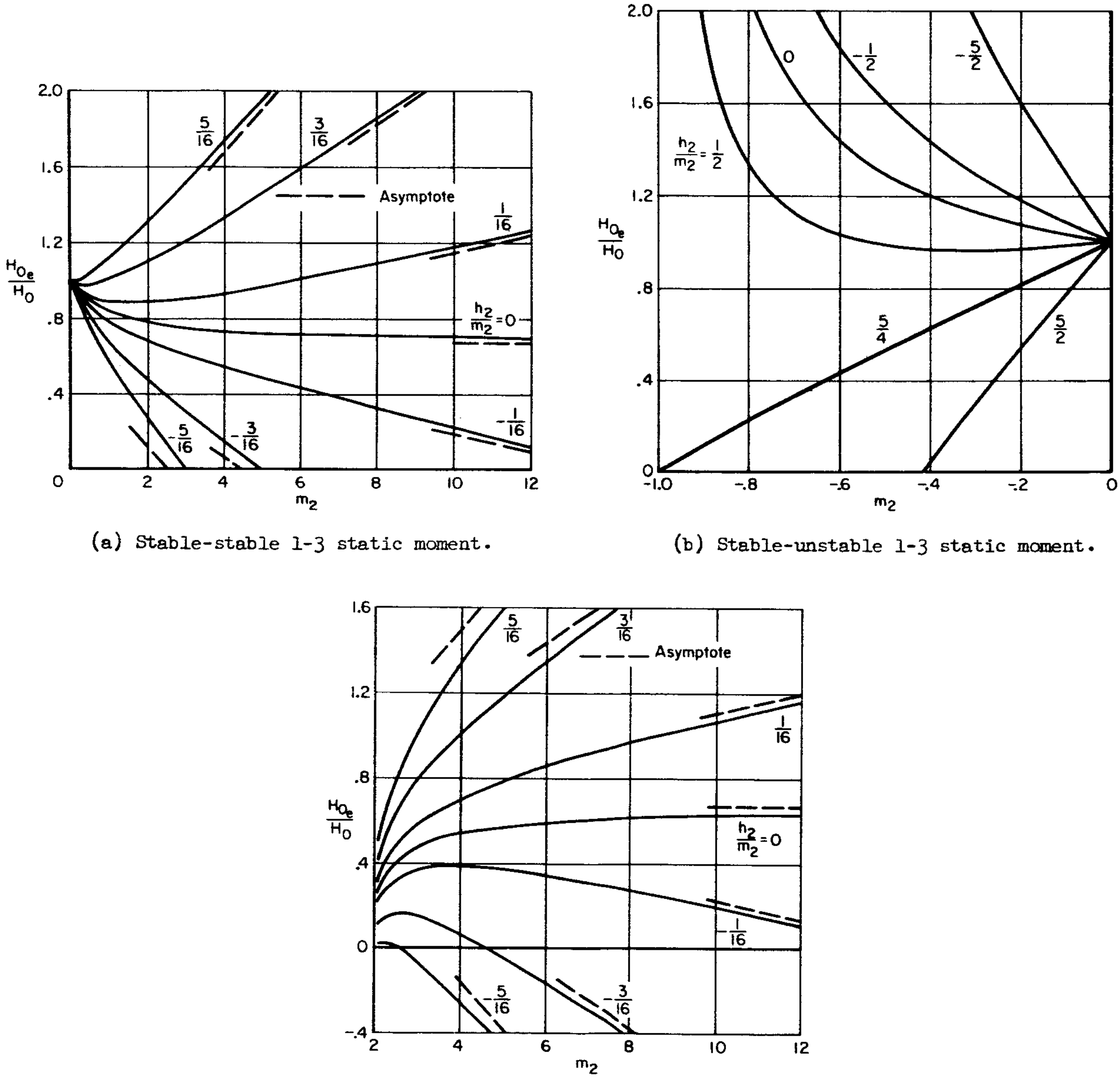
(c) Unstable-stable 1-3 static moment.

Figure 3.- Carpet plot of effective linear damping.

Another method of plotting illustrates how H_{o_e}/H_o varies with amplitude for given values of the damping- and static-moment parameters. We thus plot H_{o_e}/H_o as a function of m_2 and set $h_2 \equiv (h_2/m_2)m_2$ so that

$$\frac{h_2}{m_2} = \frac{H_2}{4H_o}\frac{\nu^2}{\overline{M}_2} = \frac{H_2}{4H_o}\frac{|\overline{M}_o|}{\overline{M}_2}$$

can be varied as a parameter that does not depend on amplitude for a given oscillator. These plots are illustrated in figures 4(a), (b), and (c). The curves denoted by $h_2/m_2 = 0$ were given by Murphy and Hodes (ref. 6). The



(a) Stable-stable 1-3 static moment.

(b) Stable-unstable 1-3 static moment.

(c) Unstable-stable 1-3 static moment.

Figure 4.- Effective linear damping versus m_2.

curves in figures 4(b) and (c) are singular for the stable-unstable moment at $m_2 \equiv \overline{M}_2 y_n^2/\overline{M}_0 = -1$ and for the unstable-stable moment at $m_2 \equiv \overline{M}_2 y_n^2/|\overline{M}_0| = 2$. The dashed lines are the asymptotes for $m_2 \to \infty$. It is possible to show by an expansion for large m_2 that the asymptotes are given by

$$\left(\frac{H_{o_e}}{H_o}\right)_{asymp} = \left[\frac{2}{3} + \left(-\frac{4}{3} + 2c + \frac{5}{16}c^2\right)\frac{m_o h_2}{m_2}\right] + c\left(\frac{h_2}{m_2}\right)m_2$$

where

$$c = \frac{32}{5}\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]^2, \qquad \Gamma = \text{gamma function}$$

or

$$\left(\frac{H_{o_e}}{H_o}\right)_{asymp} = \left(0.667 + 0.296\,\frac{m_o h_2}{m_2}\right) + 0.7311\left(\frac{h_2}{m_2}\right)m_2 \tag{75}$$

Likewise, for the stable-stable and stable-unstable moments, the variation as $m_2 \to 0$ is given by

$$\frac{H_{o_e}}{H_o} = 1 + \left(-\frac{5}{16} + \frac{h_2}{m_2}\right)m_2 + O(m_2^2) \tag{76}$$

Another interesting aspect of these curves is the appearance of maxima and minima in some cases. If these curves were established by experiment, this behavior might be considered perplexing without theoretical background. They portray the effects of strong interactions of the nonlinear damping and static moments. Still another interesting feature is the curve labeled $h_2/m_2 = 5/4$ in figure 4(b). Detailed examination of equation (74) shows that this value is the dividing point between cases where a limit cycle can or cannot exist, given a stable-unstable static moment. For $h_2/m_2 < 5/4$, there is no possibility of a limit cycle.

All curves shown in figure 4 were compared with results obtained from numerical integrations of equation (1). In these numerical integrations, various values of the parameter $h \equiv H_o/|\overline{M}_o|^{1/2}$ were chosen up to a value of about 0.1. This encompasses the practical range of damping encountered in ballistic-range testing. In all cases, the results of the numerical integrations agreed almost exactly with the results from formula (71a) or (74a) shown in figure 4. To the scale these curves are plotted, no differences can be shown.

When $H_o < 0$, a limit cycle exists when $H_{o_e} = 0$. The amplitude of the limit cycle may be found by setting $H_{o_e} = 0$ in equation (71a) or (74a). Using equation (74a) we get

$$(-h_2)_l \equiv \left(-\frac{H_2 y_a^2}{4H_o}\right)_l$$

$$= \frac{1}{2(2m_o+m_2)} \frac{\displaystyle\int_0^{\pi/2} \frac{d\tau}{(\sin^2\tau + m_o\cos^2\tau) + \sqrt{(\sin^2\tau + m_o\cos^2\tau)^2 + \mu\cos^4\tau}}}{\displaystyle\int_0^{\pi/2} \frac{\cos^2\tau\, d\tau}{\left[(\sin^2\tau + m_o\cos^2\tau) + \sqrt{(\sin^2\tau + m_o\cos^2\tau)^2 + \mu\cos^4\tau}\right]^2}} \tag{77}$$

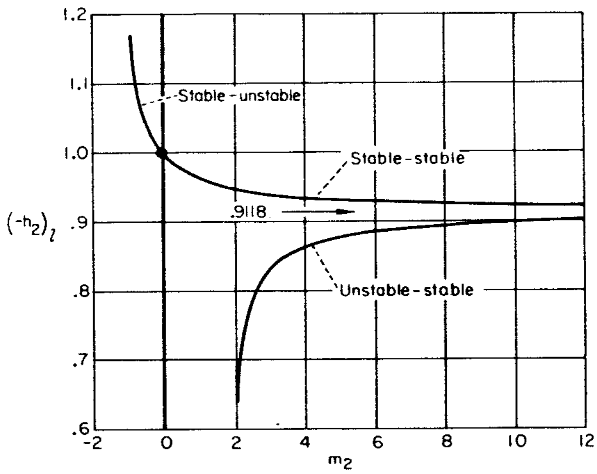


Figure 5.- Amplitude of limit cycle versus m_2.

An alternate formula is obtained from equation (71a), which is equivalent to the result found by Murphy and Hodes (ref. 6) by investigating the singular nature of the limit cycle in an amplitude plane. The effect of the cubic static moment on the amplitude of the limit cycle is demonstrated in figure 5. Except in the region of the singularities, the effect of m_2 is not large, yielding a value of $(-h_2)_l = 0.9118$ as $m_2 \to \infty$ contrasted to the well-known value $(-h_2)_l = 1$ when $m_2 \to 0$. It will be shown later that for quintic and higher order static moments, the effects are much greater.

Arbitrary One-Term Static Moment

The quadratures in equation (69) can be evaluated in terms of tabulated functions for the special case of an arbitrary one-term static moment. This case yields the asymptotic values that are obtained as the highest order term in an arbitrary polynomial static moment tends to large values.

We now consider $\overline{M}_s \neq 0$ for $s = n$, where n is an even integer. Hence we choose $\nu^2 = \overline{M}_n y_n{}^n > 0$ so that $m_s = 0$, $s \neq n$, and $m_n = 1$. Expression (69) now appears as

$$\frac{H_{o_e}}{H_o} = \frac{1}{\pi}\frac{\omega}{\nu}\left(\frac{2}{2+n}\right)^{1/2} \int_0^1 \frac{\left(2 + nY_o^{\frac{n+2}{2}}\right)\sum_{s\geq 0} h_s Y_o^{s/2}\, dY_o}{\sqrt{Y_o\left(1 - Y_o^{\frac{n+2}{2}}\right)}} \tag{78a}$$

and

$$\frac{\nu}{\omega} = \frac{1}{\pi}\left(\frac{2 + n}{2}\right)^{1/2} \int_0^1 \frac{dY_o}{\sqrt{Y_o\left(1 - Y_o^{\frac{n+2}{2}}\right)}} \tag{78b}$$

With the substitution $x = Y_o^{(n+2)/2}$, these quadratures can be written in the form of beta functions, which in turn can be evaluated in terms of gamma functions, Γ. For equation (78a), we thus obtain

$$\frac{H_{o_e}}{H_o} = \sum_{s \geq 0} \gamma_{sn} h_s = \frac{4}{n + 4} + \sum_{s \geq 2} \gamma_{sn} h_s \tag{79a}$$

where

$$\gamma_{sn} = \frac{4(2 + s)}{2s + 4 + n} \frac{\Gamma\left(\frac{s + 1}{n + 2}\right)\Gamma\left(\frac{1}{n + 2} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{n + 2}\right)\Gamma\left(\frac{s + 1}{n + 2} + \frac{1}{2}\right)} \qquad s, n \geq 0 \tag{79b}$$

Although formula (79) was derived for a one-term static moment, it also represents the asymptotic value of H_{o_e}/H_o for a polynomial static moment in which the nth term, m_n, is dominant and stable and tends to infinity. For purely linear damping it is interesting that the asymptotic value is simply $H_{o_e}/H_o = 4/(n + 4)$. On the other hand, for a linear static moment, $n = 0$, equation (79a) can be written

$$\frac{H_{o_e}}{H_o} = 1 + 2 \sum_{s \geq 2} \frac{1 \cdot 3 \cdot 5 \; . \; . \; . \; (s - 1)}{2 \cdot 4 \cdot 6 \; . \; . \; . \quad s} h_s \tag{79c}$$

Redd et al. (ref. 9), considering only a linear static moment, computed the same value as equation (79c) up to $s = 8$ by a different method. Their conclusion, however, that a nonlinear static moment does not significantly alter the formula (79c), except near unstable trim, is not substantiated by the present analysis. Here it is found that equation (79c) is a special case of a more general formula (79a), and each type of nonlinear static moment may indeed affect H_{o_e}/H_o significantly.

Expression (79a) may be used to examine the effects of the nonlinear static moment on the value of the limit amplitude. Consider a two-term damping moment given by $H_s = 0$ when $s \neq 0$ or k (k even) so that the first term is linear and the second term is nonlinear of arbitrary order. The limit cycle occurs when $H_o < 0$ and may be determined by setting $H_{o_e} = 0$ to obtain

$$(-h_k)_l = \left[\frac{-H_k}{(2 + k)H_o} y_a{}^k\right]_l = \frac{4}{(n + 4)\gamma_{kn}} \tag{80}$$

The value of $(-h_k)_l$ is a function of both k and n, and some of the values of $(-h_k)_l$ are given in the following table:

n \ k	2	4	6	8
0	1.000	1.333	1.600	1.829
2	.912	1.167	1.368	1.540
4	.870	1.084	1.250	1.392
6	.845	1.034	1.178	1.300
8	.829	1.001	1.130	1.238
∞	.750	.833	.875	.900

It can be seen from this table that the value of $(-h_k)_l$ decreases for a given value of k as n increases. Here we recall that $n = 0$ represents a purely linear static moment.

Expansion for Small Static-Moment Nonlinearities

Consider the value of H_{oe}/H_o when the nonlinearities in the static moment are small. Here we assume $\overline{M}_o > 0$ so that $m_o = 1$, and we assume that m_s is small so that higher order products and cross-products can be neglected. With these conditions, it is possible to expand equation (69) for small m_s and evaluate each integral term by term. The result can be written as follows

$$\frac{H_{oe}}{H_o} = \sum_{k \geq o} h_k \left[2I(k - 1) - \sum_{s \geq 2} b_{ks} m_s\right] + \text{higher order products of } m_s \tag{81a}$$

where

$$b_{ks} = \frac{2}{2 + s} [(k + 2)I(s + k + 1) + (s - k)I(k - 1) - (s + 2)I(k - 1)I(s + 1)] \tag{81b}$$

$$\begin{aligned} I(n) &= \frac{1}{\pi}\int_o^1 \frac{z^{n/2}\, dz}{(1 - z)^{1/2}} \\ &= 1\,, \qquad\qquad n = -1 \\ &= \frac{1 \cdot 3 \cdot 5 \ . \ . \ . \ \ n}{2 \cdot 4 \cdot 6 \ . \ . \ . \ (n + 1)}\,, \qquad n \text{ odd} \end{aligned} \tag{81c}$$

Some of the values of b_{ks} for various values of k and s are tabulated as follows:

s \ k	0	2	4	6	8
2	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{21}{128}$	$\frac{1}{8}$	$\frac{105}{1024}$
4	$\frac{11}{12}$	$\frac{37}{96}$	$\frac{33}{128}$	$\frac{19}{96}$	$\frac{125}{768}$
6	$\frac{279}{256}$	$\frac{121}{256}$	$\frac{657}{2048}$	$\frac{127}{512}$	$\frac{13415}{65536}$
8	$\frac{193}{160}$	$\frac{171}{320}$	$\frac{939}{2560}$	$\frac{1169}{4096}$	$\frac{7745}{32768}$
∞	2	1	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{35}{64}$

Expression (76) for the cubic damping and cubic static moment is a special case of (81a). Formula (81a) will be of use in the construction of a general approximate formula for H_{o_e}/H_o.

APPROXIMATE GENERAL FORMULA FOR H_{o_e}/H_o

Although the formal representation (69) is a very general result, it is difficult to evaluate for arbitrary combinations of the damping and static moments. Even for the cubic case, the evaluation in terms of elliptic integrals is rather complicated. Based on the special cases (79) and (81), however, it is possible to construct an approximate formula that is extremely accurate.

A perusal of the formal solution (69) indicates that H_{o_e}/H_o can be expressed by

$$\frac{H_{o_e}}{H_o} = \sum_{k\geq o} \frac{\sum\limits_{s\geq o} \alpha_{ks} m_s}{\sum\limits_{s\geq o} \beta_{ks} m_s} h_k \tag{82}$$

where α_{ks} and β_{ks} are functions of the parameters $m_o, m_2, m_4, \ldots$ The functions α_{ks} and β_{ks} depend only weakly on these parameters, however, and thus we wish to find approximate values of α_{ks} and β_{ks} that are independent of $m_o, m_2, m_4, \ldots$ We can do this by requiring that equation (82) have the proper behavior when the nonlinear static moment is very large and stable and when the nonlinear static moment is very small; that is, equation (82) should have the same behavior as (79) and (81) in these particular limits.

Formula (82) is thus guaranteed to be valid when the dominant term $m_o, m_2, m_4, \ldots \to \infty$ and also when the terms $m_o, m_2, m_4, \ldots \to 0$.

We begin, somewhat arbitrarily, by requiring that $\beta_{ko} = 1$. We do this in order to agree with the leading term of the sum $\sum_{s \geq o} m_s$ that appears in the denominator of equation (69). It can be shown that if equation (82) is to reduce to both (79) and (81) in these respective limits, then α_{ks} and β_{ks} are given by

$$\left.\begin{aligned} \alpha_{ko} &= 2I(k-1), \qquad \beta_{ko} = 1 \\ \alpha_{ks} &= \frac{b_{ks}\gamma_{ks}}{\alpha_{ko} - \gamma_{ks}}, \qquad \beta_{ks} = \frac{b_{ks}}{\alpha_{ko} - \gamma_{ks}}, \qquad s \geq 2 \end{aligned}\right\} \tag{83}$$

where γ_{ks}, b_{ks}, and $I(n)$ are defined by equations (79b), (81b), and (81c). Some values of α_{ks} and β_{ks} are presented in the following tables.[4]

α_{ks}

s \ k	0	1	2	3	4	5	6	7	8
0	2.0000	1.2732	1.0000	0.8488	0.7500	0.6791	0.6250	0.5821	0.5469
1	1.5349	1.0095	.8065	.6913	.6143	.5582	.5147	.4797	.4510
2	1.2500	.8398	.6797	.5875	.5250	.4788	.4428	.4136	.3895
3	1.0568	.7208	.5894	.5130	.4608	.4218	.3912	.3663	.3453
4	.9167	.6323	.5213	.4565	.4118	.3784	.3519	.3301	.3118
5	.8102	.5639	.4679	.4118	.3731	.3439	.3206	.3014	.2851
6	.7266	.5092	.4249	.3756	.3415	.3156	.2950	.2779	.2633
7	.6589	.4645	.3894	.3456	.3152	.2920	.2735	.2581	.2450
8	.6031	.4272	.3596	.3202	.2928	.2720	.2552	.2413	.2294
∞	0	0	0	0	0	0	0	0	0

[4] Note that α_{ks} and β_{ks} are given for both even and odd values of k and s. We are still considering only symmetric oscillations, but can extend the analysis to cases where k and/or s are odd by insisting that both the static moment and damping moment be odd functions of the dependent variable. This is done by employing absolute values where needed. For instance, a quadratic static moment would be written as $M = M_o y + M_1 y|y|$.

β_{ks}

s \ k	0	1	2	3	4	5	6	7	8
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.9593	.9575	.9542	.9505	.9468	.9434	.9399	.9364	.9337
2	.9375	.9346	.9297	.9243	.9188	.9135	.9084	.9038	.8997
3	.9247	.9211	.9155	.9091	.9027	.8964	.8905	.8850	.8797
4	.9167	.9127	.9067	.8998	.8929	.8861	.8797	.8735	.8677
5	.9115	.9073	.9011	.8940	.8868	.8798	.8729	.8665	.8603
6	.9082	.9039	.8975	.8904	.8831	.8758	.8688	.8621	.8558
7	.9060	.9016	.8953	.8882	.8808	.8735	.8664	.8596	.8530
8	.9047	.9003	.8940	.8868	.8795	.8722	.8650	.8582	.8516
∞	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Comparison for a Cubic Moment

For a cubic moment, equation (82) looks as follows

$$\frac{H_{O_e}}{H_O} = \frac{m_O + 0.6250m_2}{m_O + 0.9375m_2} + \frac{m_O + 0.6797m_2}{m_O + 0.9297m_2} h_2 + \ . \ . \ . \tag{84}$$

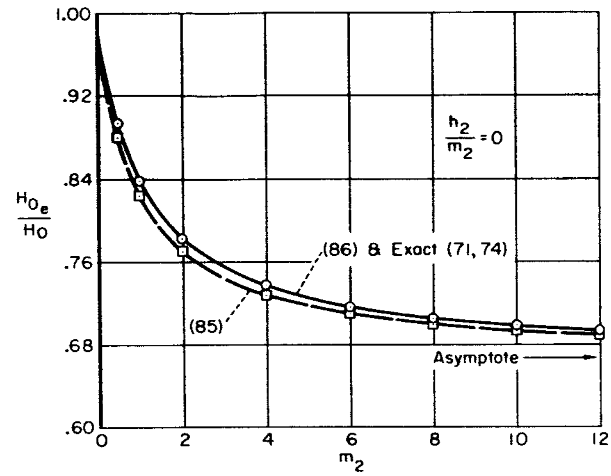
The accuracy of this expression can be demonstrated in several ways. First, we will compare equation (84) with the exact solution (71) or (74) when only linear damping is present ($h_2 = h_4 = \ . \ . \ . = h_n = 0$). In this way we can also compare an approximate formula that is given by Murphy in reference 7, which, in our notation, is

$$\frac{H_{O_e}}{H_O} = \frac{m_O + (3/4)m_2}{m_O + (9/8)m_2} \tag{85}$$

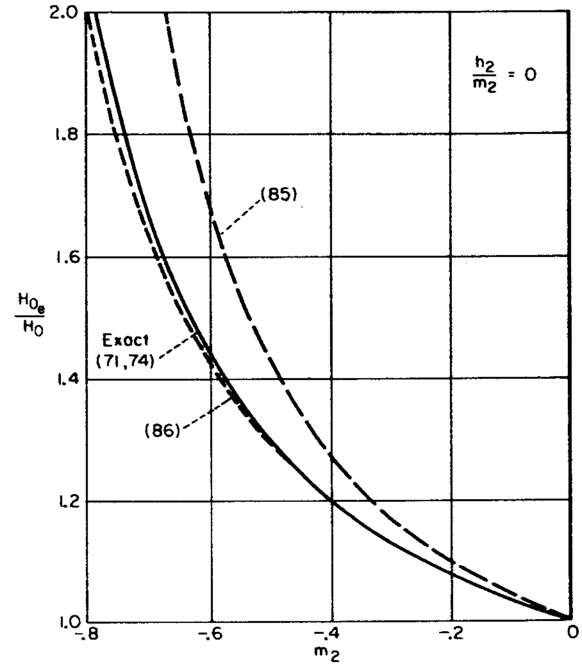
For the sake of comparison, we write the corresponding case of (84) with fractions as

$$\frac{H_{O_e}}{H_O} = \frac{m_O + (5/8)m_2}{m_O + (15/16)m_2} \tag{86}$$

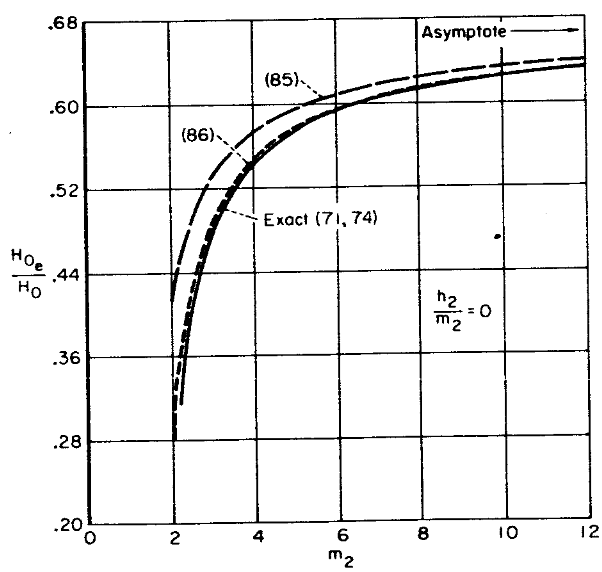
Both of these formulas have the correct limit as $m_2 \to \infty$ $(H_{O_e}/H_O = 2/3)$, but when $m_O = 1$, formula (85) does not quite have the correct behavior as $m_2 \to 0$, yielding $H_{O_e}/H_O = 1 - (3/8)m_2 + O(m_2^2)$ instead of the exact value $H_{O_e}/H_O = 1 - (5/16)m_2 + O(m_2^2)$ of the present analysis. For the stable-stable moment the approximate formula (86) is practically indistinguishable from the exact solution when plotted on the same graph, and (85) is also a good approximation for this case. These results are shown in figure 6(a). The main deviations occur in the stable-unstable and unstable-stable moments near the singular points, which we illustrate in figures 6(b) and (c). The approximate



(a) Stable-stable 1-3 static moment.



(b) Stable-unstable 1-3 static moment.

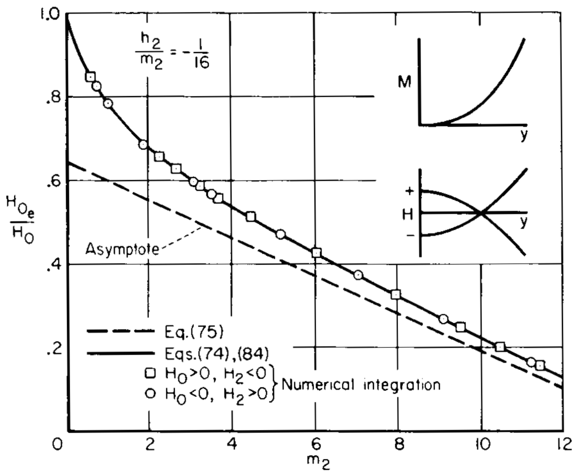


(c) Unstable-stable 1-3 static moment.

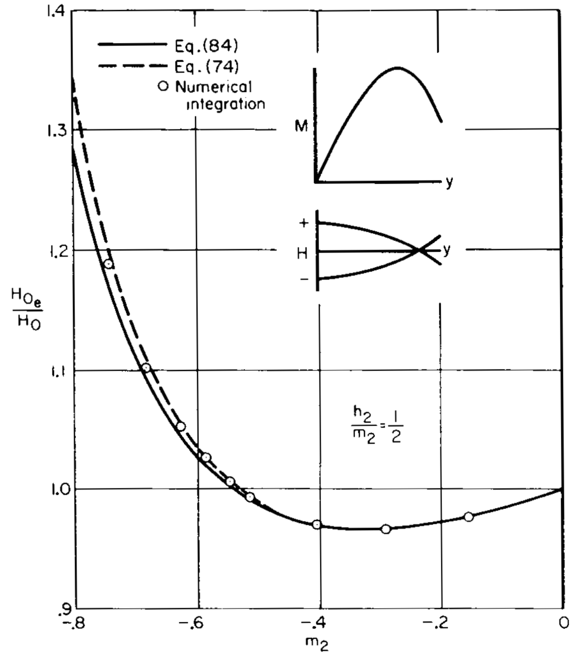
Figure 6.- Comparison of approximate solutions with exact solution.

formula (86) of the present method is seen to agree very well with the exact curve except in the immediate vicinity of the singular points, which are $m_2 = -1$ for the stable-unstable moment and $m_2 = 2$ for the unstable-stable moment. The present approximation (86) is obviously superior to the approximate formula (85).

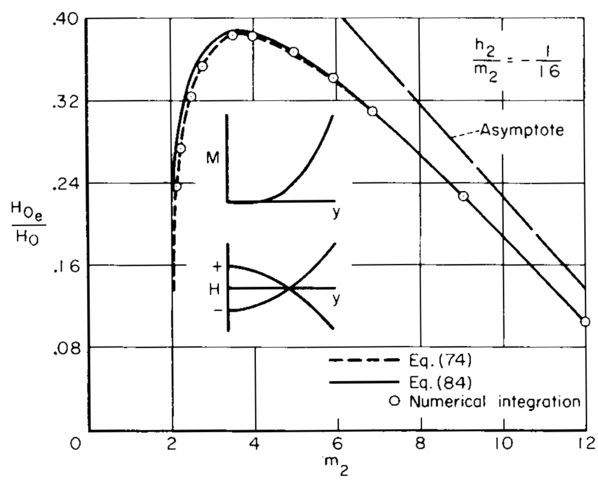
The next check that we can make on the accuracy of equation (84) is to allow h_2 to be nonzero. A number of different cases were investigated, and one example for each type of linear-cubic moment is shown in figure 7. Figure 7(a) shows results for a stable-stable moment. The magnitude of the nonlinearities being considered on this plot can be visualized by observing



(a) Stable-stable 1-3 static moment; stable-unstable or unstable-stable 1-3 damping moment.



(b) Stable-unstable 1-3 static moment; stable-unstable or unstable-stable 1-3 damping moment.



(c) Unstable-stable 1-3 static moment; stable-unstable or unstable-stable 1-3 damping moment.

Figure 7.- Comparison of present analysis with numerical integrations.

the small-scale plots of static and damping moments that appear. These are accurate representations of the nonlinearities, not sketches, and indicate a wide departure from a linear system. The first thing to note in this figure is that equation (84) agrees to this scale exactly with equation (71) or (74), which in turn agrees exactly with numerical integration of equation (1). The second thing to note is that a negative h_2/m_2 (recall that the parameter h_2/m_2 is a constant, independent of amplitude, for a given oscillator) can be obtained in two ways, H_o negative and H_2 positive or H_o positive and H_2 negative. The first case leads to a limit cycle, the second at a high enough amplitude to an unstable situation. The surprising thing is that H_{oe}/H_o is the same for both of these cases, a fact that was revealed in the analysis but is not intuitively obvious.

Also indicated in figure 7(a) is the asymptotic behavior of H_{oe}/H_o as given by equation (75). It can be shown that expanding equation (84) for large m_2 (with $h_4, h_6, \ldots = 0$) yields the same values as equation (75), except that the coefficient of $m_o h_2/m_2$ is 0.289 instead of the more exact value 0.296.

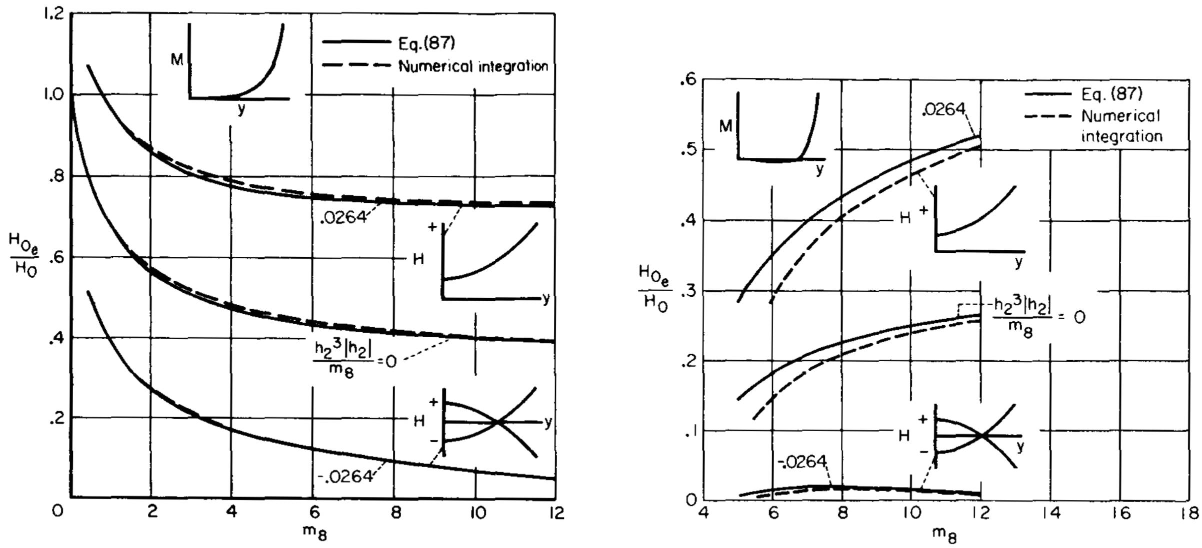
Figures 7(b) and (c) show representative cases for the stable-unstable and unstable-stable static moments. Once again the magnitude of the nonlinearities being covered is shown by the small-scale plots. For both of these cases, the accuracy of equation (84) is not as good as it was for the stable-stable moment, but the agreement is still excellent except in the immediate vicinity of the singular points. The final point to be made is that the curves for the limit amplitude, $(-h_2)_l$, presented in figure 5, can be closely duplicated by equation (84) except, again, in the immediate vicinity of the singular points.

Comparison for a 1-9 Moment

A more extreme nonlinear case was next investigated to check the adequacy of the approximate solution (82). A 1-9 static moment and a 1-3 damping moment ($M_o, M_8, H_o, H_2 \neq 0$) were considered. For this case, equation (82) appears as

$$\frac{H_{oe}}{H_o} = \frac{m_o + 0.3016m_8}{m_o + 0.9047m_8} + \frac{m_o + 0.3596m_8}{m_o + 0.8940m_8} h_2 \tag{87}$$

In this case, the closed-form solution (69) involves hyperelliptic integrals and cannot be evaluated by simple means, but a comparison can be made with results of numerical integrations of equation (1). These comparisons are made in figure 8 for a stable-stable and unstable-stable static moment. (The stable-unstable case is not too interesting in that the moment is essentially linear almost to the singular point and then suddenly goes unstable.) Instead of the parameter h_2/m_2, which was pertinent for the cubic moment, the parameter that remains constant for a given oscillator is now $h_2{}^3|h_2|/m_8$. The comparisons show that the approximate solution is not so good as it was for a linear-cubic static moment. However, for the stable-stable case, agreement is still excellent. For the unstable-stable case, agreement is satisfactory except close to the singular point (i.e., $m_8 = 5$).



(a) Stable-stable 1-9 static moment; 1-3 damping moment.

(b) Unstable-stable 1-9 static moment; 1-3 damping moment.

Figure 8.- Comparison of present analysis with numerical integrations.

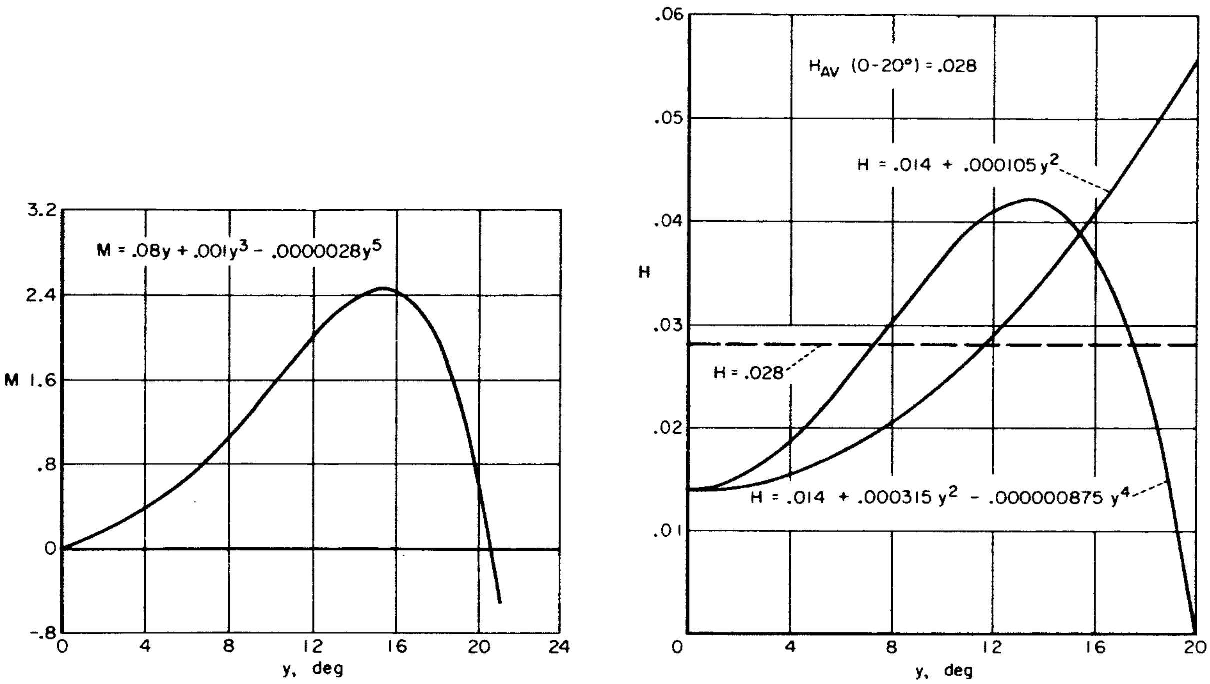
Comparison for a 1-3-5 Moment

The final comparison that was made was for a 1-3-5 static moment in conjunction with several different damping moments, the most nonlinear of which was also 1-3-5. For this case, equation (82) appears as

$$\frac{H_{0_e}}{H_0} = \frac{m_0 + 0.6250m_2 + 0.4583m_4}{m_0 + 0.9375m_2 + 0.9167m_4} + \frac{m_0 + 0.6797m_2 + 0.5213m_4}{m_0 + 0.9297m_2 + 0.9067m_4} h_2$$

$$+ \frac{0.7500m_0 + 0.5250m_2 + 0.4118m_4}{m_0 + 0.9188m_2 + 0.8929m_4} h_4 \tag{88}$$

Too many possibilities exist to cover this case in general. Hence, a particular 1-3-5 static moment was selected as being very nonlinear, a stable-stable-unstable moment. This moment is shown in figure 9(a). Three different damping moments were then chosen, consisting of the following: (1) linear term only, (2) linear-cubic, and (3) linear-cubic-quintic. The three damping moments are shown in figure 9(b). Note that they were selected so that the average value of the damping moment between 0° and 20° was the same in each case.



(a) 1-3-5 static moment.

(b) Damping moments.

Figure 9.- Moments used for more complex comparisons.

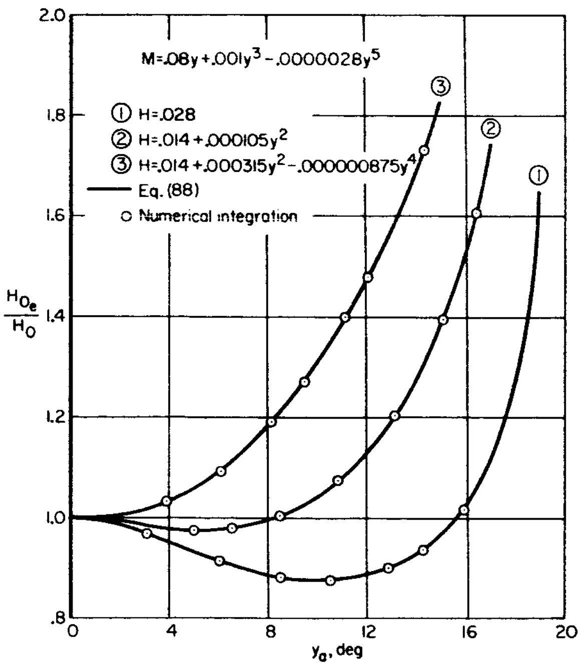


Figure 10.- Comparison of present analysis with numerical integrations; 1-3-5 static moment; various damping moments.

Equation (88) is compared with numerical integrations of equation (1) for the three cases in figure 10. To the scale plotted and for the range covered, no differences existed.

All of these comparisons that have been presented indicate that equation (82) is surprisingly accurate over a wide range of nonlinearities in both the static and damping moment. With this established, we will next concern ourselves with determination of the coefficients $\overline{M}_s$ and H_s from data.

Formula (82) with (83) can be useful in determining the nonlinear damping parameters of an oscillator from a set of observed oscillations. Let us assume that a given oscillator is governed by the differential equation (1). We wish to infer from observed oscillations the appropriate values of $H_o, H_2, H_4, \ldots$ and $M_o, M_2, M_4, \ldots$.

We assume that the damping is small so that $h \equiv H_o/\nu \ll 1$ (or more appropriately, since $\nu = |\overline{M}_n y_a{}^n|^{1/2}$ is actually arbitrary, we could also say $H_o/\omega \ll 1$, where ω is the frequency and is proportional to ν). This assumption is normally satisfied in ballistic-range testing. In addition, we assume that the frequency and maximum amplitudes for each half cycle of the motion can be accurately determined from the data. The effective linear damping for each half cycle is then computed by

$$H_{o_e} \equiv \frac{2}{\pi} \omega \ln \left| \frac{y_n}{y_{n+1}} \right| \tag{89}$$

Using an approximation for the frequency quadrature (eq. (67b)) that is developed in reference 13, we can write an expression for the frequency as a function of the parameters $\overline{M}_o, \overline{M}_1, \overline{M}_2, \ldots$ as

$$\omega^2 = \overline{M}_o + \sum_{n>o} A_n \overline{M}_n y_a{}^n \tag{90a}$$

where

$$A_n \equiv \frac{4}{n+2}\left[1 - \left(\frac{1}{2}\right)^{\frac{n+2}{2}}\right] \tag{90b}$$

This formula is a good approximation for large and small nonlinearities, except in the vicinity of the singular points of static instability. The understanding and use of equation (90a) makes it possible to extract the appropriate values of $\overline{M}_n$ by fitting equation (90a) to a set of frequency versus amplitude data that has been measured for a given oscillator. For instance, if an oscillator is governed by a linear-cubic static moment, then equation (90) yields

$$\omega^2 = \overline{M}_o + \frac{3}{4} \overline{M}_2 y_a{}^2$$

The frequency squared then gives a straight line when plotted against $y_a{}^2$. The slope of the curve yields the value of $\overline{M}_2$ and the ω^2 intercept yields $\overline{M}_o$. When $h \ll 1$, we note that $\overline{M}_n = M_n[1 + O(h^2)]$. Hence, for small damping, the values of the actual static parameters (M_n) are determined. The cubic moment is the simplest case; for more general cases the static moment polynomial (eq. (90a)) is terminated so that an adequate fit of the frequency versus maximum amplitude data is obtained (ref. 13).

A similar but slightly more complicated situation exists for determining the damping parameters H_n. In this case, the static parameters play an important role, whereas for the determination of the static-moment parameters M_n, the effect of the damping parameters can be neglected. The formula for H_{o_e} as a function of $H_o, H_1, H_2, \ldots$ can be written as

$$H_{o_e} = B_o H_o + \sum_{n>o} B_n H_n {y_a}^n \tag{91a}$$

where equation (69) is used to determine B_n as

$$B_n \equiv \frac{\frac{2}{\pi}\frac{\omega}{\nu}}{(2 + n)\sum\limits_{s\geq o} m_s} \int_o^1 \frac{Y^{n/2} \sum\limits_{s\geq o} \frac{m_s}{2 + s}\left(2 + sY^{\frac{s+2}{2}}\right) dY}{\sqrt{Y \sum\limits_{s\geq o} \frac{2m_s}{2 + s}\left(1 - Y^{\frac{s+2}{2}}\right)}} \tag{91b}$$

or, from equations (82) and (83), approximately as

$$B_n \equiv \frac{1}{2 + n} \frac{\sum\limits_{s\geq o} \alpha_{ns} \overline{M}_s {y_a}^s}{\sum\limits_{s\geq o} \beta_{ns} \overline{M}_s {y_a}^s} \tag{91c}$$

Expression (91a) is analogous in form to (90a). In this case, however, the coefficients B_n are functions of $\overline{M}_o, \overline{M}_1, \overline{M}_2, \ldots$ and y_a, whereas the coefficients A_n are constants. In the determination of the damping, however, the static parameters can be regarded as already determined by the use of (90). Hence, the coefficients B_n are functions of y_a insofar as determination of the damping is concerned.

It is now useful to divide both sides of equation (91a) by B_o, obtaining

$$\frac{H_{o_e}}{B_o} = H_o + \sum_{n>o} \frac{B_n}{B_o} H_n {y_a}^n \tag{92}$$

Because $B_o, B_1, B_2, \ldots$ each vary with y_a in a similar manner, the ratios B_n/B_o vary slowly with y_a. Equation (92) can now be used to study a set of damping data in a manner analogous to studying frequency data with equation (90).

As a simple example, assume that a set of data for a given oscillator has been obtained and that the static moment, completely arbitrary, has been determined with the use of formula (90). Let it be assumed that the damping moment can be described by a linear term and a single arbitrary nonlinear term so that (92) appears as

$$\frac{H_{o_e}}{B_o} = H_o + \frac{B_n}{B_o} H_n {y_a}^n \tag{93}$$

where n is not known. Choosing a particular value of n determines B_n/B_o. Then one can plot H_{o_e}/B_o versus $(B_n/B_o){y_a}^n$. When the right value of n is chosen, the data will fall on a straight line. The slope of the line yields H_n, and the intercept of the line with the H_{o_e}/B_o axis yields H_o. This will be true regardless of the particular form of the static moment.

For an arbitrary damping moment described by a given set of data over an amplitude range, there may be several or many combinations of the parameters $H_o, H_1, H_2, \ldots$ that will fit the data. This situation also occurs in determining the static-moment parameters (ref. 13). One must often settle for a member of a class of moments that gives a good fit. Thus, a certain amount of experience may be useful in analyzing the data. For a set of data including small angles of y_a as well as large, one should plot the damping data as H_{o_e}/B_o versus $(B_2/B_o){y_a}^2$ (or as a first try against ${y_a}^2$ since B_2/B_o varies slowly with y_a). If the data fall on a straight line, the damping moment is a cubic as described by (93) with $n = 2$. If the data deviate from a straight line, which might be expected to occur at larger values of y_a, one can estimate the magnitude and sign of the next higher order damping term needed to fit the data. In such a manner the appropriate form of damping polynomial tends to suggest itself.

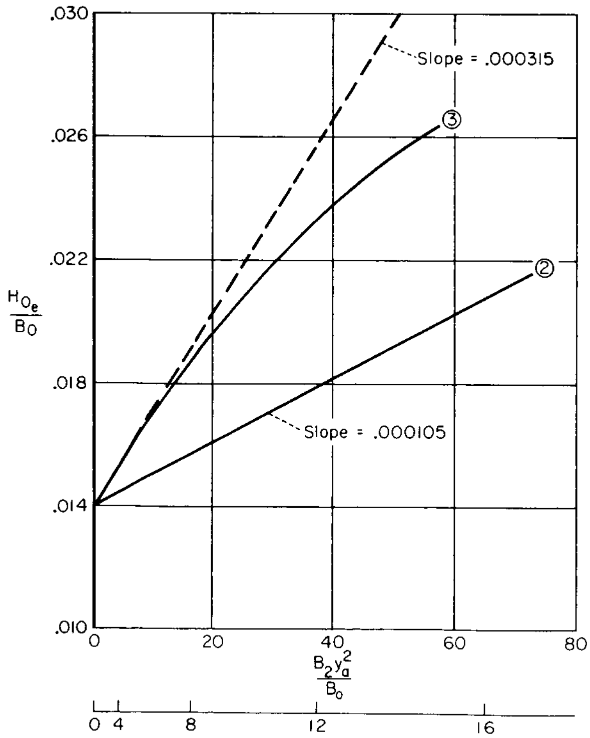


Figure 11.- Form of data to reveal type of nonlinearity in damping.

The more data that are available, the better the damping moment can be defined. Finally the data can be fitted by some curve-fitting technique for various combinations of damping parameters. The combinations that fit the data best are the appropriate ones to represent the damping moment.

In figure 11 the results of figure 10 are plotted in the manner suggested. Results for the linear-cubic damping moment fall on a straight line as expected. The deviation of the 1-3-5 damping data from a straight line indicates that the damping moment is more complicated than linear cubic; the fact that the results curve downward from a straight line indicates that the next higher order term required is a destabilizing term, which is, in fact, the case.

CONCLUDING REMARKS

The preceding investigation has considered the effects of nonlinearities on damped oscillations. It was found that this investigation could be conducted analytically in two ways. With large damping nonlinearities and small "effective" static-moment nonlinearities, an approximate solution could be established by means of an integral equation. On the other hand, when the damping is small but the static nonlinearities arbitrarily large (a common situation in aerodynamics), an approximate solution could be obtained by means of an equivalent first-order differential equation.

When the damping is small and the static nonlinearities large, it is convenient to study the characteristics of the nonlinear oscillations by means of a parameter called the "effective linear damping." This parameter is readily obtainable from experiment and offers a means of deducing the nonlinear damping characteristics of an oscillator from a set of observed oscillations. Comparisons with some exact solutions and with numerical solutions showed that the approximate formula derived herein for the effective linear damping is extremely accurate over a wide range of nonlinearities. The limit cycle of a nonlinear oscillation occurs when the "effective linear damping" is zero, and the limiting amplitude is easy to obtain for various combinations of the nonlinear damping and static moments. The concept of an "effective linear damping" is thus very useful in analyzing nonlinear damping.

The integral equation (36) and the first-order equation (39) are believed to be novel formulations for nonlinear oscillation problems. It is possible that further investigation of these equations would be fruitful.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., June 27, 1966
124-07-02-11-21

APPENDIX A

MODIFIED EQUATIONS OF KRYLOFF AND BOGOLIUBOFF

Here we wish to modify the approximation methods of Kryloff and Bogoliuboff (ref. 4) for nonlinear oscillations so that the pertinent equations fit conveniently into the framework of the present analysis. We shall forego the Fourier series expansions of the usual Kryloff-Bogoliuboff analysis and find forms of the amplitude and frequency equations that are amenable to appropriate transformations that in turn yield a basic exact solution as a special case.

Following Kryloff and Bogoliuboff, consider a nonlinear equation of the following form:

$$\frac{d^2x}{dt^2} + \nu^2 x = -f\left(x, \frac{dx}{dt}\right) \tag{A1}$$

where f is an arbitrary function of x and dx/dt. If $f \equiv 0$, then the solution is given by

$$x = a \cos(\nu t + \varphi) \tag{A2}$$

where a and φ are arbitrary constants. For the general equation (A1), we assume equation (A2) to be a solution with $a = a(t)$ and $\varphi = \varphi(t)$ suitably determined functions of t.

Let us choose $a(t)$ and $\varphi(t)$ by following the method of variation of parameters for linear systems. The derivative of equation (A2) is

$$\frac{dx}{dt} = \frac{da}{dt} \cos(\nu t + \varphi) - a\left(\nu + \frac{d\varphi}{dt}\right) \sin(\nu t + \varphi) \tag{A3}$$

Since we have two functions to determine, $a(t)$ and $\varphi(t)$, we have two conditions at our disposal. One, of course, will be the original differential equation (A1). The other we choose as

$$\frac{da}{dt} \cos(\nu t + \varphi) - a \frac{d\varphi}{dt} \sin(\nu t + \varphi) = 0 \tag{A4}$$

so that equation (A3) now reads simply

$$\frac{dx}{dt} = -a\nu \sin(\nu t + \varphi) \tag{A5}$$

just as for the case $f \equiv 0$. Differentiating equation (A5) and substituting into the original equation (A1), we get

$$\frac{da}{dt}\sin(\nu t + \varphi) + a\frac{d\varphi}{dt}\cos(\nu t + \varphi) = \frac{1}{\nu} f[a \cos(\nu t + \varphi), -a\nu \sin(\nu t + \varphi)] \tag{A6}$$

Expressions (A4) and (A6) are two equations for da/dt and $d\varphi/dt$. Solving for da/dt and $d\varphi/dt$, we obtain

$$\frac{da}{dt} = \frac{1}{\nu} f[a \cos(\nu t + \varphi), -a\nu \sin(\nu t + \varphi)]\sin(\nu t + \varphi) \tag{A7a}$$

$$\frac{d\varphi}{dt} = \frac{1}{a\nu} f[a \cos(\nu t + \varphi), -a\nu \sin(\nu t + \varphi)]\cos(\nu t + \varphi) \tag{A7b}$$

These are the basic equations of Kryloff and Bogoliuboff.

Equations (A7a, b) are coupled and thus must be solved simultaneously. Equation (A7a) can be uncoupled, however, if we introduce, in place of t, a new independent variable τ defined as follows:

$$\tau = \nu t + \varphi \tag{A8a}$$

It follows that

$$\frac{d\tau}{dt} = \frac{\nu}{1 - \frac{d\varphi}{d\tau}} \tag{A8b}$$

Equations (A7a, b) may now be written in terms of τ as follows:

$$\frac{da}{d\tau} = \frac{\frac{1}{\nu^2} f(a \cos \tau, -a\nu \sin \tau)\sin \tau}{1 + \frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau} \tag{A9a}$$

$$\frac{d\varphi}{d\tau} = \frac{\frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau}{1 + \frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau} \tag{A9b}$$

$$\frac{d\tau}{dt} = \nu\left[1 + \frac{1}{a\nu^2} f(a \cos \tau, -a\nu \sin \tau)\cos \tau\right] \tag{A9c}$$

We shall term equations (A9a, b, c) the modified Kryloff-Bogoliuboff equations. Equations (A9a) and (A9c) are the pertinent ones since the solution to equation (A1) is given by $y = a(\tau)\cos \tau$. Equation (A9a) is a first-order nonlinear equation for a as a function of τ, and it is uncoupled from the other two. When it has been solved, the other equations can be solved at least by quadratures.

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