DYNAMICS AND EVOLUTION OF CLOSE BINARY SYSTEMS

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1. INTRODUCTION

Since the discovery of the periodicity of the change in light of Algol (β Persei) in 1782 by John Goodricke in England and his correct interpretation for the variability of light of this star as due to the "interpositions of a large body revolving around Algol," the study of light variation together with the accompanying variation in radial velocity, known respectively as the light curve and the (radial) velocity curve, has provided us with the sole means for the empirical study of close binary system. On the other hand, the close binary system can also be understood theoretically from Newtonian mechanics. Thus, our function is to reconcile the observational results in the form and light and velocity curve with the prediction based on the dynamical principles and thereby to derive a consistent picture of the close binary system. Indeed, the history of our study of close binaries shows clearly that, like any field in science where theories and experiments mutually help to make progress, observations of light and velocity curves and calculations based on dynamics act as two facets of a single process of successive approximations that lead finally to the present understanding of the nature of close binaries. Thus, the general form of the light and velocity curve suggests as a first approximation that dynamically the two components behave like two mass points and geometrically look like two spheres. Any analysis of light curves and velocity curves that is based on this assumption which we call the gross, analysis will be discussed in §2 and is included here for those who are not closely associated with observational astronomy.

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Obviously this assumption is far from being true. Indeed, the prediction based on this assumption fails to agree completely with the observed light curve. Hence, the deviation of the observed results from the predicted ones opens up a way to improve the dynamical model of close binaries. Thus, the component stars may not be regarded as point masses and their shape not spherical. Also, other body or bodies than the two components themselves may be present in the system. Such refined treatments are given in the later sections.

Whether these complications can all be determined unambiguously from the photometric and spectroscopic observation with the aide of dynamical principles is not known at present. Is the empirical information—when conditioned by Newtonian mechanics and other laws of physics—comprehensive enough so that a unique determination of the geometrical configuration and physical state of the system can be performed at least in principle? This fundamental question has, to my knowledge, never been asked before, let alone answered. Undeniably this is a question of high mathematical complexity. Whatever the answer may be to this question, we are safe in predicting that such a unique determination is unlikely in practice because it is too difficult to disentangle one effect from many others in a single light and velocity curve, which has a limited accuracy imposed by measuring means. Such a situation which is common to all branches of observational science in general is particularly evident in astrophysics.

If we cannot treat the various deviations from the binary problem in an overall manner, we may nevertheless study them individually and compare the results of each individual effect with observation. In this way, we can still test the consistency between observation and theory but the uniqueness of interpretation is not warranted by this kind of approach. By necessity it is this kind of approach which astrophysicists heretofore have cultivated in their study of close-binary systems and which we will follow in our treatment.

Next we consider the Roche model and the related problem of mass overflow from the stellar surface. The mass motion resulting from the overflow is then discussed according to different approaches. In the last two sections the origin and evolution of close binary stars are examined in the light of what we have learned from this article as well as from the present understanding of evolution of single stars.

Finally, we would like to add that the main purpose of the present article is to help understand the physical nature of close binary systems. It is intended to serve as an introduction to the basic physics of close binaries. Consequently, theoretical studies of a pure mathematical nature which cannot be matched in the foreseeable future with observation will not be discussed here. Similarly observational results of individual close binaries are mentioned only when they have some bearing either on general principles or on the common nature of the binary stars. Because of a limitation of space we shall not be able to review here many physical problems such as models of peculiar systems, physical processes in the gaseous streams, atmospheric eclipses, etc. as well as many technical problems such as the distortions of velocity and light curves due to various causes and the methods of correction, accurate determination of the absolute dimension of eclipsing systems.

For surveys and observations the reader is referred to Kukarkin and Parenago's (1963) article.

While close binaries are not necessarily undergoing eclipse as seen by us, we are interested in only those that do, because the light variation caused by eclipse reveals much information about the nature of the system. It is true that the mere fact of undergoing eclipse does not necessarily warrant a close binary, but its chance of being one is very high. For this reason a theoretical study of close binaries is intimately associated with the observational investigation of eclipsing variables.

For definiteness we shall regard rather arbitrarily all binaries with separations less than 10 times the mean radius of the two component stars as close. Thus, the close binaries can be further classified into three categories: (1) detached systems, (2) semi-contact systems and (3) contact systems as we shall see in §5.

2. OBSERVATIONAL BACKGROUND

In this section we assume that the binary motion may be treated as a two-body problem, i.e. each component acts as a mass point and that the circular disk of each star follows a given law of limb-darkening. Under these simplifying assumptions, the physical state of the two component stars is specified by their masses (M_1, M_2), their radii (R_1, R_2) and their luminosities (L_1, L_2). Altogether there are six independent parameters to be determined for component stars themselves. In accordance with convention, we have denoted here by subscript 1 the quantity that is related to the primary component and by subscript 2 that related to the secondary component. Needless to say, the principal component is the brighter one of the two but is not necessarily the more massive one, although in most cases the brighter one is also the more massive one. From the radii and luminosities, the effective temperatures (T_1, T_2) or equivalently the surface brightness (J_1, J_2) can be readily computed.

The state of a binary can be specified under the assumption just made by the period, P, the eccentricity, e, the semi-major axis, a, of the relative orbit, the inclination i of the orbital plane, and the longitude of the periastron ω as defined by the angular distance of the periastron from the ascending node measured in the direction of orbital motion, and T the time of periastron passage. Altogether there are also six independent parameters to be determined for the binary itself. All these parameters are often called the orbital elements.

In the course of analysis of the velocity curve, it happens that we can also determine the radial velocity—called the γ velocity—of the entire binary system. Although the γ velocity is not connected with the internal properties of a binary system, it is nevertheless regarded also as an orbital element by convention.

In most cases of close binaries the two components move around each other in circular orbits. This gives $e=o$ which is sometimes assumed in the analysis of light curves of eclipsing binaries. Since the periastron loses its meaning when $e=o$, the binary motion is then fixed by the time of the deeper minimum of the light curve instead of the time of periastron passage.

The determination of the orbital elements from the light curve—a process perfected by H. N. Russell, J. E. Merrill, etc.—has been discussed in detail elsewhere in this series (Irwin 1962). While the procedure itself is a very tedious one, the underlying principle and what can be derived from it can be simply stated. First the law of limb-darkening is assumed to be a priori given for each component and is usually written as

$$J=J_c(1-x+x\cos\theta) \qquad (2.1)$$

where J_c is the surface brightness at the center of the disk, x is the coefficient of limb-darkening, and θ is the angle between the line of sight and the stellar radius vector to the point in question. Needless to say, both J_c and x vary with the wavelengths in which the light curve is obtained.

The period of the binary can be obtained very accurately from the long observation of the times of light minimum of the eclipsing binary. Then the shape of the light curve, obtained by observations and all reduced to one single cycle, depends upon the following elements, e, ω, R_1/a, R_2/a, i, $L_1/(L_1+L_2)$, x_1 and x_2. Conversely the observed light curve determines these orbital elements. If the binary satisfies the simplifying assumptions we have made and the light curve is determined in all phases, such a determination is unique in most cases and can be performed in practice, although it is quite complicated when e deviates appreciably from zero and i from $\pi/2$. Usually only the value of $e\cos\omega$ is determined with a high accuracy from the intervals of the minima in such cases. Also, under some circumstances, considerable ranges of various orbital elements lead, within the accuracy of observation, to the identical light curve.

For example, sometimes one cannot even tell from the light curve whether the eclipse is total or annular, consequently we cannot decide which one of the two component stars has a larger radius than the other. The intrinsic difficulty of determination of orbital elements however, arises in the first place from the various effects which are not included in our simplifying assumptions, and in the second place from the imperfection of the observed light curve.

As has been emphasized by Irwin (1962), only in the most favorable cases and with the best observations can we determine x_1 and x_2 from the analysis of the light curve. In general, these two parameters can be assigned beforehand in accordance with the prediction by theory of stellar atmospheres (e.g. Chandrasekhar 1950).

In any case the light curve which gives only the sizes of the stars in terms of their separation does not reveal the complete information about the nature of the binary system. In order to derive the dimension of the system in an absolute measure we depend upon the velocity curve. References to the analysis of velocity curves may be found for example in articles by Struve and Huang (1958), by Petrie (1962) and by others.

For some binaries spectra of both components may be seen on a spectrogram but for others only one is visible, depending upon the relative magnitude between the components and their colors (Hynek 1951). If both components are visible, two velocity curves —one from each component star—give $M_1\sin^3 i$, $M_2\sin^3 i$, $a_1\sin i$, $a_2\sin i$, where a_1 and a_2 are the semi-major axes of the orbit of the two components in the rest frame of reference, i.e.,

$$a=a_1+a_2 \qquad (2.2)$$

The velocity curve also gives $e\cos\omega$, $e\sin\omega$, time of periastron passage, T, and the γ velocity. Since the factor, $\sin i$, is persistently associated with the measured radial velocity, we cannot get rid of it from the spectroscopic results as we have seen. Here i determined from the light curve supplies the missing information. Hence, the spectroscopic results combined with the photometric data determine the properties of the binary system completely, if the two component stars obey our simplified assumption.

If the spectrum of only one component is visible, we can determine $e\cos\omega$, $e\sin\omega$, T, and γ without difficulty just as in the double-lined spectroscopic binary. However, concerning the mass and dimension of the system, we can now obtain from the velocity curve (of the primary component alone) only the quantity,

$$f_1(M)=\frac{M_2^3 \sin^3 i}{(M_1+M_2)^2}, \qquad (2.3)$$

known as the mass function and $a_1 \sin i$. When the results of analysis of both the light curve and the velocity curve are combined, $M_2^3/(M_1+M_2)^2$ and a_1 are obtained. In such a case we cannot determine M_1 and M_2 separately. The missing link is the mass ratio expressed by either

$$\alpha=\frac{M_2}{M_1} \quad \text{or} \quad \mu=\frac{M_2}{M_1+M_2}. \qquad (2.4)$$

Thus for an eclipsing binary showing only the spectrum of one component, other means must be sought in order to determine the mass ratio. The means, if available, varies from one case to another and has no general rule. Basically it consists of an estimate from various physical arguments any one of the following parameters, m_1, m_2, α, a, R_1, and R_2 since only one additional parameter is needed to specify completely a single-spectrum eclipsing binary. The fact that a knowledge of either R_1 or R_2 can determine the masses of two components and their mean separation may be seen if we remember that R_1/a and R_2/a can be empirically obtained from the light curve. This gives a from which we can calculate M_1+M_2 from the following well-known relation

$$\left(\frac{P}{2\pi}\right)^2=\frac{a^3}{G(M_1+M_2)}, \qquad (2.5)$$

where G is the gravitional constant. Consequently M_1 and M_2 may be separately determined.

In this way the physical and dynamical state of many an eclipsing binary has been painstakingly derived by investigators in the past few decades. Those binaries with apparent photographic magnitudes brighter than 8.5 at maximum have been compiled and studied by Plaut (1950, 1953). Other catalogues of the elements of eclipsing binary systems have been provided recently by Kopal and Shapley (1956) and Wood (1963).

3. DYNAMICAL CONSIDERATIONS OF THE DEVIATION FROM THE TWO-BODY PROBLEM

There are several reasons for which the motion of component stars cannot be regarded as a problem of two bodies. The component stars in a close pair obviously cannot have a spherical symmetry in density distribution. The mass of each star may not be constant. A third body or bodies in the form of stars, planets or resisting medium composed of gases, dust, and larger particles may be present in the binary system. All these effects disturb the orbit of the binary star from what would be expected from Kepler's law of binary motion.

If we regard Kepler's law as the zeroth order approximation to the orbital motion of close binaries, each of those effects mentioned above produces a corresponding perturbation which may be computed in principle to high order of approximation. It is self-evident that starting at the second order approximation, cross terms each arising from two effects will appear in the final result. While it is interesting to investigate the effect of each cause to a high degree of approximation, it is only the result of the first order approximation that the astrophysicist has actually sought in most cases because his main purpose is to identify among various perturbing factors the one that is dominant in any given binary. Needless to say, the result of higher order approximation for a single effect is often useful but we have to be very careful in applying its result quantitatively to the interpretation of observed phenomena because of the presence both of the cross terms just mentioned and of the first-order term due to some effect which has not been considered.

In what follows we shall discuss each of these perturbing factors separately, emphasizing physical significance at the expense of mathematical completeness.

3.1. Departure of Component Stars from the Spherical Symmetry

Because of the rotational and tidal distortions the component star in a close binary system can never be treated as having a spherical symmetry in its density distribution. Thus, the mutual attractive force between the two stars can be resolved into two components: one is the dominant force resulting from the mutual attraction of two point masses M_1 and M_2 and the other is the perturbing force arising from the deviations from the spherical symmetry. Since the perturbing force is very small compared with the dominant force, the orbital motion should follow closely, with only a slight deviation, to what is predicted in the two-body problem. In other words, the component star may be approximated, in each short time interval, as exercising Keplerian motion. Because of the small perturbing force it is obvious that the orbits defined by two short time-intervals which are themselves separated by a long time-interval, cannot be identical. Thus, we may describe the motion as elliptical but the orbital ellipse changes with time. Obviously the rate of change for each of the orbital elements depends upon the perturbing force and indeed can be expressed in terms of it. The mathematical procedure underlying this physical conception is the method of variation of parameters which is often used to study the motion of a planet under the dominant force of the sun and the perturbing force of other planets. Its application to the study of close binaries was due to Russell (1928) whose calculation was later improved by Sterne (1939 a,b,c). Also, Cowling (1938) studied this problem by an entirely different approach. The presentation adopted in the following discussion follows that given by Sterne (1939 a).

The calculation consists of three steps: (1) to find the disturbing potential field, at the location of the primary due to the undistorted secondary component and due to the primary's own axial rotation, (2) to find both the deformation of the primary due to the disturbing potential obtained in the first step and the potential field at the location of the secondary due to the deformed primary and (3) to find the change of orbital elements because of the potential field obtained in the second step. Since the potential field derived in the first step distorts the primary and the distorted primary in turn produces in the second step a disturbing potential to effect the binary motion, the first two steps may be regarded as a process of successive approximation.

Let S_1 and S_2 be respectively the centers of mass of the primary and the secondary which are at a distance, r, apart. If r_1 and r_2 are respectively the distances of any point from S_1 and S_2, the undistorted secondary gives rise to a potential field (negative values throughout) equal to GM_2/r_2. At point P in the neighborhood of S_1, $1/r_2$ can be expanded in a series. Thus,

$$\frac{GM_2}{r_2}=\frac{GM_2}{r}\sum_{n=0}^{\infty}\left(\frac{r_1}{r}\right)^n P_n\,(\cos\theta) \qquad (3.1)$$

where P_n is the Legendre coefficient of order n and θ represents the angle PS_1S_2. If the separation between S_1 and S_2 be fixed, the first term is a constant of no consequence while the second term produces an acceleration GM_2/r^2 equal in magnitude for all elements of the primary along S_1S_2. This acceleration is simply the orbital acceleration of the primary as a whole. The remaining terms

$$V_t=\frac{GM_2}{r}\sum_{n=2}^{\infty}\left(\frac{r_1}{r}\right)^n P_n\,(\cos\theta) \qquad (3.2)$$

constitute the tidal potential and produce tidal distortion in the primary.

In an elliptical orbit, r is actually a function of the time. Thus, a rigorous treatment would be very complicated. The complications have been described by Sterne (1939a). However, as Cowling (1938) pointed out, the time of adjustment of a star to an external gravitational field is of the order of the period of free adiabatic oscillations of the star, which is much smaller than the orbital period of its companion around it. Therefore, it appears probable that the form of the star at any time would approach a state very close to equilibrium form in which the distortion is instantaneously adjusted to the external gravitational field given by equation (3.2). (Thus the longest axis of both component stars is always along the line joining the centers of the two components.)

Now the stars will be assumed to rotate with uniform angular velocities ω_1 and ω_2 about axes normal to the orbital plane. Such an assumption appears to be consistent with observations in general. Rotation of star introduces a disturbing potential which arises from the centrifugal force. Apart from a term which is symmetric with respect to the center of mass of the primary and consequently does not distort the star from a spherical configuration, the disturbing potential, V_r,

$$V_r=-\tfrac{1}{3}r_1^2\omega_1^2P_2\,(\cos\theta') \qquad (3.3)$$

where θ' is the co-latitude.

Therefore, the total disturbing potential, V_d imposed on the primary is

$$V_d = V_t + V_r \qquad (3.4)$$

and can be expanded in a series of terms of the form

$$c_{n,m} r_1^n P_n^m(\theta,\varphi) \qquad (3.5)$$

where (θ,φ) are polar coordinates taken with respect to S_1, P_n^m is a tesseral surface function of (θ, φ) and $c_{n,m}$ is a constant of expansion. This completes the first step of calculation.

The problem of deformation of a sphere which cannot maintain a shearing stress, and consequently of the potential arising from the deformed body, as the result of an imposed disturbing function V_d has been studied by Clairaut, Legendre, Laplace and others. A summary of their works can be found in Tisserand (1891). Here, we shall only outline the general idea of the approach to the problem, omitting detailed calculations.

The equation of the surface of equal density in the distorted primary can be written as

$$r_1 = \xi(1 + \sum_{n,m} Y_n^m), \qquad (3.6)$$

where ξ is a parameter (the mean radius) characterizing the surface in question. Thus, the density ρ of the primary is a function of ξ alone—a fact that makes ξ well suited to replace r_1 as the independent variable in the present problem. Thus, the value of the independent variable ξ at the stellar surface is R_1, the mean radius of the primary itself. The Y_n^m's in equation (3.6) are tesseral surface functions with respect to θ and φ and are also functions of ξ. Thus the deformation of the primary will be completely determined if we can find the ξ-dependence of Y_n^m and know the ρ-dependence of ξ.

The total gravitational potential Ψ at any point in the primary is the sum of the gravitational potential, U, due to the deformed primary itself and the imposed disturbing potential, v_d, i.e.,

$$\Psi = U + V_d \qquad (3.7)$$

It can be shown that the condition of hydrostatic equilibrium leads to a total differential equation

$$dp = \rho d\Psi \qquad (3.8)$$

where p is the pressure. It follows from equation (3.8) that both p and ρ must be functions of Ψ alone. In other words Ψ = constant over any surface of equal density and equal pressure in the primary. Furthermore, since ρ is a function of ξ alone, Ψ must be also a function ξ alone, i.e., independent of θ and φ.

Now U at any point p inside the primary consists of two parts, arising respectively from the mass (1) inside the spherical surface passing through p. (exterior potential) and (2) outside the spherical surface (interior potential). Since the equal-density surfaces are given by equation (3.6), both parts can be expressed in terms of Y_n^m. Actually when we carry out the lengthy calculation, we will find that each part of U can be expressed as a summation of terms involving Y_{nm} and other factors (like ρ, ξ) which do not contain the indices n and m (i.e., independent of θ and φ), φ), if we assume that the distortion given by Y_n^m's is small so that only first order terms of them are retained in the calculation. We have already mentioned that V_d can be expressed as a series whose typical term is given by (3.5). Hence, we can write down Ψ as a sum of three series, two involving Y_n^m and the other involving P_n^m. All terms are indexed by n and m (i.e., indicating their θ and φ dependence). The condition that Ψ is independent of θ and φ makes it necessary that in the expression of Ψ the terms associated with each pair of indices n and m must vanish identically. In this way we derive a differential equation for each Y_n^m. (It is a differential equation because of the transformation from r_1 to ξ as given by equation (3.6).) If we define

$$\eta_n = \left(\frac{\xi}{Y_n^m}\right)\frac{\partial Y_n^m}{\partial \xi} \qquad (3.9)$$

and

$$\rho_m = \frac{3}{\xi^3}\int_0^\xi \rho \xi^2 d\xi. \qquad (3.10)$$

which is simply the mean density interior to ξ, we can transform the differential equation of Y_n^m into the following form:

$$\xi\frac{d\eta_n}{d\xi} + \eta_n(\eta_n - 1) - n(n+1) + \frac{\partial\rho}{\rho_m}(\eta_n + 1) = 0. \qquad (3.11)$$

One reason that we introduce η_n instead of dealing directly with Y_n^m is to eliminate the (θ, φ)

dependence of Y_n^m which is simply p_n^m. Hence equation (3.11) together with p_n^m determines completely the deformation of the primary if we know the values of Y_n^m and η_n at any one point inside the star. This can be easily obtained for the point $\zeta=0$. There is no distortion at the central point. Hence $Y_n^m(0)=0$. In order to derive $\eta_n(0)$ we assume that $\rho(0)$ has a finite value. Setting $\zeta=0$ in equation (3.11) we obtain

$$\eta_n(0)=\eta-2. \qquad (3.12)$$

This completes the determination of deformation of the primary.

In the process of determining the deformation we have already derived the potential U that the deformed primary produces at the external points as a series of Y_n^m. Corresponding to the (n, m) component of the distorting potential given by (3.5), the deformed primary produces a component $U_{n,}{}^m$ to the potential

$$U_{n,m}=c_{n,m}\frac{n+1\eta_n(R_1)}{n+\eta_n(R_1)}\ \frac{R_1^{2n+1}}{r_1^{n+1}}P_n{}^m(\theta,\varphi) \qquad (3.13)$$

at external points. Here $\eta_n(R_1)$ is the value of η_n at $\xi=R_1$ (i.e., at the surface). Since $\eta_n(R_1)$ is obtained by integrating equation (3.11) which involves p/ρ_m, the (n,m) component of the exterior potential depends upon the structure of the star.

The result contained in equation (3.13) can now be applied to the present case of tidal and rotational disturbance. The terms of the indices $n=2(m=o)$ in the disturbing potential $v_d-v_r+v_t$ given by equations (3.2) and (3.3) are

$$V_d=\frac{GM_2}{r^3}r_1^2p_2(\cos\theta)-\frac{1}{3}r_1^2\omega_1^2p_2(\cos\theta'). \qquad (3.14)$$

Since the result given by (3.13) is independent of the choice of coordinate systems, the primary deformed by Vf of equation (3.14) produces an external gravitational potential field

$$U_2=2\frac{R_1^5}{r_1^3}k_{2,1}\left[\frac{GM_2}{r^3}p_2(\cos\theta)-\frac{\omega_1^2}{3}p_2(\cos\theta')\right], \qquad (3.15)$$

where

$$k_{2,\,1}=\frac{3-\eta_2(R_1)}{2[2+\eta_2(R_1)]} \qquad (3.16)$$

computed for the primary to which the second subscript is κ is referred to.

The acceleration of the secondary due to U_2 *directed towards* S_1 can be obtained by setting $r_1=r$, $\theta=o$ and $\theta'=\pi/2$ $in-\alpha U_2/\delta,r$, and is given by

$$\frac{R_1^5}{r^4}k_{2,\,1}\left(\frac{6GM_2}{r^3}+\omega_1^2\right). \qquad (3.17)$$

If we now regard r as the independent variable, the acceleration given by equation (3.17) corresponds to a potential function,

$$\Phi_2=R_1^5k_{2,\,1}\left(\frac{GM_2}{r^6}+\frac{\omega_1^2}{3r^3}\right) \qquad (3.18)$$

If the relative orbit is considered, $(1+m_2/m_1))\Phi_2$ will be identical to the disturbing function R of perturbation theory in celestial mechanics. Since the perturbing force is confined to the orbital plane which coincides with the equatorial planes of both components, the change in ω (e.g., Brower and Clemence 1961) is given by

$$\frac{d\omega}{dt}=\left[\frac{1-e^2}{G(M_1+M_2)a}\right]^{\frac{1}{2}}\frac{1}{e}\ \frac{\partial R}{\partial e} \qquad (3.19)$$

In order to derive $\partial R/\partial e$ we have to expand R in terms of e. Since we are interested in the secular motion of the apsides, only non-periodic terms are needed in the evaluation. A detailed calculation by Sterne (1939a) thus derives the secular motion arising from the distortion of order $n=2$ of the primary as

$$k_{2,\,1}\frac{R_1^5}{M_1}\left(\frac{M_1+M_2}{Ga}\right)^{\frac{1}{2}}\left[15\frac{GM_2}{a^4}f_2(e)+\frac{\omega_1^2}{a^3}g_2(e)\right] \qquad (3.20)$$

where $f_2(e)$ and $g_2(e)$ are series, convergent for any value of e less than inity,

$$f_2(e)=1+\frac{13}{2}e^2+\ldots,\qquad g_2(e)=1+2e+\ldots$$

Both $f_2(e)$ and $g_2(e)$ can also be expressed in closed expressions (Sterne 1939a) and have been computed by Plavec (1960).

Similarly, the secondary will be distorted by its own rotation and by the attraction of the primary. The perturbation produced by the distorted secondary on the primary can be easily obtained, because of symmetry, by interchanging M_1 and M_2

as well as replacing respectively R_1 and $k_{2,1}$, by R_2 and $k_{2,2}$, in (3.20). Thus the resultant motion of the apsides as compared with the mean orbital motion is

$$\frac{p}{p'}=k_{2,1}\left(\frac{R_1}{a}\right)^5\left[15\frac{M_2}{M_1}f_2(e)+\frac{a^3\omega_1^2}{GM_1}g_2(e)\right]+k_{2,2}\left(\frac{R_2}{a}\right)^5\left[15\frac{M_1}{M_2}f_2(e)+\frac{a^3\omega_2^2}{GM_2}g_2(e)\right], \quad (3.21)$$

where p and p' are the orbital and apsidal periods respectively.

If the orbital motion and axial rotation of both components in a close binary are synchronized (Swings 1936), equation (3.21) reduces to

$$\frac{p}{p_1}=k_{2,1}\left(\frac{R_1}{a}\right)^5\left\{\frac{M_2}{M_1}[15f_2(e)+g_2(e)]+g_2(e)\right\}+k_{2,2}\left(\frac{R_2}{a}\right)^5\left\{\frac{M_1}{M_2}[15f_2(e)+g_2(e)]+g_2(e)\right\}. \quad (3.22)$$

The contributions arising from the third and fourth order harmonic (tidal) distortion can be similarly obtained. The results can be expressed in terms of $k_{3,i}$ and $k_{4,i}$ $(i=1,2)$ which are functions of $\eta_3(R_i)$ and $\eta_4(R_i)$ respectively in a similar way as $k_{2,i}$ being a function of $\eta_2(R_i)$.

The values of k's can be readily found for any model by integrating (3.11). Thus, values of k_2 have been tabulated for a series of models by Russell (1928). For homogeneous stars, k_2 has a value 3/4; for completely concentrated stars all the k's are zero. In the case of certain polytropic models, the k_n's have been obtained by Chandrasekhar (1933) and later more extensively by Brooker and Olle (1955) for $\eta=2$ to $\eta=7$. The k's for other stellar models have been obtained by Kellar (1948), by Motz (1941, 1950, 1953), by Pike (1955), by Härm and Rogenson (1955), and by Kushwaha (1957).

Table 1 gives the k_2 values for some polytropic gaseous spheres from Brooker and Olle's paper. It illustrates its relation to two other parameters that also depend upon the density distribution in the sphere. The two parameters are the ratio of the central density ρ_c to the mean density $\bar{\rho}$ and the radius of gyration of the sphere. The values $\rho_c/\bar{\rho}$ given in Table 1 are taken from Chandrasekhar's (1939) book. The radius of gyration

TABLE 1.—*Relation between the Apsidal Motion Constant and other Parameters that Relate to the Density Distribution in the Polytropic Gaseous Sphere*

η	k_2	$\rho c/\bar{P}$	$I/(MR^2)$
0	.75000000	1	.4
1.0	.25990728	3.28987	.26138
1.5	.14327923	5.99071	.20502
2.0	.07393839	11.40254	.15704
2.5	.03485234	23.40646	.11203
3.0	.01444298	54.1825	.07583
3.25	.00869160	88.153	----------
3.5	.00491907	152.884	.04558
4.0	.00119488	622.408	.02358
4.5	.00031609	6189.47	----------
5	0	∞	0

given in the form of I/MR^2) in the table where I denotes the moment of inertia follows Motz's (1952) calculation. According to the latter. H. N. Russell has suggested that the apsidal motion constant k_2 might depend in a simple way on the radius of gyration of a star. Indeed, in the case of polytropic gaseous spheres and main sequence stellar models, Motz (1952) has found if $\log k_2$ is plotted against $\log[I/(MR^2)]$, all points do fall on a straight line. However, he has also pointed out that the giant models do not follow this linear relationship.

The secular variation of e and a can be similarly investigated. But it is hardly necessary to follow the calculation through. The variation must vanish because the interaction we are considering does not change the total dynamical energy and the total angular momentum of the orbital motion.

The assumption that the axes of rotation of both component stars are perpendicular to the orbital plane is likely true in most close binaries. However, if either or both axes of rotation make a considerable angle with the normal of the orbital plane, the motion of stars in a close binary would be greatly complicated, because we then would have to consider, apart from the motion in the orbital plane, the motion of the orbital plane itself and the motion of the equatorial planes of both components. Brouwer (1946) studied the problem by examining the motion of two rigid spheroids. His treatment was followed by Kopal

(1959) who has included the revolution of the tidal bulge in his considerations.

Thus, if two components have similar distributions of density so as to make corresponding k's for the two components equal, we may express in general the ratio of periods in the following form (Sterne 1939b).

$$\frac{p}{p'}=\alpha_2k_2+\alpha_3k_3+\alpha_4k_4+\ldots \qquad (3.23)$$

where α's can be computed from the orbital elements alone, while k's depend upon the stellar model. For each one-parameter family of stellar models, such as the polytropic family, the observed value of p/p' determines uniquely the parameter, such as the polytropic index. Therefore, it is frequent practice to express the observed motion of apsides in terms of the equivalent index of polytrope for the component stars. If, however, there are many plausible families of stellar models, the apsidal motion does not give any information as to which is the physically correct family. Hence, what the apsidal motion can provide for the astrophysicist is only a consistency check of the computed model.

A large number of papers have been published on the observed motion of apsides of various close binaries. The reader may find the references to these papers in several I.A.U. reports on eclipsing binaries by Kopal (1954, 1957), by O'Connell (1960, 1962) and by Merrill (1964). Also, the catalogues of orbital elements of binaries provided by Kopal and Shapley (1956) and by Wood (1963) give the observed apsidal motion where ever available. The observed results of apsidal motion of some binaries have been analyzed by Luyten, Struve, and Morgan (1939), by Sterne (1939b) and by Schwarzschild (1958).

3.2 The Perturbation Caused by a Third Body

In order to study the perturbation of the orbital elements of a close binary, caused by the presence of a third body, we have to assume that all three bodies behave like point masses. If the three bodies should move at distances of the same order of magnitude from one another, no orbital elements could be defined. Intuitively, such a state of motion is perhaps not stable. Indeed, few such systems have been found observationally What we have actually observed of triple systems usually consists of a close pair accompanied by a third companion which is at a relatively large distance from the close pair. In these cases, the orbit of the close pair and that of the distant companion are well defined and perturbation theories developed for the motion of the moon can be applied directly because the perturbation of the earth-moon system by the sun is, in many respects, similar to the perturbation of the close pair by the distant companion in a triple system. Thus, Slavenas (1927), Lyttleton (1934), Brown (1936, 1937) and Martynov (1948) have respectively applied different lunar theories to the stellar case. In principle, they all derive the change of orbital elements from the disturbing function, but their detailed calculations are so involved that it is impractical to be given in this short article. It will suffice to say that the elements of the close orbits that show secular changes are the line of apsides, the line of nodes and the mean longitude. If the p and p' are respectively the period of close binary and the orbital motion of the third body respectively, the periods of revolution of the apsidal and nodal lines p'' are given by

$$\frac{p''}{p}\sim\left(\frac{p'}{p}\right)^2 \qquad (3.24)$$

in order of magnitude.

Actually, the discovery of triple systems does not depend upon the results of these calculations. Observationally it is the change in the γ velocity obtained from spectroscopic data as well as the change in the period obtained mainly from photometric data that leads to the identification of the third body.

Since the semi-major axis of the relative orbit of the close pair suffers neither secular nor long-periodic perturbations by the third body, its period will remain constant. However, the apparent period as observed between two successive light minima will not be strictly constant because of the motion of apsides and nodes. The variation arising from these causes is small however, since the periods of revolution of the apsidal and nodal lines are very long, as one can see from equation (3.24). The actual observed variation in the apparent period is due to the continuous change in distance of the close pair from the observer. Since the velocity of light is finite, the change in distance means also a variation in the time-interval between two successive minima.

Consider a rectangular coordinate system with its origin, 0, at the center of mass of the triple system and with its Z-axis coinciding with the line of sight. The distance of the center of mass S^{12} of the eclipsing pair from the XY plane is evidently

$$r \sin i \sin (v+\omega),$$

where r represents the radius vector from 0 to any point on the orbit of S_{12}, i, the angle of inclination of this orbit to the celestial sphere, v the true anomaly of S_{12}, and ω the longitude of the periastron as defined before. If the space motion of the entire triple system produces a γ velocity—denoted by γ_o—in the z-direction, the distance z between the eclipsing system and the observer at any time becomes

$$z=z_o+\gamma_o(t-t_o)+r \sin i \sin(v+\omega), \quad (3.25)$$

where z_o denotes the initial value of z at time t_o of a light minimum of the eclipsing system. Thus, a term $(z-z_o)/c$, where c is the velocity of light, will be introduced into the ephemeris of the light minima.

Any apparent difference in time, arising purely from the light propagation, of some cosmic event is known in astronomy as the time equation or light-time. For example, there is the light-time for converting observed time on the earth to the heliocentric time, because the light carrying the news of the event does not reach the earth and the sun at the same time as the result of a difference in distance. This converting factor can be computed from our knowledge of the earth's position with respect to the sun's. In the present case, the light-time results from the orbital motion of the eclipsing pair around a third body or bodies. Thus, the light-time in this case is the difference (0–C) between the time of observed light minimum and the computed time based on the constancy of the period of the eclipsing pair. A light-time curve can therefore be plotted which, like the velocity curve, yields information as regards the nature of orbit on which the center of mass, S_{12}, of the eclipsing pair move.

The light-time curve—a plot of 0–C against the number of eclipsing cycles—of the Algol system has been extensively studied by many investigators because its records can be traced back nearly two hundred years. Eggen (1948) has shown that the system is composed of four stars, namely the eclipsing pair themselves, a third component causing wriggles in the light-time curve with an orbital period of 1.873 years and a fourth component causing the slow but dominant variations in 0–C with an orbital period of 188.4 years. This interesting system is continuously being studied with the purpose to make a better determination of the periods.

The determination of the (light-time) orbit from the light-time curve, which has been discussed by Woltjer (1922), Martynov (1948), Irwin (1952, 1959) and Kopal (1959) is very similar to one of determining orbital elements from the velocity curve. Indeed, the two problems are basically identical because if we differentiate (3.25) with respect to t, we obtain the radial velocity of S_{12} in its course of orbital revolution around 0. Thus, the light-time curve of an eclipsing pair in a triple system is equivalent in principle to a velocity curve of the motion of the center of mass, S_{12}, of the eclipsing pair around the center of mass, 0, of the entire system.

Actually the velocity curve of S_{12} can be directly determined from observation. Since the motion of S_{12} may be regarded as constant during a few cycles of the orbital motion of the close pair, the γ-velocity determined from the velocity curve during these few cycles of one or both components of the close pair represents the radial motion of S_{12}. Thus, a velocity curve of S_{12} may be obtained by plotting the γ-velocity of the close pair in different epochs. Therefore, we can derive all those orbital elements that are derivable in any single-lined spectroscopic binary. For example, the component, with a period of 1.873 years in the Algol system, mentioned previously, was first detected in this way by McLaughlin (1934).

Both the presence of a third body and the tidal and rotational distortions of component stars produce the effect of apsidal motion. It would be difficult to separate the causes of the observed motion, if there were no other observable criteria. Fortunately, because the presence of the third body in a binary system can be independently detected by the change in γ-velocity or by the light equation, there will be little ambiguity in interpreting the motion of apsides in a close binary. Thus, if the central condensation of the star is to be studied, we can always choose those close binaries in which no third component is detectable.

3.3 Perturbation by Galactic Objects

A binary system is continually perturbed by the encounters with other objects in the galaxy. As a result the energy is fed into the binary system, thereby increasing the separation of its two components. Ambarzumian (1937) has estimated the effectiveness of the tidal forces, due to the neighboring stars, in modifying the orbital elements of a binary. He has used the two-body approximation of stellar encounters to evaluate the tidal effects of the nearby stars. As Chandrasekhar (1944b) has pointed out, "the essentially characteristic features of the problem are ignored if an attempt is made to evaluate the differential effects of the neighboring stars on the components of a binary along the conventional lines of treating stellar encounters as a series of independent two-body problems."

In view of some recent investigations it becomes apparent that encounters with interstellar clouds play an important role in modifying the peculiar velocities of single stars (Spitzer and Schwarzschild 1951) and in feeding energy into a cluster (Spitzer 1958). On the same ground we may expect that encounters of a binary with interstellar clouds would increase the separation of the binary system. A rigorous treatment of the encounter of a binary with another object belongs to the well-known problem of three bodies in celestial mechanics and a complete solution is still lacking. A brief review of some recent investigations along this line, however, may be found in the book by Leimanis and Minorsky (1958).

In the meantime, a general statistical method has been proposed by Chandrasekhar (1943, 1944a and b) for solving many problems in stellar dynamics. The method first analyzes the nature of the force acting on a star by the rest of the stellar system and then by Markoff's process derives the probability distribution of the force field.

In the case of a binary system it is apparent that the forces exerted by the neighboring stars on the two components differ by a certain amount because of their different positions in space. For any given separation between the components there exists a definite distribution function, $(W(\vec{F}_1, \vec{F}_2)$, which governs the probability that forces of intensities $\vec{F}_1$ and $\vec{F}_2$ respectively, will act simultaneously on the two components of the binary and which can be obtained by Markoff's method (Chandrasekhar 1944a). Thus, the difference,

$$\Delta\vec{F} = \vec{F}_1 - \vec{F}_2, \tag{3.26}$$

represents the differential force which tends to accelerate the star "1" relative to the star "2." The differential force may be resolved into two components, a parallel component given by where

$$\Delta F_{\parallel} = (\vec{F}_1 - \vec{F}_2) \cdot \vec{k}_1, \tag{3.27}$$

$\vec{k}_1$ is a unit vector parallel to the direction of $\vec{F}_1$ and a perpendicular component. Since the perpendicular component is of a random character, it produces no net effect during a time scale, τ, long compared to the periods of the elementary fluctuations in $\vec{F}$. Therefore, the average net increase in the velocity of the star "1" relative to the star "2" during τ is given by

$$\overline{\Delta\nu_{1,2}} = \Delta F_{\parallel}\, \tau. \tag{3.28}$$

If the orbit is circular, it follows from the definition of $W(\vec{F}_1, \vec{F}_2)$ that

$$\Delta\nu_{1,2} = \tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\vec{F}_1 - \vec{F}_2) \cdot \vec{k}_1 W(\vec{F}_1, \vec{F}_2) d\vec{F}_1 d\vec{F}_2. \tag{3.29}$$

The evaluation of this integral is very involved and we can only refer the reader to the original paper by Chandrasekhar (1944a and b). When the separation between the two components, i.e., a, is small compared with the average distance between stars, the result may be approximately written in a simple form as follows:

$$\overline{\Delta\nu_{1,2}} = 4\pi G N M a \tau, \tag{3.30}$$

where M denotes a certain average mass for the field stars and N the number of stars per unit volume.

Instead of the time of dissociation of a wide binary system that Chandrasekhar is interested, we set ourselves to calculate the slow increase in separation of a close binary due to galactic perturbation.

Since $\overline{\Delta\nu_{1,2}^2}$ is the increase in the relative velocity of one component with respect to the other, $\overline{\Delta\nu_{1,2}^2}/2$ represents the increase in the dynamical energy per unit reduced mass E of the binary, namely

$$\Delta E = \tfrac{1}{2}\overline{\Delta\nu_{1,2}^2} \qquad (3.31)$$

while E itself is given by

$$E = -\frac{G(M_1+M_2)}{2a}. \qquad (3.32)$$

It follows from equations (3.30)--(3.32) that the new separation, denoted by a', after a time τ is given by

$$G(M_1+M_2)\left(\frac{1}{a}-\frac{1}{a'}\right) = (4\pi GNMa\tau)^2. \qquad (3.33)$$

This equation reduces to Chandrasekhar's equation for determining the time scale of dissociation, τ_d, when we set $a' \to \infty$.

$$\tau d = \frac{(M_1+M_2)^{1/2}}{4\pi G^{1/2}NMa^{3/2}}. \qquad (3.34)$$

Assuming, in the neighborhood of the sun, $N=0.1$ star/(parsec)^3 and setting $M=0.5M_\odot$, $M_1+M_2=1M_\odot$, Chandrasekhar has obtained from equation (3.34) the following numerical results

$$\tau_d = 2.22\times 10^{15} a^{-3/2} \text{ years} \qquad (3.35)$$

if the separation, a, is expressed in astronomical units.

It becomes obvious from equation (3.35) that dissociation of close binaries does not take place in the galactic time scale, say 2×10^{10} years. What we would like to calculate is the small increase in separation during this time. For this purpose let us write

$$a' = a + \Delta a \qquad (3.36)$$

Since Δa is small, it can be shown as a first approximation that

$$\frac{\Delta a}{a} = \left(\frac{\tau}{\tau_d}\right)^2 \qquad (3.37)$$

which gives numerically

$$\frac{\Delta a}{a} = 2.03\times 10^{-14} \text{ in } 10^{10} \text{ years} \qquad (3.38)$$

if a is 1/10 A.U. Therefore, we may conclude that close binaries are not effected by the perturbation due to neighboring stars in any time scale relevant in astronomical disscussions.

Chandrasekhar's investigation of dissolution of binaries forced by the fluctuating field of stars has been followed by Takase (1953) who considers the effect of the interstellar matter instead of stars. He assumes that the interstellar medium may be regarded as a continuous mass distribution with strong density fluctuations in space (Chandrasekhar and Münch 1952). The density fluctuation may be measured by the mean square deviation of density in the medium, denoted by $\overline{(\delta\rho)^2}$. It may be assumed that the scale of fluctuations (i.e., the size of density irregularities) is much larger than the distance between two components of a binary. Then using the relation derived by Osterbrock (1952) between the fluctuating force field and the fluctuating density field and following Chandrasekhar's reasoning, Takase obtains a formula for the time of dissociation of the binary system due to interstellar matter. His formula can be written down by simply replacing $N^2 m^2$ in equation (3.34) by $\overline{(\delta\rho)^2}/\beta$. Since $\overline{(\delta\rho)^2}$ is about the same order of magnitude as $N^2 m^2$, Takase's result does not change our previous conclusion obtained from Chandrasekhar's analysis.

3.4 Resisting Medium

The effect of the resisting medium is to reduce the dynamical energy and angular momentum of a binary system that is embedded in it. Historically, this problem has created interest among astronomers mainly through the study of the origin of our planetary system. As is evident, the matter in the solar nebula in the early phase of the solar system cannot all have condensed into planets, whatever the mechanism of their formation is. A certain amount of gas and dust must have remained in the space around the sun (especially near the fundamental plane). This remnant formed a resisting medium that the new-born planets were submerged. More recently the resisting medium has again been discussed in connection with the problem of accretion of matter by stars (McCrea 1953). But the discussion has been limited only to single stars.

As an illustration of the effect of a resisting medium on the binary motion, let us assume that the medium exerts simply a retarding couple on the binary system. For simplicity, we shall denote by ℓ the retarding couple that acts on unit reduced mass of the system. Hence, the couple acting on the entire system is $M_1M_2\ell/(M_1+M_2)$. We further denote by h and E respectively the angular momentum and dynamical energy per unit reduced mass of the binary system, thus

$$h=r^2\frac{d\theta}{dt} \qquad (3.39)$$

while E has already been given in equation (3.32). We obtain

$$\frac{dh}{dt}=\ell \qquad (3.40)$$

and

$$\frac{dE}{dt}=\ell\frac{d\theta}{dt}. \qquad (3.41)$$

from our assumption. Needless to say, the polar coordinates (r,θ), represent the position of one component star with respect to the other, θ being measured from the periastron.

Since it can be shown from the two-body problem that

$$h^2=G(M_1+M_2)a(1-e^2) \qquad (3.42)$$

we find from equations (3.32) and (3.41) that

$$\frac{1}{a^2}\frac{da}{dt}=\frac{2\ell}{G(M_1+M_2)}\frac{d\theta}{dt} \qquad (3.43)$$

and from equations (3.39), (3.40) and (3.42) that

$$\frac{de}{dt}=\frac{\ell}{h}[2\cos\theta+e(1+\cos^2\theta)]. \qquad (3.44)$$

As the retarding couple acts against the orbital motion, ℓ is negative. Therefore, the semi-major axis, and consequently the period of the orbit decrease in the resisting medium according to equation (3.43) as would be expected. If the resistance to the motion should increase with the speed of stars, their orbital motion would encounter the strongest resistance near the periastron (i.e., $\theta=o$) and the least resistance near the apastron (i.e., $\theta=\pi$). Then de/dt integrated over a complete cycle would be negative, and the eccentricity decreases in the medium. Or if the resistance is independent of the speed, the same result follows because of the second term in the bracket in equation (3.44), the first term in the bracket being zero on the average. However, if the resistance should decrease with increasing speed, the eccentricity could either increase or decrease depending upon the exact law of resistance.

Jeffreys (1918, see also Jeans 1928) who advocated the collisional theory for the formation of the planetary system used this result to explain why planets formed from the matter ejected from the sun by the tidal action of the colliding star should finally settle into nearly circular orbits. According to the current view, the planets were formed in a medium that was already revolving around the sun. Hence, the planet would be rotating with about the same velocity as its surrounding medium from which it had emerged and the medium was no longer a resisting one.

The previous example concerning the motion of new-born planets illustrates very clearly the difficulty of treating the problem of binary motion in a gaseous medium. It shows that the effect of the medium on the binary motion depends critically upon the dynamical state of the medium itself.

Recently, two papers (Fesenkov 1956; Kiang 1963) have appeared which deal with the motion of planets in a medium that is not static (with respect to the sun). Actually, Kiang has considered three plausible cases for the state of the resisting medium: (1) static, (2) rotating freely and (3) rotating uniformly and calculated the effect in a great detail for each case. However, the basic idea in his calculations is identical to what we have described here.

The effect of a resisting medium on the eccentricity of a binary has been recently rediscussed by Varsavsky (1962) who has followed the treatment of Poincaré (1911) and has been able to explain why binaries of shorter and shorter periods have in general, smaller and smaller eccentricities for their orbits. However it should be noted that Poincaré has assumed that the resistance increases with an increase of relative velocity of two components and with a decrease of separation between them. Obviously this law of resistance has no physical ground. Consequently, Varsavsky's quantitative analysis may not be regarded as realistic although the general idea may be valid.

The most difficult factor in treating the binary motion in a resisting medium is the variation of masses of stars themselves. As we shall see later, the medium may have been created as a result of mass ejected from the component stars. Or the stars may accret mass from the medium in which they are embedded, and which could exercise infinite varieties of laminar or turbulent motion that we can conceive of. All these factors only indicate that a medium surrounding the binary system can give widely divergent results. Indeed if the medium is rotating faster than the binary motion, the orbit will expand instead of shrinking. Therefore, mathematically it may be interesting to study the effect on the binary motion by assuming a particular state of motion for the medium (with or without an accompanying change in stellar masses), the result will represent only one of a large number of possible cases that may or may not actually confront us. What is important astrophysically is to find out the actual circumstance under which a particular binary is found: Is the binary system embedded in a medium of its own creation? Is the medium resisting or accelerating? Does either of the component stars lose or gain mass? While we cannot answer all these questions from the scanty data we have accumulated for any binary system, these are astrophysically interesting questions. It follows that it is not urgent to make any elaborate calculation for a few possible cases. Rather we should consider the problem in a general way with the aid of some simple model. For only in this way can we expect to obtain some physical insight to the problem by combining theoretical studies with observational ones. Otherwise, theoretical calculations can never meet the challenge of observations.

Since the resisting (or accelerating) medium is intimately related to the change in mass of component stars, we shall come back to this problem in the following section.

3.5 Change in Mass of the Component Stars

The effect of mass loss of component stars on the orbital elements has been discussed by Krat (1950) and Wood (1950). The following discussions, however, will be based on Huang's (1963) presentation.

Because the orbital elements—period, P, semi-major axis, a, and eccentricity, e, — are related to the masses of the two components, angular momentum, h, and dynamical energy, E, per unit reduced mass of the system by the well-known formulae in the two-body problem, it can be easily shown that

$$\frac{1}{a}\frac{da}{dt}=\frac{1}{M_1+M_2}\frac{d(M_1+M_2)}{dt}-\frac{1}{E}\frac{dE}{dt}, \qquad (3.45)$$

$$\frac{1}{p}\frac{dp}{dt}=\frac{1}{M_1+M_2}\frac{d(M_1+M_2)}{dt}-\frac{3}{2}\frac{1}{E}\frac{dE}{dt}, \qquad (3.46)$$

$$\frac{e}{1-e^2}\frac{de}{dt}=\frac{1}{M_1+M_2}\frac{d(M_1+M_2)}{dt}-\frac{1}{2}\frac{1}{E}\frac{dE}{dt}-\frac{1}{h}\frac{dh}{dt}, \qquad (3.47)$$

where E and h are given respectively by equations (3.32) and (3.39) or (3.42). Thus, the changes in orbital elements are not uniquely determined by the rate of mass variation but are dependent also upon dE/dt and dh/dt. This illustrates clearly the importance of the mode with which the variation of mass takes place because both dE/dt and dh/dt depend critically upon the mode of mass variation. Since the mass of the component star can lose or gain in an infinite variety of ways, this makes the problem somewhat uncertain.

The quantity, h, is related to the angular momentum per unit mass, h_o by

$$h_o=\frac{M_1M_2}{(M_1+M_2)^2}h \qquad (3.48)$$

and a similar relation exists between E and E_o the latter being dynamical energy per unit mass Eliminating dE/dt from equations (3.46) and (3.47) and combining the resulting equation with equation (3.48) we obtain

$$\frac{1}{p}\frac{dp}{dt}=\frac{4}{M_1+M_2}\frac{d(M_1+M_2)}{dt} -3\left(\frac{1}{M_1}\frac{dM_1}{dt}+\frac{1}{M_2}\frac{dM_2}{dt}\right) +3\frac{1}{h_o}\frac{dh_o}{dt}+\frac{3e}{1-e^2}\frac{de}{dt} \qquad (3.49)$$

This is a very useful relation because dE/dt (or equivalently dE_o/dt) which is unknown in most cases do not appear in it.

As in any problem that has multitudinous choices of possibilities we may idealize a few simple modes of mass variation that have a physical significance and then examine their effect on the orbital elements. In this way three modes of mass loss from the component stars have been isolated: (1) slow mode, (2) intermediate mode and (3) fast (or Jean's) mode. We shall discuss these three modes here under the simplified assumption that there is no interchange of angular momentum between axial rotation and orbital revolution of the component stars. A treatment of coupling between these two kinds of motion can be found in Huang's (1963b) paper.

1. Slow Mode

This represents an extreme case of losing mass. When the ejection velocities from the stars are low, the ejected matter will not be expected to escape from the binary system. Consequently, the total angular momentum of the system will be conserved but the total dynamical energy of the binary can vary in either way.

We may subdivide this mode into two cases: (a) the particles ejected fall back either to the original star or to its companion and (b) the particles ejected from the less massive component form, after many collisions among themselves, a rotating ring around the more massive component. The second case is proposed in accordance with observational results (Joy 1942, 1947; also Sahade 1960a). Formation of a rotating ring around the less massive component from matter ejected from the more massive component does not seem common because not a single case has been observationally established without dispute. Theoretically it is difficult to form and maintain such a ring (Huang and Struve 1953; Huang 1957a) but it is by no means prohibitive.

For a long time scale, the gaseous rings are expected to dissipate since they are continuously perturbed by the less massive component of the system. Hence, case (b) represents only a transient stage of mass loss. In either case we may set as a first approximation

$$\frac{d(M_1+M_2)}{dt}=0 \qquad (3.50)$$

In adopting this approximation we may regard the rings in case (b) as a part of the more massive component.

In case (a) we have

$$\frac{dh_o}{dt}=0 \qquad (3.51)$$

which yields, after its combining with equations (3.49) and (3.50),

$$\frac{1}{p}\frac{dp}{dt}=-3\left(\frac{M_2-M_1}{M_1M_2}\right)\frac{dM_1}{dt}+\frac{3e}{1-e^2}\frac{de}{dt}. \qquad (3.52)$$

For binaries of small eccentricities, the second term on the right-hand side of equation (3.52) may be neglected. Thus, we arrive at the conclusion that a transfer of mass from the more massive component to the less massive component (i.e., $M_1>M_2$, $dM_1/dt<0$) makes the period decrease with time and a transfer of mass in the reverse direction (i.e., $M_2>M_1$, $dM_1/dt<0$) results in an increasing period. Woolf has privately pointed out that the increase in period of β Lyrae may be precisely due to this mechanism, supporting the proposition (Huang 1962, Woolf 1962) that the primary component of this system is less massive than the secondary component.

More elaborate calculation of the variations of orbital elements with mass exchange has been recently carried out by Piotrowski (1964). He has considered separately the effect on the orbital elements of ejection, of flight and of landing of each mass element. While the calculations are mathematically interesting, it is hard to evaluate their significance in connection with observed data because of the infinite varieties of ways that an individual mass element can be transferred from one component to another. Also the results will be seriously upset by collisions of particles during the flight (Cf., §6).

In case (b) h_o is no longer constant because a part of angular momentum is absorbed in the rotating rings. If we approximate the ring (formed out of matter ejected by M_2) as rotating *directly* in a circular orbit of radius, a_t, around, and under the gravitational attraction of the more massive component, M_1, alone, we obtain

$$\frac{1}{p}\frac{dp}{dt}=\alpha\frac{1}{M_2}\frac{dM_2}{dt}+\frac{3e}{1-e^2}\frac{de}{dt} \tag{3.53}$$

where

$$\alpha=3\left(\delta_1-1+\frac{M_2}{M_1}\right)\text{ and }\gamma_1^2=\frac{(M_1+M_2)a_t}{M_1a(1-e^2)} \tag{3.54}$$

Numerical values of α for a few cases of a_t/a and M_2/M_1 have been given in Huang's (1963b) paper.

Finally, it is to be noted that (a) and (b) represent two physically different cases of idealized circumstances. When we mentioned that (a) represents the ultimate situation of the slow mode, we do not mean that case (b) will reduce to case (a) if the ring coalesces with the stellar surface, because that would result a transfer of angular momentum from orbital revolution to axial rotation—a situation precluded to the present discussion. In such cases we must include the coupling between orbital motion and axial rotation into our considerations.

2. Intermediate Mode

By the intermediate mode we mean that the ejection velocities are large enough to overcome the attraction of both components so that the ejected particles can penetrate the inner contact surface (Cf 5) of the system but not large enough to justify the neglect altogether of interaction between the ejected particles and the binary system. Actually the change of orbital elements under this mode of mass loss is most uncertain because we have no way to estimate the angular momentum carried away by escaped mass.

This mode may also be divided into two cases: (a) ejected particles escape to infinity directly and (b) particles form an envelope around the entire binary system, such as found in β Lyrae (Struve 1941, 1958). If the envelope is a quasi-stable rotating ring (or disk) around; and in the plane of, the binary system, the change in period may be evaluated in terms of the size of the ring (Huang 1963b). However, observationally there is no clear-cut evidence to conclude that such is actually the case.

Finally, we should again point out that case (b) represents only a transient stage of this mode since eventually the envelope will be dissipated.

3. Fast (or Jeans's) Mode

This is an extreme case, for it is assumed that there is no net reaction of the escaped mass on the binary itself. We call it Jeans's mode because Jeans (1924, 1925) was the first to treat this case when he examined the effect of radiation loss (which is equivalent to mass loss) on the binary system. Since then it has been applied to cases involving direct loss of mass (Huang 1956, Boersma 1961). The physical circumstance of actual mass ejection that may be approximated by Jeans's mode must satisfy two conditions: (1) the ejection of mass has a statistically spherical symmetry and (2) the velocities of ejection must be very high to insure a negligible interaction with the binary system. According to these conditions the loss of mass resulting from a supernova explosion (Blaauw 1961) and from ordinary nova outburst (Ahnert 1959) would closely approximate this mode. Mass loss due to corpuscular radiation may also fall into this mode if emission of particles is spherically symmetric. That corpuscular radiation may be important has been pointed out by Fesenkov (1949) and Masevich (1949) in their theory of stellar evolution. Tidman (1958) has proposed on the other hand a theory that suprathermal particles may be produced in the binary system if the expanding coronas collide each other, thus creating a region of violent motion favorable for accelerating charged particles. In any case, corpuscular radiation from component stars may be the reason, according to Huang (1958), why a luminosity anomaly exists in many binary stars.

It can be shown (Huang 1956) that Jeans's mode of ejection leads to

$$\frac{dh}{dt}=0\text{ and }\frac{1}{E}\frac{dE}{dt}=\frac{2}{M_1+M_2}\frac{d(M_1+M_2)}{dt} \tag{3.55}$$

which, when combined with equations (3.45)--(3.47), yield the following results due originally to Jeans:

$$\frac{1}{a}\frac{da}{dt}=-\frac{1}{M_1+M_2}\frac{d(M_1+M_2)}{dt},$$
$$\frac{1}{p}\frac{dp}{dt}=\frac{2}{M_1+M_2}\frac{d(M_1+M_2)}{dt} \tag{3.56}$$
$$\frac{e}{1-e^2}\frac{de}{dt}=0$$

It follows from equation (3.48) and a similar equation between E and E_o, we may write equations (3.55) in the following form:

$$\left.\begin{aligned}\frac{1}{h_o}\frac{dh_o}{dt}&=\frac{1}{M_1}\frac{dM_1}{dt}+\frac{1}{M_2}\frac{dM_2}{dt}\\&-\frac{2}{M_1+M_2}\frac{d(M_1+M_2)}{dt},\\\frac{1}{E_o}\frac{dE_o}{dt}&=\frac{1}{M_1}\frac{dM_1}{dt}+\frac{1}{M_2}\frac{dM_2}{dt}.\end{aligned}\right\} \tag{3.57}$$

Thus, in Jeans's mode the angular momentum per unit mass changes, although h maintains constant. Also, the dynamical energy per unit mass of the system has increased as a result of this mode of ejection.

An extreme case of Jeans's mode of ejecting matter is the sudden explosion either with a spherical symmetry or with an axial symmetry as found by Weaver (1958) in his study of Nova Aquilae (1918). If the mass ejected during the explosion exceeds a critical amount of $r(M_1+M_2)/(2a)$ where r represents the separation of the two stars at the instant when the instantaneous mass ejection takes place, it would result in a complete separation of the two component stars, as was pointed out by Blaauw (1961). Indeed he proposed that this is how the high-velocity O and B type stars sometimes observed in the galaxy had acquired their velocities from orbital motion.

So far we have discussed only the case of mass loss of component stars. It first appears that we may treat the problem of mass gain in a similar manner. Actually, the problem is much more complicated than the case of mass loss because first of all, we have to assign the state of motion of the medium from which stars may accret mass. In general, the problem of changes in orbital elements depends upon the following factors: (1) the relative motion of the center of mass of the binary system with respect to the medium, (2) the state of internal motion of the medium, such as rotation, turbulence, etc. and (3) the binary motion itself. From these factors we are supposed to evaluate the rate of accretion as well as the force (resisting or accelerating depending upon cases) that each component experiences. That this is a difficult task may be seen from a simple case that the medium possesses no net angular momentum with respect to any point inside. In such a case the total rate of accretion by both components may be roughly estimated by treating the system at large distances as a single body and thereby deriving the accretion column (McCrea 1953) of the entire system. However, how can we estimate the proportion at which the two components divide their loot in the accretion column? Since they move in the same region, the more massive component which is gravitationally stronger is expected to accret matter more effectively than the less massive one. On the other hand if we consider the size and location of the orbits we may expect the reversed situation, namely the less massive component is more effective. Here we see the intrinsic difficulty of the problem.

Observationally, we can determine only the period of a binary with any accuracy that justifies the theoretical investigations. Indeed, we have found variations in orbital period in many eclipsing binaries, although it should be remembered that the variations in period can be due to many causes other than the mass variations. For example, any perturbation due to the departure from the idealized binary composed of two mass points could produce complicated variation in observed period which is the time-interval between two consecutive principal (or secondary) eclipses. Actually, we have indeed found that periods of many binaries change in irregular ways, the same binary may have its period increasing and decreasing at different epochs in a seemingly unpredicted manner. This leads Wood (1950 see also Schneller 1962) to suggest that the changes may be due to ejection of gas jet by the component star sometime in the direction of its motion and at other times in the opposite direction. Wood's view has been criticized by Kopal (1959). The present writer inclines to regard

Wood's view to have its merit. We have found that the changes in period depend upon the modes of ejection as well as the rate of mass loss. Since there is a continuous range of possible modes and rates of ejection, it is apparent that there will be statistical fluctuations with respect to time, producing thereby corresponding fluctuations in period variations. Especially we should point out that if the velocities of ejection are high, mass could be thrown out of the star in any direction, not necessarily limited to the channel through the Lagrangian points L_1 and L_2. However, the cause of variations in period observed in eclipsing binaries is not uniquely determined. Most likely many factors are involved in producing the observed variations.

Extensive collections and analyses of observed period variations of eclipsing binaries have been given by Tung and Chang (1957), by Kwee (1958), by Prikhod'ko (1961), by Wood (1953) and by others (See references given by Kopal 1954, 1957; O'Connell 1960, 1962; and Merrill 1964).

4. RELATIVISTIC EFFECT

The advance of perihelion of the orbit of a planet around the sun resulting from the relativistic considerations provides one of the crucial tests of Einstein's theory of gravitation. Gravitation in the general theory of relativity is closely interwoven with the four dimensional space-time continuum. Using the language of geometry, we may state that the absence of gravitational field corresponds to a four-dimensional Euclidean (flat) space and the presence of any permanent gravitational field distorts the space-time continuum into a curved one. Indeed, the principles of relativistic mechanics are mainly contained in what is known as the field equation of Einstein (e.g. Eddington 1923), which connects the physical situation (i.e., the energy-momentum tensor which describes the distribution of matter and energy) with the geometry of space-time and which may be regarded as the relativistic analogue of Poisson's equation. Also, according to the theory, the space-time trajectory of a free particle is given by the geodesic, which, as is well known in geometry, directly depends upon the property of the space-time continuum and consequently on the gravitational field which shapes it. Since the property of space-time continuum corresponding to an empty space surrounding a gravitational point mass is known, the equations for a geodesic can be written down immediately and provides the basis for the treatment of the motion of planets around the sun. When these equations are reduced to an ordinary polar coordinate (r, φ) they yield the equation (e.g. Eddington 1923)

$$\frac{d^2u}{d\varphi^2}+u=\frac{GM}{(ch)^2}(1+3h^2u^2) \tag{4.1}$$

as compared with the equation of a Newtonian orbit

$$\frac{d^2u}{d\varphi^2}+\mu=\frac{GM}{(ch)^2} \tag{4.2}$$

where $u-1/r$, M is the mass of the central body, ch is a constant of integration, and c is the velocity of light in empty space.

The ratio, $3\,h^2u^2$, of the second term to the first term on the right hand side of equation (4.1) can be shown to be practically equal to three times the square of the transverse velocity of the planet measured in unit of the velocity of light. For example, the ratio for the earth is 0.000,000,03. Thus, the difference between the relativistic and Newtonian orbits must be slight. It can be shown (e.g. Eddington 1923) that the perturbation caused by the presence of this small term produces an advance of the perihelion equal to

$$\Delta\omega=\frac{6\pi GM}{c^2a(1-e^2)} \tag{4.3}$$

radians per orbital revolution.

The previous result is derived from the assumption that the planet has a negligible mass. The case of two bodies of comparable masses has been studied by Levi-Civita (1937) who shows that as a first approximation the same formula for the advance of periastron holds true in the case of binary stars as in the case of an infinitesimal planet moving around a central mass having the total mass of the binary system. Thus, except by replacing M by M_1+M_2 we have equation (4.3) for the advance of periastron in a binary system. If we write p' as the period that the periastron advances a complete cycle of 2π, equation

(4.3) can be written as

$$\frac{p'}{p}=\frac{c^2p^2(1-e^2)}{12\pi^2a^2} \tag{4.4}$$

If we now write $K=K_1+K_2$ where K_1 and K_2 are respectively the semi-amplitude of the velocity curves of the two components the previous equation can be expressed in terms of directly observable quantities, thus:

$$\frac{p'}{p}=\frac{1}{3}\left(\frac{c \sin i}{K}\right)^2. \tag{4.5}$$

Since K rarely exceeds 300 km/sec while c $=300{,}000$ km/sec, p'/p is of the order of 3×10^5. Hence, the advance of periastron as a result of relativistic correction must be very small. Luyten, Struve and Morgan (1939) have computed the motion of apsides according to equation (4.5) and found that the computed values are less than the uncertainty in the observed values for all the seven binaries whose apsidal motion they have studied.

It is evident from (4.3) that apsidal motion due to relativistic correction should be most pronounced in massive binaries with short separations. After calculations based on various assumptions as to the dimensions and internal structure of the components, Rudkjbing (1959) has concluded that in the eclipsing binary, DI Hercules, the relativistic part of the apsidal motion may well be larger than the part due to deformation of the components. However, according to O'Connell (1962) "no indication of apsidal motion has in fact been observed as yet in this system."

Levi-Civita (1937) has furthermore pointed out that aside from the slow precession of the relative orbit, the general theory of relativity also predicts "the absolute motion in the sky of any double star system." This result comes naturally from the fact that general relativity does not include, as a rigorous law, the principle of action and reaction which plays such an important role in Newtonian mechanics. Thus, one of the most important consequences of this principle, namely, the uniform motion of the center of mass of a system in absence of external forces, becomes invalid.

According to the calculations by Levi-Civita, the secular acceleration of the center of mass of a binary is directed along the major axis towards the periastron of the more massive component, and the increase ΔV of the velocity of the center of mass in a century is given by

$$\Delta V=12.55\ \mu(1-\mu)(1-2\mu)\frac{e}{(1-e^2)^{3/2}} \frac{M_1-M_2}{M_\odot}\frac{1}{p_d^2}\ \text{km/sec}. \tag{4.6}$$

where p_d is the orbital period in days, $M_\odot$ the mass of the sun and μ is the mass ratio defined by (2.4). Because of the factor $\mu(1-\mu)$ $(1-2\mu)$, $\Delta\nu$ has a maximum with respect to the mass ratio at $\mu=(1-3^{-\frac{1}{2}})/2$, which roughly corresponds to two stars respectively containing ¼ and ¾ of the total mass of the system. The maximum value of $\mu(1-\mu)(1-2\mu)$ is about 0.1. Since e is usually small for close binaries, the change in velocity, even in a century, is too minute to be detected by present means of spectroscopic study. Actually, I doubt that this effect could be detected in ordinary cases even after we have observed a close binary for tens of centuries, because the apsidal motion resulting from the rotational and tidal distortion obliterates this effect completely.

As two stars revolve around each other in a binary, it is obvious that the gravitational field they produce varies with their motion. According to Newtonian theory, the effect is instantaneously felt everywhere. Consequently, the notion of gravitational waves never arises. As the relativity theory presupposes that no casual effect can travel faster than light, we might anticipate that the change in the gravitational field travels out into space with the speed of light. A moving disturbance thus propagated out may be called the gravitational wave. Such is the physical reason for the existence of gravitational waves (e.g. Eddington 1923; Synge 1960). A realistic evaluation however could be made only after empirical detection (which is now actually attempted) should be successful.

We may expect that the emission and absorption of gravitational waves carry, to be sure, very small amounts of energy. Hu (1947) has computed the very small damping forces due to the emission of these waves. Actually, the rate of energy radiated in the form of gravitational wave by a binary star (in circular motion) as given by

Landau and Lifshitz (1951) is

$$\frac{dE}{dt}=-\frac{32}{5}\frac{G^4}{c^5}\frac{(M_1+M_2)M_1^2M_2^2}{a^5} \quad (4.7)$$

From equation (4.7) Kraft, Mathews and Greenstein (1962) have found that for the repeating nova WZ Sagittae (1913, 1946) the rate is in the range of $10^{32}-10^{35}$ ergs/sec if their mass ratio μ is between 0.1 and 0.5. However, they have pointed out that the gravitational flux reaching the earth is quite small and its detection very difficult. But they went on to suggest that the period change due to energy dissipation by gravitational wave is probably detectable. As we have seen in §3, variations in period may arise from many causes. Thus, even if the period change in this star were detected, it would remain a difficult problem to prove that the variation is indeed due to dissipation of energy through gravitational waves. This kind of uncertainty is intrinsic to observational science and strongly contrasts to the high precision of controllable investigations in experimental science, such as physics and chemistry. But nowhere in the entire field of astronomy do we encounter more frustration than in the attempt to detect relativistic effects in binary systems as we have seen in this section.

5. THE ROCHE MODEL

5.1 Stellar Surfaces According to the Roche Model

The shape of single stars without axial rotation must be a sphere since there is no one particular direction that is physically different from the other. When it rotates, the shape will no longer be spherical because of the existence of a preferred direction, namely the axis of rotation. As a result, we would expect that a rotating body, like the earth, Jupiter, etc., will be flattened to become an oblate spheroid.

In general, the component star (in the close binary) rotates in synchronism with its orbital revolution (Swings 1936, Plaut 1959). Hence, the centrifugal force does play a role in shaping the component star. However, a more serious complication arises from the force field due to its companion. What would be the shape of stars thus influenced by its close companion? We have investigated this problem in §3.1. But the result is too involved to be useful for the present purpose. Here we shall consider a simple model such that the gravitational potential due to each component star may be regarded as equivalent to a mass-point. Then the potential field of the entire system can be easily calculated and the shape of star determined. Such is the Roche model of close binary stars.

Let us assume that the two component stars in a system revolve around each other in circular orbits. Furthermore, we choose as the unit of mass the total mass of the system, i.e.,

$$M_2=\mu \qquad M_1=1-\mu \quad (5.1)$$

Also, we choose the separation between two components as the unit of length. If we take $P/2\pi$ as the unit of time (i.e., the angular velocity of orbital motion is one), we shall have unity for the gravitational constant. This can be easily seen from equation (2.5).

We may easily write down the equations of motion of a test particle in the gravitational field of the binary system. It is equivalent to the restricted three-body problem in celestial mechanics and can be most advantageously expressed in a coordinate system rotating with the component stars (e.g., Moulton 1914, Plummer 1918). The rotating coordinate system (x, y, z) may be so chosen that its origin locates at the center of mass of the binary, that its z-axis is perpendicular to the orbital plane and that its x-axis coincides with the line joining the two component stars. If we denote by r the radius vector from the origin to the test particle (x, y, z) and by r_1 and r_2 the respective distances of the test particle from the $1-\mu$ component at $(-\mu, o, o)$ and from the μ component at $(1-\mu, o, o)$, the equation of motion becomes

$$\frac{d^2\vec{r}}{dt^2}=-2\vec{k}\times\frac{d\vec{r}}{dt}+\text{grad } U \quad (5.2)$$

where

$$U=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2} \quad (5.3)$$

and $\vec{k}$ represents a unit vector in the positive z-direction. Taking the scaler product of $d\vec{r}/dt$ and equation (2) and integrating the resulting equation over t we obtain an integral for the test particle

$$2U-\left|\frac{d\bar{r}}{dt}\right|^2=C \tag{5.4}$$

where C is a constant of integration and is therefore a characteristic value for the test particle of a given set of initial conditions. In celestial mechanics it is often called the Jacobian integral.

The equation (5.4) relates the kinetic energy with the coordinate of the test particle in the rotating frame of reference. Consequently negative U may be regarded as a potential function in this reference system and we may plot a series of equipotential surfaces by setting $2U=C$, namely

$$x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}=C \tag{5.5}$$

where C now serves as a parameter labeling the surfaces. However, it may be noted that because of the first term on the right side of equation (5.2), U does not behave exactly like a potential in an inertial system. It is for this reason that the surfaces defined by equation (5.5) are not called the equipotential surfaces in celestial mechanics.

For a given value of C, which is fixed by the initial conditions, equation (5.4) determines the speed $|dr/dt|$ of the test particle at different points in space. Therefore, we can find all the points at which the speed will vanish. The locus of these points then define the so-called zero-velocity surface in celestial mechanics. It is obvious from equation (5.4) that the zero-velocity surface thus defined is identical to the one given by equation (5.5) which we have previously called the equipotential surface.

In order to see the importance of the zero-velocity surface let us consider $|d\bar{r}/dt|^2$ of a test particle of a given value of C as a function of the coordinate according to equation (5.4). On the zero-velocity surface this function vanishes everywhere. Hence, in general, the function would be positive on one side and negative on the other. Since $|d\bar{r}/dt|^2$ cannot be negative, it only shows that the test particle cannot penetrate the zero-velocity surface into the other side. Because of this property, it is useful to know zero-velocity surfaces corresponding to different values of C (or different initial conditions). Finally, we may add that in some special cases, the values of C may be positive on both sides of a zero-velocity surface. This happens when the function i.e., $|d\bar{r}/dt|^2$ in the present case, has an extremum equal to zero at some points in space. Indeed, such an occurrence gives rise to a few interesting and critical cases of zero-velocity surface that will be discussed later.

It is to be noted that we have used C sometimes to characterize the condition of the test particle according to equation (5.4) and at other times to label the zero-velocity surfaces according to equation (5.5). Thus, a particle of a certain value of C cannot penetrate the zero-velocity surface labelled by the same value of C.

Let us now examine the general behavior of the zero-velocity surfaces defined by equation (5.5). For very large values of C, say C_l, we have three possibilities: (1) large x^2+y^2, (2) small r_1 and (3) small r_2, corresponding to three separate surfaces for a single value of C. If x^2+y^2 is large, other terms on the left side of equation (5.5) may be neglected. Consequently, the surface represents a large circular cylinder with its axis coinciding with the z-axis. Similarly, when r_1 (or r_2) is small, other terms become negligible. We then obtain a small sphere around the 1-μ component (or around the μ component). If we do not neglect small terms, we will obtain three surfaces slightly distorted from what have been described. Therefore, space around the binary can be divided into four regions, one outside the nearly cylindrical surface, two inside the two closed surfaces respectively around the two stars, and the fourth between these three surfaces. It is easy to see from equation (5.4) that the test particle whose C value is equal to C_l can be either inside the two small closed surfaces or outside the large cylindrical surface. But it is forbidden to enter space between these three surfaces. Thus, the test particle (of C_l) can move in any one of the three permitted regions but cannot jump from one to the other.

In a close system the shape of each component star that satisfies the Roche model (i.e., a highly centrally condensed star with an envelope of a negligible mass) is determined by the closed zero-velocity surface around it, since, as we have seen in equation (3.8), the equal pressure surfaces in a star must follow the equipotential surfaces as a result of the hydrostatic equilibrium under which negative U will now behave exactly like a potential function in the rotating coordinate system.

As C decreases, the two closed surfaces around the two stars increase in size and the outer cylindrical-like surface shrinks. Eventually, these surfaces will come into contact. Actually, the contact occurs first between the two closed surfaces at a certain value of C which will hereafter be denoted by C_1. This critical zero-velocity surface is often called the innermost contact surface (e.g., Kuiper 1941), or the S_1 surface as we shall call it later. The point of contact L_1 is one of the Lagrangian points (or double points), which are special solutions of the restricted three-body problem. The contact point illustrates the case we have mentioned earlier that in some special cases $|d\bar{r}/dt|^2$ can be positive on both sides of a zero-velocity surface. This is true at L_1, say along the x-axis.

In Figure 1 we have illustrated several zero-velocity surfaces including the critical ones for the case $\mu=1/3$. Two cross-sections for each surface have been drawn, one in the xy-plane (in the lower diagram) and the other in the xz-plane (in the upper diagram). Because of symmetry we have shown here only one half of each cross-section, namely for positive y and positive z only. The case $C=4.1$ represents the situation before surfaces come into contact. Thus, this value of C corresponds to three distinct surfaces mentioned before and all shown in the figure. The case $C=C_1=3.946$ corresponds to the innermost contact (S_1) surface. It consists of two lobes which we shall call the primary lobe (around the $1-\mu$ component) and the secondary lobe (around the μ component). A large distorted cylindrical surface associated with the same C_1 is also shown in the figure.

The importance of the contact point is the fact that it forms a channel through which a test particle with C only slightly less than C_1 may move from the permitted region around one star to that around the other. As a result of this property we can conclude that the two lobes of the S_1 surface provides respectively an upper limit to the size of individual component stars because if either of them touches the surface, the matter will leak out at the point of contact into the other lobe. We may find examples for this situation in the Algol-type eclipsing binary systems in which one star (the less massive secondary) has usually a size comparable with the corresponding lobe of the S_1 surface while the other star (the more massive primary) is invariably small compared with its lobe of the same surface.

In reality the star must be smaller than the corresponding lobe of the contact surface as the particles (atoms, ions, electrons) in the surface

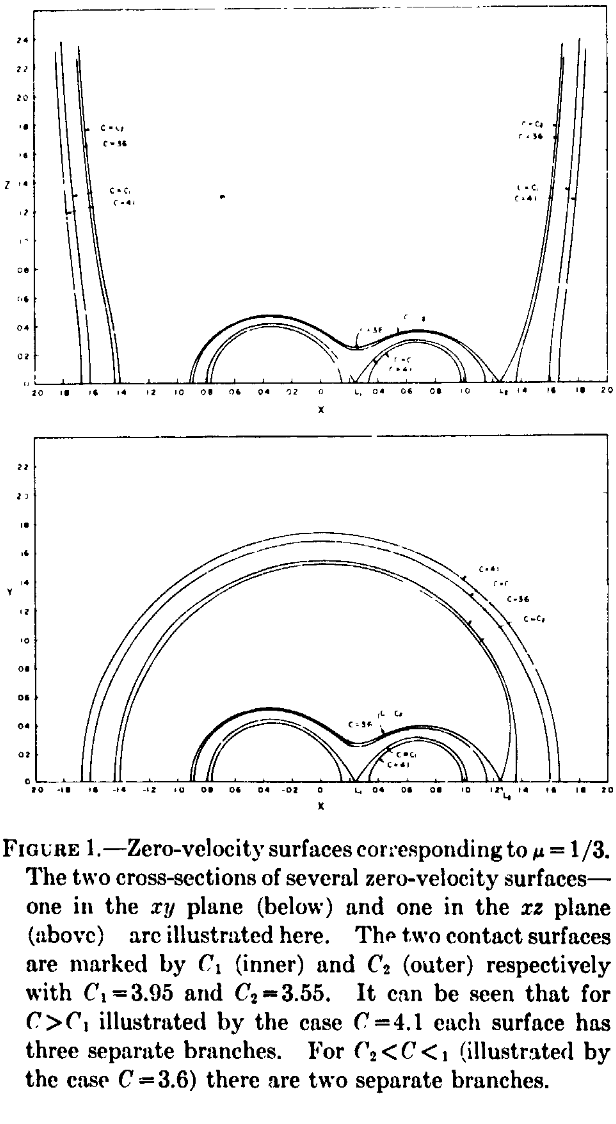

FIGURE 1.—Zero-velocity surfaces corresponding to $\mu=1/3$. The two cross-sections of several zero-velocity surfaces—one in the xy plane (below) and one in the xz plane (above) are illustrated here. The two contact surfaces are marked by C_1 (inner) and C_2 (outer) respectively with $C_1=3.95$ and $C_2=3.55$. It can be seen that for $C>C_1$ illustrated by the case $C=4.1$ each surface has three separate branches. For $C_2<C<C_1$ (illustrated by the case $C=3.6$) there are two separate branches.

layer of a star are exercising all kinds of motion, thermal, turbulent or else. Therefore, they can escape the inner contact surface even before the stellar surface has reached the contact surface. However, the velocities in these motions (except of course, corpuscular radiation) may in general be regarded as small in the present consideration because the unity velocity in the present unit system is equal to $[G(M_1+M_2)/a]^{1/2}$ and amount to a few hundred km/sec in close binaries. Therefore, a velocity of a few tens of km/sec can be neglected in equation (5.4), thus making the S_1 surface the actual limit of individual stars in a close system.

When C decreases from C_1, the two lobes of the innermost contact surface coalesce into one single dumb-bell like surface (corresponding to $C=3.6$ in Figure 1). They represent the appearance of two stars in actual contact such as in *W UMa* stars. Such a binary, though consisting of two stars, has a common envelope. A series of the contact configuration is possible, corresponding to different values of C. The smaller the value of C, the greater is the size of this dumb-bell like surface. On the other hand, the outer cylindrical like surface shrinks with decreasing C. Thus, C cannot decrease indefinitely without encountering another drastic change of the behavior of the zero-velocity surfaces. This time it is the contact between the dumb-bell like surface and the outer cylindrical-like surface. Kuiper has called it the outermost contact surface (or the S_2 surface) whose C value will be hereafter denoted by C_2 as is shown in Figure 1. The point of contact L_2 is another Lagrangian point. With this second contact, the inner region is connected to the outer region and the test particle with C only slightly less than C_2 can move without restriction due to energy. Thus, particles with C between C_1 and C_2 can move either inside the dumb-bell like surface or outside the distorted cylindrical surface. This leaves a forbidden region between the two surfaces. In Figure 1, this situation is illustrated by the case $C=3.6$. When $C<C_2$, there will not be any forbidden region for the particle from the consideration of the energy integral. Therefore, the S_2 surface represents the maximum size of a stable contact configuration of a close binary. For if the contact binary has reached this size, mass will continuously flow out of the system through L_2 and the binary is no longer stable.

With the aid of the S_1 surface we can now classify close binaries into three groups. If both components are well inside the S_1 surface, their system is said to be a detached one. If one component fills up the corresponding lobe of S_1 surface but the other is not, the two form a semi-contact (or semi-detached) binary. In the third group, i.e., the contact binaries, two components are in physical contact. More elaborate classification of close binaries may be found in Kopal's (1959) and Sahade's (1960 a and b) paper.

Finally, it may be noted that since the zero-velocity surfaces are defined in the rotating frame of reference, the Roche model predicts that axial rotation of stellar envelope and orbital revolution are synchronized, i.e., the two component stars revolve in their respective orbits face to face. This result appears to be valid in most close binaries (Swings 1936, Plaut 1959 Struve 1950). In a few cases, such as β Lyrae (Kopal 1959), where one or both component stars undergo secular expansion as a result of internal evolution, this synchronization could be temporarily violated.

We can now summarize the result briefly in the following way. The closed zero-velocity surfaces around two stars for $C>C_1$ represents the shape of two components that are detached from each other. The smaller the radius, the close to a sphere is the surface. What we are interested in, however, is only when the star approaches the size of the inner contact surface and is strongly distorted from a spherical shape. Thus, at $C=C_1$ each lobe of the S_1 surface represents the limiting size of the individual component. The star in a binary that is found actually in this limiting configuration is usually ejecting mass through L_1 towards its companion if the latter is still small compared with its own lobe of contact surface. For $C_1>/C>C_2$ the dumb-bell like surface of equation (5.5) represents the configuration of two component stars in physical contact. The limiting case of the contact configuration is given by the outer contact surface ($C=C_2$). Binary stars that are in this limiting configuration lose mass to outer space through the point, L_2. It can be seen from Figure 1 that the range of sizes of the contact systems (from the S_1 to S_2 surface) is quite small. Therefore, any binary that is in physical contact is usually losing mass through L_2. The problem of losing mass from a star through the points L_1 and L_2 was first extensively discussed by Kuiper (1941) (Cf. 6). Such an instability at the surface of the close binary stars gives rise to a number of observable phenomena which Martynov (1957 also Krat 1960) has summed up in a review together with examples of stars that show these various phenomena.

Because of their importance in the study of close binaries both S_1 and S_2 surfaces for different values of μ have been computed by Kuiper (1941); by Kopal, whose result was first published in 1954 and is now included in his book (1959); and by Kuiper and Johnson (1956) and more recently by Szebehely and Williams (1964). Table 2 gives C_1 and C_2 for 9 values of μ taken from Kuiper and Johnson's paper. Table 3 and 4 gives the intersection of the two contact surfaces with the xy-plane (first and second column) and with the xz-plane (first and third column) for those values of μ listed in Table 2. They are partly taken also

TABLE 2.—*Labelling Constants for the Contact Surfaces*

μ	C_1 (inner)	C_2 (outer)
.5	4	3.4567962
.4	3.9809086	3.5189346
1/3	3.9455706	3.5474582
2/7	3.9074840	3.5589342
.25	3.8706588	3.5611940
.20	3.8046532	3.5523932
.10	3.5969532	3.4666844
.05	3.4204164	3.3543942
.02	3.2523262	3.2257332

from Kuiper and Johnson's paper but partly computed especially for the use of the present occasion by Mr. C. Wade, Jr., of Goddard Space Flight Center. The case of $\mu=1/3$ is furthermore illustrated in Figure 1 as we have described. For diagrammatic illustrations of the contact surfaces corresponding to other values of μ we refer the reader to Kuiper's (1941) paper.

Finally, it should be noted that the basic size of a star, whether single or in a close binary, is determined by its internal structure. What we have said previously is only about the external shape of its envelope. That the envelope is all that we can observe of a star lies significance of the zero-velocity surfaces in the study of close binaries.

The Roche model of an infinitely centralized star represents only one end of a series of stellar models. At the other end there is the liquid model of a homogeneous density. Darwin (1906) and Jeans (1919) have studied the binary configurations and their stability based on the latter idealization. Darwin's and Jeans' results have recently been examined by Chandrasekhar (1964).

5.2 Departure from the Roche Model

An actual star is built neither on the infinite centralization nor on the homogeneous distribution of density. We have noted in 3.1 that the rate of apsidal motion decreases with the degree of central condensation. For component stars built on the Roche model the binary would show no apsidal motion at all. The fact that the apsidal motions in many binaries have been observed in relatively a short time of a few decades only shows how approximate the model is. However, for a prediction of the stellar surfaces the Roche model perhaps gives a good approximation. Indeed, according to a recent estimate by Plavec (1958), the departure of stars from the infinite central condensation of the Roche model does not produce a serious modification of the zero-velocity surfaces.

On the other hand, Plavec has pointed out in the same paper, that a deviation from the synchronization of axial rotation and orbital revolution would cause an important change in the shape of the contact surface. He has treated the problem of non-synchronization by simply modifying that term in U (given by equation [5.3]) which corresponds to the centrifugal force. In this way he has found that it is very simple to compute the modified contact surfaces. Unfortunately Plavec's process may not be regarded as legitimate because the centrifugal force arising from orbital motion and that arising from axial rotation do not have the same axis. This same problem has later been studied by Limber (1963) and Kruszewski (1963). The works by these two investigators are parallel and suffer the same difficulties.

In order to see the difficulties let us assume that axial rotation, say of the $1-\mu$ component, is not synchronized to the orbital motion, although its axis is still assumed to be perpendicular to the orbital plane. In considering the motion of a test particle in the envelope of the $1-\mu$ component we first translate the origin of the (xyz) coordinate system along the x-axis to the center of the $1-\mu$ component star by making the simple transformations:

$$x'=x+\mu,\ y'=y,\ z'=z, \qquad (5.6)$$

and then rotate the $(x'y'z')$ system by a second transformation so that in the end it will be in

TABLE 3.—*Innermost Contact Surfaces for Different Values of μ*

x	$\pm y$	$\pm z$
	$\mu=0.5$	
±0.1	0.1456	0.1370
±0.2	0.2520	0.2388
±0.3	0.3205	0.3050
±0.4	0.3596	0.3426
±0.5	0.3740	0.3561
±0.6	0.3647	0.3465
±0.7	0.3288	0.3112
±0.8	0.2549	0.2398
±0.9	0.0595	0.0555
±0.9050	0.0000	0.0000
	$\mu=0.4$	
−0.8415	0.0000	0.0000
−0.8	0.1773	0.1650
−0.7	0.3065	0.2871
−0.6	0.3712	0.3494
−0.5	0.4037	0.3814
−0.4	0.4122	0.3902
−0.3	0.3987	0.3777
−0.2	0.3625	0.3431
−0.1	0.2996	0.2827
0.0	0.2025	0.1900
0.1	0.0649	0.0606
0.1416	0.0000	0.0000
0.2	0.0872	0.0819
0.3	0.2045	0.1941
0.4	0.2797	0.2670
0.5	0.3221	0.3081
0.6	0.3376	0.3227
0.7	0.3272	0.3120
0.8	0.2872	0.2727
0.9	0.2001	0.1878
0.9696	0.0000	0.0000
	$\mu=1/3$	
−0.8013	0.0000	0.0000
−0.8	0.0329	0.0303
−0.7	0.2772	0.2574
−0.6	0.3656	0.3415
−0.5	0.4144	0.3886
−0.4	0.4370	0.4110
−0.3	0.4380	0.4126
−0.2	0.4182	0.3940
−0.1	0.3759	0.3538
0	0.3068	0.2878
0.1	0.2028	0.1893
0.2	0.0591	0.0550
0.2374	0.0000	0.0000
0.3	0.0919	0.0865
0.4	0.2022	0.1925
0.5	0.2701	0.2585
0.6	0.3051	0.2925
0.7	0.3123	0.2990
0.8	0.2912	0.2779
0.9	0.2337	0.2216
1.0	0.0831	0.0781
1.0120	0.0000	0.0000
	$\mu=2/7$	
−0.7743	0.0000	0.0000
−0.7	0.2482	0.2288
−0.6	0.3578	0.3318
−0.5	0.4187	0.3901
−0.4	0.4513	0.4220
−0.3	0.4619	0.4328
−0.2	0.4524	0.4243
−0.1	0.4224	0.3960
0	0.3691	0.3453
0.1	0.2867	0.2672
0.2	0.1663	0.1543
0.3	0.0114	0.0106
0.3072	0.0000	0.0000
0.4	0.1283	0.1214
0.5	0.2205	0.2107
0.6	0.2729	0.2618
0.7	0.2939	0.2810
0.8	0.2861	0.2738
0.9	0.2453	0.2335
1.0	0.1455	0.1373
1.0415	0.0000	0.0000
	$\mu=.25$	
−0.7554	0.0000	0.0000
−0.7	0.2215	0.2027
−0.6	0.3502	0.3228
−0.5	0.4208	0.3899
−0.4	0.4609	0.4286
−0.3	0.4784	0.4460
−0.2	0.4761	0.4444
−0.1	0.4543	0.4241
0	0.4112	0.3834
0.1	0.3426	0.3185
0.2	0.2405	0.2225
0.3	0.0976	0.0903
0.3607	0.0000	0.0000
0.4	0.0587	0.0551
0.5	0.1738	0.1655
0.6	0.2422	0.2323
0.7	0.2751	0.2642
0.8	0.2776	0.2662
0.9	0.2482	0.2369
1.0	0.1704	0.1613
1.0628	0.0000	0.0000

TABLE 3.—*Innermost Contact Surfaces for Different Values of μ — Continued*

x	$\pm y$	$\pm z$
	$\mu=.2$	
−0.7320	0.0000	0.0000
−0.7	0.1758	0.1590
−0.6	0.3388	0.3090
−0.5	0.4235	0.3885
−0.4	0.4740	0.4367
−0.3	0.5008	0.4628
−0.2	0.5079	0.4702
−0.1	0.4965	0.4600
0	0.4659	0.4314
0.1	0.4136	0.3822
0.2	0.3339	0.3075
0.3	0.2178	0.1997
0.4	0.0619	0.0571
0.4381	0.0000	0.0000
0.5	0.0883	0.0831
0.6	0.1867	0.1784
0.7	0.2398	0.2304
0.8	0.2583	0.2482
0.9	0.2444	0.2340
1.0	0.1895	0.1801
1.0909	0.0000	0.0000
	$\mu=.1$	
−0.7050	0.0000	0.0000
−0.7	0.0758	0.0659
−0.6	0.3332	0.2930
−0.5	0.4434	0.3931
−0.4	0.5125	0.4572
−0.3	0.5559	0.4982
−0.2	0.5797	0.5211
−0.1	0.5862	0.5278
0	0.5762	0.5192
0.1	0.5493	0.4945
0.2	0.5034	0.4522
0.3	0.4344	0.3887
0.4	0.3343	0.2977
0.5	0.1906	0.1698
0.6	0.0147	0.0134
0.6090	0.0000	0.0000
0.7	0.1151	0.1092
0.8	0.1819	0.1747
0.9	0.2037	0.1960
1.0	0.1855	0.1775
1.1	0.1072	0.1012
1.1349	0.0000	0.0000

x	$\pm y$	$\pm z$
	$\mu=.05$	
−0.7178	0.0000	0.0000
−0.7	0.1510	0.1256
−0.6	0.3733	0.3152
−0.5	0.4859	0.4149
−0.4	0.5595	0.4818
−0.3	0.6085	0.5271
−0.2	0.6385	0.5555
−0.1	0.6524	0.5689
0	0.6511	0.5684
0.1	0.6347	0.5537
0.2	0.6023	0.5243
0.3	0.5515	0.4781
0.4	0.4778	0.4118
0.5	0.3724	0.3187
0.6	0.2189	0.1875
0.7	0.0257	0.0232
0.7152	0.0000	0.0000
0.8	0.1024	0.0971
0.9	0.1554	0.1494
1.0	0.1572	0.1508
1.1	0.1002	0.0947
1.1412	0.0000	0.0000
	$\mu=.02$	
−0.7585	0.0000	0.0000
−0.7	0.2858	0.2231
−0.6	0.4534	0.3616
−0.5	0.5563	0.4508
−0.4	0.6273	0.5143
−0.3	0.6765	0.5593
−0.2	0.7085	0.5891
−0.1	0.7255	0.6053
0	0.7288	0.6088
0.1	0.7184	0.5997
0.2	0.6939	0.5776
0.3	0.6539	0.5415
0.4	0.5956	0.4893
0.5	0.5136	0.4174
0.6	0.3973	0.3187
0.7	0.2224	0.1780
0.8	0.0058	0.0052
0.8035	0.0000	0.0000
0.9	0.0990	0.0945
1.0	0.1197	0.1149
1.1	0.0671	0.0360
1.1258	0.0000	0.0000

TABLE 4.—*Outermost Contact Surfaces for Different Values of μ*

x	$\pm y$	$\pm z$
	$\mu=0.5$	
0	0.3261	0.2911
±0.1	0.3488	0.3129
±0.2	0.3948	0.3571
±0.4	0.4674	0.4247
±0.6	0.4819	0.4329
±0.8	0.4325	0.3763
±1.0	0.3019	0.2431
±1.1	0.1873	0.1382
±1.1984	0.0000	0.0000
	$\mu=0.4$	
−0.9800	0.0000	0.0000
−0.9	0.2705	0.2292
−0.8	0.3761	0.3271
−0.6	0.4794	0.4298
−0.4	0.5016	0.4561
−0.2	0.4580	0.4175
0	0.3544	0.3200
0.1	0.3046	0.2736
0.2	0.3000	0.2709
0.3	0.3387	0.3082
0.5	0.4120	0.3792
0.7	0.4255	0.3873
0.9	0.3678	0.3235
1.1	0.2137	0.1708
1.2	0.0670	0.0470
1.2308	0.0000	0.0000
	$\mu=1/3$	
−0.9165	0.0000	0.0000
−0.9	0.1251	0.1059
−0.8	0.3174	0.2754
−0.7	0.4108	0.3625
−0.5	0.5021	0.4526
−0.3	0.5150	0.4688
−0.1	0.4612	0.4200
0.1	0.3427	0.3092
0.2	0.2841	0.2556
0.3	0.2765	0.2510
0.4	0.3143	0.2889
0.6	0.3820	0.3538
0.8	0.3823	0.3493
1.0	0.3027	0.2646
1.1	0.2224	0.1850
1.2	0.0954	0.0717
1.2490	0.0000	0.0000
	$\mu=2/7$	
−0.8714	0.0000	0.0000
−0.8	0.2597	0.2246
−0.7	0.3829	0.3367
−0.5	0.5007	0.4497
−0.3	0.5314	0.4824
−0.1	0.4945	0.4500
0.1	0.3878	0.3505
0.2	0.3132	0.2816
0.3	0.2582	0.2328
0.4	0.2668	0.2440
0.5	0.3085	0.2857
0.6	0.3438	0.3199
0.8	0.3610	0.3332
1.0	0.2962	0.2634
1.2	0.1085	0.0848
1.2597	0.0000	0.0000
	$\mu=0.25$	
−0.8404	0.0000	0.0000
−0.8	0.2016	0.1737
−0.7	0.3581	0.3138
−0.6	0.4443	0.3941
−0.4	0.5294	0.4770
−0.2	0.5386	0.4887
0	0.4806	0.4359
0.2	0.3498	0.[illegible]146
0.3	0.2712	0.2434
0.4	0.[illegible]97	0.2177
0.5	0.2708	0.2501
0.7	0.3365	0.3141
0.9	0.3271	0.3004
1.1	0.2227	0.1928
1.2	0.1145	0.0916
1.2659	0.0000	0.0000
	$\mu=.2$	
−0.8014	0.0000	0.0000
−0.8	0.0395	0.0338
−0.7	0.3179	0.2766
−0.5	0.4933	0.4386
−0.3	0.5569	0.5011
−0.1	0.5504	0.4974
0.1	0.4761	0.4292
0.3	0.3238	0.2892
0.4	0.2393	0.2145
0.5	0.2208	0.2019
0.6	0.2589	0.2410
0.8	0.3106	0.2907
1.0	0.27[illegible]7	0.2507
1.2	0.1169	0.0963
1.2710	0.0000	0.0000

TABLE 4.—*Outermost Contact Surfaces for Different Values of μ — Continued*

x	$\pm y$	$\pm z$
	$\mu=.1$	
−0.7452	0.0000	0.0000
−0.7	0.2311	0.1954
−0.6	0.3966	0.3401
−0.4	0.5536	0.4837
−0.2	0.6147	0.5424
0	0.6104	0.5402
0.2	0.5416	0.4778
0.4	0.3897	0.3403
0.5	0.2739	0.2388
0.6	0.1708	0.1526
0.7	0.1772	0.1643
0.8	0.2175	0.2051
0.9	0.2355	0.2223
1.0	0.2248	0.2101
1.1	0.1815	0.1653
1.2	0.0926	0.0791
1.2597	0.0000	0.0000
	$\mu=.05$	
−0.7424	0.0000	0.0000
−0.7	0.2347	0.1913
−0.6	0.4136	0.3429
−0.5	0.5170	0.4342
−0.3	0.6328	0.5404
−0.1	0.6745	0.5805
0.1	0.6569	0.5657
0.3	0.5766	0.4932
0.5	0.4087	0.3444
0.6	0.2734	0.2300
0.7	0.1340	0.1179
0.8	0.1390	0.1295
0.9	0.1759	0.1667
1.0	0.1788	0.1689
1.1	0.1418	0.1306
1.2	0.0464	0.0393
1.2281	0.0000	0.0000
	$\mu=0.02$	
−0.7717	0.0000	0.0000
−0.7	0.3177	0.2446
−0.6	0.4739	0.3735
−0.4	0.6420	0.5212
−0.2	0.7213	0.5945
0.0	0.7410	0.6139
0.2	0.7066	0.5832
0.4	0.6102	0.4967
0.6	0.4188	0.3322
0.7	0.2562	0.2030
0.8	0.0849	0.0746
0.9	0.1133	0.1069
1.0	0.1306	0.1241
1.1	0.0934	0.0861
1.1801	0.0000	0.0000

synchronization with the axial rotation of the star. Thus:

$$x'=\xi\cos\omega t-\eta\sin\omega t,$$
$$y'=\xi\sin\omega t+\eta\cos\omega t, \qquad (5.7)$$
$$z'=\zeta,$$

where ω represents the rotational angular velocity of the star in the $(x'y'z')$ coordinate system. Therefore, the (ξ,η,ζ) coordinate system follows both the orbital revolution and axial rotation of the $1-\mu$ component. The equations of motion of the test particle in the (ξ,η,ζ) system can be easily derived by applying successively two transformations given by equations (5.6) and (5.7) to equation (5.2). If we denote by $\vec{\omega}$ the rotational angular velocity-vector, by $\vec{k}$ a unit vector in the same direction and by $\vec{r}_1$ the radius vector of the test particle from the origin o′ the (ξ,η,ζ) system, the final result may be given by

$$\frac{d^2\vec{r}_1}{dt^2}+2(\vec{\omega}+\vec{k})\times\frac{d\vec{r}_1}{dt}=\text{grad }U_1+\mu\vec{s}, \qquad (5.8)$$

where

$$U_1=\tfrac{1}{2}(\omega+1)^2(\xi^2+\eta^2)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}, \qquad (5.9)$$

and $\vec{s}$ represents a time-dependent unit vector which has the following components:

$$-\cos\omega t,\ \sin\omega t,\ o. \qquad (5.10)$$

It is now apparent that non-synchronization cannot be properly treated by simply replacing U in equation (5.2) by U_1 in equation (5.9) because equation (5.8) is not identical in form to equation (5.2). Because of the time-dependence of the vector $\vec{s}$ and of r_2 which is now given by

$$r_2^2 = (\xi - \cos \omega t)^2 + (\eta + \sin \omega t)^2 + \zeta^2, \quad (5.11)$$

we cannot derive a simple relation like equation (5.4) in the present coordinate system. Consequently, the potential function U_1 given by equation (5.9) predicts neither the equilibrium surfaces of the star in general nor its limiting surface in particular. A detailed discussion on this point and a proper treatment of non-synchronization will be given in a later paper.

Actually, the non-synchronization represents only one of several possibilities that will destroy the existence of zero-velocity surfaces. Indeed, under no circumstance can we define zero-velocity surface if the orbital motion of the binary is not circular or if a third star is present in the neighborhood (Huang 1964). In these cases the envelope of the component stars become unstable and stars themselves may not be expected to have stationary surfaces.

6. GASEOUS MOTION IN THE CLOSE BINARY SYSTEM

As a result of spectroscopic observations by Struve (1941, 1945 et seq. 1949a, 1950), Abhyankar (1959a, 1960) and many others, gaseous flow originating from stellar surfaces in the close binary system has been fully established. In the meantime Kuiper (1941) has examined the gaseous flow in the binary system from a theoretical point of view and introduced the idea of ejection of mass by the component stars through the Lagrangian points L_1 and L_2 discussed in the previous section. We may therefore, conclude that the components of some close binaries have touched the S_1 surface and their atmospheres become unstable such that gaseous particles are flowing out of the star either at L_1 or L_2. The conclusion is further strengthened by the photometric observations (Wood 1946, 1957; also Kopal 1959) which indicates that the relative radii of component stars in such binaries are indeed comparable to what would be expected from the two lobes of the S_1 surface for their respective mass ratios.

The establishment of gaseous flow in close binaries opened up a new field of theoretical study. Several approaches have since been advanced, namely (1) the orbital approach, (2) the hydrodynamic approach and (3) a rudimentary statistical approach. Needless to say, if the binary stars possess magnetic fields, a magnetohydrodynamic approach to the problem would be in order. However, unless we clearly know the nature of the magnetic field, we cannot proceed with our study. Consequently, no one has seriously treated the gaseous flow in the binary system as a magnetohydrodynamic problem although magnetic activities have been suggested rather quantitatively to account for the peculiar behavior of some binaries (Huang 1959). However, one of the effects of magnetic activities may be predictable and that is the braking of binary motion. Several mechanisms have been proposed for braking stellar axial rotation (e.g. Huang and Struve 1960) but the one proposed recently by Schatzman (1962) may be regarded as most effective. According to him, the angular momentum of axial rotation of a star is transferred outwards as mass is ejected out during strong magnetic activities. Similarly, we may propose that the same mechanism can dissipate the orbital angular momentum. If so, we wonder whether the many close binaries are indeed the result of braking due to magnetic activities in the early phase of their evolution.

6.1 The Orbital Approach

The motion of a particle in the binary system is computed according to equation (5.2), hence, the particle is treated as if it were a celestial body in classical mechanics. Kuiper (1941) studied the problem of gaseous motion in this way and his paper gives a simple but illuminating discussion that has often been referred to in the literature.

Kuiper first emphasizes the symmetry of the equation of motion with respect to the xy-plane. At least this symmetry will not be disturbed if an initial symmetry in the distribution of matter is assumed. Consequently the principal current should be symmetrical with respect to the plane. Furthermore, since two symmetrical currents in the z-direction would lead to a dissipation of energy not present in currents parallel to the xy-plane, Kuiper concludes that very probably the latter are the most important currents in the binary system. In what follows we shall consider only the motion in the xy-plane which incidently is also the plane in which observations may be made in eclipsing binaries.

Kuiper next points out that because of the Coriolis force (the first term on the right side of equation (5.2)), a flow near L_1 in the positive x-direction will be deflected in the $-y$ direction. This can be seen easily from the vector product if we set the initial velocity $d\vec{r}/dt$ to be pointing in the $+x$ direction. The current will therefore go principally in the direction of $\alpha, \beta, \ldots$ as is shown in Figure 2.

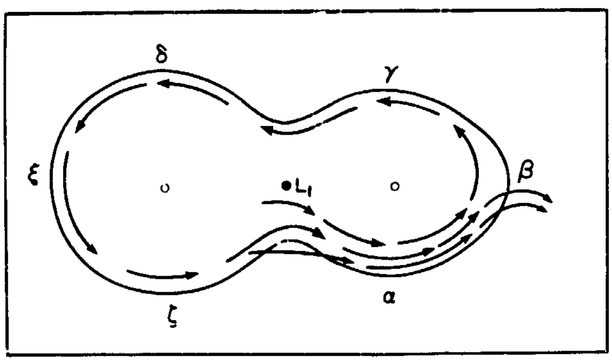

FIGURE 2.—Gaseous flow in the orbital plane expected in a close binary system (adopted from Kuiper's 1941 paper).

Numerical integrations of gaseous motion in a contact binary by Kuiper furthermore "show that if, the speed is below a certain value, the matter will go around the companion in the sense, $\alpha, \beta, \gamma \ldots$. If, however, the speed is large, the matter will fly off near β. The returning stream (in the $-x$ direction) will experience a Coriolis force in the $+y$ direction." The current would continue in the direction of δ, ξ, ζ as is shown in the figure with perhaps decreasing strength. Similarly, if the current starts in the negative x direction near L_1, the same flow pattern will result. Such a flow pattern is indeed consistent with what has been observed, say in β Lyrae.

Now if only one component of the binary is in contact with the S_1 surface, the gaseous flow pattern following the ejection of matter at L_1 will be affected by the Coriolis force in the same way as we have described for the contact binary. With frequent collisions among the ejected particles, such a tendency of motion may bring them to form a rotating ring or rings around the other component star as Kuiper has suggested. Indeed Kuiper's prediction was soon verified by Joy's (1942) discovery of a rotating gaseous ring around the primary component in the R W Tauri system —one of the Algol-type eclipsing binaries. Joy unaware of Kuiper's investigation, deduced the existence of the gaseous ring from the emission lines. This discovery induced Struve (e.g. 1950) and others to initiate an extensive study of the emission structure of many Algol type binaries. Emission rings have been since discovered in many systems. A detailed discussion of rings has been given by Struve and Huang (1957a) and a up-to-date list of such binaries may be formed in Sahade's (1960a) paper.

If the ejection velocity in the rotating system is small, the motion of particles near the point of ejection (namely L_1 and L_2) may be treated by linearized theory for the stability of orbits in the neighborhood of the Lagrangian points (Moulton 1914). Physically, we may imagine the region near the ejection point as a nozzle where the density is high and collisions are frequent. Consequently, we would not expect that any result derived from such a consideration will have any practical significance. If the ejection velocity is large, only direct numerical integration of equation (5.2) is possibie. Abhyankar (1959b) has studied the motion in the neighborhood of these points in this way.

Kuiper has also investigated the motion of a particle farther away from the ejection points L_1 and L_2 by numerical integration. His pioneer work in this direction has since been followed by Kopal (1956, 1957b, 1959), Gould (1957, 1959) and many others after the introduction of the high speed digital computer. They have computed extensive series of orbits according to equation (5.2). Many orbits thus obtained are difficult to interpret because of their seemingly erratic behavior with loops, cusps, etc. which obviously will all be erased by collisions. A few, however, yield some interesting results. For example, some of the computed orbits indicate quite convincingly that particles ejected from the less massive component may indeed coalesce into a rotating ring if we properly take into account in our mental process the effect of collisions among the particles themselves.

6.2 The Hydrodynamic Approach

In spite of some success in the orbital approach, the motion of gaseous stream in a binary system is not faithfully described by equation (5.2) since it does not include collisions and pressure. A more realistic treatment of this problem should at least take the collisions into consideration. Prendergast (1960) has examined the problem along this line. The hydrodynamic equation for the flow velocity $\vec{u}$ in space takes the following form in the rotating (x, y, z) coordinate system:

$$\frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\text{grad}\,\vec{u}+2\vec{k}\times\vec{u}=-\frac{1}{\rho}\text{grad}\,p+\text{grad}\,U \quad (6.1)$$

where U is given by equation (5.3) while p and ρ are respectively pressure and density in the gaseous medium. The unit vector $\vec{k}$ has the same meaning as in equation (5.2). The hydrodynamic equation of motion is supplemented by the equation of continuity.

$$\frac{\partial \rho}{\partial t}+d\omega(\rho\vec{u})=o \quad (6.2)$$

A comparison of equations (5.2) and (6.1) is helpful in our understanding of the difference between the two approaches. In the first place we have now introduced in equation (6.1) a new term grad p/ρ due to the pressure, which does not appear in the orbital study. Secondly, we now consider a continuous velocity field $\vec{u}(x,y,z,t)$ instead of $d\vec{r}/dt$ of individual particles. Thus, $d\vec{r}/dt$ is replaced by the first two terms in equation (6.1). We may take the continuity of the velocity field as a consequence of collisions among particles. This illustrates how by introducing the pressure and collisions we derive the hydrodynamic equation from equation (5.2). This point of view helps explain the statistical point of view of the problem that will be discussed in §6.3.

By considering a steady state $(\partial\vec{u}/\partial t=\partial\rho/\partial t=o)$ and by neglecting the pressure term, Prendergast has obtained an approximate solution for the two dimensional case (in the xy-plane). He has assumed that the velocity component at right angles to the zero-velocity curve is small and to be neglected whenever convenient. Therefore, his solution is a very special one. But because of this assumption he can advantageously choose a coordinate system based on the zero-velocity curves themselves. Thus, he has found that the solution predicts circulatory currents around each of the two components inside the S_1 surface and around both components outside the S_2 surface. In this way it has been shown that the gaseous rings around the primary component of many Algol-type binaries is indeed a solution of the hydrodynamic equation.

6.3 A Statistical Point of View

In the present section we shall present a point of view from a statistical consideration. Its scope lies in the middle between the orbital approach (which neglects both pressure and collisions) and the hydrodynamic approach (which includes both). It may also be given some physical insight to the problem that cannot be seen by simply integrating equation of motion—be it equation (5.2) or (6.1). Also, it may develope into a stochastic method that treats the gaseous particles in the binary system like stars in a cluster (Chandrasekhar 1943) and eddies in turbulence (e.g., Chandrasekhar 1949). However, this is only a remote possibility.

At present, we can only give some elementary properties of gaseous particles in the binary system according to a recent paper by Huang (1965).

Let us consider n particles of mass, m_i $(i=1, 2...n)$ moving in a binary system. Furthermore, the particle m_i corresponds at a given instant to a definite value of C_i for the Jacobian constant defined in equation (5.4). Because of collisions the individual C_i's of the particles change continually. However, since the particles are of atomic and subatomic sizes, the colliding particles at the instant of collision may be regarded as occupying the same point in space. If the total kinetic energy of the colliding particles is furthermore conserved during the collisions, it follows from the definition of C given by equation (5.4) that the new Jacobian constants denoted by C'_i for these particles after collisions should satisfy the following equation

$$\sum_{i=1}^{n} m_i C'_i = \sum_{i=1}^{n} m_i C_i \quad (6.3)$$

Thus, if we define an average $<C>$, we have

$$<C> = \frac{\sum_{i=1}^{n} m_i C}{\sum_{i=1}^{n} m_i} = \text{constant} \quad (6.4)$$

under the processes of elastic collisions. However, it may be noted that the dispersion of C_i's from their average value will in general change with collisions.

For inelastic collisions an equation connecting C_i's and C'_i's can always be obtained from the energy consideration if we know the detailed process of the collision. In the present consideration we shall assume that the collisions that take place among atomic particles in the binary system are statistically elastic, i.e., endoergic collisions balancing exoergic ones.

As a result of the constancy of $<C>$ during collision, the problem of gaseous flow is considerably simplified because we have now a macroscopic quantity, $<C>$, to deal with instead of following the courses of numerous particles in the system. This situation resembles the introduction of the concept of temperature and pressure which simplifies the study of the chaotic motion of molecules in gases in free space. Therefore, whatever is the nature of ejection that occurs on the stellar surface, the mean value of C_i's of ejected particles and their dispersion may serve as two of the most characteristic indices for the physical mode of ejection as regards the course of their subsequent motion.

It should be noted, however, that although the gaseous particles maintain a constant $<C>$, the mean flow does not follow the orbit derived from equation (5.2).

In this way, we can treat the problem of gaseous streams as a statistical problem of C_i's. One of the possibilities for further investigations along this line of thought is to link the statistical properties of C_i's from the theoretical consideration with the strengths of spectral lines arising from the gaseous streams observed at different phases of the binary motion.

Our immediate purpose is, however, to show that an intermediate approach between the orbital and hydrodynamic one may be obtained from a statistical consideration of C_i's as well as of the angular momentum (the z-component) per unit mass, h, of each particle in the system. Since a particle is moving in a two-centered field of force, its h, which is given by

$$h=x^2+y^2+x\frac{dy}{dt}-y\frac{dx}{dt}, \tag{6.5}$$

varies with time. However, it can be shown from equations (6.5) and (5.2) that

$$\frac{dh}{dt}=\mu(1-\mu)y\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right), \tag{6.6}$$

which is a function of coordinates of the particle only, being independent of its velocity. What is more important is that the total angular momentum is conserved among the colliding particles if the collision may be regarded as taking place instantaneously.

We can now derive the properties of a continuous gaseous medium in the binary system from those derived from a consideration of discrete particles. Let us illustrate it by the two-dimensional case. Since the average values of C and h do not change by collision, we may write in a steady state

$$\vec{u}\cdot\nabla C=o \tag{6.7}$$

from equation (6.4) and

$$\vec{u}\cdot\nabla h=\mu(1-\mu)\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right) \tag{6.8}$$

from equation (6.6) if we follow the streamline. While we have obtained these two equations by physical arguments, they can be easily derived from the hydrodynamic equation by first neglecting the pressure term. Naturally, equations (6.7) and (6.8) yield the same approximate solution of circulatory motions as Prendergast has obtained.

From equations (6.7) and (6.8) we can also see clearly the range of validity of the approximate solutions. As expected, the solution gives a good approximation for the circulatory motion close to each of the component stars. It has been illustrated by numerical calculation that wherever the solution represents a good approximation, the predicted motion from equations (6.7) and (6.8) approaches the periodic orbits of the restricted three-body problem (Huang 1964a). Thus, the gaseous ring observed in many binaries may be regarded equivalently either as a hydrodynamic flow or as motions of particles in a continuous series of periodic orbits that exist around each of the component stars.

Since we have emphasized the two physical parameters, h and C it is interesting to note that at some points in space certain combinations of values for these two quantities are incompatible. In other words, with a given pair of values for h and C a particle cannot go into a certain region of space which may be called the forbidden zone (Huang 1965). Since h is not a constant of motion in the restricted three-body problem, the forbidden zone is not as important as the zero-velocity surfaces. However, it may serve some useful purposes in depicting the trend of motion in general, especially when it is applied in combination with equation (6.6).

7. THE ORIGIN OF CLOSE BINARIES

Essential difficulties encountered in various hypotheses of the formation of binary systems in general, have been discussed by Hynek (1951). Many of his arguments can be applied to the close binaries and are indeed followed in the present discussion. From a general ground we may state that the close binaries can be formed by one or more of the three possibilities. They were formed (1) as a result of the fission process in rapidly rotating single stars, (2) from distant binaries and (3) directly as close binaries in the beginning. We shall briefly discuss the plausibility of these three processes.

Jeans (1919, 1928, 1944) has spent many years in studying the fission process of rapidly rotating stars. He has found that when the rotational velocity is small, the shape of a flattened spheroid is common to all rotating bodies whether they are composed of gases, liquids or plastic material. However, if the rotational velocity is high, their shape depends greatly upon their internal constitution and especially the degree of their central condensation. According to him fission would occur in massive liquid bodies in which there is no appreciable central concentration of mass. But for a gaseous body of an extreme central condensation (i.e., the Roche model), rapid rotation only makes it flatten more and more with an accompanying loss of mass at its equator. No fission will result. Jeans further shows that all bodies having less than a certain critical degree of central condensation behave very much like those made of incompressible liquid while all bodies having more than this critical amount of central condensation behave very much like those of an infinite central condensation. Since actual stars have a central condensation much higher than the critical value, fission does not occur in them (Jeans 1944).

The fission theory encounters another difficulty in that it cannot explain the existence of binaries of large separations. Because of the conservation of angular momentum, it can be easily seen that two components of a binary resulting from the fission process must be very close together. Thus, in order to account for binaries of wider separations other mechanism of formation has to be derived. While it is not prohibitive to have two different mechanisms for the formation of one kind of objects, the smooth distribution of separations and other statistical behavior of binaries throws serious doubt, according to Kuiper (1935a and b), on any theory that cannot explain the formation of all binaries, except perhaps those really wide ones with separations of the order of interstellar distances (e.g. Van Biesbroeck 1957), by a single mechanism.

Next, let us examine the possibility of converting distant binaries into close ones. In proposing this possibility, the presence of distant binaries is assumed. Then two components are supposed to drift together due to a resisting medium such as we have seen in 3.4 or due to other energy dissipating mechanisms. In order to make the theory complete we must first of all answer the question of how these distant binaries are formed. It has often been suggested that they may be formed by star capture. However, stars are far apart in space and are moving with an average velocity of about 10–20 km/sec with respect to the local centroid. Thus, if one star by chance approaches another, each will move on an hyperbolic orbit. Consequently, the two will recede from each other to large distances unless their energy can be dissipated during the encounter. The agency which absorbs the dissipated energy can be either a third star that happens to be in the immediate neighborhood or a resisting medium. But the chance of a three-star encounter in the galaxy is vanishingly small and the interstellar medium is in general too tenuous to be effective in reducing stellar velocities (McCrea 1953; Dodd 1954).

For the same reason of low interstellar densities and high orbital velocities, it is very doubtful that the resisting force of an ordinary interstellar cloud can greatly reduce the separation of a binary in a time scale, say of 10^{10} years. Therefore, the possibility of first forming binaries by star capture and then converting them into close ones is also ruled out.

This leaves us only the third possibility, i.e., the binaries—both close and distant—were formed as they are. It appears that we have not found any serious objection to this possibility. On the contrary many observed facts are consistent with it. For example, according to Kopal (1959) the relative frequency of close binaries found in the general galactic field is about 0.1 percent and is of about the same order of magnitude as that found in galactic clusters and stellar associations where stars are supposed to be formed. Indeed many eclipsing binaries have been identified to be members of galactic clusters (e.g., Sahade and Frieboes 1960) and of stellar associations (e.g., Kraft and Landolt 1959; Semeniuk 1962). If binaries of different separations were formed as they are, their high abundance in the galaxy strengthens the suggestion made frequently (e.g., Roberts 1957) that stars (at least Pop. I stars) are formed in groups. As a result of his statistical study Batten (1960) gives reasons to suggest that many binaries may have formed together in space. However, statistics also give us some puzzling results. Jaschek and Jaschek (1957, 1959) have found that although the frequency occurrence of spectroscopic binaries is uniform among the young stars (about 20 percent) whether they are in associations, in clusters or in the general field, the percentages of spectroscopic binaries in old groups of stars (ages from 3 x 10^9 to 6 x 10^9 years) decreases with age.

From the standpoint of star formation in general we also find the emergence of binaries quite natural. Two condensations just happened to be formed near together in the primeval medium and they will evolve to become a binary if their relative velocity is not large enough to escape from each other. Since the two adjacent condensations evolve separately on their own, they will become two stars as if they were chosen from a random sample. This prediction is again consistent with the observed result (Kuiper 1935 a and b). Here we may mention the interesting argument in favor of the simultaneous formation of two components of a close binary advanced by Krat (1952). He observed that if their formation did not occur at the same time, then the second component could never have been formed in the neighborhood of the first one because of the latter's strong tidal force and intense radiation both of which incline to disrupt the gaseous condensation that is to become the second star.

If the binary systems were formed as they are, it is inevitable to conclude that two components of a binary must have the same age. Previously this consequence was regarded as a difficulty, because one might argue, as with Jeans (1944) for example, that components of binaries like Sirius, Procyon, and others show signs of very different ages. Both Sirius and Procyon contain a white dwarf as one component together with an early-type main-sequence companion. Since the white dwarf is an old object while the early-type main-sequence star is regarded as young, it is difficult to believe that they could be coeval. However, it should be noted that whether a star is young or old refers only to its own evolutionary sequence and has nothing to do with time in the absolute measure. For example, a massive star of $10M_\odot$ or more passes through all of its evolutionary stages perhaps in a fraction of time that a star of $1M_\odot$ remains on the main sequence. Thus with a loss of mass at late stages either continuously (Deutsch 1956) or cataclysmically, there is no reason to object why one star has reached the white dwarf while the other of the same age is still at the main-sequence stage. This is a fortiori true for two components in a close binary, because, as we have seen, one component can accret mass from its companion. Since we may regard mass accretion as a rejuvenation process while mass loss as an aging process, such an exchange of mass will enhance the apparent age difference between two components in a close binary. More will be said about mass exchange and its effect on evolution of close binary stars in the next section.

Although we have concluded that binaries were formed in the way they now appear, we still do not know for sure the detailed mechanism how they actually emerged from the primeval nebula, as the study of formation of binaries, like that of single stars, is still at the speculative stage. Here we shall review briefly three suggestions regarding the physical processes that shape the binaries in general and close binaries in particular. As we will see presently, these three suggestions are not mutually exclusive because each deals only one aspect of the many faceted problem of binary formation.

Dodd (1954) has advanced a model of binary formation after he has found that star capture in the present state of interstellar space is untenable. He takes for granted that stars are forming around various centers of condensation in the originally homogeneous medium where there are no local variations in the gravitational field. If two condensations happen to be nearby, the material in the cylindrical volume between the two condensations, according to Dodd, will collapse on the axis joining them mainly under the gravitational influence of the condensations themselves. Because the collapse on to the axis is cylindrical, energy dissipation is much more effective than would be the case of a spherical collapse. This cylindrical collapse results in a column of comparatively small cross-section extending between two condensations. When the latter moves towards each other under their gravitation, they sweep the mass between them. In acquiring mass in this way the condensations also meet resistance which dissipates the dynamical energy of the system. In the end two condensations become two component stars in a binary. It appears that Dodd has assumed the condensations to be at rest in the medium in the beginning. They form a binary instead of a single star only by a further assumption that the condensations will be deflected from the straight collision course by perturbation due to neighboring stars. Perhaps by making these two assumptions, Dodd has proposed an unrealistic model since without initial relative motions among the various condensations, one can only expect the collapse of all condensations into one big mass.

Kuiper (1935b, 1957) has studied the problem of angular momentum of binaries both observationally and theoretically. If one considers the proto-star as a single hydrodynamical unit, its Reynolds number is high and consequently motion inside it must be turbulent. Thus, its angular momentum may be computed from the random motion of large eddies as Kuiper has done. Since the motion is random, each of the three components of the total angular momenta of stars must be given by the Gaussian curve. In other words the magnitudes of the total angular momenta of proto-stars follow the Maxwellian distribution of velocity magnitudes, from which Kuiper is able to derive a theoretical distribution for the separations between two components in binaries. He has found that the theoretical distribution curve indicates a less dispersion than the observed one but has given several convincing reasons to show why this should be expected.

Parenthetically it may be noted that unaware of Kuiper's investigation, McCrea (1959 see also Struve and Zebergs 1962) has used practically the same arguments to derive the angular momenta of stars.

Finally, Huang (1957b) has suggested that solid prestellar nuclei may serve as the basis for stellar condensations. If a medium possesses a large amount of angular momentum and is composed of only gas, its contraction will lead to a rotational break-up at the equator. It will not form a binary as we have already mentioned in connection with the fission problem. However, the situation will be different if there are large solid bodies embedded in the gaseous medium. These large bodies must move very slowly in comparison with gaseous molecules for the same reason that the Brownian movement is slow compared with molecular motions. Consequently, they are gravitationally less stable than the gaseous substratum. It could therefore happen that these bodies coalesce into two revolving nuclei which serve later as two centers of condensation. As a result, a binary instead of a rapidly rotating single star is formed.

All these ideas are qualitative in nature and function no more than some vague suggestions. Much work needs to be done before any of them will develop into a complete and consistent theory. Furthermore, other ideas may come up in the future. Therefore, we cannot avoid the impression that the problem of binary formation is a wide open field for further investigations.

Let us now put aside the question of physical processes of formation and turn our attention to the phenomenological side of the problem. In his study of separations of binaries Kuiper (1935a and b) has realized that the distances of the major planets from the sun fall nearly in the mid-range of the binary separations from 10^{-2} to 10^{5} A.U. with a median value at about 20 A.U. Therefore, Kuiper (1951) believes that the binaries and the planetary system form a single uniform group. In his way he has argued that the number of planetary systems in the galaxy must be considerable because binaries are numerous. His view is now generally accepted.

At the same time Struve (1949b, 1950; also Struve and Huang 1958) has raised some interesting points on the relation between binaries and planetary systems. Especially, he has suggested from a consideration of angular momentum that planetary systems may result from the mass dissipation of *W* Urase Majoris systems.

Whether Struve's intuition will turn out right we cannot tell at this moment. One point we are now certain is that the formation of close binaries and single stars are events with no intrinsic difference. We suggest that only a few varying parameters in the identical process of formation make all the different catagories of stars. One of the important parameters doubtless is the angular momentum. If a condensation has a low value of angular momentum, single stars neither rotating rapidly nor possessing planetary systems will be formed. If the angular momentum is high, rapidly rotating single stars, stars possessing planetary systems of close binaries will be formed. Which one of these possibilities will actually be followed depends presumably upon other parameters, among them we may mention density, total mass, turbulence, magnetic activities, and other (Huang 1965).

8. EVOLUTION OF CLOSE BINARY STARS

Concerning the evolution of close binary stars we are faced with many peculiar systems but with few tangible leads that may indicate their stages of evolution. Consequently, we have a wide choice of theories for the evolutionary scheme of close binary stars. While it is interesting to know the fascinating ideas about binary evolution advanced in the past decades, we shall follow a conservative course here in preparing this section not only because we do not have enough space but also because most of these ideas are tentative at best.

The difficulty in understanding the evolution of close binary stars can be easily seen from the fact that in spite of recent successes in the study of stellar evolution (e.g. Burbidge and Burbidge 1958; Schwarzschild 1958; Hayashi, Hōshi and Sugimoto 1962) we are still not certain about evolution of single stars after the red-giant stage. The reason for this uncertainty lies in the first place with the complications of various nuclear reactions that take place in different layers of the star. But a more serious one is the rate of loss of mass from the star at different phases at and after the red-giant stage. As a result of investigations by Deutsch (1956, 1960, 1961) there is no doubt that red giant or supergiant stars lose mass continuously but the exact amount is not certain. Coming back to the close binary stars we know even less about the rate of mass loss or gain at different stages.

Because of this uncertainty we shall discuss evolution of close binaries here only on some general ground. In the first place the component stars must follow the natural sequence of evolution of every star because of the continuous outflow of energy which is derived either from their gravitational potentials or from thermonuclear reactions. Since two components of the same binary are coeval as we have concluded in the previous section, the more massive component departs from the zero-age main sequence at a faster rate because of its higher luminosity than the less massive companion. According to Smak (1959) this is the reason why the primary components in close binaries are systematically greater than the secondaries of the same mass. However, unlike single stars we would expect that stars even at the main-sequence stage will lose mass if they are components of close binaries. This is true not only for the components in contact and semi-contact binaries but also for those in completely detached systems if prominence activities like those taking place on the solar surface are active there. Indeed the variations in period found in the detached and semi-detached systems (Kwee 1958) may well be due to this kind of mass variations.

When the star is well inside the S_1 surface, the loss of mass is slow and its internal structure will not be seriously different from that of a single star of the equivalent mass. Therefore, its structure may be derived by linear perturbation as has been performed by Morton (1960). The situation becomes quite different when one or both components reach the size of the corresponding lobe of the S_1 surface, such as in several important categories of close binaries like β Lyrae variables, *W* Ursae Majoris variables, Algol-type variables, etc. In these cases, the rate of mass variation is presumably high and the structure of the overflowing component star at any time perhaps will find no parallel among single stars. For example, it is doubtful that the subgiant component of the Algol-type binaries has any resemblance in their internal structure to the single subgiant stars (Struve 1954). Reddish (1957) has suggested some special models for red giants in order to account for the process of mass loss.

If we cannot derive the structure of component stars of contact and semi-contact systems by applying perturbation to the normal single stars, can we calculate directly their sequence of evolution by assuming a given rate of mass loss? The prospect of doing so is not good. There are several intrinsic difficulties involved in this kind of calculations. In the first place the mass loss has an effect not only on the star itself but also on the separation and the contact surfaces which limit externally the radius of the overflowing component star. In other words, the structure of the component stars is coupled to the orbit of the binary itself. Now the contact surface is not uniquely determined by the amount of mass lost from the star. It depends critically on the mode of ejection that can vary greatly from case to case as we have already seen in 3.5. Thus, starting from one configuration of contact or semi-contact we do not know exactly what will be the configuration at the next moment even if we assume a definite rate of mass loss.

We also encounter difficulty basic to the calculation of stellar structure itself. The effect of mass loss produces not only a pressure inbalance but also a thermal inbalance in the interior. The time of adjustment to the equilibrium condition is therefore measured by the Kelvin scale of gravitational contraction instead of the much shorter time scale of pulsation (Crawford and Kraft 1957; Morton 1960; Schwarzschild 1962). In other words we no longer have the simple relation of energy loss (at the surface through radiation) balanced exactly by energy production in the interior by thermonuclear reactions. It follows that the stellar structure in the present case becomes a time-dependent problem instead of a problem of equilibrium.

Or we may put our arguments differently. According to the Vogt-Russell theorem (e.g., Chandrasekhar 1939), if the pressure, the opacity and the rate of generation of energy are functions of the local values of the density, the temperature and the chemical composition only, then the structure of a star is uniquely determined by the mass of the star and its chemical compositions. In other words, among other structural properties of a star, the radius is determined internally by the mass and the chemical composition. Now in the contact and semi-contact binaries, the radius of the overflowing component is further limited by the contact surface which is imposed externally. Therefore, in general, we cannot find a stellar model that can fulfill both internal and external condition for its radius. This is equivalent to say that no solution can be obtained from the consideration of equilibrium. This explains why we cannot derive the evolutionary sequence of models by computation for the overflowing component as we have done for the single stars.

After what has been said it becomes quite apparent that discussions on the evolution of binary stars are necessarily qualitative and highly speculative. We should always remember that these discussions represent only the preliminary probing of the possibilities but not the final verdict.

Since the secondary components of many *W* Ursae Majoris stars are over-luminous with respect to their masses by more than two magnitudes on the average, Kitamura (1959, 1960) has suggested that the extra energy radiated away is derived from the gravitational contraction of mass that is being accumulated on the secondary surface at the expense of the primary. Accordingly, he has built a series of gravitationally contracting stellar models that are accreting mass. However, he has not given any reason why the secondary captures mass from the primary instead of the reversed course, namely the primary captures mass from the secondary, for after all the gases are flowing around both components of these contact binaries. Therefore, the present writer personally inclines to regard Kuiper's (1948) interpretation of the departure, from the mass-luminosity relation of *W* Ursae Majoris stars still worthy of our attention.

Because of the slowness of energy transfer in the star, the loss of mass at the surface will change the internal structure and the luminosity only slowly (on the Kelvin time scale as we have said) Thus, we arrive as with Krat (1957) at the conclusion that the star evolves under the conditions of a constant luminosity and decreasing mass. Using a simple model Krat has shown that the process of mass ejection under these conditions will be a self-accelerating process. Hence, he went on to suggest that the ejection velocity will increase. In this way he has tried to understand the high velocities of ejected matter from the Wolf-Rayet stars According to him "the evolution of massive stars proceeds from the stage of stable hot supergiants through the stage of Wolf-Rayet stars and red giants to the main sequence (thus approaching solar-type stars)" as a result of mass loss. It appears that Krat's view may have overstretched the idea of mass loss. However, he is not alone in putting the Wolf-Rayet stars before the main sequence (e.g., Sahade 1958).

Among the many interesting views concerning the evolution of close binary stars one that is appealing is about mass exchange between two components and the consequence as regards their evolution, proposed independently by Crawford (1955) and by Kopal (1955) for the Algol-type binaries. Consider in a detached close binary two components of different masses. Since the luminosity is proportional to about 4th power of the mass (Russell and Moore 1940), the more massive one evolves faster and consequently reaches the stage of hydrogen-exhaustion in the central core earlier than its companion. According to the current understanding of stellar evolution, once the hydrogen has been exhausted in the central core, its envelope expands. Eventually, it will reach the S_1 surface. Mass that flows out from the expanding component at near the point L_1 because of the contract condition will be collected at least partly by the other component. In this way the more massive component loses while the less massive one gains mass. According to Crawford this process can proceed until the originally more massive component becomes the less massive of the two. In this way, he explains why in the Algol-type binaries it is the less massive component that now fills up one lobe of the S_1 surface while the more massive component is small and stable.

While this is an appealing idea there remain some difficulties in applying it to the Algol-type binaries as was initially suggested by Crawford and Kopal. Let us first consider the difficulty raised by Struve and Huang (1958). If M_1 and M_2 are respectively the present masses of the primary and secondary component and if in their evolution, mass of an amount ΔM has been transferred from the secondary to the primary and mass of an amount δM has been dissipated into outer space by the secondary, the following inequality can be easily established from the condition that the initial mass of the secondary must be greater than that of the primary:

$$2\Delta M+\delta M>M_1-M_2 \tag{8.1}$$

Now the primary of the Algol-type binaries is usually a normal main-sequence star of spectral type A and may be assigned a mass of $M_1=3M_\odot$. In some cases, M_2 has been found to be $0.2M_\odot$ or less (Sahade 1945, 1949). If we assume $M_2=0.2M_\odot$, we find from the inequality (8.1) that $\Delta M>1.4M_\odot$ if all mass dissipated by the secondary has been collected by the primary and $\Delta M=\delta M>14/15M_\odot$ if equal amounts are collected by the primary and escaped to outer space. Thus, eight-tenths to nine-tenths of the initial mass of the secondary must have been lost. If we remember that when the envelope of a star expands during its evolution after the main sequence the central core of more than one-tenth of the total mass of the star contracts, it is hard to see a star to lose eight-tenths to nine-tenths of its mass by the mechanism described above.

Indeed Kopal (1959) has finally rescinded this interpretation. In addition to the mass consideration, he has also called attention to the fact that there is a complete lack of binaries whose more massive component fills up the lobe of S_1 surface. However, the present writer does not regard this as a serious difficulty because the Kelvin time scale is short and the chance of discovery is small (Morton 1960). Kopal has further pointed out that there is not enough energy to lift the mass from the hydrogen-depleted central core of the secondary to the level of the primary surface, if more than eight-tenths of the total mass is to be transported. Therefore, his second objection arises from the difficulty of too much mass to be transferred which was discussed before. Kopal has not provided a new explanation for the Algol-type binaries. However, he presumes on the observational ground that the less massive secondaries in these systems for some unknown reason begin to expand at a certain stage of their evolution before the more massive primaries will do so.

From these considerations this idea of mass transfer between two components does not appear promising for the interpretation of the Algol-type binaries. On the other hand, the mechanism is a plausible one because, as we have seen in 3.5, it is a necessary consequence of the slow mode of mass ejection. Indeed, we now have reasons to believe that it may be working in β Lyrae (Huang 1963a).

Thus far we have discussed only evolution of component stars in close binaries. There remains the question of evolution of the binary system itself. One can easily see that the two problems are closely related. Theoretically we cannot attack the problem of evolution of the binary system without first understanding evolution of its component stars. The difficulty encountered in the latter makes a theoretical investigation of the former impractical. Observationally, data are too scanty to lead us to any definite and plausible scheme of binary evolution. Struve (1950) once made an interesting suggestion along this line and Sahade (1960b) has recently rediscussed some of Struve's original ideas of deriving *W* Ursae Majoris stars from early-type (B or A) systems.

Another fascinating problem connected with close binaries concerns the nature of various peculiar objects in the galaxy. Struve often expressed the view that Wolf-Rayet stars, symbiotic stars, and many other peculiar objects may all be binaries and moreover derive their peculiarities from being binaries. As a result of recent investigations by Abt (1961) the metallic line stars may now be added to the list of peculiar objects which appear to exist only in binaries. Although Struve's suggestions have not been accepted by all astrophysicists, they have inspired many to look at the problem. After the discovery of the binary nature of Nova DQ Hercules (1934) by Walker (1954, 1956), the question has again been raised (Struve 1955) as to whether the novae and nova-like objects are always components of close binaries. This question has led to many interesting studies about the relationship between these exploding stars and binaries (Crawford and Kraft 1956; Huang 1956; Kraft 1962, 1964; Schatzman 1958). A review of extensive works in this field is beyond the scope of the present article.

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