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ERROR ANALYSIS OF A WEIGHTED LEAST SQUARES PROCESS TO DETERMINE THE LUNAR GRAVITATIONAL FIELD

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SUMMARY

An error analysis has been made to determine how accurately the coefficients of the spherical harmonics of the lunar gravitational potential function can be estimated by using earth-based range and range-rate measurements of a lunar satellite. Parametric analyses were made to study the effects of certain of the orbital elements on the standard deviations of the gravitational coefficients. The covariance matrices in which the standard deviations are given were obtained by forming and inverting the normal matrices in a weighted least squares process. Standard deviations of some of the gravitational coefficients were strongly dependent on the semimajor axis, inclination, nodal position, and eccentricity of the lunar satellite orbit. Also some of the high correlations associated with the estimates of the coefficients were found to be dependent either on the length of the tracking-data arc or on certain of the orbital elements. The two data types, range and range-rate, having weights of 10 meters and 0.002 meter per second, were found to be similar or essentially the same data types in the sense that they produce similar correlation matrices. There were no significant differences in the values of the determinants of these matrices, and combining the two data types did not lead to significant reductions in high correlations. From the analyses it appears that the best tracking schedule would be one in which the allowable tracking time is distributed over various times of the month. When tracking data from lunar satellites having different orbital inclinations are combined to estimate the gravity coefficients, the condition of the normal matrix for inversion is improved and high correlations between the even zonal coefficients are reduced.

INTRODUCTION

An important contribution to the scientific community would be an improvement in the present knowledge of the lunar gravitational field. This information could be used in the planning of future manned missions to the moon. For instance, close lunar satellites will be used to take pictures of the moon which, in turn, will aid in the choosing of landing sites for the manned spacecraft. An accurate knowledge of the lunar gravitational field is necessary for calculating a precise ephemeris of the photographic satellite which is needed for correct interpretation of the photographic data. Precise information about the lunar gravitational field would also aid those concerned with the figure of the moon and its internal structure.

The gravitational field of the earth has been approximated through analysis of the tracking data from close earth satellites and it is expected that similar analyses of tracking data from close lunar satellites can be used to estimate the lunar gravity field. An approximation of this gravity field can be made by determining a sufficient number of coefficients, $C_{n,m}$ and $S_{n,m}$, in the expansion of the lunar gravitational potential function U in spherical harmonics:

$$U = \frac{\mu}{r}\left[1 + C_{0,0} + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^n P_{n,m}(\sin \phi)\left(C_{n,m} \cos m\lambda + S_{n,m} \sin m\lambda\right)\right] \tag{1}$$

where μ is the gravitational constant of the moon, r is the distance from the center of the moon to the satellite, ϕ is the selenographic latitude of the satellite, λ is the selenographic longitude of the satellite, R is the mean radius of the moon, and $P_{n,m}$ are the associated Legendre polynomials (n is degree and m is order). The coefficient $C_{0,0}$ represents a deviation of the mass of the moon from a nominal value assumed in μ.

At the present time, knowledge of the lunar gravitational field is very limited. A brief discussion is given in reference 1 concerning present estimates of the mass of the moon and the two second-degree gravity coefficients, $C_{2,0}$ and $C_{2,2}$. Uncertainties in the current estimates of $C_{2,0}$ and $C_{2,2}$ are given as 1×10^{-5} or 2×10^{-5} in a nominal value of -2.071×10^{-4} for $C_{2,0}$ and about 1×10^{-5} in a nominal value of 2.072×10^{-5} for $C_{2,2}$. Estimates, such as those given in references 2 and 3, have also been made of some of the higher degree coefficients. It appears, however, that present estimates of the gravity coefficients are not very accurate and it is of interest to analyze a method of determining the coefficients to a higher degree of accuracy.

The purpose of this paper is, therefore, to perform an error analysis in which a method for determining the gravity coefficients is analyzed. Two general approaches to the problem of determining the gravitational field were considered for use in this error analysis. These approaches are referred to as the short period or direct method and the long period and secular method. Both methods may be used in a differential correction process. In the direct method, the partial derivatives of the observables with respect to the coefficients are obtained from a direct formulation of the observables as functions of the coefficients. In the long period and secular method, the observables or tracking data types are used to determine sets of intermediate elements. These elements are then used as observables in a differential correction process where the long period and secular variations in the elements are analyzed to determine the gravity coefficients. See references 1 and 4 for further discussion of the distinction between these two methods.

The gravitational coefficients are usually referred to as either zonal or tesseral coefficients. The zonal coefficients are those with $m = 0$ and the tesserals are those coefficients with $m \neq 0$. A subset of the tesseral coefficients, where $n = m$, are sometimes referred to as sectorial coefficients. The zonal coefficients in the gravitational field of the earth are best determined by the long period and secular method, whereas a direct method is the only way of determining the tesseral coefficients. In the determination of the gravitational field of the moon, however, both the tesseral and zonal coefficients are determinable by the long period and secular method. (See ref. 1.) Therefore, for a determination of the lunar gravitational field, a choice is available between the direct method and the long period and secular method. In this paper the direct method of determining the gravity coefficients is analyzed.

A parametric error analysis was made to investigate the correlations between the estimated gravity coefficients, the condition of the normal matrices for inversion, and the accuracies of estimating the gravity coefficients. The coefficients were never actually determined and in particular only the covariance matrix for the coefficients was needed. This covariance matrix was obtained by forming and inverting the normal matrices in a weighted least squares process.

SYMBOLS

A	matrix containing partial derivatives of a given data type with respect to the gravitational coefficients
a	semimajor axis of lunar satellite orbit
$C_{n,m}$	gravitational coefficient (n is degree of the coefficient and m is the order)
D	distance from center of earth to center of moon
DET	determinant of correlation matrix
e	eccentricity of lunar satellite orbit
i	inclination of orbital plane of lunar satellite to earth-moon plane

m	order of a particular gravitational coefficient
n	degree of a particular gravitational coefficient
$P_{n,m}$	associated Legendre polynomials (n is degree, m is order)
q	order of correlation matrix
R	mean radius of moon
r	distance from center of moon to lunar satellite
$S_{n,m}$	gravitational coefficient (n is the degree of the coefficient and m is the order.)
t_o	time of periapsis passage
U	gravitational potential function defined in equation (1)
v	true anomaly of the lunar satellite
W	weighting matrix
X,Y,Z	coordinates axes with origin at center of earth (The X-axis is positive in the direction from the center of the earth to the center of the moon, the Y-axis is positive in the direction of rotation of the moon, and the Z-axis is positive in such a direction that it forms a right-handed axis system.)
α_i	parameter to be estimated (a particular gravity coefficient); $i = 1,2, \ldots j$
$\Lambda\alpha$	covariance matrix of estimated parameters
λ	selenographic longitude of the lunar satellite
λ_{min}	minimum eigenvalue of correlation matrix
λ_{max}	maximum eigenvalue of correlation matrix
μ	gravitational constant of moon

Ω	longitude of ascending node of lunar satellite orbital plane measured in earth-moon plane in direction of rotation of moon from positive X-axis
Ω_0	longitude of ascending node of lunar satellite orbital plane at the beginning of the tracking interval
ω	argument of periapsis, angle measured in lunar satellite plane from ascending node to periapsis
ϕ	selenographic latitude of the lunar satellite
ρ	range or distance from center of earth to position of lunar satellite
$\dot{\rho}$	range rate or radial velocity of lunar satellite with respect to center of earth
$\rho_{\alpha_1\alpha_2}$	correlation coefficient; denotes correlation between α_1 and α_2
σ	standard deviation or one-sigma uncertainty (When this symbol appears with a subscript, it is taken to mean the one-sigma uncertainty of the variable indicated by the subscript.)
σ_{ob}	standard deviation or one-sigma uncertainty in an observation or data type
$\theta=\omega + v$	

GENERAL DISCUSSION

The tracking geometry for a close lunar satellite is considerably different from that of a close earth satellite. For instance, close earth satellites pass over the tracking station very rapidly, and, consequently, the station can view the satellite for only a limited time and over only a short segment of the orbit. In general, for orbits with high inclination, an equatorial tracking station has at most two viewing periods per day as the rotation of the earth carries the station through the orbital plane. Hence in order to obtain a good sampling of the gravity field and a suitable determination of the gravitational coefficients, tracking data must be obtained from a number of tracking stations located at various latitudes. For a close lunar satellite, an earth-based tracking station can view a lunar sattile from moon rise to moon set except for the time when the satellite is occulted by the moon. Thus, with exception for the time the satellite is occulted by the moon, two tracking stations properly situated on the earth can obtain almost complete coverage of each orbital period of the lunar satellite.

Even with such complete coverage, one might anticipate difficulty in trying to determine coefficients with tracking data which cover a short time span because in this case the tracking geometry is approximated by the so-called Stationary-Moon geometry. The Stationary-Moon geometry is defined as the case when the line from the tracking station to the center of the moon remains inertially fixed during the tracking interval. It has been shown that with this tracking geometry the state of the lunar satellite cannot be completely determined. (See ref. 5.) In reality the moon is not stationary and the line from the tracking station to the center of the moon is not fixed but rotates about the Z-axis at a rate of about 0.55 degree per hour. Therefore in practice, due to this small angular rate, the tracking geometry approaches that of the stationary moon when short data arcs are considered.

In reference 6 it was found that, after 1 orbit of tracking, the normal matrix formed to solve for the state of the lunar satellite could not be inverted with single-precision, 8-decimal arithmetic; however, double-precision, 16-decimal arithmetic, proved to be adequate. Hence one would not be surprised to encounter similar difficulties in trying to determine the gravity coefficients from short tracking intervals, and it is of interest to investigate the length of the tracking interval required for a first determination of the coefficients. This will be discussed in a subsequent section of this paper.

For a close lunar satellite, the distance from the tracking stations to the satellite is extremely large compared with that of a close earth satellite, 380 000 kilometers as compared with 4 200 kilometers. The hour angle and declination of the satellite can be measured by earth-based radar to about 0.183 degree. (See ref. 7.) These measurements lead to a very poor determination of the position of the satellite and these angular measurements cannot be used effectively in either the orbit-determination process or the process of determining the gravity coefficients. Therefore, only range and range-rate measurements were considered for use in the present analysis.

ANALYSIS

Certain simplifying assumptions were made in order to reduce the complexity of the equations involved. First, it was assumed that the moon revolved about the earth in a circular orbit. The earth-moon plane was chosen as the fundamental plane of reference and all the orientation angles of the lunar satellite orbital plane are given relative to this fundamental plane. The geometry of the problem is illustrated in figure 1. Throughout this analysis only one tracking station was considered; this station was taken to be at the origin of the geocentric coordinate system shown in figure 1. Range and range-rate measurements were simulated from this station with the assumptions that all measurement errors of a given data type were uncorrelated, unbiased, and of equal weight. Here "measurements were simulated" means that the measurements were only assumed to have been made. No real tracking data were used nor were any tracking data generated because, in this type of analysis, neither is necessary, as will be shown subsequently. Standard deviations of range and range rate were assumed to be independent of time and to have values of 10 meters and 0.002 meter per second, respectively. These accuracies are estimates of those applicable to the NASA deep space network (DSN) tracking system. (See ref. 8.) Note that, by assuming the tracking station to be located at the geocenter, it is never occulted by the earth; however, occultation of the satellite by the moon was accounted for. In reality, a topocentric tracking station would sometimes be occulted by the earth and tracking would be lost as the station went over the horizon, but, in general, another station would come up and it would be possible to continue tracking without loss of coverage due to occulation of the tracking station. This is the same type of coverage which was assumed by having the tracking station at the geocenter. In general, a given topocentric tracking station would be able to view a lunar satellite orbit at slightly different angles because of the parallax of the earth and this would normally result in a better estimate of the orbit and the gravity coefficients. Therefore, the standard deviations of the estimates of the gravity coefficients presented herein are considered to be conservative because of the assumption of a geocentric tracking station.

The gravitational coefficients were assumed to have been estimated by using the simulated range and range-rate measurements in a weighted least squares process. Normal matrices were formed and inverted in this process to obtain the covariance matrix associated with the estimates of the coefficients. Because the measurements were assumed to be uncorrelated and of equal weight, the weighting matrix was diagonal with equivalent terms on the diagonal, and hence the covariance matrix for one data type can be written as

$$\Lambda_\alpha = \left(A^T W A\right)^{-1} = \sigma_{ob}^2 \left(A^T A\right)^{-1} = \begin{vmatrix} \sigma_{\alpha_1}^2 & \rho_{\alpha_1 \alpha_2} \sigma_{\alpha_1} \sigma_{\alpha_2} & \cdots & \rho_{\alpha_1 \alpha_j} \sigma_{\alpha_1} \sigma_{\alpha_j} \\ \rho_{\alpha_1 \alpha_2} \sigma_{\alpha_1} \sigma_{\alpha_2} & \sigma_{\alpha_2}^2 & \cdots & \rho_{\alpha_2 \alpha_j} \sigma_{\alpha_2} \sigma_{\alpha_j} \\ \vdots & \vdots & & \vdots \\ \rho_{\alpha_1 \alpha_j} \sigma_{\alpha_1} \sigma_{\alpha_j} & \rho_{\alpha_2 \alpha_j} \sigma_{\alpha_2} \sigma_{\alpha_j} & \cdots & \sigma_{\alpha_j}^2 \end{vmatrix} \tag{2}$$

where (A^TWA) is called the normal matrix, A is the matrix containing the partial derivatives of the data type with respect to the coefficients to be determined, and σ_{ob} is the standard deviation of the data type. The standard deviation of the estimates of the coefficients is denoted by σ subscripted with the appropriate variable whereas the correlation coefficients are denoted by ρ subscripted with the appropriate variables. When the covariance matrix for the simultaneous use of the two data types is desired, one simply adds the normal matrix for each data type and then inverts the sum. No actual measurements are ever needed in this process; only the statistics of the measurements and the partial derivatives in the A matrix are required. Because of the special form of the weighting matrix, it can be seen from equation (2) that, for one data type, the standard deviations of the estimates of the coefficients are proportional to the standard deviation of the data type.

In subsequent discussions the term "correlation matrix" will be referred to frequently. This is a symmetric matrix having diagonal elements equal to 1 and the correlations between the gravity coefficients as the off diagonal elements. The elements of this matrix are dimensionless.

For the direct method referred to previously, the observations are related directly to the gravitational coefficients given in equation (1) through the integrals of the equations of motion of the spacecraft. Hence, in order to form the partial derivatives of the measurements with respect to the gravitational coefficients, which are needed to form the normal matrix, the equations of motion of the lunar satellite must be integrated. For this part of the analysis, a first-order general-perturbation method similar to that discussed in reference 9 was utilized to obtain the desired integrals of motion. The only difference between the equations of motion in reference 9 and those used in this analysis was in the choice of the element used to represent the position of the satellite in the orbital plane. The perturbed mean anomaly is used in reference 9 whereas the mean anomaly at epoch is used in this analysis. The necessary partial derivatives were then obtained by the so-called chain rule. First, the derivatives of the observables with respect to the state of the satellite at each observation time are obtained from purely geometrical relations. These derivatives are given in reference 6. Next, the derivatives of the state with respect to the gravitational coefficients at the observation times are obtained by direct differentiation of the integrals of motion. The partial derivatives of the observables with respect to the coefficients of interest can then be obtained by applying the chain rule to the appropriate derivatives.

To make a parametric study of the effects of a given element on the accuracy of determining the gravitational coefficients, five elements of a nominal orbit were held constant and the sixth was varied over a given range. One exception to this procedure was the eccentricity variation, where, instead of using the nominal value for the semimajor axis, a value of 5 000 kilometers was used. A large semimajor axis was necessary to insure that, over the range of eccentricities investigated, the pericentron of the orbit was never less than the radius of the moon. Nominal values of the set of elements used in this analysis are:

$a = 2\ 686$ kilometers
$i = 15^{\circ}$
$\Omega_0 = 30^{\circ}$
$\omega = 0^{\circ}$
$e = 0.336$
$t_o = 0$ second

In this study, the effects of varying ω and t_o on the accuracy of determining the gravitational coefficients were not considered. It is recognized that a variation in ω would cause a different portion of the lunar satellite orbit to be occulted by the moon from the tracking station. However, it was found from occultation studies that, for orbits with small to medium eccentricity, the covariance matrices did not change significantly with changes in ω. Also, whenever the orbital element ω is used in the partial derivatives, it always appears as an angle added to the true anomaly in the argument of either a sine or cosine function. Because the true anomaly rotates through 360° each orbit, the value of the sine or cosine changes through 1 period, regardless of the value of ω. Hence, the effects of a variation in ω on the accuracy of estimating the gravitational coefficients are negligible as long as integral orbits of tracking are considered. Likewise, the effects of a variation in the orbital element t_o are negligible for integral orbits of tracking because the parameter is used only to determine the location of the satellite at any given time in the orbit, and hence the measurements which are equally spaced in time over the orbit are independent of t_o.

RESULTS AND DISCUSSION

All results presented herein are for the simultaneous estimation of a set of 11 gravitational coefficients. This set includes the mass of the moon, all the second-degree coefficients, the zonal coefficients through degree five, and two fourth-degree second-order tesseral coefficients. Unless otherwise noted normal matrices were formed by assuming that 26 equally spaced observations, less the points omitted because of occultation, were made each orbit over a lunar satellite orbit having the nominal orbital elements given in the previous section. The 26 data points were usually reduced by 15 to 20 percent because of occultation of the satellite by the moon. As mentioned previously, the standard deviations in the range and range-rate measurements were assumed to be 10 meters and 0.002 meter per second, respectively.

Effects of Tracking Time

The variation of the standard deviation of estimating the set of gravitational coefficients with the number of orbits tracked is shown in figure 2. Here the normal matrices for each data type and for the simultaneous use of both data types have been summed and inverted after each orbit of tracking. After about 3 orbits of tracking, it appears that range-rate data have a slight advantage over range data as far as the accuracy of determining the coefficients is concerned. However there are no very large differences between the results for either range or range rate. The simultaneous use of range and range rate reduced the standard deviations by a factor of approximately $\sqrt{2}$, as might be expected, since the number of data points was doubled. Of the 11 coefficients estimated, the two odd zonals were determined with the best accuracy because these two coefficients have no high correlations with the other coefficients considered. The standard deviations of the remaining 9 coefficients decrease very rapidly because of the reduction in the correlations between certain coefficients as the tracking interval increases. In general, lower correlations usually result in lower standard deviations and a better separation of the coefficients in the solution vector.

The correlation matrices associated with the estimation discussed in the previous paragraph are shown in figure 3 after the first and tenth orbits. The high correlations, that is correlations greater than 0.90, have been divided into two categories. First, there are those correlations due to the short data arc, and, second, those correlations which are strongly dependent on certain of the orbital elements. In the first category, the correlations are very high, 0.99 or larger, after 1 orbit of tracking and decrease to reasonable values after a few orbits of tracking. Several examples of this type of correlation can be found in figure 3. For instance, in the range-rate matrix the correlation between $C_{2,1}$ and $S_{2,1}$ is -0.9992 after only one orbit and then decreases to -0.42 after the tenth orbit. In the second category there are correlations which are strongly dependent on the nodal position and inclination and to a lesser extent on the eccentricity. These high correlations do not change much over a tracking time of a day or so. The nodal-dependent correlations are usually between pairs of tesseral coefficients whereas the inclination-dependent correlations are usually between the zonal coefficients. Examples of these correlations are pointed out in subsequent discussion.

One of the primary concerns in any real operational estimation procedure is the condition of the normal matrix for inversion. The determinant of the correlation matrix, which must be greater than 0 and equal to or less than 1, gives some indication of the condition of the normal matrix. (See theorem 4.1, p. 14 of ref. 10.) It has been a common practice to analyze the condition of the covariance matrix on the basis of the ratio of maximum to minimum eigenvalues. If this ratio is equal to 1, the matrix is diagonal and hence can be inverted trivially without loss of significant figures. However, the eigenvalues of the covariance matrix are dependent on the dimensionality of the parameters being estimated, and therefore a diagonal covariance matrix could have an arbitrarily large eigenvalue ratio if the dimensions are chosen appropriately. Hence one could not examine the eigenvalue ratio in this case and conclude that the covariance matrix was diagonal. It is more appropriate to consider the eigenvalue ratio for the correlation matrix as an indicator of the condition of the matrix for inversion because in this case no dimensions are involved. This ratio is not readily available in the computer program used in this analysis, but the determinant of the correlation matrix is available and is used to indicate the condition of the matrix for inversion. Actually, the determinant contains some information about the eigenvalues since the product of the eigenvalues is equal to the determinant and the sum of the eigenvalues of the correlation matrix is equal to the order of the matrix. If q is the order and DET is the determinant of the correlation matrix, then it can be shown that

$$\frac{\mathrm{DET}}{4} \leq \frac{\lambda_{\min}}{\lambda_{\max}} \leq \frac{\sqrt[q-1]{\mathrm{DET}}}{q + (1 - q)\sqrt[q-1]{\mathrm{DET}}}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum values of the eigenvalues. This inequality gives a set of bounds on the eigenvalue ratio, however very little information is contained in the upper bound when the order of the determinant is large.

After 1 orbit of tracking, the determinants of the correlation matrices in figure 3 are extremely small and indicate nearly singular matrices. The smallness of these determinants is largely due to the short-data-arc problem mentioned previously. At the end of 10 orbits of tracking, the determinants have increased significantly but they are still so small that the associated normal matrices cannot be inverted with single precision or 8-decimal arithmetic. However, it appears that by using double-precision 16-decimal arithmetic, a first estimate of the 11 gravity coefficients considered herein can be obtained with just 1 orbit of tracking data.

Comparison of Data Types

As stated previously, two basic tracking data types are considered herein — range and range rate. It is of interest to compare the relative advantages of these two data types. One such comparison was given in the discussion of figure 2 where it was pointed out that the differences between the standard deviations in the coefficients obtained by using either data type alone or simultaneously are not significant. An equally important comparison is that between the corresponding correlation matrices. This comparison was made after each orbit up to 10 orbits and it was found that, after the third orbit, the correlation matrices for the two data types and their simultaneous use were similar.

That is, gravity coefficients which were highly correlated in one matrix were usually highly correlated in the other matrix, and, as expected, combining the two data types did not lead to any significant reduction in the correlations. A comparison of the determinants of the three correlation matrices was made after each orbit and it was found that, after 3 orbits of tracking, they were essentially the same. The matrices given in figure 3 after 10 orbits of tracking are typical of these results. It was concluded for the values of σ_{ρ} and $\sigma_{\dot{\rho}}$ and the set of 11 gravity coefficients considered herein, that range and range-rate measurements are basically the same data types in the sense that they produce estimates of the gravity coefficients which are not significantly different and that they result in similar correlation matrices having essentially the same determinental values and that no significant reduction in correlations due to the combination of data types was noted. A similar conclusion was drawn in reference 6 in the case of estimating the lunar satellite state.

Since it has been shown that range and range rate are similar data types, the results presented in the remainder of this paper are for the use of range rate only. The relative advantage of range and range rate is affected by the values of the orbital elements, primarily a. For larger values of the semimajor axis, range has a slight advantage over range rate as far as the accuracy of estimating the coefficients is concerned.

Effect of Semimajor Axis

The variation of the standard deviation in the estimates of the gravity coefficients after 10 orbits of tracking with the semimajor axis are presented in figure 4. In order to obtain the standard deviations of the estimates of these 11 coefficients, normal matrices were formed and inverted after each orbit, up to 10 orbits, for each value of the semimajor axis investigated with the elements i, Ω, ω, e, and t_o held constant. The semimajor axis was varied from 2 686 kilometers to 5 000 kilometers. Even though the period of the orbits changed as the semimajor axis changed, the number of range-rate measurements per orbit was held to 26. These measurements were assumed to be equally spaced in time over the orbital period and to have a standard deviation of 0.002 meter per second. Many of the curves in figure 4 have several local maximums and minimums which are due to changes in correlations between certain of the coefficients. In general, the standard deviation for a particular gravity coefficient increased when there was a significant increase in the correlation coefficients for that particular gravity and decreased with significant decreases in the correlations. There are several pairs of curves in figure 4 which are very similar, for instance those representing $C_{2,2}$ and $S_{2,1}$. This similarity is due to the high correlations between the two gravity coefficients. In this particular case, the two coefficients remained highly correlated over the entire range of semimajor axis.

With few exceptions the standard deviations presented in figure 4 increased as the semimajor axis increased. This trend is as expected since the partial derivatives which comprise the A matrix are inversely proportional to the n + c power of the semimajor axis where n is the degree of the coefficient and c is a constant. As the semimajor axis increased, the numerical values of the derivatives decreased and resulted in less sensitive derivatives; hence, each observation contains less information about the gravity coefficients. Because the information about the coefficients decreased, it is natural to expect a less accurate knowledge of the coefficients. Exceptions to the trend of increasing standard deviations can be seen in figure 4 and these exceptions are due to the previously noted changes in correlations between certain of the gravity coefficients. A good example of these exceptions is the curves representing $C_{2,1}$ and $S_{2,2}$. Note that even these curves increase up to about 4 000 kilometers, drop off sharply between 4 000 kilometers and 4 750 kilometers and then begin to increase again. The sharp dropoff or steep negative slope is due to a large reduction in the correlation between these two parameters.

The slope of the curves in figure 4 would also be expected to increase as the degree of the coefficient increases because the partial derivatives in the A matrix are inversely proportional to the n + c power of the semimajor axis. This particular trend can be noted in figure 4.

Although a few correlations increased or decreased as the semimajor axis was varied over the given range, there were only a few very high correlations which changed appreciably. Also a comparison of the determinants of the correlation matrices after each orbit of tracking for each value of the semimajor axis investigated showed no significant differences, and, hence, the condition of the normal matrix for inversion did not change appreciably over the entire range of semimajor axes.

Effects of Inclination

The inclination i of the satellite orbital plane with respect to the earth-moon plane has proven to be a significant parameter in the estimation of certain of the gravitational coefficients; namely, the even zonal coefficients. The reader is reminded that the assumption has been made that the earth-moon plane and the lunar equatorial plane coincide. The standard deviations in the estimates of the 11 coefficients after 10 orbits of tracking are plotted as functions of inclination in figure 5. Again the data presented in this figure were obtained by holding five of the orbital elements (a, Ω, ω, e, and t_0) at the nominal values and varying the element of interest, in this case i. Note that the accuracy of estimating the two odd zonals $C_{3,0}$ and $C_{5,0}$ is relatively independent of the inclination. The rapid decreases exhibited in some of the curves, especially the curves for the even zonal coefficients, are due to the reduction in correlations between certain of the coefficients with inclination. Likewise, increases in some of the curves such as those for certain of the tesseral coefficients and for the even zonal $C_{0,0}$ are due to increases in the correlations between certain coefficients with inclination. Although some of the tesserals may have correlations which are dependent on inclination, it is usually the correlation between pairs of even zonal coefficients which are most significantly influenced by the inclination. Examples of these correlations are shown in figure 6 where the correlation matrices after 10 orbits of tracking are shown for inclination angles of 2°, 30°, and 60° and for a combination of 2° and 60°. Note that the correlations between certain pairs of even zonal coefficients such as $C_{2,0}$ and $C_{4,0}$ go through zero at some inclination angle between 30° and 60° which is an indication that there exists some optimum angle between these limits for separation of these coefficients in the solution vector. This result leads one to expect that the coefficients which have correlations that are highly dependent on inclination angle can be separated in the solution vector by combining tracking data from various satellites having different orbital inclinations.

A comparison of the determinant of each of the correlation matrices presented in figure 6 shows that there also exists an optimum inclination angle, as far as the condition of the normal matrix is concerned. That is, there exists an inclination angle at which the determinant of the correlation matrix is a maximum. For the results presented, this angle is in the neighborhood of 40°. However, the important point is that the normal matrix might be better conditioned for inversion by combining data from various satellites having different orbital inclinations.

A comparison of the determinants of the correlation matrices obtained by combining tracking data from satellites having different inclinations with those obtained by using tracking data from a single satellite is given in figure 7. Here it was assumed that 26 range-rate measurements were made each orbit for 10 consecutive orbits of two different satellites having orbital inclinations of 2° and 60°. The curve representing the results for the combined tracking data was obtained by adding the normal matrices for the two satellites after each orbit and inverting the resulting matrix to obtain the combined covariance matrix. For example, 10 orbits represented by the combined-data curve would include 5 orbits at $i = 2^{\circ}$ and 5 orbits at $i = 60^{\circ}$. It should be pointed out that the time interval of tracking in this case would be one-half that for 10 orbits of a single satellite. The results given in figure 7 are a clear indication that, in most cases, the normal matrix can be better conditioned for inversion by combining tracking data taken from satellites having different orbital inclinations. For instance, at the end of 10 orbits of tracking, the determinant of the correlation matrices for 2° or 60° is of the order 10^{-15} whereas the combined correlation matrix determinant is of the order 10^{-9}. This represents a very significant increase in the determinant and hence a better conditioned normal matrix. A point of interest is that the determinant associated with the 60° inclination curve decreases with tracking time. This result was unexpected and is partly due to the increasing correlations of the second-degree tesseral coefficients with certain of the other coefficients. The correlation matrix for the combined set of tracking data is given in figure 6. By comparing this matrix with the others in figure 6 one can see that those correlations which have been pointed out as being dependent on inclination, the even zonals, are greatly reduced when the tracking data are combined. For example, the correlation between $C_{0,0}$ and $C_{2,0}$ at 2° and 60° inclination is 0.98 and -0.87, respectively, but with the combined tracking data, this correlation becomes 0.14. From these results it appears, as expected, that one way to reduce high correlations between certain of the gravitational coefficients and to obtain a better conditioned normal matrix would be to combine data from satellites having different orbital inclinations.

Effect of Nodal Position

One of the most important parameters in the determination of the gravitational coefficients is the node angle, which is the angle between the earth-moon line and the line of intersection of the satellite orbital plane with the earth-moon plane. The standard deviations in the estimates of the coefficients after 10 consecutive orbits of tracking at various nodal positions are presented in tabular form in figure 8. These data were not plotted because the interval between the values of Ω was not small enough to allow a smooth curve to be fitted to the points. It should be noted that the value of Ω shown in figure 8 is the value at the beginning of the tracking interval and that at the end of 10 orbits Ω has decreased approximately 19°. Of the 11 coefficients, the standard deviations for the two fourth-degree coefficients varied by approximately one order of magnitude whereas the remaining coefficients varied less than one order of magnitude. However, if the standard deviations are plotted as functions of the nodal positions, the resulting curves will contain a number of local maximums and minimums. This result is due to the strong dependence of the correlations between certain of the gravitational coefficients on the nodal positions. In regions of local maximums one would expect these gravitational coefficients to be highly correlated, whereas in the regions of the local minimums the correlations have been reduced considerably. Examples of nodal-dependent correlations can be seen in figure 9 where the correlation matrices after 10 orbits of tracking are presented for two different nodal positions. For instance note that when Ω equals 30° the correlation between $C_{2,1}$ and $S_{2,2}$ and the correlation between $C_{2,2}$ and $S_{2,1}$ are -0.97 and 0.91, respectively, whereas for the case of Ω equal 110° these two correlations have been reduced to 0.15 and 0.12, respectively. Although the correlations between these pairs of tesseral coefficients decreased considerably when Ω was increased by 80°, there are correlations which increase considerably under the same circumstances, for example, the correlation between $S_{2,1}$ and $S_{2,2}$. Usually the nodal-dependent correlations are between pairs of tesseral coefficients. One implication of these nodal-dependent correlations is that perhaps tracking should be spread throughout the lunar month to take advantage of viewing the lunar satellite orbit from all angles.

In any real orbit determination process it is unlikely that a month of continuous tracking data from one particular satellite will be available. A more likely tracking schedule will be one which includes intermittent tracking throughout the month and, hence, tracking at various nodal positions. Since it has been shown in these analyses that the nodal position during the tracking interval is a very important parameter, it is of interest to compare the standard deviations of the gravity coefficients and the determinant of the correlation matrix for different tracking schedules. One such comparison is given in figure 10 where the determinant of the correlation matrix is plotted as a function of the number of orbits tracked for five different tracking schedules. In all five cases it was assumed that the tracking station would be available for tracking a lunar satellite for 20 orbits during a period of 1 month. Curve (1) represents the results obtained by assuming the satellite had been tracked 1 orbit every 36 hours for a month while curve (2) is for a tracking schedule of 4 orbits every seventh day for a month, and curve (3) is for 2 orbits every third day for a month. For curves (4) and (5) it was assumed that the vehicle was tracked for 20 consecutive orbits with the nodal position at the beginning of the tracking interval being 45° and 120°, respectively. The curves in figure 10 resulting from 20 consecutive orbits of tracking at nodal positions of 45° and 120° represent approximately the upper and lower bounds for similar curves at all other nodal positions. The frequency of observations was 15 range-rate measurements per orbit. As far as the condition of the normal matrix is concerned, it is clear from figure 10 that tracking for 1 orbit every day and a half for a month is better than 4 orbits every 7 days or 2 orbits every 3 days for a month or for 20 consecutive orbits. However at the end of 20 orbits, tracking schedules (1), (2), and (3) result in nearly the same correlation matrix determinant which is approximately six orders of magnitude larger than the other two. This difference is significant and indicates that, by choosing the tracking schedule properly, the condition of the normal matrix for inversion can be improved.

For tracking intervals less than 10 orbits, the standard deviations in the gravity coefficients associated with tracking schedule (1) are smaller than those associated with the other four tracking schedules. But again, as in the case of the determinant of the correlation matrix, the standard deviations after 20 orbits are not very different for tracking schedules (1), (2), and (3). However, the standard deviations associated with tracking schedules (4) and (5) are from one to three orders of magnitude larger than those of the other two schedules. Therefore, for a fixed allowable tracking time, the results indicate that the best tracking schedule would be one in which the observations are distributed over various nodal positions, as opposed to concentrating all the tracking at one nodal position.

Effect of Eccentricity

The accuracy of estimating the gravitational coefficients was found to be a very strong function of the eccentricity. In figure 11 the standard deviations in the estimates of the 11 gravitational coefficients after 10 consecutive orbits of tracking have been plotted against the eccentricity. The same tracking schedule as previously noted was assumed, and again five of the orbital elements were fixed and the sixth, eccentricity, was varied over the given range. However, in this case, the nominal value of the semimajor axis was assumed to be 5 000 kilometers as opposed to 2 686 kilometers (which was used in all other cases). The large value of the semimajor axis was necessary to insure that, over the range of eccentricities investigated, the radius of periapse was never less than the radius of the moon. With the exception of the second-degree tesserals, the standard deviations in the estimates of the gravity coefficients varied by more than an order of magnitude and in the case of the fifth-degree zonal by three orders of magnitude over the given range of eccentricities. Some of the large variations in the standard deviations shown in figure 11 are due to reductions in the correlations between certain of the gravity coefficients. However most of them are due to the complicated way in which the eccentricity enters into the equations for the partial derivatives of the observations with respect to the gravity coefficients. For an eccentricity larger than approximately 0.2, all the gravity coefficients were better estimated more accurately as the eccentricity increased. A point of interest which was noted in the analysis of the eccentricity variation was that, as far as the accuracy of estimating the gravity coefficients is concerned, range data had a slight advantage over range-rate data. Similar results were noted in reference 6 in the case of estimating the lunar satellite state, and they are due to the large value of the semimajor axis chosen for the nominal orbit. As was noted previously, at large semimajor axes range is a slightly better data type than range rate.

Although it has been stated that certain correlations change with eccentricity, only a few high correlations have been noted to be dependent on eccentricity. In figure 12 the correlation matrices after 10 orbits of tracking are shown for an eccentricity of 0.01 and 0.6. Several examples of eccentricity-dependent correlations are given in figure 12, for instance the correlation between $C_{2,0}$ and $C_{4,0}$ and the correlation between $C_{2,2}$ and $S_{2,1}$. The correlations between these two pairs of gravity coefficients have also been pointed out as being dependent on inclination and nodal position, respectively. The correlations which were dependent on inclination were usually between pairs of zonal coefficients whereas the nodal-dependent correlations were between pairs of tesseral coefficients. No such consistency has been noted for the eccentricity-dependent correlations. Again, one would expect that certain of the gravity coefficients which are highly correlated could be separated in the solution vector by combining tracking data taken over satellite orbits having different eccentricities.

Normal matrices were formed and inverted by using combined sets of assumed range-rate measurements of lunar satellites having different orbital eccentricities. This was done to determine whether combining the range-rate measurements from two different eccentric orbits would reduce high correlations and improve the condition of the normal matrix for inversion. The normal matrices were formed and inverted after each orbit up to 10 orbits with 26 range-rate measurements per orbit assumed for each satellite. Hence, after 10 orbital periods for each satellite, the normal matrix would contain 20 orbits of data. For the cases investigated, the condition of the normal matrix was not significantly improved for inversion, and no large reductions in correlations between highly correlated coefficients were noted. In fact some of the correlation coefficients increased when the tracking data were combined, and the determinant of the correlation matrix for the combined data sets was sometimes smaller than for the individual set. In several cases it was noticed that the standard deviations associated with the smallest determinant were smaller than those associated with a larger determinant. It appears that combining tracking data taken from orbits having different eccentricities is not a very effective method of reducing high correlations or of improving the normal matrix for inversion.

CONCLUDING REMARKS

In the real case of determining the lunar gravitational coefficients by using actual range and range-rate measurements of a lunar satellite, analyses of the types given in this report would be useful when choosing the set of gravity coefficients which can be best estimated with the available tracking data. The results presented herein should aid in pointing out some of the more important parameters for future research in the area of gravitational-parameter determination.

The direct method of analysis for determining the gravitational coefficients of the spherical harmonics of the lunar potential function has been analyzed and seems to be a useful tool. A parametric study was made to determine the effects of the orbital elements on the accuracy of estimating the lunar gravitational coefficients and the condition of the normal matrix for inversion. The standard deviations in the estimates of certain of the gravity coefficients were very strong functions of the nodal position, eccentricity, inclination, and the semimajor axis. Some of the high correlations associated with these estimates were found to be dependent either on the length of the tracking-data arc or on certain of the orbital elements. Usually the high correlations which were dependent on inclination were between pairs of even zonal coefficients whereas the nodal-dependent correlations were between pairs of tesseral coefficients. Several eccentricity-dependent correlations were noted but they were not associated with any particular set of coefficients. A comparison of the results obtained by using range and range-rate measurements alone and by using these measurements simultaneously was made to determine the relative advantages of the two data types. The correlation matrices resulting from either data type alone or from their simultaneous use were similar, and the determinant of the combined normal matrix was not significantly different from the individual normal matrices. Hence, it was concluded that range and range rate are basically the same data types. It appears that the best tracking schedule is one in which the allowable tracking time is distributed throughout the month. Finally, it was found that the condition of the normal matrix for inversion could be improved, and high correlations between the even zonal coefficients could be reduced when tracking data from lunar satellites having different orbital inclinations are combined to estimate the gravity coefficients.

It should be noted that the results presented herein are restricted to the particular set of 11 gravity coefficients investigated and if this set were altered, the results may be different.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 24, 1966.

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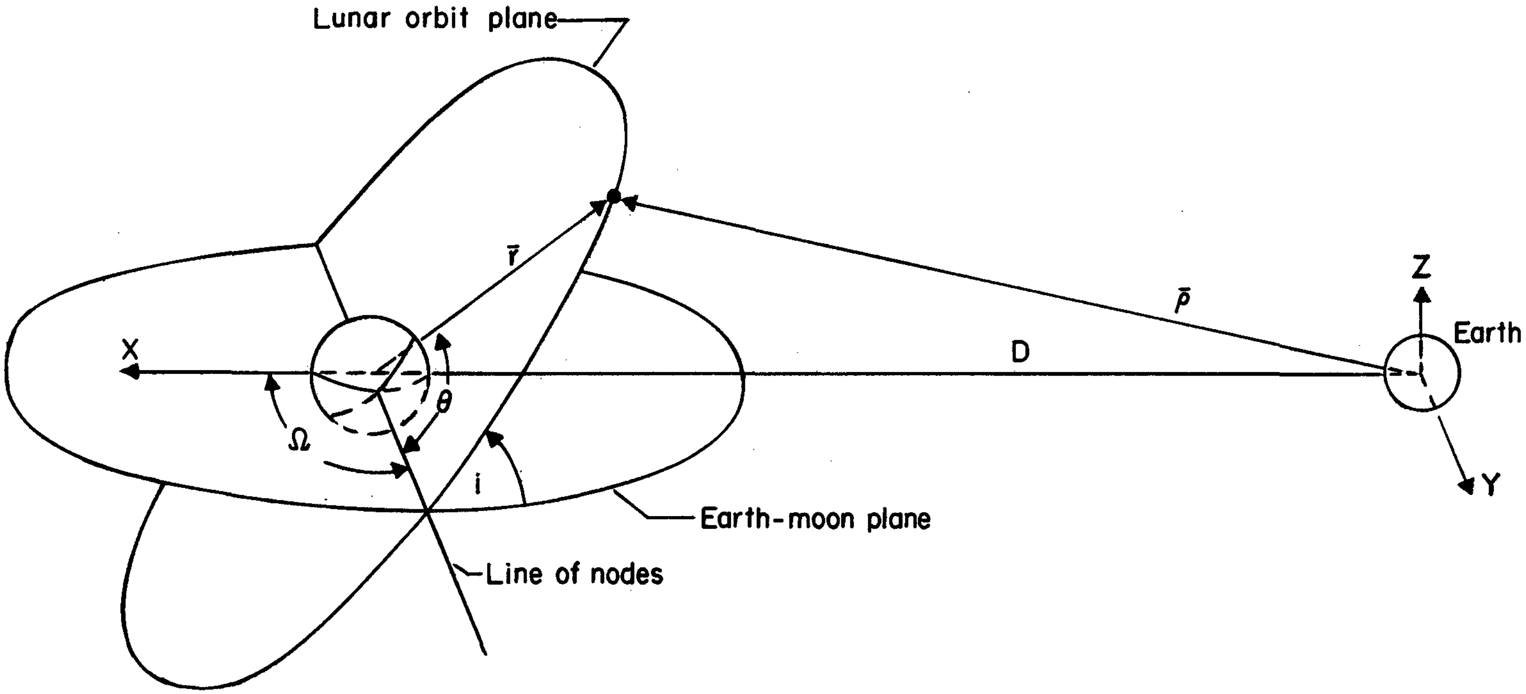

Figure 1.- Illustration of coordinate system and angular parameters.

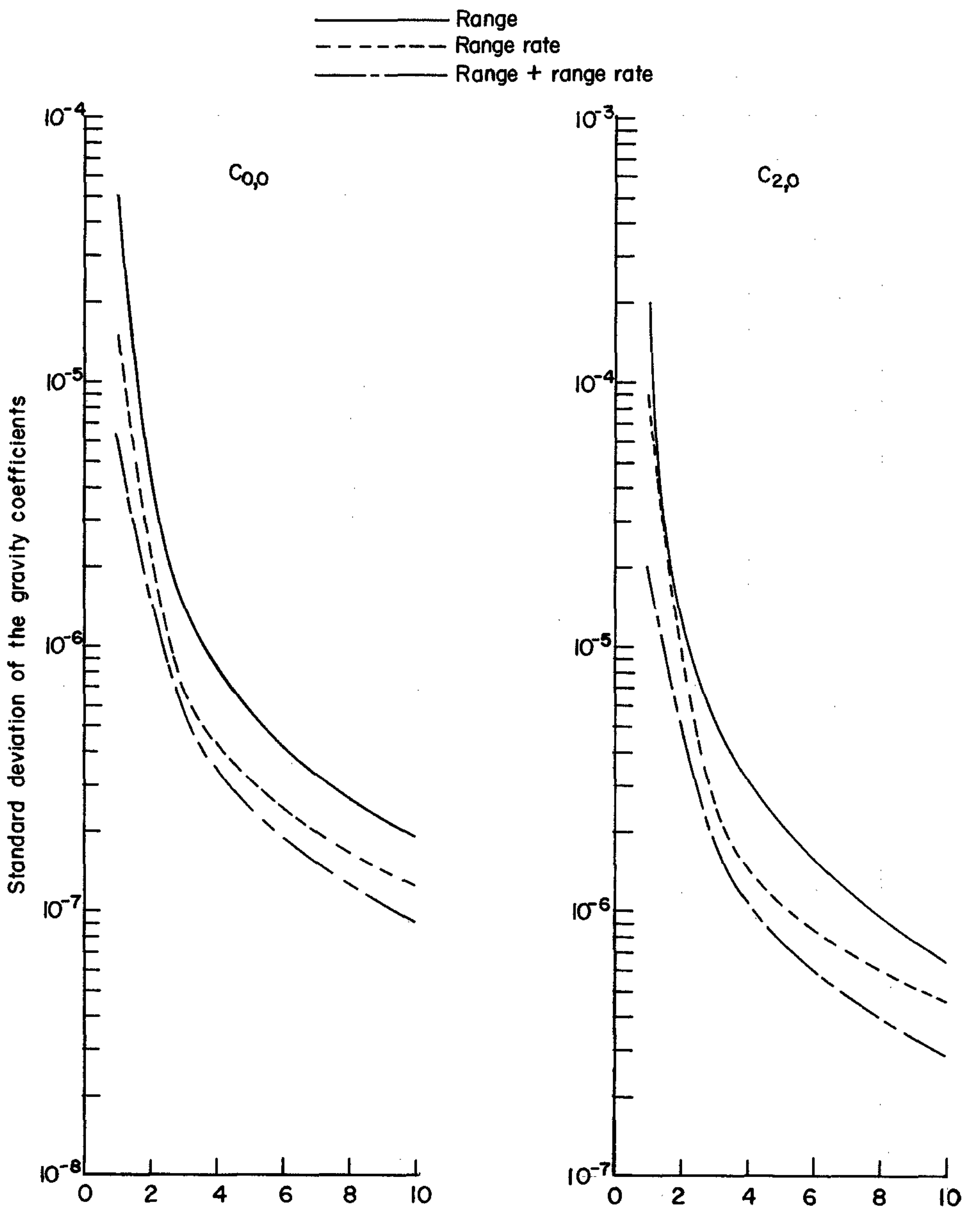

Figure 2.- Variation of standard deviation of gravity coefficients with number of orbits assumed to have been tracked.

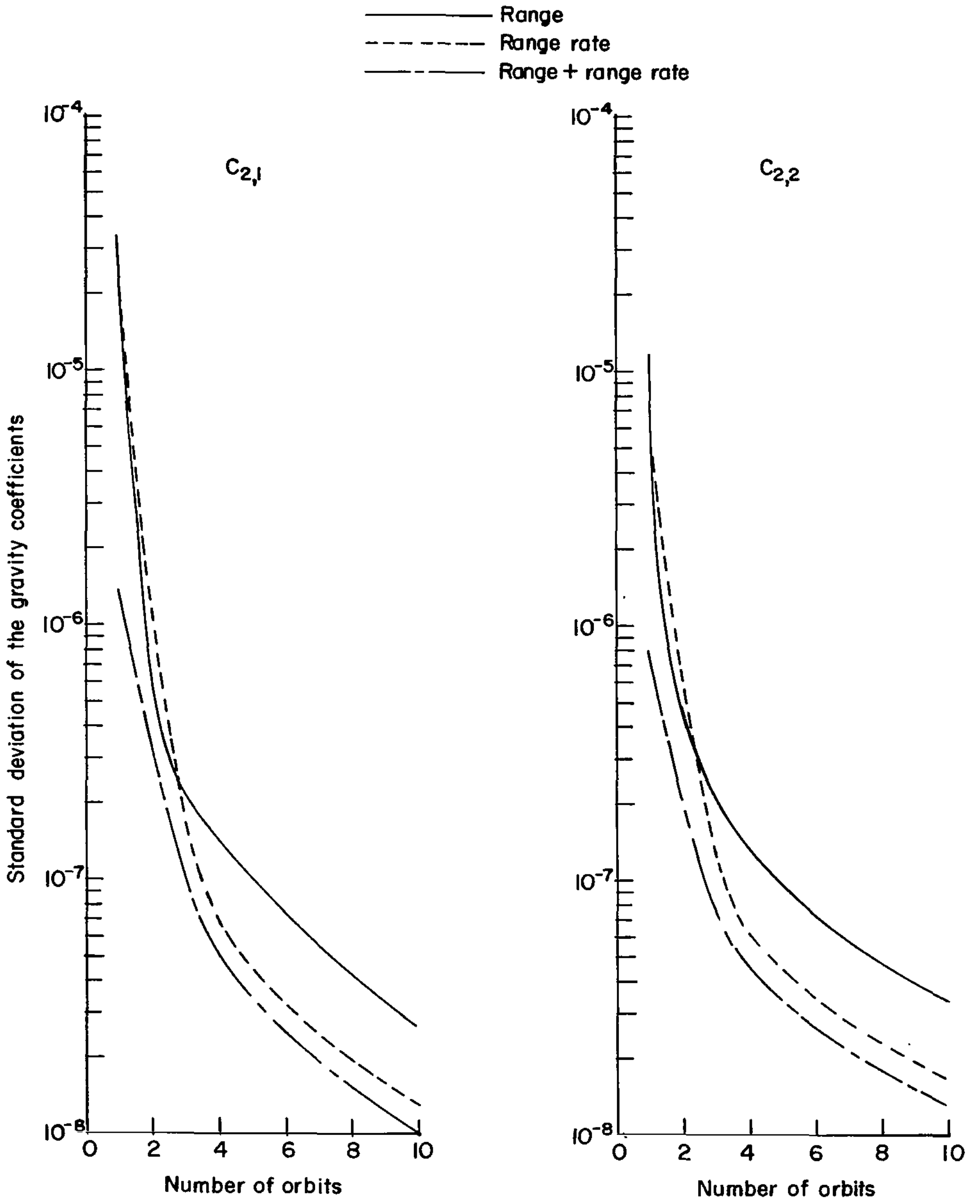

Figure 2.- Continued.

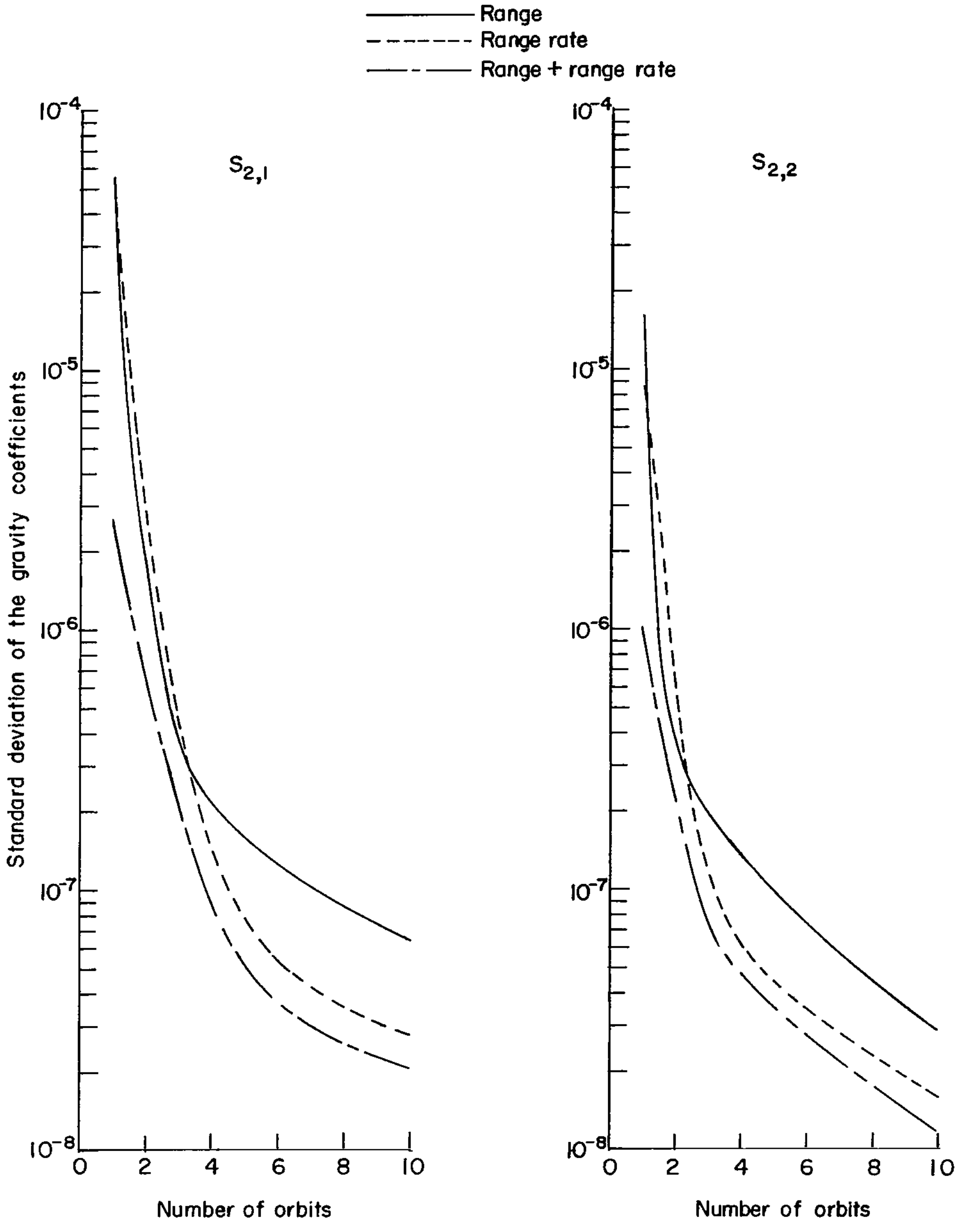

Figure 2.- Continued.

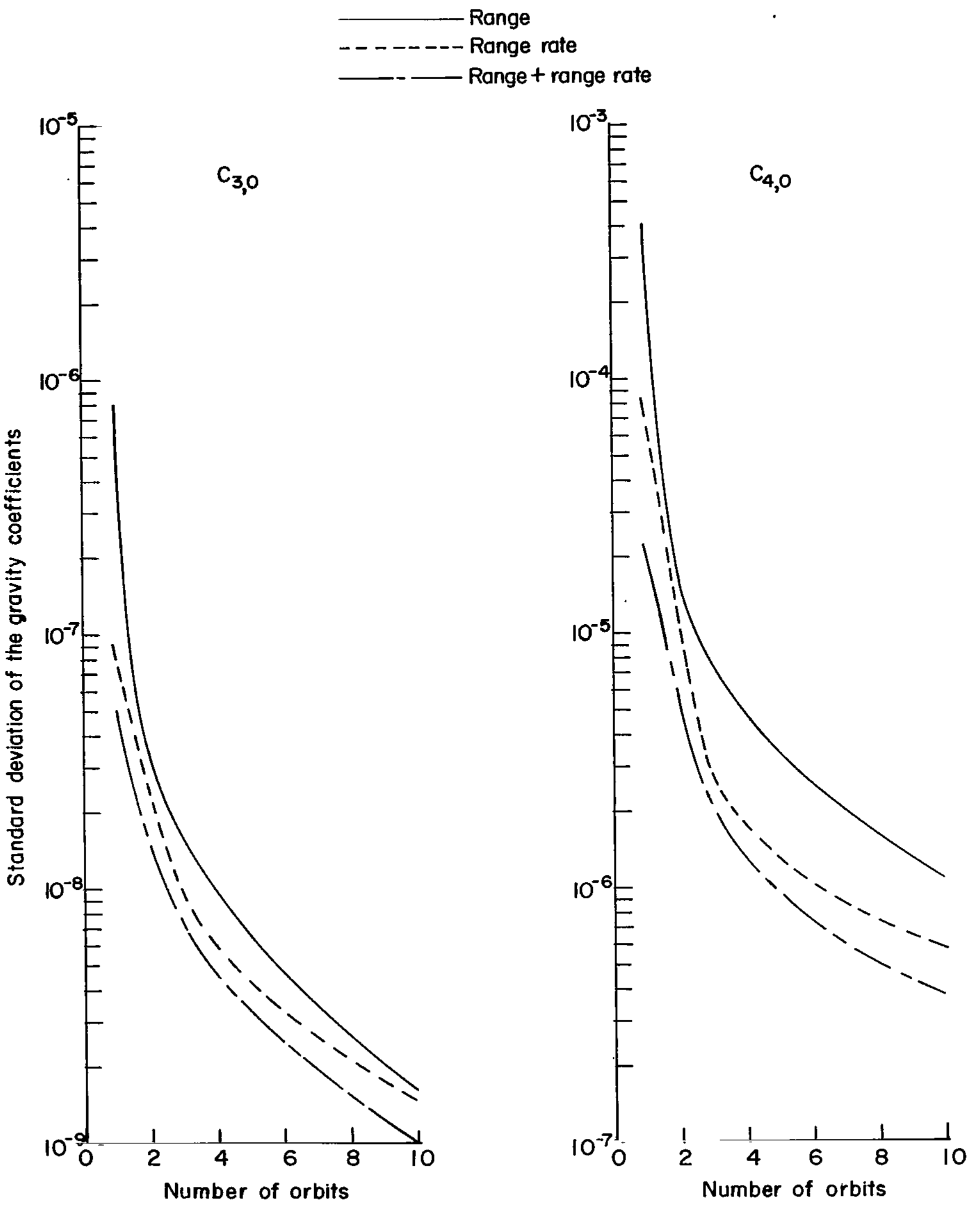

Figure 2.- Continued.

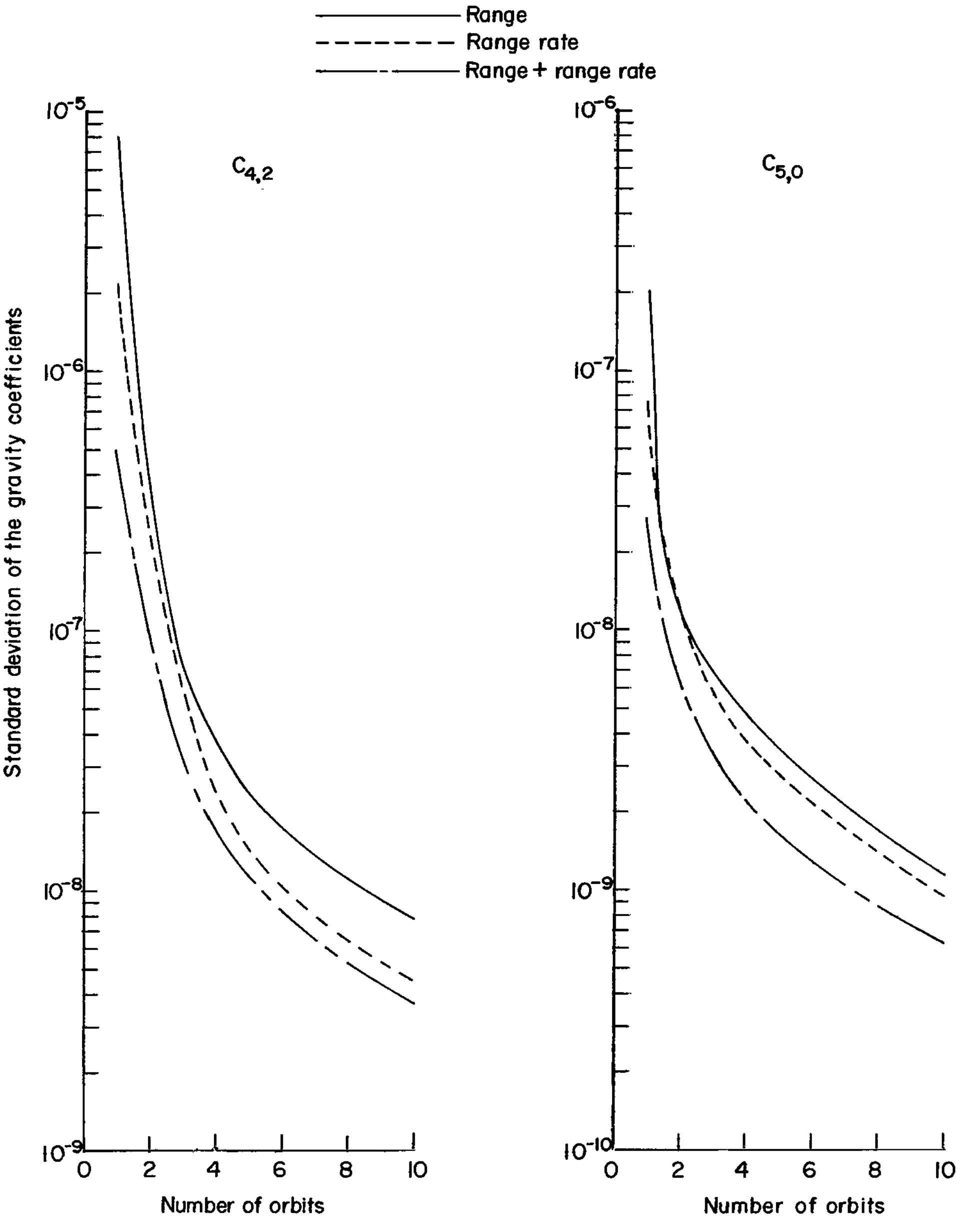

Figure 2.- Continued.

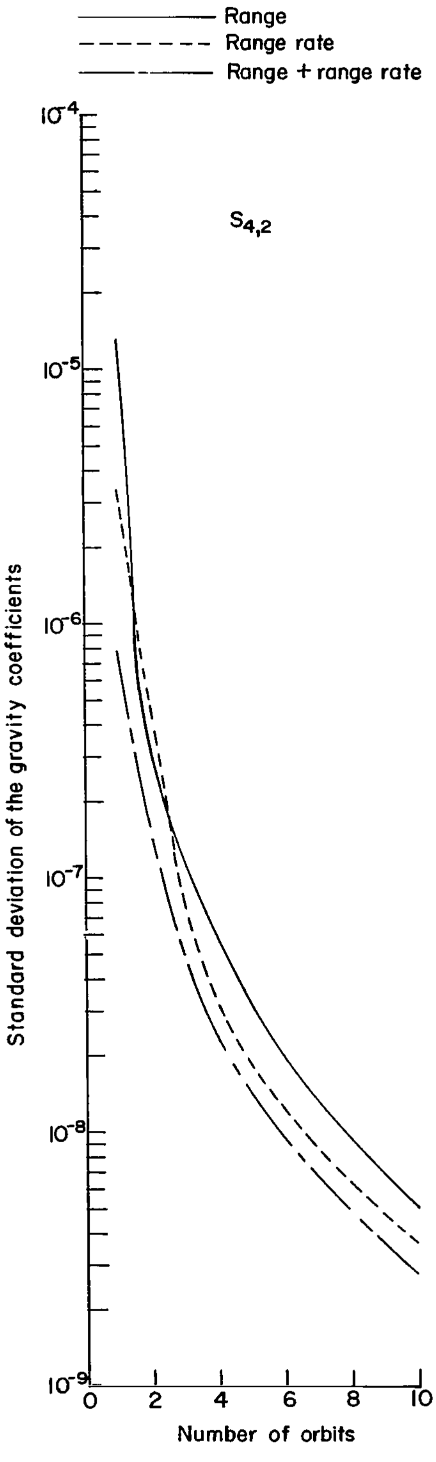

Figure 2.- Concluded.

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.98	.35	.51	.80	.95	.79	.85	-.19	.08	.81
$C_{2,0}$		1	.22	.67	.85	.99	.90	.88	-.04	.22	.91
$C_{2,1}$			1	-.51	.44	.30	-.19	.45	-.98	-.90	-.16
$C_{2,2}$				1	.53	.63	.92	.52	.68	.79	.90
$C_{3,0}$					1	.92	.74	.98	-.25	-.08	.75
$C_{4,0}$						1	.88	.93	-.12	.13	.89
$C_{4,2}$							1	.74	.37	.58	.9994
$C_{5,0}$								1	-.26	-.08	.76
$S_{2,1}$									1	.96	.34
$S_{2,2}$										1	.55
$S_{4,2}$											1

Range

Determinant = $.6 \times 10^{-31}$

1 orbit

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.79	-.46	.73	-.14	.91	.94	-.07	.49	.68	.93
$C_{2,0}$		1	-.91	.99	-.20	.96	.84	.35	.92	.98	.88
$C_{2,1}$			1	-.93	.21	-.77	-.58	-.54	-.9992	-.96	-.64
$C_{2,2}$				1	-.30	.94	84	.33	.94	.996	.88
$C_{3,0}$					1	-.26	-.38	.35	-.22	-.31	-.38
$C_{4,0}$						1	.95	.16	.79	.92	.97
$C_{4,2}$							1	-.06	.61	.79	.997
$C_{5,0}$								1	.52	.38	-.01
$S_{2,1}$									1	.97	.67
$S_{2,2}$										1	.84
$S_{4,2}$											1

Range rate

Determinant = $.6 \times 10^{-28}$

1 orbit

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.993	-.70	.89	.60	.99	.93	.27	.73	.88	.93
$C_{2,0}$		1	-.75	.90	.59	.993	.93	.26	.77	.90	.93
$C_{2,1}$			1	-.88	-.36	-.74	-.79	-.01	-.990	-.89	-.80
$C_{2,2}$				1	.59	.91	.96	.20	.92	.95	95
$C_{3,0}$					1	.57	.52	.46	.44	.41	47
$C_{4,0}$						1	.96	.25	.76	.92	.95
$C_{4,2}$							1	.19	.82	.97	.997
$C_{5,0}$								1	.06	.09	.14
$S_{2,1}$									1	.89	.81
$S_{2,2}$										1	.98
$S_{4,2}$											1

Range + range rate

Determinant = $.6 \times 10^{-19}$

1 orbit

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.58	-.10	.37	.21	.12	.41	.05	.27	.13	.24
$C_{2,0}$		1	.58	-.44	.42	.88	.37	.38	-.60	-.52	.38
$C_{2,1}$			1	-.43	.62	.78	.20	.05	-.70	-.99	.01
$C_{2,2}$				1	.004	-.72	.31	-.56	.93	42	-.07
$C_{3,0}$					1	.40	.21	.30	-.21	-.62	-.13
$C_{4,0}$						1	.27	.40	-.88	-.73	.36
$C_{4,2}$							1	-.22	.06	-.08	.76
$C_{5,0}$								1	-.45	-.03	.09
$S_{2,1}$									1	.67	-.17
$S_{2,2}$										1	.13
$S_{4,2}$											1

Range

Determinant = $.8 \times 10^{-14}$

10 orbits

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.82	.27	.07	-.08	.58	.10	-.002	-.06	-.29	-.20
$C_{2,0}$		1	47	-.42	-.14	.94	-.08	.22	-.56	-.49	-.29
$C_{2,1}$			1	-.08	.53	.54	-.24	.04	-.42	-.97	-.61
$C_{2,2}$				1	.41	-.62	.49	-.27	.91	.13	.18
$C_{3,0}$					1	-.10	.18	.28	.16	-.43	-.11
$C_{4,0}$						1	-.12	.31	-.77	-.55	-.25
$C_{4,2}$							1	.04	.39	.37	.79
$C_{5,0}$								1	-.27	.05	.14
$S_{2,1}$									1	.42	.26
$S_{2,2}$										1	.73
$S_{4,2}$											1

Range rate

Determinant = $.1 \times 10^{-12}$

10 orbits

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.78	.16	.34	.14	.41	.35	-.01	.20	-.13	.08
$C_{2,0}$		1	.42	-.22	.18	.89	.20	.21	-.41	-.39	.04
$C_{2,1}$			1	.01	.66	.53	-.10	-.04	-.36	-.98	-.49
$C_{2,2}$				1	.29	-.52	.46	-.36	.90	.02	.05
$C_{3,0}$					1	.20	.12	.35	.02	-.61	-.26
$C_{4,0}$						1	.11	.29	-.72	-.49	.05
$C_{4,2}$							1	-.06	.30	.24	.76
$C_{5,0}$								1	-.32	.10	.15
$S_{2,1}$									1	.34	.08
$S_{2,2}$										1	.63
$S_{4,2}$											1

Range + range rate

Determinant = $.3 \times 10^{-12}$

10 orbits

Figure 3.- Correlation matrices after first and tenth orbit.

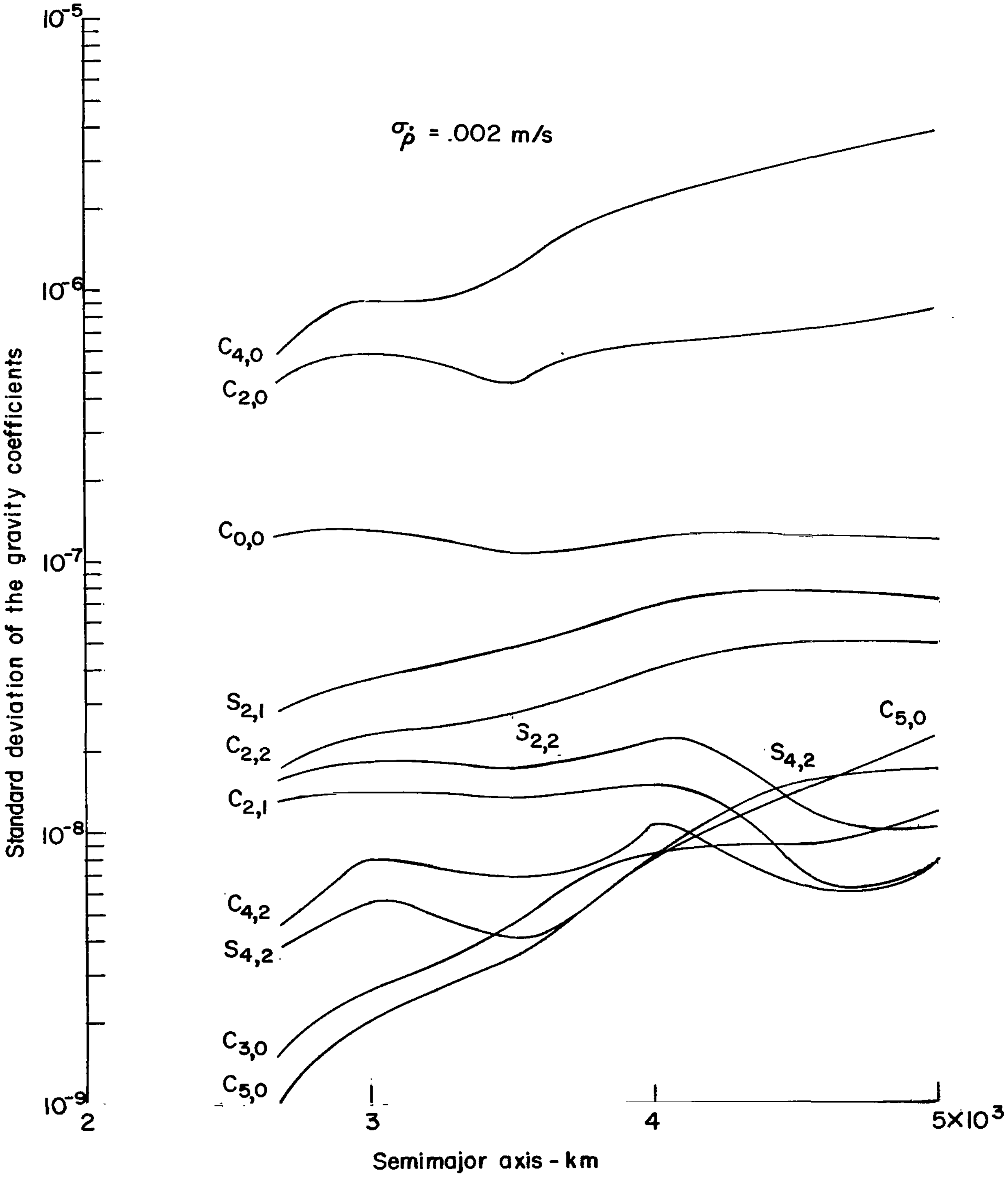

Figure 4.- Variation of standard deviation of gravity coefficients after 10 orbits of tracking with semimajor axis.

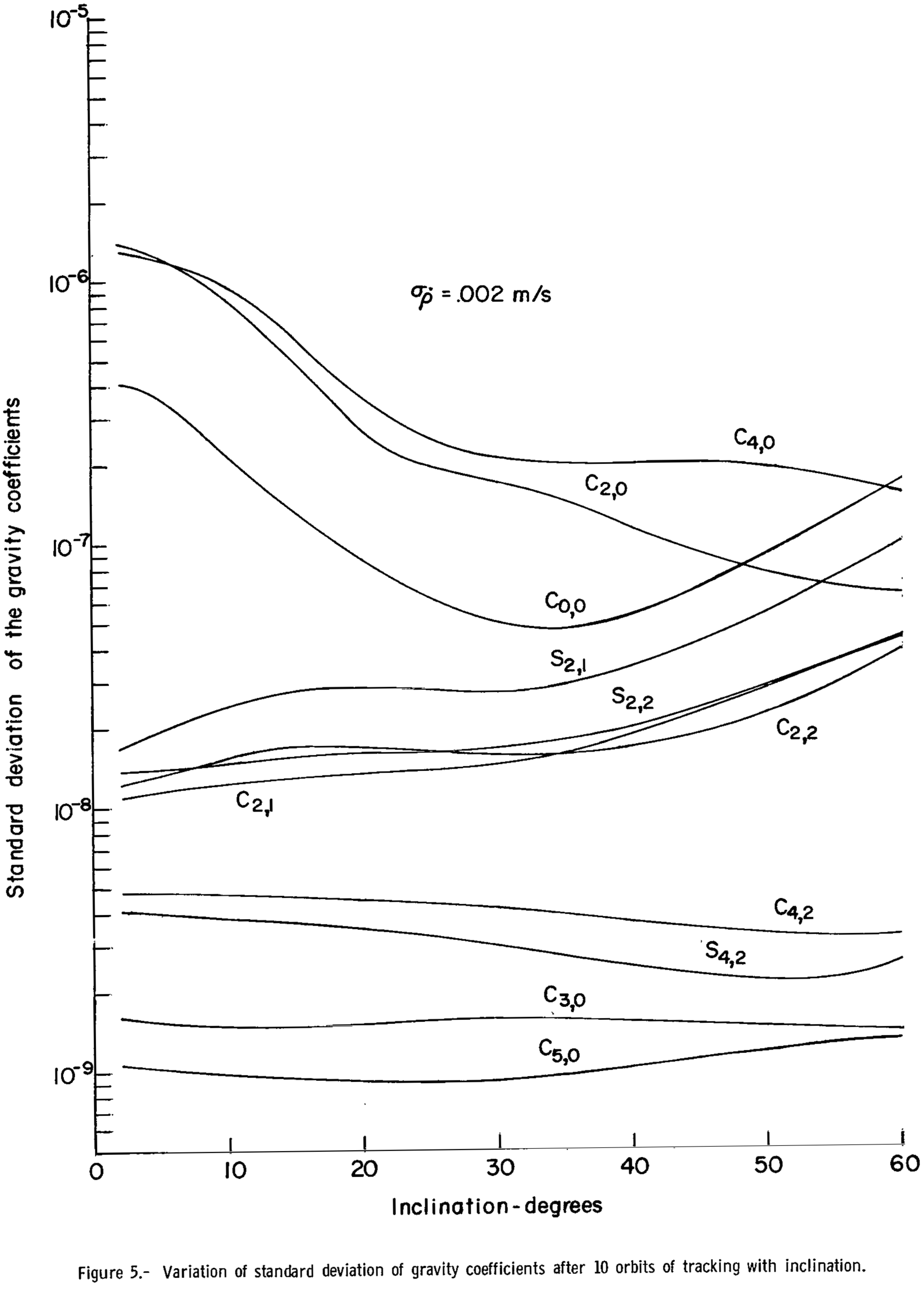

Figure 5.- Variation of standard deviation of gravity coefficients after 10 orbits of tracking with inclination.

CORRELATION MATRICES AFTER TEN ORBITS

I = 2°

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.98	.14	-.24	.32	.95	.04	.42	-.35	-.07	.09
$C_{2,0}$		1	.17	-.36	.36	.992	-.05	.47	-.47	-.10	.08
$C_{2,1}$			1	.17	.67	.20	-.40	-.05	-.25	-.97	-.68
$C_{2,2}$				1	.17	-.41	.67	-.21	.86	-.13	.11
$C_{3,0}$					1	.40	.02	.48	-.16	-.53	-.12
$C_{4,0}$						1	-.10	.49	-.53	-.12	.07
$C_{4,2}$							1	.06	.70	.50	.75
$C_{5,0}$								1	-.22	.20	.28
$S_{2,1}$									1	.21	.27
$S_{2,2}$										1	.81
$S_{4,2}$											1

Range rate

Determinant = $.3 \times 10^{-14}$

I = 30°

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.81	.45	.18	-.03	.66	.12	-.13	-.07	-.44	-.20
$C_{2,0}$		1	.24	.02	-.32	.83	-.35	.07	-.01	-.33	-.51
$C_{2,1}$			1	.03	.36	.52	.05	-.12	-.42	-.97	-.55
$C_{2,2}$				1	.51	-.37	.40	-.15	.86	.08	.31
$C_{3,0}$					1	-.35	.33	.20	.26	-.23	.13
$C_{4,0}$						1	-.36	.13	-.49	-.63	-.70
$C_{4,2}$							1	-.11	.17	.10	.69
$C_{5,0}$								1	-.06	.16	.0007
$S_{2,1}$									1	.48	.39
$S_{2,2}$										1	.70
$S_{4,2}$											1

Range rate

Determinant = $.2 \times 10^{-11}$

CORRELATION MATRICES AFTER TEN ORBITS

I = 60°

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	-.87	.98	-.93	-.45	.62	.06	.09	-.98	-.98	-.38
$C_{2,0}$		1	-.84	.78	.29	-.23	.08	.15	.83	.86	.64
$C_{2,1}$			1	-.88	-.45	.65	.07	.09	-.95	-.996	-.37
$C_{2,2}$				1	.48	-.69	.15	-.23	.98	.89	.43
$C_{3,0}$					1	-.53	-.04	.19	.49	.45	.06
$C_{4,0}$						1	.10	.56	-.69	-.64	.12
$C_{4,2}$							1	.01	.03	-.01	.62
$C_{5,0}$								1	-.18	-.06	.33
$S_{2,1}$									1	.96	.40
$S_{2,2}$										1	.44
$S_{4,2}$											1

Range rate

Determinant = $.4 \times 10^{-15}$

I = 2° & 60°

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.14	.47	-.27	.12	-.29	-.24	-.31	-.67	-.23	-.20
$C_{2,0}$		1	-.23	-.21	-.17	.86	.19	.19	-.13	.49	.54
$C_{2,1}$			1	.31	.45	-.49	-.55	-.20	-.53	-.85	-.77
$C_{2,2}$				1	.47	.03	.42	.002	.57	-.42	-.06
$C_{3,0}$					1	-.17	.02	.38	.06	-.45	-.31
$C_{4,0}$						1	.46	.29	.34	.58	.74
$C_{4,2}$							1	.22	.62	.55	.76
$C_{5,0}$								1	.08	.39	.29
$S_{2,1}$									1	.17	.42
$S_{2,2}$										1	.88
$S_{4,2}$											1

Range rate

Determinant = $.6 \times 10^{-9}$

Figure 6.- Correlation matrices after 10 orbits for different inclinations.

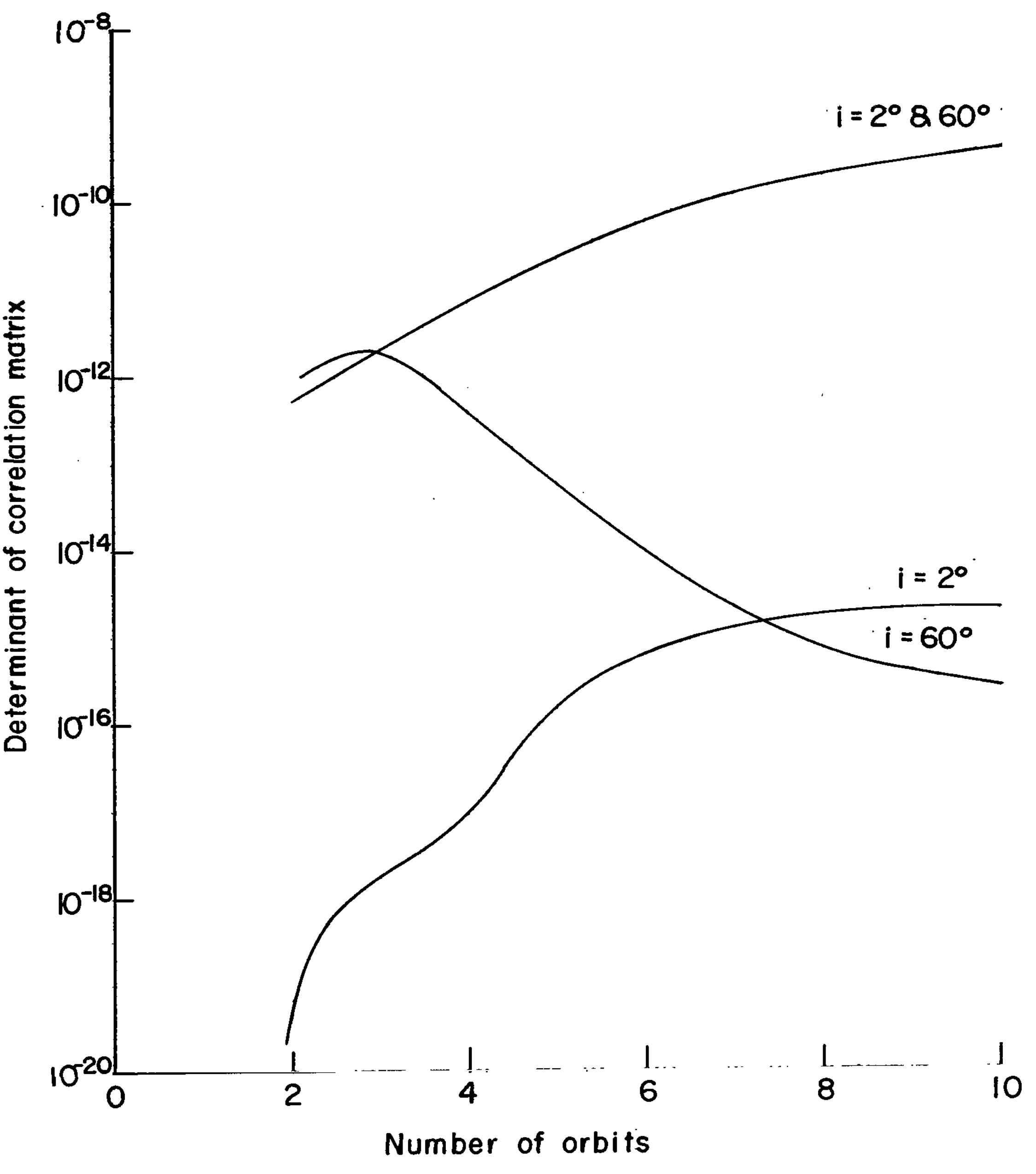

Figure 7.- Determinant of correlation matrix for two different lunar satellite orbital inclinations and for the combination of the two.

STANDARD DEVIATIONS AFTER TEN ORBITS, $\sigma \times 10^8$

COEFFICIENTS	LONGITUDE OF NODE, Ω											
	0°	30°	45°	90°	110°	150°	180°	210°	225°	270°	290°	330°	360°
$C_{0,0}$	14.28	12.36	22.43	17.86	15.52	7.792	7.024	5.929	7.318	10.98	16.59	19.42	14.28
$C_{2,0}$	52.56	46.17	78.43	45.40	43.24	17.78	17.55	13.47	14.67	30.00	47.59	68.60	52.56
$C_{2,1}$	1.226	1.285	1.401	1.930	2.361	.9028	.4255	.3406	.4668	1.169	1.970	1.608	1.226
$C_{2,2}$	2.184	1.691	1.265	1.059	1.309	.3273	.6525	.6999	.4647	.5585	1.097	1.103	2.184
$C_{3,0}$	.1686	.1474	.1743	.1217	.1150	.0867	.0776	.0820	.0739	.1104	.1149	.1260	.1686
$C_{4,0}$	75.76	58.55	82.82	35.20	41.00	15.34	18.10	15.24	12.05	28.48	46.26	73.68	75.76
$C_{4,2}$	.6209	.4544	.1822	.0705	.0554	.0847	.0433	.0445	.0685	.0561	.0535	.0727	.6209
$C_{5,0}$	.1127	.0950	.0815	.0433	.0402	.0405	.0344	.0313	.0341	.0347	.0339	.0401	.1127
$S_{2,1}$	3.718	2.790	1.752	.7076	.5506	.7489	1.350	1.336	.9723	.5779	.5832	1.674	3.718
$S_{2,2}$	1.474	1.551	1.198	.7841	.6221	.6198	.4616	.3837	.4578	.6579	.7151	1.197	1.474
$S_{4,2}$	.3387	.3720	.4563	.0985	.0854	.0342	.0613	.0618	.0435	.0751	.0867	.1541	.3387

Figure 8.- Standard deviations of estimates of lunar gravitational coefficients after 10 orbits of tracking at various nodal positions throughout month.

CORRELATION MATRICES AFTER TEN ORBITS

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.82	.27	.07	-.08	.58	.10	-.01	-.06	-.29	-.20	1	.96	-.05	-.05	.11	.74	.13	-.14	-.13	-.03	-.15
$C_{2,0}$		1	.47	-.42	-.14	.94	-.08	.22	-.56	-.49	-.29		1	.21	.20	.03	.90	.13	-.16	-.05	.04	-.21
$C_{2,1}$			1	-.08	.53	.54	-.24	.04	-.42	-.97	-.61			1	.99	-.49	.60	.23	-.24	.23	.15	-.37
$C_{2,2}$				1	.41	-.62	.49	-.27	.91	.13	.18				1	-.56	.58	.25	-.29	.12	.04	-.43
$C_{3,0}$					1	-.10	.18	.28	.16	-.43	-.11					1	-.12	-.61	.80	.44	.54	.88
$C_{4,0}$						1	-.12	.31	-.77	-.55	-.25						1	.11	-.19	.14	.18	-.29
$C_{4,2}$							1	.04	.39	.37	.79							1	-.62	-.51	-.55	-.53
$C_{5,0}$								1	-.27	.05	.14								1	.38	.43	.96
$S_{2,1}$									1	.42	.26									1	.99	.34
$S_{2,2}$										1	.73										1	.40
$S_{4,2}$											1											1

Ω = 30°

Determinant = 10^{-13}

Ω = 110°

Determinant = 10^{-16}

Figure 9.- Correlation matrix after 10 consecutive orbits for two different nodal positions.

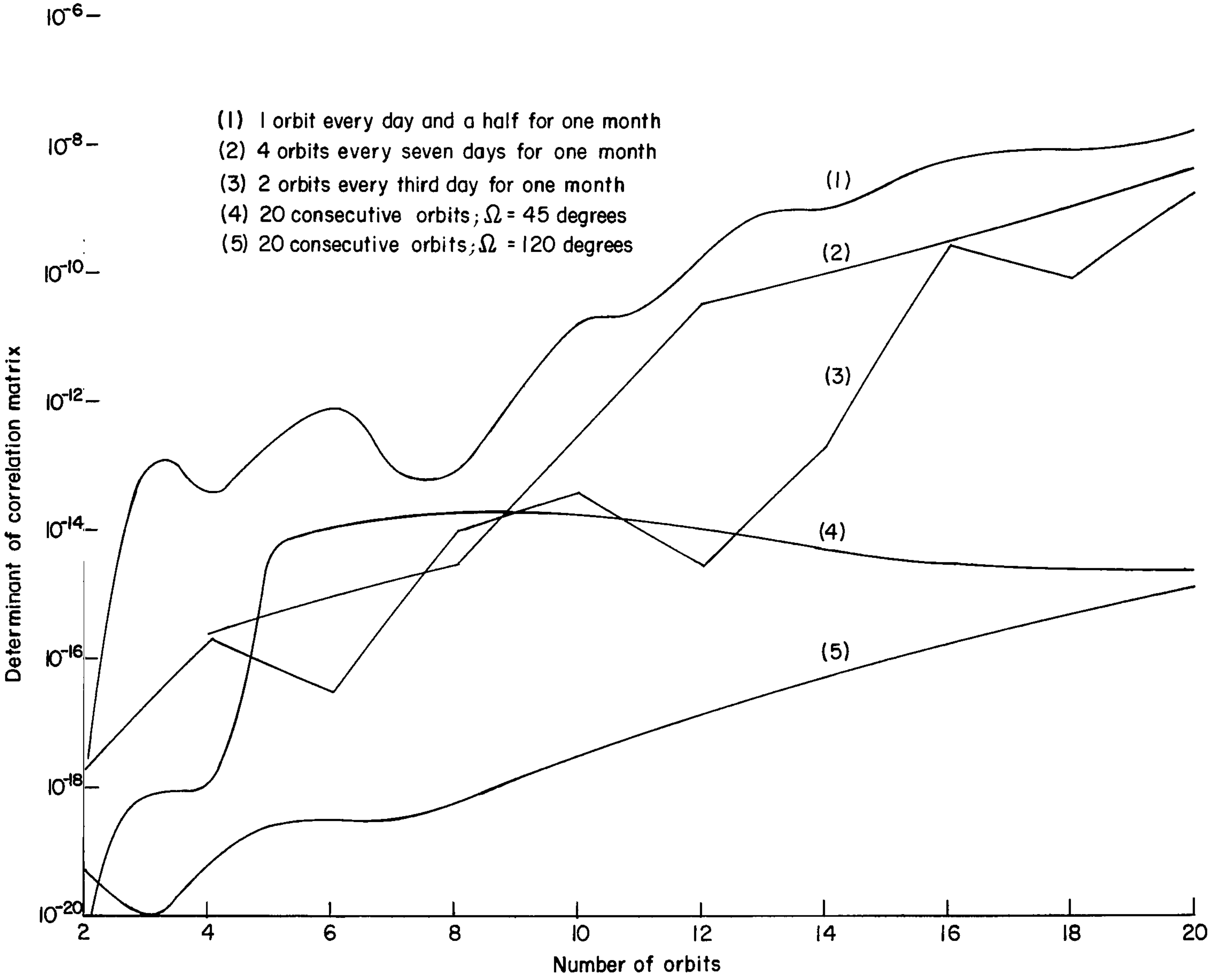

Figure 10.- Variation of determinant of correlation matrix for five different tracking schedules.

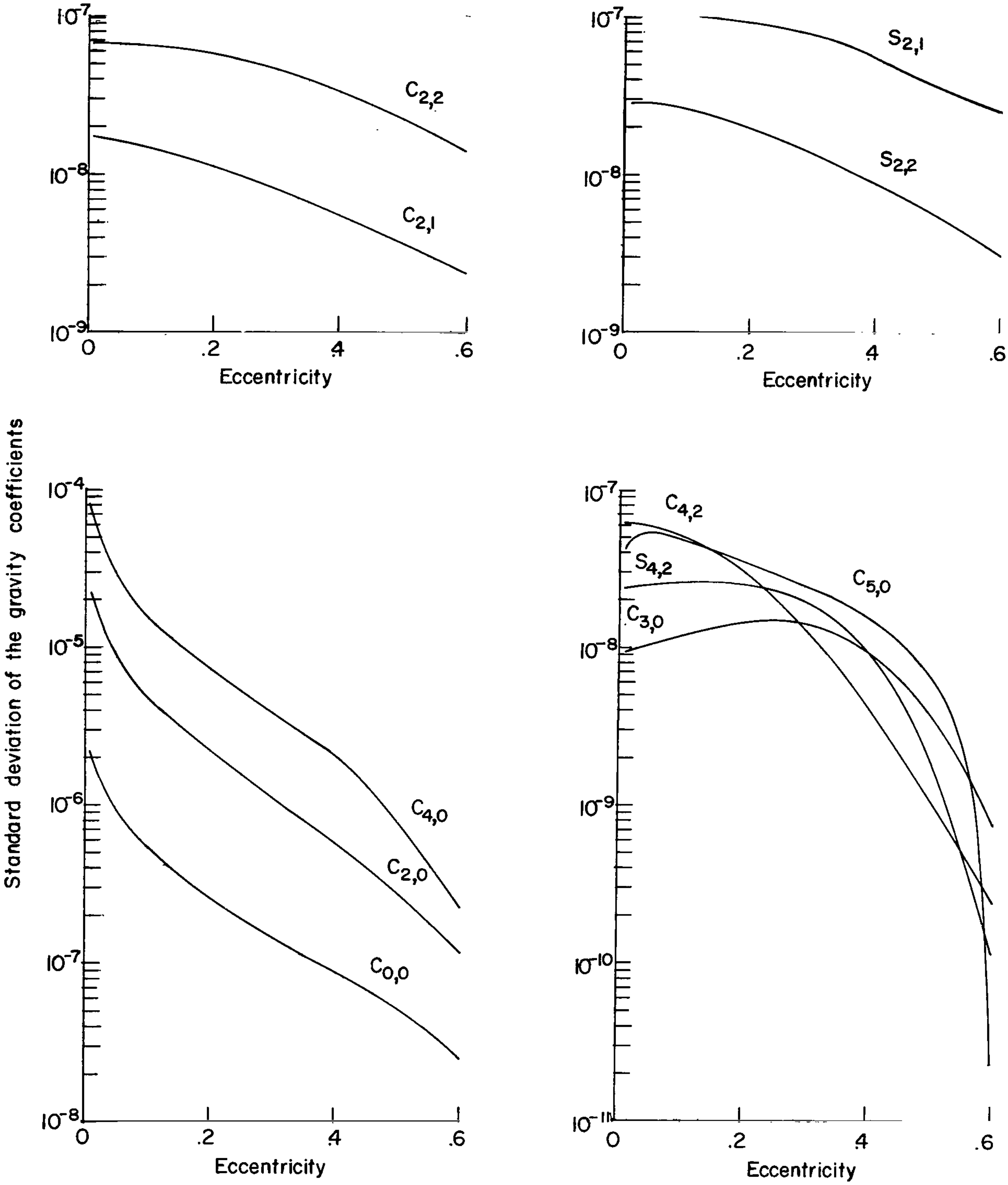

Figure 11.- Variation of standard deviation of gravity coefficients after 10 orbits of tracking with eccentricity.

NASA-Langley, 1966 L-5028

CORRELATION MATRICES AFTER TEN ORBITS

	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$	$C_{0,0}$	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{3,0}$	$C_{4,0}$	$C_{4,2}$	$C_{5,0}$	$S_{2,1}$	$S_{2,2}$	$S_{4,2}$
$C_{0,0}$	1	.997	-.45	.01	.07	.98	-.18	-.19	-.04	.12	.18	1	.83	.43	.26	.03	.34	-.04	-.16	.27	-.38	.03
$C_{2,0}$		1	-.46	-.07	.07	.994	-.19	-.20	-.10	.14	.17		1	.42	-.24	-.22	.80	-.05	-.34	-.22	-.31	-.16
$C_{2,1}$			1	.15	-.02	-.46	.35	-.03	.29	-.76	-.58			1	.28	.25	.25	.11	.03	.30	-.97	.14
$C_{2,2}$				1	-.01	-.16	.57	.09	.93	-.10	-.07				1	.40	-.73	.05	.16	.9993	-.37	.33
$C_{3,0}$					1	.07	-.02	-.68	-.02	-.05	.05					1	-.37	-.24	.49	.41	-.48	.26
$C_{4,0}$						1	-.21	-.21	-.18	.16	.15						1	-.05	-.34	-.72	-.11	-.30
$C_{4,2}$							1	.07	.76	-.09	-.47							1	-.11	.06	-.03	-.48
$C_{5,0}$								1	.09	.17	.03								1	.16	-.17	-.12
$S_{2,1}$									1	-.15	-.21									1	-.39	.33
$S_{2,2}$										1	.11										1	-.21
$S_{4,2}$											1											1

$e = .01$
Range rate
Determinant = $.6 \times 10^{-12}$

$e = .6$
Range rate
Determinant = $.3 \times 10^{-11}$

Figure 12.- Correlation matrices for two different eccentricities.

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