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RM-5009-NASA
SEPTEMBER 1966

ADAPTIVE AGE REPLACEMENT

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PREFACE

This Memorandum stems from RAND's continuing interest in optimal maintenance policies. It reports the results of research on adaptive age replacement.

The Memorandum is addressed to mathematical statisticians and operations research personnel concerned with age replacement under conditions where it would be unrealistic to assume that the failure distribution is known precisely. A subsequent study will present numerical results obtained using a computer code now in preparation.

This research was undertaken as a part of the reliability assessment study that RAND is conducting for the Apollo Reliability and Quality Office, Headquarters NASA, under contract NASr-21(11).

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SUMMARY

Under certain parametric assumptions, a Bayesian approach to adaptive age replacement is treated via dynamic programming. We prove that the adaptive policy has an important asymptotic optimality property: viz., that the replacement intervals set converge (w.p.1) to the one we would use if we knew the true parameter value. Various other asymptotic results are obtained. An important suboptimal policy is partially characterized.

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ACKNOWLEDGMENTS

The material in this Memorandum is based in part on a dissertation written under the supervision of Dr. R. E. Barlow and submitted in partial satisfaction of the requirements for the Ph.D. degree at the University of California, Berkeley. It is a pleasure to thank Dr. Barlow for encouragement and advice. I am indebted to E. V. Denardo and R. E. Strauch for comments on a later version, which resulted in substantial improvements.

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CONTENTS

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1. INTRODUCTION

Under an age replacement policy, we replace at failure or at the end of a specified time interval, whichever occurs first. Age replacement makes sense when a failure replacement costs more than a planned replacement and the failure rate is strictly increasing. We assume an infinite horizon and continuous discounting, with the loss incurred at the time of replacement and the total loss equal to the sum of the discounted losses incurred on the individual stages. (A stage is the period starting just after one replacement and ending just after the next replacement.) The cost of a planned (failure) replacement is $c_1(c_2)$, where $0 < c_1 < c_2$. Suppose that a stage starts at time t and we set a replacement interval a, chosen from the extended half-line $[0,\infty]$. If replacement actually occurs at t+x, then the loss incurred on that stage is

$$(1) \qquad L(a,x,t) = \begin{cases} c_1 \, e^{-\alpha(t+a)} , & \text{if } x = a \\ c_2 \, e^{-\alpha(t+x)} , & \text{if } x < a \end{cases}$$

where α is a positive discount rate.

We shall make a strong parametric assumption: viz., that the failure distribution belongs to the family

$$(2) \qquad F_\lambda(y) = \begin{cases} 1 - e^{-\lambda y^k} , & y \geq 0 \\ 0 , & \text{elsewhere} \end{cases}$$

$k > 1$ and known.

For fixed λ, we have a Weibull distribution with known shape parameter and strictly increasing failure rate $\lambda k y^{k-1}$. We assume that λ has a fixed (but unknown) value λ^o. We further assume that we have at hand a prior distribution G with specified parameters which we modify after each stage according to Bayes's rule. If G has density

$$(3) \qquad g(\lambda;b,c) = \begin{cases} b^c \lambda^{c-1} e^{-b\lambda}/\Gamma(c) \ , \ \lambda \geq 0, \\ 0, \text{ elsewhere} \end{cases}$$

the posterior density in case of planned replacement at a [failure replacement at x] is again a gamma density $g(\lambda;b+a^k,c)$ [$g(\lambda;b+x^k,c+1)$]. Thus, we have a natural conjugate prior distribution [9].

The loss structure (1) was considered earlier in [5]. This is not the usual model considered in the literature. Several authors (see, e.g., [1]) have treated the case where the failure distribution is known and the criterion is expected cost per unit time. In that case, the optimal replacement interval to set is found as an elementary application of renewal theory. Note that, with unknown failure distribution, if the loss were (literally) undiscounted cost per unit time, the problem of finding a suitable adaptive policy effectively reduces to the preceding case, since we could ignore the loss in any finite transient period while we learned about the failure distribution. With discounting, there is a tradeoff between minimizing expected loss with respect to one's current prior distribution for λ as if future information obtained about the failure distribution were to be ignored, and acquiring maximal information about the failure distribution so as to minimize future losses. We take account of this intuitive consideration in a precisely defined way via dynamic programming.

An alternative procedure would be to act as if our current estimate of λ were the true value; as our estimate is updated from stage to stage, we would modify the replacement interval set accordingly. It would seem intuitively desirable to bias the replacement intervals set on the high side, but it is not clear precisely how to do this.

2. DYNAMIC PROGRAMMING FORMULATION

Suppose a replacement interval a is set and replacement occurs at t+x. We assume that replacing a unit takes an interval Δ. Now we apply Bayes's rule: to obtain the new state variables, make the transformation

$$(b,c,t) \rightarrow \begin{cases} (b+a^k,\ c,\ t+a+\Delta), & \text{if } x = a \\ (b+x^k,\ c+1,\ t+x+\Delta), & \text{if } x < a. \end{cases} \tag{4}$$

Note that c increases in steps of 0 or 1. Dependence on t can be suppressed, since we shall restrict the class $\mathcal{C}$ of policies to those Baire functions mapping $\Omega = \{(b,c): b > 0,\ c > 0\}$ into $[\xi,\infty]$, where except for Theorem 2 we take $\xi = 0$. Note that $\mathcal{C}$ is the set of non-randomized, stationary policies.

The expected loss from the next replacement when the state is (b,c,0) is

$$\begin{aligned} \phi(a,b,c) &= c_1\, e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c \\ &\quad + c_2\, kcb^c \int_0^a \frac{e^{-\alpha x}\, x^{k-1}\, dx}{(b+x^k)^{c+1}}, \end{aligned} \tag{5}$$

since $\int_0^\infty F_\lambda(x)\, dG(\lambda; b,c) = 1 - [b/(b+x^k)]^c$. Toward describing the future-stages cost, consider an arbitrary Baire function v from Ω to the nonnegative reals. It is convenient to think of v(b,c) as the aggregate discounted cost obtained from proceeding in some (unspecified) way from the state (b,c). If this were the case, the future-stages cost

T(a,b,c,v) would be given by

(6) $$T(a,b,c,v) = \left(\frac{b}{b+a^k}\right)^c e^{-\alpha a} v(b+a^k, c)$$

$$+ kcb^c \int_0^a v(b+x^k, c+1) \frac{e^{-\alpha x} x^{k-1} dx}{(b+x^k)^{c+1}} .$$

As is often the case in dynamic programming problems, it is useful to introduce notation describing the effect of using policy π for one stage with terminating cost function v. Hence, we define the one-stage cost function Ψ_π and the operators τ_π and H_π by

$$\Psi_\pi(b,c) = \phi[\pi(b,c), b,c]$$

$$(\tau_\pi v)(b,c) = T[\pi(b,c), b,c,v]$$

$$H_\pi v = \Psi_\pi + \beta \tau_\pi v,$$

where $\beta = e^{-\alpha\Delta} \in [0,1]$ is a discount factor.

The one-stage optimization operator A and the minimal risk function R are now defined by

(7) $$(Av)(b,c) = \inf_{\pi \in \mathcal{C}} (H_\pi v)(b,c)$$

(8) $$R(b,c) = \inf_{\pi \in \mathcal{C}} v_\pi(b,c),$$

where, with $\vec{0}$ as the zero function on Ω, we define

(9) $$v_\pi = \lim_{n\to\infty} H_\pi^n(\vec{0}) .$$

THEOREM 1. _If_ $\beta < 1$,

(i) R _is the unique bounded fixed point of_ A;

(ii) v _is a bounded Baire function_ $\Rightarrow A^n v \to R$ _uniformly on_ Ω;

(iii) R _is continuous_;

(iv) _the infima of_ $H_\pi R$ _and_ v_π _are attained_;

(v) R _is minimal over all policies_.

PROOF. $v_\pi < c_2/(1 - \beta)$. Since A is a contraction mapping, the theorem follows from the fact that the uniform limit of continuous functions is continuous and from results in [3]; viz., Corollary 2, Theorem 3 and Theorem 5.||

If $\beta = 1$, it can be shown that (i) remains true with "unique" replaced by "smallest positive;" we conjecture that (ii)-(v) remain true, except that "uniformly" is to be replaced by "pointwise." The fact that AR = R also follows from the heuristically derived principle of optimality; see, e.g., [2]. An alternative argument for (iv) uses the fact that the policy space is compact with respect to the topology of pointwise convergence [7]. In (v), the phrase "all policies" includes randomized, nonstationary policies.

REMARK. When the failure distribution is known, it is easily shown that the optimal planned replacement age a^* does not depend on the age of the item we start with (provided that it is less than a^*) -- a result that is perhaps intuitively obvious. A. F. Veinott (personal communication) conjectured (correctly) that with the above Bayesian set-up this translation invariant property no longer holds. At each stage we assume that we start with a unit of age zero.

Let π be a minimizer for the functional equation $v = \min\{H_\sigma v: \sigma \in \mathcal{C}$, v bounded$\}$; then the policy π is optimal by Theorem 1. If we knew R, we could find π. Although no solution to AR = R is apparent, we shall find a sequence $\{R_N(b,c)\}$ such that $R_N(b,c) \to R(b,c)$. There are many sequences converging to R, but $\{R_N\}$ has an important asymptotic optimality property to be described in the sequel (Theorem 2).

Let us at first proceed heuristically. Instead of following an adaptive policy indefinitely, suppose we were to do so for N stages; from the (N+1)-st stage onward we would set the same replacement interval that we did on the N-th stage. Call the minimal "risk" when we are to adapt for exactly n more stages $R_n(b,c)$. (We compute $R_n(b,c)$ as if the value of (b,c) remained fixed after the N-th stage. Thus, $R_n(b,c)$ may really not be the "true" minimal risk, but it turns out to be a useful fiction.) Either directly or using the optimality equation, we can readily show that

$$(10) \qquad R_1(b,c) = \min_{a \geq \xi}\left[\frac{\phi(a,b,c)}{1-\beta\delta(a,b,c)}\right],$$

where

$$(11) \qquad \delta(a,b,c) = e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c + kcb^c \int_0^a \frac{e^{-\alpha x}\, x^{k-1}\, dx}{(b+x^k)^{c+1}} .$$

We obtain recursively

$$(12) \qquad R_n(b,c) = \min_{a \geq \xi}\{\phi(a,b,c) + \beta T[a,b,c,R_{n-1}]\},\ n > 1 .$$

Let $\pi_n(b,c)$ be a minimizing a in (12) [in (10), if n = 1]. Although the original policy of adapting only for N stages is not stationary, we shall take a particular N and use the stationary policy $(\pi_N, \pi_N, \ldots)$.

Nothing in the sequel depends on the interpretation of R_n. In operator notation, $R_{n+1} = A^n R_1$. It is easily seen that R_n depends continuously on ξ; hence as $\xi \to 0$, we approach the unconstrained minimum. Note that both R_n and π_n depend on ξ, although the notation does not explicitly indicate this dependence.

3. ASYMPTOTIC PROPERTIES

Let

$$R_\lambda(a) = \frac{\phi_\lambda(a)}{1-\beta\delta_\lambda(a)} \tag{13}$$

where

$$\phi_\lambda(a) = c_1 e^{-\alpha a}[1-F_\lambda(a)] + c_2\int_0^a e^{-\alpha x}\, dF_\lambda(x) \tag{14}$$

$$\delta_\lambda(a) = e^{-\alpha a}[1-F_\lambda(a)] + \int_0^a e^{-\alpha x}\, dF_\lambda(x) . \tag{15}$$

When λo is known, a stationary policy is one for which each decision is the same independent of previous decisions, number of replacements, and transition times (replacement ages). Note that the sequence $\{(b_n,c_n)\}$ is a function of the initial value of (b,c), the policy, and the sequence of independent and identically distributed random variables $\{Y_n\}$, where Y_i has distribution $F_{\lambda o}$. Throughout the sequel, convergence with probability one (w.p.1) is with respect to the measure induced by $F_{\lambda o}$ on the sequence space $X_{i=1}^{\infty}(0,\infty)$. We now state our main results.

THEOREM 2 (asymptotic optimality). _If_ $\beta < 1$,

(i) $R_{\lambda o}(a)$ _has a finite unique unconstrained minimizer, say_ a^*;

(ii) _When_ λ^o _is known, there exists a stationary, nonrandomized optimal policy with risk_ $R_{\lambda o}(a^*)$;

_if_ $a^* \geq \xi > 0$, _then for_ $N = 1, 2, \ldots$

(iii) $\lim_{n\to\infty} R_{\lambda o}[\pi_N(b_n, c_n)] = R_{\lambda o}(a^*)$ w.p.1;

(iv) $\lim_{n\to\infty} \pi_N(b_n, c_n) = a^*$ w.p.1.

THEOREM 3. Assume $\beta < 1$; then $R_N \to R$ uniformly over Ω and $\pi_{N_i} \to \sigma \Rightarrow \sigma$ is optimal for R. $\{\pi_N(b,c)\}$ has a convergent subsequence, for (b,c) fixed.

Denote the expected (undiscounted) cost up to time t when a replacement interval a is set at each stage by C(t,a;b,c). We define expected cost per unit time by lim C(t,a;b,c)/t, where renewal theory can be used to show that the limit exists.

THEOREM 4. With the constraint on the replacement intervals set that they must all be equal, the minimal risk when the criterion is expected cost per unit time is $\lim_{\alpha \downarrow 0} \alpha R_1(b,c;\alpha)$.

Let us first prove Theorem 2. Parts (i) and (ii) follow from results in [5] and [3], respectively. Noting that (i) and (iii) $\Rightarrow$ (iv), it remains to prove (iii). For this we need some preliminary results.

LEMMA 1. Let $X = \min(a,Y)$, where a is a constant. The variance of X is increasing in a. If Y has distribution F_λ, the variance of X^k is maximized at $a = \infty$ and is finite.

PROOF. Generalizing a result of the author, R. Strauch (personal communication) has supplied a proof that, whatever the distribution of Y, truncation reduces the variance. The proof goes as follows: Set $D = Y-X$. $\text{Var } Y = E(X+D)^2 - [E(X+D)]^2 = \text{Var } X + \text{Var } D + 2(EXD-EXED)$. But $EXD = aED$ since $D \neq 0 \Rightarrow X = a$; in addition, $EX \leq a$. Hence $\text{Var } Y \geq \text{Var } X$. The second assertion of the lemma now follows from the fact that, in our case, Y^k is exponentially distributed. ||

LEMMA 2. <u>As</u> $b,c\to\infty$, <u>with</u> c/b <u>remaining bounded</u>,

$$R_N(b,c) = R_1(b,c) + o(1),\ N = 2,3, \ldots$$

PROOF. From inspection of (5) and (10)-(12), we have

$$T[a,b,c,R_1] = \delta(a,b,c)R_1(b,c) + o(1)\ . \tag{16}$$

Note that (10) implies

$$R_1(b,c) = \min_a[\phi(a,b,c) + \beta\delta(a,b,c)R_1(b,c)]. \tag{17}$$

Hence

$$\begin{aligned}(18)\qquad R_2(b,c) &= \min_a[\phi(a,b,c) + \beta\delta(a,b,c)R_1(b,c) + o(1)] \\ &\qquad\qquad \text{by (12) and (16)} \\ &= R_1(b,c) + o(1) \text{ by (17) (comparing minimands)} \\ &= \min_a[\phi(a,b,c) + \beta\delta(a,b,c)R_2(b,c) + o(1)].\end{aligned}$$

By induction on the hypothesis $R_N(b,c) = R_1(b,c) + o(1)$,

$$R_N(b,c) = \min_a[\phi(a,b,c) + \beta\delta(a,b,c)R_N(b,c) + o(1)], \tag{19}$$

$N=1,2, \ldots$, and so

$$\begin{aligned}(20)\qquad R_N(b,c) &= \min_a\left[\frac{\phi(a,b,c)}{1-\beta\delta(a,b,c)}\right] + o(1) \\ &= R_1(b,c) + o(1),\end{aligned}$$

completing the proof.||

LEMMA 3. Under any policy for which a_n, the n-th replacement interval set, must be at least $\epsilon > 0$, $n=1,2,\ldots$, $c_n/b_n \to \lambda^o$ w.p.1..

PROOF. For $n=1,2,\ldots$, let $X_n = \min(a_n, Y_n)$ and $U_n = 1$ if $Y_n < a_n$, 0 otherwise, where Y_n has distribution F_{λ^o}. An easy calculation shows that $E(X_n^k | a_n) = \rho(a_n)/\lambda^o$ and $E(U_n | a_n) = \rho(a_n)$, where $\rho(a) = 1-\exp(-\lambda^o a^k)$.

By Lemma 1, Var X_n^k is uniformly bounded over n for all $(a_1, a_2, \ldots)$; the bound is achieved for $(\infty, \infty, \ldots)$. Therefore, by a standard martingale convergence theorem [8, p. 387, Theorem E],

$$[b_n - \sum_1^n E(X_i^k | X_1, \ldots, X_{i-1})]/n \to 0$$

w.p.1, which is equivalent to $[b_n - \sum_1^n \rho(a_i)/\lambda^o]/n \to 0$ w.p.1. Similarly, we find that $[c_n - \sum_1^n \rho(a_i)]/n \to 0$ w.p.1. Combining these relations yields $(c_n - \lambda^o b_n)/n \to 0$ w.p.1. Hence, $(b_n/n)(c_n/b_n - \lambda^o) \to 0$ w.p.1. Since $\epsilon > 0 \Rightarrow \liminf b_n/n > 0$ w.p.1, $c_n/b_n \to \lambda^o$ w.p.1.||

LEMMA 4. Under any policy for which $a_n \geq \epsilon > 0$, $\forall n$,

$$\lim_{n\to\infty} G(\lambda; b_n, c_n) = \begin{cases} 0, & \lambda < \lambda^o \\ 1, & \lambda \geq \lambda^o \end{cases}$$

w.p.1; in addition,

(21) $$\phi(a, b_n, c_n) \to \phi_{\lambda^o}(a) \quad \text{w.p.1}$$

(22) $$\delta(a, b_n, c_n) \to \delta_{\lambda^o}(a) \quad \text{w.p.1.}$$

PROOF. The prior distribution at stage n is a gamma distribution with mean and variance, respectively, c_n/b_n and c_n/b_n^2. By Lemma 3, $c_n/b_n \to \lambda^o$ and $c_n/b_n^2 \to 0$ w.p.1. The first assertion of the lemma follows. Hence, we have (21) and (22) by the Helly-Bray theorem [8].||

PROOF OF THEOREM 2. In the sequel, $\{(b_n, c_n)\}$ denotes the sequence of random variables generated by the policy π_N. Recalling that $\pi_N(\bullet,\bullet) \geq \xi$, we apply Lemmas 2 and 3, obtaining

$$\lim_{n\to\infty} R_N(b_n, c_n) = \lim_{n\to\infty} \min_{a\geq\xi}\left[\frac{\phi(a,b_n,c_n)}{1-\beta\delta(a,b_n,c_n)} + o_p(1)\right], \tag{23}$$

where $o_p(1)$ vanishes w.p.1 as $n\to\infty$. Since the minimand in (23) converges uniformly for $\beta < 1$, we may interchange lim and min (see, e.g., [6]). Applying Lemma 4, we have w.p.1

$$\lim_{n\to\infty} R_N(b_n, c_n) = \min_{a\geq\xi}\left[\frac{\phi_{\lambda^o}(a)}{1-\beta\delta_{\lambda^o}(a)}\right] \tag{24}$$

$$= R_{\lambda^o}(a^*), \text{ if } a^* \geq \xi.$$

Let

$$\Delta_n(a) = \left|\frac{\phi_{\lambda^o}(a)}{1-\beta\delta_{\lambda^o}(a)} - \frac{\phi(a,b_n,c_n)}{1-\beta\delta(a,b_n,c_n)}\right|. \tag{25}$$

Since by (21) and (22) $\Delta_n(a)$ converges w.p.1 to 0 uniformly for $\beta < 1$, $\lim_{n\to\infty} \Delta_n[\pi_N(b_n,c_n)] \leq \lim_{n\to\infty} \sup_{a\geq\xi} \Delta_n(a) = \sup_{a\geq\xi} \lim_{n\to\infty} \Delta_n(a) = 0$ w.p.1 and hence, using Lemma 2 and the continuity of $T(\bullet,b,c,R_N)$,

(26) $$\lim_{n\to\infty} |R_{\lambda^o}[\pi_N(b_n, c_n)] - R_N(b_n, c_n)| = 0$$

w.p.1. Combining (26) with (24) completes the proof.||

To prove Theorem 3, it suffices to take in particular $v = R_1$ in the second assertion of Theorem 1. The first assertion of Theorem 3 follows immediately. The remainder of the proof is routine and is omitted.

REMARKS. Let $\gamma_o[\gamma_N]$ minimize $H_\sigma(\vec{0})[H_\sigma(A^{N-1}(\vec{0})), N > 1]$. It is easily seen that the stationary policy γ_N is not asymptotically optimal in the sense of Theorem 2, N = 1, 2, ..., because, when λ^o is known, the optimal action at the first stage of a finite horizon problem is not a^*. It is likely that $\{A^N R_1\}$ converges faster than $\{A^N\vec{0}\}$. We conjecture that in Theorem 2 we can drop the condition $\xi > 0$, although, since ξ can be taken arbitrarily small (but positive), this is not of practical concern.

To prove Theorem 4 we may use a standard Tauberian theorem [10], p. 192, paralleling the proof for an analogous theorem in [5].

4. PARTIAL CHARACTERIZATION OF Π_1

Based on geometric considerations, we shall prove

THEOREM 5.

(i) $b \geq [c(c_2 - c_1)/c_1\alpha]^k (k-1)^{k-1} \Rightarrow \pi_1(b,c) = \infty$.

(ii) Suppose that the inequality in (i) does not hold. Let $a_S(b,c)$ denote the smaller positive root of $\frac{\partial}{\partial a}\phi(a,b,c) = 0$. Then $\phi(a_S,b,c) \geq \phi(\infty,b,c) \Rightarrow \pi_1(b,c) = \infty$.

(iii) Suppose that neither of the preceding inequalities holds. Let $a_L(b,c)$ denote the finite root of $\phi(a,b,c) = \phi(\infty,b,c)$ that is larger than $a_S(b,c)$. Then either $\pi_1(b,c) = \infty$ or $a_S(b,c) < \pi_1(b,c) < a_L(b,c)$.

PROOF. It will be shown later that $\phi(a,b,c)$ looks like one of the five possibilities in Fig. 1. We shall also show that the number of positive zeros (N.P.Z.) of $\frac{\partial}{\partial a}\phi(a,b,c)$ satisfies

$$\text{(27)} \qquad \text{N.P.Z.} = \begin{cases} 2, & \text{if } b < Q \\ 0, & \text{if } b > Q \\ 1, & \text{if } b = Q \end{cases}$$

where

$$\text{(28)} \qquad Q = [c(c_2 - c_1)/c_1\alpha]^k (k-1)^{k-1}.$$

Referring to Fig. 1, case 4[5] holds if N.P.Z. = 0[1]; case 3 holds if N.P.Z. = 2 and $\phi(a_S,b,c) \geq \phi(\infty,b,c)$. Using simple dominance arguments, the theorem follows from the fact that $1-\beta\delta(a,b,c)$ is increasing in a and from inspection of Fig. 1.||

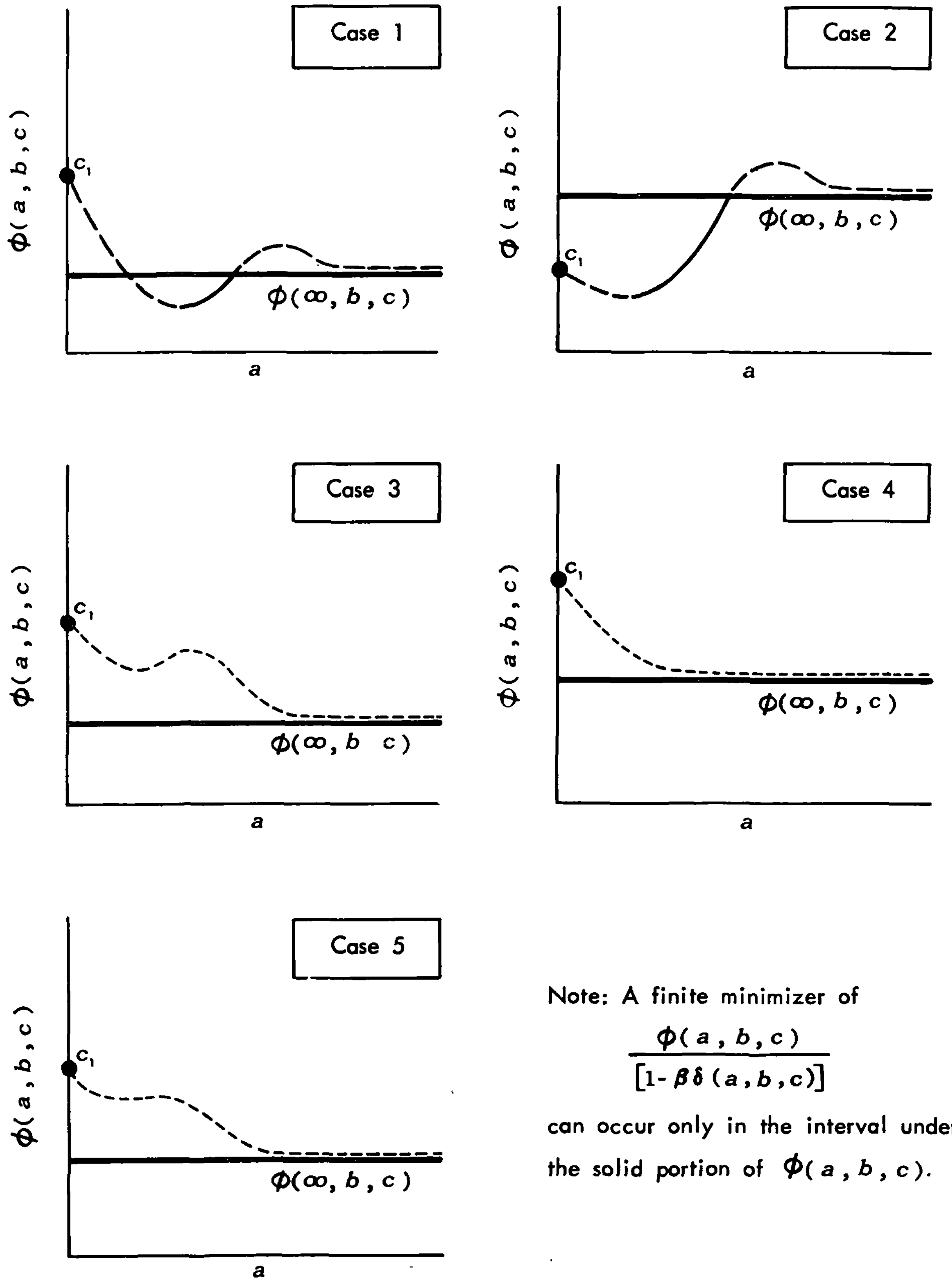


Fig. 1 -- Possibilities for $\phi(a,b,c)$

To see that one of the cases shown in Fig. 1 must hold, we note that the slope of $\phi(a,b,c)$, given by

$$(35) \qquad \frac{\partial}{\partial a}\phi(a,b,c) = e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c \left[\frac{kc(c_2-c_1)\ a^{k-1}}{b+a^k} - c_1\alpha\right],$$

is negative at $a = 0^+$ and for all sufficiently large a. The case that occurs depends on the number of positive zeros of $\frac{\partial}{\partial a}\phi(a,b,c)$. The correspondence is

N.P.Z	case
2	(1)-(3)
0	(4)
1	(5)

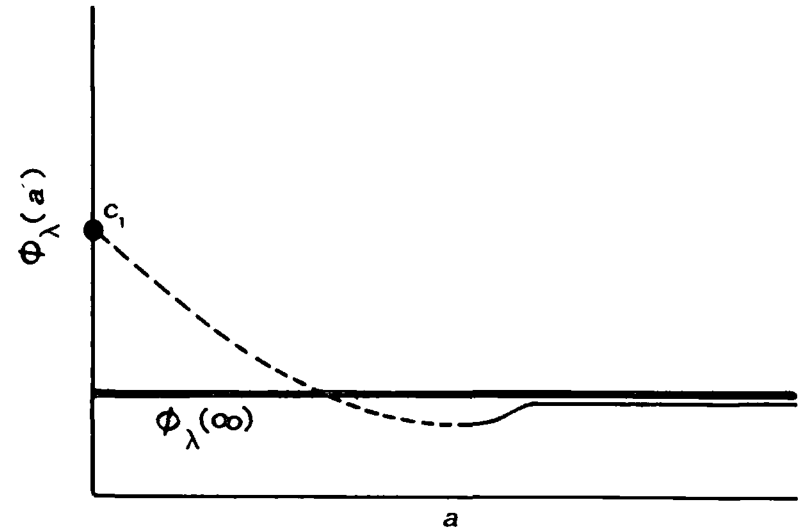
where N.P.Z. = 2,0,1 as

$$\alpha \begin{matrix} < \\ > \\ = \end{matrix} \max_a \left[\frac{kc(c_2-c_1)\ a^{k-1}}{c_1(b+a^k)}\right].$$

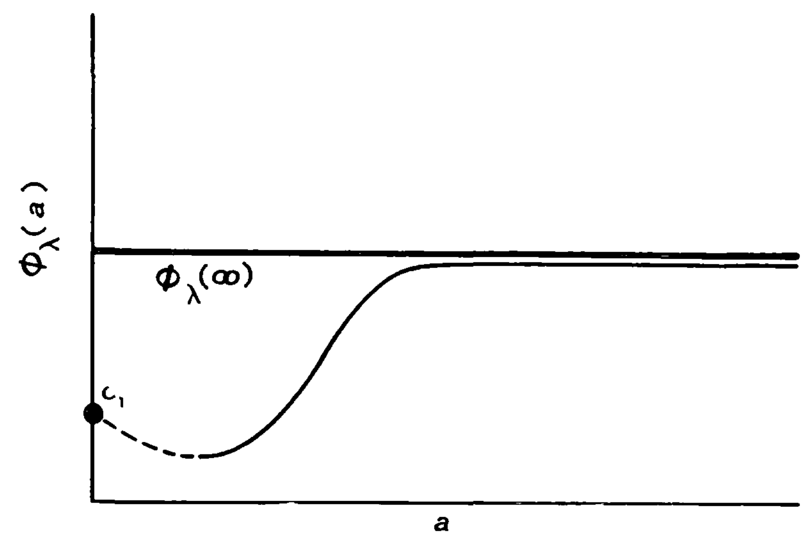
By setting the derivative of the maximand equal to zero, we find that the maximum occurs at $a = [(k-1)b]^{1/k}$ and hence (27) follows. Since the derivative of the maximand has exactly one zero, $\frac{\partial}{\partial a}\phi(a,b,c)$ can cross the abscissa at most twice; i.e., $\frac{\partial}{\partial a}\phi(a,b,c)$ has at most two positive zeros.

For the case when λ^o is known, we get a strikingly different picture (Fig. 2). Since the optimal replacement interval is finite when λ^o is known, it is remarkable that there are values of (b,c) such that

Case (1)



Case (2)



Note: The minimizer of $R_\lambda(a)$ is finite. It ie, in the interval under the solid portion of $\phi_\lambda(a)$.

Fig. 2 -- Possibilities for $\phi_\lambda(a)$

$\pi_1(b,c) = \infty$. An explanation for this phenomenon may be the fact that a mixture [corresponding in our case to weighting λ by $g(\lambda;b,c)$] of increasing failure rate (IFR) distributions is not necessarily IFR (see [1]). For our mixture, the failure rate is $kcx^{k-1}/(b+x^k)$, which increases in the interval $(0,q)$ and decreases in (q,∞), where $q = [(k-1)b]^{1/k}$. As an immediate corollary to a result in [4], we obtain

THEOREM 6. _Either_ $\pi_1(b,c) = \infty$ _or_ $\pi_1(b,c) \in (0,q]$.

Theorems 5 and 6 can be used together to expedite the search for $\pi_1(b,c)$.

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