A UNIFIED APPROACH TO ONE-DIMENSIONAL ELASTIC WAVES BY THE METHOD OF CHARACTERISTICS

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ABSTRACT

A large number of elastic wave problems which involve one space variable are treated, in a unified manner, by a system of second-order hyperbolic partial differential equations, with the generalized displacements as dependent variables. This system of n equations is analyzed by the method of characteristics, yielding closed form equations for the physical characteristics, the characteristic equations, and the propagation of discontinuities. Procedures for numerical integration along the characteristic curves are established. Among the elastic wave problems that may be represented by this unified approach are the Timoshenko beam, plates, bars, and sheets, all including the lateral inertia and shear effects. Various approximate shell equations may also be represented. Results of numerical calculations are in agreement with those obtained by other methods.

SYMBOLS

c_b - bar velocity $= (E/\rho)^{1/2}$

c_d - dilatational (or irrotational) velocity $= \{(\lambda+2G)/\rho\}^{1/2}$

c_e - equivoluminal (or distortional) velocity $= (G/\rho)^{1/2}$

c_i - general wave velocities as defined in text

c_p - plate velocity $= \{E/\rho(1-\nu^2)\}^{1/2}$

c_s - shear velocity $= kc_e$

D - flexural rigidity $= Eh^3/12(1-\nu^2)$

E - Modulus of Elasticity

G - Shear Modulus $= E/2(1+\nu)$

k^2 - shear correction factor

M - bending moment

N - normal stress resultant averaged across sheet

P - bar stresses

Q - shear stress resultant

r - radial distance

S_i - generalized stresses

t - time

u_i - generalized displacements

ν - Poisson's ratio

ρ - density

λ - Lame Constant of Elasticity $= \nu E/(1+\nu)(1-2\nu)$

α_{ij}, β_{ij}, a_{ij}, b_i - coefficients as defined in text

$[\;]$ - bracket represents jump in the enclosed variable

I. INTRODUCTION

In the theoretical analysis of the elastic wave propagation there are in general three methods available; namely, the Laplace transform method, the method of mode superposition, and the method of characteristics. Due to inversion difficulties the Laplace transform method is usually limited to simple wave equations. In the method of mode superposition, the phase velocity of different fundamental modes of motion at different wave lengths can be calculated for steady wave motion. However, it is not suitable for the study of transient problems with prescribed initial and boundary conditions, especially for those inputs involving steep wave fronts. On the other hand, from the method of characteristics many important features, such as the wave propagation velocities and the equation governing the propagation of discontinuities, can be obtained in closed form without any difficulty.

The governing equations, either exact or approximate, for linear wave motions can be expressed as equations of motion in terms of displacement components; this will be called the displacement formulation. Alternatively, the governing equations can be expressed as the equations of motion in terms of displacements <u>and stresses</u>, along with the stress-displacement relations; this will be called the stress-displacement formulation. In the displacement formulation the governing equations are second order equations; while in the stress-displacement formulation the governing equations are first order equations. Because the boundary conditions are sometimes prescribed in terms of stress, it has been customary in the application of the method of characteristics to use the stress-displacement formulation, such as in References 1 to 4. It will be shown in this paper that the displacement formulation is much more useful. The wave velocities, as well as the parameters that govern the propagation of discontinuities, appear explicitly in the equations of the displacement formulation.

This paper begins with a general mathematical study of a system of n hyperbolic second-order differential equations with two independent variables. The physical characteristics, as well as the characteristic equations, are derived. The equations governing the propagation of discontinuities in the first derivatives of the dependent variables are also established. A numerical procedure is then developed for the calculation of the distribution of the dependent variables behind the wave fronts for problems with two distinct wave speeds. The procedures for numerical integration for problems involving one displacement variable are quite well known; e.g., recently, in [1], a numerical procedure has been applied to the cylindrical and spherical wave problems. Leonard and Budiansky [2] have solved the wave propagation in a Timoshenko beam which involves two displacement variables. However, they only treated the case where the two wave speeds are equal. Plass [3] solved the Timoshenko beam problem with two different wave speeds; but he did not include any loading which excites a discontinuity along the slower of the two wave fronts. The procedure developed in this paper, which is an improved version of that given in [4], can handle discontinuities across both the first and the second wave fronts.

It is shown that a large number of elastic wave problems can be treated as special cases of the general mathematical problem with n governing equations. The n dependent variables in each of the elastic wave problems are generalized displacements. Among the examples with one displacement variable are simple dilatational and irrotational waves in cylindrical or spherical coordinates. Problems with two displacement variables include: the Timoshenko beam [5], the motion of a plate incorporating shear effect and rotary inertia by Uflyand [6] and Mindlin [7], the corresponding bar problem incorporating lateral inertia by Mindlin and Herrmann [8], and the sheet problem by Kane and Mindlin [9]. Numerical calculations were performed for many of the n = 2 problems and the results compared with known solutions.

Examples of problems with three displacement variables (n = 3) include the various theories for thin cylindrical shells, [10], [11], and [12]. For n = 4, we have the thick cylindrical shell equations, such as those derived by Mirsky and Herrmann [13]. For n = 6, we have the problem of wave propagation in helical springs by Wittrick [14].

II. METHOD OF CHARACTERISTICS FOR A SYSTEM OF SECOND ORDER EQUATIONS

A. Physical Characteristics and Characteristic Equations

Let us consider the following system of n second order partial differential equations for the n dependent variables u_i and two independent variables, x and t,

$$\frac{\partial^2 u_i}{\partial x^2} - \frac{1}{c_i^2}\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{n} \left(\alpha_{ij} u_j + \beta_{ij}\frac{\partial u_j}{\partial x}\right) \equiv R_i \qquad (1)$$

$$i = 1, 2, 3, \ldots n$$

where c_i, α_{ij} and β_{ij} are continuous, and functions of x only. (The Einstein summation convention will not be used in this paper.) We shall limit our discussions to continuous functions u_i, although the derivatives of u_i may be discontinuous. For regions in the physical plane (x,t-plane) where the first partial derivatives of u_i are continuous, we may write

$$d(u_{i,x}) = \frac{\partial}{\partial x}(u_{i,x})\,dx + \frac{\partial}{\partial t}(u_{i,x})\,dt \qquad (i = 1, 2, \ldots n) \qquad (2)$$

$$d(u_{i,t}) = \frac{\partial}{\partial x}(u_{i,t})\,dx + \frac{\partial}{\partial t}(u_{i,t})\,dt \qquad (i = 1, 2, \ldots n) \qquad (3)$$

where

$$u_{i,x} = \frac{\partial u_i}{\partial x}, \qquad u_{i,t} = \frac{\partial u_i}{\partial t}$$

Equations (1) to (3) form a system of 3n hyperbolic equations which may be used to solve for the 3n second derivatives of u_i, if the distribution of u_i, together with their first derivatives are known along a certain curve.

Along certain directions in the physical plane, however, the specification of u_i, $u_{i,x}$, and $u_{i,t}$ produces solutions which are of indeterminate form. These directions will be called characteristic directions, and lines along these directions will be called the physical characteristics, or simply characteristics. In general, the characteristics are curved lines except for the case of constant c_i, where the characteristics are straight lines. Across these characteristics, the second derivatives of u_i may be discontinuous.

Solving the system of 3n equations, (1) to (3), for $\partial^2 u_1/\partial x^2$, we obtain

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{N_1}{M} \tag{4}$$

where

$$M = \begin{vmatrix} 1 & 0 & -1/c_1^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ dx & dt & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & dx & dt & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/c_2^2 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & dx & dt & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 0 & dx & dt & 0 & \cdots & & 0 \\ \cdots \\ \cdots \\ 0 & & & \cdots & & & & 1 & 0 & -1/c_n^2 \\ 0 & & & \cdots & & & & dx & dt & 0 \\ 0 & & & \cdots & & & & 0 & dx & dt \end{vmatrix} \tag{5}$$

and

$$N_1 = \begin{vmatrix} R_1 & 0 & -1/c_1^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d(u_{1,x}) & dt & 0 & 0 & 0 & & \cdots & & & 0 \\ d(u_{1,t}) & dx & dt & 0 & & & \cdots & & & 0 \\ R_2 & 0 & 0 & 1 & 0 & -1/c_2^2 & 0 & \cdots & & 0 \\ d(u_{2,x}) & 0 & 0 & dx & dt & 0 & 0 & \cdots & & 0 \\ d(u_{2,t}) & 0 & 0 & 0 & dx & dt & 0 & \cdots & & 0 \\ \cdots & & & & & & & & & \\ \cdots & & & & & & & & & \\ R_n & 0 & 0 & & \cdots & & & 1 & 0 & -1/c_n^2 \\ d(u_{n,x}) & 0 & 0 & & \cdots & & & dx & dt & 0 \\ d(u_{n,t}) & 0 & 0 & & \cdots & & & 0 & dx & dt \end{vmatrix} \quad (6)$$

This second derivative is indeterminate if both M and N_1 vanish. The vanishing of M yields, after applying the Laplace expansion technique for determinants,

$$\{c_1^2 - (\frac{dx}{dt})^2\} \{c_2^2 - (\frac{x}{dt})^2\} \quad \cdots \quad \{c_n^2 - (\frac{dx}{dt})^2\} = 0 \quad (7)$$

The vanishing of each of the braces in (7) leads to two families of physical characteristics, e.g., from the first brace,

$$\frac{dx}{dt} = \pm c_1$$

which will be called the C_1^+ and C_1^- characteristics. Altogether, (7) produces 2n families of physical characteristics C_i^+ and C_i^-, where along C_i^+ and C_i^-,

$$\frac{dx}{dt} = \pm c_i \quad (8)$$

respectively.

It is customary to call the c_i's the wave velocities.

The vanishing of N_1 yields

$$\{c_1^2 R_1 (dt)^2 - d(u_{1,x})dx + d(u_{1,t})dt\} \ \{c_2^2 - (\frac{dx}{dt})^2\} \ \ldots \ \{c_n^2 - (\frac{dx}{dt})^2\} = 0 \qquad (9)$$

Assuming that c_1 is not equal to any other c_i, we observe from (9) that along the directions $dx/dt = \pm c_1$,

$$d(u_{1,t}) \mp c_1 (du_{1,x}) \pm c_1 R_1 dx = 0 \qquad (10)$$

These two will be called the characteristic equations along the C_1^+ and C_1^- characteristics, respectively. It can be shown by a limiting process that (10) is true even when c_1 is equal to one or more of the other c_i's. The solution for $\partial^2 u_i/\partial x^2$ from eqs. (1) to (3) yields the characteristic equations

$$d(u_{i,t}) \mp c_i d(u_{i,x}) \pm c_i R_i dx = 0, \qquad i = 1, 2, \ldots n \qquad (11)$$

along $(dx/dt) = \pm c_i$, respectively. The vanishing of the denominators and numerators of the solutions of $\partial^2 u_i/\partial x \partial t$ and $\partial^2 u_i/\partial t^2$ yields identical results as (8) and (11). Since only continuous u_i are being considered, we may write

$$du_i = u_{i,x} dx + u_{i,t} dt , \qquad i = 1, 2, \ldots n \qquad (12)$$

along any direction. In regions in the physical plane where the first derivatives of u_i are continuous, (11) and (12), which consist of 3n equations, may be used to solve for the 3n variables u_i, $u_{i,x}$, and $u_{i,t}$, if proper boundary and initial conditions are specified.

B. Propagation of Discontinuities

Along the physical characteristics, the variables u_i, $u_{i,x}$, and $u_{i,t}$ are governed by the characteristic equations (11). Across the physical characteristics, the second derivatives of u_i may be discontinuous; these discontinuities do not affect the applicability of (11), which does not contain second derivatives of u_i. Discontinuities in the first derivatives of u_i may also exist across the physical characteristics, but these will not be governed by (11). Discontinuities in $u_{i,x}$ and $u_{i,t}$ occur when a finite step input (or jump input) in these variables is applied at a particular x. The equations for the propagation of these discontinuities will now be derived.

First, let us demonstrate that lines of discontinuity in the first derivatives of u_i are necessarily characteristics. Consider a line DE that is not a characteristic. Assume that discontinuities in the first derivatives of u_i exist across DE, or between FG and DE when FG $\to$ DE, as shown in Fig. 1. Further assume that within each of the two regions divided by DE, all functions are continuous. Integration of (11), with lower signs, along any C_i^- characteristic from A to B, where A is a point on DE, yields

$$u_{i,t}(B) - u_{i,t}(A) + \int_A^B c_i d\,(u_{i,x}) = \int_A^B c_i R_i dx \tag{13}$$

If we allow FG to approach DE, and B to approach A, (13) becomes

$$\left[u_{i,t}\right] + c_i \cdot \left[u_{i,x}\right] = 0 \tag{14}$$

where the bracket designates the value of discontinuity (or jump) in the variable it encloses, e.g.,

$$\left[u_{i,t}\right] = u_{i,t}(B) - u_{i,t}(A), \qquad \text{as } B \to A$$

In writing (14) we also assumed that R_i is bounded and c_i continuous; therefore, the right hand side of (13) vanishes as dx approaches zero. Integration along C_i^+, as C approaches A, yields

$$\left[u_{i,t}\right] - c_i \left[u_{i,x}\right] = 0 \qquad (15)$$

Combining (14) and (15), we obtain

$$\left[u_{i,t}\right] = \left[u_{i,x}\right] = 0$$

From this we conclude that discontinuities of first derivatives cannot exist across a line that is not a characteristic.

Now, let us consider discontinuities in $u_{i,x}$ and $u_{i,t}$ across one particular characteristic, C_k^+, where c_k is not equal to any other c_i's. Write (11), with the lower signs, and integrate it along the C_k^- characteristic from A to B, as shown in Fig. 2. As B approaches A, or as C_k^+ (2) approaches C_k^+ (1), we have

$$\left[u_{k,t}\right] + c_k \left[u_{k,x}\right] = 0 \qquad (16)$$

Integration from C_k^+ (1) to C_k^+ (2) along the other C_i^- characteristics yields the same results, or

$$\left[u_{i,t}\right] + c_i \left[u_{i,x}\right] = 0 \qquad \text{across } C_k^+ \text{ ; } i = 1, 2, \ldots n \qquad (17)$$

Since c_k is not equal to any of the other c_i's, every C_ℓ^+ characteristic passing through point A must intersect the C_k^+ (2) characteristic, where $\ell = 1, 2, \ldots k-1, k+1, \ldots n$. Integration along C_ℓ^+ gives

$$\left[u_{\ell,t}\right] - c_\ell \left[u_{\ell,x}\right] = 0 \qquad \text{across } C_k^+ \text{ , } \ell \neq k \qquad (18)$$

Combining (17) and (18), and assuming that c_ℓ, $u_{\ell,t}$, and $u_{\ell,x}$ are continuous along C_k^+ (2), we obtain

$$\left[u_{\ell,t}\right] = \left[u_{\ell,x}\right] = 0 \qquad \text{across } C_k^+ \text{ ; } \ell = 1, 2, \ldots k-1, k+1, \ldots n \qquad (19)$$

This indicates that across C_k^+, discontinuities in $u_{\ell,t}$ and $u_{\ell,x}$, $\ell \neq k$, cannot exist. Thus, discontinuities in $u_{k,t}$ and $u_{k,x}$ are not coupled with discontinuities in other $u_{i,t}$ and $u_{i,x}$; therefore they can be treated separately.

The relations governing the magnitude of the jumps $[u_{k,x}]$ and $[u_{k,t}]$ across C_k^+ as they propagate along C_k^+ are obtained by writing (11) twice, both with the upper signs, once along C_k^+ (2) and the other along C_k^+ (1), and subtracting one from the other. Thus, as C_k^+ (2) approaches C_k^+ (1), we have

$$d[u_{k,t}] - c_k\, d[u_{k,x}] = -c_k \sum_{j=1}^{n} \{\alpha_{kj}[u_j] + \beta_{kj}[u_{j,x}]\}\, dx \tag{20}$$

Since u_i are continuous throughout, $[u_j] = 0$, by inserting (16) into (20), and utilizing (19), we obtain

$$[u_{k,x}]\, \frac{dc_k}{c_k} + 2d[u_{k,x}] = \beta_{kk}[u_{k,x}]dx \tag{21}$$

or

$$\frac{d[u_{k,x}]}{[u_{k,x}]} = \frac{1}{2}\{\beta_{kk}\, dx - \frac{dc_k}{c_k}\}$$

This may be integrated to give

$$[u_{k,x}] = K_k\, c_k^{-1/2} \exp \int (\beta_{kk}/2)\, dx, \quad \text{along } C_k^+ \tag{22}$$

where K_k is a constant. From (22) and (16), we have

$$[u_{k,t}] = -K_k\, c_k^{1/2} \exp \int (\beta_{kk}/2)\, dx, \quad \text{along } C_k^+ \tag{23}$$

Following the same procedure, it can be shown that the propagation along C_k^- of discontinuities is governed by

$$\left.\begin{aligned} [u_{k,x}] &= K_k\, c_k^{-1/2} \exp \int (\beta_{kk}/2)\, dx \\ [u_{k,t}] &= +K_k\, c_k^{1/2} \exp \int (\beta_{kk}/2)\, dx \end{aligned}\right\} \text{along } C_k^- \tag{24}$$

C. Problems with Two of the c_i's Equal

If two of the c_i's are equal, (22) to (24) are not applicable. Let us assume c_{k+1} equal to c_k, then C^+_{k+1} coincides with C^+_k and C^-_{k+1} with C^-_k; the equations governing the jumps $[u_{k,x}]$ and $[u_{k,t}]$ will be derived below. Following the same procedure as in the previous section, it can be shown that (17) and the corresponding equations for C^-_k are still valid in the present case, or,

$$[u_{i,t}] \pm c_i [u_{i,x}] = 0 \qquad i = 1, 2, \ldots n \tag{25}$$

where the upper sign is for discontinuities across C^+_k and the lower sign for those across C^-_k, respectively. Furthermore, analogous to (19), we can show that

$$[u_{\ell,t}] = [u_{\ell,x}] = 0 \qquad \text{across } C^+_k \text{ and } C^-_k$$
$$\ell = 1, 2, \ldots k-1, k+2, \ldots n \tag{26}$$

In place of (20), the following equations may be written

$$d[u_{k,t}] \mp c_k d[u_{k,x}] = \mp c_k \{\beta_1 [u_{k,x}] + \beta_2 [u_{k+1,x}]\} dx \tag{27}$$

$$d[u_{k+1,t}] \mp c_k d[u_{k+1,x}] = \mp c_k \{\beta_3 [u_{k,x}] + \beta_4 [u_{k+1,x}]\} dx \tag{28}$$

where

$$\beta_1 = \beta_{kk}, \quad \beta_2 = \beta_{k,k+1}, \quad \beta_3 = \beta_{k+1,k}, \quad \beta_4 = \beta_{k+1,k+1} \tag{29}$$

Eliminating $[u_{k,t}]$ and $[u_{k+1,t}]$ from (27) and (28) by using the two equations obtained from (25) with $i = k$ and $i = k+1$, respectively, and solving the resulting two equations for $[u_{k,x}]$, we obtain

$$\frac{d^2}{dx^2}\left[u_{k,x}\right] + \left\{\beta_2 \frac{d}{dx}\left(\frac{1}{\beta_2}\right) + \frac{1}{c_k}\frac{dc_k}{dx} - \frac{\beta_1}{2} - \frac{\beta_4}{2}\right\}\frac{d}{dx}\left[u_{k,x}\right]$$

$$+ \left\langle \frac{1}{2c_k}\frac{d^2c_k}{dx^2} + \frac{\beta_2}{2}\left\{\frac{d}{dx}\left(\frac{1}{\beta_2 c_k}\right) + \frac{1}{2c_k^2\beta_2}\frac{dc_k}{dx} - \frac{\beta_1}{2c_k\beta_2} - \frac{\beta_4}{2\beta_2 c_k}\right\}\frac{dc_k}{dx}\right.$$

$$\left. - \frac{\beta_2}{2}\frac{d}{dx}\left(\frac{\beta_1}{\beta_2}\right) - \frac{\beta_2\beta_3}{4} + \frac{\beta_4\beta_1}{4}\right\rangle \left[u_{k,x}\right] = 0 \qquad (30)$$

This second order equation for $[u_{k,x}]$ may be integrated readily if the values of $[u_{k,x}]$ and $[u_{k+1,x}]$ at one point are given, since the value of $\frac{d}{dx}[u_{k,x}]$ at this point can be obtained from (27) and (25). Once $[u_{k,x}]$ has been determined, $[u_{k,t}]$ may be obtained from (25).

D. Generalized Stresses

In stress wave problems, the functions u_i correspond to generalized displacement variables and the $u_{i,t}$ correspond to generalized velocities, as will be shown in a later section. In these problems, certain generalized stress variables are also of practical importance and some of these stress variables may be prescribed as boundary conditions. The generalized stresses will be designated as S_m and are defined as

$$S_m = b_m u_{m,x} + \sum_{j=1}^{n} h_{mj} u_j \qquad m = 1, 2, \ldots \qquad (31)$$

In a given problem the number of generalized stresses S_m is either equal to, or greater than, the number of generalized displacements u_i ; although the number of S_m that can be prescribed as boundary conditions is usually equal to the number of generalized displacement.

When there are jumps in $u_{m,x}$, the generalized stresses will also have jumps. Consider the case of a jump in $u_{m,x}$ across a C_m^+ (or C_m^-) characteristic. Writing (31) twice along the two sides of this characteristic and subtracting one from the other, we obtain

$$\left[S_m\right] = b_m \left[u_{m,x}\right] \qquad \text{across } C_m^+ \text{ or } C_m^- , \qquad m = 1, 2, \ldots \qquad (32)$$

where the conditions of $[u_j] = 0$ are used. The variation of $[S_m]$ as it ... C^+ ... C^- can be obtained from (32), (22), and (24); or from

E. Initial and Boundary Conditions

The governing equations (1) are of second order in both x and t; therefore, two initial conditions and two boundary conditions must be specified for each of the variables u_i . The specification of all $u_{i,x}$ and $u_{i,t}$ functions along the initial line t = 0 constitutes a properly posed initial condition. Note that the specification of $u_{i,x}$ along the initial line t = 0 is equivalent to specifying u_i along t = 0.

Along each of the boundary lines $x = x_1$ and $x = x_2$, one boundary condition for each u_i must be specified. One properly posed boundary condition is to specify all u_i's along $x = x_1$ and $x = x_2$. Any of the generalized stress, instead of the corresponding displacement, may also be specified along these lines. For a particular value of i, say i = k, either u_k or S_k, but not both, may be specified. If the number of generalized stresses is greater than n, usually only n of the stresses can be prescribed as boundary conditions, the rest are not feasible from a practical engineering point of view.

Properly posed initial and boundary conditions are those which assure a unique solution of the equations. Uniqueness of solutions to eqs. (1) will be discussed in another paper.

III NUMERICAL PROCEDURES

Once the characteristic equations of a system of hyperbolic differential equations are known, they can be integrated readily by numerical means. For linear equations, the numerical integration is equivalent to a straight forward solution of simultaneous algebraic equations and involves no iteration process. We shall limit our discussion to numerical procedures for the case of two dependent variables (n = 2) and constant wave velocities.

A. Continuous Boundary Conditions

In this section we shall establish the procedure for the calculation of regions where u_1 and u_2 have continuous first derivatives. The general case of $n = 2$ and $c_1 \neq c_2$ will be considered; whereas problems with $c_1 = c_2$, and those with $n = 1$ can be treated as special cases without any difficulty.

In performing the numerical calculations, the physical plane is first divided into a network by the characteristic lines; the characteristic and continuity equations are then written in finite-difference form in terms of the values of the dependent variables at the mesh points of the network. For problems with $n = 2$, there are four families of characteristic lines in the physical plane, with each characteristic intersecting every one of the other three characteristic families. The resulting network contains too many irregular mesh points to be practical for numerical calculations. For simplicity, only C_1^+ and C_1^- characteristics are used as the main network, where $c_1 > c_2$, as shown in Fig. 3; and only at the mesh points of this network will the dependent variables be calculated. Values of the variables u_1, $u_{1,x}$, $u_{1,t}$, u_2, $u_{2,x}$, and $u_{2,t}$ at a typical interior point 1 may be calculated if the corresponding values at neighboring points 2, 3, and 4 are known from previous calculations. To accomplish this, draw C_2^+ and C_2^- characteristics from point 1, intersecting the C_1^- and C_1^+ characteristics that pass through point 4 at points 5 and 6, respectively. Values of the variables at points 5 and 6 are obtained from those at points 2, 3, and 4 by linear interpolation. The finite-difference form of the characteristic equation (11), with $i = 1$ and the upper signs, is

$$\{u_{1,t}(1) - u_{1,t}(2)\} - c_1\{u_{1,x}(1) - u_{1,x}(2)\}$$

$$= -c_1 \sum_{j=1}^{2} \{\alpha_{1j}(1,2)\, u_j(1,2) + \beta_{1j}(1,2)\, u_{j,x}(1,2)\}\, \{x(1) - x(2)\}$$

where a single numeral in a parenthesis indicates the point at which the variable is evaluated, a double numeral within a parenthesis designates the average of the variable between the two points. Three other finite-difference equations may be written for the characteristic equations along C_1^- between points 1 and 3, along C_2^+ between points 1 and 5, and along C_2^- between 1 and 6. These finite-difference characteristic equations may be written as

$$\Delta(u_{i,t}) \mp c_i\, \Delta(u_{i,x}) \pm c_i \sum_{j=1}^{2} (\bar{\alpha}_{ij}\bar{u}_j + \bar{\beta}_{ij}\bar{u}_{j,x})\, \Delta(x) = 0 \qquad \text{along } \frac{dx}{dt} = \pm c_i \tag{34}$$

where $\Delta(\ )$ represents difference, and a bar over a letter designates average.

The continuity equation for u_1 and u_2 are written in finite-difference form along C_1^- and C_2^- , respectively, as

$$u_1(1) - u_1(3) = u_{1,x}(1,3)\ \{x(1) - x(3)\} + u_{1,t}(1,3)\ \{t(1) - t(3)\} \tag{35}$$

$$u_2(1) - u_2(6) = u_{2,x}(1,6)\ \{x(1) - x(6)\} + u_{2,t}(1,6)\ \{t(1) - t(6)\} \tag{36}$$

The four characteristic equations together with the two continuity equations, (35) and (36), constitute six equations for the six unknowns u_1, $u_{1,x}$, $u_{1,t}$, u_2, $u_{2,x}$, and $u_{2,t}$ at point 1.

For mesh points on the left boundary line $x = x_1$, two of the characteristics, C_1^+ and C_2^+, are absent. If u_1 and u_2 are specified along $x = x_1$, the remaining four equations are sufficient for finding the remaining four unknowns $u_{1,x}$, $u_{1,t}$, $u_{2,x}$, and $u_{2,t}$. If S_1 and S_2 are specified along $x = x_1$, then the two finite-difference equations obtained from (31) with $m = 1$ and $m = 2$, replace the finite-difference characteristic equations along C_1^+ and C_2^+, and the system of six equations necessary for the determination of the six variables is again complete.

B. Discontinuities in the First Derivatives

When the input at $x = x_1$ involves discontinuities (jumps) in $u_{1,x}$, $u_{1,t}$, or S_1, these discontinuities propagate along the C_1^+ line, which has an equation $x = x_1 + t'$, where $t' = c_1 t$. The propagation of these jumps is governed by (22), (23), and (32), with $k = 1$; no special difficulties will be encountered in the numerical integration procedure, as discussed in [1]. However, when the input at $x = x_1$ involves jumps in $u_{2,x}$, $u_{2,t}$, or S_2, a different situation arises. These discontinuities propagate along the C_2^+ line which has an equation $x = x_1 + \mu t'$, where $\mu = c_2/c_1$. In general, this line does not intersect the main network at the mesh points, as shown in Fig. 4a. This line may be replaced by a "zig-zag line" with discontinuous slope but passing through the regular mesh points [4]. Numerical results indicated that although the treatment by this approximate "zig-zag line" gave overall good qualitative results, the accuracy was less than satisfactory. In this paper, the exact C_2^+ line, $x = x_1 + \mu t'$, is used, and is not replaced by any approximate lines. At each point where this line intersects lines of the regular network, values of the dependent variables will be calculated. Details of this procedure, which is similar to the one used in [15] for the plate bending problems, will now be given below.

We shall call the line $x = x_1 + \mu t'$ the jump line, and introduce a new coordinate system (α,β) which consists of the C_1^+ and C_1^- characteristics as shown in Fig. 4a. The finite-difference network is then composed of constant α and constant β lines, with constant increment δ in both α and β. The point of intersection between the jump line and a particular

α = constant line, say $\alpha = m\delta$ line, where m is an integer, is at

$$\beta = \left(\frac{1-\mu}{1+\mu}\right) m\delta \tag{37}$$

In general, this β is not an integer; therefore the intersection is not located at a regular mesh point. In calculating the values of the variables at a regular mesh point adjacent to the jump line, three types of net may be encountered. A net is defined as a square in the network with each side of length δ. A net is called type I if the jump line intersects both of the α = constant lines of the net and does not intersect the β = constant lines of the net. The net ABCD in Fig. 4a, used for the calculation of values at point B, is of type I. If the jump line, while proceeding upwards, intersects a β = const. line first and then an α = const. line of a net, then this net is of type II, such as net KLGB. If it intersects first an α = const. line and then a β = const. line, then the net is of type III, e.g., net BGHC. The detection as to the type of a net may be accomplished as follows. For a net with sides $\alpha = m\delta$, $\alpha = (m+1)\delta$, $\beta = n\delta$ and $\beta = (n+1)\delta$, where m and n are integers, it is type I if

$$\begin{aligned} \gamma m &= n + \varepsilon_1, \quad 0 \le \varepsilon_1 < 1 \\ \gamma(m+1) &= n + \varepsilon_2, \quad 0 \le \varepsilon_2 < 1 \end{aligned} \tag{38}$$

where $\gamma = (1-\mu)/(1+\mu)$. The net is type II, if

$$\begin{aligned} \gamma m &= n - 1 + \varepsilon_3, \quad 0 \le \varepsilon_3 < 1 \\ \gamma(m+1) &= n + \varepsilon_4, \quad 0 \le \varepsilon_4 < 1 \end{aligned} \tag{39}$$

The net is type III, if

$$\gamma m = n + \varepsilon_5 \,, \quad 0 \le \varepsilon_5 < 1$$
$$\gamma(m+1) = n + 1 + \varepsilon_6 \,, \quad 0 \le \varepsilon_6 < 1 \tag{40}$$

At a mesh point on the jump line, each of the variables, $u_{2,x}$, $u_{2,t}$, assumes two values, e.g., $u_{2,x}$ (unjumped value) and $u_{2,x} + [u_{2,x}]$ (jumped value), etc., where $[u_{2,x}]$ and $[u_{2,t}]$ are calculated from (22) and (23). We shall now discuss the finite-difference solution of the governing equations for a type-I net. Referring to Fig. 4b for the type-I net ABCD, it is assumed that values of the variables at points A, E, D, and C are known from previous calculation. Values at point F will now be determined. Draw line F-3, through point F parallel to the C_2^- family of curves; line F-2 is also drawn through point F parallel to the C_1^+ family. The values at point 2 and 3 are obtained by linear interpolation. We now apply the six equations, (35), (36), and four of the form of (34), evaluated at proper points, to obtain the six unknowns $u_{1,x}$, $u_{1,t}$, $u_{2,x}$, $u_{2,t}$, u_1, and u_2 at point F. For the characteristic equation along F-2, the jumped values at point F must be used; for the equation along FC, unjumped values at F are used; for the equation along FE at both points F and E, the jumped values of $u_{2,x}$ and $u_{2,t}$ are used; for the characteristic equation along F-3 as well as the two continuity equations, the unjumped values of $u_{2,x}$ and $u_{2,t}$ at point F must be used. Having obtained the values of the variables at F, we may now determine those at B. Fig. 4c shows the network necessary for the calculation of $u_{2,x}$, $u_{2,t}$, u_2, $u_{1,x}$, $u_{1,t}$, and u_1 at point B which is a regular mesh point. Again the system of six equations is utilized, where jumped values at point F must be used. Values at points 5, 6, and 2 are, again, determined by linear interpolation.

For the other two types of nets, similar procedures are adapted. The proper initial conditions for this case require the specification of $u_{1,t}$, $u_{2,t}$, $u_{1,x}$, and $u_{2,x}$ along $t = 0$. For all the example problems solved in this paper, the initial conditions are

$$u_{1,x}(x,0) = u_{2,x}(x,0) = u_{1,t}(x,0) = u_{2,t}(x,0) = 0 \qquad x_1 \leq x \leq x_2 \tag{41}$$

At $x = x_1$, properly posed boundary conditions require the specification u_1 or S_1; and, u_2 or S_2. The same can be said for $x = x_2$; however, in many of the problems where $x_2 = \infty$ we will require regularity of the two variables at infinity.

The region between $x = x_1 + c_1 t$ and $t = 0$ in the physical plane contains the trivial solutions of vanishing derivatives of u_1 and u_2. Along the line $x = x_1 + c_1 t$ these derivatives are also zero if the boundary condition at $x = x_1$, $t = 0$, does not include discontinuities in the functions $u_{1,x}$, $u_{1,t}$, or S_1. When discontinuities in these variables occur at $x = x_1$, $t = 0$, they will propagate along the line $x = x_1 + c_1 t$ according to (22) and (23) for $k = 1$.

When discontinuous functions of $u_{2,x}$, $u_{2,t}$, or S_2 are prescribed at $x = x_1$, $t = 0$, these discontinuities will propagate along the line $x = x_1 + c_2 t$, according to (22) and (23) with $k = 2$. Within the region between the lines $x = x_1 + c_2 t$ and $x = x_1 + c_1 t$, the derivatives of u_2 are in general different from zero, although they vanish on the line $x = x_1 + c_1 t$.

For problems with $c_1 = c_2$, no special difficulties will be encountered since all characteristics intersect at regular mesh points. For jump inputs, the solution of eq. (30), instead of (22) and (23) should be used.

IV. APPLICATION TO ELASTIC WAVE PROBLEMS

A large number of problems in linear elastic wave propagation and vibration can be arranged in the form of eqs. (1). From these unified equations, the wave propagation velocities, c_i, and the parameter β_{ii}, which governs the propagation of discontinuities, are immediately known. In the following, we shall discuss some examples in elastic wave problems in relation to the unified equations. For the cases of n = 2, comparison of the results from our numerical calculation with those obtained by others will also be included. No discussion will be given on the derivation of the various approximate wave equations; emphasis will be placed on the analysis and solution of these equations.

A summary of some of the problems with n = 1 and n = 2 are given in Table I and II, respectively. In these tables, the first row gives the name of the physical problem; the second row gives the authors whose notations, with minor modifications, have been adopted here; the rest of the rows list the coefficients in eqs. (1) and (31) that each of the physical problems assumes. Certain notations, such as modulus of elasticity E, Lame's constants λ and G, shear correction factor k^2, plate modulus D, are standardized for all cases. The radial space variable in cylindrical or spherical coordinates is represented by r.

A. Problems with One Displacement Variable (n = 1)

For the cylindrical and spherical dilatational waves, the governing equations for homogeneous media such as eq. (10) of [1], are well known.

Results of numerical calculations by the method of characteristics are also given in [1]. The cylindrical rotary equivoluminal (shear) waves in homogeneous materials are treated by Goodier and Jahsman [16]. The corresponding problem for nonhomogeneous media are solved by Sternberg and Chakravorty [17] by the Laplace transform method; and by Chou and Schaller [18] by the method of characteristics. Solutions of the cylindrical longitudinal equivoluminal waves may be found in [19]. In Table I, the corresponding equations for all these cases with variable spatial distribution of elastic properties (nonhomogeneous) are presented.

B. Problems with Two Displacement Variables ($n = 2$)

Only two generalized stresses, S_1 and S_2, are listed for each case in Table II. These are the two that may be prescribed as boundary conditions. Additional generalized stresses, such as M_θ in the plate problem, usually (except the beam case) appear in the stress equations of motion; however, they may not be prescribed as boundary conditions and they are not needed for the solution of the problem in terms of generalized displacements.

After the elimination of one displacement variable, the two equations of motion of any of the $n = 2$ problems may be expressed as one fourth order equation. However, from this single fourth order equation the wave velocities and the factors β_{11} and β_{22} cannot be detected readily.

In all the numerical calculations, 150 space points are used; which required an average computing time of 20 to 30 minutes on an IBM 7040 computer.

Timoshenko Beams

The governing equations, the wave velocities, and the equation governing the discontinuities for beams with variable cross-sectional area and variable elastic properties are in agreement with those obtained by Leonard and Budiansky [2]. In [2], they also solved the beam problem with two wave speeds equal by both the method of characteristics and the method of Laplace transform. In particular, they obtained closed form solutions for infinite beams with either step velocity or step moment input applied at the end. The case of a uniform cantilever beam subjected to a step velocity at the root was calculated by the present technique; the relative difference between our numerical results and eq. (C14) of [2] is less than 0.05%.

Boley and Chao [20] presented the Laplace transformation solutions to four types of loadings applied to a semi-infinite beam. These loadings applied at x = 0 are:

a. Step velocity and zero bending moment,

b. Step moment and zero displacement,

c. Step angular velocity and zero shear force, and

d. Step shear force and zero rotation.

These problems were solved by the present technique; our numerical results were found to be in good agreement with the curves of [20] except in case (b), where a slight discrepancy in moment exists, as shown in Fig. 5.

Plass [13] presented solutions to eleven types of loadings applied to a semi-infinite beam, by using a numerical procedure similar to the present one. He applied various types of support conditions and impact conditions where in every case the impact is a pulse in the form of a half-sine wave. The problems presented in Figs. 1, 4, 6, 10, 11, 12, and 13 of [3] were solved by the present technique. The solutions were found to be in good agreement with those of [3] except in one case. For the case of half-sine rotation impact with zero shear, our resulting moment distributions are one-fourth in magnitude of those presented in Fig. 13 of [3].

Plates

The equations in Table II for plane and cylindrical waves in plates are based on the two-dimensional equations derived by Mindlin [7]. Chou and Koenig [4] calculated cylindrical waves due to various axisymmetrical loadings of a plate with a circular hole. The numerical results in [4] are satisfactory except for the case with a jump shear force input at the hole; in which case the procedure for the calculation of jumps across the second characteristic is not accurate. Improved results for plates using the present technique are given in [15]. A few curves showing the response of a plate due to jump shear input are reproduced in Fig. 6. for easy reference.

Bars

The equations for nonhomogeneous bars are based on the work by Mindlin and Herrmann [8]. Miklowitz [21] presented the Laplace transform solution to the problem of a semi-infinite bar with step axial stress and zero velocity applied at $x = 0$. His solution was also successfully duplicated by the present technique.

Sheets

Equations governing the propagation of dilatational waves in a plate incorporating the lateral inertia effect were derived by Kane and Mindlin [9] for two-dimensional problems. The corresponding equations for cylindrical waves were presented by Jahsman [22]. In Table II, corresponding equations for plane and cylindrical waves in nonhomogeneous material are given.

C. Problems with More Than Two Displacement Variables ($n > 2$)

Several sets of approximate equations, all incorporating the rotary inertia and shear effect, for thin cylindrical shells can all be arranged into the form of eqs. (1), with $n = 3$. For instance, the equations derived by Herrmann and Mirsky [1] reduce to our unified form, if, in [1], the first of (18) is multiplied by $1/R$ and combined with the third multiplied by minus one, the resulting equation contains second derivatives of u only; the first of (18) minus the third multiplied by $(-h^2/12R)$ gives the corresponding equation for ψ_x; while the second of (18) is already in the form of our (1). The three wave velocities detected from these equations, and the equations governing the propagation of discontinuities are in agreement with those obtained by Spillers [23], who used the corresponding set of first order equations. Detailed discussion of the approximations involved in different shell theories, in terms of the present unified approach, as well as numerical calculations, will be given in a forthcoming paper.

One example for the $n = 4$ case is the thick cylindrical shell equations derived by Mirsky and Herrmann [13]. Here, if the first two equations of (22) in [13] are multiplied by proper constants and combined, two equations, one containing second derivatives in ψ_x only, the other containing second derivatives in u only, may be obtained. Similarly, the last two equations of (22) in [13] may be combined to give two equations each with second derivatives of one variable only.

Another interesting problem that may be represented by the present unified approach is the wave propagation in helical springs [14]. In this case there are six generalized displacement variables ($n = 6$), three components of displacement and three components of rotation of the cross-section of the spring. The theory is essentially an extension of the Timoshenko theory for straight beams. In [14], Wittrick obtained six stress-displacement relations, (49) and (50), and six equations of motion in terms of stresses and displacements, (51) and (52). Substituting his eqs. (49) and (50) into his (51) and (52), we obtain six equations of the form of our eqs. (1). It is interesting to note that for these equations there are only three distinct wave velocities for the cases where the cross-section of the spring is either square or circular.

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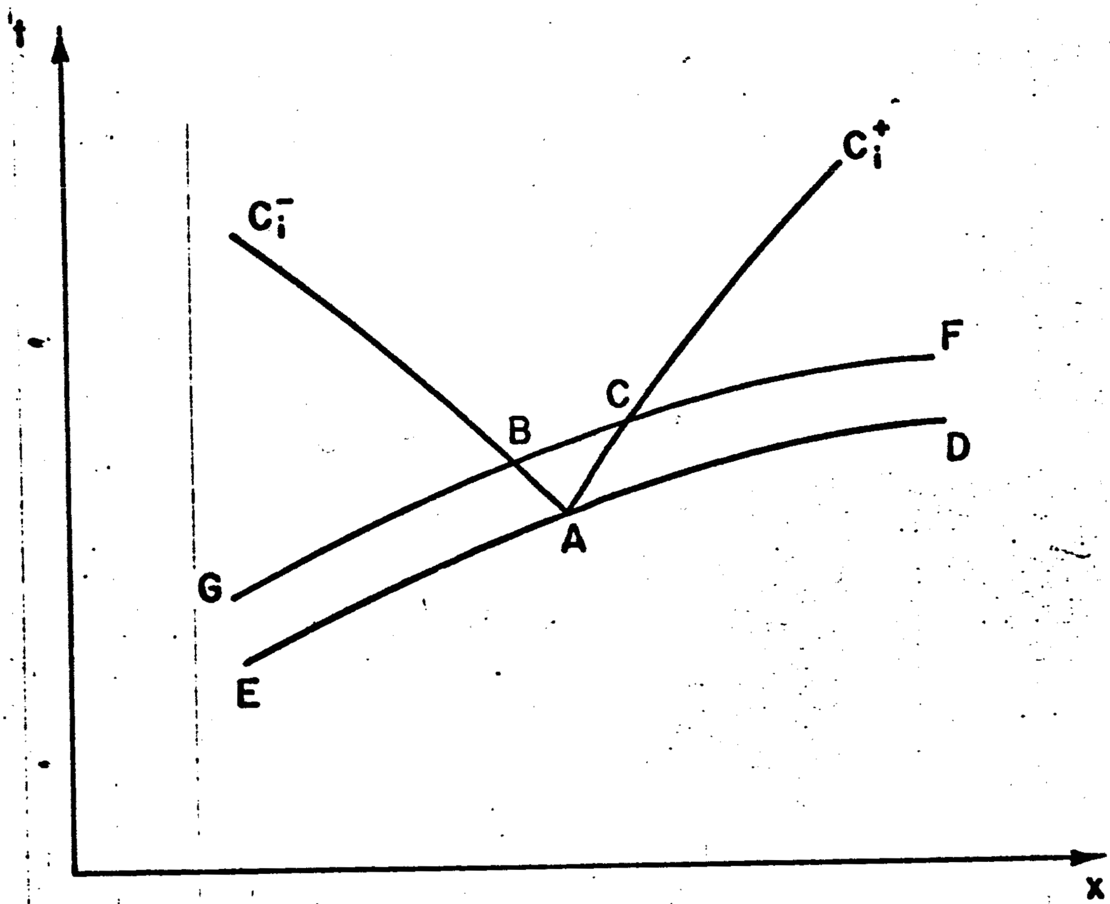


Figure 1 Discontinuity Cannot Exist Across a Line DE Which Is Not a Characteristic

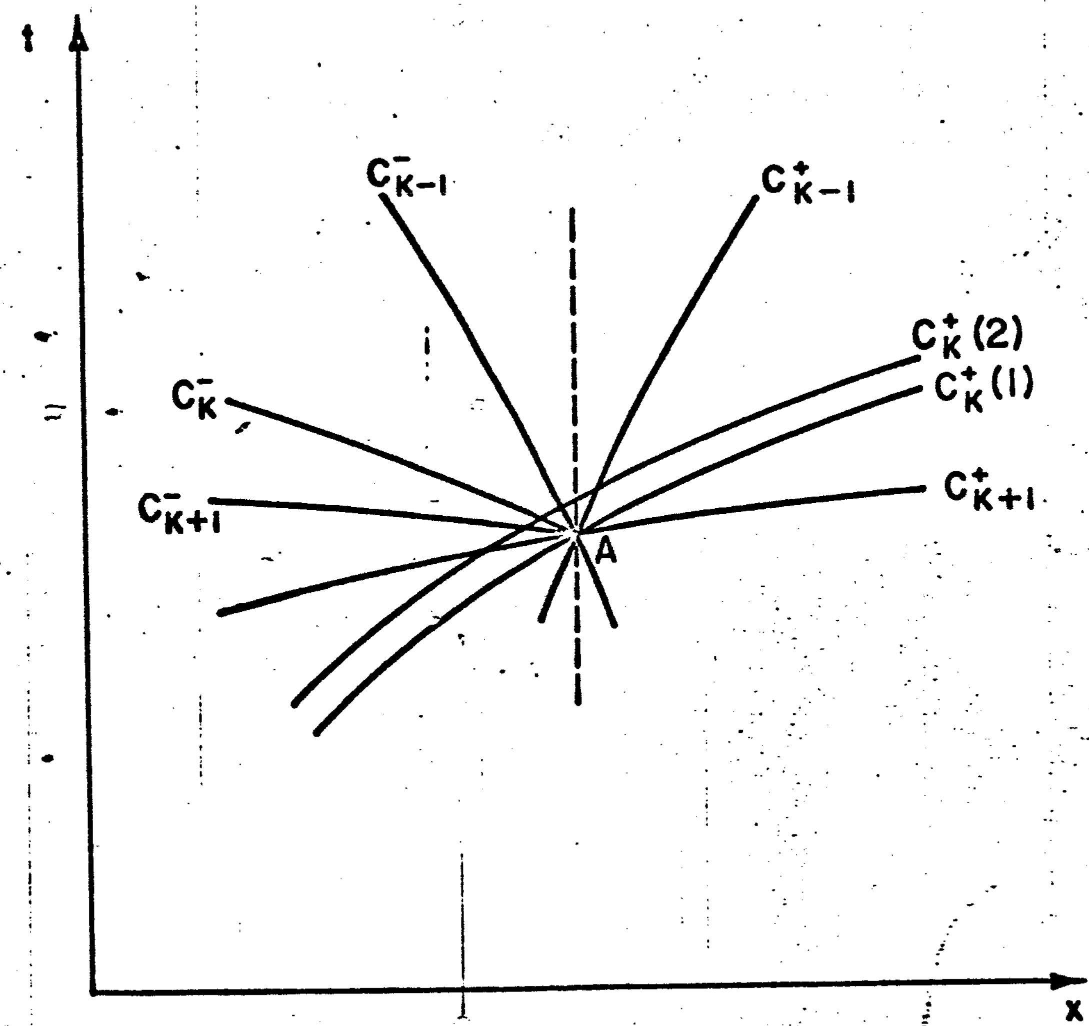
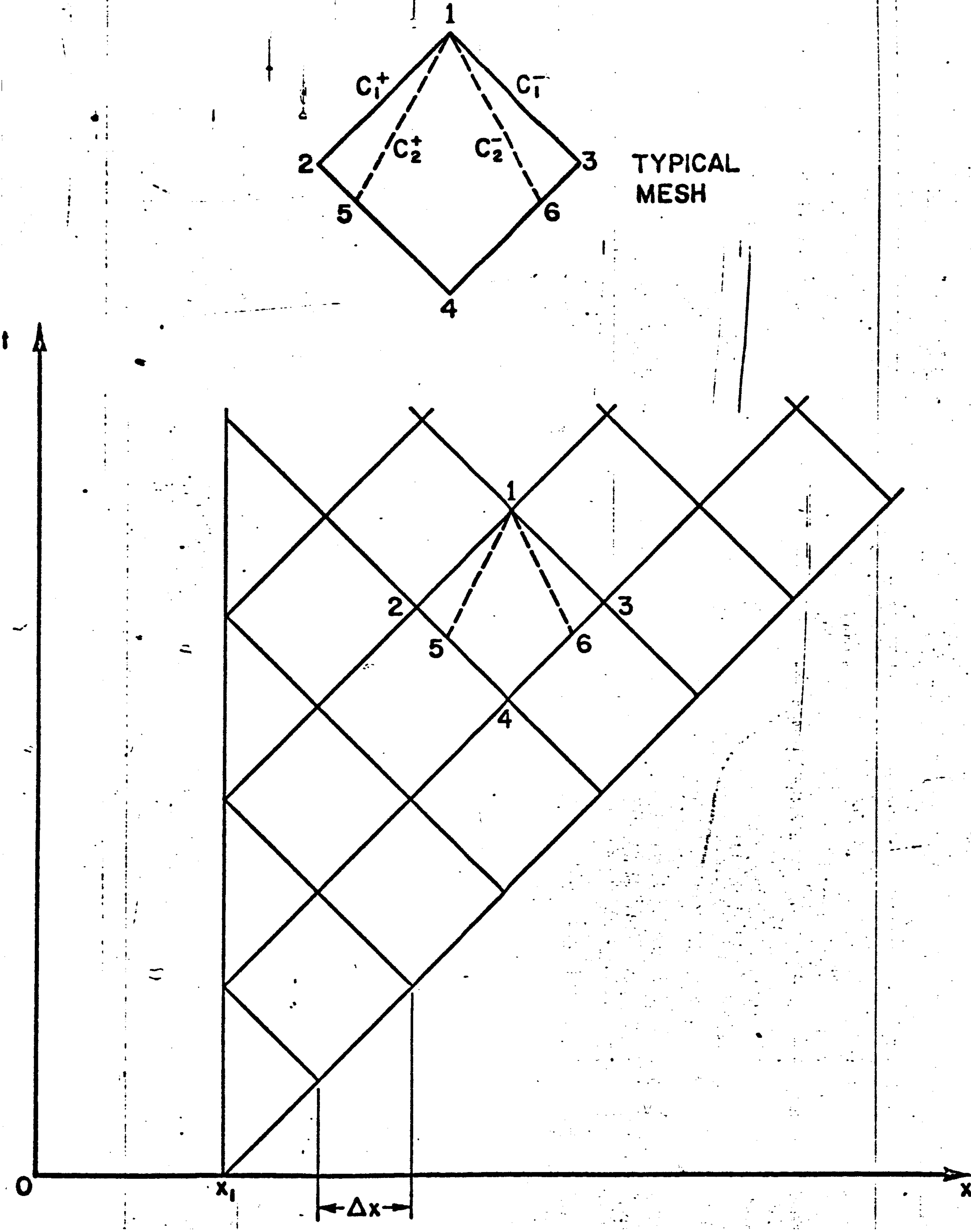
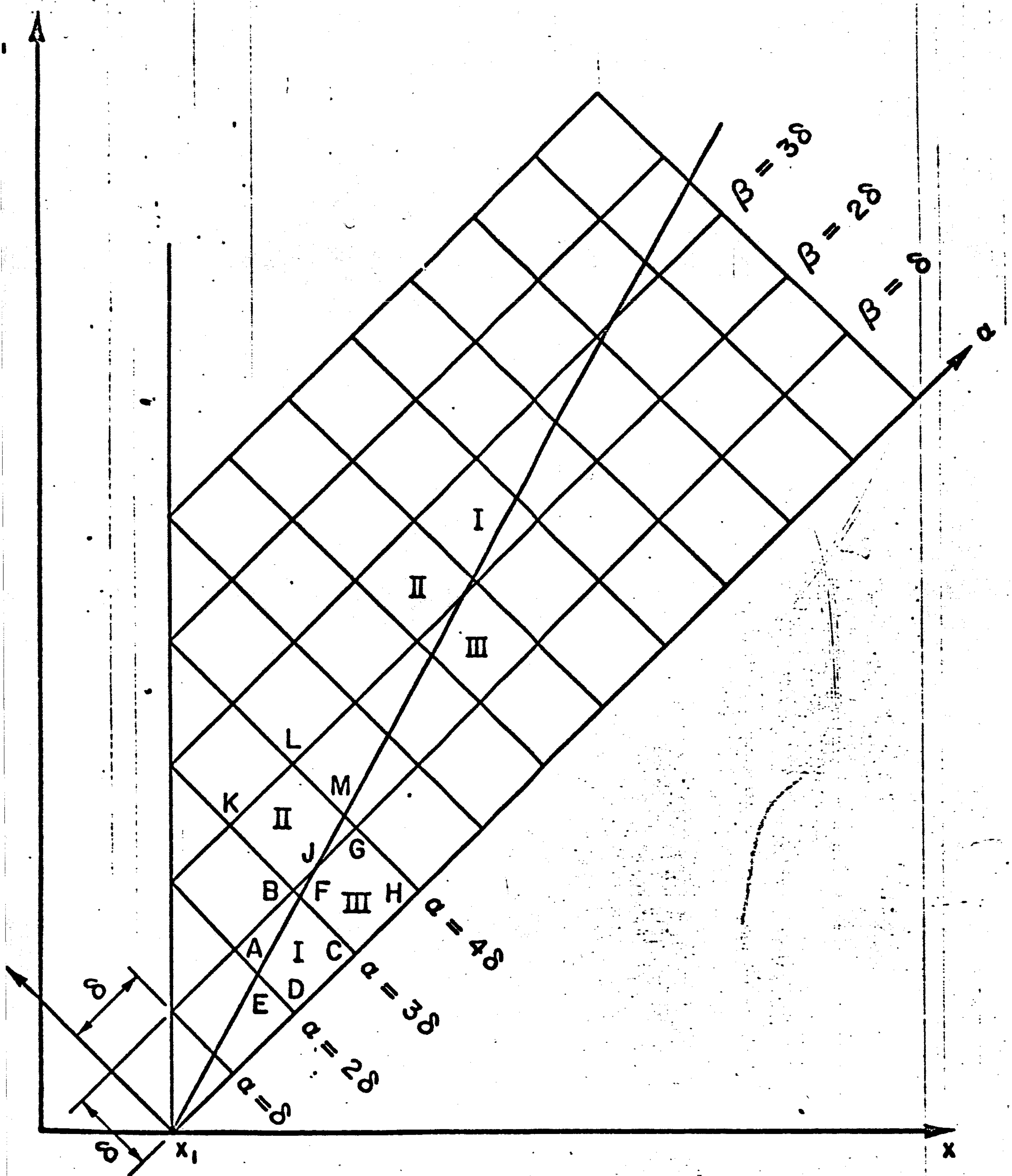


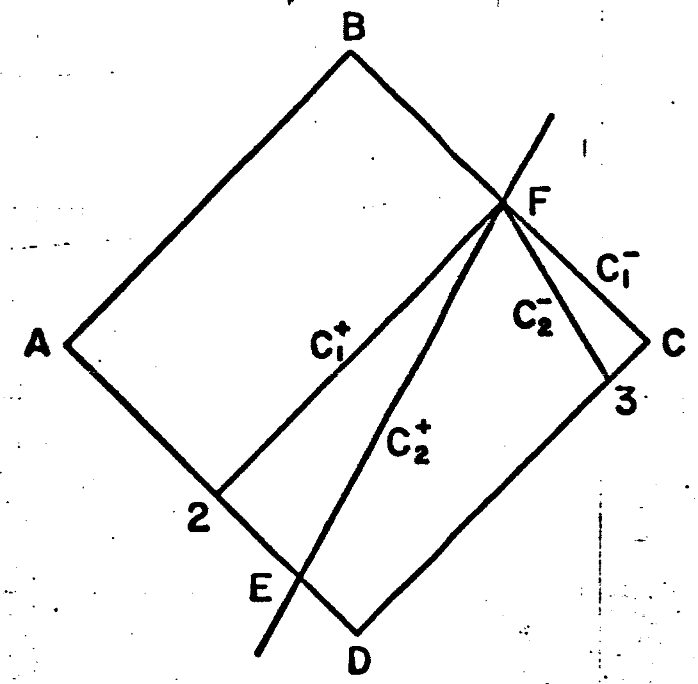
Figure 2 Propagation of Discontinuities Along a C^+_k Characteristic


1
C_1^+
C_1^-
C_2^+
C_2^-
2
3
5
6
4
TYPICAL MESH
t
1
2
3
5
6
4
O
x_1
Δx
x

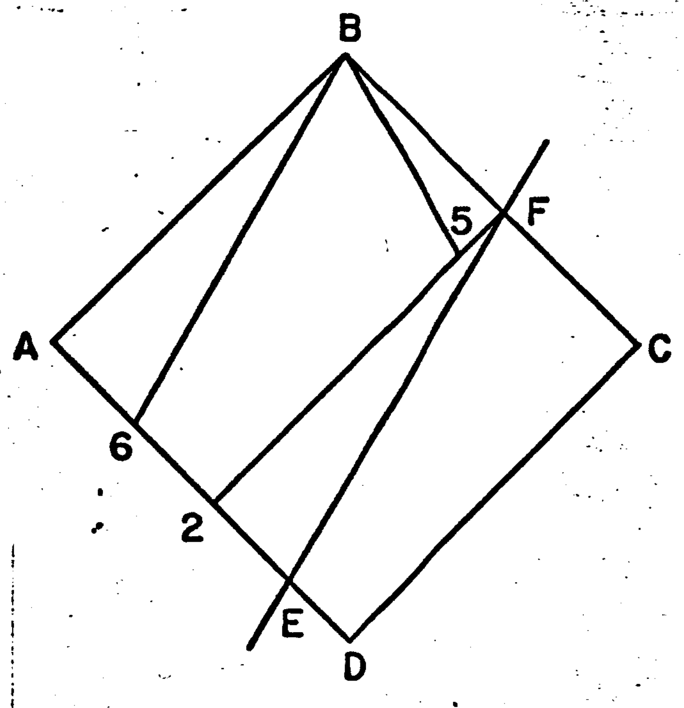


a. The Three Types of Nets

Figure 4 The "Jump Line" in the Physical Plane



b. Determination of Values at Point F



c. Determination of Values at Point B

Figure 4 The "Jump Line" in the Physical Plane

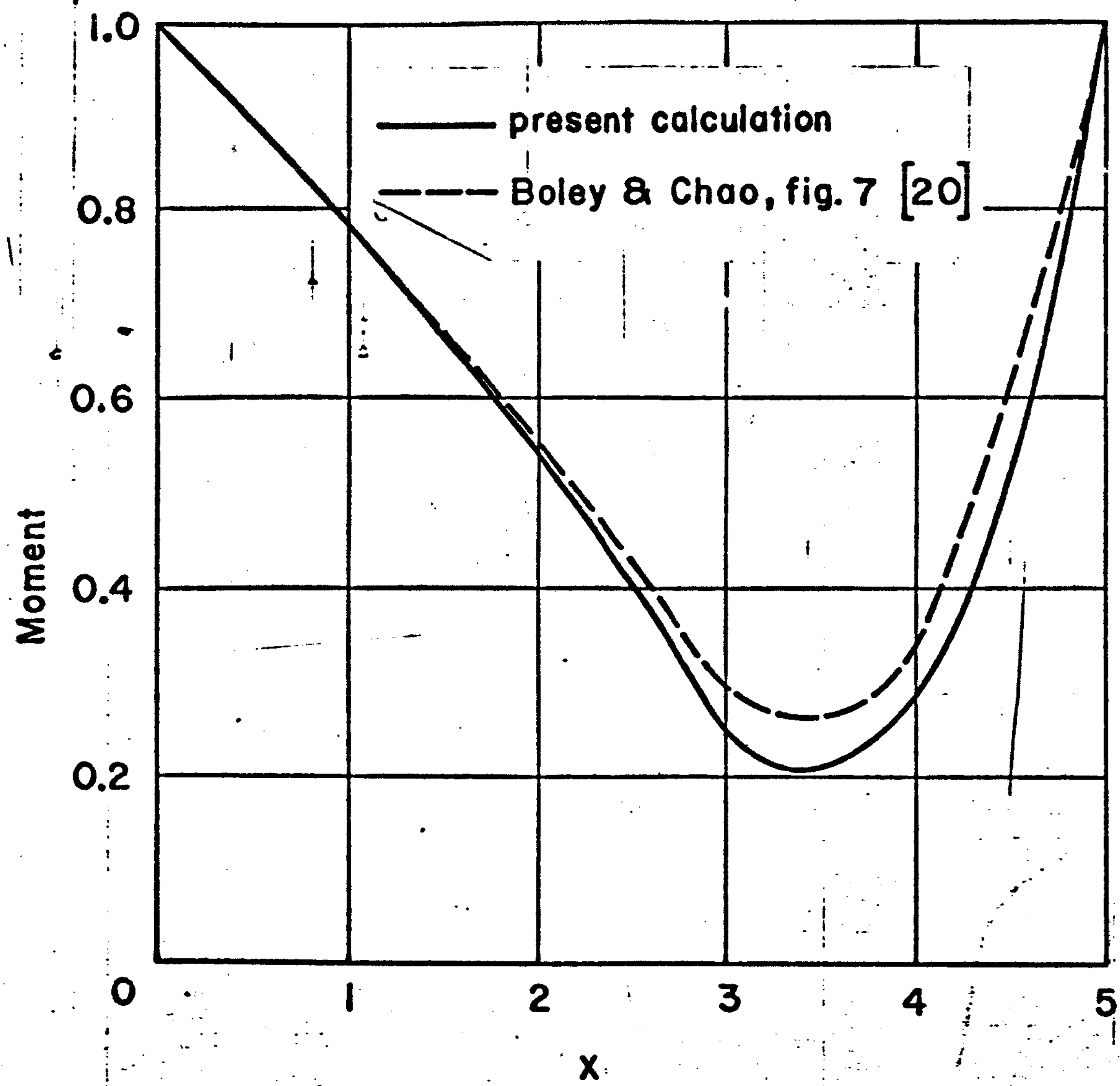
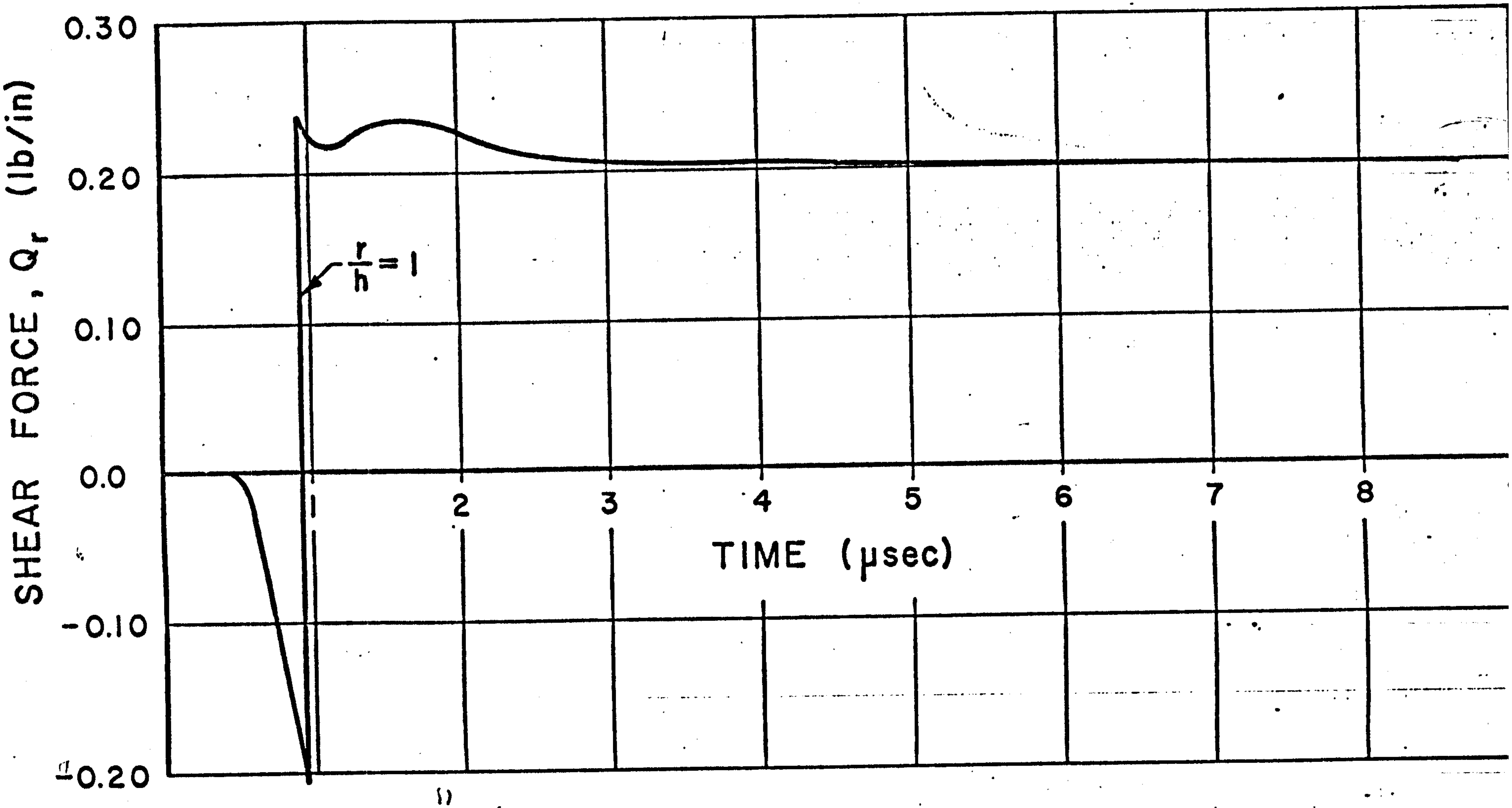
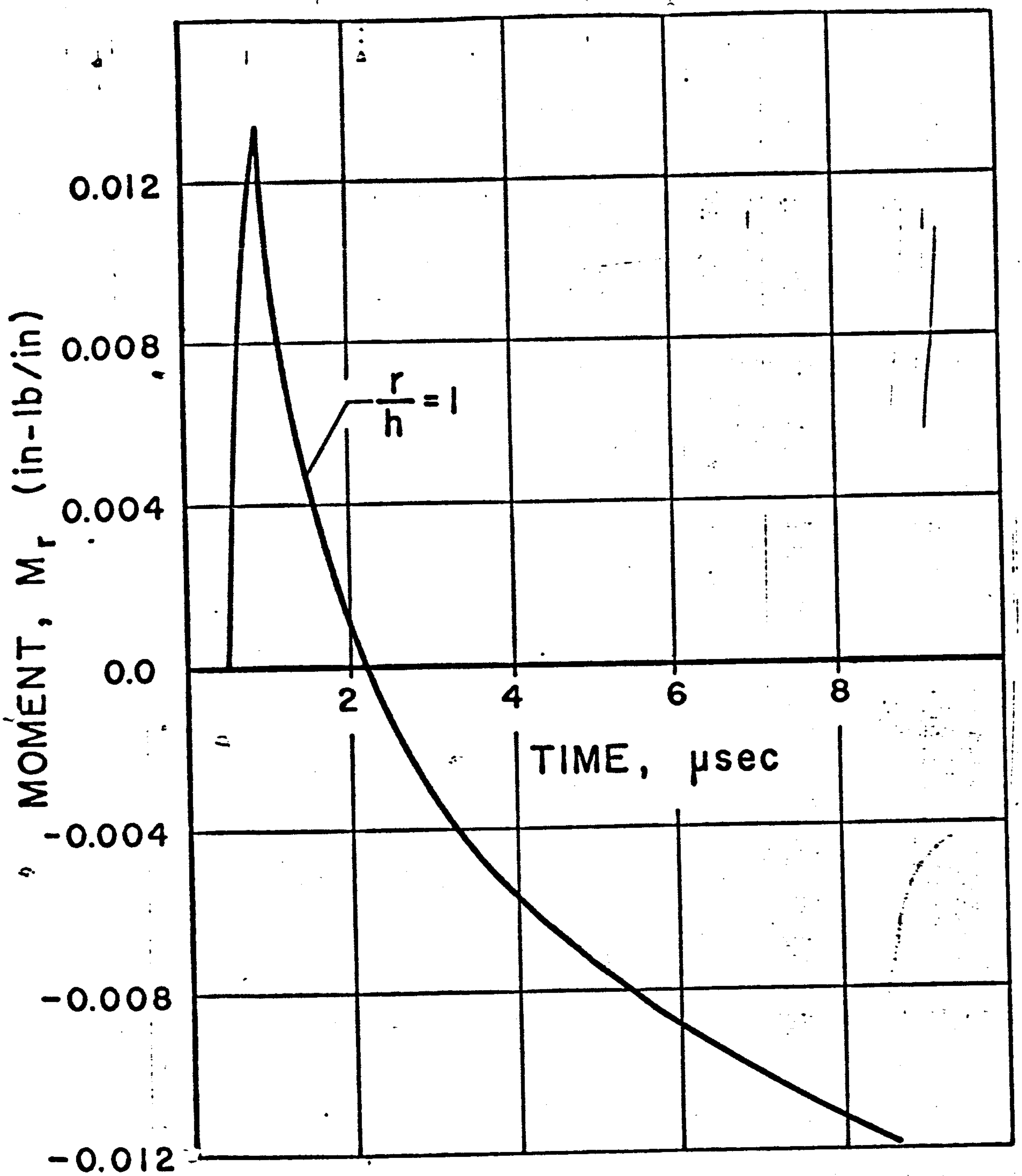


Figure 5 Comparison of Moment Distribution Given in [20] With Present Calculation, for a Beam Under a Unit Step Moment and Zero Displacement Loading at $x = 0$; $t = 5$



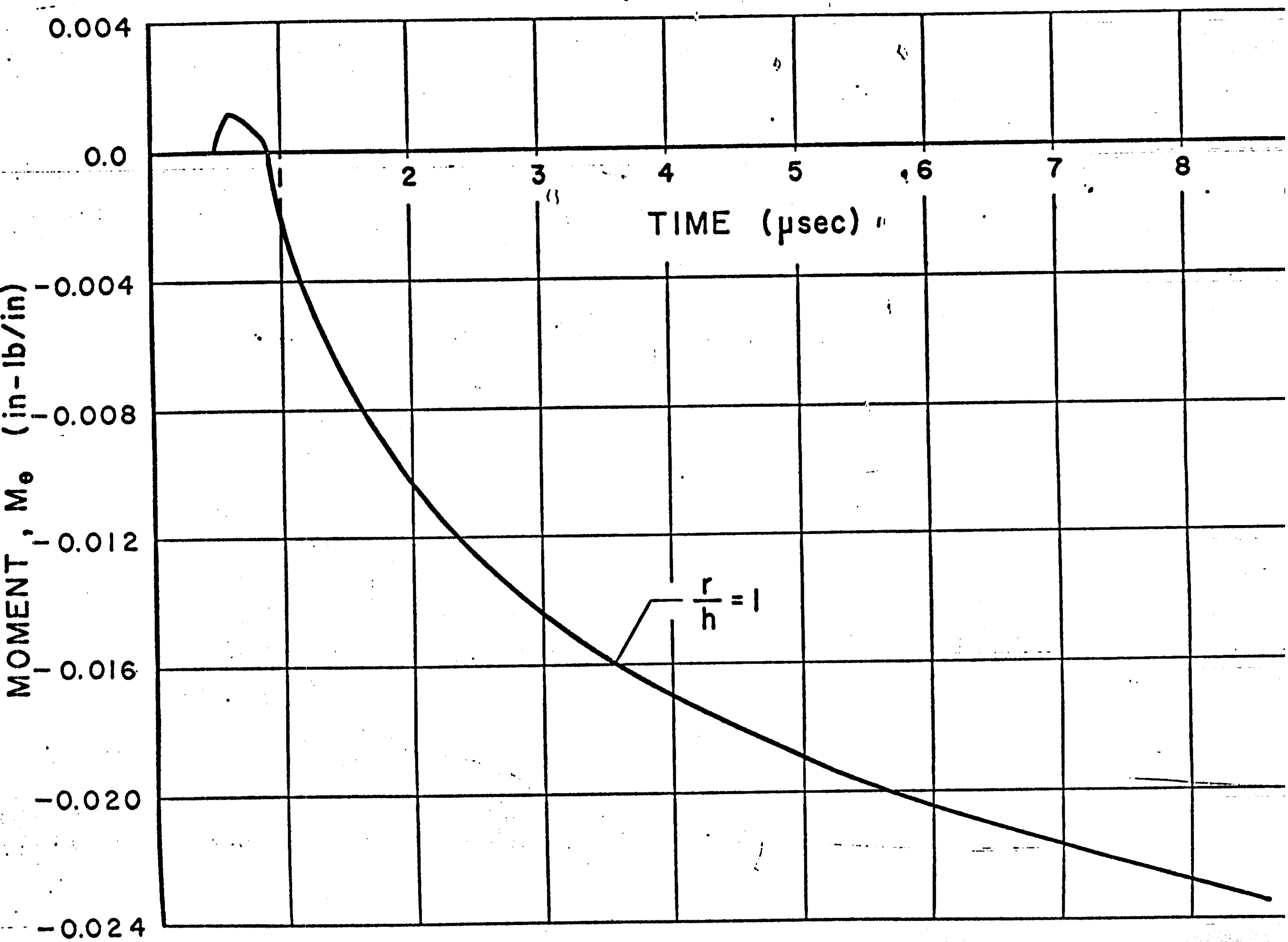
a. Shear Force Q_r Versus Time

Figure 6 Response of a Plate Under a Step Q_r, Zero Moment Input at $r/h = 0.2$; $E = 28 \times 10^6$ psi, $\nu = 0.3$, and $\rho = 7.41 \times 10^4$ lb-sec^2/in^4



b. Moment M_r Versus Time

Figure 6 Response of a Plate Under a Step Q_r, Zero Moment Input at $r/h = 0.2$; $E = 28 \times 10^6$ psi, $\nu = 0.3$, and $\rho = 7.41 \times 10^{-4}$ lb-sec^2/in^4


0.004
0.0
-0.004
-0.008
-0.012
-0.016
-0.020
-0.024
MOMENT, M_θ (in-lb/in)
1
2
3
4
5
6
7
8
TIME (μsec)
$\frac{r}{h} = 1$

OBLEM	CYLINDRICAL DILATATION (PLANE STRESS)	CYLINDRICAL DILATATION (PLANE STRAIN)	SPHERICAL DILATATION	SHEAR (ROTARY)	SHEAR (LONGITUDINAL)
ERENCE	[1]	[1]	[1]	[18]	[19]
u_1	u	u	u	v	w
c_1	c_p	c_d	c_d	c_e	c_e
α_{11}	$\frac{1}{r^2} - \frac{1}{\rho r c_p^2}\frac{d}{dr}\{\rho\nu c_p^2\}$	$\frac{1}{r^2} - \frac{1}{\rho r c_d^2}\frac{d}{dr}\{\frac{\rho\nu c_d^2}{1-\nu}\}$	$\frac{2}{r^2} - \frac{2}{\rho r c_d^2}\frac{d}{dr}\{\frac{\rho\nu c_d^2}{1-\nu}\}$	$\frac{1}{r^2} + \frac{1}{r}\frac{d}{dr}\{\ell n(\rho c_e^2)\}$	0
β_{11}	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(\rho c_p^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(\rho c_d^2)\}$	$-\frac{2}{r} - \frac{d}{dr}\{\ell n(\rho c_d^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(\rho c_e^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(\rho c_e^2)\}$
S_1	σ_r	σ_r	σ_r	$\tau_{r\theta}$	τ_{zr}
a_{11}	$\frac{E\nu}{r(1-\nu^2)}$	$\frac{\lambda}{r}$	$\frac{2\lambda}{r}$	$-\frac{G}{r}$	0
b_{11}	$\frac{E}{1-\nu^2}$	$\lambda + 2G$	$\lambda + 2G$	G	G

TABLE I

UNIFIED REPRESENTATION OF VARIOUS ELASTIC WAVE PROBLEMS WITH VARIABLE ELASTIC PROPERTIES; $n = 1$

(For notations, see eqs. (1) and (31))

α_{11}	$\frac{AGk^2}{EI}$	$\frac{hGk^2}{D}$	$\frac{1}{r^2} + \frac{k^2Gh}{D} - \frac{1}{rD}\frac{d(\nu D)}{dr}$	0	0	$\frac{1}{r^2} - \frac{1}{rc_d^2\rho h}\frac{d}{dr}(\frac{c_d^2\rho h\nu}{1-\nu})$
α_{12}	0	0	0	$-\frac{2}{a^2\rho c_d^2}\frac{d(a\lambda)}{dx}$	$-\frac{1}{\rho hc_d^2}\frac{d(K\lambda)}{dx}$	$-\frac{1}{c_d^2\rho h}\frac{d}{dr}(\frac{\rho K_1\nu c_d^2}{1-\nu})$
α_{21}	$\frac{d}{dx}\{\ell n(A\rho c_s^2)\}$	$-\frac{d}{dx}\{\ell n(h\rho c_s^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(h\rho c_s^2)\}$	0	0	$\frac{24K_1\nu}{h(1-2\nu)r}$
α_{22}	0	0	0	$\frac{8K_1^2(\lambda+G)}{a^2Gk^2}$	$\frac{3K^2}{h^2}(\frac{c_d}{c_e})^2$	$\frac{24K_1^2(1-\nu)}{h^2(1-2\nu)}$
β_{11}	$-\frac{d}{dx}(\ell nEI)$	$-\frac{d}{dx}(\ell nD)$	$-\frac{1}{r} - \frac{d}{dr}(\ell nD)$	$-\frac{d}{dx}\{\ell n(a^2\rho c_d^2)\}$	$-\frac{d}{dx}\{\ell n(h\rho c_d^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(h\rho c_d^2)\}$
β_{12}	$-\frac{AGk^2}{EI}$	$+\frac{hGk^2}{D}$	k^2Gh	$-\frac{2\lambda}{a\rho c_d^2}$	$-\frac{K\lambda}{h\rho c_d^2}$	$-\frac{\nu K_1}{h(1-\nu)}$
β_{21}	1	-1	-1	$\frac{4\lambda K_1^2}{aGk^2}$	$\frac{3\lambda K}{hG}$	$\frac{24K_1\nu}{h(1-2\nu)}$
β_{22}	$-\frac{d}{dx}\{\ell n(A\rho c_s^2)\}$	$-\frac{d}{dx}\{\ell n(h\rho c_s^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(h\rho c_s^2)\}$	$-\frac{d}{dx}\{\ell n(a^2\rho c_s^2)\}$	$-\frac{d}{dx}\{\ell n(h^2\rho c_e^2)\}$	$-\frac{1}{r} - \frac{d}{dr}\{\ell n(h^2\rho c_e^2)\}$
S_1	M	M_x	M_r	P_x	N_x	N_r
a_{11}	0	0	$\frac{D\nu}{r}$	0	0	$\frac{h\lambda}{r}$
a_{12}	0	0	0	$a\lambda$	$2K\lambda$	$K_1\lambda$
b_1	-EI	D	D	$\frac{a^2(\lambda+2G)}{2}$	$2h(\lambda+2G)$	$h(\lambda+2G)$
S_2	Q	Q_x	Q_r	Q	R_x	S_{rz}
a_{21}	$-k^2AG$	k^2hG	k^2Gh	0	0	0
b_2	$+k^2AG$	k^2hG	k^2Gh	$\frac{k^2a^2G}{4}$	$\frac{2}{3}h^2G$	$\frac{Gh^2}{12}$

TABLE II

UNIFIED REPRESENTATION OF VARIOUS ELASTIC WAVE PROBLEMS WITH VARIABLE ELASTIC PROPERTIES; $n = 2$

(For notations, see eqs. (1) and (31))

[redacted] LI[redacted]	[redacted] EA[redacted]	[redacted] TE [redacted] (PLANE)	[redacted] (CYLINDRICAL)	[redacted] R [redacted]	[redacted] SHE[redacted] (PLANE)	[redacted] ET [redacted] (CYLINDRICAL)
FERENCE	TIMOSHENKO [5]	MINDLIN [7]	CHOU AND KOENIG [4]	MINDLIN AND HERRMANN [8]	KANE AND MINDLIN [9]	JAHSMAN [22]
u_1 u_2	ψ y	ψ_x $\bar{w}$	ϕ w	w u	v_x v_z	u y
c_1 c_2	c_b c_s	c_p c_s	c_p c_s	c_d c_s	c_d c_e	c_d c_e
α_{11}	$\frac{AGk^2}{EI}$	$\frac{hGk^2}{D}$	$\frac{1}{r^2} + \frac{k^2Gh}{D} - \frac{1}{rD}\frac{d(\nu D)}{dr}$	0 $- \frac{2}{[illegible]}\frac{d(a\lambda)}{[illegible]}$	0 $- \frac{1}{[illegible]}\frac{d(K\lambda)}{dr}$	$\frac{1}{r^2} - \frac{1}{rc_d^2\rho h}\frac{d}{dr}(\frac{c_d^2\rho h\nu}{1-\nu})$ $- \frac{1}{[illegible]}\frac{d}{dr}(\frac{\rho K_1 \nu c_d^2}{1-\nu})$